On Interest Rate Policy and Asset Bubbles*

Franklin Allen
Imperial College London

Gadi Barlevy
Federal Reserve Bank of Chicago

Douglas Gale
New York University

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Abstract

In a provocative paper, Gali (2014), showed that a policymaker who raises interest rates because of concerns about a potential asset bubble will paradoxically make the bubble bigger. In this paper, we argue that Gali's framework abstracts from the possibility that a policymaker who raises rates can end up crowding out resources that would have otherwise been used to buy the asset. We show that when we introduces this possibility, a policy of raising rates can in fact be effective in dampening a bubble if one is present. This suggests policymakers who raise rates with the aim of dampening bubbles should look to evidence of reduced savings or changes in the composition of the portfolio held by the public as signs of whether their intervention was likely to have been effective.

Preliminary and Incomplete

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Introduction

In a provocative paper, Gali (2014) argued that a policymaker who raises interest rates because she is concerned about the possibility of an asset bubble – a policy sometimes known as "leaning against the wind" – will paradoxically make the bubble larger if one is present. Given the ongoing debate among policymakers as to whether central bankers should act to provide financial stability, Gali's result represents a significant challenge to those who advocate that central banks should act in response to the possibility of bubbles in asset markets for the sake of financial stability.

Gali's result certainly defies conventional wisdom among policymakers on the effect of raising interest rates. While policymakers have long been hesitant to act in response to the prospect of asset bubbles, this largely reflected concern about their ability to identify bubbles and the view that interest rates were too blunt a policy tool that affect asset prices as well as economic activity and inflation. See Bernanke and Gertler (1999) for a formalization of this view. However, it has long been taken for granted that raising rates would depress bubbles, and the question was only whether policymakers wanted to exercise this option. Gali's analysis suggests that either this conventional wisdom is flawed, or, alternatively, that the framework he uses misses elements that support the conventional wisdom on the impact of interest rates on bubbles. Gali was aware of the latter possibility, and cautioned as much in his paper. For example, he noted that he assumed all agents are rational, while in practice policymakers may be concerned about bubbles driven by irrational exuberance that may be incompatible with his framework.

In this paper, we argue that the environment Gali analyzed precludes a more elementary argument for why raising rates can be effective in mitigating bubbles, even when agents are rational. In particular, we show that Gali's framework rules out the possibility that tighter policy crowds out resources that would otherwise be spent on the bubble asset. The logic of Gali's result is as follows. In his model, higher interest rates are associated with higher savings in equilibrium. But since Gali assumes the bubble asset is the only financial instrument agents can use to save, higher savings requires agents to spend more on the bubble asset, driving up its price. But this result is not robust. First, if agents can invest in multiple assets, the amount they allocate to any one particular asset can fall even if their overall savings rises or stays unchanged. If the policy intervention that drives up interest rates requires the public to hold more of some other asset, e.g. government debt, it might leave agents with fewer free resources to spend on the bubble asset. Second, to the extent that tighter policy induces a temporary contraction in output, agents might not increase their savings even when the interest rate rises. In this case, agents would spend less on the bubble asset even if it were the only financial instrument agents could use. Both scenarios serve to depress the bubble by discouraging agents from spending resources to buy up the bubble asset.

Our purpose in this paper is to demonstrate that policy interventions which lead to higher interest rates can push asset prices closer to their fundamental value, i.e. to show that raising rates can dampen bubbles. It is not our aim in this paper to further argue that mitigating bubbles is desirable or that interest rate policy is the most effective way to do so. For this reason, we rely on simple models to illustrate our results,

even though these simplifications may preclude a role for policy in improving welfare. For example, in most of the paper we assume the asset that exhibits a bubble is available in fixed supply. While this makes it easy to solve for equilibrium prices, it implies that an intervention meant to influence the price of bubble assets will either have no effect on welfare or will result in a redistribution; it cannot make all agents better off. But the absence of a role for policy is not inherent to models in which interest rate policy can mitigate bubbles. In models where new assets can be created, acting to mitigate a bubble can discourage the creation of assets that is either wasteful or exposes agents to greater aggregate risk. While we do not develop these ideas in detail here, we do sketch out how these issues can potentially be integrated into our analysis.

The paper is organized as follows. In Section 1, we lay out our model and use it to demonstrate Gali's original result that a higher real interest rate can be associated with a larger bubble. In Section 2, we allow agents to hold more than one asset, and construct an example in which a policy that drives up interest rates also reduces the size of the bubble. This example is based on fiscal policy, specifically an increase in outstanding government obligations that leaves agents with fewer resources to spend on the bubble. In Section 3, we argue that similar results can be constructed based on monetary policy interventions. Illustrating this requires modifying the model to encompass both money and nominal rigidities.

While these results show that higher interest rates can dampen bubbles, the bubbles that emerge in our examples fail to capture the key reasons policymakers are concerned with bubbles. In practice, policymakers seem concerned not with whether assets are overvalued, but with the prospect that asset prices might collapse, especially if asset purchases are financed with debt and a fall in prices would trigger massive defaults. For this reason, in Section 4 we develop a model of credit-driven bubbles in which bubbles arise because agents borrow to buy assets, intent on profiting if things work out and defaulting if they do not. These bubbles differ in important ways from bubbles due to a shortage of assets for savings, but a policy that increases rates can still dampen a credit-driven bubble. We then conclude in Section 5.

1 Gali's Original Result

We begin by replicating Gali's result. The environment we present differs from the one Gali described along some dimensions, but it will be analytically more convenient to work with. Nevertheless, we will argue that it captures the key features that underpin Gali's results.

Time is discrete and indexed $t = 0, 1, 2, ...$ There is a single consumption good available at each date. This good can be stored at no cost, allowing agents to convert a unit of consumption goods at date t into a unit of consumption goods at date $t + 1$.

We assume overlapping generations of agents with two-period lives. Agents are risk neutral. They derive utility only from the goods they consume in the second period of life. Specifically, the utility of agents born

at date t from consumption c_t and c_{t+1} at dates t and $t+1$, respectively, is given by

$$u(c_t, c_{t+1}) = c_{t+1} \tag{1}$$

The cohort born at date t is endowed with $e_t > 0$ units of consumption goods when young and none when old. The amount of resources successive cohorts are endowed with grows over time according to

$$e_t = (1+g)^t e_0 \tag{2}$$

The key problem agents in this environment face is that they need to convert the goods they are endowed with when young into goods they can consume when old. They can do this on a one-for-one basis by storing their endowment. However, they might be able to do even better by exchanging their endowment for assets when young and then back into goods when old.

Gali essentially assumes there is only one asset that agents can trade.[1] We begin with this case as well, although importantly we eventually relax this assumption. The asset is available in fixed supply, normalized to 1, and yields a constant dividend flow of $d \geq 0$ consumption goods per period. Gali assumed $d = 0$, rendering the asset intrinsically worthless. For now, we also assume $d = 0$, although we will eventually allow $d > 0$. Assuming $d > 0$ is not only more realistic given that the historical episodes often described as bubbles involve that yield positive dividends, but it matters for the indeterminacy of equilibria.

All shares in the asset are initially endowed to those who are old at date 0. We use p_t to denote the price of a share of the asset at date t, measured in units of consumption goods. Given our normalization, this is also the total market value of the asset.

Asset market clearing in our overlapping generations economy requires that at each date t, the old agree to sell all of their asset holdings to the young. This ensures all assets will be owned by someone each period. Hence, an equilibrium is a path of prices $\{p_t\}_{t=0}^{\infty}$ such that for each date t, the old are willing to sell their asset holdings for a price p_t and the young are willing to buy them at this price. In principle, $\{p_t\}_{t=0}^{\infty}$ can be a sequence of random variables. However, we restrict attention to deterministic price paths.

Let r_t denote the rate of return that those who buy the asset at date t expect to earn from it. Since we are assuming the asset yields no dividends, then

$$1 + r_t = \frac{p_{t+1}}{p_t} \tag{3}$$

We will refer to r_t as the real interest rate. The presence of a storage technology implies that $r_t \geq 0$ for all t in equilibrium. Otherwise, the asset market would fail to clear since the young at date t would not want to hold the asset and would prefer storage.

[1] In principle, Gali allows agents to buy the asset and borrow and lend to one another. But only young agents can borrow or want to lend. Since all young agents are identical in the model, there is no borrowing or lending between agents in equilibrium. Hence, even though Gali allows for both the asset and private debt, only the asset will ever trade.

To characterize the equilibrium paths $\{p_t\}_{t=0}^{\infty}$, consider first the possibility that the real interest rate r_t at date t is positive. In this case, the cohort born at date t would strictly prefer buying the asset to storage. Since we normalized the supply of the asset to 1, this implies that

$$p_t = e_t$$

Hence, in any period t in which the interest rate $r_t > 0$, the price of the asset is uniquely determined.

If the real interest rate were instead equal to zero at date t, i.e. $r_t = 0$, the young would be indifferent between storing their endowment and using it to buy the asset. As a result, the equilibrium price of the asset p_t is no longer uniquely determined. However, in this case there is still something we can say about the path of both prices and interest rates from date t on. If $r_t = 0$, then (3) implies $p_{t+1} = p_t$. Since $p_t \leq e_t$, then the fact that $r_t = 0$ implies the price in the next period p_{t+1} will necessarily be less than the endowment of agents, since

$$p_{t+1} = p_t \leq e_t < e_{t+1}$$

But this implies $r_{t+1} = 0$, since if agents at date $t+1$ do not use all their endowment to buy the asset, storage must not be dominated. Hence, a zero real interest rate is absorbing: Once the real interest rate falls to 0, it will remain there indefinitely. Moreover, since the price of the asset grows at the rate of interest, it follows that once $r_t = 0$, from then on the price of the asset will remain equal to its value at date t. These observations allow us to provide a full characterization of all deterministic equilibrium price paths:

Proposition 1: Suppose $d = 0$. A deterministic path $\{p_t\}_{t=0}^{\infty}$ is an equilibrium iff there exists a cutoff date t^* with $0 \leq t^* \leq \infty$ such that

$$p_t = \begin{cases} e_t & \text{if } t < t^* \\ p_{t^*} & \text{if } t > t^* \end{cases} \tag{4}$$

and p_{t^*} can assume any value in $[e_{t^*-1}, e_{t^*}]$.

The equilibrium price of the asset is thus indeterminate: For any date t and for any value $\rho \in [0, e_t]$, we can find an equilibrium in which $p_t = \rho$. However, any equilibrium path involves two phases, one in which the price of the asset is equal the endowment e_t and one at which the price is constant and does not change.

Next, define the fundamental value of the asset, denoted by f_t, as the value of its dividends discounted to date t. When $d = 0$, what discount rate we use is irrelevant, since f_t would equal 0 for any discount rate. We will say the asset is a bubble whenever $p_t \neq f_t$, and we define the size of the bubble as

$$\Delta_t \equiv p_t - f_t \tag{5}$$

An intrinsically worthless asset will have a fundamental value f_t equal to 0, and so the case where $p_t > 0$ corresponds to a bubble of size $\Delta_t = p_t$. Armed with this definition, we can ask how a move by a policymaker to increase the real interest rate r_t at some date t would affect the size of the bubble Δ_t.

Let us set aside for the moment the question of how a policymaker affects interest rates. We are interested in how moving from one interest rate path $\{r_t\}_{t=0}^{\infty}$ to another affects the path for the price of the asset $\{p_t\}_{t=0}^{\infty}$. As Gali notes in his paper, the fact that the model admits multiple equilibria raises some conceptual issues for this comparative static exercise. His argument can be framed as follows. The equilibrium condition (3) only governs the rate at which the price of the asset p_t grows over time. Hence, given a path $\{r_t\}_{t=0}^{\infty}$, the path for asset prices p_t will be uniquely determined only up to an initial price p_0 that is not pinned down by (3). The fact that the initial price p_0 is an equilibrium does not pin this price down either, since the model admits multiple equilibria. If we want to compare equilibrium prices under two different interest rate paths, then, what initial price p_0 should we assign each of the two paths? While working with an incomplete partial equilibrium version of the model, Gali suggested requiring p_0 to be the same for the two paths in order to avoid policy from affecting the bubble indirectly through some "indeterminacy" channel. This may have given some the impression that Gali's result relies on some arbitrary selection of equilibria. But as we shall now argue, both our model and the general equilibrium model Gali developed later in his paper provide additional structure that restricts the value of p_0 for a given interest rate path, rendering Gali's restriction unnecessary and potentially in contradiction with other features of the model.

Figure 1 plots all pairs (p_0, r_0) that correspond to an initial price and an interest rate for some deterministic equilibrium path described in Proposition 1. As evident from the figure, any equilibrium in which the initial price of the asset p_0 is below e_0 will be associated with a unique equilibrium interest rate, namely $r_0 = 0$. Likewise, any equilibrium in which the interest rate r_0 is positive will be associated with a unique initial equilibrium price for the asset, namely $p_0 = e_0$. If we think of a policymaker as selecting an equilibrium from the set of possible equilibria, then in choosing an equilibrium with a certain interest rate, the policymaker is effectively also choosing the (possibly unique) price of the asset associated with that interest rate in equilibrium.[2] It should be clear from Figure 1 that in our model, a move by the policymaker to select an equilibrium with a higher initial interest rate r_0 will either leave the associated equilibrium price p_0 unchanged or cause it to increase. The latter is only possible when the policymaker increases the interest rate from 0. Hence, a higher interest rate r_0 will either increase the bubble at date 0 or leave it unchanged, and will unambiguously increase at date 1 since (3) implies the bubble will grow at the rate r_t. The model makes a clear prediction without resorting to any arbitrary selection of equilibria: If the policymaker selects a higher interest rate, it will increase the bubble and magnify the extent to which the asset is overvalued.

The intuition behind this result is as follows. A higher interest rate leads agents to rely less on storage and use financial markets instead. In our model, this occurs discontinuously as agents shift from possibly holding no assets when $r_0 = 0$ to exchanging all of their endowment for assets when $r_0 > 0$. Gali's model exhibits a similar feature. He assumes no storage, but allows agents to value consumption when young at a diminishing rate. Agents in his model must allocate their endowment between two uses, just as in ours, only in his setting they choose between consumption and saving rather than storage and saving. A higher

[2] This is essentially what Gali assumes as well. He allows the policymaker to choose the equilibrium interest rate for each realization of some shock that is determined after producers set prices. This amounts to choosing the interest rate state-by-state, subject to a constraint that the average interest rate coincides with what price-setters expect.

equilibrium interest rate will induce young agents to allocate more resources to buying assets in both cases. Since agents can only buy one asset, this asset necessarily becomes more valuable. Note that Gali's setup does not exhibit the same discontinuities as our model does, so each initial interest rate r_0 in his model is associated to a unique price for the asset p_0, i.e. the analog of Figure 1 for his model is a strictly upward-sloping curve. Hence, in his model, a higher interest rate at date 0 will imply a larger bubble starting at date 0, contrary to what he assumes in his partial equilibrium model.

We can thus reproduce Gali's result that a higher real interest rate will be associated with a larger bubble. Specifically, our model features multiple equilibria, as does Gali's, and across these equilibria, those that feature a higher initial interest rate are associated with a (weakly) larger bubble. Hence, if a policy intervention corresponds to selecting a different equilibrium from the set of all possible equilibria, an intervention that leads to a higher interest rate will also lead to a larger bubble. Our discussion suggests this result leans on the fact that agents are assumed to save using only one asset. In the next section, we modify the model above to remove both indeterminacy and the assumption of a single asset. Given the equilibrium is unique, we will naturally focus on a different form of intervention than selecting equilibria. Within this setting, we demonstrate how we can overturn Gali's result so that if a policymaker takes a particular action that results in a higher interest rate, the equilibrium bubble will be smaller.

2 Overturning Gali

We begin by modifying our model from Section 1 to get rid of the indeterminacy of equilibria it implies. One simple way to ensure that equilibrium prices are unique is to assume $d > 0$, so the asset yields positive dividends. To see this leads to a unique equilibrium, observe that we can rule out $r_t = 0$ as an equilibrium. If r_t did equal 0, then (3) would imply

$$p_{t+1} = p_t - d < p_t$$

Since $p_t \leq e_t$, it follows that $p_{t+1} < e_t < e_{t+1}$, which implies that $r_{t+1} = 0$. By this logic, the price would continue to decline by d at dates $t+1$, $t+2$, and so on, until the price would eventually turn negative. But this cannot be an equilibrium, since at a negative price the cohort that owns them would prefer not to sell them. The only candidate deterministic equilibrium price path is one with positive asset prices.

Since agents always have access to storage, it is still the case that $r_t \geq 0$ in equilibrium. Hence, we must have $r_t > 0$. But in that case, buying the asset dominates storage and young agents will exchange all of their endowment for the asset. The price of the asset will thus be uniquely determined in every period. We confirm this in the next proposition. The proof of this and other propositions are contained in an appendix.

Proposition 2: Suppose $d > 0$. Then the unique equilibrium path $\{p_t\}_{t=0}^{\infty}$ is given by

$$p_t = e_t \tag{6}$$

and the unique equilibrium interest rate is given by

$$r_t = \frac{d}{e_t} + g \tag{7}$$

Note that our finding the equilibrium price is unique implies that if this equilibrium exhibited a bubble, an asset bubble would be unavoidable, as there would be no equilibrium without a bubble. To those well familiar with models that feature intrinsically worthless assets, this may suggest that the asset can no longer be a bubble, since in those models there is always an equilibrium without a bubble. However, we will now argue that this is not the case here: The equilibrium in Proposition 2 is indeed a bubble.

To determine whether the equilibrium in Proposition 2 corresponds to a bubble, we first need to compute the fundamental value of the asset, f_t. The natural rate at which we should discount dividends to arrive at f_t is the market interest rate, i.e.

$$f_t = \sum_{j=1}^{\infty} \left(\prod_{i=0}^{j-1} \frac{1}{1 + r_{t+i}} \right) d \tag{8}$$

The reason for using the market interest rate is that if we take resources from a young agent at date t, he would demand $1 + r_t$ units at date $t+1$ to remain equally well off. The interest rate thus faithfully captures the way society trades off resources between adjacent dates.

From Proposition 2, we know that $r_t > g > 0$ for all t. Hence, (8) is finite, specifically

$$f_t \leq \sum_{j=1}^{\infty} \left(\frac{1}{1+g} \right)^j d = \frac{d}{g}$$

Since $\lim_{t\to\infty} r_t = g$, we can further conclude that $\lim_{t\to\infty} f_t = d/g < \infty$. At the same time, the price of the asset grows without bound, since $\lim_{t\to\infty} p_t = \lim_{t\to\infty} (1+g)^t e_0 = \infty$. Hence, the asset price will eventually exceed its fundamental value. Thus, this economy still features a bubble, at least asymptotically. But we can show that the equilibrium price of the asset exceeds the fundamental value at all dates, not just asymptotically. To see this, note that the equilibrium interest rate r_t in Proposition 2 implies that

$$p_t = \frac{d + p_{t+1}}{1 + r_t}$$

At the same time, the fundamental value f_t in (8) satisfies

$$f_t = \frac{d + f_{t+1}}{1 + r_t}$$

Subtracting the latter expression from the former reveals that the difference $\Delta_t \equiv p_t - f_t$ must satisfy

$$\Delta_t = \frac{\Delta_{t+1}}{1 + r_t}$$

By repeated substitution, it follows that $\Delta_0 = \left(\prod_{t=0}^{T-1} \frac{1}{1+r_t} \right) \Delta_T$ for any $T > 0$. But since we just argued $\Delta_T > 0$ as $T \to \infty$, it follows that $\Delta_0 > 0$. From the expression for r_t in Proposition 2, we know that

$1 + r_t > 0$ for all t. Hence, the price of the asset exceeds its fundamental value at all dates, just as in the case where $d = 0$. The asset should is thus overvalued in equilibrium even though it yields positive dividends and even though its price is uniquely determined.

As we noted above, given there is a unique equilibrium, we can no longer appeal to comparisons across equilibria to determine whether higher interest rates are associated with less overvalued assets. Instead, we need to modify the model to allow policy to directly affect the equilibrium interest rate. We therefore introduce government debt as another asset that agents can purchase. Introducing government bonds not only admits a natural channel by which policy can affect interest rates, but it also allows for multiple assets, which we previously argued was an important limitation of the Gali setup.

Formally, consider a government that issues real one-period debt obligations, each of which sells for one unit of real resources at date t and pays $1 + r_t$ units of consumption at date $t + 1$. The government can finance its obligations using lump sum taxes levied on the young in period $t + 1$. Let b_t denote the number of bonds issued at date t and τ_t denote the lump sum tax collected at date t. A lump-sum tax τ_t is feasible only if $\tau_t \leq e_t$, i.e. the government cannot collect more resources than agents are endowed with. The flow government budget constraint is given by

$$b_{t+1} = (1 + r_t)\, b_t - \tau_{t+1} \tag{9}$$

That is, any part of the government's debt obligation not covered by tax revenue must be financed by new debt. More generally, we could have allowed the government to purchase goods with the revenue it collects, but we ignore this possibility in what follows. The initial obligation of the government to the old at date 0 is denoted $(1 + r_{-1})\, b_{-1}$. Feasibility requires that the obligation $(1 + r_{-1})\, b_{-1}$ not exceed e_0, i.e. the government cannot transfer to the old more than it can collect in the same period.

Market clearing requires that at each date t, the young are willing to buy all of the debt issued by the government, b_t, as well as all shares of the asset. The old must be willing to sell all of their shares of the asset. In equilibrium the interest rate on government bonds and the asset be equal, i.e.

$$1 + r_t = \frac{d + p_{t+1}}{p_t} \tag{10}$$

Using the same argument as before, we can conclude that $r_t > 0$ for $t = 0, 1, 2, ...$ This implies storage is dominated, and so the equilibrium price of the asset is once again uniquely determined:

$$p_t = e_t - \tau_t - b_t$$

We are now ready to analyze the effect of policy interventions on interest rates and asset prices relative to fundamentals. Advocates of leaning against the wind in the face of a potential asset bubble often frame their policy recommendation in terms of monetary policy. Hence, the most natural direction to take the model would be to introduce a monetary authority that can intervene in the market for government debt with open market operations and study the effect of those interventions. However, incorporating this would

require us to introduce money, as well as additional features that ensure money is non-neutral. For ease of exposition, we defer on this for now and instead consider interventions that affect the amount of outstanding government obligations. These should be understood as changes in fiscal policy. Although such interventions do not correspond to the types of policies people typically consider as possible responses to a bubble, this analysis is still useful for drawing out the relevant intuition. In the next section, we discuss monetary policy and show how monetary policy interventions are related to our results here.[3]

To study the effects of changes in fiscal policy, we first need to characterize fiscal policies. In our setting, we can fully characterize a policy using both the initial obligation of the government $(1+r_{-1})\,b_{-1}$ to the old at date 0 and the path for lump-sum taxes $\{\tau_t\}_{t=0}^{\infty}$ from date 0 on. That is, the government begins with some initial obligation and determines how much resources it wants to collect each period. Any shortfall it faces must be financed by issuing government debt (or, in principle, lending to the public if it collects more than its obligations). As we now show, the government's initial obligation and subsequent path for taxes imply a unique path for both the interest rate $\{r_t\}_{t=0}^{\infty}$ and debt issuance $\{b_t\}_{t=0}^{\infty}$ given the the intertemporal budget constraint (9) and the equilibrium condition (10).

Lemma 1: Suppose $d > 0$. Given an initial debt obligation $(1+r_{-1})\,b_{-1} \leq e_0$ and tax path $\{\tau_t\}_{t=0}^{\infty}$ where $\tau_t \leq e_t$ for all t, there is a unique path $\{b_t, r_t\}_{t=0}^{\infty}$ that satisfies both the intertemporal budget constraint (9) and the equilibrium interest rate condition (10).

Beyond requiring that the government collect no more resources in any period than young agents are endowed with, we find it natural to restrict the path of fiscal policy to ensure that the government sector doesn't fully crowd out private economic activity. That is, we want to make sure that young agents are not asymptotically handing over all of their endowment to the government, either in tax payments or purchases of government bonds. Formally, we assume the initial debt obligation $(1+r_{-1})\,b_{-1}$ and subsequent tax payments $\{\tau_t\}_{t=0}^{\infty}$ are such that

$$\lim_{t\to\infty} \frac{\tau_t + b_t}{e_t} < 1 \tag{11}$$

where b_t is the debt issuance implied by the path of fiscal policy that is summarized in Lemma 1. Our next result provides expressions for the unique equilibrium asset price p_t and interest rate r_t given condition (11).

Proposition 3: Suppose $d > 0$. Given an initial debt obligation $(1+r_{-1})\,b_{-1}$ and a path $\{\tau_t\}_{t=0}^{\infty}$, the equilibrium price of the asset and the interest rate are uniquely determined each period and given by

$$\begin{aligned} p_t &= e_t - \tau_t - b_t \\ 1 + r_t &= \frac{d}{e_t - \tau_t - b_t} + \frac{(1+g)\,e_t - \tau_{t+1} - b_{t+1}}{e_t - \tau_t - b_t} \end{aligned}$$

where $\{b_t\}_{t=0}^{\infty}$ is the unique path for debt issuance in Lemma 1. If (11) is satisfied, $\lim_{t\to\infty} p_t = \infty$.

[3] Although policy interventions against bubbles are often framed in terms of monetary policy, issuing more debt has been proposed for when a bubble indicates a shortage of assets. For example, Diamond (1965) shows that the introduction of public debt can potentially restore dynamic efficiency to an economy that would otherwise be vulnerable to bubbles.

To study the effects of changes in fiscal policy, it will be convenient to restrict attention to policies that can be summarized with a single parameter. In particular, consider paths for fiscal policy under which the initial government obligation to the old at date 0 and its debt issuance at all subsequent dates can be summarized with a single parameter b such that

$$\begin{aligned} (1+r_{-1})\,b_{-1} &= (1+r_{-1})\,b \\ b_t &= b \text{ for } t=0,1,2,... \end{aligned} \tag{12}$$

Under this parameterization, a higher b implies an increase in government liabilities at all dates, including its initial obligation to the old at date 0. It is straightforward to compute the unique path for taxes $\{\tau_t\}_{t=0}^{\infty}$ needed to sustain such paths for debt. Our next result shows that a policy that increases government liabilities at all dates would result in a higher interest rate r_t and a lower price for the asset p_t.

Proposition 4: Suppose fiscal policy is given by (12) and $r_{-1} > 0$. Then at all dates t, the equilibrium interest rates r_t are increasing in b and the equilibrium asset prices p_t are decreasing in b.

The intuition behind our result is as follows. An increase in b at all dates requires lump sum taxes τ_t to be higher at all dates.[4] With more taxes to pay and more bonds to hold, each cohort will have fewer available resources with which to buy the asset. As a result, the price of the asset will fall. Simply put, issuing more bonds serves to crowd out savings in the bubble asset. This crowding out will also lead to higher rates of return. The reason is that agents invest fewer resources but the dividends they earn are fixed. Hence, the dividend yield d/p_t on the asset rises. Of course, the return on the asset also depends on how the intervention affects the price of the asset in the future. For policies that stipulate a constant amount of debt issued each period, the effect of more debt on the interest rate is the same as on the dividend yield.[5]

Remark: One way to understand the implication of our model for interest rates is to observe that each cohort of young agents in our model gives away all of its endowment e_t either to pay taxes or to buy financial instruments. Since the government consumes no resources, each cohort will consume the endowment of the next cohort e_{t+1} when it is old, together with the dividend d on the asset. They receive this endowment either when they sell their asset to the next cohort, or when the government repays their outstanding obligations by taking resources from the next cohort, either by taxing them or selling them new debt. The return on private savings for the cohort born at date t is thus

$$1 + r_t = \frac{e_{t+1} + d}{e_t - \tau_t} \tag{13}$$

What ultimately determines interest rates, then, is the path of lump-sum taxes τ_t. If lump-sum taxes are higher, agents have less disposable income they can save. But regardless of how much they save, they will

[4] Here the assumption that $r_{-1} > 0$ is important, since it implies the government's initial obligation increases more than any new obligation the government takes on at date 0, forcing the government to collect more taxes at date 0.

[5] If agents value bonds differently from other assets as in Krishnamurthy and Vissing-Jorgensen (2012), interest rates might rise for reasons not captured in our model. That is, if we allowed assets to be imperfect substitutes, issuing more government debt would naturally drive the return on bonds up to compel agents to hold more of these assets.

receive the same amount from saving, namely e_{t+1}. Hence, the rate of return on their savings will be higher the more they must pay in taxes. The interest rate r_t thus depends on the tax rate τ_t. By contrast, the price of the asset p_t depends on both τ_t and b_t. A policy that increases the government liability $(1 + r_{t-1}) b_{t-1}$ due at date t will always drive down the price of the asset, since

$$\begin{aligned} p_t &= e_t - \tau_t - b_t \\ &= e_t - (1 + r_{t-1}) b_{t-1} \end{aligned}$$

But the same policy could lower the interest rate if τ_t is lower and the liability is largely financed with new debt. That said, the government budget constraint implies that higher debt obligations will have to be financed by higher taxes eventually. Hence, increasing government obligations would have to be associated with higher interest rates at some point even if not immediately. This reasoning also reveals that changing b has no effect on welfare: Regardless of policy, each cohort consumes the entire endowment of the next cohort together with the dividend d. The policy only affects whether resources are transferred through the market or through the government. Issuing more debt forces young agents to give their resources to the old indirectly by giving it first to the government rather than directly exchange it with the old for their assets. This will matter for market prices such as p_t and r_t, but it will not matter for welfare.

Proposition 4 establishes that there exists a policy intervention that increases interest rates and drives down the price of the asset. However, with $d > 0$, it is no longer obvious that a lower price corresponds to a smaller bubble; since f_t as defined in (8) will also fall as interest rates rise, it is possible that the fundamental falls by the same amount or even more than the price. Our next result confirms that under assumption (11), if policy is given by (12), an increase in government liabilities will reduce the size of the bubble $\Delta_t = p_t - f_t$. Thus, the policy intervention we consider increases the real interest rate and reduces the bubble, mitigating the extent to which the asset is overvalued.

Proposition 5: Suppose fiscal policy is given by (12). Given (11), then $\Delta_t \equiv p_t - f_t$ is decreasing in b for all t.

Thus, we have presented an example in which, contrary to Gali's result, a policy intervention that leads to higher interest rates also lowers assets prices and reduces the extent to which assets are overvalued. At the core of our example is the fact that agents can save using more than one financial instrument. This allows a policymaker intent on shrinking a bubble to induce the public to hold other assets instead of the bubble and thus depress its price. However, reducing bubbles this way serves no useful purpose in our model, since allocations are unaffected by the policy interventions we consider. Essentially, the government uses one bubble – government debt – to replace a bubble on another asset.

While intervention is irrelevant in our particular model, reducing a bubble can still serve a useful purpose more generally. To see this, suppose agents could create additional assets, according to a technology subject to an increasing marginal cost of production. If the price of the asset exceeds its fundamental value, too many assets would be created than is socially desirable: Agents would be better served relying on the

government to provide savings instruments than wasting resources creating assets that achieve the same thing. For example, issuing bonds would presumably be better than building unoccupied houses that only serve as savings vehicles. While this would offer a possible justification for intervening against bubbles, in practice policymakers do not seem concerned about bubbles because of the resource misallocation they might cause. Instead, policymakers seem most concerned about the possibility that a bubble may burst, in contrast to the deterministic bubbles we've analyzed so far, especially when asset purchases are financed by debt and the collapse in asset prices is likely to result in large-scale defaults. In Section 4 below we show how risky leverage-based bubbles can emerge in our framework. Although these bubbles differ in various ways from the type of bubbles driven by a desire for savings that we have discussed so far, we show that interventions that raise interest rates can still act to dampen credit-driven bubbles as well.

3 Monetary Policy

In the previous section, we analyzed a policy intervention based on fiscal policy. But advocates of leaning against the wind typically look to monetary authorities to increase rates in order to quell bubbles. In his analysis, Gali also focused on monetary policy interventions. We therefore turn in this section to the implications of our model for monetary policy interventions. We first show that when prices are flexible, a contractionary monetary policy that succeeds in driving down the price level can force the government to issue more debt. This is because such a policy would increase the value of any outstanding nominal obligations previously issued by the government. As in the case of higher debt due to a fiscal change, such an intervention would crowd out resources from the bubble asset. It would also only generate higher interest rates if the fiscal authority intervened to increase taxes. However, if we assume prices are temporarily inflexible and cannot respond to changes in monetary policy, then even when fiscal policy is unchanged, an unanticipated monetary policy intervention can lead to both higher interest rates and lower bubbles. This is because such policies lead to temporarily lower output, leaving agents with fewer resources with which to buy the bubble even if no more new debt is issued. Since output is only lower temporarily, real interest rates would rise while the economy contracts. Although the full details of the analysis are left to Appendix B, we try to provide some sense of the argument here.

To analyze the role of monetary policy, we need to modify the model from Section 2 in two ways: We need to introduce money, and we need to assume agents are endowed with productive labor inputs rather than with a fixed amount of goods. This allows monetary policy to affect the quantity of goods produced.

We begin with money. We continue to assume that young agents who wish to save can store goods, trade them for the asset, or trade them for government debt. But we now also allow them to exchange goods for money, i.e. a non-interest bearing asset issued by the government. Let M_t denote the amount of money circulating at date t, so that $M_t - M_{t-1}$ corresponds to the amount of money issued or removed from circulation at date t. We will assume these injections or withdrawals are distributed lump sum to the old who already hold money. Let P_t denote the price of one unit of consumption goods in terms of

money. Conversely, $1/P_t$ represents the real price of one unit of money, and $m_t = M_t/P_t$ denotes real money balances. Let Π_t denote the gross inflation rate between dates t and $t+1$, i.e. $\Pi_t = P_{t+1}/P_t$. It will be convenient to define

$$x_{t+1} = \frac{M_{t+1} - M_t}{P_{t+1}}$$

as the injection of money between dates t and $t+1$ measured in terms of how much this amount could buy at date $t+1$. If young individuals opt to set aside m_t of their original wealth to exchange for money, their real money balances at date $t+1$ will equal

$$m_{t+1} = \Pi_t^{-1} m_t + x_{t+1}$$

where agents view x_{t+1} as fixed and unaffected by their actions. We continue to denote the real rate of return on government debt by $1 + r_t$, and we use $1 + i_t$ to denote the nominal interest rate $(1 + r_t)\,\Pi_t$.

If we assume money is just another asset that can be used as a store of value, money would have to offer the same real interest rate as bonds or the asset in equilibrium, or else it would be dominated. As Gali observes, there would be limited scope in this setting for monetary policy to influence inflation or interest rates. We therefore need a reason for agents to value money beyond its store of value. Rather than model the function money plays, we follow the approach followed by Gali (see Appendix 3 of his paper) in assuming that real balances enter preferences directly.[6] This specification is meant to be a stand-in for the liquidity value of holding money, since it makes agents willing to hold money even when it offers a lower return.

The second modification we introduce is to assume young agents are endowed not with final goods but with inputs that can be used for production. That is, individuals are endowed with one unit of labor while young. Those born at date t can allocate an amount $n_t \in [0, 1]$ to be used in production, but doing so incurs them some disutility since this time cannot be used for leisure. Production is linear, i.e. n_t units of labor can produce $A_t n_t$ units of output, where productivity A_t grows over time according to

$$A_t = (1 + g)^t A_0$$

Gali also assumes agents are endowed with labor. However, he assumes agents do not value leisure and supply labor inelastically regardless of policy. We instead follow Adam (2003) in assuming agents value time spent not working so that the amount of labor employed in equilibrium can vary with policy. We also follow Adam in terms of who we assume agents sell their labor to, as we discuss in Appendix B.[7] Here, though, we will proceed as if agents are yeoman farmers who operate their own technology rather than sell their labor services to others. This simplifies the exposition.

[6] Waldo (1985) offers another example of a monetary OLG economy where agents value real balances directly.

[7] In particular, we assume the young sell their labor endowment to other young agents, while Gali assumes the young sell their labor endowment to the old. As a result, even though output does not vary with policy in Gali's model, the amount of income that accrues to those who buy assets does. And it is the latter which matters for asset prices. Thus, Gali's model does not preclude the possibility that tight policy reduces the income available for agents to buy the asset, but the shocks he considers that policy responds to seem to offset this channel.

Formally, we replace the preferences in (1) with

$$u\left(c_t, c_{t+1}, m_t, n_t\right) = c_{t+1} + \theta \ln\left(m_t\right) - v_t\left(n_t\right)$$

That is, agents derive log utility from the real balances they set aside while young, M_t/P_t. Under log utility, we could have equally assumed that agents cared about the value of the nominal balances M_t they hold evaluated at P_{t+1}, i.e. the price of goods at date $t+1$. What we do rule out is that individuals obtain liquidity value from the money injection they receive when old , x_{t+1}, or if they do then it is separable from the liquidity value they obtain from money balances set aside at date t. The disutility from working, $v_t\left(n_t\right)$, is allowed to differ across cohorts. In particular, it will be convenient to assume that $v_t\left(n_t\right) = A_t v\left(n\right)$ for some common function $v\left(\cdot\right)$. This specification implies that even though later cohorts are more productive, their cost of effort is also growing, preventing them from allocating all of their labor to working. We assume $v\left(\cdot\right)$ is concave and that $\lim_{n\to 0} v'\left(n\right) = 0$ and $\lim_{n\to 1} v'\left(n\right) = \infty$ to ensure an interior solution.

Agents are subject to the budget constraint

$$c_{t+1} = \left(1 + r_t\right)\left(A_t n_t - m_t - \tau_t\right) + \Pi_t^{-1} m_t + x_{t+1}$$

In equilibrium, both the government bond and the asset will yield the same return r_t, and so the agent's portfolio decision boils down to choosing between how much to save in interest-bearing instruments and how much to save in money. The first order conditions of the problem above with respect to labor effort and real balances are given by

$$\begin{aligned} v'\left(n_t\right) &= 1 + r_t & (14)\\ m_t &= \frac{\theta \Pi_t}{\left(1 + r_t\right)\Pi_t - 1} = \frac{\theta \Pi_t}{i_t} & (15) \end{aligned}$$

Finally, we need to modify the intertemporal government budget constraint in (9) to take into account money injections and withdrawals. This constraint is now given by

$$\begin{aligned} b_t &= \left(1 + r_{t-1}\right) b_{t-1} - \tau_t - \frac{M_t - M_{t-1}}{P_t}\\ &= \left(1 + r_{t-1}\right) b_{t-1} - \tau_t - m_t + \frac{m_{t-1}}{\Pi_{t-1}} & (16) \end{aligned}$$

Our first observation concerns the effect of policies that are contractionary in the sense that they drive down the price level of goods. Consider the price p_t at date $t = 0$. Since agents will not rely on storage, and since they only enjoy consumption when old, they will exchange all of the $A_t n_t$ goods they produce for financial assets. Thus, they will use all the resources they have after paying taxes τ_t, buying up debt issued by the government b_t, and buying up the real money balances of the old, M_t/P_t, to buy the asset. Since we assume there is single unit of the asset, this implies the price of the asset p_0 is given by

$$p_0 = A_0 n_0 - \tau_0 - b_0 - \frac{M_0}{P_0} \quad (17)$$

From the intertemporal government budget constraint in (16), we know that

$$b_0 + \tau_0 + \frac{M_0}{P_0} = (1 + r_{-1})\, b_{-1} + \frac{M_{-1}}{P_0}$$

Substituting this into (17) reveals

$$p_0 = A_0 n_0 - (1 + r_{-1})\, b_{-1} - \frac{M_{-1}}{P_0} \tag{18}$$

Note that $(1 + r_{-1})\, b_{-1}$ and M_{-1} are predetermined. In Appendix B, we argue that when prices are flexible, changes in monetary policy on their own will have no effect on the amount of labor supplied in any period t, including date 0, so n_0 will be independent of monetary policy. Hence, the only way monetary policy can affect the initial real price of the asset p_0 is by affecting the initial price level, P_0. The reason changes in the price level P_0 matter for the real price of the asset is that a higher price level P_0 reduces the seniorage revenue the government collects for a given m_t, forcing it to issue more debt to satisfy its intertemporal budget constraint. More generally, if we assumed the initial debt $(1 + r_{-1})\, b_{-1}$ was nominal rather than real, a lower price level would have forced the government to raise more real resources today to cover its higher obligation. As we argued in Section 2, if the government issues more debt, it crowds out resources that would have been used to buy the asset, pushing down the price of the asset. Hence, a monetary policy that succeeds in lowering the price level would also reduce the real price of the asset.

However, monetary policy on its own would have no effect on the real interest rate $1 + r_t$. To see this, observe that the equilibrium interest rate must still satisfy the condition

$$(1 + r_t)\, p_t = d + p_{t+1}$$

Substituting in for the expression for p_t given that agents allocate all of their available resources not already spent on money or bonds to buying the asset, we can rewrite this as

$$(1 + r_t)\, (A_t n_t - \tau_t - b_t - m_t) = d + (A_{t+1} n_{t+1} - \tau_{t+1} - b_{t+1} - m_{t+1})$$

Using the intertemporal budget constraint (16), we can reduce this expression to

$$\begin{aligned}
(1 + r_t)\, (A_t n_t - \tau_t) &= d + \left(A_{t+1} n_{t+1} + \left(1 + r_t - \Pi_t^{-1}\right) m_t\right) \\
&= d + \left(A_{t+1} n_{t+1} + i_t \frac{m_t}{\Pi_t}\right) \\
&= d + (A_{t+1} n_{t+1} + \theta)
\end{aligned}$$

where the last step uses the money demand rule (15). Hence, the interest rate is given by

$$1 + r_t = \frac{A_{t+1} n_{t+1} + \theta + d}{A_t n_t - \tau_t} \tag{19}$$

Given our claim that n_t is unaffected by monetary policy when goods prices are flexible, the only way the real interest rate would change is if τ_t changes, i.e. if the change in monetary policy triggers a change in fiscal policy. This accords with our earlier results for fiscal policy, namely that issuing more debt raises the

interest rate only when the government resorts to higher taxes. If we assume fiscal policy does not respond to monetary policy, the monetary authority could drive down the price of the asset without raising interest rates by lowering the price level. However, the monetary authority would have to raise more seniorage revenue at some future date to offset the increase in debt. Note that if the real interest rate is unaffected, the present discounted value of dividends evaluated at real market interest rates would be unchanged, so a policy that lowered the real price of the asset would reduce the size of the bubble.

The fact that monetary policy has no effect on the real interest rate is due both to our assumption that agents have log utility over real money balances and the fact that prices are flexible. But it is well known that when prices are not fully flexible, monetary policy can affect the real interest rate even when it would have been neutral under fully flexible prices. In Appendix B, we describe a version of the model where produces explicitly set prices to study the effects of monetary policy under nominal price rigidity. The chief insight of our analysis is that an unexpected monetary contraction can raise the interest rate r_t and reduce equilibrium employment n_t. Intuitively, a monetary contraction reduces the ability of the old to buy goods if the price of goods cannot respond to policy. The real wage would then adjust to discourage workers from putting in effort, and output would fall temporarily. As evident from (19), this would cause the real interest rate to rise. In our model, this is because agents at date t give up fewer resources today to buy the asset but receive the same amount of output next period. If we assumed agents valued consumption even when young, the intertemporal Euler equation would imply that low consumption today that is expected to be undone next period must be associated with a higher interest rate.

We further show in Appendix B that an unexpected monetary contraction will drive down the price of the asset. The logic once again involves the crowding out of resources that would have been used to buy the asset. This can be seen from (18). If prices are rigid so the initial price level P_0 cannot respond to changes in monetary policy but n_0 falls, the real price of the asset would fall as agents would be left with fewer resources with which to buy the asset. However, this intervention crowds out resources by reducing the income agents have to buy the asset rather than by forcing them to hold more of some other asset. If instead we assumed prices are partially flexible, in the sense that some but not all suppliers can adjust their prices down in response to the policy, the effect of an unexpected monetary contraction would be to both leave agents with fewer resources with which to buy the asset and more bonds they had to hold.

Finally, we show in Appendix B that when prices are rigid, an unexpected monetary shock not only raises rates and lowers asset prices, but also reduces the extent to which the asset is overvalued. That is, under sticky prices, there is a monetary intervention that increases interest rates and dampens the bubble, contrary to what Gali finds in his setting. This is because Gali's model effectively abstracts from both of the channels that arise here. First, his model assumes away government debt, and so does not have a similar mechanism whereby the government must issue more debt when the price level falls. Second, while Gali assumes that total output and therefore total income is unaffected by monetary policy, his model does allow the income of young agents to fall, and these are the ones who buy the bubble asset. But he assumes that the shocks which trigger tighter monetary policy are those which make young households wealthier,

which tends to offset any negative impact contractionary policy might have on their earnings. We would therefore argue that notwithstanding Gali's results, there are realistic environments in which contractionary monetary policy in the face of price rigidity can be effective in dampening asset bubbles.

4 Credit-Driven Risky Bubbles

So far, the bubbles we have considered were deterministic, implying an asset this is permanently overvalued. When $d > 0$, this is in fact the unique equilibrium, and additional equilibria in which the asset price may eventually revert to the fundamental value of the asset can be ruled out. But when $d = 0$ and the asset is intrinsically worthless, there are additional equilibria beyond those we characterized in Proposition 1 in which the bubble is stochastic. In particular, we can construct equilibria in which there is positive probability each period that the asset price will revert to the fundamental value of the asset. In other words, there are equilibria in which the bubble can burst at random. Previous work have exploited this fact to explore the possibility of price crashes, which arguably speak more to the reason policymakers worry about bubbles. However, this approach to analyzing stochastic bubbles has several drawbacks. First, it only arises for intrinsically worthless assets. Moreover, such bubbles can be constructed absent any borrowing, and policymakers appear to be particularly concerned that the collapse of credit-driven bubbles will trigger a cascade of defaults.[8] This raises the question of whether our insights on the effect of interest rates on bubbles extend to credit-driven bubbles. In this section, we modify the environment to show how credit-driven bubbles can arise in our model. Although such bubbles turn out to be different in important respects from the ones we analyzed above, we argue that there can still be policy interventions that increase interest rates and dampen asset bubbles.

In what follows, we return to the framework in Section 2, i.e. we abstract from money once again. To facilitate comparisons with the previous environment, we choose to continue to assume that the dividends the asset yields are deterministic. That is, we continue to assume the asset yields a fixed dividend d per period. In addition, we assume $d > 0$, i.e. the asset yields a positive dividend. We view this as more realistic, but it also eliminates the problem of multiple equilibria that arises with intrinsically worthless assets. Since dividends are fixed, we need some additional source of uncertainty that could trigger the collapse in the bubble that will arise in our framework. We therefore allow for uncertainty about aggregate economic conditions. Specifically, we assume there is a risk that the endowment growth that sustains rising asset prices might come to a stop. In modern parlance, the economy faces a risk of secular stagnation, i.e. a regime shift in which the economy grows more slowly and the real interest rate collapses.[9]

Formally, suppose the endowment of successive cohorts e_t keeps growing at rate g only until some random

[8] See, for example, Mishkin (2008, 2011).

[9] The exact source of uncertainty is not essential for bubbles. In Appendix C, we discuss an example in which the aggregate endowment is nonstochastic but dividends are random. Bubbles arise in that environment as well.

date T. Thus, we replace the endowment process in (2) with

$$e_t = \begin{cases} (1+g)^t e_0 & \text{if } t < T \\ (1+g)^{T-1} e_0 & \text{if } t \geq T \end{cases} \tag{20}$$

One convenient specification for the distribution of T is geometric, i.e.

$$\Pr(T = k) = (1-\pi)^{k-1} \pi \text{ for } k = 1, 2, 3, ...$$

In this case, there is a constant probability π that the endowment will cease growing each period.

Once again, we begin our analysis with the case where agents can only buy one asset, just as in Gali's specification. As long as $d > 0$, the interest rate in this economy will be positive. Hence, once the endowment ceases to grow, the interest rate will exceed the growth rate of the economy. This condition rules out the possibility that a bubble can survive once the economy stops growing. To state this more formally, it will help to adopt the convention of using a variable with a hat to capture the value of a variable once growth has already stopped. Thus, $\widehat{p}_t$ denotes the price of the asset after the endowment stopped growing, i.e. if date t exceeds the realized value of T at which growth ceases, and $\widehat{f}_t$ denotes the fundamental value of the asset in this case. Our claim is that $\widehat{p}_t = \widehat{f}_t$, meaning no bubble is possible after date T. To see that this must be the case, note that since the interest rate in this economy is positive, storage will be dominated. Young agents will trade all of their endowment for the asset, and we would have

$$\widehat{p}_t = e_t = e_{T-1}$$

It then follows that the risk-free interest rate at date $t \geq T$ is given by

$$1 + \widehat{r}_t = \frac{d + \widehat{p}_{t+1}}{\widehat{p}_t} = 1 + \frac{d}{e_{T-1}}$$

We can then readily calculate the fundamental value of the asset after growth stops:

$$\widehat{f}_t = \sum_{j=1}^{\infty} \left(\prod_{i=0}^{j-1} \frac{1}{1 + \widehat{r}_{t+i}} \right) d = \sum_{j=1}^{\infty} \left(\prod_{i=0}^{j-1} \frac{1}{1 + d/e_{T-1}} \right) d = e_{T-1}$$

Note that the fundamental value of the asset depends on the realization of T even though the dividends are constant and as such independent of T. The reason the fundamental value depends on the realization of T is because the equilibrium interest rate r_t will depend on the realization of T.

When we use a variable without a hat, then unless noted otherwise we will assume it captures the value of that variable before growth has stopped. Thus, p_t denotes the price of the asset when the endowment has continued to grow up through date t. Since storage remains dominated, young agents will exchange all of their endowment for the asset. This implies $p_t = e_t$. Since agents are risk neutral, the return on a risk-free instrument r_t would have to equal the return on the asset, i.e.

$$1 + r_t = \frac{d + E[p_{t+1}]}{p_t}$$

Substituting in for p_t reveals that

$$r_t = (1-\pi)\, g + \frac{d}{e_t}$$

We now argue that no bubble will emerge in our model even when $t < T$, i.e. while the economy is growing.

Proposition 6: Suppose $d > 0$. If e_t is given by (20), no bubble can emerge in equilibrium, i.e. $p_t = f_t = e_t$ for $t < T$ and $\widehat{p}_t = \widehat{f}_t = e_T$ for $t \geq T$.

The fact that a bubble cannot exist beyond T does not on its own lead the bubble to unravel given T is random with unbounded support. That is, while it is true that the non-existence of a bubble beyond a fixed date can be used to prove the impossibility of a bubble beforehand by a backwards-induction argument, this is not true when the date at which the bubble cannot exist is potentially arbitrarily large. Recall that Blanchard and Watson (1982) develop examples of bubbles that burst with constant probability and no finite end date. And indeed, bubbles can arise when the endowment growth is stochastic if we allowed the asset to be intrinsically worthless, i.e. if we set $d = 0$. In that case, the fundamental value of the asset would equal 0, yet it is easy to confirm that $p_t = e_t > 0$ is an equilibrium in that case. When dividends are positive, by contrast, such equilibria break down.[10]

Allowing for government debt as an additional asset that agents can buy would not change the fact that no bubble is possible when growth can stop at random date. That is, suppose we introduced a government authority with an initial obligation $(1 + r_{-1})\, b_{-1}$ to the old at date 0 and allowed it to levy lump-sum taxes on the young each period. Then we can still rule out the possibility of a bubble. As before, the tax cannot exceed the endowment of the young at the same date, i.e. $\tau_t \leq e_t$. However, since the endowment is uncertain, we must add a restriction to ensure that the government not have to default on its obligation in the next period. We will require that $(1 + r_t)\, b_t \leq e_t$ for all t, where r_t denotes the risk free rate on government debt. This ensures that even if growth stops, the government can always collect enough resources to pay its debt in full.

In sum, the fact that agents want to buy assets in order to convert their endowment when young to consumption when old will not give rise to a bubble once we allow for the possibility that growth will stop at some stochastic date. However, work by Allen and Gorton (1993), Allen and Gale (2000), and Barlevy (2014) has argued that if agents can purchase assets with funds obtained from others, bubbles can arise even if they would not occur otherwise. The argument relies on risk-shifting: If agents can finance their purchases of risky assets using limited liability contracts, they can shift the losses if their purchases turn unprofitable onto their creditors. These truly represent credit-driven bubbles, since it is the possibility of risk-shifting to a lender that allows bubbles to arise. Absent credit, bubbles would not arise. In our setting,

[10] Note that the fundamental value changes discontinuously as d tends to 0 while e_t is held fixed, from $f_t = e_t > 0$ when $d > 0$ to $f_t = 0$ when $d = 0$. The reason for this discontinuity is that as d tends to 0, the interest rate after date T also tends to 0. Thus, for small values of d, we have an infinite stream of vanishingly small dividends discounted at a vanishginly small rate. This allows the fundamental value to remain bounded above 0 even when d is arbitrarily small but positive.

agents who borrow can in principle use those funds to gamble on when the endowment stops growing by buying the asset. This is because the price of the asset would be higher if the endowment continued growing. We now examine whether the possibility of speculation allows for the possibility of asset bubbles, and if so whether an intervention that increases interest rates could still dampen them.

The first step to allowing agents to speculate with borrowed funds is to introduce a credit market into our model. Suppose agents can make one-period loans. Only young agents would ever agree to lend, since old agents would not be around to collect on their loans. Young agents who want to save for old age now have the option to store their endowment, trade it for risk-free government bonds, trade it for the asset, or lend it out. Let R_t denote the interest rate they charge on loans, as distinct from the return r_t on government bonds. Young agents would only agree to lend to other young agents who will be around next period to repay their debt. The question is whether any young agent would want to borrow given their wish to save for when they are old. In principle, young agents might want to borrow to purchase more assets. For riskless government bonds, this would be pointless: Lenders can buy the same bonds themselves, and so would demand a return on loans R_t that is at least as high as the riskless rate of return r_t. Borrowing to buy government bonds thus cannot be profitable. Similarly, there would be no point in buying the asset after date T, at which point the asset is also riskless. Before date T, however, the realized return on the asset is random and depends on whether the endowment next period will be higher or not. Thus, agents might seek to profit by borrowing and gambling on high growth, defaulting if growth turns out low. Lenders understand this, and if they could monitor what borrowers do would set R_t equal to the high rate of return on the asset for those who borrow to buy the asset. This would again leave borrowers with no profits.

However, if the borrowers who bought risky assets could blend in with other borrowers with less risky investments, buying risky assets could be profitable, as previous work on risk-shifting has demonstrated. This is because competition among lenders would drive the interest rate on loans below the maximal return on the risky asset. To model this, we follow Barlevy (2014). That is, we introduce an additional group of agents that is endowed with no resources. Their presence thus adds nothing to the aggregate endowment of the economy. However, with no endowment, they also have nothing to lose from betting on risky assets, since they have no resources that could be seized if their gamble fails to pay off. If some of these agents buy assets, they will need to blend in with others who make less risky investments. We therefore assume that agents with no endowment also have access to a productive technology that only they can operate. This technology can convert one unit of resources today into $y > 1$ units of resources tomorrow, where y is sufficiently large that agents would want to undertake production.[11] To ensure not all agents produce, we assume the technology can only be operated up to some finite capacity $\kappa < e_0$: Once κ units of resources are

[11] The assumption that speculators can blend in with entrepreneurs who can achieve high returns may be appropriate for new technologies, where some applications are genuinely productive while others are speculative and outsiders have a hard time distinguishing the two. It is obviously not a good description of the housing market, where borrowers and the assets they purchase are closely scrutizined. But in that case, Barlevy and Fisher (2014) show that for housing, speculators can blend in with genuine home buyers who have a stronger preference for home ownership. In that case, both types buy the same asset, but the price of that asset can still exceed the value of an additional unit of housing.

invested, any additional unit of resource invested will yield a return of 0. Hence, some agents will borrow resources to operate the technology while others will borrow to buy the asset.[12] Once the economy stops growing, borrowing to buy the asset will no longer be profitable given the asset has no upside. In this case, agents should be indifferent about buying assets, and there is no loss of generality in proceeding as if agents borrow to buy the asset at all dates.

If the group of agents who wish to borrow is large, speculators will earn zero profits in equilibrium from buying risky assets. This requires that they earn nothing in the best scenario, i.e. if the endowment continues to grow. That is, R_t must be such that

$$(1 + R_t)\, p_t \geq p_{t+1} + d \tag{21}$$

Competition will ensure that (21) holds with equality. When economy stops growing, agents who borrowed to buy assets would be forced to default and hand over the asset to their creditors. Agents who borrowed to produce will instead earn a return y, which we assume is large enough that they can pay back R_t in full. Note that the interest rate on loans R_t is the same as when lenders can observe that their borrowers are buying risky assets, i.e. R_t is equal to the highest possible return on the asset. The effect of cross-subsidization in this case will show up not in the interest rate on loans R_t, but in other equilibrium prices.

Since $d > 0$, we can show once again that storage will be dominated. Hence, all of the endowment each period will be used to pay taxes, buy bonds, finance production, and purchase the asset. From this, we can deduce the equilibrium price of the asset at each date:

$$p_t = e_t - b_t - \tau_t - \kappa$$

Combining this with (21), we can derive an expression for the equilibrium loan rate R_t when $t < T$, i.e. before growth stops:

$$1 + R_t = \frac{d + p_{t+1}}{p_t} = \frac{d + (1+g)\, e_t - b_{t+1} - \tau_{t+1} - \kappa}{e_t - b_t - \tau_t - \kappa} \tag{22}$$

Turning to the risk-free rate r_t, let us suppose that young agents who wish to save do not buy the asset directly, and instead loan their funds to others who buy the asset. As we will see, young agents who wish to save will indeed prefer lending to buying the asset in equilibrium. In this case, total lending will equal $\kappa + p_t$ to cover both production and asset purchases. Since creditors could have purchased riskless government bonds with these resources and earned $1 + r_t$, and they are risk-neutral, they will charge an interest rate on loans R_t that will yield the same expected return as buying government debt, i.e.

$$(1 + r_t)\,(\kappa + p_t) = (1 + R_t)\,(\kappa + (1 - \pi)\, p_t) + \pi\,(d + \widehat{p}_{t+1})$$

[12] This suggests we need some way to ration who produces in a way that ensures both types would be able to receive credit. For example, if borrowers arrive sequentially and then choose which project to run, creditors would only loan κ units of resources. However, if the order in which agents arrive to borrow is not the same as the order in which they choose how to deploy their funds, all agents will be able to borrow and those who get to choose projects first will produce.

Rearranging this equation, we obtain

$$\begin{aligned} R_t - r_t &= \frac{\pi\left(p_{t+1} - \widehat{p}_{t+1}\right)}{\kappa + p_t} \\ &= \frac{\pi g e_t}{e_t - b_t - \tau_t} > 0 \end{aligned} \tag{23}$$

In words, there is a spread between the rate agents charge on loans R_t and the risk-free rate paid on government bonds r_t. This spread is meant to compensate lenders for losses they would incur on their loans if the economy were to stop growing.

To see that this economy exhibits a bubble, note that if we combine (23) with the fact that $1 + R_t = \left(d + p_{t+1}\right)/p_t$, we can solve for the risk-free rate r_t while the economy is growing, i.e. $t < T$:

$$1 + r_t = \frac{\left(1-\pi\right)p_{t+1} + \pi\widehat{p}_{t+1} + d}{p_t} + \frac{\pi\kappa g e_t}{p_t\left(p_t + \kappa\right)} \tag{24}$$

Hence, as long as $\kappa > 0$, the risk-free rate r_t will strictly exceed the expected return on buying the asset. As an aside, this observation confirms that young agents who wish to save their endowment for old age will strictly prefer to buy government bonds or lending out their endowment to buying the asset directly. Thus, the only agents who buy the asset before date T will be those who borrow to buy it. Let us define $1 + r_t^A$ as the expected return to buying the asset at date t, i.e.

$$1 + r_t^A \equiv E\left[\frac{p_{t+1} + d}{p_t}\right] = \begin{cases} \dfrac{d + \left(1-\pi\right)p_{t+1} + \pi\widehat{p}_{t+1}}{p_t} & \text{if } t < T \\ 1 + r_t & \text{if } t \geq T \end{cases}$$

Prior to date T the expected return on the asset captures the uncertainty in asset prices. But after date T, when there is no uncertainty, the return on the asset r_t^A will necessarily be equal to r_t. Proposition 6 shows that if we discount dividends at the rate $1 + r_t^A$, the expected present discounted value of dividends must equal the price, i.e.

$$p_t = E\left[\sum_{t=1}^{\infty}\left(\prod_{i=0}^{s}\frac{1}{1 + r_t^A}\right)d\right]$$

where the expectation above is taken with respect to the realization of T. Although Proposition 6 assumes there is no government sector, so $b_t = \tau_t = 0$, the same argument would follow if we imposed (11). Hence, the equilibrium price of the asset p_t corresponds to the expected present discounted value of dividends when we discount future dividends at the rate $1 + r_t^A$. By contrast, the equilibrium fundamental value of the asset f_t is defined as the expected present discounted value of dividends discounted at the riskless rate $1 + r_t$, since the dividends on the asset are riskless. Since (24) implies that $r_t^A < r_t$ for $t < T$, the price of the asset will exceed the fundamental value while the economy continues to grow. The asset is thus a bubble.

Intuitively, the fact that lenders cannot monitor borrowers creates a situation in which profitable borrowers who deploy their funds to produce cross-subsidize speculators by lowering the interest charged to all borrowers. This makes speculators more eager to borrow to buy the asset. Since the amount of resources borrowers obtain is fixed in equilibrium, rather than bid up the price of assets, the demand by speculators

translates into a lower expected return on the asset. The fact that the return on the asset is low implies it is overvalued relative to the risk-free rate an agent could earn.

The model thus features a stochastically bursting bubble. The price of the asset exceeds its fundamental value while the economy is growing. When the economy stops growing, the risk free rate falls and the bubble collapses to zero, i.e. $\widehat{p}_t$ will equal $\widehat{f}_t$ from date T on. Note that the bubble bursts because the fundamental value of the asset rises at date T when the real interest rate r_t falls to r_t^A, not because the price of the asset collapses. If we assumed the endowment fell rather than stopped growing at date T, the price of the asset would have fallen when the bubble burst.

We can once again use our framework to study the effect of increasing government obligations at all dates on the size of the bubble. Our proof of Proposition 4 carries through as before. That is, if we restrict fiscal policies to paths as in (12) that can be parameterized by a single parameter b that governs the size of debt obligations, then increasing debt will drive down the price of the asset and drive up the risk-free rate.

Proposition 7: Suppose fiscal policy is given by (12), where $r_{-1} > 0$. Then the equilibrium interest rate r_t and loan rate R_t are both increasing in b and the equilibrium asset price p_t is decreasing in b for all t.

Although an increase in government obligations depresses the price of the asset, p_t, the argument that it also makes the bubble smaller is more involved, since higher rates not only drive down the price but also the fundamental value of the asset. In fact, for credit-driven bubbles, it is no longer the case that issuing more debt will reduce the bubble. This highlights an important difference between a risk-shifting bubble and the bubbles we described earlier that arose because of a scarcity of assets for savings. Recall that we can think of the price of the asset and the fundamental value of the asset as expected present discounted value of dividends using different discount rates – the price corresponds to discounting at rate $1 + r_t^A$, the expected return on the asset, and the fundamental corresponds to discounting at rate $1 + r_t$, the return on risk-free assets. How a policy intervention affects the bubble depends on how it affects these two rates. By contrast, in the deterministic case we analyzed earlier where there was no risk, there was a single interest rate. In that case, we can think of the fundamental value of the asset as the present discounted value of the dividends d it yields, discounted at the risk-free rate, while the price of the asset corresponded to the present discounted value at the same discount rate of a shadow dividend that exceeds d and captures the value the bubble provides as a superior savings instrument. In that case, there was no ambiguity about how an intervention affected the bubble: Raising rates lowers the discounted value of both dividends and shadow dividends. But since the latter is larger, the effect of higher rates will be larger on the price than on fundamentals. This logic does not carry over to credit-driven bubbles.

Calculating the fundamental value f_t in our model to determine the effect of increasing the interest rate r_t on Δ_t turns out to be unwieldy. However, we can obtain analytical insights if we assumed that rather

than assume a geometric distribution, T could only take on two values, i.e.

$$T = \begin{cases} 1 & \text{with probability } \pi \\ 2 & \text{with probability } 1-\pi \end{cases}$$

In other words, the endowment will either grow for one period or not at all. Speculators at date 0 can gamble on whether this growth materializes or not, but there is no scope for gambling beyond date 0. In this case, we have the following result:

Proposition 8: Suppose fiscal policy is given by (12), where $r_{-1} > 0$. Then there exists a value κ^* where $0 < \kappa^* < e_0$ such that an increase in b will lead to a larger bubble at date 0 if $0 < \kappa < \kappa^*$ and a smaller bubble at date 0 if $\kappa^* < \kappa < e_0$.

In words, as long as there are enough safe borrowers to cross-subsidize speculation, a policy intervention that induces agents to hold more government debt will increase rates, drive down the price of the asset, and reduce the extent to which the asset is overvalued. Intuitively, a higher value of κ not only implies more cross-subsidization, but it also implies fewer resources will be used to purchase the asset, since by assumption production is assumed to carry a high return. This will drive down both the price and the fundamental value of the asset. When the fundamental value of the asset is low, a policy that increases the interest rate will have a small effect on the fundamental value given it is already low. But the effects of crowding out resources from the asset are not blunted when the price is low. In this case an intervention that induces agents to increasingly hold more assets will depress the bubble.

We conclude with a discussion of welfare. Although we now have a model with risky credit-driven bubbles in which agents default when the economy stops growing and the bubble bursts, the welfare implications of intervention of our setup remain uninteresting. The only effect of intervention is to redistribute resources from the agents who undertake production to the endowed agents who finance them. To see this, observe that each cohort of young agents seeking to save for old age spends $e_t - \tau_t$ to buy bonds and make loans. They earn $(1 + r_t) b_t$ on their bond holds, $(1 + R_t) \kappa$ from producers, and they expect to earn $\left(1 + r_t^A\right) p_t$ from lending to speculators to buy the asset, i.e. their expected return is

$$\frac{(1 + r_t) b_t + (1 + R_t) \kappa + \left(1 + r_t^A\right) p_t}{e_t - \tau_t}$$

Using the expressions for p_{t+1} and $\widehat{p}_{t+1}$, we can deduce that the return for these young agents is equal to

$$\frac{(1 + (1 - \pi) g) e_t + d + R_t \kappa}{e_t - \tau_t}$$

Hence, an increase in R_t makes young agents who wish to save better off at the expense of those young agents who undertake production and must hand over more of the y units of output they produce to creditors. Speculators who buy risky assets always earn zero profits in equilibrium, regardless of policy. But beyond this redistribution, an intervention that raises r_t and drives down p_t will have no effects on welfare. In particular, the risk from the uncertain growth rate for the economy is always fully borne by young agents

who wish to save, regardless of b. Reducing the bubble cannot eliminate this risk, since it is inherent to the endowment of agents. Dampening the bubble in our model will therefore not be inherently desirable.

Once again, though, there are alternative formulations in which an intervention to mitigate the bubble can matter for welfare. In Appendix C, we describe a variation of the model of credit-driven bubbles above with two key differences. First, rather than assume dividends are deterministic and that the source of uncertainty concerns the growth rate of the endowment, we allow dividends to be risky. In this case, we can assume the endowment never grows, since this uncertainty no longer plays an important role. If lenders are unable to monitor borrowers, agents will be able to profit from buying risky assets to gamble on whether the dividends they pay out are high or low. The second important difference is that we allow agents to produce assets rather than assume the supply of assets is fixed. As the price of the asset rises, more will be created. In Appendix C, we provide a numerical example that shows the asset is a bubble even though new units can be produced, i.e. the fact that the supply of assets is fixed is not essential for allowing the price to exceed the asset's fundamental value. We also show that there exists an intervention that drives up the real interest rate, reduces the size of the bubble, and decreases the number of assets created. Since assets are risky, the number of assets created determines the aggregate amount of risk in the economy. Thus, our example shows that an intervention can not only dampen the bubble but also reduce the extent of aggregate risk in the economy. Since we continue to assume agents are risk-neutral, our formulation is not useful for analyzing the welfare implications of reducing aggregate risk and assessing when this policy might be desirable. But the example suggests how one might use our results to construct scenarios in which an intervention to dampen bubbles can be welfare improving, in a way that is consistent with they type of arguments policymakers invoke in practice: Acting to dampen the bubble can reduce the extent of aggregate risk and contribute to financial stability.

5 Conclusion

In this paper, we argue that policymakers who wish to act against a potential asset bubble can turn to policies that involve raising interest rates. This result stands in contract to recent work by Gali (2014) which showed that a policy that led to higher rates only served to increase the bubble if one were in fact present. We began by replicating his result in a slightly different setting. In particular, we constructed a similar environment that features multiple equilibria price paths, and showed that those equilibria with higher interest rates were also associated with larger bubbles. To the extent that a policymaker setting rates is choosing among these equilibria, a policymaker who chooses to raise rates will also end up with a larger bubble. We then showed that introducing a reasonable and seemingly inconsequential modification to our model – assuming the bubble asset is not intrinsically worthless but yields actual dividends – can eliminate the multiplicity of equilibria, forcing us to think of policy interventions not in terms of equilibrium selections but in terms of some direct intervention in financial markets such as issuing or selling government bonds. We construct a few examples of interventions by a policymaker that can simultaneously raise the interest rate and reduce the extent to which the bubble asset is overvalued. Thus, the notion that a policymaker

intent on mitigating a bubble would act in a way that raised interest rates can be plausible, notwithstanding the findings laid out by Gali.

We conclude our analysis with a few observations. First, one of the points Gali emphasizes in his paper is that theoretically, higher interest rates ought to be associated with more rapid asset price appreciation. In subsequent work, Gali (2015) provides empirical evidence in support of this proposition. Our model, which also features rational agents, is consistent with this implication. However, whether a policymaker who intervenes to raise rates succeeds in depressing a bubble depends not on how interest rates affect the growth rate of asset prices, but how they affect asset prices as compared to fundamentals. Unfortunately, since this requires measuring the fundamental value of assets, the latter is difficult if not impossible to resolve empirically. But our analysis nevertheless shows that the case in which higher rates affect asset prices more than they do fundamentals is a plausible theoretical possibility.

Even if empirical evidence on how interest rates impact prices and fundamental is hard to come by, our analysis does suggest empirical measures that policymakers can use to assess whether contractionary policies are likely to have their intended effect on bubbles, at least in line with the channels we emphasize here. The reason contractionary policy mitigates bubbles in our model is that it crowds out resources from overvalued assets. In fact, our analysis suggests that crowding out is more essential for mitigating bubbles than raising rates per se, and in some of our examples a policymaker can crowd out resources from the asset market without changing interest rates or even allowing them to fall. This suggests that for an interest rate hike to depress bubbles through the channel we describe in this paper, it should be accompanied by a decline in savings or in the share of the bubble asset in the aggregate wealth portfolio of investors. Absent these, our model would suggest an interest rate hike would not be successful in reining in bubbles. Of course, there may be other channels through which a higher interest rate might depress bubbles that our model fails to capture. For example, one view that is sometimes articulated in policy circles is that low interest rates trigger a search for yield that induces agents to take on more risks. Our model does not admit such a channel. We also ignore any role for interest rate policy to affect bubbles by affecting credit markets. Although we do describe a model in Section 4 where agents borrow to speculate on assets, the amount speculators borrow in our model in unaffected by the interest rate. In principle, though, a higher interest rate might make speculation unprofitable, even if a higher rate increases the expected rate of appreciation. Formalizing this reasoning would presumably require a richer model of credit spreads than we develop here.

Finally, although we largely follow Gali in exploring models of bubbles that arise because of a shortage of assets to meet the savings needs of agents, in the last part of the paper we examine a version of the model in which bubbles arise because agents speculate with borrowed funds rather than because of a shortage of assets for savings. This setup admits the possibility of a stochastically bursting bubble even though the price of the asset is uniquely determined, i.e. it does not involve the use of sunspots and multiple equilibria. Our setup also incorporates the possibility of default and losses to lenders when speculation proves unsuccessful. As we detail in an appendix, our setup suggests that a bubble might lead to greater aggregate risk in the economy, which might conceivably create a role for policy intervention. While it is not

obvious that increasing interest rates is necessarily the best way to mitigate excessive aggregate risk, our setup suggests an environment in which one might explore such questions. We leave this to future work.

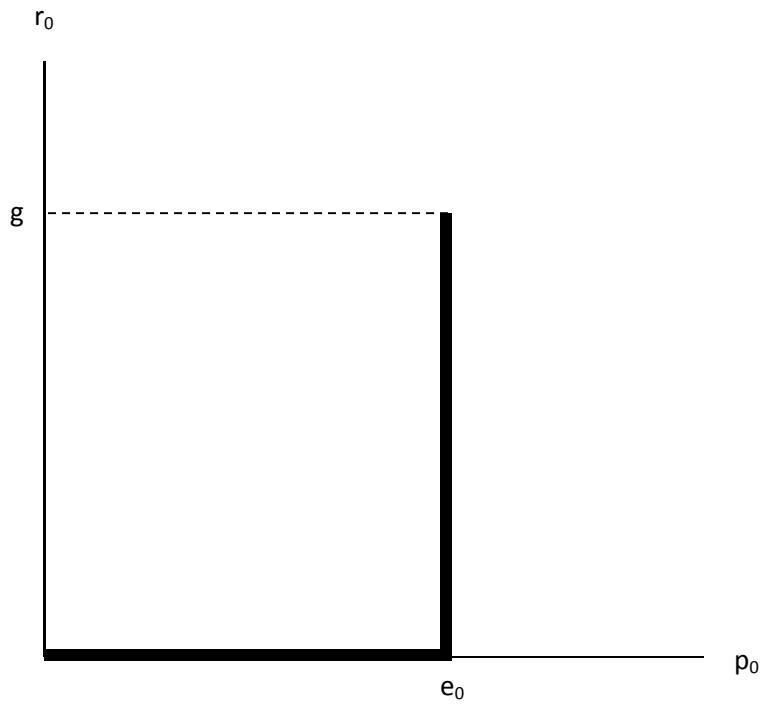


Figure 1: Set of possible equilibrium values (p_0,r_0) at date 0

Appendix A: Proofs of Propositions

Proof of Proposition 1: In the text, we argued that $r_t = 0$ is an absorbing state. That is, there exists a t^* where $0 \leq t^* \leq \infty$ such that $r_t > 0$ for $t < t^*$ and $r_t = 0$ for $t \geq t^*$. For $t < t^*$, we know that $r_t > 0$ implies storage is dominated, and so $p_t = e_t$ for $t < t^*$. Since $r_t = 0$ for $t \geq t^*$ we can use (3) to conclude that $p_{t+1} = p_t$ for $t \geq t^*$, and by induction we can infer $p_t = p_{t^*}$ for all $t \geq t^*$.

The last step is to show that at date t^*, any $x \in [e_{t^*-1}, e_{t^*}]$ and only those values can be an equilibrium. Since $r_t \geq 0$, $p_{t^*} \geq p_{t^*-1} = e_{t^*-1}$. Since $p_t \leq e_t$ at all dates t, this is also true for date t^*. For any $x \in (e_{t^*-1}, e_{t^*}]$, the rate of return on the asset to those purchasing the asset at date $t^* - 1$ will be positive, so they would buy the asset at this price at date $t^* - 1$ and sell all their holdings at date t^*. Since $r_{t^*} = 0$, the young at date t^* are indifferent between storage and buying the asset, so they would be willing to buy any amount the old sell. Hence, we can construct an equilibrium where $p_{t^*} = x$ for any $x \in (e_{t^*-1}, e_{t^*}]$. ■

Proof of Proposition 2: In the text, we argued that $r_t > 0$. Since storage is dominated, the price $p_t = e_t$, since at any other price the young would be using storage which is inferior or saving more resources than they are endowed with. The interest rate is then pinned down by the requirement that

$$1 + r_t = \frac{d + p_{t+1}}{p_t}$$

and substituting in $p_t = e_t$ and $p_{t+1} = (1 + g)\, e_t$. ■

Proof of Lemma 1: The proof is by induction. Suppose we are given histories $\{b_{-1}, ..., b_{t-1}\}$ and $\{r_{-1}, ..., r_{t-1}\}$ up to but not including date t. We want to show that these histories imply a unique candidate pair b_t and r_t that satisfy both the government budget constraint and the equilibrium interest rate condition. From the government's flow budget constraint, we have

$$b_t = (1 + r_{t-1})\, b_{t-1} - \tau_t$$

Since τ_t, b_{t-1}, and r_{t-1} are all given to us, b_t is uniquely determined. Next, consider the equilibrium condition for interest rates (10). We already argued that when $d > 0$, the equilibrium interest rate r_t must be positive or else the price of the asset will eventually turn negative. Hence, storage will be dominated, and agents will exchange all of their endowment net of taxes for bonds and the asset, i.e.

$$p_t = e_t - \tau_t - b_t$$

Substituting in for p_t and p_{t+1} into (10), we get

$$(1 + r_{t-1})\, b_{t-1} - b_t = (1 + r_{t-1})\, (e_{t-1} - \tau_{t-1}) - (e_t - \tau_t) - d$$

But from (9), we know that

$$(1 + r_{t-1})\, b_{t-1} - b_t = \tau_t$$

and so we have

$$\tau_t = (1 + r_{t-1})(e_{t-1} - \tau_{t-1}) - (e_t - \tau_t) - d$$

which reduces to

$$d = (1 + r_t)(e_t - \tau_t) - e_{t+1} \tag{25}$$

Since e_t, e_{t+1}, and τ_t are given, r_t is uniquely determined. The claim then follows by induction. ■

Proof of Proposition 3: We know from Lemma 1 that the path $\{r_t, b_t\}_{t=0}^{\infty}$ is uniquely determined in equilibrium. Since $r_t > 0$, we know that storage is dominated, so the young will use all of their available endowment to purchase assets. This is equal to $e_t - \tau_t$. Since they must allocate b_t to government debt, the amount spent on the asset is equal to $p_t = e_t - \tau_t - b_t$. The expression for r_t then follows from the fact that in equilibrium, $(1 + r_t) p_t = d + p_{t+1}$. Finally, (11) implies that

$$\lim_{t \to \infty} \frac{\tau_t + b_t}{e_t} = \theta$$

for some $\theta < 1$. Hence,

$$\begin{aligned} \lim_{t \to \infty} p_t &= \lim_{t \to \infty} e_t - \tau_t - b_t \\ &= \lim_{t \to \infty} e_t - \theta e_t \\ &= (1 - \theta) \lim_{t \to \infty} e_t = \infty \end{aligned}$$

where the last step uses the fact that $\theta < 1$. ■

Proof of Proposition 4: At $t = 0$, we have $\tau_0 = r_{-1} b$ which is increasing in b. From the intertemporal budget constraint (9) and the equilibrium condition (10), we know that

$$d = (1 + r_t)(e_t - \tau_t) - (1 + g) e_t$$

Since τ_0 is increasing in b, it follows that r_0 is increasing in b as well. Next, suppose we know $r_0, ..., r_{t-1}$ are increasing in b. Then we have $\tau_t = r_{t-1} b$ and by the same logic as above, we can conclude that τ_t and r_t are increasing in b. To finish the claim, we need to prove that p_t is decreasing in b. But from Proposition 4, we know that

$$p_t = e_t - b - \tau_t$$

which establishes the result. ■

Before proving Proposition 5, we establish the following lemma:

Lemma A1: Suppose fiscal policy is given by (12) and satisfies (11). Then $\lim_{t \to \infty} r_t = g$.

Proof of Lemma A1: Recall from (25) that

$$d = (1 + r_t)(e_t - \tau_t) - e_{t+1}$$

Using the fact that $e_{t+1} = (1+g)\, e_t$ and the fact that when $b_t = b$ for all t,

$$\begin{aligned} \tau_t &= (1+r_{t-1})\, b_{t-1} - b_t \\ &= (1+r_{t-1})\, b - b = r_{t-1} b \end{aligned}$$

we can rewrite (25) as

$$d = (r_t - g)\, e_t - (1+r_t)\, r_{t-1} b$$

which, upon rearranging, yields a difference equation that relates r_t to r_{t-1}:

$$r_t = \frac{d + g e_t + r_{t-1} b}{e_t - r_{t-1} b} \equiv h_t\left(r_{t-1}\right) \tag{26}$$

Since $r_{t-1} b = b_t + \tau_t$, we can rewrite this as

$$r_t = \frac{d + g e_t + b_t + \tau_t}{e_t - b_t - \tau_t}$$

From (11), we know that

$$\lim_{t\to\infty} b_t + \tau_t = \lim_{t\to\infty} \theta e_t$$

for some $\theta < 1$. Hence, taking limits as t tends to infinity, we get

$$\begin{aligned} \lim_{t\to\infty} r_t &= \lim_{t\to\infty} \frac{d + g e_t + b_t + \tau_t}{e_t - b_t - \tau_t} \\ &= \lim_{t\to\infty} \frac{d + (g+\theta)\, e_t}{(1-\theta)\, e_t} \\ &= \frac{g+\theta}{1-\theta} \end{aligned}$$

where the last expression is finite. But if $\lim_{t\to\infty} r_t$ is finite, then

$$\begin{aligned} \lim_{t\to\infty} r_t &= \lim_{t\to\infty} \frac{d + g e_t + r_{t-1} b}{e_t - r_{t-1} b} \\ &= \lim_{t\to\infty} \frac{g + \frac{d + r_{t-1} b}{e_t}}{1 - \frac{r_{t-1} b}{e_t}} \\ &= g \end{aligned}$$

where the last equality follows from the fact that if $\lim_{t\to\infty} r_t$ is finite, so is $\lim_{t\to\infty} r_t b$. Hence, (11) together with (12) imply that $\lim_{t\to\infty} \frac{b_t + \tau_t}{e_t} = \theta = 0$ and $\lim_{t\to\infty} r_t = g$, as claimed. ■

Proof of Proposition 5: By repeated substitution, we can relate Δ_t to any Δ_{t+h} for any horizon h:

$$\Delta_t = \left(\prod_{j=0}^{h-1} \frac{1}{1+r_{t+j}} \right) \Delta_{t+h}$$

From Lemma A1, we know that $\lim_{t\to\infty} r_t = g$. Hence, for any value of b, we have

$$f_t \equiv \lim_{t\to\infty} \sum_{j=0}^{\infty} \left(\prod_{i=1}^{j} \frac{1}{1+r_{t+j}} \right) d \to \frac{d}{g}$$

By contrast, $p_t = e_t - r_{t-1}b$, which tends to $e_t - gb$. This implies that the bubble term $\Delta_t = p_t - f_t$ tends to $e_t - gb - \frac{d}{g}$, which is decreasing in b. In other words, there exists a T such that for $t \geq T$, Δ_t is decreasing in b. Since r_{t+j} is increasing in b by Proposition 4, it follows that $\Delta_t = \left(\prod_{j=0}^{h-1} \frac{1}{1+r_{t+j}}\right) \Delta_{t+h}$ will be decreasing in b. ■

Proof of Proposition 6: We show the statement holds for the fundamentals at date 0. The argument for other dates is similar. The fundamental value of the asset at date 0 if growth stops until a random date T_n is given by

$$f_0^n = \sum_{T=1}^{n} \frac{(1-\pi)^{T-1}\pi}{1-(1-\pi)^n} \left(\sum_{s=1}^{T-1} \left(\prod_{i=0}^{s} \frac{1}{1+r_i} \right) d + \left(\prod_{i=0}^{T-1} \frac{1}{1+r_i} \right) e_T \right)$$

Here we use the fact that $f_t^n = e_t$ for $t \geq n$. The above summation can be written as two sums:

$$f_0^n = \sum_{T=1}^{n} \frac{(1-\pi)^{T-1}\pi}{1-(1-\pi)^n} \sum_{t=1}^{T-1} \left(\prod_{i=0}^{s} \frac{1}{1+r_i} \right) d + \sum_{T=1}^{n} \frac{(1-\pi)^{T-1}\pi}{1-(1-\pi)^n} \left(\prod_{i=0}^{T-1} \frac{1}{1+r_i} \right) e_T$$

Since for $t < n$, we have

$$r_t = (1-\pi)\, g + \frac{d}{e_t} > (1-\pi)\, g > 0$$

we can easily establish that the first term converges. As for the second term, once again using the expression for r_t when $t < n$, we have

$$\begin{aligned}
\sum_{T=1}^{n} \frac{(1-\pi)^{T-1}\pi}{1-(1-\pi)^n} \left(\prod_{i=0}^{T-1} \frac{1}{1+r_i} \right) e_T &= \sum_{T=1}^{n} \frac{(1-\pi)^{T-1}\pi}{1-(1-\pi)^n} \left(\prod_{i=0}^{T-1} \frac{1+g}{1+(1-\pi)\, g + d/e_i} \right) e_0 \\
&= \sum_{T=1}^{n} \frac{\pi}{1-(1-\pi)^n} \left(\prod_{i=0}^{T-1} \frac{(1-\pi)\,(1+g)}{1+(1-\pi)\, g + d/e_i} \right) \frac{e_0}{1-\pi} \\
&< \sum_{T=1}^{n} \frac{\pi}{1-(1-\pi)^n} \left(\prod_{i=0}^{T-1} \frac{(1-\pi)\,(1+g)}{1+(1-\pi)\, g} \right) \frac{e_0}{1-\pi}
\end{aligned}$$

Since the last term converges whenever $\pi > 0$, we know that $\lim_{n\to\infty} f_0^n$ exists. Moreover, we know that $f_0^n = e_0$. This can be computed directly, but it also follows from a simple unravelling argument for bubbles with finite horizons. Hence, $f_0 = \lim_{n\to\infty} f_0^n = e_0$. From this, we can conclude that $f_0 = e_0$. A similar argument can be applied at any date. ■

Proof of Proposition 7: The implications of higher b for r_t and p_t can be established in the same way as in Proposition 4. From (23), we have that

$$\begin{aligned}
R_t &= r_t + \frac{\pi g e_t}{e_t - b_t - \tau_t} \\
&= r_t + \frac{\pi g e_t}{e_t - (1+r_{t-1})\, b}
\end{aligned}$$

Since both r_t and $(1+r_{t-1})\, b$ are increasing in b, we can conclude that R_t is increasing in b as well. ■

Proof of Proposition 8: Observe that the price of the asset at date 0 is given by

$$\begin{aligned} p_0 &= e_0 - b_0 - \tau_0 - \kappa \\ &= e_0 - (1 + r_{-1})\, b - \kappa \end{aligned}$$

Next, from (24), we have

$$(1 + r_0)\,(e_0 - b_0 - \tau_0 - \kappa) = (1 - \pi)\, g e_0 + e_0 - b_1 - \tau_1 - \kappa + d + s_0 \kappa$$

where

$$s_0 = \frac{\pi g e_0}{e_0 - b_0 - \tau_0}$$

Using the parameterization (12), we have

$$\tau_t = r_{t-1} b$$

and so

$$(1 + r_0)\,(e_0 - b - r_{-1} b - \kappa) = (1 - \pi)\, g e_0 + e_0 - (1 + r_0)\, b - \kappa + d + s_0 \kappa$$

Rearranging, we have an expression for r_0 in terms of b and other primitives.

$$1 + r_0 = \frac{(1 - \pi)\, g e_0 + r_{-1} b + d + s_0 \kappa}{e_0 - r_{-1} b - \kappa}$$

The fundamental value at date 0 is given by

$$f_0 = \frac{d + (1 + (1 - \pi)\, g)\, e_0 - (1 + r_0)\, b - \kappa}{1 + r_0}$$

Substituting in for r_0 and evaluating $\Delta_0 = p_0 - f_0$ yields

$$\Delta_0 = \frac{\pi g e_0 \kappa\,(r_{-1} b - e_0 + \kappa)}{(1 + r_{-1})\, b\,(d + (1 + (1 - \pi)\, g)\, e_0 - \kappa) - e_0\,(d + e_0\,(1 + (1 - \pi)\, g) - \kappa\,(1 - \pi g))}$$

Differentiating Δ_0 with respect to b and simplifying (with Mathematica) yields

$$\frac{d\Delta_0}{db} = \frac{(1 + (1 - \pi)\, g)\, e_0^2 - (2 + (1 + g)\, r_{-1} + g\,(1 - \pi))\, e_0 \kappa + (1 + r_{-1})\, \kappa^2 + (e_0 - (1 + r_{-1})\, \kappa)\, d}{\left[(1 + r_{-1})\, b\,(d + (1 + (1 - \pi)\, g)\, e_0 - \kappa) - e_0\,(d + e_0\,(1 + (1 - \pi)\, g) - \kappa\,(1 - \pi g))\right]^2} \pi g e_0 \kappa$$

The sign of this derivative corresponds to the sign of the numerator in the fraction above, which is independent of b, and represents a quadratic in κ. This quadratic is convex, since the coefficient on κ^2 is $1 + r_{-1} > 0$. Evaluating the quadratic at $\kappa = 0$ yields

$$(1 + (1 - \pi)\, g)\, e_0^2 + e_0 d$$

which is positive, while evaluating the quadratic at $\kappa = e_0$ yields

$$-e_0\,(d + g e_0)\, r_{-1}$$

which is negative. This tells us that the roots of the quadratic are both real, with one between 0 and e_0 and the other above e_0. It follows that we can find a $\kappa^* \in (0, e_0)$ such that $\frac{d\Delta_0}{db} > 0$ if $\kappa \in (0, \kappa^*)$ and $\frac{d\Delta_0}{db} < 0$ if $\kappa \in (\kappa^*, e_0)$. ■

Appendix B: Monetary Policy

In this Appendix, we describe a monetary OLG economy, building on the framework we sketch out in Section 3. The key difference between the model here and the one sketched out in Section 3 is that rather than assuming workers are yeoman farmers who operate their own technology, we now introduce producers who price their goods and hire labor, allowing us to incorporate the possibility of price rigidity. Our framework borrows elements from both Adam (2003) and Appendix 3 in Gali (2014). As in Gali, we assume agents hold money because they derive utility directly from money holdings without modelling why. However, we follow Adam in allowing for variable labor supply and in assuming that the entrepreneurs who hire workers are young rather than old. The implication of this is that all output generated within the period will be used to buy assets and not just a fraction of the output that accrues to the young.

Following Adam (2003), suppose each cohort consists of a unit mass of workers and a unit mass of entrepreneurs. This is in contrast to the model described in Section 3, where we assume there is only a single unit mass of individuals in each cohort. When agents are young, those who are workers supply their labor services to entrepreneurs who know how to deploy labor to produce goods. When they turn old, neither type is productive any more and each must rely on previous earnings to consume. We assume that the two cohorts equally bear the burden of lump sum taxes, so that when the government collects τ_t from the young, it collects $\frac{1}{2}\tau_t$ from workers and entrepreneurs.

As in the text, we use M_t to denote the amount of money circulating at date t, Π_t to denote the gross inflation rate P_{t+1}/P_t, and

$$x_{t+1} = \frac{M_{t+1} - M_t}{P_{t+1}}$$

as the injection of money between dates t and $t+1$ measured in terms of how much this amount could buy at date $t+1$. We assume the injection is split equally between old workers and old entrepreneurs, so each expects to receive $\frac{1}{2}x_{t+1}$ when old.

We start with workers. Each worker is endowed with one unit of time and must decide how to allocate it. The cost of providing effort n_t for the cohort born at date t is given by a convex function $v_t(n_t)$. As in the text, we assume $v_t(n_t) = A_t v(n_t)$ where $\lim_{n\to 0} v'(n) = 0$ and $\lim_{n\to 1} v'(n) = \infty$. In addition to caring about consumption when old and leisure when young, workers derive utility from their money holdings as in Gali (2014). Specifically, we replace (1) with

$$u\left(c_t^w, c_{t+1}^w, m_t^w, n_t\right) = c_{t+1}^w + \frac{\theta}{2}\ln\left(m_t^w\right) - v_t(n_t) \tag{27}$$

The budget constraint of a worker is given by

$$c_{t+1}^w = (1+r_t)\left(\frac{W_t}{P_t}n_t - m_t^w - \frac{1}{2}\tau_t\right) + \Pi_t^{-1}m_t^w + \frac{1}{2}x_{t+1} \tag{28}$$

where W_t denotes the nominal wage per unit labor. Since government bonds and the asset are perfect substitutes, agents will be indifferent between them. Hence, workers face only two non-trivial choices: How

hard to work and how much of their wealth to hold as money. The optimal effort level n_t will satisfy

$$A_t v'(n_t) = (1 + r_t) \frac{W_t}{P_t}$$

and the optimal level of money demand will satisfy

$$m_t^w = \frac{\theta}{2\left((1 + r_t) - \Pi_t^{-1}\right)} = \frac{\theta \Pi_t}{2i_t}$$

Next, we turn to entrepreneurs. They are endowed not with labor but with the knowledge of how to deploy labor to produce goods. Each entrepreneur $i \in [0, 1]$ can produce a different intermediate good i. These goods are sold to competitive final goods producers who produce the goods old agents consume. Entrepreneurs derive utility from consuming when old and from holding money, i.e.

$$u\left(c_t^e, c_{t+1}^e, m_t^e\right) = c_{t+1}^e + \frac{\theta}{2} \ln\left(m_t^e\right) \tag{29}$$

The budget constraint of an entrepreneur is given by

$$c_{t+1}^e = (1 + r_t)\left(\rho_{it} - m_t^e - \frac{1}{2}\tau_t\right) + \Pi_t^{-1} m_t^e + \frac{1}{2} x_{t+1} \tag{30}$$

where ρ_{it} denote the profits of entrepreneur i after paying the workers he hires. As will become clear below, entrepreneurs also face two non-trivial choices: How much of their earnings to hold as money, just like workers, and what price to charge for the particular good they produce. Their demand for money is the same as workers, i.e.

$$m_t^e = \frac{\theta \Pi_t}{2i_t}$$

As for what price to set, we first need to specify the production technology for both intermediate and final goods. Suppose that if entrepreneur $i \in [0, 1]$ hires n_{it} units of labor to work at date t, he will produce

$$y_{it} = A_t n_{it}$$

units of good i, where $A_t = A_0 (1 + g)^t$. These goods can be combined to produce final goods according to a Dixit-Stiglitz production function, i.e. y_{it} of each good $i \in [0, 1]$ combine to yield Y_t of final goods, where

$$Y_t = \left(\int_0^1 y_{it}^{1-\sigma} di\right)^{\frac{1}{1-\sigma}}$$

The production of final goods is competitive. Final goods producers will therefore choose intermediate goods y_{it} to produce the Y_t final goods at the lowest possible cost. That is, they solve

$$\max P_t \left(\int_0^1 y_{it}^{1-\sigma} di\right)^{\frac{1}{1-\sigma}} - \int_0^1 P_{it} y_{it} di$$

The first-order condition with respect to y_{it} for final goods producers yields the demand for each intermediate good as follows:

$$y_{it} = Y_t \left(\frac{P_t}{P_{it}}\right)^{\frac{1}{\sigma}} \tag{31}$$

where recall Y_t denotes the output of final goods in equilibrium. Substituting in for y_{it}, we can compute the cost of producing a unit of final goods. Since the market for final goods is competitive, the price of final goods P_t must equal this cost. Equating the two yields the familiar Dixit-Stiglitz price index:

$$P_t = \left(\int_0^1 P_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}$$

Each entrepreneur chooses the price P_{it} of his good to maximize profits ρ_{it} given demand (31) for his good,

$$\left(P_{it} - \frac{W_t}{A_t} \right) Y_t \left(\frac{P_t}{P_{it}} \right)^{\frac{1}{\sigma}}$$

From the first-order condition for this problem, we have

$$P_{it} = \frac{W_t}{(1-\sigma) A_t} \tag{32}$$

Since the price P_{it} will determine demand, the choice of price will fully determine how much labor effort entrepreneur i will hire. The equilibrium nominal wage W_t will equate this demand with the amount workers are willing to supply.

Finally, we turn to the government sector. The government budget constraint still corresponds to (16). By repeated substitution, we get

$$\begin{aligned} (1 + r_{t-1}) b_{t-1} &= \sum_{s=0}^{\infty} \left(\prod_{i=0}^{s-1} \frac{1}{1+r_{t+i}} \right) \left[\tau_{t+s} + \left(\frac{M_{t+s} - M_{t+s-1}}{P_{t+s}} \right) \right] \\ &= \sum_{s=0}^{\infty} \left(\prod_{i=0}^{s-1} \frac{1}{1+r_{t+i}} \right) \left[\tau_{t+s} + \frac{1}{P_t} \frac{M_{t+s} - M_{t+s-1}}{\prod_{i=0}^{s-1} \Pi_{t+i}} \right] \end{aligned} \tag{33}$$

Equation (33) states that the outstanding government liability $(1 + r_{t-1}) b_{t-1}$ at date t must equal the present discounted value of taxes and seniorage revenue the government is set to collect. We define a monetary intervention as a change in the path $\{M_t\}_{t=0}^{\infty}$ holding the path of lump-sum taxes $\{\tau_t\}_{t=0}^{\infty}$ fixed.

We can now examine the effects of monetary policy. Let p_t denote the real price of the asset relative to the final good. An equilibrium is a path for prices $\{p_t, P_t, W_t, r_t\}_{t=0}^{\infty}$ and quantities $\{n_t, c_t^w, c_t^e\}_{t=0}^{\infty}$ such that agents optimize and markets clear. We first consider the case where prices are flexible, i.e. where entrepreneurs set prices P_{it} knowing the full path of $\{M_t\}_{t=0}^{\infty}$ and the implied equilibrium nominal wage W_t associated with this path. We then contrast this with the case where entrepreneurs set their prices before the monetary authority and thus before seeing the realized nominal wage W_t.

Case I: Flexible Prices

We begin with the case where prices are flexible. Since intermediate goods producers all choose the same price according to (32), the price of final goods will be given by

$$P_t = \left(\int_0^1 P_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}} = \frac{W_t}{(1-\sigma) A_t}$$

The real wage will then equal $W_t/P_t = (1-\sigma) A_t$, and so the equilibrium amount of labor will solve

$$A_t v'(n_t) = (1+r_t)(1-\sigma) A_t$$

Since entrepreneurs each set the same price and face the same technology, they will hire the same amount of labor in equilibrium. Since there is a unit mass of entrepreneurs, this implies each entrepreneur will hire n_t units of labor where n_t denotes aggregate employment. The total output of final goods will be given by

$$Y_t = A_t \left(\int_0^1 n_t^{1-\sigma} di \right)^{\frac{1}{1-\sigma}} = A_t n_t$$

We now argue that when prices are flexible, the equilibrium real interest rate r_t, employment n_t, output of final goods $Y_t = A_t n_t$, and total consumption $C_t = c_t^e + c_t^w$ are all independent of monetary policy, i.e. they will not depend on M_{-1} or $\{M_t\}_{t=0}^{\infty}$. This feature follows from our assumption that agents have log preferences over real money balances. These preferences ensure that even though changes in $\{M_t\}_{t=0}^{\infty}$ affect equilibrium real money balances, changes in real balances do not influence consumption and employment decisions.

Formally, the equilibrium real return on the asset must equal the return on government debt $1+r_t$, i.e.

$$(1+r_t) p_t = d + p_{t+1}$$

Since agents will not use storage in equilibrium, we can substitute in for the price of the asset p_t, i.e.

$$(1+r_t)(A_t n_t - \tau_t - b_t - m_t) = d + (A_{t+1} n_{t+1} - \tau_{t+1} - b_{t+1} - m_{t+1})$$

where aggregate real money balances under log utility are given by

$$m_t = m_t^e + m_t^w = \frac{\theta \Pi_t}{i_t}$$

Substituting the government budget constraint (16) into the equilibrium condition above implies

$$\begin{aligned}
(1+r_t)(A_t n_t - \tau_t) &= d + \left(A_{t+1} n_{t+1} + \left(1 + r_t - \Pi_t^{-1} \right) m_t \right) \\
&= d + \left(A_{t+1} n_{t+1} + i_t \frac{m_t}{\Pi_t} \right) \\
&= d + (A_{t+1} n_{t+1} + \theta)
\end{aligned}$$

and so

$$1 + r_t = \frac{d + A_{t+1} n_{t+1} + \theta}{A_t n_t - \tau_t} \tag{34}$$

From the worker's first order condition, we know that they will choose n_t so that

$$v'(n_t) = (1+r_t)(1-\sigma) \tag{35}$$

Conditions (34) and (35) together yields a difference equation in n_t:

$$\frac{d + A_{t+1} n_{t+1} + \theta}{A_t n_t - \tau_t} = \frac{v'(n_t)}{1-\sigma}$$

In the limit as $t \to \infty$, this converges to the condition

$$n_{t+1} = \frac{v'(n_t)\, n_t}{(1+g)(1-\sigma)}$$

This condition has a fixed point at $n_* = 0$ and one other fixed point at $n^* = v'^{-1}[(1+g)(1-\sigma)]$, and the latter is an unstable fixed point. Hence, the limiting condition $\lim_{t\to\infty} n_t = n^*$ provides a boundary condition for the difference equation in n_t. It follows that n_t is independent of $\{M_t\}_{t=0}^{\infty}$, and from (35) is as well.

Next, we turn to the variables that do respond to monetary policy. Consider first the price of goods relative to money, P_t. From the demand for money balances, we have

$$\frac{M_t}{P_t} = \frac{\theta \Pi_t}{i_t} = \frac{\theta \Pi_t}{(1+r_t)\,\Pi_t - 1}$$

Since $\Pi_t = P_{t+1}/P_t$ for all t, we can rearrange this condition into a difference equation where P_{t+1} can be expressed as a function of P_t:

$$P_{t+1} = \frac{1}{(1+r_t)/P_t - \theta/M_t}$$

The boundary condition for this difference equation comes from the government budget constraint (16) evaluated at date 0. That is, given a path for inflation $\{\Pi_t\}_{t=0}^{\infty}$, the initial price P_0 must be such that the amount of seniorage revenue just offsets the difference between the initial obligation $(1+r_{-1})\, b_{-1}$ and the present discounted of lump-sum taxes the government collects. This allows us to solve for the path of prices $\{P_t\}_{t=0}^{\infty}$. We then can easily back out nominal wages and the real price of the asset:

$$\begin{aligned} W_t &= (1-\sigma)\, A_t P_t \\ p_t &= A_t n_t - \tau_t - b_t - \frac{M_t}{P_t} \end{aligned}$$

In short, when prices are flexible, monetary policy has no effect on the real interest rate, employment, or output, but it can affect the real price of the asset p_t. Consider the price p_t at date $t = 0$. From the intertemporal government budget constraint in (16), we have

$$b_0 + \tau_0 + \frac{M_0}{P_0} = (1+r_{-1})\, b_{-1} + \frac{M_{-1}}{P_0}$$

Substituting this into the expression for p_t, the real price of the asset at date 0 is given by

$$p_0 = A_0 n_0 - (1+r_{-1})\, b_{-1} - \frac{M_{-1}}{P_0}$$

Since n_t is independent of monetary policy, while $(1+r_{-1})\, b_{-1}$ and M_{-1} are fixed, the effect of monetary policy on the price of the asset at date 0 works entirely through the initial price level P_0. In particular, a policy that lowers the nominal price of goods, or alternatively that increases the real value of money, will lower the real price of the asset. This is also what we argue in the text.

Note that a monetary policy intervention that drives down the price level P_0 will also depress any bubble in the asset. This is because the real interest rate r_t is independent of monetary policy. Hence, the

fundamental value of the asset, i.e. the present discount value of the dividends it yields evaluated at the market real interest rate, will be unchanged, and so the gap between the price and the fundamentals will be lower.

Reducing the price level arguably represents a contractionary monetary policy, since it makes money more valuable. However, this policy will not result in a higher real interest rate, since recall the real interest rate $1 + r_t$ is independent of $\{M_t\}_{t=0}^{\infty}$ when prices are flexible. It has ambiguous implications for the nominal rate: (33) suggests that the price level generally depends on the entire future path of money, so a lower initial price level P_0 could in principle be associated with either a higher or lower nominal interest rate $1 + i_0$. To generate an example of contractionary monetary policy that both raises the real interest rate and depresses the bubble, we now turn to the case where goods prices are set in advance of monetary policy.

Case II: Sticky Prices

Suppose entrepreneurs must set their prices P_{it} before they get to observe monetary policy, based only on their expectation of the nominal wage W_t. We want to study what would happen if at date t the monetary authority unexpectedly announced a different path for money from what entrepreneurs expected. For ease of exposition, we will refer to any variables after the intervention with a hat whenever these variables might differ from what would have happened without the change in monetary policy. Hence, the new path for monetary policy will be denoted by $\{\widehat{M}_t\}_{t=0}^{\infty}$. The change in path is unanticipated as of date t, but is perfectly anticipated from date t on. Hence, any changes in the path of money beyond date t are anticipated and incorporated into entrepreneurs' pricing decisions.

We want to show that there exists an unanticipated monetary policy intervention starting from date t that both raises the real interest rate and reduces the bubble. In particular, suppose the government shrinks the amount of money by a small amount Δ at date t, i.e.

$$\widehat{M}_t = M_t - \Delta$$

From the government budget constraint (16), we know that

$$\widehat{b}_t = (1 + r_{t-1})\, b_{t-1} - \tau_t - \frac{M_t - \Delta - M_{t-1}}{P_t}$$

Since $(1 + r_{t-1})\, b_{t-1}$, τ_t, M_{t-1}, and P_t are all preset variables, it follows that as a result of this intervention, the government at date t must issue an additional Δ/P_t units of real debt, i.e.

$$\widehat{b}_t = b_t + \frac{\Delta}{P_t} \tag{36}$$

We then assume that the remaining path $\{\widehat{M}_{t+1}, \widehat{M}_{t+2}, ...\}$ will be set to ensure that the real price of the asset at date $t + 1$ is unchanged, i.e. they will be set to ensure

$$\widehat{p}_{t+1} = p_{t+1}$$

To appreciate what this condition implies for the path of money balances, recall that the equilibrium price of the asset is given by

$$p_t = An_t - \tau_t - b_t - m_t$$

Since $\{\widehat{M}_{t+1}, \widehat{M}_{t+2}, ...\}$ is anticipated, our analysis of the case where prices are flexible applies. This implies $\widehat{n}_{t+1} = n_{t+1}$, i.e. employment at date $t+1$ would be the same as if there was no surprise change in money. The lump-sum tax τ_{t+1} is unaffected by monetary policy by assumption. Hence, $\widehat{p}_{t+1} = p_{t+1}$ implies

$$b_{t+1} + m_{t+1} = \widehat{b}_{t+1} + \widehat{m}_{t+1}$$

Substituting in from (36), we can rewrite this as

$$\widehat{m}_{t+1} - m_{t+1} = (r_t - \widehat{r}_t)\, b_t - (1 + \widehat{r}_t)\, \frac{\Delta}{P_t}$$

We will argue below that $\widehat{r}_t > r_t$, so this implies that real money balances at date $t+1$ must be lower than they would have been absent the shock.

Given a path $\{\widehat{M}_t, \widehat{M}_{t+1}, ...\}$ that satisfies these conditions, we can now characterize the effect on certain equilibrium prices at date t. Since the old exchange all of their resources for goods, they will want to consume

$$\frac{M_t - \Delta}{P_t} + (1 + r_{t-1})\, b_{t-1} + (\widehat{p}_t + d)$$

Since the old are the only ones who consume, the amount of final goods Y_t produced at date t must equal the above amount net of the d units of consumption available as a dividend. That is,

$$A_t \widehat{n}_t = \frac{M_{t-1} - \Delta}{P_t} + (1 + r_{t-1})\, b_{t-1} + \widehat{p}_t \tag{37}$$

Note that most of the variables on the right-hand side of (37) are pre-determined, including the price level P_t. The only endogenous variable on the right-hand side is the price of the asset $\widehat{p}_t$. But we can rewrite this as

$$\begin{aligned} A_t \widehat{n}_t &= \frac{M_{t-1} - \Delta}{P_t} + (1 + r_{t-1})\, b_{t-1} + \frac{d + \widehat{p}_{t+1}}{1 + \widehat{r}_t} \\ &= \frac{M_{t-1} - \Delta}{P_t} + (1 + r_{t-1})\, b_{t-1} + (d + \widehat{p}_{t+1}) \left(\frac{A_t n_t - \tau_t}{A_{t+1} n_{t+1} + d + \theta} \right) \end{aligned}$$

where the second equation uses the fact that $1 + \widehat{r}_t$ must satisfy (34) whether prices are flexible or not. Since young agents at date $t+1$ must be able to afford the asset, we know $\widehat{p}_{t+1} \leq A_{t+1} n_{t+1}$. Hence,

$$\frac{d + \widehat{p}_{t+1}}{A_{t+1} n_{t+1} + d + \theta} < 1$$

Since we considered a path that ensures $\widehat{p}_{t+1} = p_{t+1}$, then $\widehat{p}_{t+1}$ is by construction independent of Δ. This implies we have

$$\left[1 - \frac{d + \widehat{p}_{t+1}}{A_{t+1} n_{t+1} + d + \theta} \right] A_t \widehat{n}_t = \frac{M_t - \Delta}{P_t} + (1 + r_{t-1})\, b_{t-1} - \frac{\tau_t\, (d + \widehat{p}_{t+1})}{A_{t+1} n_{t+1} + d + \theta} \tag{38}$$

where all terms above are independent of Δ. From this, it follows that $\widehat{n}_t$ is decreasing in Δ, since

$$\frac{d\widehat{n}_t}{d\Delta} = -\left[1 - \frac{d + \widehat{p}_{t+1}}{A_{t+1}n_{t+1} + d + \theta}\right]^{-1} \frac{P_t}{A_t} < 0$$

This shows that an unexpected reduction in money, followed by an adjustment in money balances that sterilizes the affect on asset prices at date $t+1$, would reduce employment and thus output. Intuitively, the shock leaves old agents with fewer resources to consume, and so less will be produced to meet their demand.

From (34), we can further deduce that the real interest rate $1 + \widehat{r}_t$ at date t will be higher than $1 + r_t$. This is because all date $t+1$ variables will be unchanged, while $\widehat{n}_t < n_t$. As for the price of the asset, (38) reveals that output $A_t\widehat{n}_t$ falls more than one for one with Δ/P_t, implying the real price of the asset $\widehat{p}_t < p_t$. Appealing to a similar argument as in the case of fiscal policy, we can use the fact that any bubble would not change asymptotically as a result of this policy, and yet the interest rate $1 + \widehat{r}_t$ is higher than $1 + r_t$, to argue that the bubble at date t must fall. Hence, an unexpected monetary contraction would not only temporarily raise the real interest rate, but it would temporarily depress the bubble. This is precisely what we wanted to demonstrate.

Finally, we note that from the equation for labor supply, we have

$$v'\left(\widehat{n}_t\right) = \left(1 + \widehat{r}_t\right) \frac{\widehat{W}_t}{\widehat{P}_t}$$

Since $\widehat{n}_t < n_t$ while $1 + \widehat{r}_t > 1 + r_t$, we can conclude that the real wage $\widehat{W}_t/\widehat{P}_t < W_t/P_t$. That is, since employment must fall when goods prices are sticky, the real wage must fall to induce workers to put in less effort. Since $\widehat{P}_t = P_t$ when prices are set in advance, it follows that the nominal wage $\widehat{W}_t < W_t$, i.e. the monetary policy we consider will cause the nominal wage to fall. But this means that if some producers could respond to monetary policy, they would want to lower their prices. This suggests that if some producers had flexible prices while others were not, then the policy we consider would lead the price level to fall. Thus, when some prices are flexible, we would expect both a fall in output, meaning young agents have fewer resources with which to buy the asset, and an increase in the amount of debt issued to rise as the value of previous nominal government obligations increased. Thus, in a version where some but not all prices were flexible, contractionary monetary policy would depress the bubble both because there are fewer resources agents can use to buy the asset and because there is more public debt that agents must hold instead of the asset.

Appendix C: Credit-Driven Bubbles and Variable Supply of Assets

In this Appendix, we provide an example of credit-driven bubbles in which an intervention that raises the interest rate and mitigates a bubble can affect the amount of aggregate risk in the economy. This example requires two departures from the model we analyzed in Section 4. First, we need to modify the model so that additional units of the asset can be created. This allows an intervention that mitigates the bubble to affect the quantity of bubble assets created. Second, for a change the number of assets created to influence aggregate risk, we need the source of risk in the economy to come from the asset rather than the aggregate endowment. If the source of risk in the economy concerns the aggregate endowment as in the model we analyze in Section 4, the amount of assets created would have no effect on the risk the economy is exposed to.

Our formulation follows the setup in Section 4 in which young agents who want to save can buy government bonds, buy the asset, or lend to a combination of entrepreneurs or speculators. We consider two modifications. First, we assume $g = 0$, i.e. $e_t = e_0$ for all t and there is no uncertainty about next period's endowment. Second, rather than assume that the initial old are endowed with a fixed stock of assets that yields a fixed dividend d, we assume that the initial old can create assets by converting output into assets, and that the dividend on these assets is stochastic. The production technology for assets features increasing marginal cost. In particular, we assume that creating the q-th unit of the asset requires $c(q)$ units of output. For simplicity, suppose $c(q) = q^a$ for some constant $a > 0$. As before, we use p_0 to denote the real price of the asset at date 0. Since producers earn p_0 on each unit they produce, they will create assets up to the point where the cost of the last asset is equal to the price at which they can sell the asset, i.e.

$$p_0 = c(q) = q^a$$

We continue to assume the asset yields a constant dividend. However, the value of this dividend is only revealed at date $t = 1$, when those who purchased the asset at date 0 are old. Thus, only the initial cohort who buys the asset is uncertain about its payoff. To keep things simple, suppose the dividend d has a binomial distribution and will equal $\underline{d} > 0$ with probability $1 - \pi$ and $\overline{d} > \underline{d}$ with probability π.

Since $\underline{d} > 0$, agents who are young at date 0 will not rely on storage. Since we assume the return y on production is large, all entrepreneurs will wish to borrow. They will therefore receive an amount κ of the resources available at date 0. This implies

$$e_0 - b_0 - \tau_0 - \kappa = p_0 q$$

That is, total spending on the asset at date 0 is equal to the endowment of agents at date date net of the amount they use to pay taxes, buy bonds, and finance entrepreneurs. Substituting in $q^a = p_0$ yields an expression for the initial price of the asset as well as the quantity produced:

$$\begin{aligned} p_0 &= (e_0 - b_0 - \tau_0 - \kappa)^{\frac{a}{a+1}} \\ q &= (e_0 - b_0 - \tau_0 - \kappa)^{\frac{1}{a+1}} \end{aligned}$$

The higher is a, the more an increase in $p_0 q$ will be reflected in a higher price as opposed to a higher quantity. When $a = 1$, the two effects are equal.

To show that the model still gives rise to a bubble, let us begin at date 1 when the value of the dividend d is revealed. The total amount spent on the asset at this date will equal

$$p_1 q = e_0 - b_1 - \tau_1 - \kappa$$

This amount will be the same whether $d = \overline{d}$ or $d = \underline{d}$. However, the realization of dividends will instead affect the equilibrium interest rate. In particular, for $t \geq 1$, we have

$$1 + r_t = \frac{d + p_{t+1}}{p_t} \tag{39}$$

where $d \in \left\{\underline{d}, \overline{d}\right\}$. Thus, if dividends are low, the interest rate will be low as well. Indeed, the interest rate will fall by exactly the amount needed to push up the present discounted value of dividends to be the same as when dividends are high. In other words, if we denote $1 + \underline{r}_t$ as the interest rate at date t if $d = \underline{d}$ and $1 + \overline{r}_t$ if $d = \overline{d}$, then

$$\sum_{t=1}^{\infty} \left(\prod_{s=1}^{t} \frac{1}{1 + \underline{r}_s} \right) \underline{d} = \sum_{t=1}^{\infty} \left(\prod_{s=1}^{t} \frac{1}{1 + \overline{r}_s} \right) \overline{d} = p_1 = \frac{e_0 - b_1 - \tau_1 - \kappa}{q}$$

To prove this result formally, we use (39) to obtain

$$\begin{aligned} f_1 &= \sum_{t=1}^{\infty} \left(\prod_{s=1}^{t} \frac{1}{1 + r_s} \right) d \\ &= d \left[\frac{p_1}{p_2 + d} + \frac{p_2}{p_3 + d} \frac{p_1}{p_2 + d} + \frac{p_3}{p_4 + d} \frac{p_2}{p_3 + d} \frac{p_1}{p_2 + d} + \cdots \right] \\ &= d \times \frac{p_1}{d} \times \left[\frac{d}{p_2 + d} + \frac{p_2}{p_3 + d} \frac{d}{p_2 + d} + \frac{p_3}{p_4 + d} \frac{p_2}{p_3 + d} \frac{d}{p_2 + d} + \cdots \right] \\ &= d \times \frac{p_1}{d} \times \left[\frac{d}{p_2 + d} + \frac{d}{p_3 + d} \frac{p_2}{p_2 + d} + \frac{d}{p_4 + d} \frac{p_3}{p_3 + d} \frac{p_2}{p_2 + d} + \cdots \right] \end{aligned}$$

We now argue that the last expression in brackets is equal to 1. Consider a random variable that describes the date of an arrival, where the arrival can occur at any date $t \geq 2$, and where the probability of arrival at date t conditional on no arrival up to date t is equal to $\dfrac{d}{p_t + d} \in (0, 1)$. Then the expression in brackets is the probability of an arrival at any date, which is 1. Hence, $f_1 = p_1$, regardless of the realization of d.

At date 0, the fundamental value of the asset is equal to

$$f_0 = \frac{E\left[d + f_1\right]}{1 + r_0} = \frac{E\left[d\right] + p_1}{1 + r_0}$$

We now want to show that the price of the asset at date 0 exceeds f_0. To see this, note that because there is free entry into speculation, the interest rate on loans R_0 must ensure speculators earn zero profits from borrowing if the high state is realized, i.e.

$$1 + R_0 = \frac{\overline{d} + p_1}{p_0}$$

Since creditors could have purchased riskless government bonds and earned $1 + r_0$ on the resources they lend out, and since lenders are risk-neutral, the expected profits from lending out resources must yield the same return as the risk-free rate, i.e.

$$(1 + r_0)(\kappa + p_0 q) = (1 + R_0)(\kappa + (1 - \pi) p_0 q) + \pi (\underline{d} + p_1) q \tag{40}$$

By adding and subtracting $\pi \left(\overline{d} + p_1\right) q$ to the RHS implies

$$\begin{aligned}(1 + r_0)(\kappa + p_0 q) &= (1 + R_0)(\kappa + (1 - \pi) p_0 q) + \pi \left(\overline{d} + p_1\right) q - \pi \left(\overline{d} + p_1\right) q + \pi (\underline{d} + p_1) q \\ &= (1 + R_0)(\kappa + p_0 q) - \pi \left(\overline{d} - \underline{d}\right) q\end{aligned}$$

where the second step uses the fact that $1 + R_0 = \left(\overline{d} + p_0\right)/p_0$. Rearranging, we can derive an expression for the risk free interest rate on government debt r_0 in terms of the loan rate R_0:

$$r_0 = R_0 - \frac{\pi \left(\overline{d} - \underline{d}\right) q}{\kappa + p_0 q}$$

This implies $R_0 > r_0$. Next, observe that from (40), we have

$$\begin{aligned}(1 + r_0)(\kappa + p_0 q) &= \frac{\overline{d} + p_1}{p_0}(\kappa + (1 - \pi) p_0 q) + \pi (\underline{d} + p_1) q \\ &= \frac{\overline{d} + p_1}{p_0}\kappa + (1 - \pi)\left(\overline{d} + p_1\right) q + \pi (\underline{d} + p_1) q \\ &= (1 + R_0)\kappa + [E[d] + p_1] q\end{aligned}$$

Dividing both sides by $1 + r_0$ and rearranging the above equation yields

$$(p_0 - f_0) q = \frac{R_0 - r_0}{1 + r_0}\kappa$$

Since we argued above that $R_0 > r_0$, this implies that if $\kappa > 0$, there will be a bubble in the asset market.

Finally, we want to study the effects of intervention. As before, we parameterize policy so that $b_t = b$ for $t = -1, 0, 1, ...$ From the government budget constraint, we have

$$b_0 + \tau_0 = (1 + r_{-1}) b_{-1} = (1 + r_{-1}) b$$

Substituting this into the expression for the price of the asset at date 0 yields an expression in terms of b and other primitives:

$$p_0 = (e_0 - (1 + r_{-1}) b - \kappa)^{\frac{a}{a+1}} \tag{41}$$

Next, we turn to the real interest rate. Using the expression for the spread between R_0 and r_0 above, as well as the expression for R_0 above, we can solve for r_0 as the value which solves

$$\begin{aligned}1 + r_0 &= \frac{\overline{d} q + p_1 q}{p_0 q} - \frac{\pi \left(\overline{d} - \underline{d}\right) q}{\kappa + p_0 q} \\ &= \frac{d_1 q + (e_0 - (1 + r_0) b - \kappa)}{e_0 - (1 + r_{-1}) b - \kappa} - \frac{\pi (d_1 - d_0) q}{e_0 - b - r_{-1} b} \\ &= \frac{d_1 (e_0 - (1 + r_{-1}) b - \kappa)^{\frac{1}{a+1}} + (e_0 - (1 + r_0) b - \kappa)}{e_0 - (1 + r_{-1}) b - \kappa} \\ &\quad - \frac{\pi (d_1 - d_0)(e_0 - (1 + r_{-1}) b - \kappa)^{\frac{1}{a+1}}}{e_0 - b - r_{-1} b}\end{aligned}$$

The analytical expression for r_0 in terms of primitives is messy, and differentiating it with respect to b does not offer much insight. However, we can easily produce numerical examples in which an increase in b will raise the risk-free interest rate r_0, reduce the initial price of the asset p_0, lower the bubble $\Delta_0 = p_0 - f_0$, and reduce the quantity of the asset supplied q. For example, suppose we set the initial endowment $e_0 = 1$. For the asset, we assume $\overline{d} = 0.1$ and $\underline{d} = 0.05$, each equally likely so $\pi = 0.5$. We set $\kappa = 0.99$, so most lending finances entrepreneurial activity rather than speculation. This is consistent with our finding in Proposition 8 that a high κ is more compatible with bond issuance reducing the size of the bubble. We set the initial interest rate at $r_{-1} = 0.15$. Finally, we set $a = 9$ so the effect of bonds on total spending on the asset primarily affects the price of the asset p_0 rather than the quantity of the asset created q. In this case, we can compute numerically that for all values of b that ensure the asset price p_0 and p_1 will be positive, a higher b will in fact raise the real interest rate, reduce the price and quantity of the asset produced at date 0, and reduce the bubble component at date 0.