On the optimal design of a Financial Stability Fund*

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Abstract

A Financial Stability Fund set by a union of sovereign countries can improve countries' ability to share risks, borrow and lend, with respect to the standard instrument used to smooth fluctuations: sovereign debt financing. Efficiency gains arise from the ability of the fund to offer long-term contingent financial contracts, subject to limited enforcement (LE) and moral hazard (MH) constraints. In contrast, standard sovereign debt contracts are uncontingent and subject to untimely debt roll-overs and default risk. We develop a model of the *Financial Stability Fund* (*FSF*) as a long-term partnership with LE and MH constraints. We quantitatively compare the constrained-efficient *FSF* economy with the incomplete markets economy with default. In particular, we characterize how (implicit) interest rates and asset holdings differ, as well as how both economies react differently to the same productivity and government expenditure shocks. In our economies, 'calibrated' to the euro area 'stressed countries', substantial efficiency gains are achieved by establishing a well-designed *Financial Stability Fund*; this is particularly true in times of crisis. Our theory provides a basis for the design of an *FSF* - for example, beyond the current scope of the *Euroepan Stability Mechanism* (ESM) - and a theoretical and quantitative framework to assess alternative risk-sharing (shock-absorbing) facilities, as well as proposals to deal with the euro area 'debt overhang problem'.

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1 Introduction

> "For all economies to be permanently better off inside the euro area, they also need to be able to share the impact of shocks through risk-sharing within the EMU."

This quote from the *Five Presidents' Report* (2015) recognizes a widely accepted fact: without a Federal Budget, or an institutional framework with similar fiscal automatic stabilizers for the euro area, it is unlikely that it will efficiently exploit its capacity for risk-sharing local or country risks with only private risk sharing and the existing EMU institutions[1]. In the aftermath of the financial and euro crises, with the subsequent upsurge of social unrest and discontent, the Five Presidents' call seems timely and urgent[2].

We develop a dynamic model of a *Financial Stability Fund* (*FSF*) as a long-term partnership addressing three features that are usually seen as the most problematic for a risk-sharing institution to be sustainable. First, risk-sharing transfers should not become persistent, or permanent, transfers, beyond the level of redistribution that partners would accept at any point in time, i.e. *ex-post* not only *ex-ante*. Second, as in any insurance contract, the *FSF* must take into account *moral hazard* problems, e.g. by avoiding current political costs, governments may increase future social and economic risks, and it is not possible to make the *Financial Stability Fund* contract conditional on the effort a country makes to reduce future liabilities and risks[3]. Third, risk-sharing among *ex-ante* equal partners without debt liabilities is relatively easy to design and achieve but, for example, this is not the case among European countries – in particular, the euro crisis has left a 'debt overhang problem' that aggravates the euro area divide. Our *Financial Stability Fund* accounts for the first two features by taking them as constraints, and accounts for the third by having country-specific long-term contracts and accounts, which can adapt to the EU diversity and may be able to solve existing 'debt overhang problems' . In sum, the *Financial Stability Fund* is a *constrained efficient mechanism* that can enhance countries' ability to share risks, borrow and lend, with respect to the standard instrument used to smooth fluctuations: sovereign debt financing.

In this paper we develop a model of *Financial Stability Fund* contracts between a risk-averse, relatively small and impatient borrower (the sovereign country) and a risk-neutral lender (the fund itself), and we evaluate its quantitative performance by calibrating the model to 1980-2015 data from the euro area 'stressed countries', to see how they would have performed during this period had they been in the fund. However, to assess the efficiency of the *FSF* it must be in relation to some other risk-sharing mechanism. We use, as a benchmark, an incomplete markets model where sovereign countries issue long-term defaultable debt in order to smooth their consumption.

In order to properly compare the *FSF* economy with the incomplete markets economy with defaultable debt (IMD), we 'decentralize' the fund contract generating the appropriate prices. For example,

[1] For example, **?** estimate that the percentage of non-smoothed GDP shocks is 20% in Germany, 25% in the United States, but 70% in the Euro Area (15) 1978 - 2010. Using their methodology, M. Lanati estimates that it is 83% non-smoothed for EA(19), 1995 - 2015. **?** has performed the counterfactual exercise of having the United States being Independent States. He finds, using a ' Semi-Structural Methodology' that, if the employment rate's cross-state standard deviation was 2.6% in 2010, it would have been 3.5% had it not been a fiscal union.

[2] However, almost surprisingly, they leave the task for a future date:

> "In the medium term, as economic structures converge towards the best standards in Europe, public risk-sharing should be enhanced through a mechanism of fiscal stabilisation for the euro area as a whole."

[3] Although 'austerity programs' , to gain financial assistance in the euro crisis, attempted this.

both in the *incomplete markets* economy with default, and in the two-sided limited-enforcement *FSF* economy, interest rates may differ from the risk-free rate. In the former, the *positive spreads* reflect the risk of default, while in the latter the *negative spreads* reflect the risk that the lender's participation – or, with moral hazard, borrower's incentive compatible – constraints become binding. Lower interest rates deter the lender from lending, thus implementing the *FSF* lender's enforcement constraint, which in our simulations is a tight constraint: 'at any point and state the lender's expected profits must be non-negative'. In both regimes, default is costly, resulting in autarky, with only a small probability of being able to join the incomplete markets economy and issue long-term defaultable debt.

It is interesting to note how the *FSF* mechanism compares with – *de facto* defaultable – long-term uncontingent sovereign debt contracts, currently in place, when the risk-averse borrowing country is subject to similar shocks to those to which the euro area 'stressed countries' have been exposed, in the last ten years. Without debt crises, the real euro crisis would not have been so severe, nor would it have turned into a recession; consumption smoothing and, therefore, the welfare of the borrowing country would have been higher, even if ex-post permanent transfers from the risk-neutral fund were set to zero.

We are not the first to address these issues, and there are many proposals for how risks could be shared in a monetary union, as there are for dealing with sovereign debt overhang problems. For example, as an implicit criticism of different proposals to issue some form of joint-liability eurobonds, **?** emphasises the asymmetry issue: the optimal (one-period) risk-sharing contract with two symmetric countries is a joint liability debt contract serving as a risk-sharing mechanism, while the optimal contract between two countries with very different distress probabilities is a debt contract with a cap and no joint liability, where the cap depends on the extent of solidarity, which is given by the externality cost of debt default on the lender. With long-term relationships – as they are among sovereign countries that form a union – better contracts can be implemented: the *FSF* contracts are *constrained efficient* and can be implemented as long-term bonds with state-contingent coupons.

On the more practical side, a positive development within the euro crisis was the creation of the *European Stability Mechanism* in 2012, which treaty (Ch. 4 Art. 12.1) establishes as its first principle that:

> If indispensable to safeguard the financial stability of the euro area as a whole and of its Member States, the ESM may provide stability support to an ESM Member subject to strict conditionality, appropriate to the financial assistance instrument chosen.

While this first principle assesses the need to have contingent contracts, it also limits its funding to *extreme* events. Conditionality is a property of the optimal long-term contract that we characterize, but in contrast with the *ESM*, the *FSF* is designed as a risk-sharing mechanism, not as a crisis-resolution mechanism, and its conditionality is based on *ex-post* realisations, not on *ex-ante* promises to reform, which often require *ex-post* renegotiations.

Our model of the *FSF* as a partnership builds on the literature on dynamic optimal contracts with enforcement constraints (e.g. **?**), as well as on the related literature on price decentralization of optimal contracts (e.g. **?**, **?**). Our benchmark incomplete markets economy with long-term debt with default, builds on the model of **?**, who extend the sovereign default models of **?** and **?** to long-term debt.

The paper is organized as follows. Section 2 presents the economy with incomplete markets and

sovereign defaultable long-term debt. Section 3 develops the *FSF* mechanism and Section 4 shows how to decentralize the FSF contract with state-contingent long term bonds. Section 5 discusses the calibration and data sources. Section 6 quantitatively compares the IMD and FSF regimes *without moral hazard*, concluding with a welfare comparison and showing the ability of the FSF to confront 'debt overhang' problems. Section 7 extends the calibrated model to account for *moral hazard*, showing how allocations and bond prices change when *incentive compatibility* constraints are introduced. Section 8 shows how a simpler – less state contingent – Fund contract can be designed and decentralized. Section 9 concludes.

2 The economy and the benchmark case of sovereign debt financing

We consider a standard infinite-horizon representative agent economy, where the agent has preferences for current leisure, $l = 1 - n$, consumption, c, and effort, e, represented by $U(c, n, e) := u(c) + h(1 - n) - v(e)$ and discounts the future at the rate β. We make standard assumptions on preferences[4]. The agent has access to a decreasing returns labor technology $y = \theta f(n)$, where $f' > 0$, $f'' < 0$ and θ is a productivity shock, assumed to be Markovian; $\theta \in (\theta_1, ..., \theta_N)$, $\theta_i < \theta_{i+1}$. The economy is a small open economy in a world with no uncertainty with interest rate r satisfying $1/(1 + r) \geq \beta$; an inequality that, in general, we will assume to be strict. In order to borrow and save, the agent, which we also identify with the government of the country, may have access to different financial technologies, which will define different regimes, which we also call different economies.

The country also faces government expenditure shocks $G = G^c + G^d$, which together with the productivity shock defines the exogenous state, denoted by $s = (\theta, G)$. G^c takes discrete values from $G^c \in \{G^c_1, \ldots, G^c_{N_G}\}$ and is a Markov process with transition probability $\pi^{G^c}(G'|s, e)$, and G^d is i.i.d. over time with continuous distribution ν over $G^d = \big[-\bar{m}, \bar{m}\big]$. In addition, G^c and G^d are independent with each other. The interpretation is that G^c are government expenditures and the distribution of next period expenditures depend on the policies that the government implements in the current period. In particular, the government can have a a better distribution of tomorrow's expenditures if it exercises sufficient effort in the current period (e.g. politically costly reforms are more likely to result in lower government expenditures). G^d is a residual shock that cannot be affected by government actions[5]. More precisely, we assume that given the current state, $s = (\theta, G)$, the next period realizations of θ and G are independent and only the latter depends on effort. That is

$$\pi(s'|s, e) = \pi^{\theta}(\theta'|\theta)\pi^{G}(G'|G, e)$$

We assume that the cost of this effort are expressed in utility terms given by $v(e)$. We assume that high effort increases the probability of lower government expenditure, in this sense we can think about effort as 'austerity' measures with utility costs which reduce primary deficit. We assume that

[4]In particular, we assume that $(c, n, e) \in \mathbb{R}^3_+, n \leq 1$, and u, h, v are differentiable, with $u'(c) > 0, u''(c) < 0, h'(c) > 0, h''(c) < 0$ and $v'(c) > 0, v''(c) > 0$.

[5]The introduction of G^d is for technical reasons, as in Chatterjee and Eyigungor (2012). Notice that the composite G shock admits a Markov structure as well, with state space $G = \cup_i \big[G^c_i - \bar{m}, G^c_i + \bar{m}\big] \subset \mathbb{R}$ and transition kernel $\pi^G = \pi^{G_c} \otimes \nu$.

both $v(\cdot)$ and $\pi^G(G'|s,e)$ are continuous and twice differentiable in effort, moreover we assume that $v(\cdot)$ is convex.

2.1 The incomplete market model with long-term bond financing

The incomplete market model is a quantitative version of the seminal model by **?**. We integrate three modeling advances in the recent literature, namely endogenous labor and output, long-term bonds, and an asymmetric default penalty, to achieve a more complete description of the business cycle dynamics of an small open economy with sovereign debt. We detail the specification of the baseline incomplete market model in this section.

In the incomplete market model, the borrower can issue or purchase *long-term* bonds, which promise to pay constant cash flows across different states. We model the long-term bond in the same way as **?**.

A unit of long-term bond is parameterized by (δ, κ), where δ is the probability of continuing to pay out the coupon in the current period, and κ is the coupon rate. Alternatively, $1-\delta$ is to the probability of maturing in the current period, and this event is independent over time. The size of each bond is infinitesimal and the payment of each bond is independent in cross-section. As a result, on average one unit of bond (δ, κ) will repay $(1-\delta)+\delta\kappa$ in the current period for sure. It also follows that the bond portfolio has a recursive structure, in which only the size of total outstanding debt b matters, regardless when a particular issue of the bond enters into the portfolio. Moreover, δ directly captures the duration of the bond: if $\delta = 0$ the bond becomes the standard one-period debt, and in general, the average maturity of the bond equals to $1/(1-\delta)$, which is increasing in δ. The coupon rate κ provides a flexible way to capture the coupon payment: $\delta\kappa$ equals to the coupon payment on each unit principal of outstanding debt.

For an outstanding bond portfolio of size b, its cash flow stream is given by $(1-\delta)b + \delta\kappa b$, $\delta(1-\delta)b + \delta^2\kappa b, \ldots$. When there is no default, the price of a unit of a *riskless* long-term bond (δ, κ), given a constant discount rate r, is:

$$\mathfrak{q} = \sum_{t=0}^{\infty}[(1-\delta)+\delta\kappa]\frac{\delta^t}{(1+r)^{t+1}} = \frac{(1-\delta)+\delta\kappa}{r+1-\delta},$$

with the corresponding risk free yield to maturity:

$$r = \frac{(1-\delta)+\delta\kappa}{\mathfrak{q}} - (1-\delta)$$

2.2 The budget Constraint and default

Let b_t denote the size of the bond portfolio (δ, κ) held by the borrower at the beginning of time t. Following the convention in the literature, $b_t \geq 0$ means holding assets while $b_t < 0$ means having debt. The borrower first makes a decision on whether to default on the promised bond payment of the entire bond portfolio b_t.

No default When the borrower chooses not to default, then the bond payment $(1-\delta)b_t + \delta\kappa b_t$ will be settled as promised: if $b_t \geq 0$, then the bond payment is part of the borrower's time t income; else if $b_t < 0$, then the borrower will make the required payment to the lender. Choosing not to default

allows the borrower to stay in the bond market, so that the borrower may choose the bond holding position b_{t+1} for the next period. The difference between b_{t+1} and the remaining principal δb_t is the net issuance at time t. Due to the recursive structure of the long-term bond, the cash flows starting from $t+1$ onward of both b_{t+1} and δb_t are proportional, and therefore the same unit bond price applies to both. As to be explained below, the bond price is a function of the exogenous shock s_t and the bond position b_{t+1} for the next period, thus we use $q(s_t, b_{t+1})$ to denote this function. It follows that when the borrower chooses not to default, the budget constraint is as follows:

$$c_t + q(s_t, b_{t+1})(b_{t+1} - \delta b_t) \leq \theta_t n_t^\alpha - G_t + (1 - \delta + \delta\kappa) b_t.$$

Default Upon choosing default, the borrower is excluded from the bond market immediately and enters into autarky. As a result, the time t consumption is given by

$$c_t = \theta^p(\theta_t)\, n_t^\alpha - G_t.$$

The exclusion lasts for a random number of periods. If the borrower is excluded from the market in the previous period, then with probability $\lambda < 1$ the borrower regains access to the bond market in the current period, and with remaining probability $1-\lambda > 0$ the borrower stays in autarky. Moreover, upon regaining access to the bond market, the borrower starts from a zero bond position.

Besides the exclusion from the bond market, the borrower also suffers from a productivity penalty in autarky. As in **?**, the penalty takes the following form:

$$\theta^p(\theta) = \begin{cases} \bar{\theta}, & \theta \geq \bar{\theta} \\ \theta, & \theta < \bar{\theta} \end{cases} \quad \text{with } \bar{\theta} = \psi \mathbb{E}\theta,$$

which is *asymmetric* in the sense that the magnitude of the penalty is zero for lower than average" productivity states, while it is equal to $\theta - \bar{\theta}$—increasing in θ—for higher than average productivity states. The level of the penalty is parameterized by $\psi > 0$. Given that $0 < \theta_1 < \cdots < \theta_{N_\theta}$, on the one hand the penalty becomes a benefit if $\psi \geq \theta_{N_\theta}/\mathbb{E}\theta$; and on the other hand, the penalty ceases to be effective if $\psi < \theta_1/\mathbb{E}\theta$, since the borrower can always choose to have zero debt while enjoying higher productivity levels. An asymmetric penalty is crucial for the quantitative performance of models with sovereign debt and default. When the penalty is properly specified, it creates incentives for the borrower to borrow more in good states while deterring default temptation by harsh punishment, and these high levels of debt then induce the borrower to choose default in bad states where the penalty is zero.

2.3 Recursive Formulation

If b the size of the long-term bond portfolio held by the borrower at the beginning of a period[6], then (s, b), $s = (\theta, G)$, is the state. Let $V_n^b(b,\, s)$ denote the value function of the borrower, in the incomplete market economy, at the beginning of a period before any decisions are made. The value function when

[6]We assume that $b \in \mathcal{B} = [b_{\min}, b_{\max}]$, with $-\infty < b_{\min} < 0 \leq b_{\max} < \infty$., where we will choose $b_{\min}$ and $b_{\max}$ so that in equilibrium the bounds are not binding.

the borrower chooses not to default satisfies

$$V_n^b(b,\, s) = \max_{c,n,e,b'} \left\{U(c,n,e) + \beta \mathbb{E}\left[V^b(b',\, s') \mid s, e\right]\right\} \tag{1}$$
$$\text{s.t.} \quad c + G + q(s,b,b')(b' - \delta b) \leq \theta f\,(n) + (1 - \delta + \delta\kappa)b,$$

where, taking into account that default can occur next period,

$$V^b(b,\, s) = \max\{V_n^b(b,\, s), V^a(s)\}, \tag{2}$$

and $V^a(s)$ is the value upon default, given by

$$V^a(\, s) = \max_{n,e}\{u\left(\theta^p(\theta) f\,(n) - G\right) + h(1-n) - v(e) \tag{3}$$
$$+ \beta \mathbb{E}\left[\left(1 - \lambda^i\right) V^a(s') + \lambda^i V^b\,(0,\, s') \mid s, e\right],$$

where λ^i is the probability to come back to the market and be able to borrow again. We will denote the choices in autarky, given by (3), by $(n^a(s), e^a(s))$.

The reason the bond price $q(s,b,b')$ depends on b is that e, which it is assumed that it is not observable or contractable, affects the distribution of $G^{c\prime}$. The bond price has also a recursive structure. Let the default decision be given by

$$D(\, s, b) = 1 \text{ if } V^{ai}(\, s) > V_n^{bi}(b,\, s) \text{ and 0 otherwise;}$$

therefore, the expected default rate is $d(s,b,b') = \mathbb{E}\left[D(s',\, b') \mid\ s, e^*(s,b)\right]$ The equilibrium bond pricing function $q(s,b,b')$ satisfies the following recursive equation:

$$q(s,b,b') = \frac{\mathbb{E}\left[(1 - D(s',b'))\left[(1-\delta) + \delta\left[\kappa + q(s',\, b',\, b''(s',\, b'))\right]\ \right] \mid\ s, e^*(s,b)\right]}{1+r},$$

which can also be expressed as:

$$\begin{aligned} q(s,b,b') &= \frac{(1-\delta) + \delta\kappa}{1+r}\left(1 - d(s,b,b')\right) \\ &+ \delta \frac{\mathbb{E}\left[(1 - D(s',\, b'))\, q(s',\, b',\, b''(s',\, b') \mid\ s, e^*(s,b)\right]}{1+r}, \end{aligned} \tag{4}$$

For the one-period bond ($\delta = 0$), this reduces to $q(s,b,b') = \frac{1 - d(s,b,b')}{1+r}$. The implied interest rate (i.e. yield to maturity) of the long-term bond is given by

$$r^i(s,b,b') = \frac{(1-\delta) + \delta\kappa}{q(s,b,b')} - (1-\delta)\,,$$

resulting in a *positive spread* $r^i\,(s,b,b') - r \geq 0$, which is strictly positive if $d(s,b,b') > 0$.

In order to keep track of debt flows, it is useful to define the *primary surplus* – or *primary deficit* if negative – which is given by

$$q(s,b,b')(b' - \delta b) - (1 - \delta + \delta\kappa)b = \theta f\,(n) - (c + G)$$

2.4 The effort decision

The optimal policies $c^b(b,s)$, $n^b(b,s)$, $b'(b,s)$ and $n^a(s)$ are standard dynamic programming solutions to (1) and (3), respectively. The effort policy function when there is no default in state $s=(\theta,G)$, $e^b(b,s)$, is given by

$$v'(e^b(b,s)) = \beta \sum_{s'} \pi^\theta(\theta'|\theta) \frac{\partial \pi^G(G'|G, e^b(b,s))}{\partial e} V^b(b', s'),$$

where b' is the optimal choice of debt in (1). Similarly, the effort policy function when there is default in state $s=(\theta,G)$, $e^a(s)$, is given by

$$v'(e^a(s)) = \beta \sum_{s'} \pi^\theta(\theta'|\theta) \frac{\partial \pi^G(G'|G, e^a(s))}{\partial e} \left[(1-\lambda^i)V^a(s') + \lambda^i V^b(0,s')\right],$$

since in (3) the choice of debt is predetermined to be zero.

3 The *Financial Stability Fund* as a long-term contract

An economy with a *Financial Stability Fund* (FSF) is modeled as a long-term contract between a fund, or *FSF* (also called lender), who can freely borrow and lend in the international market, and an individual partner (also called country or borrower), who is 'the representative agent' of the small open economy. We assume that the manager cannot observe the effort of the partner – or, simply, that the effort is not contractable,– which implies that the long term contract will have to provide sufficient incentives for the country to implement a (constrained) efficient level of effort. In the *fund contract*, the country consumes c and the resulting transfer to the FSF manager is $\tau = \theta f(n) - (c+G)$; i.e. when $\tau < 0$ the country is effectively borrowing. We consider that there is two-sided limited enforcement; that is, both the FSF manager and the lender can renege of their contract and pursue their outside options at any time-state.

In state $s^t = (s_0, \ldots, s_t)$, the outside value for the borrower country is the value of being in the incomplete market economy after default – that is $V^a(s^t)$, given by (3). In other words, once a country has joined the fund if it ever quits, or does not fulfil the *FSF* contract, the country is not allowed back and goes into autarky and then with probability λ^i is able to borrow in the private market. The outside option of the lender is $Z \leq 0$, at any s^t, which is determined by the willingness of the FSF (the lender) to accept some level of redististribution or to avoid that the country breaks away[7]. Whether it is *e*x-post altruisitic or self-interested – as in **?** – *solidarity*, Z is an important parameter when assessing the efficiency gains of establishing a FSF; in particular, if $Z=0$ the FSF may still be superior to other mechanisms, since it can still provide some level of risk-sharing and for the impatient borrower can always be a better 'borrowing mechanism'. In the next Section we show how Z constraints the paths of FSF transfers and its effect on prices.

With *two-sided limited enforcement*, denoted (2S), an optimal fund contract is a solution to the

[7]Our characterisation easily generalises to the case that the outside value of the manager (lender) is time-state dependent.

following problem:

$$\max_{\{c(s^t),n(s^t),e(s^t)\}} \mathrm{E}\left[\mu_{b,0}\sum_{t=0}^{\infty}\beta^t\left[U(c(s^t),\, n(s^t), e(s^t))\right] + \mu_{l,0}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \tau(s^t) \mid s_0\right]$$

$$\text{s.t.} \qquad \mathrm{E}\left[\sum_{r=t}^{\infty}\beta^{r-t}\left[U(c(s^r),\, n(s^r),\, e(s^r))\right] \mid s^t\right] \geq V^a\left(s_t\right), \tag{5}$$

$$v'(e(s^t)) = \beta\sum_{s^{t+1}|s^t}\pi^{\theta}(\theta'|\theta)\frac{\partial\pi(s^{t+1}|s_t,\, e(s^t))}{\partial e(s^t)}V^{bf}(s^{t+1}), \tag{6}$$

$$\mathrm{E}\left[\sum_{r=t}^{\infty}\left(\frac{1}{1+r}\right)^{r-t}\tau(s^r) \mid s^t\right] \geq Z, \tag{7}$$

$$\text{and} \qquad \tau(s^t) = \theta(s^t)f\left(n(s^t)\right) - c(s^t) - G(s^t),\ \forall s^t, t \geq 0,$$

where the first two constraints (5) and (7) are the intertemporal participation constraints for the borrower and the lender, respectively, and (μ_{b0}, μ_{l0}) are initial Pareto weights. Here the notation is explicit about the fact that expectations are conditional on the implemented effort sequence as it affects the distribution of the shocks. The constraint (6) is the incentive compatibility constraint with respect to effort[8], where $V^{bf}(s^{t+1})$ is the value of the *FSF* contract to the borrower in state s^{t+1}.

By imposing equality in (6) we have implicitly assumed that effort is interior, that is $e > 0$. The interpretation of this constraint is standard: the marginal cost of increasing effort has to be equal to the marginal benefit. The latter is measured as the change in life-time utility due to the change in the distribution of future shocks as a result of the increasing effort. Note that $V^{bf}(s^{t+1})$ can also be written explicitly as the continuation life-time utilities of the borrower for all continuation states from next period on. In particular, (6) can also be written as:

$$v'(e(s^t)) = \beta\sum_{s^{t+1}|s^t}\pi^{\theta}(\theta'|\theta)\frac{\partial\pi(s^{t+1}|s_t,\, e(s^t))}{\partial e(s^t)}\mathbb{E}\left[\sum_{r=t+1}^{\infty}\beta^{r-(t+1)}\left[U(c(s^r),\, n(s^r),\, e(s^r))\right] \mid s^{t+1}\right].$$

It is known from **?** and **?** that we can rewrite the general fund contract problem as a saddle-point

[8]Note that we have used the first-order condition approach here, that is, we have replaced by the agent's full optimization problem by its necessary first-order conditions of optimality. According to the results of Rogerson (1988), the first-order conditions are also sufficient if the $\pi^G(G'|s,e)$ functions satisfy the monotone likelihood ratio and the convex distribution function conditions described below.

MLR. The probability shifting function $\pi^G\left(G' \mid s, e\right)$ has the *monotone likelihood ratio* property if, for each $e \geq 0$ and s, the ratio $\frac{\partial\pi^G(G'|s,e)/\partial e}{\pi^G(G'|s,e)}$ is non-increasing in G'.

CDF. The functions $\pi^G(G'|s,e)$ satisfy the convex distribution function condition if $\frac{\partial^2 F_{\tilde{G}}(s,e)}{\partial e\partial e}$ is non-negative for every e, s and $\tilde{G}$ where $F_{\tilde{G}}(s,e) = \sum_{G'\leq\tilde{G}}\pi^G(G'|s,e)$.

problem [9]:

$$\begin{aligned}
\text{SP} \min_{\{\gamma_{b,t},\gamma_{l,t}\,\xi_t\}} \max_{\{c_t,n_t,e_t\}} \mathbb{E}\Bigg[\sum_{t=0}^{\infty}\beta^t\left(\mu_{b,t}U(c_t,\,n_t,\,e_t)-\xi_t v'(e_t)\right.\\
\left.+\gamma_{b,t}\left[U(c_t,\,n_t,\,e_t)-V^a(s_t)\right]\right)\\
+\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t\left(\mu_{l,t+1}\left[\theta_t f(n_t)-G_t-c_t\right]-\gamma_{l,t}Z\right)\mid s_0\Bigg]\\
\mu_{b,t+1}=\mu_{b,t}+\gamma_{b,t}+\xi_t\frac{\partial\pi(s_{t+1}|s_t,e_t)/\partial e}{\pi(s_{t+1}|s_t,e_t)},\quad\text{with }\mu_{b,0}\text{ given, and}\\
\mu_{l,t+1}=\mu_{l,t}+\gamma_{l,t},\quad\text{with }\mu_{l,0}\text{ given}
\end{aligned}$$

Here $\beta^t\pi(s^t|s_0)\gamma_b(s^t)$, $\beta^t\pi(s^t|s_0)\gamma_l(s^t)$ and $\beta^t\pi(s^t|s_0)\xi(s^t)$ are the Lagrange multipliers of the limited enforcement constraints (5), (7) and incentive compatibility constraint (6), respectively, in state s^t. That is, with one-sided limited commitment $\gamma_l(s^t)=0$, $\forall t\geq 0$. Notice that, by construction, $\frac{\partial\pi(s_{t+1}|s_t,e_t)/\partial e}{\pi(s_{t+1}|s_t,e_t)}=\frac{\partial\pi^G(G_{t+1}|G_t,e_t)/\partial e}{\pi^G(G_{t+1}|G_t,e_t)}$; that is, with the incentive compatibility constraint (6), the co-state $\mu_{b,t+1}$ is a vector $\mu_{b,t+1}(G_{t+1}\mid G_t)$, while without (6) it is a number.

We will use a convenient **normalization**, in order to minimise the dimension of the co-state vector. Let $\eta\equiv\beta(1+r)\leq 1$ and normalize multipliers: $v_{i,t}=\gamma_{i,t}/\mu_{i,t}, i=b,l$, $\widetilde{\xi}_t=\frac{\xi_t}{\mu_{b,t}}$ and

$$\varphi_{t+1}(G_{t+1}\mid G_t,e_t)=\widetilde{\xi}_t\frac{\partial\pi(G_{t+1}|G_t,e_t)/\partial e_t}{\pi(G_{t+1}|G_t,e_t)};\tag{8}$$

then, a new co-state vector is recursively defined as:

$$x_0=\mu_{b,0}/\mu_{l,0}\text{ and }x_{t+1}=\frac{1+v_{b,t}+\varphi_{t+1}}{1+v_{l,t}}\eta x_t$$

With this normalization, $v_{b,t}$ and $v_{l,t}$ become the multipliers of the limited enforcement constraints, corresponding to (5) and (7), and φ_t the multiplier of the incentive compatibility constraint, corresponding to (6). With this normalization, the state and co-state vector is given by (x,s) and the saddle-point Bellman equation is given by

$$\begin{aligned}
FV(x,s)=\text{SP}\min_{\left\{v_b,v_l,\widetilde{\xi}\right\}}\max_{\{c,n,e\}}\Big\{x\left[(1+v_b)\,U(c,\,n,\,e)-v_bV^a(s)-\widetilde{\xi}v'(e)\right]\\
+\left[(1+v_l)\left(\theta f(n)-G-c\right)-v_lZ\right]+\frac{1+v_l}{1+r}\mathbb{E}\left[FV(x',s')\mid s\right]\Big\}\qquad(9)\\
\text{where }x'=\frac{1+v_b+\varphi(G'\mid G,e)}{1+v_l}\eta x\text{ and }\varphi(G'\mid G,e)=\widetilde{\xi}\frac{\partial\pi(G'|G,e)/\partial e}{\pi(G'|G,e)}.
\end{aligned}$$

Furthermore (see **?**), the *FSF* policy function takes the form:

$$FV(x,s)=xV^{bf}(x,s)+V^{lf}(x,s),\quad\text{with}$$

[9] Following **?**, we only consider saddle-point solutions and their corresponding saddle-point multipliers; that is, given $F(a,\lambda)$, (a^*,λ^*) solves $\text{SP}\min_\lambda\max_a F(a,\lambda)$ if and only if $F(a^*,\lambda)\leq F(a^*,\lambda^*)\leq F(a,\lambda^*)$, for any feasible action a and Lagrangian multiplier λ.

$$V^{bf}(x,s) = U(c(x,s),\, n(x,s)\, e(x,s)) + \beta \mathbb{E}\left[V^{bf}(x',s') \mid s\right], \text{ and}$$

$$V^{lf}(x,s) = \tau^b(x,s) + \frac{1}{1+r}\mathbb{E}\left[V^{lf}(x',s') \mid s\right];$$

where $\tau^b(x,s) = \theta f(n^b(x,s)) - G - c^b(x,s)$. The policy functions defining the *FSF* contract are given by the first-order conditions of (9). In particular, $c^b(x,s)$ and $n^b(x,s)$ satisfy

$$u'(c^b(x,s)) = \frac{1+v_l(x,s)}{1+v_b(x,s)}\frac{1}{x} \quad \text{and} \quad \frac{h'(1-n^b(x,s))}{u'(c^b(x,s))} = \theta f'(n^b(x,s))$$

The effort policy $e^b(x,s)$ is more complex since the first-order condition with respect to e is:

$$\begin{aligned} & x\left[(1+v_b(x,s))v'(e(x,s) + \widetilde{\xi}(x,s)v''(e(x,s)\right] \qquad (10)\\ = \;& \frac{1+v_l(x,s)}{1+r}\sum_{s'|s}\pi^\theta(\theta'|\theta)\frac{\partial \pi^G(G'|G,e(x,s))}{\partial e}\left[x'V^{bf}(x',s') + V^{lf}(x',s')\right] \\ & + \frac{1}{1+r}\sum_{s'|s}\pi^\theta(\theta'|\theta)\pi^G(G'|G,e)\eta\widetilde{\xi}(x,s)x\left[\frac{\partial^2\pi^G(G'|G,e(x,s))/\partial e\partial e}{\pi^G(G'|G,e(x,s))} - \frac{\left(\partial\pi^G(G'|G,e(x,s))/\partial e\right)^2}{\pi^G(G'|G,e(x,s))^2}\right]V^{bf}(x',s'). \end{aligned}$$

Notice that if the *incentive constraint* is not binding (i.e. $\widetilde{\xi}(x,s) = 0$), then (10) reduces to:

$$x'v'(e(x,s) = \frac{1}{1+r}\sum_{s'|s}\pi^\theta(\theta'|\theta)\frac{\partial\pi^G(G'|G,e(x,s))}{\partial e}\left[x'V^{bf}(x',s') + V^{lf}(x',s')\right]. \qquad (11)$$

Dividing (10) by x and rearranging terms becomes:

$$\begin{aligned} & (1+v_b(x,s))v'(e(x,s) + \widetilde{\xi}(x,s)v''(e(x,s) \\ = \;& \sum_{s'|s}\pi^\theta(\theta'|\theta)\frac{\partial\pi^G(G'|G,e(x,s))}{\partial e}\left[\beta\left(1+v_b+\varphi(G'\mid G,e)\right)V^{bf}(x',s') + \frac{1}{1+r}\frac{1+v_l(x,s)}{x}V^{lf}(x',s')\right] \\ & + \beta\sum_{s'|s}\pi^\theta(\theta'|\theta)\pi^G(G'|G,e)\widetilde{\xi}(x,s)\left[\frac{\partial^2\pi^G(G'|G,e(x,s))/\partial e\partial e}{\pi^G(G'|G,e(x,s))} - \frac{\left(\partial\pi^G(G'|G,e(x,s))/\partial e\right)^2}{\pi^G(G'|G,e(x,s))^2}\right]V^{bf}(x',s') \end{aligned}$$

The left-hand side of (10) is the social marginal cost of effort, while the right-hand side are the conditional rewards (or punishments) corresponding to the different states $s'(\cdot, G')$. The contract must stablish a system of punishments and rewards such that (10) is satisfied, while the *intertemporal incentive constraint* (6) is also satisfied. Therefore, if we substitute (6) into (10) the remaining equation must also be satisfied; that is,

$$\begin{aligned} & \widetilde{\xi}(x,s)v''(e(x,s) \\ = \;& \sum_{s'|s}\pi^\theta(\theta'|\theta)\frac{\partial\pi^G(G'|G,e(x,s))}{\partial e}\left[\beta\varphi(G'\mid G,e)V^{bf}(x',s') + \frac{1}{1+r}\frac{1+v_l(x,s)}{x}V^{lf}(x',s')\right] \\ & + \beta\sum_{s'|s}\pi^\theta(\theta'|\theta)\pi^G(G'|G,e)\widetilde{\xi}(x,s)\left[\frac{\partial^2\pi^G(G'|G,e(x,s))/\partial e\partial e}{\pi^G(G'|G,e(x,s))} - \frac{\left(\partial\pi^G(G'|G,e(x,s))/\partial e\right)^2}{\pi^G(G'|G,e(x,s))^2}\right]V^{bf}(x',s'), \end{aligned}$$

which, using the definition of $\varphi(G' \mid G, e)$, simplifies to the following equality between the 'non-accounted' marginal cost of effort and the 'non-accounted' expected marginal benefit of effort:

$$\begin{aligned}
& \text{NMC(s)} \\
\equiv\ & \widetilde{\xi}(x,s)v''(e(x,s) \\
=\ & \frac{1}{1+r}\sum_{s'|s}\pi^{\theta}(\theta'|\theta)\pi^{G}(G'|G,e)\left[\widetilde{\xi}(x,s)\eta\frac{\partial^2\pi^{G}(G'|G,e(x,s))/\partial e\partial e}{\pi^{G}(G'|G,e(x,s))}V^{bf}(x',s')\right. \\
& \left.+\frac{1+v_l(x,s)}{x}\frac{\partial\pi^{G}(G'|G,e(x,s))/\partial e}{\pi^{G}(G'|G,e(x,s))}V^{lf}(x',s')\right] \\
\equiv\ & \frac{1}{1+r}\mathbb{E}\left[\text{NMB(s')}|\text{s}\right]
\end{aligned}$$

In order to calibrate the model, we provide more structure by assuming that, given current government liabilities G^c, there are two possible distributions of tomorrow's liabilities, $\pi^b(\cdot|G^c)$ and $\pi^g(\cdot|G^c)$, and $\pi^g(\cdot|G^c)$ first-order stochastically dominates $\pi^b(\cdot|G^c)$ for all G; in particular, there is $\zeta(e)$ with $\zeta'(e) < 0$, such that $\pi^G(G'|G^c, e) = \zeta(e)\pi^b(G'|G^c) + (1-\zeta(e))\pi^g(G'|G^c)$. Therefore,

$$\frac{\partial\pi^{G}(G'|G,e)}{\partial e} = -\zeta'(e)\left[\pi^g(G'|G^c) - \pi^b(G'|G^c)\right]$$

Furthermore, we also assume that $v(e) = \omega e^2$ and $\zeta(e) = \exp(-\rho e)$. In this case[10] (12) becomes:

$$\begin{aligned}
& \widetilde{\xi}(x,s)2\omega \\
=\ & \frac{1}{1+r}\sum_{s'|s}\pi^{\theta}(\theta'|\theta)\pi^{G}(G'|G,e)\rho\exp(-\rho e)\frac{\pi^g(G'|G)-\pi^b(G'|G)}{\pi^{G}(G'|G,e(x,s))}\left[\frac{1+v_l(x,s)}{1+r}\frac{1}{x}V^{lf}(x',s') - \rho\eta\widetilde{\xi}(x,s)V^{bf}(x',s')\right]
\end{aligned}$$

4 Decentralization of the fund contract

We now show how to decentralize the optimal contract as a competitive equilibrium with endogenous borrowing constraints, which will allow us to compare the different fund contracts with the debt contract of the economy with incomplete markets. We build on the work of Alvarez and Jermann (2000) and Krueger, Lustig and Perri (2008). To make it more comparable with the incomplete market model we consider that agents trade **in** *state-contingent bonds (assets)*; that is, agent trade portfolios of S **securities** parameterized by (δ, κ, s), where (δ, κ) denote the common coupon and duration probability but, in contrast with **the** long-term bond (δ, κ) **the** – Arrow type – security (δ, κ, s) only pays the coupon or the maturity value if the state is s. As in the incomplete markets model, it is assumed that agents hold a continuum of these portfolios, resulting that at period and state (t, s), $1-\delta$ securities mature, although only the (δ, κ, s) redeem the value and only δ of them pay the coupon κ.

4.1 The competitive equilibrium

In the market equilibrium, the borrower has a home technology that produces $\theta(s)f\left(n(s)\right)$ with his own labor. The borrower has access to long term state-contingent assets and solves the following

[10] Notice that then, $\frac{\partial\pi^{G}(G'|G,e(x,s))}{\partial e} = \rho\exp(-\rho e)\left(\pi^g(G'|G) - \pi^b(G'|G)\right)$ and $\frac{\partial^2\pi^{G}(G'|G,e(x,s))}{\partial e\partial e} = -\rho^2\exp(-\rho e)\left(\pi^g(G'|G) - \pi^b(G'|G)\right)$.

dynamic programming problem:

$$
\begin{aligned}
&W^b(a,\, s) = \max_{(c,\, n,\, e,\, a'(s'))} \left\{U(c,\, n,\, e) + \beta \mathbb{E}\left[W^b(a', s') \mid s\right]\right\} \\
&\text{s.t. } c + \sum_{s'|s} q\left(s'|s\right)\left(a(s') - \delta a(s)\right) + \tau^e(s) \leq \theta(s) f(n) - G(s) + \left(1 - \delta + \delta\kappa\right) a(s) + \tau^r(s|\ s_-) \\
&\quad a'(s') \geq A_b\left(s'\right)
\end{aligned}
$$

where $a_b\left(s'\right)$ the amount of the asset (δ, κ, s') bought by the borrower, in state s, at the price $q\left(s'|s\right)$, while $A_b\left(s'\right)$ is an endogenous borrowing limit defined below. We assume, without loss of generality, $a_b\left(s_0\right) = a_l\left(s_0\right) = 0$. Furthermore, $\tau^e(s)$ and $\tau^r(s|\ s_-)$ are Pigouvian taxes and rewards, respectively; where $\tau^r(s|\ s_-)$ denotes a reward in state s conditional on the state the previous period being s_- and the following non-arbitrage condition is satified: $\tau^e(s) = \sum_{s'|s} Q(s'|s)\tau^r(s'|s)$, where $Q(s'|s) = \frac{q(s'|s)}{1-\delta+\delta\kappa+\delta q(s')}$ and $q(s') = \sum_{s''|s'} q(s''|s')$. We assume $\tau^r(s_0|\ s_{-1}) = 0$. These taxes are designed to align the individual and social value of effort, given that the individual choice is determined by:

$$
v'(e) = \beta \sum_{s'|s} \frac{\partial \pi(s'|s,\, e)}{\partial e} W^b(a'(s'), s').
$$

The choice of consumption and assets determines the following Euler condition:

$$
q\left(s'|s\right) \geq \beta \pi\left(s'|s\right) \frac{u'\left(c\left(s'\right)\right)}{u'\left(c\left(s\right)\right)} \left[1 - \delta + \delta\kappa + \delta \sum_{s''|s'} q\left(s''|s'\right)\right],
$$

with equality if $a_b\left(s'\right) > A_b\left(s'\right)$. Notice that the last component on the left-hand side accounts for the fact that long-term assets are part of the portfolio next period, even when their state does not realize. Switching back to the time-notation, the transversality condition guarantees that

$$
\lim_{t\to\infty} \sum_{s^t} \beta^t \pi\left(s^t\right) u'\left(c\left(s^t\right)\right)\left[a_b\left(s^t\right) - A_b\left(s^t\right)\right] \leq 0
$$

The lender receives the coupon and can trade long-term assets and receives the net Piguvian revenues:

$$
\begin{aligned}
&W^l(a,\, s) = \max_{(c,\, a'(s'))} \left\{c + \frac{1}{1+r}\mathbb{E}\left[W^l(a', s') \mid s\right]\right\} \\
&\text{s.t. } c + \sum_{s'|s} q\left(s'|s\right)\left(a(s') - \delta a(s)\right) = \left(1 - \delta + \delta\kappa\right) a(s) + \tau^e(s) - \tau^r(s|\ s_-) \\
&\quad a'\left(s'\right) \geq A_l\left(s'\right)
\end{aligned}
$$

The corresponding Euler conditions is:

$$
q\left(s'|s\right) \geq \frac{1}{1+r} \pi\left(s'|s\right) \left[1 - \delta + \delta\kappa + \delta \sum_{s''|s'} q\left(s''|s'\right)\right]
$$

with equality if $a_l\left(s'\right) > A_l\left(s'\right)$ and, by the transversality condition,

$$
\lim_{t\to\infty} \sum_{s^t} \left(\frac{1}{1+r}\right)^t \pi\left(s^t\right)\left[a_l\left(s^t\right) - A_l\left(s^t\right)\right] \leq 0
$$

In particular, in equilibrium

$$\begin{aligned} q(s'|s) &= \frac{1}{1+r}\pi(s'|s)\max\left\{\frac{u'(c(s'))\eta}{u'(c(s))}\left[(1-\delta+\delta\kappa)+\delta\sum_{s''|s'}q(s''|s')\right],\left[(1-\delta+\delta\kappa)+\delta\sum_{s''|s'}q(s''|s')\right]\right\} \\ &= \frac{1}{1+r}\pi(s'|s)\left[(1-\delta+\delta\kappa)+\delta\sum_{s''|s'}q(s''|s')\right]\max\left\{\frac{u'(c(s'))\eta}{u'(c(s))},1\right\} \\ &= \frac{1}{1+r}\pi(s'|s)\left[(1-\delta+\delta\kappa)+\delta q(s')\right]\max\left\{\frac{u'(c(s'))\eta}{u'(c(s))},1\right\} \\ &\equiv \bar{q}(s'|s)\max\left\{\frac{u'(c(s'))\eta}{u'(c(s))},1\right\} \end{aligned}$$

Note that:

$$q(s)=\sum_{s'|s}q(s'|s)\geq\frac{1}{1+r}\left[(1-\delta+\delta\kappa)+\delta q(s')\right],$$

which iterating the right-hand side forward, using iterative expectations, results in:

$$q(s)\geq\bar{q}(s)\equiv\frac{1-\delta+\delta\kappa}{1-\delta+r};$$

with the corresponding interest rate:

$$r(s)=\frac{1-\delta+\delta k}{q(s)}-(1-\delta)$$

which results in a **negative spread**: $r(s)-r\leq 0$.

Let $c_b(a_b,s)$, $n(a_b,s)$, $e(a_b,s)$ and $a_b(a_b,s')$, and $c_l(a_l,s)$ and $a_l(a_l,s')$ be the optimal policies of the borrower and the lender, respectively. Market clearing implies that:

$$\begin{aligned} c_b(a_b,s)+c_l(a_l,s) &= \theta(s)f(n(a_b,s))-G(s) \\ a_b(a_b,s')+a_l(a_l,s') &= 0 \end{aligned}$$

We assume that the borrowing limits are properly tight in the sense that satisfy:

$$W^b(A_b(s),s) = V^a(s) \tag{12}$$

$$W^l(A_l(s),s) = Z \tag{13}$$

We conclude our characterization of competitive equilibria, with Arrow-securities and endogenous borrowing limits, by restricting them to those with allocations satisfying the *high implied interest rate condition*, namely:

$$\sum_{t=0}^{\infty}\sum_{s^t}Q\left(s^t|s_0\right)\left[c\left(s^t\right)+c_l\left(s^t\right)\right]<\infty,$$

where present value prices are defined by $Q(s_0)=\left[1-\delta+\delta\kappa+\delta q(s_0)\right]^{-1}$ and

$$Q\left(s^t|s_0\right)=Q(s_0)Q\left(s^1|s_0\right)Q\left(s^2|s^1\right)...Q\left(s^t|s^{t-1}\right).$$

4.2 Decentralization

Now we show how a *FSF* contract can be decentralized as a competitive equilibrium with long-term assets and endogenous borrowing limits. This allow us to obtain asset prices and holdings supporting the *FSF* contract, which we can compare to the debt prices and holdings of the incomplete markets economy. It will also allow us to define the Pigou taxes and rewards that implement the efficient level of effort.

Let $(c^*(x,s), n^*(x,s), e^*(x,s), \tau^*(x,s))$ be the optimal policy allocations of the *FSF*. First, we use the allocations to price the long-term assets as follows:

$$q^*(s'|s) = \frac{1}{1+r}\pi(s'|s)\frac{u'(c(s'))\eta}{u'(c(s))}\left[1-\delta+\delta\kappa+\delta\sum_{s''|s'}q^*(s''|s')\right] \text{ if } v_b(x',s')=0 \ \&\ v_l(x',s')\geq 0; \text{ while}$$

$$q^*(s'|s) = \frac{1}{1+r}\pi(s'|s)\left[1-\delta+\delta\kappa+\delta\sum_{s''|s'}q^*(s''|s')\right] \text{ if } v_l(x',s')=0 \text{ and } v_b(x',s')>0.$$

Therefore, using the FSF allocation we obtain:

$$\begin{aligned} q^*(s'|s) &= \frac{1}{1+r}\pi(s'|s)\left[(1-\delta+\delta\kappa)+\delta q^*(s')\right]\max\left\{\frac{1+v_l(x',s')}{1+v_b(x',s')}\frac{1}{1+\frac{\varphi(s'|x,s)}{1+v_b(x,s)}},1\right\} \\ &\equiv \bar{q}^*(s'|s)\max\left\{\frac{1+v_l(x',s')}{1+v_b(x',s')}\frac{1}{1+\frac{\varphi(s'|x,s)}{1+v_b(x,s)}},1\right\}. \end{aligned}$$

Since we impose borrowing limits that bind exactly when the participation constraints are binding in the optimal fund contract, asset prices $q(s'|s) = q^*(s'|s)$ satisfy the Euler conditions in the competitive equilibrium characterized above. Therefore, we obtain the ***price of a long-term bond*** $q^f(s) = \sum_{s'|s} q^*(s'|s)$, the implicit interest rate $r^f(s) = \frac{1-\delta+\delta k}{q(s)} - (1-\delta)$, and the ***negative spread***: $r^f(s^t) - r \leq 0$.

Note that both, the lender and the borrower, intertemporal participation constraints cannot be simultaneously binding, as long as there are positive expected rents to be shared. Without *moral hazard* $\varphi(s'|x,s) = 0$ and, therefore, the *negative spread* reflects the fact that the lender intertemporal participation constraint is binding; that is, $q^*(s'|s) > \bar{q}^*(s'|s)$ only if $v_l(x',s') > 0$, in which case $\tau(x',s') \geq 0$, otherwise the lender could simply relax the costly constraint by lending less. However with *moral hazard*, if s' is a bad state it may be the case $\varphi(s'|x,s) < 0$, resulting in a *negative spread* even if the the lender intertemporal participation constraint is not binding

In sum, *the negative spread,* $r^f(s) - r < 0$, reflects a *the wedge* that aligns the market price with the lender unwillingness to lend in some states of the future. Furthermore, if the lender is unconstrained, the borrower must be constrained in those states which are less likely when effort is exercised.

Note also that, given our assumptions, there is a one-to-one correspondence between the state variable x in the FSF problem and a in the decentralized problem, given by:

$$u'(c(a,s)) = \frac{1+v_l(x,s)}{1+v_b(x,s)}\frac{1}{x};$$

that is, if at s, a and x satisfy this one-to-one correspondence, then $c(a,s) = c^*(x,s)$, and, similarly, for the the value functions: $W^b(a,\, s) = V^{bf}(x,s)$.

We now define the Pigou taxes and rewards. Let:

$$\tau^e(s) = \text{NMC(s)} = \widetilde{\xi}(\text{x},\text{s})\text{v}''(\text{e}(\text{x},\text{s})$$

and

$$\begin{aligned}
\tau^r(s'|s) &= \frac{\text{NMB(s}'|\text{s})}{\max\left\{\frac{u'(c(s'))\eta}{u'(c(s))},1\right\}} \\
&= \left[\widetilde{\xi}(x,s)\eta\frac{\partial^2\pi^G(G'|G,e(x,s))/\partial e\partial e}{\pi^G(G'|G,e(x,s))}V^{bf}(x',s') + \frac{1+v_l(x,s)}{x}\frac{\partial\pi^G(G'|G,e(x,s))/\partial e}{\pi^G(G'|G,e(x,s))}V^{lf}(x',s')\right] \\
&\times \left[\max\left\{\frac{1+v_l(x',s')}{(1+v_b(x',s'))}\frac{1}{1+\frac{\varphi(s'|x,s)}{1+v_b(x,s)}},1\right\}\right]^{-1} \\
&\leq \text{NMB(s}'|\text{s}).
\end{aligned}$$

To see that the non-arbitrage condition, $\tau^e(s) = \sum_{s'|s} Q^*(s'|s)\tau^r(s'|s)$, implies $\text{NMC(s)} = \frac{1}{1+\text{r}}\mathbb{E}\left[\text{NMB(s}'|\text{s})\right]$ (i.e. condition (12)) notice that:

$$\begin{aligned}
\text{NMC(s)} = \tau^{\text{e}}(\text{s}) &= \sum_{s'|s} Q^*(s'|s)\tau^r(s'|s) \\
&= \sum_{s'|s}\frac{q^*(s'|s)}{1-\delta+\delta\kappa+\delta q^*(s')}\tau^r(s'|s) \\
&= \frac{1}{1+r}\sum_{s'|s}\pi(s'|s)\max\left\{\frac{u'\left(c\left(s'\right)\right)\eta}{u'\left(c\left(s\right)\right)},1\right\}\tau^r(s'|s) \\
&= \frac{1}{1+r}\sum_{s'|s}\pi(s'|s)\text{NMB(s}'|\text{s}).
\end{aligned}$$

Finally, we use the intertemporal budget constraints to construct the ***asset holdings*** that make the allocations in the optimal contract satisfy the present value budget, namely:

$$\begin{aligned}
a_b\left(s^t\right) &= \sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t} Q^*\left(s^{t+n}|s^t\right)\left[c^*\left(s^{t+n}\right) + \tau^e(s^{t+n}) - \left(\theta(s^{t+n})f\left(n^*(s^{t+n})\right) - G\left(s^{t+n}\right) + \tau^r(s^{t+n}|s^{t+n-1})\right)\right] \\
&= -\sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t} Q^*\left(s^{t+n}|s^t\right)\tau^*\left(s^{t+n}\right) \\
a_l\left(s^t\right) &= \sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t} Q^*\left(s^{t+n}|s^t\right)c_l\left(s^{t+n}\right) = \sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t} Q^*\left(s^{t+n}|s^t\right)\tau^*\left(s^{t+n}\right) \\
a_l\left(s^t\right) &= -a_b\left(s^t\right).
\end{aligned}$$

In this economy, binding participation constraints provide us with the borrowing limits given by

(12) and (13). More precisely,

$$\begin{aligned} A_b\left(s^t\right) &= -\sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t} Q^*\left(s^{t+n}|s^t\right)\left[\theta(s^{t+n})f\left(n_b^*(s^{t+n})\right) - G\left(s^{t+n}\right) + \tau^r(s^{t+n}|s^{t+n-1}) - \tau^e(s^{t+n})\right] \\ &= -\sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t} Q^*\left(s^{t+n}|s^t\right)\left(\tau^*\left(s^{t+n}\right) + c^*\left(s^{t+n}\right)\right) \\ A_l\left(s^t\right) &= Z \\ &= \sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t}\left(\frac{1}{1+r}\right)^t \tau^*\left(s^{t+n}\right) \end{aligned}$$

where the first equality refers to histories $\{s^{t+n}\}_{n=0}^{\infty}$ following a state s^t where the limited enforcement constraint of the borrower is binding (i.e. the borrower is indifferent between remaining in the *FSF* contract and autarky) and, similarly the last equality corresponds to histories following a state where the limited enforcement constraint of the lender, who values transfers at the risk-free interest rate, is binding.

The corresponding *recursive competitive equilibrium* for these *FSF* decentralized economies is also defined in the standard way as a set of policy functions: $c(a_b, s)$, $n(a_b, s)$, $e(a_b, s)$, $a_b'(a_b, s), \tau(a_l, s), a_l'(a_l, s)$ and value functions, W^{bf}, W^{lf}, that solve the agents problems for the corresponding Arrow security prices, $q\left(s'|s\right)$ and, finally, markets clear. In particular, as we have seen, the value functions $(W^{bf}(a_b, s),\ W^{lf}(a_l, s))$ are the mirror image of the value functions $(V^{bf}(x,\ s), V^{lf}(x, s))$, since, given that $a_l = -a_b$, the dimension of the state (co-state) is the same and, as we have seen the allocations are the same.

To conlude some *FSF* accounting is also useful. Paralleling the discussion of the *incomplete markets,* the ***primary surplus*** (primary deficit if negative) is given by

$$\sum_{s'|s} q\left(s'|s\right)\left(a(s') - \delta a(s)\right) - \left(1 - \delta + \delta k\right)a(s) = \tau^*(x, s).$$

5 Calibration

5.1 Functional Forms, Shock Processes and Parameter Values

The model period is assumed to be one year. To make the different contracts comparable, we choose the same parameter values across economies whenever this is possible.

The utility of the borrower is additively separable in consumption and leisure. In particular, we assume

$$\log(c) + \gamma\frac{(1-n)^{1-\sigma} - 1}{1-\sigma} - \omega e^2$$

The preference parameters are set to $\sigma = 0.6887$ and $\gamma = 1.4$. These are used, together with the discount factor $\beta = 0.945$, to match the average hours, in together with the volatility of consumption and hours relative to GDP. The interest rate is set to $r = 2.48\%$, the average short-term real interest rate of German. Note that this implies a different discount factor for the lender of $\frac{1}{1+r} = 0.9758$, as well as a growth rate for the relative Pareto weight of the borrower of $\eta = 0.9684$ in the optimal contract. Regarding the technology, we assume that $f(n) = n^{\alpha}$ with the labor share of the borrower

Table 1: Parameter Values

α	β	σ	γ	r	λ^i	ψ	δ	κ	Z
0.566	0.945	0.6887	1.4	0.0248	0.15	0.8099	0.814	0.083	0

Table 2: Parameters of the labor productivity process

	$\mu(\zeta)$	$\rho(\zeta)$	$\sigma(\zeta)$	P	$\zeta=1$	$\zeta=2$	$\zeta=3$
$\zeta=1$	6.35	0.93	0.02	$\zeta=1$	0.90	0.10	0.00
$\zeta=2$	6.94	0.92	0.01	$\zeta=2$	0.06	0.87	0.07
$\zeta=3$	7.09	0.81	0.02	$\zeta=3$	0.01	0.08	0.91

set to $\alpha = 0.566$ to match the average labor share across the Euro Area 'stressed' countries. The $\omega(e)$ function determining how effort increases the likelihood of good realizations of the G^c shock is $-\exp(-\rho e)$. The participation constraint of the lender in the *FSF* contract is set to $Z = 0$, a very tight level. Finally, the probability that the borrower comes back to the market upon default is set it to $\lambda^i = 0.15$ in the incomplete market model with default, while we assume that *FSF-exit is irreversible*, therefore we set $\lambda^f = 0$. Furthermore, in both models, the default penalty takes the form

$$\theta^p = \begin{cases} \bar{\theta}, & \text{if } \theta \geq \bar{\theta} \\ \theta, & \text{if } \theta < \bar{\theta} \end{cases} \quad \text{with } \bar{\theta} = \psi \mathbb{E}\theta,$$

where $\psi = 0.8099$. The latter two parameters, together with the discount factor β are chosen to match jointly the PIIGS average debt to GDP ratio, spread level and spread volatility. Finally, the parameters of the long term bond (δ, κ) are set to $\delta = 0.814$ and $\kappa = 0.083$ to match the average maturity and the average coupon rate (coupon payment to debt ratio) of long term debt. The following table summarizes all the parameter values.

The log labor productivity $\log \theta$ is assumed to be a Markov regime switching (MRS) AR(1) process. In our calibration, we fit the labor productivity $\log(\theta_{it})$ of five PIIGS countries to the following panel MRS AR(1) model:

$$\log \theta_{it} = (1 - \rho(\zeta_{it}))\mu(\zeta_{it}) + \rho(\zeta_{it}) \log \theta_{it} + \sigma(\zeta_{it})\varepsilon_{it},$$

where $\zeta_{it} \in \{1, \ldots, R\}$ denotes the regime of country i at time t, $\mu(\zeta_{it})$, $\rho(\zeta_{it})$, and $\sigma(\zeta_{it})$ are functions of the regime, and $\varepsilon_{it} \overset{\text{iid}}{\sim} N(0,1)$. The country specific regime s_{it} is independent in the cross-section, and follows a Markov chain over time, with an $R \times R$ regime transition matrix P.

Since our model does not have any capital accumulation, we first calculate time series for the labor productivity data for the 5 Euro Area 'stressed' countries. We then estimate the model above by adapting the expectations maximization (EM) algorithm outlined in **?** to our setup, combined with a more efficient procedure of **?** to calculate the smoothed probabilities of latent regimes. We set $R = 3$ for the panel MRS model in our estimation. Because the likelihood function of the model is highly nonlinear, the EM algorithm of likelihood maximization may be stuck at a local maximum. To overcome this potential deficiency, we randomize the initial point in the parameter space for 1,000 times. The estimated parameters of the Markov Switching Process are displayed below:

Finally, the process is then discretized into a 27-state Markov chain, with 9 values in each regime.

In the benchmark calibration without moral hazard, we consider a simple specification for the cyclical component G_c of the government consumption shock G. In particular, G_c has a state space of $\mathcal{G}_c = \{G_c(1), G_c(2), G_c(3)\}$, with $G_c(1) > G_c(2) > G_c(3)$, and the transition matrix for G_c is pinned down by two parameters[11]:

$$\pi^{G_c} = \begin{bmatrix} \phi & \frac{2}{3}(1-\phi) & \frac{1}{3}(1-\phi) \\ 2\eta & \phi & 1-\phi-2\eta \\ \eta & 1-\phi-\eta & \phi \end{bmatrix}$$

The parameters of the transition matrix are set to $\phi = 0.965$ and $\eta = 0.015$. These parameters, together with the state space for the shock, are used to match several moments of current government expenditures, such as the G to GDP ratio, the persistence of the observed government consumption, and the relative volatility of government consumption with respect to output. The resulting transition matrix and government shock values of G^c are given below:

$$\pi^{G_c} = \begin{bmatrix} 0.9650 & 0.0233 & 0.0117 \\ 0.0300 & 0.9650 & 0.0050 \\ 0.0150 & 0.0200 & 0.9650 \end{bmatrix}$$

$$G^c = \begin{bmatrix} 0.038 & 0.029 & 0.025 \end{bmatrix}$$

For the iid component G_d to government expenditures, we simply assume that is uniformly distributed over $[-\bar{m}, \bar{m}] = [-0.0005, 0.0005]$. In particular, we discretize G_d into $N_d = 11$ equally spaced grid points $\{G_d(1), \ldots, G_d(N_d)\}$ over the previous interval, and set $\Pr(G_d(i)) = 1/N_d$ for all i. Using G_c and the discretized values of G_d, the discretized G shock can be constructed according to:

$$G_{(i-1)N_c+j} = G_d(i) + G_c(j), \quad i = 1, \ldots, N_d, j = 1, \ldots, N_c,$$

where $N_c = 3$. Moreover, with some slight abuse of notation, we define the transition matrix π^G of the discretized G shock to be the Kronecker product of two matrices:

$$\pi^G = \pi^{G_c} \otimes \pi^{G_d},$$

where π^{G_d} is simply an $N_d \times N_d$ matrix with all entries equal to $1/N_d$. Defining π^G this way follows directly from the fact that G_c is independent of G_d. As noted by Chatterjee and Eyigungor (2012), the iid component improve considerably the convergence properties of the model with incomplete markets and default. We also use it to match, together with G^c, the relative volatility of government expenditures.

When moral hazard is effective, we assume that effort only affects the cyclical government liabilities G^c. Given the current G^c there are two possible distributions of tomorrow's $\pi^b(\cdot|G^c)$ and $\pi^g(\cdot|G^c)$, and $\pi^g(\cdot|G^c)$ first order stochastically dominates $\pi^b(\cdot|G^c)$ for all G, with $\pi^G(G'|s,e) = \pi^G(G'|G^c,e)$ and

$$\pi^G(G'|G^c,e) = \exp(-\rho e))\pi^b(G'|G^c) + (1-\exp(-\rho e))\pi^g(G'|G^c)$$

Recall that that government expenditure is ordered in a decreasing way. This implies that increasing effort, ceteris paribus, increases the probability of low government expenditure. Note that this functional form implies simple expressions for $\frac{\partial \pi^G(G'|G,e)}{\partial e}$ and $\frac{\partial^2 \pi^G(G'|G,e)}{\partial e^2}$ as follows:

$$\frac{\partial \pi^G(G'|G,e)}{\partial e} = \rho \exp(-\rho e)[\pi^g(G'|G) - \pi^b(G'|G)]$$

[11] See Appendix for further details.

and

$$\frac{\partial^2 \pi^G(G'|G,e)}{\partial e^2} = -\rho^2 \exp(-\rho e)[\pi^g(G'|G) - \pi^b(G'|G)]$$

5.2 Data Sources and Measurement

The primary data source we use is the AMECO dataset. We use annual data for the 5 Euro Area 'stressed' countries, and except for a few series, the sample coverage is 1980–2015. Table 5 provides a summary of the data sources and definitions. We construct model consistent measures based on the raw data. In what follows, we detail on the sources and measurement methods.

5.2.1 National accounts variables

For the aggregate output Y_{it} and government consumption expenditure G_{it} of each country, we use directly the corresponding data series from AMECO over 1980–2015, measured in constant prices of 2010 euros. Since there is no capital accumulation in the model, we interpret consumption in the model as standing for private absorption, and define the model consistent measure in the data as the sum of the private consumption and gross capital formation. For the aggregate labor input n_{it}, we use two series from AMECO, the aggregate working hours H_{it} and the total employment E_{it} of each country over the period 1980–2015. We calculate the normalized labor input according to $n_{it} = H_{it}/(E_{it} \times 5200)$, assuming 100 hours of allocatable time per worker per week. However, for most parts of the data moments computations, we use H_{it} directly, since the per worker annual working hours do not show a significant cyclical pattern and both the level and the trend do not affect the computation of the moments.

5.2.2 Government bond variables

We use the end-of-year government debt to GDP ratios in AMECO to measure the indebtedness of the Euro Area 'stressed' countries. The government debt is defined as the general government consolidated gross debt. This is conceptually different from the debt in the model, which corresponds to national debt more closely. nevertheless, we use the gross debt measure, as it provides a consistent measure across countries and is arguably an upper limit on the indebtedness of the government.

We use the nominal long-term bond yields in AMECO to measure the nominal borrowing costs of the Euro Area 'stressed' countries, and use short-term interest rates in German to measure the funding cost of international investors. The risk-free rate is measured as the real short-term interest rate of Germany, which equals to the average of the nominal rate minus GDP deflator from 1980–2015. To arrive at a meaningful measure of the *real* spread, i.e., a spread unaffected by expected inflation hence rightly reflect 'stressed' countries' credit risk, we split the sample into to parts by the introduction of the euro. For the first part, 1980–1998, we use spot and forward exchange rates to convert German nominal risk free rate into each stressed country's local currency, hence deriving a synthetic local currency risk free rate, and then take the difference between the local nominal long-term bond yield with the synthetic risk free rate. Since the synthetic risk free rate is denominated in the local currency as well, so it is subject to the same inflation expectations as the long-term bond yield, and consequently, the difference is equivalent to the real spread. For the second part, 1998–2015, we can directly use the spread between the 'stressed' countries' long-term bond yields and

German short-term interest rates, since all rates are denominated in the euro, hence subject to the same inflation expectation.

The information on the maturity structure of the government debt for the Euro Area 'stressed' countries is not comprehensive. We are able to find average years to maturity for the five countries from 3 sources.[12] The overall time coverage is unequal across countries: 1998–2010 and 2014–2015 for Ireland, 1998–2015 for Greece, 1991–2015 for Spain, 1990–2015 for Italy, and 1995–2015 for Portugal.

5.2.3 Fiscal positions

For the model, the theoretically consistent measure of the primary surplus is simply $y - c - G$, i.e., the total saving of the economy. Recall that the primary surplus is defined as government surplus minus interest payments. Alternatively, by the government's budget constraint, the primary surplus can be expressed as the net lending by the government, i.e., the difference between revenue of newly issued debt and payments on interests and retiring debt. For the economy with incomplete markets and default we are considering, this equals to $q_t(b_{t-1} - b_t) - (1 - \delta + \delta\kappa)b_t$, and by the economy's budget constraint, the last expression is just equal to $y_t - c_t - G_t$, which is the measure we use for primary surplus in the model.

To be consistent with the model, we also measure the primary surplus in the data according to the last expression. Since c_t is already measured as the private absorption, i.e., sum of the private consumption and gross capital formation, the empirical measure of the primary surplus is equivalent to the net export by the national accounting identity.

5.2.4 Labor share

We use various data series from AMECO to construct the labor share of annual output for each of the Euro Area 'stressed' countries over the period 1980–2015. First, we use nominal compensation to employees of the total economy in AMECO (labeled by UWCD) to measure the labor income for employees. Second, to measure the labor income for self-employed people, we take the difference between two AMECO series, UOGD and UQGD, where the former is gross operating surplus and the latter is the same measure net off imputed compensation for the self-employed population. We define the total labor income as the sum of the labor income for employees and self-employed, i.e., UWCD + UOGD − UQGD. Finally, the labor share is calculated as the ratio of labor income to nominal GDP.

5.2.5 Labor productivity

Given the production function, $y = \theta n^{\alpha}$, we measure the labor productivity of country i at time t according to $\theta_{it} = Y_{it}/H_{it}^{\alpha}$, or equivalently, $\log \theta_{it} = \log y_{it} - \alpha \log n_{it}$. Note that we use a common α for all ' Euro Area 'stressed' countries. Let $\hat{\theta}^o_{it}$, or $\log \hat{\theta}^o_{it}$, denote the original measured level for labor productivity. To compute the data moments involving the labor productivity, we use the HP-filter to detrend the sample productivity $\{\log \hat{\theta}^o_{it}\}$. Moreover, as described earlier, we use a Markov regime switching model to estimate the productivity process. Before taking the data to the model, we adjust the original sample in the following two steps:

[12] See the appendix for more details.

1. We take out a common linear time trend in the $\{\log \hat{\theta}^o_{it}\}$ series.

2. After detrending, we further standardize $\{\log \hat{\theta}^o_{it}\}$ for each i so that the resulting series has the same sample mean and volatility over i. This is to prevent the level and volatility differences in $\{\log \hat{\theta}^o_{it}\}$ across i to induce spurious regime switching behavior in the estimation process.

We denote the adjusted sample productivity by $\{\log \hat{\theta}_{it}\}$, which is then used in the estimation of the MRS model discussed earlier.

6 The IMD vs. the FSF regimes without moral hazard

This section discusses **the numerical results without moral hazard** (i.e. $v(e) = 0$). We compare the incomplete markets economy with default (**IMD**) and the economy with a *FSF* with two sided limited of commitment (**2S**). We first present calibration results in Table 3 and the *policy functions* for both economies in Figures (1) - (2). To better understand how these economies work, we show representative paths of both economies, subject to the same sequence of shocks in steady state, in Figures (5) - (6).Finally, we study how both economies respond to a combined negative shock when they are in steady state: Figures (7) - (8). TFP shocks are labeled $e_i, i = 1, \ldots, 27$ where $e_i < e_{i+1}$ and G shocks are labeled $g_j, j = 1, \ldots, 3$ where $g_j > g_{j+1}$ –that is (e_1, g_1) **is the worst combination of shocks and, increasingly,** (e_{27}, g_3) **is the best combination of shocks.**

6.1 Calibration results

The following Table 3 provides an exhaustive account of our calibration results.

Table 3: Benchmark calibration

Moments	Data	Model (IMD)
Mean		
Debt to GDP ratio	77.29%	76.56%
Real bond spread	3.88%	3.76%
G to GDP ratio	20.18%	19.62%
Percentile: 1 & 99	$[13.48\%, 32.79\%]$	$[11.56\%, 33.02\%]$
Primary surplus to GDP ratio	-0.78%	1.30%
Fraction of working hours	36.74%	37.28%
Volatility		
$\sigma(C)/\sigma(Y)$	1.49	1.47
$\sigma(N)/\sigma(Y)$	0.92	0.69
$\sigma(G)/\sigma(Y)$	0.91	0.86
$\sigma(PS/Y)/\sigma(Y)$	0.65	0.80
σ(real spread)	1.53%	0.93%
Correlation		
$\rho(C, Y)$	0.88	0.76
$\rho(N, Y)$	0.67	-0.13
$\rho(PS/Y, Y)$	-0.29	0.11
$\rho(G, Y)$	0.35	0.07
$\rho(\text{real spread}, Y)$	-0.35	-0.29
$\rho(G, \theta)$	0.13	0.00
$\rho(G_t, G_{t-1})$	0.94	0.94

6.2 Policy Functions

The core of the analysis is given by the study of the different optimal policy functions. Figures (1) and (2) display the policy functions for the main variables for the incomplete markets economy with default **IMD** and for the **FSF**, as function of the level of debt for selected (intermediate) values of shocks ($s = (\theta, G) = (e, g)$).

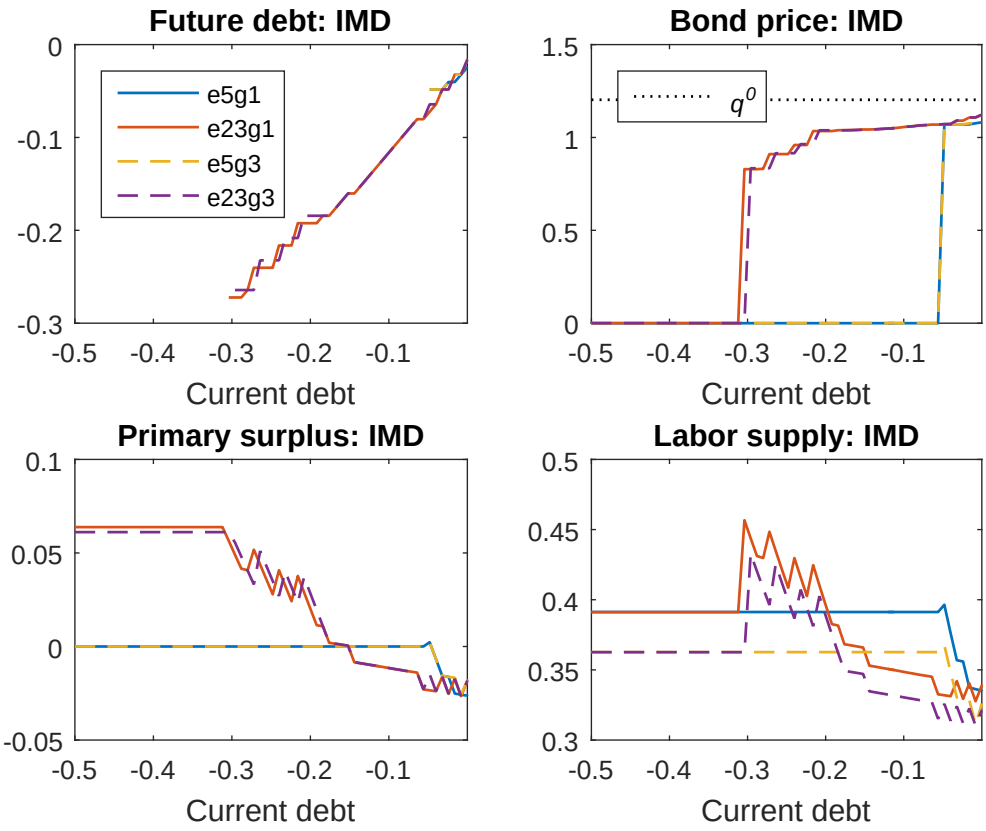


Figure 1: IMD policy functions.

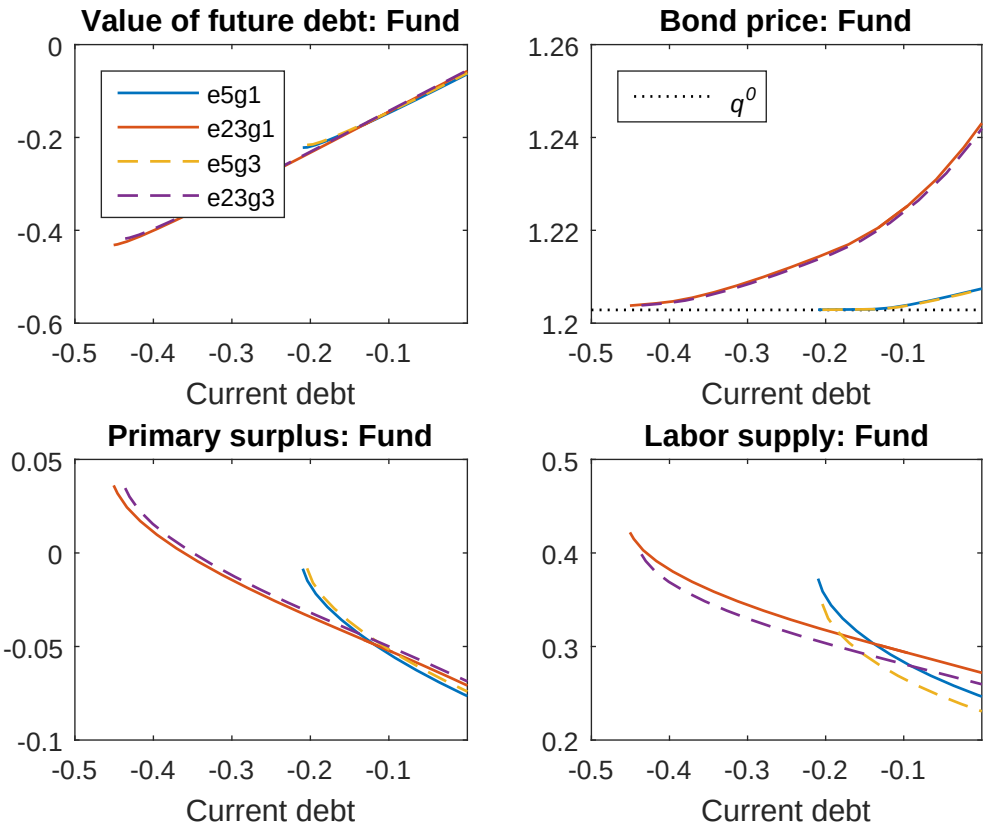


Figure 2: FSF policies as functions b.

The left panels of Figures (1) and (2) show the end-of-period debt and primary surplus policies for the **IMD** economy and the **FSF**. As it can be seen for a relatively bad state ($e5g_j$), the IMD economy only allows a minimum level of debt, while more is borrowed in the FSF, this is true at any state – as it is also true that, within a regime, more can be borrowed in better states. But it can also be seen that in the relatively good state the IMF economy requires to run a (positive) primary surplus for levels of debt for which the FSF has a primary deficit. The upper-right panel shows bond prices for both regimes, in relation to the riskless bond price q^o: *positive spreads* and price collapses, with default, in the IMD economy and, in contrast, *negative spreads* in the FSF regime. The lower-right panel shows how in the IMD economy the labor supply is distorted even at values of debt below the default threshold. It also shows that even in the FSF regime the efficient allocation of labor may be distorted (e.g. at $b = -0.2$ the supply of labor is higher when productivity is lower), although the distortion is more severe in the IMD economy. It also should be noted that expenditure shocks g do not play a major role, in comparison with the productivity shocks e.

While the **IMD** policies are given as functions of (b, s), as in Figure (1), the **FSF** functions of Figure (2) are derived – through 'decentralization'– from underlying policies as functions of (x, s). Therefore, to better understand how the FSF mechanism works it is useful to analyse these policies, which are illustrated in Figures (3) and (4) for the same (intermediate) values of shocks. The upper-left

panel of Figure (3) shows the core of the FSF mechanism. In an economy without limited enforcement or moral hazard constraints a line from the origin with slope η will determine the evolution $x \to x'$, therefore the borrower's relative Pareto weight will monotonically decrease. Such a decay – $x'(x,s) = \eta x$ – is stopped by the borrower's intertemporal participation constraints, which define the horizontal lines to the left of the 'decay line' – we denote them by $\underline{x}(s)$, i.e. $x'(\underline{x}(s),s) = \underline{x}(s)$. On the other hand, the lender's limited enforcement constraints deter x' from being too high, which define the horizontal lines to the right of the 'decay line' – we denote them by $\overline{x}(s)$ – i.e. $x'(\overline{x}(s),s) = \overline{x}(s)$. In particular, if $\underline{x}(e_{27}, g_3) > \overline{x}(e_1, g_1)$ then the support of the steady-state distribution of x is $[\underline{x}(e_1, g_1), \underline{x}(e_{27}, g_3)]$ and the lender's participation constraint is occasionally binding. The other panels show the asset holding and primary surplus policies, as functions of μ_b, as well as the bond price. The patterns of these policies can be traced back to the upper-left panel. Take, for example, the state (e_{23}, g_3), increasing x towards the value of the lender's participation constraint binds (above 0.2) the bond price jumps – i.e. *negative spread* – the primary surplus becomes a primary deficit – i.e. the borrower transfers to the lender – and, consequently, debt is drastically reduced; alternatively, decreasing x to the value that the borrower's participation constraint binds (around 0.15) gives, in the lower-left panel the maximum level of debt that the FSF can have at this state (circa -0.42). Figure (4) shows the FSF consumption and labor policies, as well as the value functions, for the same intermediate states. Again, the patterns of these policies and values can be traced back to the upper-left panel of Figure (3) – in particular, the consumption policy and, more smoothly, the value of the borrower mimic the Pareto weight policy and, obviously the value of the lender mirrors the value of the borrower since both share the surplus of the FSF.

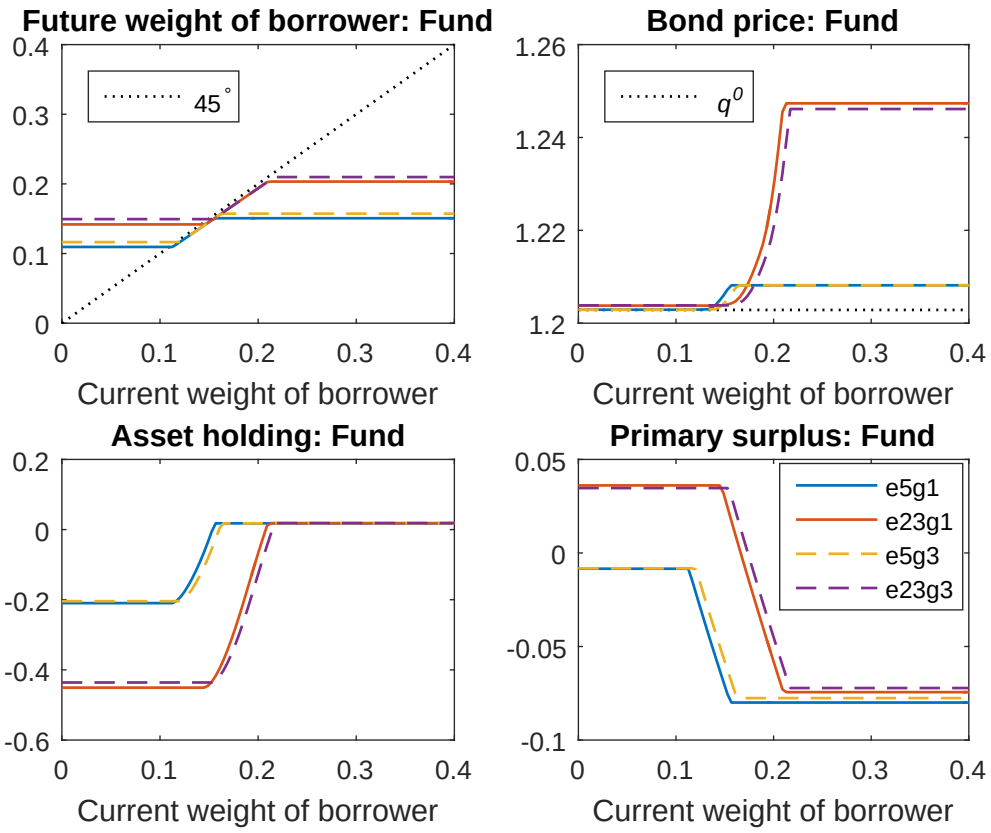


Figure 3: *FSF* policy functions: Pareto weights and assets.

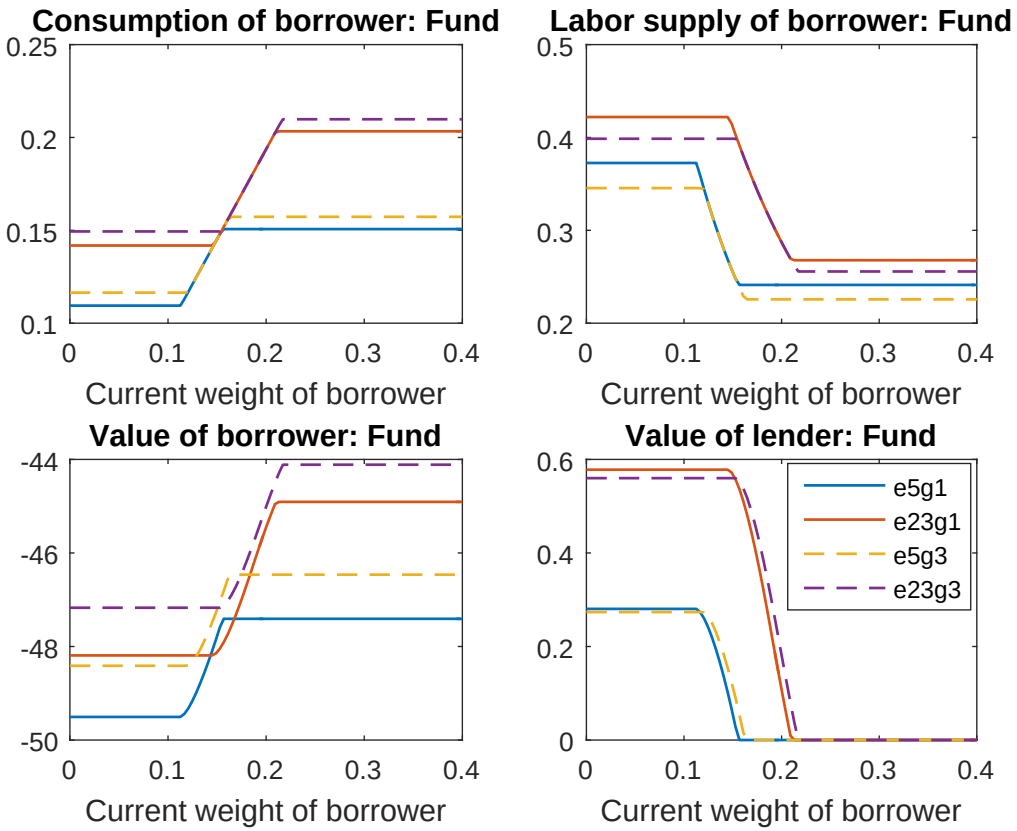


Figure 4: *FSF* policy functions: allocations and values.

6.3 Running the FSF in normal times and in times of crisis.

It is illustrative to simulate the risk-sharing outcomes of the FSF in normal times and in times of crisis. We do it here with two experiments. The first, denoted *Business Cycle Paths* – Figures (5) and (6)– is a long-run simulation at the steady state. In the second, denoted *Impulse Responses* – Figures (7) and (8) – we assume that, independently and simultaneously many independent economies are hit by negative (θ, G) shocks (e_1, g_1) but then all shocks after the initial period follow a realization of the (θ, G) stochastic process; therefore we report the average impulse response from 500 simulations. The initial endogenous conditions are randomly chosen from the stationary distribution.

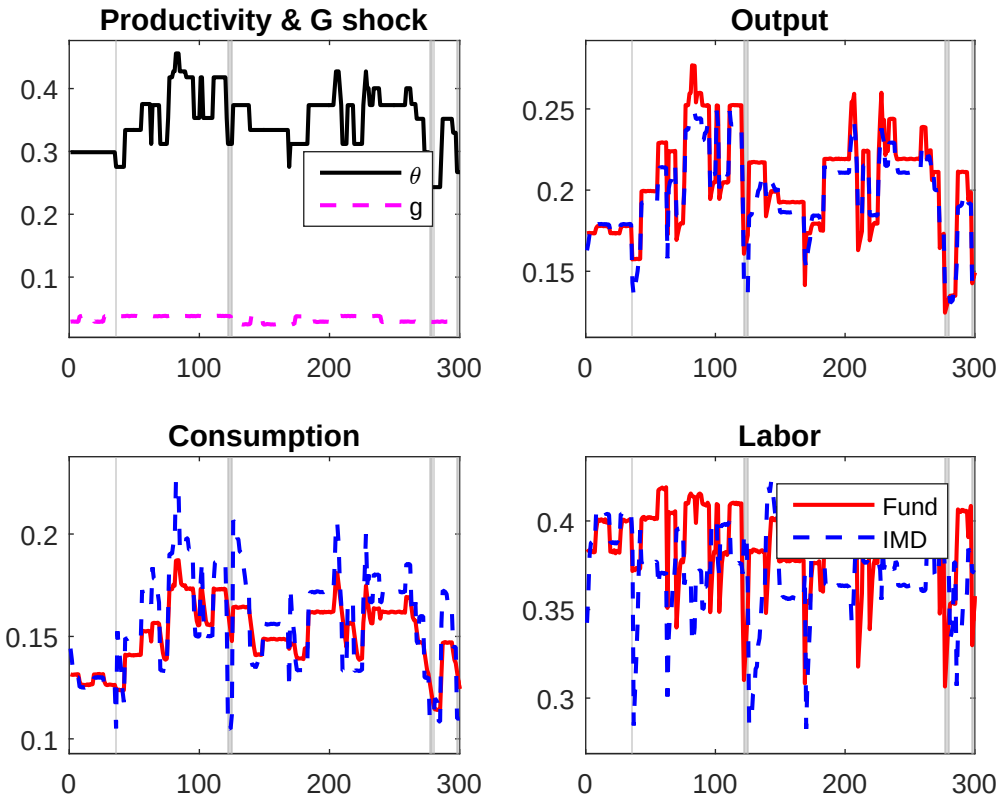


Figure 5: IMD vs. FSF Business Cycle Paths: shocks and allocations.

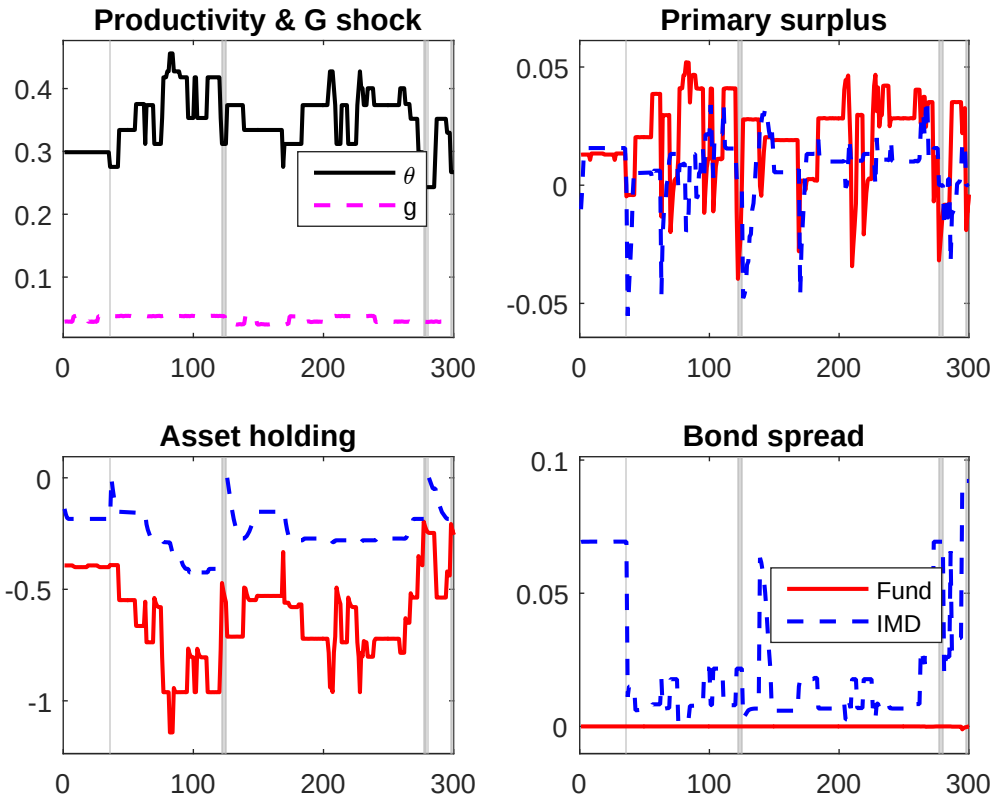


Figure 6: IMD vs. FSF Business Cycle Paths: shocks and assets.

In Figure (7), the upper-left panel shows the history of shocks for three hundred years. The grey periods correspond to periods of default in the IMD economy – defaults are associated with drops in productivity, but not all drops of productivity trigger defaults. The allocations in the IMD and FSF regimes are shown in the other panels. As it can be seen, there is more consumption smoothing in the FSF and default periods are periods of austerity where output, employment and consumption plunge. Figure (8) shows the asset allocations and prices. These panels are very revealing of why some particular productivity drops trigger defaults in the IMD economy. Just observing the evolution of shocks the first default seems puzzling since not much had happened – a small increase in government expenditures followed of a small drop in productivity, – however these were not normal times: the productivity level was medium-low but spreads were high even if there was a primary surplus: the economy was 'stressed' and transfers were countercyclical; fortunately, it was a short default episode. The other two default episodes follow a very familiar pattern: a productivity drop following relatively good years in which debt built up. Life is very different in the FSF regime: debt capacity is substantially larger and good years are years of primary surpluses; transfers from the debtor to the lender are procyclical and only a small episode of *negative spreads* happens towards the end of the series, mirroring the largest positive spread in the IMD economy.

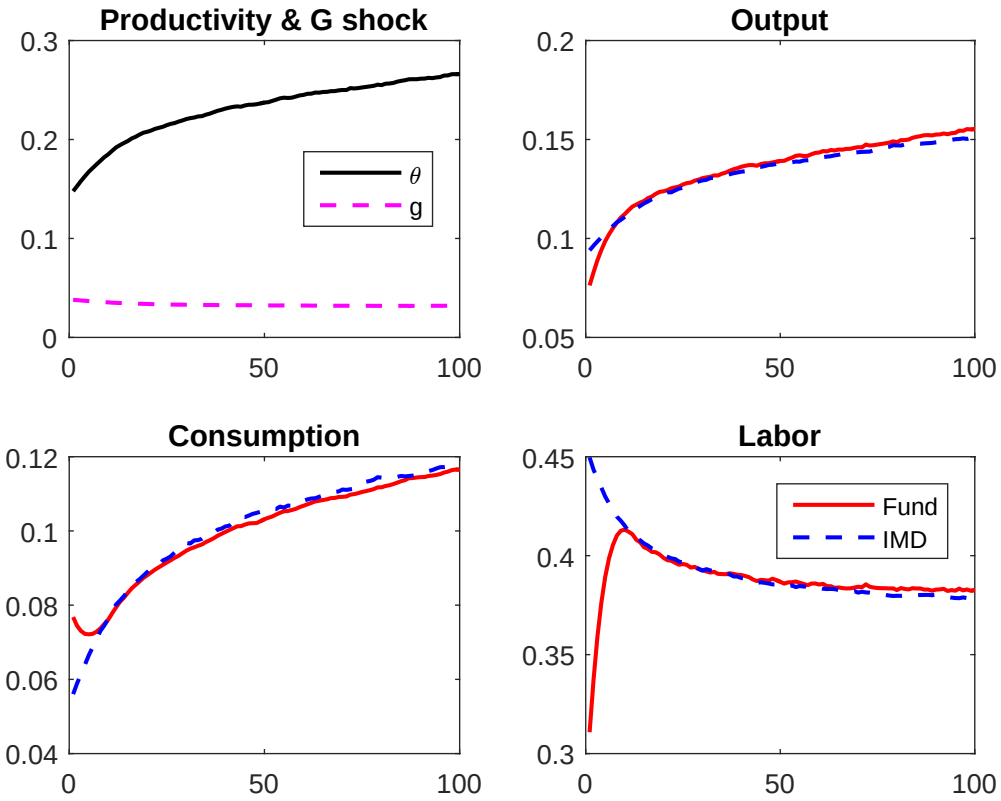


Figure 7: IMD vs. FSF: combined shock impulse-responses: allocations.

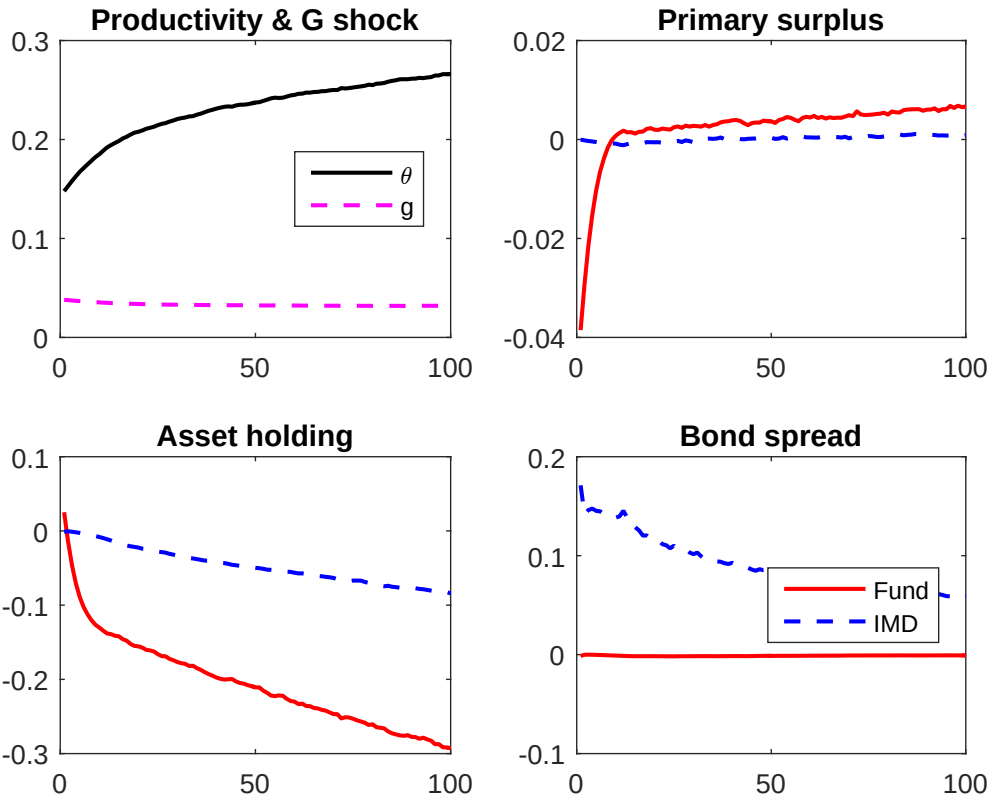


Figure 8: IMD vs. FSF: combined shock impulse-responses: assets.

As it has already been seen in default episodes, life is particularly different in times of crisis. To analyse this in more detail we induce an unexpected negative shock to our economies. Figures (7) and (8) show how the two economies react to a transitory combination of bad (θ, G) shocks. If Figure (7) everything looks very smooth is because the figure depicts average paths – for example, behind the smooth growth of output in the IMD economy there are many episodes of default, as it is reflected in the *positive spreads* of Figure (8). Nevertheless, the averages do not hide that the crisis is more severe in the incomplete markets economy with default (**IMD**) than in the economy with a *FSF*. While the crisis is a severe austerity crisis in the IMD economy, consumption is higher and labor supply lower in the FSF regime, as the immediate response to the negative shock. More remarkable is the fact that in the FSF economy a large primary deficit is allowed following the shock and, consequently, there is debt accumulation. Two facts are behind these patterns. One is the the fact that the borrower is more impatient than the lenders – being the market lenders or the FSF, – the other that the severe negative shock is a rare event. The first explains the wish to front-load consumption, which is partially satisfied with a FSF, but not in the IMD economy; the second the fact that being a rare event the borrower has ample borrowing capacity in the FSF, and none in the IMD economy, as a result in the long-run the primary surplus is higher with a FSF than in the IMD economy, showing that the former is more efficient. We now take a closer look at the relative efficiency, and the associate debt capacity,

H

Table 4: Welfare comparison at zero debt

Shocks (θ, G_c)	Welfare Gain	$(b'/y)_{\max}$: M	$(b'/y)_{\max}$: F
$(\theta_l, G_h) = (0.148, 0.038)$	8.90	1.71	97.42
$(\theta_m, G_h) = (0.299, 0.038)$	7.03	107.55	187.16
$(\theta_h, G_h) = (0.456, 0.038)$	4.68	217.43	336.77
$(\theta_l, G_l) = (0.148, 0.025)$	7.87	1.84	101.89
$(\theta_m, G_l) = (0.299, 0.025)$	6.56	111.40	187.93
$(\theta_h, G_l) = (0.456, 0.025)$	4.46	217.80	334.47
Average	**6.53**		

of these economies.

6.4 Welfare implications and confronting 'debt overhang' problems

Table 4 shows the increased capacity to absorb debts and the welfare gains of the FSF regime in comparison with the IMD economy. The table is constructed as follows:

- Welfare gains are in (annual) consumption equivalent terms at $b = 0$ (%).
- b'/y is the end-of-period debt (i.e., next period debt) to output ratio in percentage, for Market or Fund. For the fund solution, b' denotes the expected value of assets in terms of the next period goods. More specifically, we define

$$b'(s) = \frac{\sum_{s'|s} q(s'|s)a'(s')}{\sum_{s'|s} q(s'|s)}.$$

  Conceptually, this is the closest measure of country's indebtedness in the fund economy to the counterpart in the incomplete market with default economy.

Comparison of b'/y across the IMD economy and the economy with a FSF shows the difference of the debt and risk-sharing capacities between the two economic regimes; in particular $(b'/y)_{\max}$ shows the maximum amount of debt to output ratio the country can have in any of the two. Only the FSF is able to absorb close to ratios close to 100% in all the states, while the capacity to absorb debts in the IMD economy is substantially smaller particularly in bad states. Welfare gains are very substantial, a consumption equivalent steady-state average welfare gain of 6.5 is a very high number and, even more relevant, a 8.9 in the worst state. Nevertheless, this is the order of magnitude that we have obtained in many other simulations that we had run as robustness check.

Table 4 suggests the following experiment: how would a highly indebted economy behave in the two different regimes? In other words, which economy will be able to deal better if there is a 'debt overhang' problem? While the table already gives an answer to this question, the details are illustrative and can be seen, for example, in Figures (9) and (10), where we started an economy with steady-state average shocks and a 100% b'/y. The IMD economy almost immediately goes into default, getting rid of its debt liabilities, and after that there is a negative widening gap on output and consumption

in comparison with the defaultess economy with a FSF. In fact, the mirror of the pre-default in the IMD economy is a small *negative spread* and a subsequent increase in debt in the FSF regime. Telling figures!

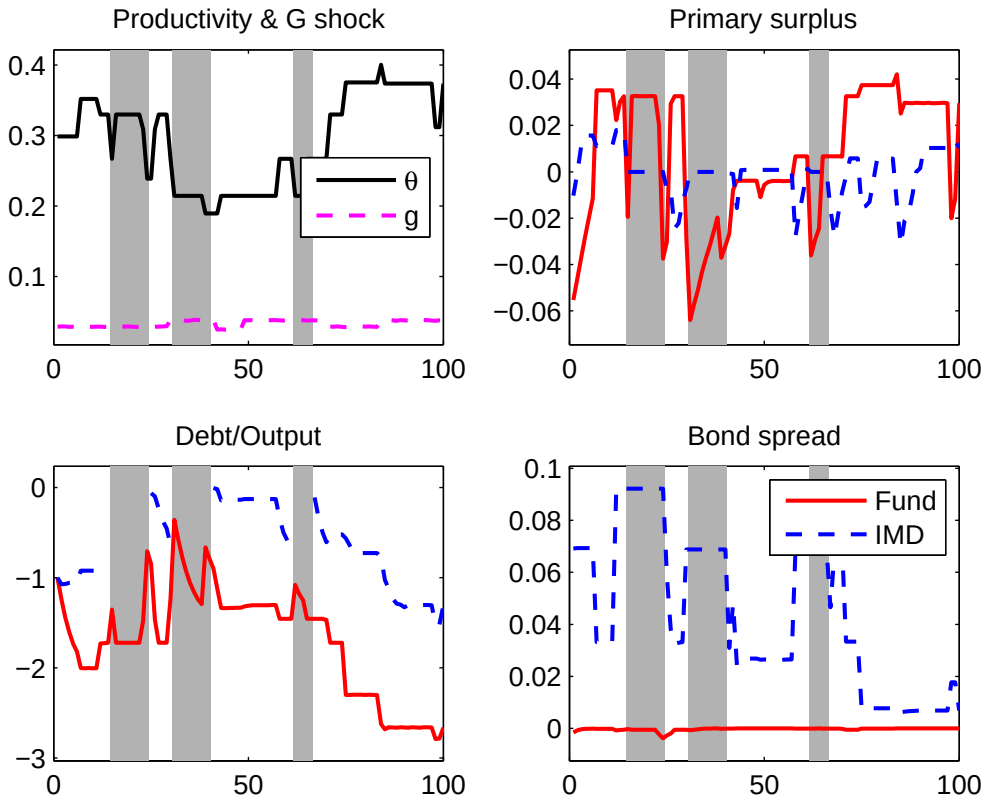


Figure 9: IMD vs. FSF in highly indebted economy: debts and spreads

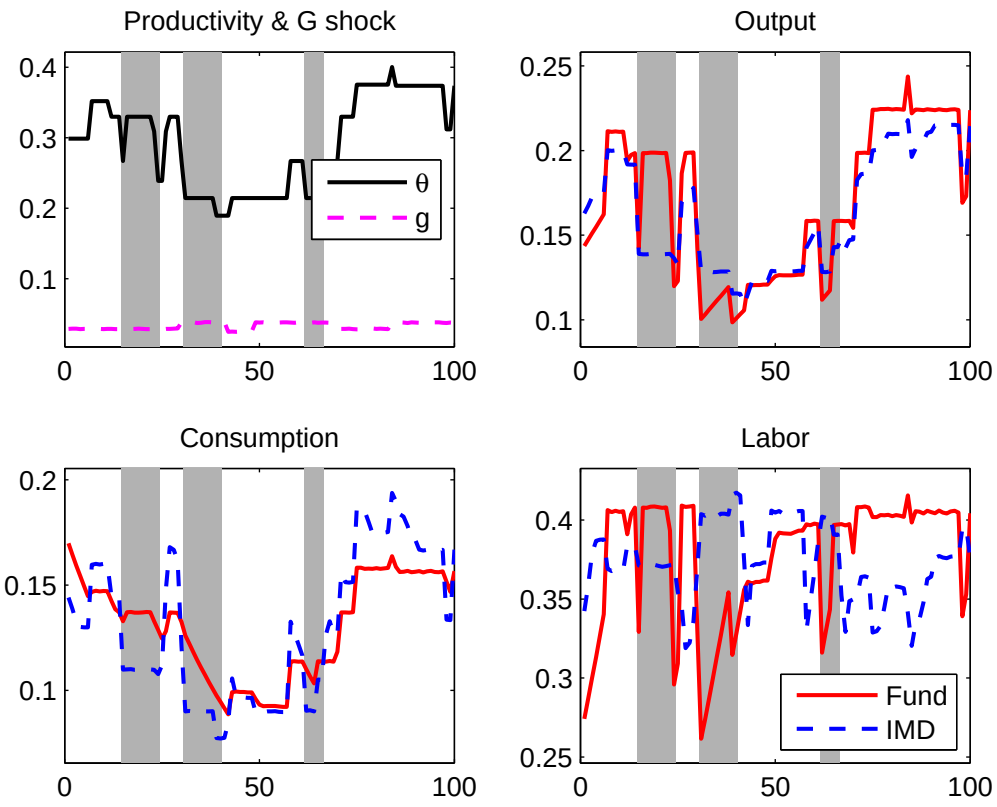


Figure 10: IMD vs. FSF in highly indebted economy: allocations

7 The IMD vs. the FSF regimes with moral hazard

In the previous simulations the distribution of the G shock was exogeneous. In this Section the distribution of G depends on the effort, e, that the borrower exercises. In particular we analyse how FSF policies change when the *moral hazard* problem is also accounted for. As we know from our calibrated exercises, that only accounted for *limited enforcement constraints*, government expenditure shocks, G, do not play a major role. Since in our formulation the unobservable effort only affects the distribution of G^c shocks, we should not expect substantial differences if the *incentive compatibility constraint* is introduced, but it is illustrative to quantify the effect. Furthermore, we want to understand how the different constraints interact and, as we know from our theory, with *moral hazard negative spreads* may be more frequent, but how much bond prices are distorted by the introduction of the *incentive compatibility* constraint? To answer these questions we study the policy functions in an economy with two-sided limited commitment (limited enforcement) with observable effort and without observable effort. Without, the economy with the FSF is the same than the economy studied in the previous Section, except in two features: first, the distribution of G is endogenous – although, the calibration is parameterised as to have similar G distributions at the steady-state, – and, second, the outside value for the borrower upon quitting the fund is autarky without the possibility to borrow in the market in

the future; however, since the default penalty is very similar the difference should not be significant[13]

Figure 11 plots the Pareto weight policies with observable and unobservable effort (i.e. with *moral hazard*). With observable effort, as in the previous analysed case of an exogenous G_t process, there is no *incentive compatibility* constraint, while with unobservable effort the contract must account for the $\varphi(G_{t+1}|G_t)$ multiplier (the effect of this is shown for $g1$ and $g3$). As it can be seen, there is a small spread for the participation constraints.

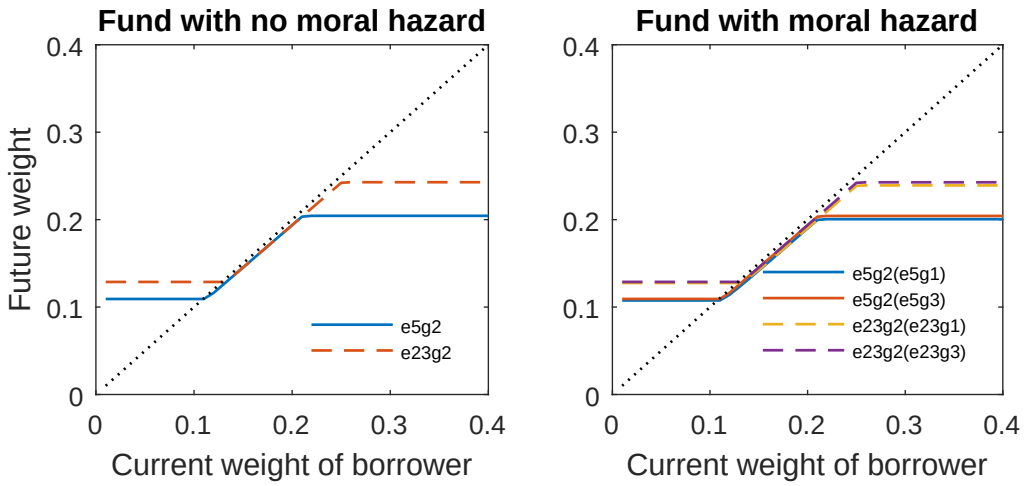


Figure 11: Pareto weight policies with exogenous and non-observable endogenous effort.

Figure 12 shows the effort policies. The differences 'at the participation constraints' are noticeable – in particular with moral hazard, effort, conditional on a relatively good ($e23g2$) and bad ($e5g2$) state, varies more when the lender's participation constraint is binding and less when the borrower's participation constraint is binding. The effect of binding *incentive compatibility* constraints – in particular, 'at, and near, the participation constraints'- translates into *negative spreads* with moral hazard that do not exist when effort is observable, as it is shown in the bottom two panels of Figure 12. In sum, the main differences of introducing *moral hazard* are the distortions created by the interplay between participation and incentive compatibility constraints and in the existence of more frequent *negative spread* episodes.

[13]This difference on default values will be eliminated in future versions.

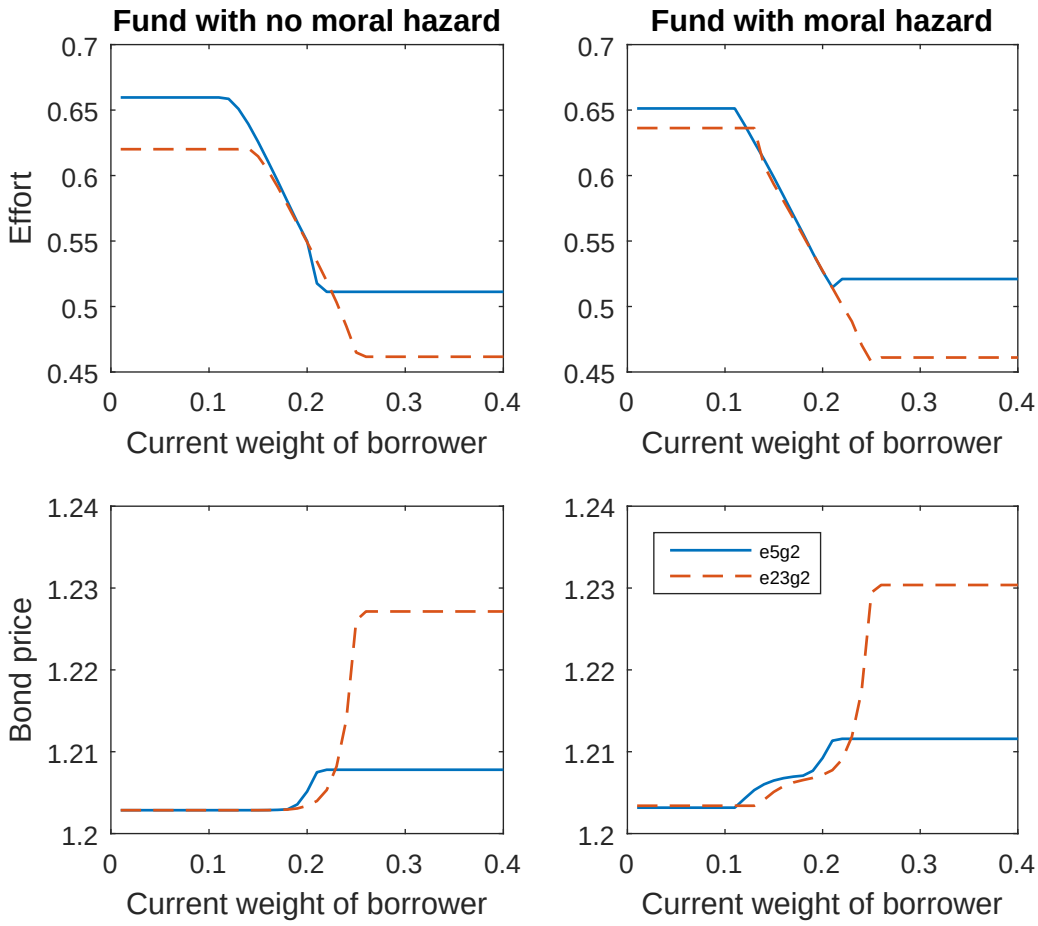


Figure 12: Effort and bond price policies: observable vs. non-observable effort

8 Simplifying the Fund contract

The *Financial Stability Fund* (FSF) described in Section 3 is complete in the sense that, subject to enforcement and incentive compatibility constraints, the contract is contingent to all shocks. However it may be difficult to implement a contract with many contingencies, even more to decentralize it. In this section we show how the Fund contract can be simplified by making it less contingent, provided that the total amount of debt – with the Fund and outside the fund – can be constrained. We also show alternative ways that the *simplified Fund contract* can be implemented. Finally, we calibrate the simpler contract to quantify the loss of efficiency due to the loss of contingency or, in other words, *limited risk-sharing*[14].

First, we consider a partition of S: $\{S_1, \ldots, S_n\}, S = \cup_{i=1}^{n} S_i; S_i \cap S_j = \emptyset$ if $i \neq j$. If $s = (\theta, G^c) \in S_i$ then we also denote S_i by $\bar{s} \equiv S_i$, i.e. $\bar{s}' = \bar{s}$ if and only if $s \in S_i$ and $s' \in S_i$ for some $i = 1, \ldots, n$; similarly, $\bar{s}^t = (s_0, \bar{s}_1, \ldots, \bar{s}_t)$. Regarding G shocks, to simplify the exposition, we will only consider

[14] Work in progress.

two extreme cases: non-contingent G shocks – $(\theta, \bar{G}_i) = (\theta, \bar{G}_j)$ if $j \neq i$ (i.e. no moral hazard) – and full-contingent G shocks – $(\theta, \bar{G}_i) \neq (\theta, \bar{G}_j)$ if $j \neq i$ (i.e. moral hazard). Unless we state otherwise, we will refer to the latter case. With this notation, the *simplified Fund contract* is the solution to the following problem:

$$\max_{\{c(s^t), n(s^t), e(s^t)\}} \mathrm{E}\left[\mu_{b,0} \sum_{t=0}^{\infty} \beta^t \left[U(c(s^t),\, n(s^t), e(s^t))\right] + \mu_{l,0} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \tau(s^t) \mid s_0\right]$$

$$\text{s.t.} \qquad \mathrm{E}\left[\sum_{r=t}^{\infty} \beta^{r-t} \left[U(c(s^r),\, n(s^r),\, e(s^r))\right] \mid s^t\right] \geq \max_{s \in \bar{s}_t} V^a(s)\,, \tag{14}$$

$$v'(e(s^t)) = \beta \sum_{s^{t+1}|s^t} \pi^{\theta}(\theta'|\theta) \frac{\partial \pi(s^{t+1}|s_t,\, e(s^t))}{\partial e(s^t)} V^{bf}(s^{t+1}), \tag{15}$$

$$\mathrm{E}\left[\sum_{r=t}^{\infty} \left(\frac{1}{1+r}\right)^{r-t} \tau(s^r) \mid s_t\right] \geq Z, \tag{16}$$

$$\text{and} \qquad \tau(s^t) = \theta(s^t) f\left(n(s^t)\right) - c(s^t) - G(s^t),\ \forall s^t, t \geq 0.$$

The borrower's *limited enforcement constraints* of the *simplified Fund contract* (14) are *robust* to deviations within subsets of states. That is, the highest outside value that the borrower can achieve within the states in $\bar{s}$ becomes the effective common outside value and, therefore, in the recursive contract formulation $\gamma_{b,t}(s) = \gamma_{b,t}(s')$ if $s' \in \bar{s}$, and we will use the notation $\gamma_{b,t}(\bar{s})$. Note that $\gamma_{b,t}(\bar{s}) \geq \gamma^*_{b,t}(s)$, where $\gamma^*_{b,t}(s)$ is the Lagrange multiplier of the *limited enforcement constraint* in the full-contingent fund contract (5). In this formulation, transfers are residual and, therefore fully state-dependent. The rest of the contract has the same characterization of the *Fund contract* discussed in Section 3. In particular, $c^b(x,s)$ and $n^b(x,s)$ satisfy:

$$u'(c^b(x,s)) = \frac{1 + v_l(x,s)}{1 + v_b(x,\bar{s})} \frac{1}{x}, \quad \frac{h'(1 - n^b(x,s))}{u'(c^b(x,\bar{s}))} = \theta f'(n^b(x,s)) \text{ and}$$

$$\tau(x,s) = \theta(s) f\left(n^b(x,s)\right) - c^b(x,s) - G(\bar{s}).$$

That is, as long as the lender's limited enforcement constraint is not binding $c^b(x,s) = c^b(x,\bar{s})$, while by construction: $G(s) = G(\bar{s})$.

8.1 Decentralization of the *simplified Fund contract*

We first show that decentralization of the *simplified Fund contract* can take the same form than the one discussed in Section 4. In particular, trading *state-contingent long-term bonds* the problem of the borrower becomes:

$$W^b(a,\, s) = \max_{(c,\, n,\, e,\, a'(s'))} \left\{U(c,\, n,\, e) + \beta \mathbb{E}\left[W^b(a', s') \mid s\right]\right\}$$

$$\text{s.t. } c + \sum_{s'|s} q\,(s'|s)\,(a(s') - \delta a(s)) \leq \theta(s) f(n) - G(s) + (1 - \delta + \delta\kappa)\, a(s)$$

$$a'(s') \geq A_b\,(\bar{s}')\,,$$

where the only apparent difference is the endogenous asset-limit constraint, which satisfies: $W^b(A_b(\bar{s}), s) = V^a(\bar{s}) \equiv \max_{s \in \bar{s}_t} V^a(s)$. As in Section 4,

$$q(s'|s) = \bar{q}(s'|s) \max\left\{\frac{u'\left(c^b(x', s')\right)\eta}{u'\left(c^b(x, s)\right)}, 1\right\} = \bar{q}(s'|s) \max\left\{\frac{1 + v_l(x', s')}{1 + v_b(x', \bar{s}')} \frac{1}{1 + \frac{\varphi(s'|x,s)}{1+v_b(x,\bar{s})}}, 1\right\}.$$

In particular, without moral hazard, as long as the lender's limited enforcement constraint is not binding $q(s'|s) = \bar{q}(s'|s)$, and borrower's (marginal utility of) consumption smoothly decays at the rate η, until the lower bound $V^a(\bar{s})$ is reached. The problem of the lender is the same than in Section 4, although equilibrium prices and transfers (i.e. lender's asset holdings) are not the same[15].

Note that without moral hazard, while consumption and asset prices are not very sensitive to the state s (but they are to $\bar{s}$) labour and transfers are[16]. This suggests a further simplification of the Fund contract: replace the lender's limited enforcement constraints (16) by:

$$\min_{s \in \bar{s}_t} \mathrm{E}\left[\sum_{r=t}^{\infty} \left(\frac{1}{1+r}\right)^{r-t} \tau(s^r) \mid s_t\right] \geq Z; \tag{17}$$

that is, (17) is binding in state $s \in \bar{s}_t$ if the worst expected discounted value of the transfers to the lender, conditional on a state of $\bar{s}_t$, is equal to Z; therefore (17) is either binding or not binding for all the states in $\bar{s}_t$. In this case, Fund policies $c^b(x, \bar{s})$ and $n^b(x, \bar{s})$ satisfy:

$$u'(c^b(x, \bar{s})) = \frac{1 + v_l(x, \bar{s})}{1 + v_b(x, \bar{s})} \frac{1}{x}, \quad \text{and} \quad \frac{h'(1 - n^b(x, s))}{u'(c^b(x, \bar{s}))} = \theta f'(n^b(x, s)).$$

We can now decentralize the Fund contract, with limited enforcement constraints (14) and (17), with long-term $\bar{s}$-state-contingent bonds (i.e. with n assets instead of $|S|$) and, for example, one period non-contingent debt contracts. The problem of the borrower becomes:

$$\begin{aligned}
&\tilde{W}^b(a, b, s) = \max_{(c,\, n,\, e,\, a'(\bar{s}'),b')} \left\{U(c, n, e) + \beta \mathbb{E}\left[\tilde{W}^b(a', b', s') \mid s\right]\right\} \\
&\text{s.t. } c + \sum_{s'|s} q(\bar{s}'|\bar{s})\left(a(\bar{s}') - \delta a(\bar{s})\right) + q^b(\bar{s}) b' \leq \theta(s) f(n) - G(s) + (1 - \delta + \delta\kappa) a(\bar{s}) + b \\
&\quad a'(\bar{s}') + b' \geq A_b(\bar{s}'),
\end{aligned}$$

where $q(\bar{s}'|\bar{s})$ is defined as before, with $\bar{s}$ instead of s and

$$q^b(\bar{s}) = \frac{1}{1+r} \sum_{\bar{s}'|\bar{s}} \pi(\bar{s}'|\bar{s}) \max\left\{\frac{1 + v_l(x', \bar{s}')}{1 + v_b(x', \bar{s}')} \frac{1}{1 + \frac{\varphi(\bar{s}'|x,\bar{s})}{1+v_b(x,\bar{s})}}, 1\right\}.$$

The problem of the lender is:

$$\begin{aligned}
&W^l(a, b, s) = \max_{(c,\, a'(s'))} \left\{c + \frac{1}{1+r} \mathbb{E}\left[W^l(a', b', \bar{s}') \mid \bar{s}\right]\right\} \\
&\text{s.t. } c + \sum_{s'|s} q(\bar{s}'|\bar{s})\left(a(\bar{s}') - \delta a(\bar{s})\right) - q^b(\bar{s}) b' = (1 - \delta + \delta\kappa) a(\bar{s}) - b \\
&\quad a'(s') - b' \geq A_l(s'),
\end{aligned} \tag{18}$$

[15] We do not account for Piguvian taxes here since are not needed to determine equilibrium prices.

[16] And with moral hazard the same insensitivity statement is true changing s and $\bar{s}$ by θ and $\bar{\theta}$ respectively.

where $W^l(A_l(s), s) = Z$.

Note that we assume that the lender buys the one-period bonds of the borrower to close the model, but in the interpretation of the lender as the Fund there is no need for the Fund to hold the debt, which can be held by the private sector. However, the lender must account for the amount of new debt in order to guarantee that the lending constraint (18) is satisfied.

9 Conclusions

By developing and computing a model of a *Financial Stability Fund* as a *constrained efficient mechanism* we have contributed to the existing literature on risk-sharing and sovereign debt, and provided a useful instrument to study how to design it – with more engineering and legal detail – and the gains of implementing it. In particular, we have quantitatively shown that the visible welfare gains can be substantial, even if we have calibrated the model to euro area 'stressed countries', and we have set a 'tight constraint' on risk-sharing transfers: the fund should always have non-zero expected profits from the FSF contract. We have also shown that accounting for *moral hazard* does not substantially change the FSF allocations, whereas incentive compatibility constraints interact with limited enforcement constraints, distorting effort and making *negative spreads* more likely to emerge. In our economies, the moral hazard problem only affects the distribution of government expenditures. If, however, it were to affect productivity shocks too (e.g. through costly structural reforms) the effect may be greater. More work needs to be done, in particular in making our *FSF* model a workable proposal within the EU or EA legal and institutional framework. As we have seen, the *FSF* can also be used to address sovereign 'debt overhang' problems. In our formulation, the self-enforcing stabilisation nature of the *FSF* is what gives it its credibility and its capacity to absorb large existing debts – or provide generous credit in times of crisis – in contrast with existing debt market instruments. On the other hand, existing crisis-resolution institutions – such as the ESM – may be able to absorb relatively large debts, but they are not usually designed as *constrained efficient mechanisms*[17]. Our work may be useful to them.

Appendix

9.1 Data sources

9.2 Solution Method

9.2.1 The Solution of the IMD

In what follows, we describe the computational algorithm to solve for the IMD model with no moral hazard.

Solving for the labor supply For given (s, b) and b', we can solve for the optimal labor from the optimality condition. If the borrower chooses not to default, the optimal labor supply n^* solves:

$$h(n) \equiv (\theta n^{\alpha} - \chi)\, n^{1-\alpha} - \vartheta(1-n)^{\sigma} = 0$$

where $\vartheta = (\theta\alpha)/\gamma > 0$ and $\chi = G - (1 - \delta + \delta\kappa)b + q(s, b')(b' - \delta b)$. Since $h(1) = (\theta - \chi)$ and $h(0) = -\vartheta < 0$, there exists an $n^* \in (0, 1)$ such that $h(n^*) = 0$ and $c^* > 0$ if and only if $\theta - \chi > 0$. It is easy to show that n^* is unique. If the borrower chooses to default, we can use the same condition with $\vartheta = \theta^p\alpha/\gamma$ and $\chi = G$.

[17] For example, as of May 2017, the ESM is holding 49.4% of Greece's sovereign debt (which amounts to 88.5% of Greece GDP) as long-term, over 30 years, unconditional debt.

Table 5: Data sources and definitions

Series	Time periods	Sources	Unit
Output	1980–2015	AMECO (OVGD)[a]	1 billion 2010 constant euro
Private consumption	1980–2015	AMECO (OCPH)	1 billion 2010 constant euro
Government consump.	1980–2015	AMECO (OCTG)	1 billion 2010 constant euro
Working hours	1980–2015	AMECO (NLHT)[b]	1 million hours
Employment	1980–2015	AMECO (NETD)	1000 persons
Government debt	1980–2015	AMECO EDP[c]	end-of-year percentage of GDP
Primary surplus	1980–2015	AMECO (UBLGIE)[d]	end-of-year percentage of GDP
Bond yields	1980–2015	AMECO (ILN)[e]	percentage, nominal
Inflation rate	1980–2015	AMECO (PVGD)	percentage, GDP deflator
Debt maturity	1990–2010	OECD[f]	years
Labor share	1980–2015	AMECO[†]	percentage

a Strings in parentheses indicate AMECO labels of data series.

b PWT 8.1 values for Greece in 1980–1982.

c General government consolidated gross debt; ESA 2010 and former definition, linked series.

d AMECO linked series for 1995–2015; European Commission *General Government Data* (GDD 2002) for 1980–1995.

e A few missing values for Greece and Portugal replaced by Eurostat long-term government bond yields.

f Differing time coverage across countries; see the text for details.

† Calculated based on various series on labor compensation; see the text for details.

In what follows, we denote by $N_{\rm nd}(s,b,b')$ the optimal labor supply in the case of no default, given the current state (s,b) and the bond choice for the next period b'; and we use $N_{\rm d}(s)$ to denote the optimal labor supply in the case of default. Here we have chosen to suppress the dependence of $N_{\rm nd}$ on the bond price $q(s,b')$ for two reasons: first, given any pricing function $q(\cdot)$, the specific value of the bond price is determined by (s,b'); and second, to enhance computational efficiency, we will rewrite $N_{\rm nd}(\cdot)$ as a function of θ and χ, where χ summarizes all the dependence of $N_{\rm nd}$ on G, b, b', and $q(s,b')$.

Solving the Bellman Equation To find a solution to the model, we combine equations (1)-(3) as well as the pricing equation in (4) into one Bellman equation of four functions: three value functions and one pricing function. We can then use backward induction to solve the functional equation. More precisely, let $V^{bi}(s,b;k-1)$, $V_n^{bi}(s,b;k-1)$, $V^{ai}(s;k-1)$, and $q(s,b';k-1)$ denote the value and pricing functions obtained in the k'th iteration. We first solve

$$V_n^{bi}(s,b;k) = \max_{c,b'} \; U(c,1-N_{\rm nd}(s,b,b';k)) + \beta\mathbb{E}\left[V^{bi}(s',b';k-1)|s\right] \tag{1}$$
$$\text{s.t. } c + q(s,b';k)(b'-\delta b) \le \theta[N_{\rm nd}(s,b,b';k)^{\alpha} - G_c - G_d + (1-\delta+\delta\kappa)b,$$

and

$$V^{ai}(s;k) = U(c,1-N_{\rm d}(s)) + \beta\mathbb{E}\left[(1-\lambda)V^{ai}(s';k-1) + \lambda V^{bi}(s',0;k-1)|s\right] \tag{2}$$
$$\text{s.t. } c = \theta^p[N_{\rm d}(s)]^{\alpha} - G_c - G_d,$$

so that

$$V^{bi}(s,b;k) = \max\{V_n^{bi}(s,b;k), V^{ai}(s;k)\}. \tag{19}$$

As explained earlier, we denote the labor supply function in the no default case by $N_{\rm nd}(s,b,b';k)$ to make explicit the dependence of $N_{\rm nd}(\cdot)$ on the bond pricing function $q(\cdot;k)$ in each iteration. This is a standard dynamic programming problem that delivers value and policy functions for consumption, labor and bond choices, as well as default decisions. Once we have these, we can update the pricing function via

$$q(s,b';k+1)=\mathbb{E}\left[(1-D(s',b';k))\frac{(1-\delta)+\delta[\kappa+q(s',b(s',b';k);k)]}{1+r}|s\right], \tag{20}$$

where $D(s,b;k)$ and $b(s,b;k)$ are the default and bond holding decisions obtained in iteration k. In general, this shows that $q(\cdot;k)$ is obtained in iteration $k-1$.

To implement the backward induction algorithm, we use discrete space value function iteration. Since (θ,G_c) is discrete by assumption, we only need to discretize G_d and b. In particular, we set G_d to be equally spaced over $[-\bar{m},\bar{m}]$ with N_d grid points, and with equal probability on each grid point for simplicity. Moreover, we discretize the bond holding space $\mathcal{B}$ with N_b grid points. We iterate on the value function and the pricing function on the discretized space $\Theta\times\mathcal{G}_c\times\mathcal{G}_d\times\mathcal{B}$ until convergece, namely, until

$$\max|V^{bi}(s,b;k)-V^{bi}(s,b;k+1)| \text{ and } \max|q(s,b';k)-q(s,b';k+1)|$$

are both smaller than some convergence criterion. Moreover, we use two parameters $\zeta_V,\zeta_q\in[0,1]$ to control the updating speed of $V^{bi}(\cdot)$ and $q(\cdot)$ as follows:

$$\begin{aligned}V^{bi}(s,b;k+1)&=\zeta_V V^{bi}(s,b;k)+(1-\zeta_V)\text{RHS of (19)},\\ q(s,b';k+1)&=\zeta_q q(s,b';k)+(1-\zeta_q)\text{RHS of (20)}.\end{aligned}$$

Setting $\zeta_q>0$ is useful for the convergence of $q(\cdot)$ as well.

Note that it is important to have a continuously distributed G_d to smooth off discrete changes in $D(s,b)$ and enhance the convergence properties of the model. In principle, we could keep G_d as a continuous state variable in the computation, and use the involved procedure of Chatterjee and Eyigungor (2012) to obtain the functions $D(\cdot,G_d,\cdot)$ and $b(\cdot,G_d,\cdot)$ accurately. Instead, we use a discrete approximation of G_d, which is straightforward to implement, and we find that such an approximation works good enough to improve the convergence properties of the algorithm to compute our model.

Note also that $q(s,b')$ does not depend on G_d, and this simplifies the iterations of $q(s,b';k)$. Also, in the backward iteration, we use the fact that $q(s',b(s',b';k);k)$ is simply the equilibrium bond price in state (s',b'), the value of which has already being computed in solving for the optimal bond choice $b(s',b';k)$. Let $q^{\rm e}(s,b;k)$ denote the equilibrium bond price in state (s,b) under optimal bond choice $b(s,b;k)$, then (20) can be simplified into

$$q(s,b';k+1)=\mathbb{E}\left[(1-D(s',b';k))\frac{(1-\delta)+\delta[\kappa+q^{\rm e}(s',b';k)]}{1+r}|s\right].$$

The above expression implies that two equivalent ways of updating the bond prices. The first is to compute $q(\cdot;k+1)$ in the k'th iteration, after obtaining the default decision $D(\cdot;k)$. The second is to compute $q(\cdot;k+1)$ in iteration $k+1$ for each bond choice b', using default decisions obtained in the previous iteration. The latter will be useful when we implement the moral hazard case.

Improving on Efficiency In the preceeding algorithm, we do the computation of optimal labor supply $N_{\rm nd}(s,b,b')$ under no default *within* the main loop. To improve on efficiency we can use an approximation of $N_{\rm nd}(s,b,b';k)$. As shown before, for given (s,b,b') and bond pricing function $q(s,b';k)$, the optimal labor n^*

can be written as a function of θ and χ. Since $G > 0$, $0 \leq q(s,b') \leq \bar{q} = \frac{1-\delta+\delta\kappa}{1-\delta+r}$ and $b_{\min} < 0 \leq b_{\max}$, we have $\chi_{\min} \leq \chi \leq \chi_{\max}$, where

$$\chi_{\min} = G_{\min} + \bar{q}[b_{\min} - (1+r)b_{\max}],$$
$$\chi_{\max} = G_{\max} + \bar{q}[b_{\max} - (1+r)b_{\min}].$$

Therefore, we can discretize the interval $[\chi_{\min}, \chi_{\max}]$ into a fine grid $\mathcal{X}$ with N_χ equally spaced points, and then solve for n^* over the grid $\Theta \times \mathcal{X}$ once and for all *outside* the main loop. Denote this solution by $N^*_{\text{nd}}(\theta, \chi)$. To evaluate $N_{\text{nd}}(s, b, b'; k)$ within the loop, we can simply interpolate N^*_{nd} for the level of χ implied by (s, b, b').

9.2.2 The Solution of the *FSF*

Using the functional forms above, the equilibrium conditions for the *FSF* can be rewritten as:

$$c\left(x,s\right) = \frac{1+\upsilon_b(x,s)}{1+\upsilon_l(x,s)}x,$$
$$c\left(x,s\right)\gamma\left(1-n\left(x,s\right)\right)^{-\sigma} = \theta\alpha n\left(x,s\right)^{\alpha-1},$$
$$x\left(s'\right) = \frac{1+\upsilon_b(x,s)+\varphi(x,s')}{1+\upsilon_l(x,s)}\eta x,$$

where $\varphi(x,s')$ is given by

$$\varphi(x,s') = \tilde{\xi}(s)\frac{\rho\exp(-\rho e)\left[\pi^g(G'|G) - \pi^b(G'|G)\right]}{\pi^G(G'|G,e)}$$

and $\tilde{\xi}(s)$ is the Lagrange multiplier of the incentive constraint in the normalized problem; i.e. $\tilde{\xi}(s) = \frac{\xi(s)}{\mu_b(s)}$. Furthermore,

$$2\omega e = \beta\rho\exp(-\rho e)\sum_{G',\theta'}\left[\pi^g(G'|G) - \pi^b(G'|G)\right]\pi^\theta(\theta'|\theta)V^{bf}(x',s'),$$

$$\begin{aligned} 0 \quad = \quad & \rho\exp(-\rho e)\sum_{G',\theta'}\left[\pi^g(G'|G) - \pi^b(G'|G)\right]\pi^\theta(\theta'|\theta)\left[\frac{1+\upsilon_l(x,s)}{(1+r)x}V^{lf}(x\left(s'\right),s') - \tilde{\xi}(x,s)\beta\rho V^{bf}(x(s'),s')\right] \\ & -\tilde{\xi}(x,s)2\omega \end{aligned}$$

$$V^{bf}(x,s) = \log(c(x,s)) + \frac{\gamma(1-n(x,s))^{1-\sigma}}{1-\sigma} - \omega e^2 + \beta\sum_{s'\in S}\pi^G(G'|G,e)\pi^\theta(\theta'|\theta)V^{bf}(x(s'),s').$$

$$V^{lf}(x,s) = \theta n(x,s)^\alpha - c(x,s) - G + \frac{1}{1+r}\sum_{s'\in S}\pi^G(G'|G,e)\pi^\theta(\theta'|\theta)V^{lf}(x(s'),s').$$

$$V^{af}(s) = \max_n\left\{\begin{array}{c}\log(\theta n^\alpha - (1-\phi)G) + \frac{\gamma(1-n)^{1-\sigma}}{1-\sigma} - \omega e^2 + \\ +\beta\sum_{s'\in S}\pi^G(G'|G,e)\pi^\theta(\theta'|\theta)V^{af}(s')\end{array}\right\}$$

The solution to this system of equations is found numerically using a policy iteration algorithm. More precisely, we discretize the relative pareto weight for the borrower x. For each grid point, we can calculate the value of autarky by solving for the optimal labor in autarky first and calculating $V^{af}(s)$ from the previous equation.

We then define the region of pareto weights between which none of the participation constraints are binding. In that region, for each shock $s = (\theta, G)$, the solution is characterized by the first full commitment solution with unobservable effort but no participation constraints:

$$\begin{aligned} c(x,s) &= x \\ c\left(x,s\right)\gamma\left(1-n\left(x,s\right)\right)^{-\sigma} &= \theta\alpha n\left(x,s\right)^{\alpha-1}, \\ x\left(s'\right) &= \left[1+\widetilde{\xi}\frac{\partial\pi(G'|G,e)/\partial e}{\pi(G'|G,e)}\right]\eta x, \end{aligned}$$

$$\begin{aligned} 0 = &\sum_{s'|s}\pi^{\theta}(\theta'|\theta)\rho\exp(-\rho e)\left(\pi^{g}(G'|G)-\pi^{b}(G'|G)\right)\frac{1}{1+r}\frac{1}{x}V^{lf}(x',s') \\ &+\widetilde{\xi}(x,s)\left[-\sum_{s'|s}\pi^{\theta}(\theta'|\theta)\rho\exp(-\rho e)\left(\pi^{g}(G'|G)-\pi^{b}(G'|G)\right)\rho\beta V^{bf}(x',s')-2\omega\right] \end{aligned}$$

$$2\omega e = \beta\rho\exp(-\rho e)\sum_{G',\theta'}\left[\pi^{g}(G'|G)-\pi^{b}(G'|G)\right]\pi^{\theta}(\theta'|\theta)V^{bf}(x',s'),$$

where

$$V^{bf}(x,s) = \log(c(x,s)) + \frac{\gamma(1-n(x,s))^{1-\sigma}}{1-\sigma} - \omega e^{2} + \beta\sum_{s'\in S}\pi^{G}(G'|G,e)\pi^{\theta}(\theta'|\theta)V^{bf}(x(s'),s').$$

$$V^{lf}(x,s) = \theta n(x,s)^{\alpha} - c(x,s) - G + \frac{1}{1+r}\sum_{s'\in S}\pi^{G}(G'|G,e)\pi^{\theta}(\theta'|\theta)V^{lf}(x(s'),s').$$

$$V^{af}(s) = \max_{n}\left\{\begin{array}{c}\log(\theta n^{\alpha}-(1-\phi)G)+\frac{\gamma(1-n)^{1-\sigma}}{1-\sigma}-\omega e^{2}+\\ +\beta\sum_{s'\in S}\pi^{G}(G'|G,e)\pi^{\theta}(\theta'|\theta)V^{af}(s')\end{array}\right\}$$

To find the region for which the participation constraint binds for the borrower, for each shock $s = (\theta, G)$, we find $c\left(x_b, s\right) = x_b$ such that $V^{bf}\left(x_b, s\right) = V^{af}\left(s\right)$. For the decentralization, using one-period Arrow securities, the bond price simplifies to:

$$\begin{aligned} q\left(s'|s\right) &= \max\left\{\beta\pi\left(s'|s\right)\frac{u'\left(c\left(x',s'\right)\right)}{u'\left(c\left(x,s\right)\right)},\left(\frac{1}{1+r}\right)\pi(s'|s)\right\} \\ &= \pi\left(s'|s\right)\max\left\{\beta\frac{c\left(x,s\right)}{c\left(x',s'\right)},\left(\frac{1}{1+r}\right)\right\} \end{aligned}$$

The price of a one period bond is then equal to:

$$q^{f}\left(s\right) = \sum_{s'\in S}q\left(s'|s\right)$$

which in turn implies a risk free rate of $r^f(s) = \frac{1\delta+\delta\kappa}{q^f(s)}$. Finally, we can recover the asset holdings numerically by iterating to find the asset holding function that satisfies:

$$\begin{aligned} a_b(x,s) &= \sum_{\theta' \in S} q(s')\, a_b(x',s') + c(x,s) - \theta f(n(x,s)) + G \\ a_l(x,s) &= -a_b(x,s) \end{aligned}$$

Moreover, we define the repayment as:

$$a_b(x',s') - \sum_{s' \in S} q(s'|s)\, a_b(x',s')$$

9.3 Further notes on the calibration procedure

On the transition matrix of the G shock. Note that this specification of the transition matrix is motivated by the one-period-crash Markov chain of Rietz (1988):

$$\pi^R = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \eta & \phi & 1-\phi-\eta \\ \eta & 1-\phi-\eta & \phi \end{bmatrix},$$

where the first state is labeled as the "crash" or "crisis" state, and the associated stationary distribution is

$$\mu^R = \begin{bmatrix} \frac{\eta}{1+\eta} & \frac{1}{2(1+\eta)} & \frac{1}{2(1+\eta)} \end{bmatrix}.$$