Earnings and consumption dynamics: a nonlinear panel data framework

Arellano, Blundell, Bonhomme

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by Nicolò Russo
This paper

- Develops a nonlinear framework to study:
  - Nature of earnings persistence
  - Impact of earnings shocks on consumption

- Provides:
  - New method of studying earnings persistence and shocks
  - New way of estimating the earnings process
  - New evidence on impact of earnings shocks on consumption
Why we should study the earnings process

• This paper takes earnings as *exogenous*, like BPP (2008)

• Why studying the earnings process:

  1. Size and persistence of income shocks influence consumption decisions

  2. Persistence of earnings affects inequality and drives much of the variation in consumption

  3. Designing optimal social insurance and taxation
The literature so far

- **Focus on linear models**
  - Permanent/transitory (BPP); AR(1) (HSV)
  - Main characteristics of linear models:
    1. All shocks are associated to same persistence
    2. Identification using covariance techniques
    3. Nonlinear transmission of shocks ruled out

- **Approaches to consumption**
  1. Take stand on the consumption smoothing mechanisms and take a fully specified model to the data ⇒ Gourinchas and Parker (2002), Kaplan and Violante (2014)
  2. Estimate the degree of “partial insurance”, without specifying the insurance mechanisms ⇒ BPP (2008)
Why getting over linear models

- Empirical evidence in favor of richer earnings dynamics
- Consumption function in BPP (2008) comes from linear approximation of the Euler equation
  - Linear approximations not always accurate
  - Precautionary saving, asset accumulation with borrowing constraints, and nonlinear persistence ruled out
Empirical Results

Working families from the PSID (1999-2009):

1. Impact of earnings shocks varies across households’ earnings histories

2. Nonlinear persistence of earnings, where:
   - “Unusual” positive shocks for low earnings households
   - “Unusual” negative shocks for high earnings households
   are associated with lower persistence

3. Asymmetries in consumption responses to earnings shocks that hit households at different points of the income distribution

4. Similar empirical patterns in Norwegian administrative data
Roadmap for today

1. Introduction
2. Quantile Methods
3. Model
4. Identification
5. Estimation
6. Results
7. Life-cycle model
8. Conclusions
A crash course on the quantile function

• Given a probability $p$, the quantile function for a random variable $Y$ $Q_p(Y)$ gives the values $y$ such that:

$$Q_p(Y) = \inf\{y \in \mathbb{R} : p \leq F(y)\}, \quad \text{for } p \in (0, 1)$$

• cdf $F(\cdot)$ continuous and monotonically increasing: $Q = F^{-1}$

• Conditional quantile function at a quantile $p$ for a random variable $Y$ given $X$ is:

$$Q_p(Y|X) = \inf\{y \in \mathbb{R} : p \leq F(y|x)\}$$
Quantile regression

- Models quantiles in the distribution of $Y$ given $X$
  - e.g. estimate how the 1st and 3rd quartiles in the distribution of $Y|X$ change with $X$

- Object of interest: conditional quantiles, not mean

- Described by:
  
  \[ y_i = x_i' \beta_q + \varepsilon_i \]

  where different choices of $q$ yield different estimated $\beta$

- Yields one vector of coefficients for every quantile analyzed

- Interpretation depends on which quantile coefficients refer to
Earnings process

“Detrended” log pre-tax labor earnings:

\[ y_{i,t} = \eta_{i,t} + \varepsilon_{i,t}, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T \]

where:

- \( \eta_{it} := \text{persistent component:} \)

\[ \eta_{it} = Q_t(\eta_{i,t-1}, u_{i,t}), \quad (u_{i,t} | \eta_{i,t-1}, \ldots) \sim U(0, 1), \quad t = 2, \ldots, T \]

where:

- \( Q_t(\eta_{i,t-1}, \tau) := \tau\)-th quantile of \( \eta_{i,t} | \eta_{i,t-1} \forall \tau \in (0, 1) \)

- \( \varepsilon_{it} := \text{transitory component:} \)

  - Zero mean, serially uncorrelated, independent of \( \eta \)

- \textbf{Special case: canonical model of earnings dynamics}

\[ y_{i,t} = \eta_{i,t} + \varepsilon_{i,t}, \quad \eta_{i,t} = \eta_{i,t-1} + \nu_{i,t} \]
Nonlinear persistence

• Quantile model allows for nonlinear dynamics of earnings

• We are interested in nonlinear persistence

• Measures of persistence of the $\eta$ component:

$$\rho_t(\eta_{i,t-1}, \tau) = \frac{\partial Q_t(\eta_{i,t-1}, \tau)}{\partial \eta}, \quad \rho_t(\tau) = \mathbb{E} \left[ \frac{\partial Q_t(\eta_{i,t-1}, \tau)}{\partial \eta} \right]$$

• **Canonical model:** $\rho_t(\eta_{i,t-1}, \tau) = 1$ always

• **Quantile model:** depends on magnitude and direction of $u_{i,t}$
Persistence of earnings in the canonical earnings model

**Figure 2.** Nonlinear persistence. Note: Graphs (a), (b), and (c) show estimates of the average derivative of the conditional quantile function of $y_{it}$ given $y_{i,t-1}$ with respect to $y_{i,t-1}$, evaluated at percentile $\tau$ shock and at a value of $y_{i,t-1}$ that corresponds to the $\tau$ init percentile of the distribution of $y_{i,t-1}$. Graph (a) is based on the PSID data, graph (b) is based on data simulated according to our nonlinear earnings model with parameters set to their estimated values, and graph (c) is based on data simulated according to the canonical random-walk earnings model. Graph (d) shows estimates of the average derivative of the conditional quantile function of $\eta_{it}$ on $\eta_{i,t-1}$ with respect to $\eta_{i,t-1}$, based on estimates from the nonlinear earnings model.

Non-normal and presents high kurtosis and fat tails. These results are qualitatively consistent with empirical estimates of non-Gaussian linear models in Horowitz and Markatou (1996) and Bonhomme and Robin (2010).

In Figure 4, we report the measure of conditional skewness in (6), for $\tau = 11/12$, for both log-earnings residuals (left graph) and the $\eta$ component (right). Panel (b) shows that $\eta_{it}$ is positively skewed for low values of $\eta_{i,t-1}$, and negatively skewed for high values of $\eta_{i,t-1}$. This is nonlinear persistence reported in Figure 2 (d): when low-$\eta$ households are hit by an unusually positive shock, dependence of $\eta_{it}$ on $\eta_{i,t-1}$ is low with the result that they have a relatively large probability of outcomes far to the right from the central part of the distribution. Likewise, high-$\eta$ households have a relatively large probability of getting outcomes far to the left of their distribution associated with low...
Evidence of nonlinear persistence of earnings

(a) PSID data

(b) Norwegian administrative data
Consumption rule in a simple life-cycle model

- Households choose consumption and savings subject to:

\[ A_{i,t} = (1 + r)A_{i,t-1} + Y_{i,t-1} - C_{i,t-1} \]

- Family log-earnings are given by:

\[ \log Y_{i,t} = \kappa_t + \eta_{i,t} + \varepsilon_{i,t} \]

- No advance information, no aggregate uncertainty, agents know all distributions.

- Bellman equation in each period:

\[ V_t(A_{i,t}, \eta_{i,t}, \varepsilon_{i,t}) = \max u(C_{i,t}) + \beta \mathbb{E}_t[V_{t+1}(A_{i,t+1}, \eta_{i,t+1}, \varepsilon_{i,t+1})] \]
Consumption rule in a simple life-cycle model

- Consumption policy rule:

\[ C_{i,t} = G_t(A_{i,t}, \eta_{i,t}, \varepsilon_{i,t}) \]

for some age-dependent function \( G_t \)

- Possible approaches:

1. Build a fully structural model and calibrate or estimate it via MSM \( \Rightarrow \) Gourinchas and Parker (2002)

2. Linearize the Euler equation and then use standard covariance based methods \( \Rightarrow \) BPP (2008)

3. Directly estimate the nonlinear consumption rule \( \Rightarrow \) This paper!
Empirical consumption rule

• Log-consumption net of age dummies:

\[ c_{i,t} = g_t(\mathbf{a}_{i,t}, \eta_{i,t}, \varepsilon_{i,t}, \nu_{i,t}) \]

\[ \nu_{i,t} := \text{unobserved arguments of the consumption function.} \]

• Assets:

\[ a_{i,t} = h_t(\mathbf{a}_{i,t-1}, \mathbf{c}_{i,t-1}, y_{i,t-1}, \eta_{i,t-1}, \nu_{i,t}) \]

\[ \nu_{i,t} := \text{i.i.d. and independent of the other arguments of } h \]
Derivative effects

• Average consumption for given assets and earnings components:

$$
\mathbb{E}[c_{i,t} \mid a_{i,t} = a, \eta_{i,t} = \eta, \varepsilon_{i,t} = \varepsilon] = \mathbb{E}[g_t(a, \eta, \varepsilon, \nu_{i,t})]
$$

• Average derivative of consumption with respect to $\eta$:

$$
\phi_t(a, \eta, \varepsilon) = \mathbb{E} \left[ \frac{\partial g_t(a, \eta, \varepsilon, \nu_{i,t})}{\partial \eta} \right]
$$

• Average derivative effect:

$$
\bar{\phi}_t(a) = \mathbb{E}[\phi_t(a, \eta, \varepsilon)]
$$

• $1 - \bar{\phi}_t(a)$: degree of consumption insurance to shocks to persistent earnings component
Identification overview and references

- We have nonlinear models with latent state variables.
- A series of papers has established conditions under which nonlinear models with latent variables are nonparametrically identified under conditional independence restrictions:
  2. D’Hautfoeuille (2011)
  3. Hu and Shum (2012)
  5. Arellano and Bonhomme (2016)
- This section covers the gist of the identification strategy, but more details are provided in the paper and in the Supplemental Material available online.
Earnings process

• Assume that the data contain $T$ consecutive periods.
• The goal is to identify the joint distributions of $(\eta_i, 1, \ldots, \eta_i, T)$ and $(\varepsilon_i, 1, \ldots, \varepsilon_i, T)$ given data from $(y_i, 1, \ldots, y_i, T)$
• These are identified under conditions which follow directly from Hu and Schennach (2008) and Wilhelm (2015).
• These conditions rely on the distributions of $(y_i, t | y_i, t-1)$ and $(\eta_i, t | y_i, t-1)$ being complete.
  • The distribution of $(y_i, t | y_i, t-1)$ is complete if the only function $h$ satisfying $\mathbb{E}[h(y_i, t | y_i, t-1)] = 0$ is $h = 0$.
• Completeness is a common assumption in nonparametric instrumental variables problems.
Consumption rule without unobserved heterogeneity

- For a generic variable $z$, let $z^t_i = (z_{i1}, \ldots, z_{it})$. Then make the following assumption:

**ASSUMPTION 1:** For all $t \geq 1$:

1. $u_{i,t+s}$ and $\varepsilon_{i,t+s}$, for all $s \geq 0$, are independent of $a^t_i$, $\eta^{t-1}_i$, and $y^{t-1}_i$. $\varepsilon_{i1}$ is independent of $a_{i1}$ and $\eta_{i1}$.
2. $a_{i,t+1}$ is independent of $(a^{t-1}_i, c^{t-1}_i, y^{t-1}_i, \eta^{t-1}_i)$ conditional on $(a_{it}, c_{it}, y_{it}, \eta_{it})$.
3. The taste shifter $\nu_{it}$ is independent of $\eta_{i1}$, $(u_{is}, \varepsilon_{is})$ for all $s$, $\nu_{is}$ for all $s \neq t$, and $a^t_i$.

- The identification argument proceeds in a sequential way:
  1. Start with the first period.
  2. Proceed to the second period using the assets.
  3. Using the second period consumption move to subsequent periods and use induction.
Consumption rule - First period

- Let $y_i = (y_{i1}, \ldots, y_{iT})$ denote the whole history of earnings for agent $i$ and $f$ denote a density function.

- Using Assumption 1.1 we have that

$$f(a_1|y) = \mathbb{E}[f(a_1|\eta_{i1})|y_i = y] \quad (1)$$

- Given that distribution of $(\eta_{i1}|y_i)$ is complete, $f(a_1|\eta_{i1})$ is identified from (1).

- Using the consumption rule and Assumption 1.3 we have that:

$$f(c_1|a_1, y) = \mathbb{E}[f(c_1|a_{i1}, \eta_{i1}, y_{i1})|a_{i1} = a_1, y_i = y] \quad (2)$$

- Given that the distribution of $(\eta_{i1}|a_{i1}, y_i)$ is complete, $f(c_1|a_1, \eta_{i1}, y_{i1})$ and $f(c_1, \eta_{i1}|a_1, y)$ are identified from (2).

- Identification of the consumption function for $t = 1$ follows.
Consumption rule - Second period

- Using Assumption 1.1 and 1.3, we have that:

$$f(a_2|a_1, c_1, y) = \int f(a_2|a_1, c_1, \eta_1, y_1)f(\eta_1|a_1, c_1, y)d\eta_1 \quad (3)$$

- Given that the distribution of $(\eta_{i1}|c_{i1}, a_{i1}, y_i)$ is complete, $f(a_2|a_1, c_1, \eta_1, y_1)$ is identified from (3).

- Using Bayes’ rule and Assumption 1.1 and 1.3 we have that:

$$f(\eta_2|a_1, a_2, c_1, y) = \int \frac{f(y|\eta_1, \eta_2, y_1)f(\eta_1, \eta_2|a_1, a_2, c_1, y_1)}{f(y|a_1, a_2, c_1, y)}d\eta_1$$

- As the density $f(\eta_1|a_1, a_2, c_1, y)$ is identified from above, and, by Assumption 1, we have that:

$$f(\eta_1, \eta_2|a_1, a_2, c_1, y_1) = f(\eta_1|a_1, a_2, c_1, y_1)f(\eta_2|\eta_1)$$

it follows that $f(\eta_2|a_1, a_2, c_1, y)$ is identified.
Consumption rule - Subsequent periods

• Consider second period’s consumption. Using Assumption 1.3 we have that:

\[ f(c_2|a_1, a_2, c_1, y) = \int f(c_2|a_2, \eta_2, y_2)f(\eta_2|a_1, a_2, c_1, y)d\eta_2 \]

(4)

• Given that the distribution of \((\eta_{i2}|a_{i1}, a_{i2}, c_{i1}, y_i)\) is complete, \(f(c_2|a_2, \eta_2, y_2)\) is identified from (4).

• By induction, using in addition Assumption 1 from the third period onward, the joint density of \(\eta\)'s, consumption, assets, and earnings is identified, provided, for all \(t \geq 1\), the distributions of \((\eta_{it}|c_i^t, a_i^t, y_i)\) and \((\eta_{it}|c_i^{t-1}, a_i^t, y_i)\) are complete in \((c_i^{t-1}, a_i^{t-1}, y_i^{t-1}, y_i, t+1, \ldots, y_i T)\).
Overview of estimation

• Estimate jointly quantile regressions of:
  1. Markovian transitions $Q$
  2. Transitory component $\varepsilon$
  3. Initial persistent component $\eta_1$

• Use these to estimate:
  1. Consumption $c$
  2. First period assets $a_1$
  3. Evolution of assets
Empirical specification - Earnings

- Quantile function of $\varepsilon_{it}$ (for $t = 1, \ldots, T$) given $age_{it}$:

$$Q_{\varepsilon}(age_{it}, \tau) = \sum_{k=0}^{K} a_{k}(\tau)\varphi_{k}(age_{it})$$

where:

- $\varphi_{k}$ for $k = 0, 1, \ldots$ := polynomial
- In practice, they will use Hermite polynomials
- $a_{k}(\varepsilon) :=$ scalars that will be estimated

- In practice:

$$Q_{\varepsilon}(age_{it}, \tau) = a_{1}(\tau) + a_{2}(\tau)age_{it} + a_{3}(\tau)(age_{it}^{2} - 1) + a_{4}(\tau)(age_{it}^{3} - 3age_{it}) + \ldots$$
Empirical specification - Earnings

- Quantile function of $\eta_{i1}$ given $age_{i1}$:

$$Q_{\eta_1}(age_{i1}, \tau) = \sum_{k=0}^{K} a_{\eta_1}^k(\tau) \varphi_k(age_{i1})$$

- Markovian transitions of persistent component:

$$Q_t(\eta_{i,t-1}, \tau) = Q(\eta_{i,t-1}, age_{it}, \tau) = \sum_{k=0}^{K} a_{Q}^k(\tau) \varphi_k(\eta_{i,t-1}, age_{it})$$

- In practice:

$$Q_t(\eta_{i,t-1}, \tau) = a_1(\tau) + a_2(\tau)\eta_{i,t-1} + a_3(\tau)age_{it} + a_4(\tau)\eta_{i,t-1} age_{it} + a_5(\tau)(\eta_{i,t-1}^2 - 1)age_{it} + a_6(\tau)\eta_{i,t-1}(age_{it}^2 - 1) + \ldots$$
Empirical specification - Consumption rule

- Empirical specification for consumption:

\[
c_{i,t} = g_t(a_{i,t}, \eta_{i,t}, \varepsilon_{i,t}, \nu_{i,t})
\]

- Conditional distribution of consumption given assets and earnings components:

\[
g_t(a_{it}, \eta_{it}, \varepsilon_{it}, \tau) = g(a_{it}, \eta_{it}, \varepsilon_{it}, \text{age}_{it}, \tau) = \sum_{k=1}^{K} b_{g_k} \tilde{\varphi}_k(a_{it}, \eta_{it}, \varepsilon_{it}, \text{age}_{it}) + b_{g_0}^g(\tau)
\]

where:

- \(b_{g_k}\) and \(b_{g_0}^g(\tau)\) are scalars to be estimated
- \(\tilde{\varphi}_k\) is a product of Hermite polynomials
Empirical specification - Assets evolution

- Distribution of initial assets $a_{i1}$ conditional on $\eta_{i1}$ and $age_{i1}$:

$$Q_a(\eta_{i1}, age_{i1}, \tau) = \sum_{k=0}^{K} b^a_k(\tau) \tilde{\varphi}_k(\eta_{i1}, age_{i1})$$

- Evolution of assets:

$$a_{i,t} = h_t(a_{i,t-1}, c_{i,t-1}, y_{i,t-1}, \eta_{i,t-1}, v_{i,t})$$

- Assets evolution is specified as:

$$h_t(a_{i,t-1}, c_{i,t-1}, y_{i,t-1}, \eta_{i,t-1}, \tau) = h(a_{i,t-1}, c_{i,t-1}, y_{i,t-1}, \eta_{i,t-1}, age_{i,t}, \tau)$$

$$= \sum_{k=1}^{K} b^h_k \tilde{\varphi}_k(a_{i,t-1}, c_{i,t-1}, y_{i,t-1}, \eta_{i,t-1}, age_{i,t}) + b^0_k(\tau)$$
Overview of the estimation algorithm

• Adaptation of the techniques developed in Arellano and Bonhomme (2016) to a setting with time-varying latent variables

• Sequential algorithm:
  1. Recover estimates of the earnings parameters $a^Q_k, a^\varepsilon_k, a^\eta_1_k$.
  2. Given the estimates of $a^Q_k, a^\varepsilon_k, a^\eta_1_k$, recover the consumption and asset parameters $b^g_0, b^h_0, b^a_k$ and $b^g_1, \ldots, b^g_K$ and $b^h_1, \ldots, b^h_K$.

• Parameters not estimated jointly, because $a^Q_k, a^\varepsilon_k, a^\eta_1_k$ are identified from the earnings process alone

• Closely related to the “Stochastic EM” algorithm (see Celeux and Diebolt (1993)), but based on quantile regression rather than on maximum likelihood
Data

- PSID for 1999-2009

- $Y_{it}$ total pre-tax household labor earnings. $y_{it}$ residual of a regression of log $Y_{it}$ on demographics

- $C_{it}$ consumption of nondurables and services. $c_{it}$ residual of a regression of log $C_{it}$ on same demographics

- $A_{it}$ sum of financial and non-financial assets, net of mortgages and debt. $a_{it}$ as residual of regression of log $A_{it}$ on same demographics

- Sample selection from Blundell, Pistaferri, and Saporta-Eksten (2016)
Persistence of earnings

(a) Earnings, PSID data
(b) Earnings, nonlinear model
(c) Earnings, canonical model
Persistence of $\eta$ - Simulated Data

In Figure 2, we report the measure of conditional skewness in (6), for $\tau = \frac{11}{12}$, for both log-earnings residuals (left graph) and the $\eta$ component (right). Panel (b) shows that $\eta$ is positively skewed for low values of $\eta_{t-1}$, and negatively skewed for high values of $\eta_{t-1}$. This is in line with the nonlinear persistence reported in Figure 2(d): when low-$\eta$ households are hit by an unusually positive shock, dependence of $\eta_t$ on $\eta_{t-1}$ is low with the result that they have a relatively large probability of outcomes far to the right from the central part of the distribution. Likewise, high-$\eta$ households have a relatively large probability of getting outcomes far to the left of their distribution associated with low...
Norwegian population register data

- In order to corroborate their findings using a different and larger data set, they use Norwegian administrative data.
- They consider a balanced sample of 2873 households in the 2000-2005 period.
  - Male, non-immigrant, residents between the age 30 and 60 and their spouses.
  - Continuously married males, with household disposable income above the threshold of substantial gainful activity ($14,000 in 2014).
- Part of Blundell, Graber, Mogstad (2015)
Norwegian population register data results

(a) Earnings, Norwegian data

(b) Earnings, nonlinear model

(c) Persistent component $\eta_{st}$, nonlinear model
Consumption response to $\eta$ - Simulated Data

![Graph showing consumption response to earnings shocks, by assets and age, model without household-specific unobserved heterogeneity.](image)

Note: Graphs (a) and (b) show estimates of the derivative of the conditional mean of $c_{it}$ with respect to $y_{it}$, given $y_{it}$, $a_{it}$, and age $i_t$, evaluated at values corresponding to their $\tau$-assets and $\tau$-age percentiles, and averaged over the values of $y_{it}$. Graph (a) is based on the PSID data, and graph (b) is based on data simulated according to our nonlinear model with parameters set to their estimated values. Graph (c) shows estimates of the average consumption responses $\phi_t(a)$ to variations in $\eta_{it}$, evaluated at $\tau$-assets and $\tau$-age.

Effects lie between 0.2 and 0.3. Moreover, the results indicate that consumption of older households, and of households with higher assets, is less correlated to variations in earnings. Figure 5(b) shows the same response surface based on simulated data from our full nonlinear model of earnings and consumption. The fit of the model, though not perfect, seems reasonable. In particular, the model reproduces the main pattern of correlation with age and assets.

In Figures S22 and S23 of the Supplemental Material, we report 95% confidence bands for $\phi_t(a)$ based on both parametric bootstrap and nonparametric bootstrap. The findings on insurability of shocks to the persistent earnings component seem quite precisely estimated. In Figure S24 of the Supplemental Material, we report estimates of the model with household unobserved heterogeneity in consumption; see (19). Estimated consumption insurability parameter lies between 0.3 and 0.4, suggesting that more than half of pre-tax household earnings fluctuations is effectively insured. Moreover, variation in assets and age suggests the presence of an interaction effect. In particular, older households with high assets seem better insured against earnings fluctuations.

In Figure S20 of the Supplemental Material, we show that the model fit to the density of log-consumption is also good. While the covariances between log-earnings and log-consumption residuals are well reproduced, the baseline model does not perform as well in fitting the dynamics of consumption, as it systematically underestimates the autocorrelations between log-consumption residuals. The specification with household unobserved heterogeneity improves the fit to consumption dynamics.

Consumption responses to transitory shocks are shown in Figure S21 of the Supplemental Material.
Consumption response to assets - Simulated Data

Figure 6.—Consumption responses to assets. Note: Estimate soft heaveraged derivative of the conditional mean of \( c_{it} \), with respect to \( a_{it} \), given \( y_{it} \) (respectively, given \( \eta_{it} \) and \( \epsilon_{it} \) in graph (c)), \( a_{it} \), and age \( it \), evaluated at values of \( a_{it} \) and age \( it \) that correspond to their \( \tau \) assets and \( \tau \) age percentiles, and averaged over the values of \( y_{it} \) (resp., over the values of \( \eta_{it} \) and \( \epsilon_{it} \) in graph (c)). Model without unobserved heterogeneity.

Consumption responses are quite similar to the ones without unobserved heterogeneity, although the nonlinearity with respect to assets and age seems more pronounced.

Consumption Responses to Assets. In addition to consumption responses to earnings shocks, our nonlinear framework can be used to document derivative effects with respect to assets. Such quantities are often of great interest, for example when studying the implications of tax reforms. Graph (c) in Figure 6 shows estimated average derivatives, in a model without unobserved heterogeneity.

The quantile polynomial specifications are the same as in Figure 5. We see that the responses range between 0.05 and 0.2, and that the derivative effects seem to increase with age and assets.

6.4. Simulating the Impact of Persistent Earnings Shocks

In this last subsection, we simulate life-cycle earnings and consumption according to our nonlinear model, and show the evolution of earnings and consumption following a persistent earnings shock. In Figure 7, we report the differences between age-specific medians of log-earnings of two types of households: households that are hit, at the same age 37, by either a large negative shock to the persistent earnings component (\( \tau \) shock = 0/period or 10), or by a large positive shock (\( \tau \) shock = 0/period or 90), and households that are hit by a median shock \( \tau \) = 0/period to the persistent component. We report age-specific medians across 100,000 simulations of the model. At the start of the simulation (i.e., age 35), all households have the same persistent component indicated by the percentile \( \tau \) init. With some abuse of terminology, we refer to the resulting earnings and consumption paths as "impulse responses."
Impact of persistent earnings shocks on earnings - Canonical model

\[(g) \tau_{\text{shock}} = .1\]

\[(h) \tau_{\text{shock}} = .9\]
Impact of persistent earnings shocks on earnings - Nonlinear model

Earnings responses change, based on the rank of the household in the income distribution and the magnitude of the earnings shock.
Impact of persistent earnings shocks on consumption - Canonical model

\[(g) \, \tau_{\text{shock}} = .1 \]

\[(h) \, \tau_{\text{shock}} = .9 \]

Different timing of shocks
Impact of persistent earnings shocks on consumption - Nonlinear model

- Consumption responses change, based on the rank of the household in the income distribution and the magnitude of the earnings shock.
Simulating a life-cycle model

- Want to study the possible implications of the nonlinearity in the earnings process.

- Simulate consumption and assets using the life-cycle model of Kaplan and Violante (2010).

- Compare the canonical linear model with a simple nonlinear earnings model, with “unusual” earnings shocks.
Some details of the simulation

• Each household enters the labor market at age 25, works until 60, and dies with certainty at 95.
• After retirement, households receive Social Security transfers $Y_{i}^{ss}$, which are functions of the entire realizations of labor income.
• Utility is CRRA.
• Single risk-free, one period bond, with constant return is $1 + r$.
• Period-by-period budget constraint.
• Natural borrowing limit (households cannot die in debt)
The earnings process

- During working years, after-tax earnings are described by:
  \[
  \log Y_{it} = \kappa_t + y_{it}
  \]
  \[
  y_{it} = \eta_{it} + \varepsilon_{it}
  \]
  where:
  - \( \kappa \) is a deterministic experience profile.
  - \( \eta \) is the persistent component of earnings, \( \varepsilon \) is the transitory one.
  - The process for the persistent component of earnings is:
    \[
    \eta_{it} = \rho_t(\eta_{i,t-1}, v_{it})\eta_{i,t-1} + v_{it}
    \]
  where two specifications are compared:
  - \( \rho_t = 1 \) and \( v_{it} \) is normally distributed in the canonical earnings process.
  - \( \rho_t \) is nonlinear and follows the rich process estimated from the PSID.
Consumption and assets

Solid lines are for the canonical earnings model, dashed lines for the nonlinear one.

(a) Consumption, age 37 by decile of $\eta_{t-1}$

(b) Average consumption over the life-cycle

(c) Consumption variance over the life-cycle

(d) Assets variance over the life-cycle
Consumption response to earnings

Figure S37.—Simulations based on the estimated nonlinear earnings model. Notes: In the top panels, dashed is based on the nonlinear quantile-based earnings process estimated on the PSID, solid is based on a comparable canonical earnings process. Panel (e): estimate of the average derivative of the conditional mean of log-consumption with respect to log-earnings, given earnings, assets, and age, evaluated at values of assets and age that correspond to their $\tau_{assets}$ and $\tau_{age}$ percentiles, and averaged over the earnings values.
Conclusions

• Develops a nonlinear framework for modeling persistence

• Reveals asymmetric persistence patterns, with “unusual” shocks associated with a drop in persistence

• Provides conditions for the nonparametric identification and develops a simulation-based quantile regression method for estimation

• Nonlinear persistence is an important feature of earnings processes

• Consumption responses vary with the position of the household in the income distribution, age, and assets
Extensions

- Combine this framework with more structural approaches.
- Extend the analysis to consider the effects of business cycles.
  - Gonzalo Paz-Pardo’s JMP (2019)
- Extend the analysis to incorporate family labor supply à la Blundell, Pistaferri, and Saporta-Eksten (2016)
- Extend the analysis to older households
Appendix
In the absence of panel data or a clear decomposition between low- and high-frequency shocks, none of these studies is able to relate the deviations in the two series to the durability of shocks (or the degree of insurance to shocks of different persistence), but the patterns they find do line up very closely with those in Figure 1. In particular, Johnson, Smeeding, and Torrey (2005) show the Gini for real equivalized disposable income rising from 0.34 to 0.40 in the period 1981 to 1985 and then up to 0.41 by 1992. The Gini for equivalized real nondurable consumption rises from 0.25 to 0.28 over the first period and then hardly at all in the second period.

Finally, Krueger and Perri (2006) document a rise in consumption inequality of a similar magnitude over this period with the variance of log consumption rising around 0.05 units over the 1980s. Their study uses data from the CEX exclusively and does not directly model the panel data dynamics of consumption and income jointly. In particular, they do not allow the degree of persistence in income shocks to vary over time.

In their ground-breaking study, Deaton and Paxson (1994) present some detailed evidence on consumption inequality and interpret this within a life-cycle model. They note that consumption inequality should be monotonically increasing with age. Figure 2 shows this is broadly true for the cohorts in our sample. It also shows the large differences in initial conditions across birth cohorts with more recent cohorts experiencing a higher level of inequality at any given age. Initial conditions for different date-of-birth cohorts are extremely important to control for in understanding inequality.

Although Figure 1, and the discussion surrounding it, identify two distinct episodes in the growth of income and consumption inequality, these overall trends do not help inform why these different episodes took place. Specifically, they do not tell us anything about the nature of the changes in the income process or the nature of insurance that may have driven a wedge between consumption and income inequality. Studies that have investigated the impact of insurance either assume some external process for income or assume a specific form of insurance, typically the

It is worth noting that the Gini and the variance of the log measures of inequality do not necessarily move in the same direction. Log normality is an exception. It is also useful to note in making these comparisons that the variance of logs is most sensitive to transfers of income at the lowest end of the distribution, whereas the Gini coefficient is most sensitive to transfers around the mode of the distribution.
Derivation of a quantile function

The cdf of Exponential($\lambda$) is:

$$F(x; \lambda) = \begin{cases} 
1 - e^{-\lambda x} & x \geq 0 \\
0 & x < 0 
\end{cases}$$

To find the quantile function we need to find the value of $x$ such that $\Pr(X \leq x) = p$. That is:

$$1 - e^{-\lambda x} = p \Rightarrow -\lambda x = \log(1 - p) \Rightarrow x = -\frac{\log(1 - p)}{\lambda}$$

which means that the quantile function is:

$$Q(p; \lambda) = -\frac{\log(1 - p)}{\lambda}$$

This means that in order to find the value of $X$ for which, say, $\Pr(X \leq x) = 0.5$, you feed $p = 0.5$ to the quantile function.
Intuition for quantile function

The quantile function, denoted as $F^{-1}$, maps probabilities to quantiles. For any probability $p$, the quantile $F^{-1}(p)$ gives the corresponding quantile $Q$ such that $\Pr(Y < Q) = p$. This function is the inverse of the cumulative distribution function (CDF), which maps quantiles to probabilities.

In the diagram, $F^{-1}(p_1)$ and $F^{-1}(p_2)$ correspond to quantiles $Q_1$ and $Q_2$, respectively, where $p_1$ and $p_2$ are the probabilities associated with these quantiles.
Orthogonal polynomials

- Hermite polynomials are an orthogonal polynomial sequence.
  - An orthogonal polynomial sequence is such that any two different polynomials in the sequence are orthogonal to each other under some inner product.
- The polynomials $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = 3x^2 - 1$ constitute a sequence of orthogonal polynomials under the inner product:

$$\langle g, h \rangle = \int_{-1}^{1} g(x)h(x)dx$$

This is because

$$\langle p_0, p_1 \rangle = \int_{-1}^{1} 1x = \frac{x^2}{2} \bigg|_{-1}^{1} = 0, \quad \langle p_0, p_2 \rangle = \int_{-1}^{1} 1 \cdot (3x^2 - 1)dx = x^3 - x \bigg|_{-1}^{1} = 0,$$

$$\langle p_1, p_2 \rangle = \int_{-1}^{1} x \cdot (3x^2 - 1)dx = \frac{3}{4}x^4 - \frac{1}{2}x^2 \bigg|_{-1}^{1} = 0$$
Hermite polynomials

- The first four probabilists’ Hermite polynomials are:
  \[ He_0(x) = 1 \]
  \[ He_1(x) = x \]
  \[ He_2(x) = x^2 - 1 \]
  \[ He_3(x) = x^3 - 3x \]

- Hermite polynomials are orthogonal with respect to a weight function \( w(x) \).
- In particular, the probabilist Hermite polynomials are orthogonal with respect to the standard normal probability density function, that is:
  \[ \int_{-\infty}^{\infty} He_m(x)He_n(x)e^{-\frac{x^2}{2}} = \sqrt{2\pi}n!\delta_{nm} \]
  where \( \delta \) is the Kronecker delta, \( \delta_{nm} = 0 \) if \( n \neq m \) and 1 otherwise.
Densities of earnings components

(a) Persistent component $\eta_{it}$

(b) Transitory component $\varepsilon_{it}$
Conditional skewness of earnings components

(a) Log-earnings residuals $y_{it}$

(b) Persistent component $\eta_{it}$
Confidence bands

(a) PSID data  (b) Nonlinear model

Parametric bootstrap

Nonparametric bootstrap
IRF’s with different timing of shocks

Earnings

\[ \tau_{\text{init}} = .9, \tau_{\text{shock}} = .1 \]

(a) Young

(b) Old

\[ \tau_{\text{init}} = .1, \tau_{\text{shock}} = .9 \]

(c) Young

(d) Old

Consumption

\[ \tau_{\text{init}} = .9, \tau_{\text{shock}} = .1 \]

(e) Young

(f) Old

\[ \tau_{\text{init}} = .1, \tau_{\text{shock}} = .9 \]

(g) Young

(h) Old