Indeterminacy and Imperfect Information*

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Abstract

We study equilibrium determination in an environment where multiple agents have different information sets: The fully informed agents know the structure of the model and observe histories of all exogenous and endogenous variables. The less informed agents observe only a strict subset of the full information set. All types of agents form expectations rationally, but agents with limited information need to solve a dynamic signal extraction problem to gather information about the variables they do not observe. We show that for parameters values that imply a unique equilibrium under full information, the limited information rational expectations equilibrium can generally be indeterminate. In a simple application of our framework to a monetary policy problem we show that limited information on part of the central bank implies indeterminate outcomes.

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1 Introduction

We study existence and uniqueness of rational expectations equilibria in environments where agents have nested information sets. Specifically, we consider an environment that contains a fully informed agent and an agent with a limited information set that is nested within the full information set. Throughout, we maintain the assumption that all agents form expectations rationally and focus on linear equilibria in models with Gaussian innovations, which implies that the agent with the limited information set employs the Kalman filter to optimally conduct inference about variables that are outside his information set. In order to provide some structure on our discussion we distinguish between cases where the signal merely reflects a combination of exogenous variables, or where the signal is endogenous in the sense that it also reflects the realization of endogenous variables.

The key finding in our paper is that equilibrium indeterminacy is substantially more prevalent in this environment. Even when the equilibrium is unique in an equivalent model where all agents have full information, the presence of a less informed agent introduces a feedback mechanism via the Kalman filter that potentially introduces additional stable dynamics into an otherwise saddle-path system. More specifically, the mechanics of the updating equations of the Kalman filter can imply stable eigenvalues that are not matched by a corresponding change in the number of expectational variables within a system of equations. Consequently, the full system of equations is characterized by indeterminacy of the rational expectations equilibrium. This outcome is generic in an environment where the limited information set extends only to exogenous variables such as the shock processes of an economic model. In this case, Kalman-updating simply adds indeterminacy-inducing dynamics.

Turning to the case where the information contains endogenous variables observed with error, we first construct one specific equilibrium that always exists. In this equilibrium, agents coordinate on a rational expectations forecast error that rules out sunspot shocks and imposes a unique parametric restriction on the effect of measurement errors and the exogenous shocks of the economic model. Under this parametrization, the prediction error from Kalman updating is zero and information on a subset of unobservable variables is fully revealed. This equilibrium does not, however, replicate the corresponding full-information equilibrium since the latter does no include measurement errors in its set of fundamental disturbances, whereas this "benchmark " equilibrium does.

Our paper touches upon a wide swath of literature that is concerned with the properties

of rational expectations equilibria and deviations therefrom. The most immediate precursor to our work is Svensson and Woodford (2004) and a related paper by Aoki (2006). They describe the optimal monetary policy problem in an information environment that is similar to ours: a central bank observes only a subset of all available information, whereas the private sector is fully informed. For a given set of first-order conditions to the optimal policy problem under imperfect information, the approach of Svensson and Woodford falls into the class of expectational linear-difference equations studied here. In contrast to our paper, Svensson and Woodford only consider an equilibrium that corresponds to a minimum-state variable solution, while explicitly assuming away issues of equilibrium indeterminacy. Svensson and Woodford (2004) are not alone in pursing a MSV approach in such models, other examples are given by Aoki (2008), or Nimark (2008). Specifically, they do not address the full set of indeterminate equilibria that can arise in this framework. We show below how our approach compares to Svensson and Woodford (2004) and how the latter maps into our set of solutions.

The paper also draws inspiration from the literature on solving linear rational expectations models, e.g., Sims (2002) whose formalism we employ. In addition, we use the solution technique for indeterminate rational expectations models developed by Lubik and Schorfheide (2003) which focuses on the properties of the rational expectations forecast error. In order to characterize we equilibrium decomposition of this error into fundamental and non-fundamental components we find it convenient to make use of the recent reinterpretation of this earlier paper by Farmer et al. (2015). We show how in an environment with limited information the loadings on the various stochastic components of the forecast error are related to the endogenous Kalman gain and which further restrictions are therefore imposed on equilibrium selection.

Finally, our paper falls into the general category of recent work that studies the implications of imperfect information on rational expectations equilibria. Recent examples include Nimark (2011), Mertens (2016), Rondina and Walker (2016), and Lubik and Matthes (2016).

Our paper is structured as follows. In the next section, we provide a simple example that details our solution approach and that highlights analytically our findings. We also show how our approach connects with and refines some earlier contributions to the literature. Section 3 discusses the general solution approach, while we apply our framework to a typical model used in the analysis of monetary policy in Section 4. Section 5 concludes with discussing our findings in the light of empirical applicability.

2 A Simple Analytical Example

We begin by developing a simple example for which analytical solutions are available to characterize the issues discussed in the introduction. Our example relies on a bare-bones model of inflation determination and monetary policy that is rich enough to develop intuition for the general case. In the first step, we describe equilibrium determination under full information rational expectations which we use as a benchmark to evaluate limited information equilibria against. Within this simple model, we then introduce our framework where agents have different information sets, specifically, where one agent is fully informed and the other agents only observes a subset of the full information set. Finally, we discuss how our framework relates to the set-up and the findings in Svensson and Woodford (2004).

2.1 Determinacy and Indeterminacy under Rational Expectations

We consider a simple model of inflation determination in the vein of the cashless economy without nominal frictions as in Woodford (2003). The model economy consists of a Fisher equation which links the nominal interest rate i_t to the real rate r_t via expected inflation $E_t\pi_{t+1}$ and a monetary policy rule that has the nominal rate respond to current inflation π_t, that is, a Taylor-rule.[1] Furthermore, the real rate of interest is exogenous and serially correlated of order one with innovation ε_t, whereby $\varepsilon_t \sim iidN(0, \sigma_\varepsilon^2)$.

The equation system is thus given by:

$$i_t = r_t + E_t\pi_{t+1}, \tag{1}$$

$$i_t = \phi\pi_t, \tag{2}$$

$$r_t = \rho r_{t-1} + \varepsilon_t. \tag{3}$$

The first equation is the Fisher equation, the second is the policy rule, while the third equation describes the evolution of the real rate as an AR(1) process. All variables can be understood as deviations from their respective steady state values. ϕ is a monetary policy parameter. The autoregressive parameter $|\rho| < 1$ to guarantee stationarity. E_t is the rational expectations operator under full information. It denotes the mathematical expectation conditional on the full information set S^t of all shocks through time t, so that for some variable x_t, $E_t x_{t+h} = E\left(x_{t+h}|S^t\right)$ and $E_t x_t = x_t$.

[1] We also considered the case of a Wicksellian policy rule: $i_t = r_t + \phi\pi_t$, with a time-varying intercept given by the real rate of interest. The steps towards deriving a solution are very much identical to the ones described in the main text. The derivations are available from the authors upon request.

We can also distinguish two agents in this economy: First, a representative private-sector agent whose behavior is characterized by the Fisher equation (1), and second, a central bank whose behavior is given by the monetary policy rule (2). Under full information rational expectations (FIRE), both agents are assumed to know S^t. This means that they observe all variables in the model without error, that they know the history of all shocks, that they understand the structure of the economy and the solution concepts. We will modify this assumption minimally in the next section.

In order to find a rational expectations equilibrium (REE), we can substitute the policy rule into the Fisher equation to derive a relationship in inflation with driving process r_t:

$$E_t\pi_{t+1} = \phi\pi_t + r_t. \tag{4}$$

The dynamic behavior of inflation depends on the value of the policy coefficient ϕ. Applying covariance stationarity as an equilibrium selection criterion, then it is well known that a unique equilibrium exists if and only $|\phi| > 1$.[2] It is straightforward to establish that the REE solution is $\pi_t = \frac{1}{\phi-\rho}r_t$ and $i_t = \frac{\phi}{\phi-\rho}r_t$. This is the case of equilibrium determinacy.

The behavior of inflation changes, however, when we look at the case $|\phi| < 1$. In this scenario, the equilibrium can be indeterminate in the sense that there are possibly infinitely many solutions that are consistent with equation (4). In order to analyze this case we find it convenient to follow Lubik and Schorfheide (2003) and introduce the rational expectations forecast error $\eta_t = \pi_t - E_{t-1}\pi_t$, whereby $E_{t-1}\eta_t = 0$ by construction. We can use this definition to substitute out the expectations term in (4). This results in the expression:

$$\pi_t = \phi\pi_{t-1} + r_{t-1} + \eta_t. \tag{5}$$

It is easily verifiable that this is a solution to the expectational difference equation (4). In this equilibrium, inflation is stationary process with autoregressive parameter $|\phi| < 1$ and driving process r_{t-1}.

What makes this equilibrium indeterminate is the fact that the solution imposes no restriction on the evolution of η_t other than it is a martingale difference sequence with $E_{t-1}\eta_t = 0$. Consequently, there can be infinitely many solutions. We can, however, provide some structure, at least in terms of economic intuition, if we are willing to link the process η_t to the shock processes of the economy. To this effect, we find it convenient to follow

[2]There has been some recent interest in equilibria that use different selection criteria, accompanied by the development of analytical tools to deal with these cases. A precursor of this literature is the regime-switching model of Davig and Leeper (2007) and the rejoinder by Farmer, Waggoner, and Zha (2009). More recently, Cochrane (2016) looks at equilibria along explosive paths.

Lubik and Schorfheide (2003) and rewrite the inflation equation in terms of expectations only. We define $\xi_t = E_t\pi_{t+1}$ and rewrite equation (4) as $\xi_t = \phi\xi_{t-1} + r_t + \phi\eta_t$.

In this specification, the forecast error η_t emerges as an innovation to the conditional expectation ξ_t. Taking an additional step, we can further decompose, without loss of generality, η_t into a fundamental component, namely the policy innovation ε_t and a non-fundamental component, the belief shock b_t; more specifically, $\eta_t = \gamma_\varepsilon\varepsilon_t + \gamma_b b_t$.[3] The coefficients γ_ε and γ_b can then be used to index specific equilibria with the set of indeterminate equilibria. In the case of FIRE the choice of these parameters is arbitrary. We will show next that when we introduce limited information in a rational expectations setting (LIRE) it is possible to restrict the set of multiple equilibria further.

2.2 Equilibrium Determinacy under Limited Information

We now introduce the idea that the central bank possesses only limited information. The structure of the model, specifically the form of the equations, remains the same, but we alter the conditioning of the expectations operator with respect to different information sets. We assume that the limited information set of the central bank, denoted Z_t is nested in S^t, the information set of the private sector. That is, the central bank knows less than the private sector, but it still forms expectations rationally.[4] The central bank's projections of any variable x_t are denoted $x_{t|t} = E\left(x_t|Z^t\right)$ and $x_{t+h|t} = E\left(x_{t+h}|Z^t\right)$. Since Z^t is spanned by S^t we can apply the law of iterated expectations to obtain: $E\left(Ex_{t+h}|Z^t\right) = x_{t+h|t}$. Therefore, the central bank can set the nominal interest rate only as a function of Z^t.

We can capture this idea by conditioning down the policy rule (2) where any input variable is replaced by its projection onto the central bank's information set:

$$i_t = \phi\pi_{t|t} \tag{6}$$

We can now apply the same conditioning to the other two equations in the system to derive a solution for the central bank's projection for the model's variables. Going through the same steps as above, we can condition down the variables in the system (1) - (3) and derive the projection for inflation. We continue to assume that $|\phi| > 1$, in which case we have $\pi_{t|t} = \frac{1}{\phi-\rho}r_{t|t}$ and $i_t = \frac{\phi}{\phi-\rho}r_{t|t}$. This solution in terms of the central bank's projections

[3] To be more specific, this is without loss of generality within the set of equilibria that are time-invariant and linear. There are other, non-linear equilibria that can be constructed for this linear model.

[4] This is a key difference to the framework in Lubik and Matthes (2016) who assume in addition that the central bank engages in least-squares learning to gain information about private-sector outcomes. In our setup, the deviation from the standard rational expectations benchmark is only minor in the sense that the central bank does not observe everything that the private sector does, but is otherwise fully informed.

for the model variables hold under any definition of the central bank's information set as long as Z^t is spanned by S^t. Under the assumption that policy follows the Taylor principle, $|\phi| > 1$, the central bank projects the equilibrium to be determinate, even if it has access to a limited information set only. However, this is not the equilibrium in the full system, which includes private sector expectations that are conditional on the full information set. We now proceed to solve for this equilibrium.

We combine the private sector Fisher equation (1) with the projected policy rule (6):

$$\phi \pi_{t|t} = r_t + E_t \pi_{t+1}. \tag{7}$$

$\pi_{t|t}$ is an endogenous variable, the central bank's projection of inflation, that has to be linked to private sector variables to derive a solution, while $E_t \pi_{t+1}$ is the private sector's expectation. Using the formalism described in the previous section, we can rewrite this expression as:

$$\pi_t = \phi \pi_{t-1|t-1} - r_{t-1} + \eta_t = \frac{\phi}{\phi - \rho} r_{t-1|t-1} - r_{t-1} + \eta_t,$$

where $r_{t-1|t-1}$ is an endogenous variable on account of the central bank's projection problem. Finding an REE therefore requires linking projections such as $r_{t-1|t-1}$ to the state variables of the model. Moreover, we need to solve out the endogenous forecast error η_t, the solution to which can be unique or indeterminate. The next step therefore requires to specify the information set of the central bank. For purposes of exposition we distinguish between exogenous and endogenous information sets: While exogenous information sets solely reflect the realisation of exogenous variables, endogenous information sets also include (possibly noisy) signals of endogenous variables.[5]

2.2.1 Equilibria with an Exogenous Information Set

Suppose that the central bank observes the real interest rate with measurement error ν_t, where $\nu_t \sim iidN(0, \sigma_\nu^2)$. Therefore, the central bank's information set is such that $Z_t = r_t + \nu_t$. It is exogenous in that the real rate is an exogenous process which does not depend on other endogenous variables. The central bank uses the Kalman filter, the optimal linear filter in this environment, to make projections. Specifically, the prediction equation is:

$$r_{t|t} = r_{t|t-1} + \kappa_r \left(r_t - r_{t|t-1} + \nu_t \right). \tag{8}$$

[5] As will be seen shortly, in either case information sets are defined as linear combinations of variables. For the purpose of our paper, the term "endogenous information" refers to cases where these linear combinations also comprise endogenous variables, which is not to be confused with other settings where the make up of the linear combination itself might be the outcome of an endogenous choice.

The central bank updates its current projection $r_{t|t}$ relative to its previous one-step ahead projection $r_{t|t-1}$ using the current information $r_t+\nu_t$. The crucial parameter in this updating equation is the (optimal) Kalman gain κ_r which we have to compute separately.

In the specific case of an exogenous information set, the Kalman filtering problem can, however, be solved independently from the rest of model — since both state and measurement equation comprise only exogenous variables — which greatly facilitates the analysis. Moreover, equilibrium indeterminacy in the case of an exogenous information set, should not be a surprising result in this simple economy: Under imperfect information, the nominal interest becomes a function of the history of the measurement variable, which is exogenous, and exogenous interest rate policies are well known to cause indeterminate outcomes in this class of models. Nevertheless, the case of an exogenous information set is also highly useful to set the stage for the analysis of model variants where the measurement variables reflect endogenous variables.

We first note that $r_{t|t-1} = \rho r_{t-1|t-1}$ so that we can derive the full equation system which describes the joint evolution of actual inflation π_t, the exogenous real rate r_t, and the central bank projection of the real rate $r_{t|t}$. The system is given by:

$$\begin{aligned} \pi_t &= \frac{\phi}{\phi-\rho} r_{t-1|t-1} - r_{t-1} + \eta_t, \\ r_{t|t} &= (1-\kappa_r)\,\rho r_{t-1|t-1} + \kappa_r \rho r_{t-1} + \kappa_r \varepsilon_t + \kappa_r \nu_t, \\ r_t &= \rho r_{t-1} + \varepsilon_t. \end{aligned} \tag{9}$$

This is a well-specified equation system in three unknowns and can be solved using standard methods for linear rational expectations models that allow for indeterminacy (Lubik and Schorfheide, 2003). Specifically, a solution would pin down the endogenous forecast error η_t, which would then determine whether the REE under limited information is determinate or indeterminate. We also note that the equation system is recursive and that the overall dynamic properties depend on the value of the $(1-\kappa_r)\,\rho$. In order to determine the size of this 'root' we need to compute the gain κ_r first.

We find it convenient to define innovations of any variable x_t as its unexpected component relative to the limited information set Z^t: $\widetilde{x}_t = x_t - x_{t|t-1}$. The Kalman gain is given by:

$$\kappa_r = \frac{cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)}{var(\widetilde{Z}_t)}, \tag{10}$$

where the tildes denote the projection innovations, e.g. $\widetilde{r}_t = r_t - r_{t|t-1}$ and $\widetilde{Z}_t = \widetilde{r}_t + \nu_t$. Given the definition of $\Sigma = var\left(\widetilde{r}_t - \widetilde{r}_{t|t}\right) = var\left(\widetilde{r}_t\right) - var\left(\widetilde{r}_{t|t}\right)$ (whereby $cov(\widetilde{r}_t, \widetilde{r}_{t|t}) =$

$var(\widetilde{r}_{t|t})$), we can find that $var\left(\widetilde{r}_t\right) = \rho^2\Sigma + \sigma_\varepsilon^2$ and that $var(\widetilde{Z}_t) = var\left(\widetilde{r}_t\right) + \sigma_\nu^2$. Similarly, $cov(\widetilde{r}_t, \widetilde{Z}_t) = var\left(\widetilde{r}_t\right)$ and we find the following expression for the Kalman gain:

$$\kappa_r = \frac{\rho^2\Sigma + \sigma_\varepsilon^2}{\rho^2\Sigma + \sigma_\varepsilon^2 + \sigma_\nu^2}, \tag{11}$$

which for positive Σ lies within the unit circle.

The variance Σ can be computed from the definition above, whereby $var\left(\widetilde{r}_{t|t}\right) = \kappa_r cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$ and where we have utilized the expression for the prediction equation $\widetilde{r}_{t|t} = \kappa_r \widetilde{Z}_t$. Substituting these expressions into the definition of Σ results in a non-linear Riccati equation:

$$\Sigma = \frac{\rho^2\Sigma + \sigma_\varepsilon^2}{\rho^2\Sigma + \sigma_\varepsilon^2 + \sigma_\nu^2}\sigma_\nu^2. \tag{12}$$

The solution to this (quadratic) equation is given by:

$$\Sigma = \frac{1}{2\rho^2}\left[-\left(\sigma_\varepsilon^2 + \left(1-\rho^2\right)\sigma_\nu^2\right) + \sqrt{\left(\sigma_\varepsilon^2 + \left(1-\rho^2\right)\sigma_\nu^2\right)^2 + 4\sigma_\varepsilon^2\sigma_\nu^2\rho^2}\right]. \tag{13}$$

Since $0 < \kappa_r < 1$, it follows that $0 < (1-\kappa_r)\,\rho < 1$ and the law of motion for $r_{t|t}$ in the full equation system is a stable difference equation that delivers a stationary solution. Since the equation for actual inflation does not depend on its own lags, we can conclude that the equilibrium is indeterminate since there is no restriction from within the equation system, such as an unstable root to pin down η_t.

Following Farmer et al. (2015) we can decompose the forecast error $\eta_t = \gamma_\varepsilon\varepsilon_t + \gamma_b b_t + \gamma_\nu\nu_t$, where the weights on the innovations are generally undetermined. Specific values index specific equilibria among the set of indeterminate equilibria as discussed in Lubik and Schorfheide (2003). The use of the Kalman filter in this LIRE setting, however, imposes an additional restriction on the equilibrium which allows us to narrow the set of multiple equilibria. Existence of a stationary Kalman filter requires consistency with the equilibrium condition $\pi_{t|t} = \frac{1}{\phi-\rho}r_{t|t}$, which implies $cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right) = \frac{1}{\phi-\rho}cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$. We show that this subspace-condition restricts the set of multiple equilibria, which differentiates the class of LIRE models from their FIRE counterparts.

We know that $cov\left(\widetilde{r}_t, \widetilde{Z}_t\right) = \rho^2\Sigma + \sigma_\varepsilon^2$. $cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right) = cov\left(\widetilde{\pi}_t, \widetilde{r}_t\right) + cov\left(\widetilde{\pi}_t, \nu_t\right)$. Using the innovation representation of the prediction equation for π_t we have

$$\widetilde{\pi}_t = -\left(\widetilde{r}_{t-1} - \widetilde{r}_{t-1|t-1}\right) + \eta_t. \tag{14}$$

After some substitution we find that $cov\left(\widetilde{\pi}_t, \widetilde{r}_t\right) = -\rho\Sigma + \gamma_\varepsilon\sigma_\varepsilon^2$. Similarly, we can find $cov\left(\widetilde{\pi}_t, \nu_t\right) = \gamma_\nu\sigma_\nu^2$. Combining all expressions then results in the following linear restriction

on the weights in the forecast error: $\eta_t = \gamma_\varepsilon \varepsilon_t + \gamma_b b_t + \gamma_\nu \nu_t$:

$$\gamma_\nu = \frac{\phi}{\phi - \rho} \frac{\Sigma}{\sigma_\nu^2} + \frac{1}{\phi - \rho} \frac{\sigma_\varepsilon^2}{\sigma_\nu^2} - \frac{\sigma_\varepsilon^2}{\sigma_\nu^2} \gamma_\varepsilon. \tag{15}$$

The solution to the simple model under LIRE with an exogenous information set can therefore be written as:

$$\begin{aligned} \pi_t &= \frac{\phi}{\phi - \rho} r_{t-1|t-1} - r_{t-1} + \gamma_\varepsilon \varepsilon_t + \gamma_b b_t + \gamma_\nu \nu_t, \\ r_{t|t} &= (1 - \kappa_r) \rho r_{t-1|t-1} + \kappa_r \rho r_{t-1} + \kappa_r \varepsilon_t + \kappa_r \nu_t, \\ r_t &= \rho r_{t-1} + \varepsilon_t. \end{aligned} \tag{16}$$

where:

$$\begin{aligned} \kappa_r &= \frac{\rho^2 \Sigma + \sigma_\varepsilon^2}{\rho^2 \Sigma + \sigma_\varepsilon^2 + \sigma_\nu^2}, \\ \Sigma &= \frac{1}{2\rho^2} \left[- \left(\sigma_\varepsilon^2 + \left(1 - \rho^2\right) \sigma_\nu^2 \right) + \sqrt{\left(\sigma_\varepsilon^2 + (1 - \rho^2) \sigma_\nu^2 \right)^2 + 4 \sigma_\varepsilon^2 \sigma_\nu^2 \rho^2} \right], \\ -\infty &< \gamma_b < \infty, -\infty < \gamma_\varepsilon < \infty, \gamma_\nu = \frac{\phi}{\phi - \rho} \frac{\Sigma}{\sigma_\nu^2} + \frac{1}{\phi - \rho} \frac{\sigma_\varepsilon^2}{\sigma_\nu^2} - \frac{\sigma_\varepsilon^2}{\sigma_\nu^2} \gamma_\varepsilon \end{aligned}$$

We can thus summarize our findings as follows. First, the limited information rational expectations equilibrium is indeterminate even though the full information counterpart has a determinate equilibrium. The optimal filtering procedure employed by the central bank introduces a stable root into the system that does not fully determine the forecast error. Despite the central bank's projection of a determinate equilibrium, the actual equilibrium in the economy is indeterminate solely on account of the central bank's filtering. Second, a consistency requirement for Kalman filtering imposes restrictions on the set of multiple equilibria, but leaves belief shocks b_t ('sunspots') unrestricted. In full information solutions under indeterminacy the set of multiple equilibria is typically unrestricted. Optimal filtering in a limited information counterpart restricts how the private agents coordinate on an equilibrium. From an empirical perspective, the FIRE solution results in a reduced-form representation for inflation that is first-order autoregressive. The LIRE solution on the other hand exhibits much richer dynamics. In particular, the resulting inflation process can be quite persistent when the signal-to-noise ratio is small as a large σ_ν^2 translates into a small Kalman gain.

This simple example may be regarded as restrictive as the central bank only observes an exogenous process with error, whereas in practice observed variables are typically endogenous (and measured with error). We therefore study the implications of the central

bank observing the inflation rate with error. The information set is endogenous in the sense that the variable contained therein is determined in equilibrium. This potentially creates additional feedback within the model.

2.2.2 An Endogenous Information Set

We now assume that the central bank observes the inflation rate with measurement error ν_t such that $Z_t = \pi_t + \nu_t$. The solution in the terms of the central bank projections is identical to the previous case. What changes is however the interdependence between the filtering problem and the equilibrium dynamics. We find it convenient to express the analysis in terms of the prediction equations for the real rate to maintain comparability with the previous case. The prediction equation is:

$$r_{t|t} = r_{t|t-1} + \kappa_r \left(\pi_t - \pi_{t|t-1} + \nu_t \right). \tag{17}$$

From this we can derive the full equation system as before, namely:

$$\begin{aligned} \pi_t &= \frac{\phi}{\phi - \rho} r_{t-1|t-1} - r_{t-1} + \eta_t & (18) \\ r_{t|t} &= (\rho + \kappa_r)\, r_{t-1|t-1} + \kappa_r r_{t-1} + \kappa_r \nu_t + \kappa_r \eta_t & (19) \\ r_t &= \rho r_{t-1} + \varepsilon_t & (20) \end{aligned}$$

This system looks superficially similar to the previous one with exogenous information. What is different is that the coefficient on the lagged real rate prediction $(\rho + \kappa_r)\, r_{t-1|t-1}$ depends now on the endogenous dynamics of π_t, which may or may not lead to an unstable root $|\rho + \kappa_r| > 1$ and thus give rise to non-fundamental belief shocks affecting equilibrium dynamics without causing non-stationary variations.

2.2.3 A Benchmark Solution with Endogenous Information

The observation that the prediction equation $r_{t|t} = (\rho + \kappa_r)\, r_{t-1|t-1} + \kappa_r r_{t-1} + \kappa_r \nu_t + \kappa_r \eta_t$ can be potentially explosive and would therefore pin down the forecast error η_t to find *an* equilibrium. Using the definition $r_t^* = r_t - r_{t|t}$, which is the error from the prediction off the current information set, we can rewrite the prediction equation as follows:

$$r_t^* = (\rho + \kappa_r)\, r_{t-1}^* + \varepsilon_t - \kappa_r \nu_t - \kappa_r \eta_t. \tag{21}$$

This is a first-order difference equation with exogenous shocks ε_t, the real rate innovation; ν_t, the measurement error, and the endogenous forecast error η_t. Now suppose that $|\rho + \kappa_r| > 1$. The solution to this explosive equation is $r_t^* \equiv 0$ and $\eta_t = \frac{1}{\kappa_r}\varepsilon_t - \nu_t$. This solution uniquely

pins down the forecast error and therefore seems to render the equilibrium determinate. Why is this equilibrium then not unique? The Kalman gain κ_r and as such, there can in general be other equilibria with a different Kalman gain. We explore this possibility in the next section The benchmark equilibrium is a full revelation solution in that it implies that $r_{t|t} = r_t$, that is, the real rate prediction using current information is exact (the equilibrium inflation rate, however, is not revealed without error to the central bank).

Existence of this solution requires that we can find a κ_r such that $|\rho + \kappa_r| > 1$ and that the subspace condition holds. We can verify that this is in fact the case by substituting the solution into the prediction equation which results in $r_t = \rho r_{t-1} + \varepsilon_t$, that is, the exogenous process for the real rate. Substituting the solution in the inflation equation yields $\pi_t = \frac{\rho}{\phi-\rho} r_{t-1} + \frac{1}{\kappa_r}\varepsilon_t - \nu_t$, which depends on the Kalman gain. Since this solution achieves full revelation for the real rate, we can hypothesize that it is consistent with the full information solution that we derived above in the following sense: $\pi_t^{FI} = \frac{\rho}{\phi-\rho} r_{t-1} + \frac{1}{\phi-\rho}\varepsilon_t$, which is void of the measurement error. Comparing the LIRE and FIRE solution we find that $\pi_t^{LI} = \pi_t^{FI} - \nu_t$, which implies $\kappa_r = \phi - \rho$. The subspace condition additionally imposes $\kappa_\pi = \kappa_r / (\phi - \rho) = 1$ and always holds.

The nature of this equilibrium is such in terms of the forecast error decomposition $\eta_t = \gamma_\varepsilon \varepsilon_t + \gamma_b b_t + \gamma_\nu \nu_t$ we have $\gamma_\varepsilon = 1/(\phi - \rho)$, $\gamma_\nu = -1$, and $\gamma_b = 0$. The latter follows since under determinacy sunspot shocks do not affect equilibrium outcomes. As a final step, we need to verify that this Kalman gain is consistent with the Riccati equation. Under this parametrization, the Riccati equation has a positive solution and a solution with $\Sigma = 0$, which implies full revelation of the real rate. This root intersects with the subspace condition at the value $\gamma_\varepsilon = 1/(\phi - \rho)$.[6] Finally, the root of the prediction equation $\rho + \kappa_r = \phi > 1$, which validates our original assumption.

2.2.4 Other Equilibria under Endogenous Information

We next turn to finding other equilibria in the model with the endogenous information set described above. As a first step, we again derive the optimal Kalman gain. After some lengthy algebra, we find that:

$$\kappa_r = \frac{cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)}{var(\widetilde{Z}_t)} = \frac{-\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2}{\Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 \sigma_b^2 + (1+\gamma_\nu)^2 \sigma_\nu^2}. \tag{22}$$

[6] In reference to Figure 1 this means that the negative root touches zero where it intersects with the hyperbola of the subspace condition.

The prediction error variance Σ can be computed from the Riccati equation

$$\Sigma = var\left(\widetilde{r}_t - \widetilde{r}_{t|t}\right) = var\left(\widetilde{r}_t\right) - var\left(\widetilde{r}_{t|t}\right) = \rho^2\Sigma + \sigma_\varepsilon^2 - \frac{-\left(\rho\Sigma + \gamma_\varepsilon\sigma_\varepsilon^2\right)^2}{\Sigma + \gamma_\varepsilon^2\sigma_\varepsilon^2 + \gamma_b^2\sigma_b^2 + \left(1+\gamma_\nu\right)^2\sigma_\nu^2}.$$

Existence of an equilibrium also requires that the subspace condition

$$cov\left(\widetilde{r}_t, \widetilde{Z}_t\right) = \left(\phi - \rho\right) cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right)$$

holds, which imposes the restriction that

$$\phi\Sigma = \left(1 - \left(\phi - \rho\right)\gamma_\varepsilon\right)\gamma_\varepsilon^2\sigma_\varepsilon^2 - \left(\phi - \rho\right)\gamma_\nu\left(1+\gamma_\nu\right)\sigma_\nu^2 - \gamma_b^2\sigma_b^2.$$

The solution to the full equation system therefore rests on whether the root $|\rho + \kappa_r| \gtrless 1$. If the root is greater than one, we recover the benchmark equilibrium described above. Using numerical simulation we found it generally difficult to find solutions for which the root is less than one. While we can compute solutions to the prediction error variance Σ from the Riccati equation, these generally turn out to be inconsistent with the sub-space condition. Figure 1 highlights the issue. For a typical parametrization and $\gamma_b = 0$ it shows the two roots of the quadratic Riccati equation and the hyperbola of the subspace condition. These values are plotted as functions of the loading on the real rate innovation γ_ε over the range $[-10, 10]$. We also report the Kalman gain for reference purposes.

Existence of a solution requires that the positive root of the Riccati equation and the subspace condition hold at the same time, but they never intersect or even touch. For this given parametrization no equilibrium exists since under these parameters the Kalman filter does not exist in the sense that the prediction equations for the variables outside of the central bank's information set are explosive and mutually inconsistent. A wide-ranging exploration of the parameter space does reveal, however, that an equilibrium exists for small ρ, high ϕ, and large σ_ν^2. In this case, the equilibrium would be indeterminate as in the case with an exogenous information set since the Kalman gain is small and negative as already indicated in Figure 1. Moreover, if the equilibrium exists, there is one non-linear restriction on three weights in the forecast error.

To summarize, in the LIRE model with an endogenous information set it is possible to find an equilibrium that (almost) replicates the FIRE equilibrium. There is higher inflation volatility on account of the presence of the measurement error. Nevertheless there are potentially other equilibria that lead to indeterminacy but based on numerical solutions we regard them as implausible for this specific model. Other models, because of different

equilibrium conditions (and as such subspace conditions) might very well have multiple equilibria with endogenous information sets for reasonable parameters. The scenario described in this section is akin to the outcome described in Lubik and Schorfheide (2003), where an indeterminate equilibrium without sunspots is observationally equivalent to a corresponding determinate equilibrium.

2.3 MSV solution akin to Svensson and Woodford (2004)

In this section we describe a minimum-state-variable (MSV) approach as in Svensson and Woodford (2004). An important difference between their work and ours is that, in the present model is described by a given rule, whereas Svensson and Woodford endevaour to characterize optimal policy. However, for a given set of first-order conditions to the optimal policy problem under imperfect information, their approach falls into the class of expectational linear-difference equations studied here as well (see Section 3 for a more general discussion). Svensson and Woodford (2004) are not alone in pursing a MSV approach in such models, other examples are given by Aoki (2008), or Nimark (2008); applied to our model, this approach begins with a guess that the equilibrium process for inflation has the following form:

$$\pi_t = g\, r_t^* + \bar{g}\, r_{t|t} \qquad \bar{g} \equiv \frac{1}{\phi - \rho} \tag{23}$$

$$= g\, r_t + (\bar{g} - g)\, r_{t|t} \tag{24}$$

For any choice of g, this guess automatically satisfies the sub-space condition $\pi_{t|t} = \bar{g} r_{t|t}$. What remains to be seen is which values for g (if any) would be consistent with the rest of the dynamic system, notably the innovations version of Fisher equation in (14). Notice that the proposed solution excludes belief shocks.

Let us proceed by deriving the dynamics for r_t^* and $r_{t|t}$ implied by (23) for a given value of g. A slight complication for setting up the Kalman filter — encountered also by Svensson and Woodford — is that the guess for inflation in (23) depends on the projected real rate, and thus on the history of measurements (Z^t) which in turn depends on the history of inflation:

$$Z_t = g\, r_t + (\bar{g} - g)\, r_{t|t} + \nu_t \tag{25}$$

However, notice that the term in $r_{t|t}$ does not add any new information to Z_t; in fact Z_t rather provides an implicit definition of an information set spanned by:

$$W_t = g\, r_t + \nu_t \tag{26}$$

in the sense that $E(x_t|Z^t) = E(x_t|W^t)$ for any variable x_t. While projections of variables onto W^t and Z^t are equivalent, the associated Kalman gains will, however, differ by a factor of proportionality.[7]

For starters, consider the Kalman gain involved in projecting the real rate onto W^t, $\tilde{r}_{t|t} = \kappa \tilde{W}_t$:

$$\kappa = \frac{\text{Cov}\,(\tilde{r}_t, \tilde{W}_t)}{\text{Var}\,(\tilde{W}_t)} = \frac{g\,\tilde{\Sigma}}{g^2\,\tilde{\Sigma} + \sigma_\nu^2} \qquad \tilde{\Sigma} \equiv \text{Var}\,(\tilde{r}_t) \tag{27}$$

and define

$$R^2 \equiv g \cdot K \qquad \Rightarrow \quad 0 \leq R^2 \leq 1. \tag{28}$$

We can then write

$$\begin{aligned}
\tilde{\pi}_t &= g \cdot \tilde{r}_t + (\bar{g} - g)\,\kappa\,\tilde{W}_t && (29)\\
&= \left(g \cdot (1 - R^2) + \bar{g} \cdot R^2\right) \tilde{r}_t + (\bar{g} - g)\,\kappa\,\nu_t && (30)\\
&= \left(g \cdot (1 - R^2) + \bar{g} \cdot R^2\right) \rho\, r^*_{t-1} + \underbrace{\left(g \cdot (1 - R^2) + \bar{g} \cdot R^2\right) \varepsilon_t + (\bar{g} - g)\,\kappa\,\nu_t}_{=\eta_t} && (31)
\end{aligned}$$

where the last line uses $\tilde{r}_t = \rho\, r^*_{t-1} + \varepsilon_t$.

In order to match (14) we can set η_t equal to the shock components of (31) as indicated above *and* we need to find a value for g that sets the loading on r^*_{t-1} in (31) equal to minus one:

$$\begin{aligned}
\left(g \cdot (1 - R^2) + \bar{g} \cdot R^2\right) \rho &= -1 && (32)\\
\Rightarrow \quad g &\leq 0 && (33)
\end{aligned}$$

where the inequality follows from $\bar{g}$, R^2 and ρ being all positive numbers. As a further condition, the solution approach espoused by Svensson and Woodford (2004) would require the roots of the characteristic equation describing the joint dynamics of π_t, $r_{t|t}$ and r_t, see (18)- (20) above, to satisfy the usual counting rule for values inside and outside the unit circle. In the present case, with only one backward-looking variables, r_t, and two forward-looking variables, π_t and $r_{t|t}$, the approach of Svensson and Woodford (2004) would rely on finding one stable and two unstable eigenvalues. However, it can be shown that in the

[7] Let K_r continue to denote the Kalman gain of r_t onto Z_t and we have

$$\tilde{Z}_t = \tilde{W}_t + (\bar{g} - g) \cdot K_r \cdot \tilde{Z}_t = \tilde{W}_t / (1 - (\bar{g} - g) \cdot K_r).$$

present example, the Kalman filter will always stabilize the dynamics of $r_t - r_{t|t}$ causing the system to have two stable and only one unstable root.

Note that the set of MSV candidate solutions — described by (23) for any given value of g — does not span the set of all candidate solutions that we have looked at so far — described by any combination of weights γ for the linear combination of shocks that make up the endogenous forecast error η_t: Furthermore, the set of SW candidates does not even span the restricted set of candidates for η_t where $\gamma_b = 0$. To see this, notice that the MSV candidate is parametrized by a single unknown coefficient, g, which places also a restriction on the weights γ_ε and γ_ν implied by the associated specification of η_t as seen in (31).

3 General Framework

Consider the following system of expectational linear-difference equations:

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} E_t \begin{bmatrix} X_{t+1} \\ Y_{t+1} \\ U_{t+1} \end{bmatrix} + J \begin{bmatrix} X_{t+1|t} \\ Y_{t+1|t} \\ U_{t+1|t} \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & 0 \\ A_{yx} & A_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_t \\ Y_t \\ U_t \end{bmatrix} + \hat{A} \begin{bmatrix} X_{t|t} \\ Y_{t|t} \\ U_{t|t} \end{bmatrix} \tag{34}$$

with

$$U_t \equiv U_{t|t} \tag{35}$$

$$X_t - E_{t-1} X_t = Q\varepsilon_t \qquad \text{Var}\,(\varepsilon_t) = I \tag{36}$$

where X_t is a vector of backward-looking variables, characterized by exogenous forecast errors $\varepsilon_t \sim N(0, QQ')$, Y_t is a vector of forward-looking variables, U_t is a vector of forward-looking variables that is restricted to lie in a space spanned by a restricted information set that will be discussed shortly.[8] J is a square matrix and may be singular. Expectations $E_t(\cdot)$ are conditional on perfect information about the history of all shocks and variables in the system.[9] Formally, for any variable v_t:

$$E_t\, v_t \equiv E(v_t|S^t) \qquad S^t = \{S_t, S_{t-1}, S_{t-2}, \ldots\} \qquad S_t \equiv \begin{bmatrix} X_t \\ Y_t \end{bmatrix} \tag{37}$$

In contrast, $v_{t|t}$ denotes a conditional expectation derived from a limited information set:

$$v_{t|t} \equiv E(v_t|Z^t) \qquad Z^t = \{Z_t, Z_{t-1}, Z_{t-2}, \ldots\} \qquad Z_t = L\, S_t \tag{38}$$

where C has not full column rank.[10]

Notice that $U_t = U_{t|t}$ is not included in S_t; by definition, U_t merely reflects information contained in S^t and thus does not add any new information.

The linear difference system in (36) does not only arise in models where policy is characterized by Taylor rule under limited information but also extends to the system of first-order conditions for optimal policy under asymmetric information derived by Svensson and Woodford (2004) and Aoki (2006). Notice further that (36) is quite different from the often-cited

[8] In our monetary-policy applications, U_t will be the vector of policymakers' control variables — typically just the nominal short-term rate. Separating out U_t from X_t and Y_t simply serves to avoid singularities once the system is written in terms of variables that are orthogonal to the policymaker's information set.

[9] Note: Q may have more rows than columns and QQ' may be only positive *semi*-definite (i.e. not necessarily positive definite), as X_t may also track lagged variables.

[10] Notice, this allows also the case of noisy observations $Z_t = \mathcal{C}S_t + \nu_t$ after appropriate redefinition of X_t (and thus S_t) to include also ν_t.

work by Pearlman, Currie and Levine (1986) who study linear, expectational difference systems where *all* expectation operators are conditioned on limited information.[11]

Apart from the term involving $X_{t|t}$ and $Y_{t|t}$, the system in (36) corresponds to the framework(s) known from Blanchard and Kahn (1980), King and Watson (1998), Klein (2000) and Sims (2002). Most of these studies even allow for "singular" systems, where some equations are purely static, without involving any forward-looking terms. For ease of notation in my derivation, it will however be convenient to assume a non-singular system as in (36).

3.1 Unique solution for central-bank projections

Throughout, it is assumed that the public information set includes also the central bank's information set — formally: Z^t lies in the span of S^t — which turns out to be highly useful for the next step: Consider conditioning down (36) onto the central bank's information set (Z^t). The resulting system can easily be "solved" in terms of a relationship between $Y_{t|t}$ and $X_{t|t}$ with any of the above-mentioned solution methods for linear rational expectation systems. The resulting system is isomorphic to the full information analogue of (36):

$$\mathcal{J}\begin{bmatrix} X_{t+1|t} \\ Y_{t+1|t} \\ U_{t+1|t} \end{bmatrix} = \mathcal{A}\begin{bmatrix} X_{t|t} \\ Y_{t|t} \\ U_{t|t} \end{bmatrix} \qquad \mathcal{A} = \begin{bmatrix} A_{xx} & A_{xy} & 0 \\ A_{yx} & A_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \hat{A} \qquad \mathcal{J} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} + J \tag{39}$$

This part of the solution is identical to the analysis of Aoki (2006), Svensson and Woodford (2004).

Notice that the presence of central-bank expectations $Y_{t+1|t}$ and $X_{t+1|t}$ in (36) does not materially alter the problem since (39) is neatly handled by the methods of, for example, Klein (2000) and Sims (2002), independently of whether J (or $\mathcal{J}$) is singular or not.[12]

Provided the conditions for a unique rational expectations under full information are met — which are henceforth assumed to hold — we obtain the following relationship between the central bank's projections of forward- and backward-looking variables:[13]

$$U_{t|t} = FX_{t|t} \qquad Y_{t|t} = GX_{t|t} \qquad X_{t+1|t} = PX_{t|t} \qquad \Rightarrow \qquad S_{t+1|t} = \underbrace{\begin{bmatrix} P & 0 \\ G\,P & 0 \end{bmatrix}}_{\equiv \mathcal{P}} S_{t|t} \tag{40}$$

[11] The framework of Pearlman, Currie and Levine (1986) would thus capture a version of (36) where $X_{t+1|t}$ and $Y_{t+1|t}$ were to replace E_tX_{t+1} and E_tY_{t+1}, respectively.

[12] In anticipation of the Kalman filtering equations introduced further below, I am however assuming that the loadings on E_tY_{t+1} and E_tX_{t+1} in (36) are non-singular such that the system can be written in the form shown in (36) where these expectations are pre-multiplied by the identity matrix.

[13] A unique RE equilibrium exists when the number of eigenvalues of $\mathcal{A}$ that lie outside the unit circle is identical to the number of forward-looking variables Y_t.

where P (and thus also $\mathcal{P}$) is stable. For any sequence of innovations to $X_{t|t}$ *under the central banks' information set* Z^t, denoted

$$\tilde{X}_{t|t} \equiv X_{t|t} - X_{t|t-1},$$

there is is thus a unique, stable solution for the central bank's projections stacked in $S_{t|t}$.[14]

In general, the following notation will be useful: For any variable v_t, define "innovations" $\tilde{v}_t$ and "residuals" v_t^* as

$$\tilde{v}_t \equiv v_t - v_{t|t-1}, \qquad\qquad v_t^* \equiv v_t - v_{t|t} = \tilde{v}_t - \tilde{v}_{t|t}. \tag{41}$$

and notice that by construction we always have $\tilde{v}_{t|t-1} = 0$.

As all shocks are normally distributed, and assuming a linear, time-invariant equilibrium exists, we can conjecture that optimal projections will be characterized by a Kalman filter (this will be verified below). The Kalman filter (if it exists) then implies

$$S_{t|t} = S_{t|t-1} + K\tilde{Z}_t \tag{42}$$

$$= \mathcal{P}\, S_{t-1|t-1} + \underbrace{K\, L\, \tilde{S}_t}_{=\tilde{S}_{t|t}} \tag{43}$$

where K is the Kalman gain, and the second equation follows from (40) and the definition of Z_t. So far, we have found a unique stable solution for the evolution of the central bank's projections, for a given sequence of innovations $\tilde{X}_{t|t} = K\, C\, \tilde{S}_t$. What remains to be shown is how $\tilde{S}_t$ is pinned down in equilibrium and whether the solution is unique.

3.2 Sub-space condition

The endogenous innovations process $\tilde{S}_t$ remains to be derived, and it remains to be seen whether this process is uniquely determined. Importantly, the solution to the system in (39), given in (40), imposes the following restriction on projections of the state innovations onto the information set, and thus on the Kalman filter:

$$\underbrace{\begin{bmatrix} G & -I \end{bmatrix}}_{\equiv \boldsymbol{G}} S_{t|t} = 0 \qquad \Rightarrow \quad \boldsymbol{G}\tilde{S}_{t|t} = 0 \qquad \Rightarrow \quad \boldsymbol{G}\, K\, L = 0 \tag{44}$$

(Notice that (40) implies $\boldsymbol{G}\,\mathcal{P} = 0$ and thus also $\boldsymbol{G}S_{t|t-1} = 0$.) More succinctly, the solution to (39) implies the following orthogonality condition:

$$E(\tilde{Y}_t - G\tilde{X}_t)\tilde{Z}_t' = 0 \tag{45}$$

[14] $\tilde{X}_{t|t}$ is a martingale difference sequence with respect to Z^t but not S^t: By construction we have $\tilde{X}_{t|t-1} = 0$ but typically not $E_{t-1}\tilde{X}_{t|t} = 0$.

3.3 The "residual system"

From now on, let's work with an expectational difference system expressed in terms of "innovations" $\tilde{S}_t$ and "projection residuals" $S_t^* = \tilde{S}_t - \tilde{S}_{t|t} = S_t - S_{t|t}$. Having solved for the innovations process $\tilde{S}_t$ we can express the solution to the entire system via (40) and (43).

Specifically, we can take expectations of all terms in (39) conditional on Z^t and subtract these on both sides of the equation to obtain[15]

$$E_t(S_{t+1} - S_{t+1|t}) = A\,(S_t - S_{t|t}) \qquad \Rightarrow \quad E_t\tilde{S}_{t+1} = A\,S_t^*. \tag{46}$$

$$\text{where} \quad A \equiv \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \tag{47}$$

Clearly, this is not quite a standard difference equation, as the left-hand side involves $\tilde{S}_{t+1}$ and not S_{t+1}^* while the left-hand side involves S_t^* (and not $\tilde{S}_t$).

Define endogenous forecast errors η_t and recall the definition of the exogenous forecast errors ε_t that affect the backward-looking variables:[16]

$$\eta_t \equiv (1 - E_{t-1})Y_t \qquad\qquad Q\varepsilon_t \equiv (1 - E_{t-1})X_t. \tag{48}$$

As argued above, the endogenous forecast errors η_t can be written as the sum of a component due to fundamental shocks and a component due to non-fundamental shocks:

$$\eta_t = S\varepsilon_t + Rb_t \qquad \text{where} \quad E(\varepsilon_t\, b_t') = 0 \qquad b_t \sim N(0, I) \tag{49}$$

Denoting the length of the vectors X_t, Y_t, and Z_t by N_x, N_y, and N_z, respectively, there is a total of $N_x \times N_y + N_z \times (N_z + 1)/2$ free parameters.[17]

The residual system (46) can then be written as the state equation of a Kalman filtering system in innovations form:

$$\tilde{S}_{t+1} = A\,(\tilde{S}_t - \tilde{S}_{t|t}) + B\,w_{t+1} \tag{50}$$

$$\tilde{Z}_t = L\,\tilde{S}_t. \tag{51}$$

$$w_{t+1} \equiv \begin{bmatrix} \varepsilon_{t+1} \\ b_{t+1} \end{bmatrix} \qquad\qquad \text{Var}\,(w_t) = I \tag{52}$$

We are interested in restrictions on the forecast errors η_t that ensure a bounded solution to (50) and (51).[18] Strikingly, the Kalman filter, if it exists, ensures bounded innovations

[15] Note that $E_t S_{t+1|t} = S_{t+1|t}$ as Z^t is in the span of S^t.

[16] Note that $(1 - E_{t-1})Y_t = (1 - E_{t-1})\tilde{Y}_t$ and $(1 - E_{t-1})X_t = (1 - E_{t-1})\tilde{X}_t$.

[17] There are $N_x \times N_y$ coefficients in D_ε as well as $N_z \times (N_z + 1)/2$ free coefficients in Ω_b.

[18] In principle, an initial condition on $\tilde{S}_0$ may also be needed to ensure a bounded outcome, but the issue will not arise here, as will be seen shortly.

$\tilde{S}_{t+1}$ (and thus also bounded residuals $S_t^* = \tilde{S}_t - \tilde{S}_{t|t}$). Furthermore, for the system above, a time-invariant Kalman filter exists under two, fairly mild conditions; see, for example, Anderson and Moore (1979).

(A, L) **must be "detectable"** which means that every (non-zero) right-eigenvector of A that lies in the nullspace of L must be associated with a stable eigenvalue.[19]

(A, B) **must be "stabilizable"** (A, B) are stabilizable when (A', B') are detectable.

To gain a better intuition for what these conditions entail, recall the notion of "canonical" variables from Blanchard and Kahn (1980). Canonical variables form the basis for solving expectational difference systems. and they are obtained from a rotation $V\tilde{S}_t$ of the original variables such that VAV^{-1} is a (block) diagonal matrix. Intuitively, detectability requires then the measurement equation to load on any unstable "*canonical*" variables of (50).[20] Stabilizability requires that at least every unstable canonical variable is affected by a shock; which is easily satisfied if $\text{Var}(w_t)$ has full rank, i.e. in the presence of non-fundamental shocks.[21]

Provided the system given by (50) and (51) is detectable and stabilizable, only the covariance restriction (44) — resulting from $Y_{t|t} = GX_{t|t}$ — can put a lid on the number of equilibria. However, (44) consists of only $N_y \times N_z$ restrictions and thus less than the number of $N_y \times (N_x + (N_y + 1)/2)$ free parameters governing the evolution of endogenous forecast errors. Moreover, as illustrated in the notes to a little model (distributed in fall), the restrictions imposed by (44) are non-linear, and may not necessarily have a solution for any combination of parameter values.

3.4 A benchmark solution

The previous section described a simple model of a cashless economy and characterized a particular benchmark solution, see Section 2.2.3. The generic properties of this solution can also be carried over to the more general framework described in this section, provided we put a little more structure on the measurement vector as well as the number of exogenous and endogenous variables: Specifically, assume that the backward-looking variables are purely exogenous,

$$X_{t+1} = A_{xx}X_t + Q\varepsilon_t$$

[19] Formally: $A\,v = \lambda\,v$ and $L\,v = 0$ only if $|\lambda| < 1$ or $v = 0$.

[20] Detectability is, for example, violated by an information set that reflects only exogenous variables.

[21] Non-fundamental shocks are orthogonal to ε_t and are also referred to as "belief shocks" (Lubik and Schorfheide, 2003).

and that the policymaker observes all endogenous variables with error:

$$Z_t = Y_t + \nu_t$$

Furthermore, let there be (at least) as many endogenous variables as exogenous variables, $N_y = N_x$, with an invertible mapping from X_t to Y_t under full information, $|G| \neq 0$, where G is defined in (40).

In this case, the following benchmark solution for Y_t is always consistent with equilibrium:

$$Y_t = GX_t - \nu_t \tag{53}$$

$$\Rightarrow \quad Z_t = GX_t \qquad \Rightarrow \quad X_{t|t} = X_t \tag{54}$$

$$\Rightarrow \quad K_x = G^{-1} \qquad K_y = I \tag{55}$$

which also satisfies the sub-space condition $K_y = G\,K_x$.

3.5 Set of indeterminate solutions in the general case

[TO BE WRITTEN]

4 Some Numerical Applications

[TO BE WRITTEN]

5 Conclusion

The existence of multiple equilibria in dynamic general equilibrium models has important effects on policy analysis and statistical inference, as well as many other applications of these models. In linear, full-information rational expectations models the existence and computation of such indeterminate equilibria is well understood (Lubik & Schorfheide, 2004).
In corresponding models with limited and asymmetric information sets, the existence of multiple equilibria has not been studied. We take a first step in this direction and show that in a linear Gaussian setting (i) multiple equilibria in asymmetric information settings can arise even if the corresponding economy under full information would deliver a unique equilibrium, (ii) when multiple equilibria exist, the set of such multiple equilibria is restricted by the structure of the economy, in particular the nature of the less informed agent's filtering problem and (iii) the structure of these multiple equilibria depends crucially on the information set of the less informed agents, in particular whether these agents have access to noisy estimates of endogenous or exogenous variables.

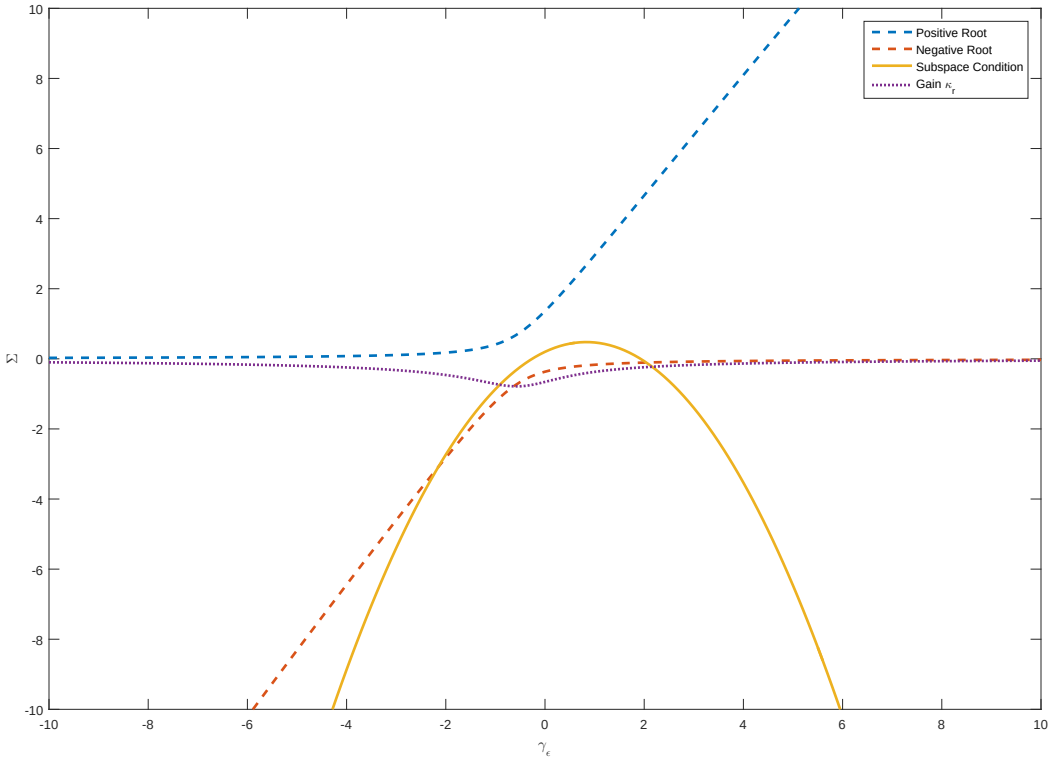


Figure 1: Equilibrium conditions with the endogenous information set

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