Reallocation of Intangible Capital and Secular Stagnation*

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Abstract

Low interest rates can hurt capital reallocation and reduce aggregate productivity and output in economies that rely strongly on intangible capital. This insight is obtained in a model in which productive credit-constrained firms can only borrow against the collateral value of their tangible assets and there is substantial dispersion in productivity. In a tangibles-intense economy with highly leveraged firms, low rates enable more borrowing and faster debt repayment, reduce misallocation, and increase aggregate output. Conversely, an increase in the share of intangible capital in production reduces the borrowing capacity and increases the cash holdings of the corporate sector, which switches from being a net borrower to a net saver. In this intangibles-intense economy, the ability of firms to purchase intangible capital using retained earnings is impaired by low interest rates, because they increase the price of capital and slow down the accumulation of corporate savings. As a result, the emergence of intangible technologies, even when they replace significantly less productive tangible technologies, may be contractionary.

Keywords: Intangible Capital, Borrowing Constraints, Capital Reallocation, Secular Stagnation
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1 Introduction

Real interest rates have decreased in the last decades, while economic growth has fallen short of previous trends, developments that have been linked to a process of 'secular stagnation' (Summers (2015), Eichengreen (2015)). At the same time, the developed world has experienced a technological change towards a stronger importance of information technology and knowledge, human and organizational capital, which has gradually reduced the reliance on physical capital (Corrado and Hulten (2010a)), and which has been linked to the significant decrease in corporate net borrowing (Falato et al (2013), Döttling and Perotti (2015)).

This paper argues that the increased reliance on intangible capital and the low real interest rates interact to hurt capital reallocation and reduce productivity and output growth. Aggregate productivity depends on an efficient reallocation of resources from declining or exiting firms to new entrants or expanding firms. The rise of intangible capital implies a growing importance of the reallocation of intangible assets such as patents, brand equity, and human and organizational capital. These assets cannot be collateralized, and their acquisition has to be financed mostly using retained earnings. As a result, the corporate sector borrows less, holds an increasing amount of cash, and switches from being a net borrower to a net saver. We show that this shift not only adds additional downward pressure on interest rates, but also alters the dynamic relationship between interest rates and efficiency in the allocation of capital. The decrease in interest rates increases the price of these intangible assets, and reduces the ability of credit constrained expanding firms to purchase them. Lower interest rates also decrease the rate at which non-investing firms can accumulate savings to finance future expansions.

This alternative explanation of secular stagnation is consistent with crucial stylized facts about recent trends in industrialized economies, such as declining interest rates, below-potential growth, and large increases in net corporate savings and asset prices over GDP, and has potentially important policy implications.

We formalize this intuition by developing a stylized model of an economy in which a productive sector uses a technology with tangible capital, intangible capital and labor as complementary factors in the production of consumption goods. We follow Kiyotaki and Moore (2012) in assuming that this sector is populated by a continuum of firms that can only invest occasionally. Firms suffer from financing constraints that prevent them from issuing equity, or

from borrowing any amount in excess of the collateral value of their holdings of tangible and intangible capital. They have finite lives, and this prevents them from accumulating enough savings to overcome their financial constraints. In equilibrium, they save as much as possible in non-investing periods, and invest all of their accumulated net savings plus their maximum available borrowing in investing periods. Any residual capital not absorbed by the firm sector is used by an unproductive alternative sector, which as a result of being the marginal buyer of capital also prices it. Therefore, aggregate productivity in this economy depends on the ability of growing productive firms to absorb the assets liquidated by the exiting firms. The consumer sector is modelled as overlapping generations of households displaying a realistic life cycle, modelled in a way that enables us to obtain an equilibrium interest rate in the steady state which is not necessarily equal to the household rate of time preference.

We first inspect the analytical solution of a simplified version of the model to describe four channels through which lower interest rates interact with the intensity of intangible capital in firms' production function to affect the steady state equilibrium of our economy. First, a *debt overhang channel* allows net borrowing firms to pay down their debt more easily when interest rates are low, and helps capital reallocation. Conversely, a *savings channel* operates when the firm sector is a net saver, and reductions in the interest rate decrease the speed of accumulation of savings and hurt capital reallocation. Third, lower interest rates that increase the price of tangible and intangible assets, reduce the amount of capital firms can purchase for a given amount of net worth and borrowing capacity, a *capital purchase price channel*. Fourth, a lower interest rate increases the present value of the collateral pledged next period, and reduces the size of the downpayment necessary to purchase capital, improving capital reallocation through a *borrowing/collateral value channel*.

Next, we implement an exogenous fall in real interest rates similar to the one observed in the United States from the 1970s until the present. We first study this process holding fixed the share of tangible capital at 100%. In such an economy, the borrowing/collateral value channel and the debt overhang channels dominate and the decrease in rates raises aggregate output by 0.6%. Intuitively, the corporate sector is highly leveraged and benefits from lower interest rates. However, when we repeat the same exercise holding fixed the share of intangible capital at 100%, we find that the two negative channels, the capital purchase price channel and the

savings channel, are much more important, and result in a 9% decline in aggregate output. Firms are unable to borrow at all, which increases the impact of capital price increases on the downpayment necessary to purchase capital. In addition, they need to accumulate retained earnings, and the return to those savings is reduced in a low interest rate environment.

Finally, we simulate our full general equilibrium model to study how the parallel developments in the household and the corporate sector have interacted to generate aggregate patterns consistent with the secular stagnation hypothesis. In the household sector, we model a progressive increase in individuals' life expectancy, and a progressive decrease in their rate of time preference. Both developments are meant to capture demographic forces that have acted to increase household sector net savings between the 1970s and the present, which puts downward pressure on the equilibrium interest rate. We interpret our exercise as a shortcut for a collection of different factors, such as population aging, wealth and income inequality, and foreign sector developments. In the corporate sector, we introduce a gradual shift in the reliance on intangible capital of firms, from the pre-1980 economy, in which intangible capital accounted only for 20% of aggregate capital, to the post-2010 value of 60% of total capital ($\mu = 0.6$) (Corrado and Hulten (2010a), Falato et al (2013), Döttling and Perotti (2015)). Since we assume that intangible capital is more productive than tangible capital, this gradual shift is consistent with the notion of the transition to intangible capital as a privately optimal choice of firms adopting more productive technologies.

We find that while the household sector developments in isolation and the corporate sector developments in isolation are both expansionary, the combination of both developments is contractionary. The increase in household net savings puts downward pressure in interest rates and, even though it encourages capital creation and increases productive firms' ability to borrow and pay down their debt, affects capital allocation negatively by increasing capital prices. Firms' increased reliance in a type of capital which attracts less external finance decreases corporate leverage and tightens firms' borrowing constraints significantly. Firms switch from being net borrowers to being net lenders, and the share of output produced by the productive sector drops significantly. The lower corporate borrowing puts downward pressure on interest rates, which amplifies the misallocation of capital through a capital purchase price channel and a savings channel. Despite the fact that capital creation increases strongly, and that the economy is shift-

ing to a higher reliance on a type of capital which is significantly more productive, the output drop caused by the combination of both developments is in excess of 1%.

We interpret this comparative static exercise as the developments in the US economy following the rise in the share of intangible capital and in net household and foreign sector savings in the last 40 years. In this respect, this simple model is remarkably consistent with a series of well documented trends during this period: i) corporate savings increased as a fraction of GDP; ii) household leverage increased as a fraction of GDP; iii) the real interest rate fell; iv) asset prices relative to aggregate output increased; and v) output and productivity progressively declined relative to their previous trends. While the importance of the rise in intangible capital for stylized facts i)-iii) has been already shown by Falato et al (2013) and Döttling and Perotti (2015), this paper is the first to show that it is potentially very important in explaining the low growth.

2 Related Literature

The secular stagnation hypothesis as an explanation of recent economic trends has been proposed, amongst others, by Summers (2015) and Eichengreen (2015). One prominent example of a formalization of these ideas is Eggertsson and Mehrotra (2014), who show how a persistent tightening of the debt limit facing households can reduce the equilibrium real interest rate and, in the presence of a zero lower bound and sticky prices, generate permanent reductions in output.[1]

Common to most of these accounts of secular stagnation is that the excess savings arise from the household or the foreign sector, but not from a decrease in the demand for those savings from the corporate sector. For example, Summers (2015) and Eichengreen (2015) mention factors such as population aging and a rise in savings of developing economies. An exception is Thwaites (2015), who explains the decrease in interest rates as a result of the decrease in the relative price of investment goods. In our model, a realistically calibrated increase in the use of intangible capital can achieve a substantial decrease in interest rates. More importantly, our paper identifies a novel misallocation effect of endogenously low real interest rates which has important policy implications, different from those of other existing secular stagnation theories.

[1] Other recent theoretical papers with alternative explanations of secular stagnation are Bachetta et al (2015) and Benigno and Fornaro (2015).

The rising use of intangible capital has been documented by Corrado and Hulten (2010a), and its relation to the decrease in corporate borrowing and the rise in corporate cash holdings has been shown empirically by Bates et al. (2009). Falato et al. (2013) and Döttling and Perotti (2015) introduce models that describe how the rise in intangibles can lower the equilibrium interest rate by decreasing firms' net borrowing. Giglio and Severo (2012) link the decrease in interest rates caused by the rise of intangibles to the appearance of asset price bubbles. Our contribution to this literature is to describe a mechanism through which the rise in intangibles can have a negative impact on aggregate reallocation and growth.

Finally, our paper is related to the literature on the determinants of aggregate investment. A broad class of investment models predict that lower interest rates reduce the user cost of capital and stimulate investment. However, a large body of empirical research finds very little evidence of this negative relation (e.g. see, Caballero (1999), and Schaller (2007)). More recently, Kothari, Lewellen and Warner (2015), using a multivariate regression framework that includes as additional determinants of investment corporate profits, stock market returns, credit spreads and GDP growth, find a positive relation between lagged risk free interest rates and aggregate investment up to 2 quarters into the future. In our general equilibrium model, aggregate capital and interest rates are both endogenous and they may correlate positively or negatively with each other depending on the relative importance of tangible and intangible factors of production, with a positive relation which prevails, because of the rise in intangibles, during the post-1980 period.

3 Model

We introduce an infinite-horizon, discrete-time economy populated by an intermediate sector which produces capital, by a final good sector in which firms use labor and capital to produce consumption goods, and by households, who provide labor and own both sectors. We describe the firm sector and firms' optimization problem in Section 3.2, and discuss the households' problem and optimal choices in Section 3.3.

3.1 The Capital-producing Sector

A representative firm in this sector chooses investment in tangible and intangible capital, respectively I_t^T and I_t^I, in order to maximize profits:

$$\max_{I^J} q_{J,t} I_t^J - b\left(\frac{I_t^J}{\varphi}\right)^{\varphi}$$

where $\varphi > 1, b > 0$, and $q_{J,t}$ is the price of type of capital $J \in \{T, I\}$. The first order condition yields $I_t^J = \varphi\left(\frac{q_{J,t}}{b}\right)^{\frac{1}{\varphi-1}}$, and profits are:

$$\pi_t^J = \frac{q_{J,t}^{\frac{\varphi}{\varphi-1}}}{b^{\frac{1}{\varphi-1}}}(\varphi - 1)$$

At the beginning of period t total capital available is $\overline{K}_t^T$ and $\overline{K}_t^I$. New capital I_t^T and I_t^I is produced and sold in period t, so that the aggregate dividends generated by the capital-producing sectors are:

$$D_t^k = \pi_t^T + \pi_t^I$$

During period t capital depreciates at the rate δ, where $0 \leq \delta < 1$, and the law of motion of aggregate capital is:

$$\overline{K}_{t+1}^T = I_t^T + (1-\delta)\overline{K}_t^T$$

$$\overline{K}_{t+1}^I = I_t^I + (1-\delta)\overline{K}_t^I$$

3.2 Final Good Sector

There are two types of final good firms, *productive* and *unproductive*. The unproductive sector is introduced to address issues of misallocation, and because it allows for a simple characterization of the equilibrium prices of tangible and intangible capital.

3.2.1 The Productive Sector

The productive sector is composed of a continuum of firms of mass 1.

Technology and financing opportunities

Productive firms produce a final good using a constant returns to scale production function which is Cobb-Douglas in labor and capital. The firms use two different types of complementary capital, tangible and intangible. For simplicity, we assume that they are perfect complements.

The production function takes the form:

$$y_t^p = z_t n_t^{(1-\alpha)} \left[\min \left(\frac{k_{T,t}}{1-\mu}, \frac{k_{I,t}}{\mu} \right) \right]^{\alpha} \tag{1}$$

where $0 < \alpha \leq 1$, $0 < \mu < 1$. The terms $k_{T,t}$ and $k_{I,t}$ represent tangible and intangible capital installed in period $t-1$ that produce output in period t. Finally, z_t is a productivity parameter, and n_t is labor. The Leontief production structure implies that in equilibrium intangible capital as a share of total capital in the productive sector is equal to μ.

The budget constraint for productive firms is given by the following dividend equation:

$$d_t = y_t^p + (1+r_t)a_{f,t} - a_{f,t+1} - q_{T,t}\left(k_{T,t+1} - (1-\delta)k_{T,t}\right) - q_{I,t}(k_{I,t+1} - (1-\delta)k_{I,t}) - w_t n_t. \tag{2}$$

where r_t is the interest rate, q_T and q_I are the prices of tangible and intangible capital, respectively, and w_t is the wage. δ is the depreciation rate of capital, which we assume to be, for simplicity, the same for tangible and intangible capital. $a_{f,t} > 0$ indicates that the firm is a net saver, and $a_{f,t} < 0$ indicates that the firm is a net borrower.

Productive firms are subject to frictions in their access to external finance. They are unable to issue equity, which means that dividends are subject to a non-negativity constraint:

$$d_t \geq 0. \tag{3}$$

They can issue one-period riskless debt, subject to the constraint that they can pledge, as collateral, the fractions θ^T and θ^I of tangible capital and intangible capital, respectively. This translates into the following borrowing constraint:

$$a_{f,t+1} \geq -\frac{\theta^T q_{T,t+1} k_{T,t+1} + \theta^I q_{I,t+1} k_{I,t+1}}{1+r_{t+1}} \tag{4}$$

where $0 < \theta^T \leq 1$ and $0 < \theta^I < \theta^T$. We conjecture, and check later, that in equilibrium firms are credit constrained and choose not to pay dividends. Imposing that $d_t = 0$ in budget constraint (2), and substituting for $a_{f,t+1}$ in (2), we can express the borrowing constraint as:

$$\left(q_{T,t} - \frac{\theta^T q_{T,t+1}}{1+r_{t+1}}\right) k_{T,t+1} + \left(q_{I,t} - \frac{\theta^I q_{I,t+1}}{1+r_{t+1}}\right) k_{I,t+1} \leq y_t^p - w_t n_t + (1+r_t)a_{f,t} + (1-\delta)q_{T,t}k_{T,t} + (1-\delta)q_{I,t}k_{I,t} \tag{5}$$

From the Leontief structure of the production function it follows that $k_{T,t} = \frac{1-\mu}{\mu} k_{I,t}$. There-

fore, from now onwards, we use this result to express all equations as a function of intangible capital only, and (5) becomes:

$$\left[\left(q_{T,t}-\frac{\theta^T q_{T,t+1}}{1+r_{t+1}}\right)\frac{1-\mu}{\mu}+\left(q_{I,t}-\frac{\theta^I q_{I,t+1}}{1+r_{t+1}}\right)\right]k_{I,t+1}\leq y_t^p-w_t n_t+(1+r_t)a_{f,t}+(1-\delta)\left(q_{T,t}\frac{1-\mu}{\mu}+q_{I,t}\right)k_{I,t}.$$

Solving for $k_{I,t+1}$, we obtain:

$$k_{I,t+1}\leq\frac{y_t^p-w_t n_t+(1+r_t)a_{f,t}+(1-\delta)\left(q_{T,t}\frac{1-\mu}{\mu}+q_{I,t}\right)k_{I,t}}{q_{T,t}\frac{1-\mu}{\mu}+q_{I,t}-\left(\theta^T\frac{q_{T,t+1}}{1+r_{t+1}}\frac{1-\mu}{\mu}+\theta^I\frac{q_{I,t+1}}{1+r_{t+1}}\right)} \tag{6}$$

The right hand side of equation (6) is the maximum feasible investment in intangible capital for a firm. The numerator of equation (6) is the total wealth available to invest. The denominator captures the downpayment necessary to purchase one unit of $k_{I,t+1}$ and $\frac{1-\mu}{\mu}$ units of $k_{T,t+1}$. The term $q_{T,t}\frac{1-\mu}{\mu}+q_{I,t}$ represents the total cost necessary to purchase these amounts of both types of capital, and the term $\theta^T\frac{q_{T,t+1}}{1+r_{t+1}}\frac{1-\mu}{\mu}+\theta^I\frac{q_{I,t+1}}{1+r_{t+1}}$ is the amount that can be financed by borrowing.

At the beginning of each period, both types of capital are predetermined and in their optimal ratio $k_{T,t}=\frac{1-\mu}{\mu}k_{I,t}$, and therefore the production function can be written as:

$$y_t^p=z_t n_t^{(1-\alpha)}\left(\frac{k_{I,t}}{\mu}\right)^{\alpha}. \tag{7}$$

Given the wage w_t and its predetermined capital $k_{I,t}$, a firm will choose the profit maximizing level of labor, which determines the optimal capital labor ratio:

$$\frac{k_{I,t}}{n_t}=\mu\left[\frac{w_t}{(1-\alpha)\,z_t}\right]^{\frac{1}{\alpha}} \tag{8}$$

After producing, the firm's technology becomes obsolete with probability ψ. In this case, the firm liquidates all its capital, and pays out as dividends all of its savings, including the liquidation value of capital, and exits.

Firms cannot invest every period. More specifically, they can only invest in a given period with probability η. This assumption, in addition to capturing the realistic feature that firms' investment is lumpy (Caballero (1999)), is meant to allow firms to have the opportunity to accumulate significant amounts of liquid savings, in line with the empirical evidence.

Let λ_t and ϑ_t be the Lagrange multipliers of constraints (3) and (6), respectively. We define the value functions conditional on investing and not investing, respectively $V^+(k_I, a_{f,t})$ and $V^-(k_I, a_{f,t})$, as follows:

$$\begin{aligned} V_t^+(k_{I,t}, a_{f,t}) &= \max_{a_{f,t+1}, k_{I,t+1}} (1+\lambda_t)d_t + \vartheta_t \left[\begin{array}{c} y_t^p - w_t n_t + (1+r_t)a_{f,t} + (1-\delta)\left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right)k_{I,t} - \\ + \left[q_{T,t}\frac{1-\mu}{\mu} + q_{I,t} - \theta^T \frac{q_{T,t+1}}{1+r_{t+1}}\frac{1-\mu}{\mu} - \theta^I \frac{q_{I,t+1}}{1+r_{t+1}}\right] k_{I,t+1} q_{I,t} \end{array} \right] \\ &+ \frac{1}{1+r_{t+1}}\left[(1-\psi)V_{t+1}(k_{I,t+1}, a_{f,t+1}) + \psi d_{t+1}^{exit}\right], \end{aligned} \tag{9}$$

and

$$\begin{aligned} V_t^-(k_{I,t}, a_{f,t}) &= \max_{a_{f,t+1}} (1+\lambda_t)d_t + \vartheta_t \left[\begin{array}{c} y_t^p - w_t n_t + (1+r_t)a_{f,t} + (1-\delta)\left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right)k_{I,t} - \\ + \left[q_{T,t}\frac{1-\mu}{\mu} + q_{I,t} - \theta^T \frac{q_{T,t+1}}{1+r_{t+1}}\frac{1-\mu}{\mu} - \theta^I \frac{q_{I,t+1}}{1+r_{t+1}}\right] k_{I,t+1} q_{I,t} \end{array} \right] \\ &+ \frac{1}{1+r_{t+1}}\left[(1-\psi)V_{t+1}(k_{I,t}, a_{f,t+1}) + \psi d_{t+1}^{exit}\right], \end{aligned} \tag{10}$$

where d_{t+1}^{exit} is the dividend in case of liquidation and exit from activity:

$$d_t^{exit} = y_t^p + (1+r_t)a_{f,t} + (1-\delta)q_{T,t}\frac{1-\mu}{\mu}k_{I,t} + (1-\delta)q_{I,t}k_{I,t} - w_t, \tag{11}$$

and $V_{t+1}(k_{I,t+1}, a_{f,t+1})$ is the value function conditional on continuation but before the investment shock is realized:

$$V_{t+1}(k_{I,t+1}, a_{f,t+1}) = \eta V^+(k_{I,t+1}, a_{f,t+1}) + (1-\eta)V^-(k_{I,t+1}, a_{f,t+1}) \tag{12}$$

The firm solves (9) or (10), subject to (2), (3) and (6).

We claim that in equilibrium the marginal return on capital for firms in the productive sector is always higher than its user cost:

$$\frac{\partial y_{t+1}^p}{\partial k_{I,t+1}} = \frac{\alpha z_{t+1} n_{t+1}^{(1-\alpha)}}{\mu}\left(\frac{k_{I,t+1}}{\mu}\right)^{\alpha-1} > \left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right) - \frac{(1-\delta)\left(q_{T,t+1}\frac{1-\mu}{\mu} + q_{I,t+1}\right)}{1+r_{t+1}}, \tag{13}$$

and, therefore, that the borrowing constraint (6) is always binding. If this is the case, then the optimal intangible capital is given by:

$$k_{I,t+1} = \frac{y_t^p - w_t n_t + (1+r_t)a_{f,t} + (1-\delta)\left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right)k_{I,t}}{\left(q_{T,t} - \frac{\theta^T q_{T,t+1}}{1+r_{t+1}}\right)\frac{1-\mu}{\mu} + q_{I,t} - \theta^I \frac{q_{I,t+1}}{1+r_{t+1}}}. \tag{14}$$

Moreover condition (13) also implies that a firm that cannot invest will not sell any of its

capital, and for that firm capital depreciates at the rate δ:

$$k_{I,t+1} = (1-\delta)k_{I,t} \tag{15}$$

Regarding the dividend and cash accumulation policy, the first order condition for cash holdings $a_{f,t+1}$ is:

$$(1+\lambda_t) = (1-\psi)\left[\eta(1+\lambda^+_{t+1}+\vartheta_t) + (1-\eta)\left(1+\lambda^-_{t+1}+\vartheta_t\right)\right] + \psi, \tag{16}$$

Substituting (16) recursively forward, it is clear that if the firm expects ϑ_t to be positive now or in the future, then $\lambda_t > 0$, and the firm will always retain all earnings and $d_t = 0$. It is important to note that this is so because there is no cost of holding cash. The general formula for cash holdings for investing and non-investing firms is obtained by substituting $d_t = 0$ in (2):

$$a_{f,t+1} = y^p_t - w_t n_t + (1+r_t)a_{f,t} + (1-\delta)\left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right)k_{I,t} - \left(q_{T,t+1}\frac{1-\mu}{\mu} + q_{I,t+1}\right)k_{I,t+1}, \tag{17}$$

and simplifies to:

$$(a_{f,t+1} \mid \text{not invest}) = y^p_t + (1+r_t)a_{f,t} - w_t n_t \tag{18}$$

for non-investing firms. Investing firms in equilibrium borrow as much as possible, and:

$$(a_{f,t+1} \mid \text{invest}) = -\left(\theta^T \frac{q_{T,t+1}}{1+r_{t+1}}\frac{1-\mu}{\mu} + \theta^I \frac{q_{I,t+1}}{1+r_{t+1}}\right)k_{I,t+1} < 0. \tag{19}$$

Equations (18) and (19) determine the wealth dynamics of firms. A firm that invested in period $t-1$ but is not investing in period t has debt equal to $-a_{f,t} = \left(\theta^T \frac{q_{T,t+1}}{1+r_{t+1}}\frac{1-\mu}{\mu} + \theta^I \frac{q_{I,t+1}}{1+r_{t+1}}\right)k_{I,t+1}$. It uses current profits $y^p_t - w_t n_t$ to pay the interest rate on debt $-r_t a_{f,t}$ and to reduce the debt itself. As long as the firm is not investing, the debt $-a_{f,t}$ decreases until the firm becomes a net saver and has $a_{f,t} > 0$. At this point, wealth accumulation is driven both by profits $y^p_t - w_t n_t$ and by interest on savings $r_t a_{f,t}$, until the firm has an investment opportunity and its accumulated wealth $(1+r_t)a_{f,t}$ is used to purchase capital (see equation (14)). This discussion clarifies that a lower interest rate r_t helps the non-investing firm to repay existing debt, but it slows down the accumulation of savings after the firm has repaid the debt. Later we will refer to these two effects as the *debt hangover channel* and the *savings channel*.

3.2.2 The Unproductive Sector

There is a mass one of identical firms in the unproductive sector. They have production functions which are linear in tangible and intangible capital, taking the form:

$$y_t^u = z_t^{u,I} k_{I,t}^u + z_t^{u,T} k_{T,t}^u.$$

This sector is assumed to be able to finance intangible capital with equity from the household sector and to pay out all profits as dividends to households every period. Their budget constraint is

$$d_t^u = y_t^u - q_{I,t}\left(k_{I,t+1}^u - (1-\delta)k_{I,t}^u\right) - q_{T,t}\left(k_{T,t+1}^u - (1-\delta)k_{T,t}^u\right) \tag{20}$$

Given its linear technology, and provided that its return on capital is lower than in the productive sector, the unproductive sector is willing to absorb all the capital not demanded by the productive sector, at a price equal to its marginal return on capital.

3.2.3 Aggregation of the Firm Sector, and Pricing of Assets

Since all productive firms produce at the optimal capital labor ratio determined by equation (8), and the production function is constant returns to scale, we can aggregate it across firms and substitute aggregate labor supply $N = 1$ to obtain:

$$Y_t^p = z_t \left(\frac{K_{I,t}}{\mu}\right)^{\alpha}. \tag{21}$$

The wage is determined in competitive markets by the marginal return of labor:

$$w_t = (1-\alpha)\, z_t \left(\frac{K_{I,t}}{\mu}\right)^{\alpha}, \tag{22}$$

Aggregate capital is determined as follows. A fraction $(1-\psi)$ of productive firms continues activity and a fraction η of those has an investment opportunity. They have a fraction $(1-\psi)\,\eta$ of total assets in the productive firm sector, and use it to buy capital following equation (14). The $(1-\eta)$ fraction of surviving firms that do not have an investment opportunity continue to hold their depreciated capital. A fraction ψ of productive firms exits, and is replaced by an

equal number of firms with an initial endowment of W_0 and no capital. Therefore:

$$K_{I,t+1} = (1-\psi)\left[\eta\frac{Y_t^p - w_t + (1+r_t)A_{f,t} + (1-\delta)\left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right)K_{I,t}}{\left(q_{T,t} - \frac{\theta^T q_{T,t+1}}{1+r_{t+1}}\right)\frac{1-\mu}{\mu} + q_{I,t} - \theta^I\frac{q_{I,t+1}}{1+r_{t+1}}} + (1-\delta)(1-\eta)K_{I,t}\right] + \psi\eta\frac{W_0}{\left(q_{T,t} - \theta^T\frac{q_{T,t+1}}{1+r_{t+1}}\right)\frac{1-\mu}{\mu} + q_{I,t} - \theta^I\frac{q_{I,t+1}}{1+r_{t+1}}}. \quad (23)$$

We define W_t as total wealth at the beginning of period t:

$$W_t \equiv Y_t^p - w_t + (1+r_t)A_{f,t} + (1-\delta)\left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right)K_{I,t} \quad (24)$$

Rearranging (23), we get:

$$K_{I,t+1}^* = \eta K_{I,t+1}^{INV} + (1-\delta)(1-\psi)(1-\eta)K_{I,t}, \quad (25)$$

where

$$K_{I,t+1}^{INV} = \frac{(1-\psi)W_t + \psi W_0}{\left(q_{T,t} - \theta^T\frac{q_{T,t+1}}{1+r_{t+1}}\right)\frac{1-\mu}{\mu} + q_{I,t} - \theta^I\frac{q_{I,t+1}^I}{1+r_{t+1}}} \quad (26)$$

is total intangible capital in the hands of investing agents at the end of period t, expressed in aggregate terms. Aggregate tangible capital of the productive sector is equal to:

$$K_{T,t+1}^* = \frac{1-\mu}{\mu}K_{I,t+1}^* \quad (27)$$

Furthermore we can aggregate firms in the unproductive sector and obtain:

$$Y_t^u = z_t^{u,I}\left(\overline{K}^I - K_{I,t}\right) + z_t^{u,T}\left(\overline{K}^T - K_{T,t}\right).$$

The marginal return of capital in the productive sector is as follows. In order obtain a marginal increase $\frac{\partial Y_t^p}{\partial K_{I,t}} = \frac{\alpha}{\mu}z_t\left(\frac{K_{I,t}}{\mu}\right)^{\alpha-1}$, the productive sector purchases one unit of intangible capital and $\frac{1-\mu}{\mu}$ units of tangible capital. The return of this investment in the unproductive sector is $z_t^{u,I} + \frac{1-\mu}{\mu}z_t^{u,T}$. The equilibrium described above requires that the productive sector have the highest return on capital:

$$\frac{\alpha}{\mu}z_t\left(\frac{K_{I,t+1}^*}{\mu}\right)^{\alpha-1} > z_t^{u,I} + \frac{1-\mu}{\mu}z_t^{u,T} \quad (28)$$

If condition 28 is satisfied, then it follows immediately that the prices of capital are:

$$q_{I,t} = z_t^{u,I} + \frac{1-\delta}{1+r_{t+j}} q_{I,t+1} \tag{29}$$

and

$$q_{T,t} = z_t^{u,T} + \frac{1-\delta}{1+r_{t+j}} q_{T,t+1}, \tag{30}$$

By substituting (29) and (30) into (28), it follows that:

$$\frac{\alpha}{\mu} z_t \left(\frac{K^*_{I,t+1}}{\mu} \right)^{\alpha-1} > q_{I,t} - \frac{1-\delta}{1+r_{t+j}} q_{I,t+1} + \frac{1-\mu}{\mu} \left(q_{T,t} - \frac{1-\delta}{1+r_{t+j}} q_{T,t+1} \right), \tag{31}$$

which implies that the claim (13) is correct.

To compute aggregate financial assets of the productive sector $A_{f,t+1}$, we take into account that, among the fraction $1-\psi$ of continuing firms, a fraction $1-\eta$ simply accumulates savings, while a fraction η borrows up to the maximum to invest. Among the fraction ψ of new firms, a fraction η borrows up to the maximum, while the rest save their initial endowment W_0 :

$$\begin{aligned} A_{f,t+1} &= (1-\psi)\left[(1-\eta)\left(Y_t^p + (1+r_t)A_{f,t} - w_t\right)\right] + \psi\,(1-\eta)\,W_0 \\ &\quad -\eta \left(\theta^T \frac{q_{T,t+1}}{1+r_{t+1}} \frac{1-\mu}{\mu} + \theta^I \frac{q_{I,t+1}}{1+r_{t+1}} \right) K^{INV}_{I,t+1}. \end{aligned} \tag{32}$$

At the aggregate level total investment $\left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right)\left(K_{I,t+1} - (1-\psi)\,K_{I,t}\right)$ is also equal to total resources available to invest:

$$\begin{aligned} \left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right)\left(K_{I,t+1} - (1-\delta)\,(1-\psi)\,K_{I,t}\right) &= \left[(1-\psi)\,\eta\left(Y_t^p - w_t + (1+r_t)A_{f,t}\right) + \psi\eta W_0\right] \\ &\quad +\eta \left(\theta^T \frac{q_{T,t+1}}{1+r_{t+1}} \frac{1-\mu}{\mu} + \theta^I \frac{q_{I,t+1}}{1+r_{t+1}} \right) K^{INV}_{I,t+1}. \end{aligned} \tag{33}$$

Substituting (33) into (32) we obtain:

$$A_{f,t+1} = (1-\psi)\left(Y_t^p - w_t + (1+r_t)A_{f,t}\right) + \psi W_0 - \left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right)\left(K_{I,t+1} - (1-\delta)\,(1-\psi)\,K_{I,t}\right), \tag{34}$$

Finally, total dividends paid out by exiting productive firms to households are equal to:

$$D_t^p = \psi \left(Y_t^p - w_t + (1+r_t)A_{f,t} + \left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right) K_{I,t} \right) - \psi W_0, \tag{35}$$

and the dividends paid by the unproductive sectors are:

$$D_t^u = Y_t^u - q_{I,t}\left[\left(\overline{K}^I - K_{I,t+1}\right) - \left(\overline{K}^I - K_{I,t}\right)\right] - q_{T,t}\left[\left(\overline{K}^T - K_{T,t+1}\right) - \left(\overline{K}^T - K_{T,t}\right)\right], \quad (36)$$

3.3 Households

We consider a life-cycle with two types of households, young and old, with measures H^y and H^o, respectively, whose sum is normalized to 1. Young households supply one unit of labor inelastically in exchange for a wage w, and receive a fraction γ of the aggregate dividends. Households remain young for N periods, and become old after $N+1$ periods, so that there is a constant fraction $\phi = \frac{1}{N}$ of young households for every age between 1 and N, and every period a measure ϕH^y of households becomes old. Old households cannot work, receive a fraction $(1-\gamma)$ of aggregate dividends, and die with probability ϱ. The measure of old households H^o is determined as follows:

$$H^o = (1-\varrho)H^o + \phi H^y, \quad (37)$$

while the measure of young households is:

$$H^y = (1-\phi)H^y + N^y, \quad (38)$$

where N^y is the constant measure of newborn households. From the assumption that $H_t^o + H_t^y = 1$ follows that $N^y = \frac{\phi\varrho}{\phi+\varrho}$, $H_t^o = \frac{\phi}{\phi+\varrho}$, and $H_t^y = \frac{\varrho}{\phi+\varrho}$.

We follow Blanchard (1985) and Yaari (1965) in assuming that households participate in a life insurance scheme when old. The insurance scheme works within a cohort, so that the survivors within a cohort pay the debt of the dying (if they are in debt), or alternatively receive the savings of the dying. An old household begins a period with net debt $(1+r_t)b_t^o$. The insurance contract specifies that the ϱ fraction of old households that die transfer their assets (or debt) $(1+r_t)b_t^o$ to the life insurer. Among the fraction $(1-\varrho)$ of households that survive, if they are net savers ($b_t^o < 0$) then they receive a return $\frac{1}{1-\varrho}(1+r_t)b_t^o$ on their assets, while if they are net debtors ($b_t^o > 0$), they make a payment of $\frac{1}{1-\varrho}(1+r_t)b_t^o$ to the life insurer.

Households have log utility. A representative old household still living at time t maximizes

the following objective function:

$$V_t^o\left(b_t^o\right) = \max_{c_t^o, b_{t+1}^o} \sum_{j=0}^{\infty} (1-\varrho)^j \beta^j \log\left(c_{t+j}\right) \tag{39}$$

subject to

$$c_t^o = b_{t+1}^o + (1-\gamma)d_t - \frac{(1+r)}{(1-\varrho)} b_t^o. \tag{40}$$

Working backwards, we next consider the optimization problem of a young agent of age N in period t, who will become old in period $t+1$[2]:

$$V_{t,N}^y\left(b_{t,N}^y\right) = \max_{c_{t,N}^y, b_{t+1}^o} u\left(c_{t,N}^y\right) + \beta(1-\varrho) V_{t+1}^o\left(b_{t+1}^o\right) \tag{41}$$

subject to

$$c_{t,N}^y = \gamma d + w - (1+r) b_{t,N}^y + b_{t+1}^o. \tag{42}$$

Then we consider the optimization problem for a young household of age $j < N$:

$$V_{t,j}^y\left(b_{t,j}^y\right) = \max_{c_{t,j}^y, b_{t+1,j+1}^y} u\left(c_{t,j}^y\right) + \beta V_{t+1,j+1}^y\left(b_{t+1,j+1}^y\right) \tag{43}$$

subject to

$$c_{t,j}^y = \gamma d + w - (1+r) b_{t,j}^y + b_{t+1,j+1}^y \tag{44}$$

We postpone the discussion of the solution of households' optimization problem and their aggregation to the steady state analysis section, given that all our exercises explore the steady state of this economy. The problem is significantly simplified by considering constant aggregate variables and prices.

4 Steady State Equilibrium

We consider a steady state equilibrium and we drop the subscript t. We can compute aggregate household borrowing as:

$$B = B^o + B^y, \tag{45}$$

[2] We assume that an agent can also die with probability ϱ in the transition between young and old.

where savings of the old B^o is:

$$B^o = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\left[\left(\frac{A}{\beta+r\beta-1}+b^{retirement}\right)\frac{(1-\varrho)}{1-(1-\varrho)\beta(1+r)}-\frac{A}{\beta+r\beta-1}\frac{1-\varrho}{\varrho}\right], \quad (46a)$$

with

$$\begin{aligned} A &\equiv \left(\frac{(1-\varrho)\,(1-(1+r)\beta)}{(\varrho+r)}\right)(1-\gamma)d, \\ b^{retirement} &\equiv \frac{\Psi\frac{(1-\gamma)d}{(\varrho+r)}-(\gamma d+w)\,\frac{1}{r}\left[(1+r)^N-1\right]}{\frac{\Psi}{1-\varrho}+1}, \\ \Psi &\equiv \left(\frac{1}{\beta}-(1-\varrho)\right)\frac{\beta^N-1}{\beta^{N-1}\,(\beta-1)}. \end{aligned}$$

Savings of the young B^y is:

$$B^y = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\left[A_1\frac{\gamma d+w}{r}-A_2c_N+A_3b^{retirement}\right]$$

where

$$\begin{aligned} A_1 &\equiv \frac{1}{r}\left(\frac{1}{(1+r)^{N-1}}+r(N-1)-1\right), \\ A_2 &\equiv \frac{\beta}{\beta-1}\left[\frac{1-(1+r)^{-N+1}}{(1+r)-1}-\frac{1-[(1+r)\,\beta]^{-N+1}}{(1+r)\,\beta-1}\right], \\ A_3 &\equiv \frac{1-(1+r)^{-N+1}}{(1+r)-1}, \\ c_N &= \left(\frac{1}{\beta}-(1-\varrho)\right)\left(\frac{(1-\gamma)d}{(\varrho+r)}-\frac{b^{retirement}}{(1-\varrho)}\right). \end{aligned}$$

We describe in detail the solution of the household sector, and the derivation of B^o and B^y, in the Appendix A.

Dividends d are given by:

$$d = D^p + D^u + D^k,$$

where

$$D^u = Y^u - q_I\delta\left(\overline{K}^I - K_I\right) - q_T\delta\left(\overline{K}^T - K_T\right)$$

$$D^p = \psi\left(\alpha z\left(\frac{K_I}{\mu}\right)^{\alpha}+(1+r)A_f+\left(q_T\frac{1-\mu}{\mu}+q_I\right)K_I\right)-\psi W_0,$$

$$D^k = \frac{q_{T,t}^{\frac{\varphi}{\varphi-1}}}{b^{\frac{1}{\varphi-1}}}(\varphi-1) + \frac{q_{I,t}^{\frac{\varphi}{\varphi-1}}}{b^{\frac{1}{\varphi-1}}}(\varphi-1)$$

Aggregate cash holdings of the productive sector in the steady state can be obtained by combining (34), (21) and (22) to obtain:

$$A_f = \frac{(1-\psi)\,\alpha z_t \left(\frac{K_I}{\mu}\right)^{\alpha} + \psi W_0 - \left(q_T \frac{1-\mu}{\mu} + q_I\right)\left[\psi + \boldsymbol{\delta}(1-\psi)\right] K_I}{\left[1-(1-\psi)\,(1+r)\right]} \tag{47}$$

Aggregate borrowing is equal to aggregate savings, or

$$A_f = B, \tag{48}$$

and by Walras' Law, the aggregate resource constraint is satisfied. In order to determine the aggregate capital of the productive sector, equation (25) in the steady state is equal to:

$$K_I = \eta \frac{(1-\psi)W + \psi W_0}{\left[q_T\left(1-\frac{\theta^T}{1+r}\right)\frac{1-\mu}{\mu} + q_I\left(1-\frac{\theta^I}{1+r}\right)\right]\left[1-(1-\boldsymbol{\delta})(1-\psi)\,(1-\eta)\right]} \tag{49}$$

where W is defined using equation (24) in steady state:

$$W \equiv \alpha z_t \left(\frac{K_I}{\mu}\right)^{\alpha} + (1+r)A_f + (1-\boldsymbol{\delta})\left(q_T \frac{1-\mu}{\mu} + q_I\right) K_I \tag{50}$$

We can also express (49) as

$$K_I = \frac{\eta(1-\psi)\left(\alpha z_t \left(\frac{K_I}{\mu}\right)^{\alpha} + (1+r)A_f\right) + \eta\psi W_0}{\left[q_T\left(1-\frac{\theta^T}{1+r}\right)\frac{1-\mu}{\mu} + q_I\left(1-\frac{\theta^I}{1+r}\right)\right]\left[\boldsymbol{\delta} + \psi\,(1-\boldsymbol{\delta})\right] - \left(q_T\frac{\theta^T}{1+r}\frac{1-\mu}{\mu} + q_I\frac{\theta^I}{1+r}\right)\eta(1-\boldsymbol{\delta})(1-\psi)}, \tag{51}$$

which has an intuitive explanation. The numerator is the aggregate amount of liquid resources of investing firms. The denominator is the downpayment necessary to support one unit of capital in the steady state. It requires the replacement of the depreciated capital and the lost capital of exiting firms (a fraction $\delta + \psi\,(1-\boldsymbol{\delta})$), and can benefit from using existing capital held by the investing firms as collateral (fraction $\eta(1-\delta)(1-\psi)$).

Finally, the prices of capital are determined by recursively iterating forward equations (29) and (30):

$$q_I = \frac{1}{r+\boldsymbol{\delta}} z^{u,I} \tag{52}$$

$$q_T = \frac{1}{r+\boldsymbol{\delta}} z^{u,T} \tag{53}$$

Investment is equal to:

$$I_t = \varphi \left(\frac{q_T}{b}\right)^{\frac{1}{\varphi-1}}$$

$$I_t^I = \varphi \left(\frac{q_I}{b}\right)^{\frac{1}{\varphi-1}}$$

And aggregate capital is equal to:

$$\overline{K}^T = \frac{I^T}{\boldsymbol{\delta}}$$

$$\overline{K}_t^I = \frac{I^I}{\boldsymbol{\delta}}$$

The steady state values of W, A_f, B, K_I, q_I, q_T, and r are jointly determined by equations (86), (47), (48), (49), (50), (52), and (53).

4.1 Characterization of the Equilibrium

To illustrate the main properties of the model, we now develop some important features of the equilibrium that can be characterized analytically. The analysis clarifies the main mechanisms through which interest rates interact with the degree of reliance on intangible capital to affect the allocation of capital and aggregate output.

4.1.1 Partial Equilibrium Analysis of Firm Investment Policies

In this section, we analyze firms' investment policies to provide a greater understanding of the *collateral value channel*, the *capital purchase price channel,* the *debt overhang channel*, and the *savings channel*, the four channels through which interest rate variations interact with firm financial constraints to affect capital reallocation. To provide clearer results, we take the interest rate r and capital prices q_T and q_I as given, assume $\alpha = 1$, so that the production function is linear in capital, and assume that total capital is fixed, so that $\delta = 0$. Combining equations (47), (49), and (50), we obtain the following expression for the total amount of steady state intangible capital K_I held by the productive firms:

$$K_I = \frac{\eta(1-\psi)\left[z\left(\frac{K_I}{\mu}\right) + (1+r)A_f\right]}{\psi\left[q_T\frac{1-\mu}{\mu}\left(1-\frac{\theta^T}{1+r}\right) + q_I\left(1-\frac{\theta^I}{1+r}\right)\right] - \left(q_T\frac{\theta^T}{1+r}\frac{1-\mu}{\mu} + q_I\frac{\theta^I}{1+r}\right)\eta\,(1-\psi)}, \tag{54}$$

where

$$A_f = \eta A_I + (1-\eta)\, A_S, \tag{55}$$

and

$$A_I = \left[(1-\psi)\, z\left(\frac{K_I}{\mu}\right) + \psi W_0 + A_f\,(1-\psi)\,(1+r)\right] - \left(q_T \frac{1-\mu}{\mu} + q_I\right)\psi K_I, \tag{56}$$

$$A_S = \left[(1-\psi)\, z\left(\frac{K_I}{\mu}\right) + \psi W_0 + A_f\,(1-\psi)\,(1+r)\right]. \tag{57}$$

The interpretation of expression (54) is as follows. The productive sector can hold in equilibrium an amount of intangible capital equal to its total wealth available to invest divided by the downpayment required to sustain one unit of intangible capital. The downpayment required to sustain one unit of K_I is equal to the fraction of K_I that needs to be replaced every period, which is the fraction ψ of capital lost to exiting firms, times the price of capital $(q_T \frac{1-\mu}{\mu} + q_I)$ minus the amount that can be borrowed. The amount that can be borrowed can be separated into two parts. One, is the debt collateralized by the new capital purchased, and is equal to $(q_T \frac{1-\mu}{\mu} \frac{\theta^T}{1+r} + q_I \frac{\theta^I}{1+r})$. The other is the debt collateralized by the existing stock of capital, and is equal to $\left(q_T \frac{\theta^T}{1+r} \frac{1-\mu}{\mu} + q_I \frac{\theta^I}{1+r}\right) \eta\,(1-\psi)$.

We can identify the four channels clearly in (54). The debt overhang and savings channels are captured by the term $(1+r)A_f$ in the numerator. If the productive firm sector is a net saver, then $A_f > 0$, and increases in the interest rate r increase the speed of accumulation of savings and help capital reallocation. This is the savings channel. Net savings are composed of the net savings A_I of the η fraction of firms that where investors in the previous period, and net savings A_S of the fraction $(1-\eta)$ that were not. Conversely, if $A_f < 0$, increases in the interest rate r lower the speed of accumulation of savings and hurt capital reallocation. This is the debt overhang channel. An inspection of expressions (54), (56), and (57) shows that the strength of both channels is negatively related to the probability of firm exit ψ, and disappear when $\psi = 1$. The intuition for this is that when $\psi = 1$ firms only live for one period and are unable to save or to carry over debt from previous periods.

The capital purchase price channel is captured by the term $\psi\left[q_T \frac{1-\mu}{\mu}\left(1 - \frac{\theta^T}{1+r}\right) + q_I\left(1 - \frac{\theta^I}{1+r}\right)\right]$ in the denominator. Lower interest rates that increase the price of tangible and intangible assets reduce the amount of capital firms can purchase for a given amount of net worth and borrowing

capacity. The collateral value channel has two components. First, a lower interest rate increases the present value of the collateral pledged next period, and reduces the size of the downpayment. Second, the increase in capital prices brought about by the low interest rate increases the value of the existing capital held by productive investing firms, which can also be used as collateral for the purchase of new capital. This is captured in the term $\left(q_T \frac{\theta^T}{1+r} \frac{1-\mu}{\mu} + q_I \frac{\theta^I}{1+r}\right) \eta (1-\psi)$.

To isolate and gain a further understanding of the *capital purchase price* and *collateral value* channels, we consider the case in which $\psi = 1$ and $\eta = 1$. This is a situation in which firms only live for one period, and all firms can invest, so the *debt overhang* and the *savings* channels are mute. Furthermore we assume that $\theta^I = 0$. A closed form solution for K_I can be obtained, which is:

$$K_I = \frac{W_0}{q_T \frac{1-\mu}{\mu} + q_I - q_T \frac{\theta^T}{1+r} \frac{1-\mu}{\mu}}. \tag{58}$$

The *collateral value channel* and the *capital purchase price channel* operate in opposite directions, so to analyze which dominates we study the sensitivity of K_I to variations in r using expression (58):

$$\frac{\partial K_I}{\partial r} = \frac{W_0 \left(- \left(\frac{\partial q_T}{\partial r} \frac{1-\mu}{\mu} + \frac{\partial q_I}{\partial r} \right) + \frac{\partial q_T}{\partial r} \frac{\theta^T}{1+r} \frac{1-\mu}{\mu} + q_T \frac{\partial \left(\frac{\theta^T}{1+r} \right)}{\partial r} \frac{1-\mu}{\mu} \right)}{\left(\left(q_T \frac{1-\mu}{\mu} + q_I \right) - q_T \frac{\theta^T}{1+r} \frac{1-\mu}{\mu} \right)^2},$$

where then sign of this derivative is given by:

$$sign \left[- \left(\frac{\partial q_T}{\partial r} \frac{1-\mu}{\mu} + \frac{\partial q_I}{\partial r} \right) + \frac{\partial q_T}{\partial r} \frac{\theta^T}{1+r} \frac{1-\mu}{\mu} + q_T \frac{\partial \left(\frac{\theta^T}{1+r} \right)}{\partial r} \frac{1-\mu}{\mu} \right] \overset{?}{\lessgtr} 0. \tag{59}$$

The first term in brackets captures the capital price channel, and is positive, given that $\partial q_T / \partial r$ and $\partial q_I / \partial r$ are both negative. The second and third terms capture the collateral value channel, and are both negative. The term inside the brackets in (59) increases in μ, meaning that the effect of higher interest rates is more likely to be expansionary when the share of intangibles is high. As μ approaches 1, the collateral value channel disappears, and the capital price channel remains. A similar effect would obtain if we decrease the ability to collateralize tangible capital, captured by θ^T.

4.1.2 Characterization of the Equilibrium

To provide a deeper understanding of how the features of the equilibrium of the economy described in Section 3 change as a result of a transition from an economy reliant on tangible capital to one in which intangible capital acquires a larger importance, we represent the equilibrium in the credit market in Figure 1. In the graph, upward sloping savings curve captures net savings of the household sector and net savings of the non-investing firm sector. Higher interest rates induce households to save more, while the savings of the non-investing firms are not sensitive to interest rates, as they simply save as much as possible. The demand for capital by the investing firms is equal to the amount borrowed by them plus (minus) the savings (debt) they carry over from the previous period. This curve can be upward or downward sloping depending on the relevance of intangible capital in the production function.

The left panel in the figure represents a "tangibles" economy with a low μ. In such an economy, an increase in aggregate savings has the effect of lowering interest rates and increasing capital purchases from expanding firms. The collateral value channel and the debt overhang channels dominate. As a result, a larger share of the capital stock is in the hands of the productive sector, and output increases. The right panel considers the case of an "intangibles" economy with a high μ. The demand for capital curve is upward sloping when an economy has a high reliance on intangible capital due to the strength of the capital price and savings channel. As interest rates rise, firms demand more capital because they have larger savings and because the price of capital is lower. In this case, an outwards shift in the savings schedule generates a decrease in equilibrium capital purchases, because the decrease in interest rates it generates hurts reallocation of capital towards productive firms.

5 Calibration

For the purpose of evaluating the qualitative and quantitative importance of the channels explained above for the real economy, we calibrate the model on US data. Our benchmark calibration, illustrated in Table 1, is meant to capture the US economy during the period immediately preceding 1980, with a small share of intangible capital and high real interest rates. In this respect we follow Falato et al.(2013) in setting $\mu = 0.2$, so that the share of intangible capital over total capital is 20%. We set the share of dividends that are paid to the working

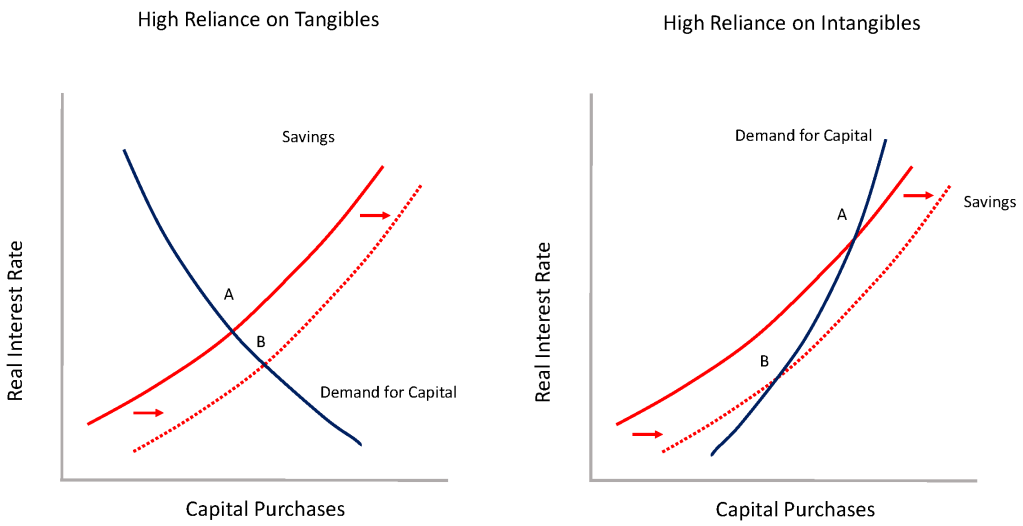


Figure 1: Equilibrium in the Credit Market: Demand for Capital and Supply of Savings in an Economy with a High Reliance on Tangible Capital (low μ) and an Economy with a High Reliance on Intangible Capital (high μ).

age population, γ, so that we obtain a real interest rate $r = 6\%$, and the elasticity of output with respect to capital α equal to 0.4.

The remaining parameters are set as follows. The pledgeability parameters of tangible capital θ^T and intangible capital θ^I directly affect the leverage of the firm and therefore determine the strength of the collateral channel. We set θ^T and θ^I equal to 0.95 and 0.1 respectively, broadly in line with Falato et al (2013). We allow for intangible capital to be partially collateralizable to capture the fact that in reality firms have other forms of external financing beyond collateralized debt, and that these sources are likely proportional to the value of firms assets.

The productivity variable z_t is modeled as follows:

$$z_t = [1 + (\mu - 0.2)\kappa]\, z,$$

where the parameter κ determines the way in which the productivity variable z_t depends on μ. For the benchmark value of $\mu = 0.2$ it follows that $z_t = z$, and a shift to a total reliance on intangible capital (μ increases from 0.2 to 1) would raise z_t by 16%. This is set as a large value to be consistent with the notion of the transition to intangible capital as a privately optimal choice of firms, and to be able to make conservative and robust statements about

Table 1: Benchmark Calibration - Parameter Choices

Parameter	**Symbol**	**Value**
Discount factor	β	0.95
Capital (tangible + intangible) share	α	0.33
Intangible share of total capital	μ	0.2
Unproductive sector tangible productivity	$z_t^{u,T}$	1
Unproductive sector intangible productivity	$z_t^{u,I}$	1
Size of young population	H^y	0.667
Size of old population	H^o	0.333
Years households remain young	N	40
Probability of death of old households	ϱ	0.05
Productivity parameter	z	40
Collateral value of tangible capital	θ^T	0.99
Collateral value of intangible capital	θ^I	0.1
Probability of an investment opportunity	η	0.1
Additional productivity of intangible capital	κ	0.2
Adjustment cost convexity parameter	φ	25
Adjustment cost linear parameter	b	10
Exit probability of productive firms	ψ	0.15
Endowment of new firms	W_0	5
Depreciation of capital	δ	0.1
Share of dividends to old households	$1-\gamma$	38%

the potential for negative effects of the shift to intangibles. The parameter z determines in equilibrium a productivity differential between productive and unproductive sectors equal to 40%. This relatively large value is low compared to the empirical evidence from the productivity literature. Syverson (2004) finds that, on average, in a 4 digit US manufacturing sector, a plant at the 90th percentile of productivity makes twice the output, with the same inputs, than a plant at the 10th percentile. These differences become even larger when comparing firms in less narrowly defined sectors.

The parameter φ of the capital-producing sector is equal to 25, and matches average capital adjustment costs equal to 5% of total investment, in line with the estimates of Abel and Eberly (2002) and Cooper and Haltiwanger (2006). Conditional on φ, the other parameter of the capital production function b determines the total supply of capital, and we calibrate it in order to have that the productive sector produce one-third of output in the economy.

The probability of having an investment opportunity η is set equal to 10%. This value is consistent with the empirical evidence from the lumpy investment literature. The probability of

an exit, ψ, is equal to 10%. This relatively high value is necessary in order to generate sufficiently high levels of dividend payouts from the corporate sector to the household sector, given that in the model only exiting firms distribute dividends. The initial endowment of newborn firms W_0 is equal to 5, which corresponds to 2% of average firm output. The depreciation factor δ is set equal to 10%, which is relatively high for tangible capital but probably appropriate for intangible capital. The value of the parameters $z^{u,T}$ and $z^{u,I}$ are normalized to 1.

6 Exogenous Decrease in Interest Rates

In our first exercise, we implement an *exogenous* fall in real interest rates similar to the one observed in the United States from the 1970s until the present. This exercise is relevant because recent empirical studies demonstrate that demand side factors such as demographic forces, higher inequality within countries, and a preference shift towards higher saving by emerging market governments, have been responsible for a large increase in savings and the declining trend in real interest rates over the last three decades (Rachel and Smith, 2015). We compare how such an exogenous fall in rates impacts an economy in which capital is entirely of a tangible nature, with one in which capital is entirely of an intangible nature. Furthermore, we assume that tangible capital is close to fully collateralizable, that intangible capital cannot attract any external finance, and that the aggregate stock of capital is fixed. This very stylized exercise is useful to highlight the main mechanisms of this paper. In the next section, we endogenize interest rates in an empirically realistic setting and allow for an endogenous aggregate stock of capital.

The results for a tangibles economy are displayed in Figure 2, in which we present the equilibrium values of selected aggregate variables in response to the exogenous decrease in interest rate from 6% to close to 0%, for three reasons that can be described intuitively in Figure 3, where we decompose the impact of each of the three channels. First, the decrease in interest rates relaxes the borrowing constraint of productive firms and increases their ability to borrow (the borrowing/collateral value channel). To study this channel we keep capital prices constant and also keep the interest rate on firms' debt constant. The interest rate is only allowed to affect the collateral value in the borrowing constraint. The "borrowing channel" line in figure 3 shows an increase in corporate leverage and a fall in Tobin's Q, which is a proxy for

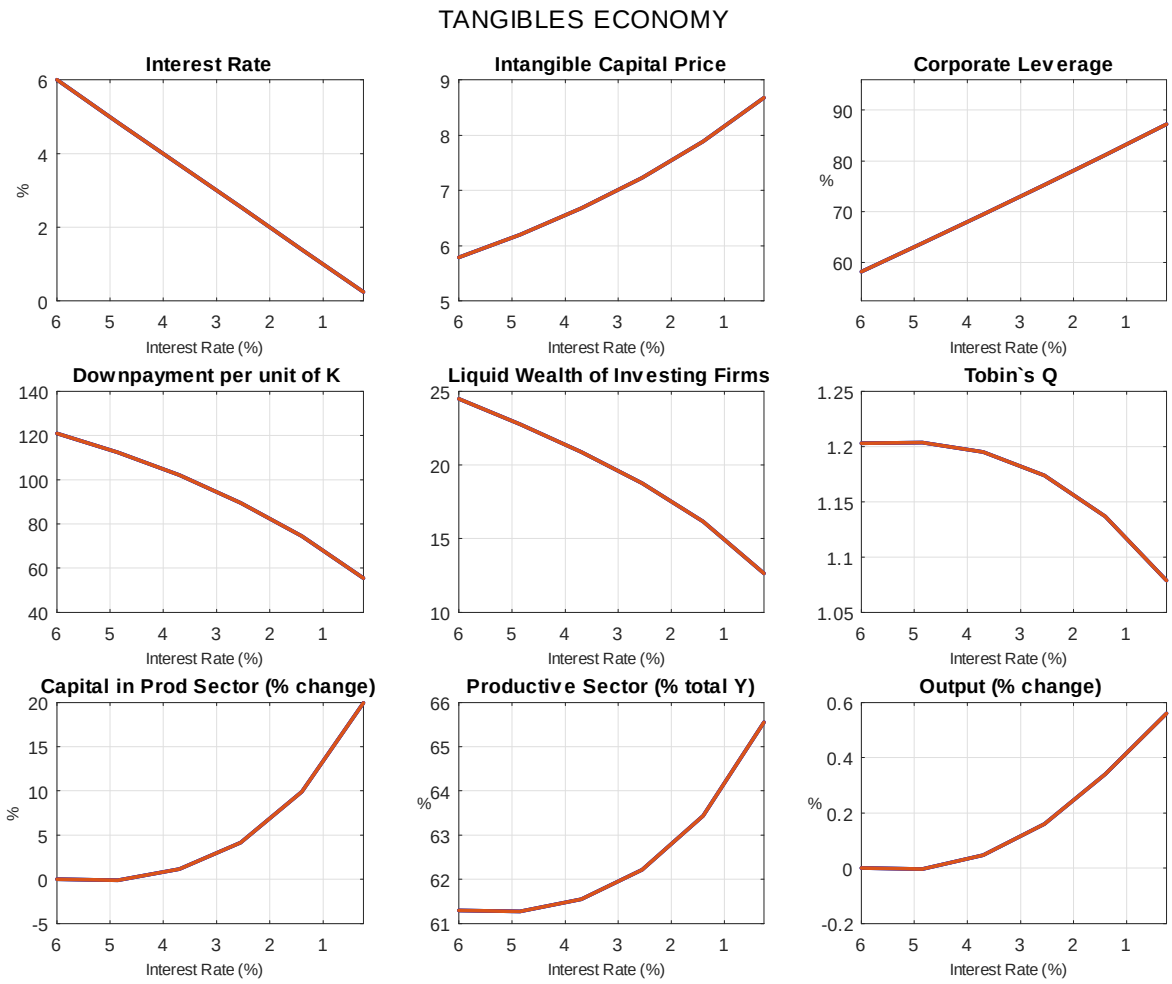


Figure 2: Equilibrium response of aggregate variables to an exogenous decrease in interest rates - tangibles economy.

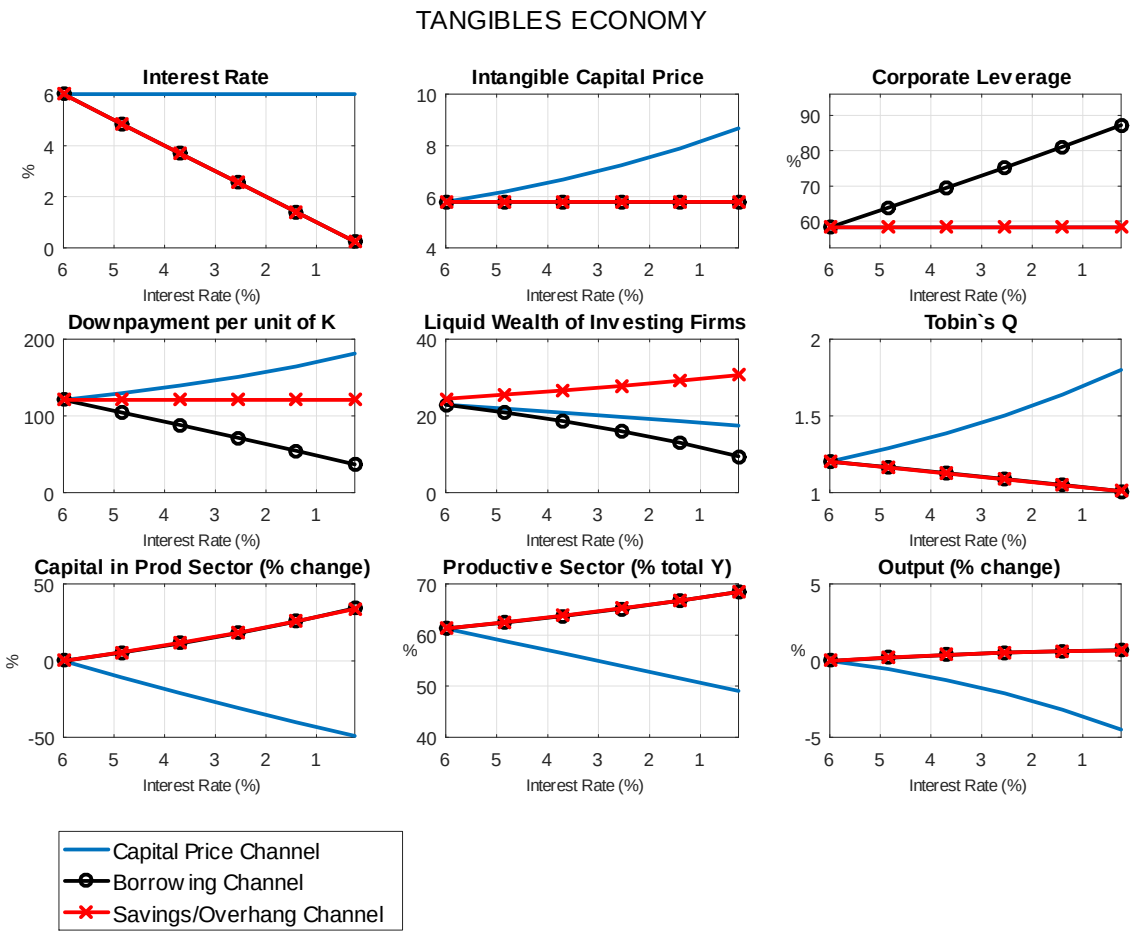


Figure 3: Equilibrium response of aggregate variables to an exogenous decrease in interest rates - tangibles economy.

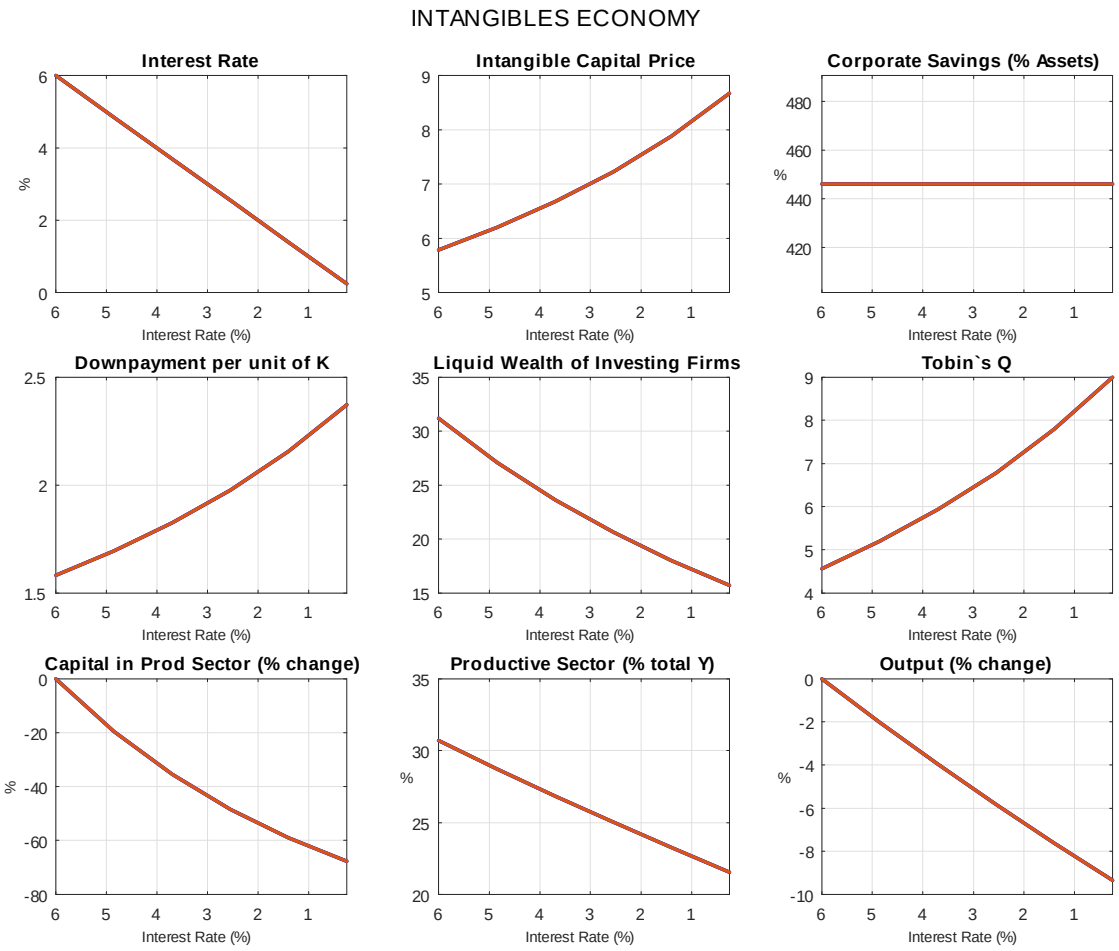


Figure 4: Equilibrium response of aggregate variables to an exogenous decrease in interest rates - intangibles economy.

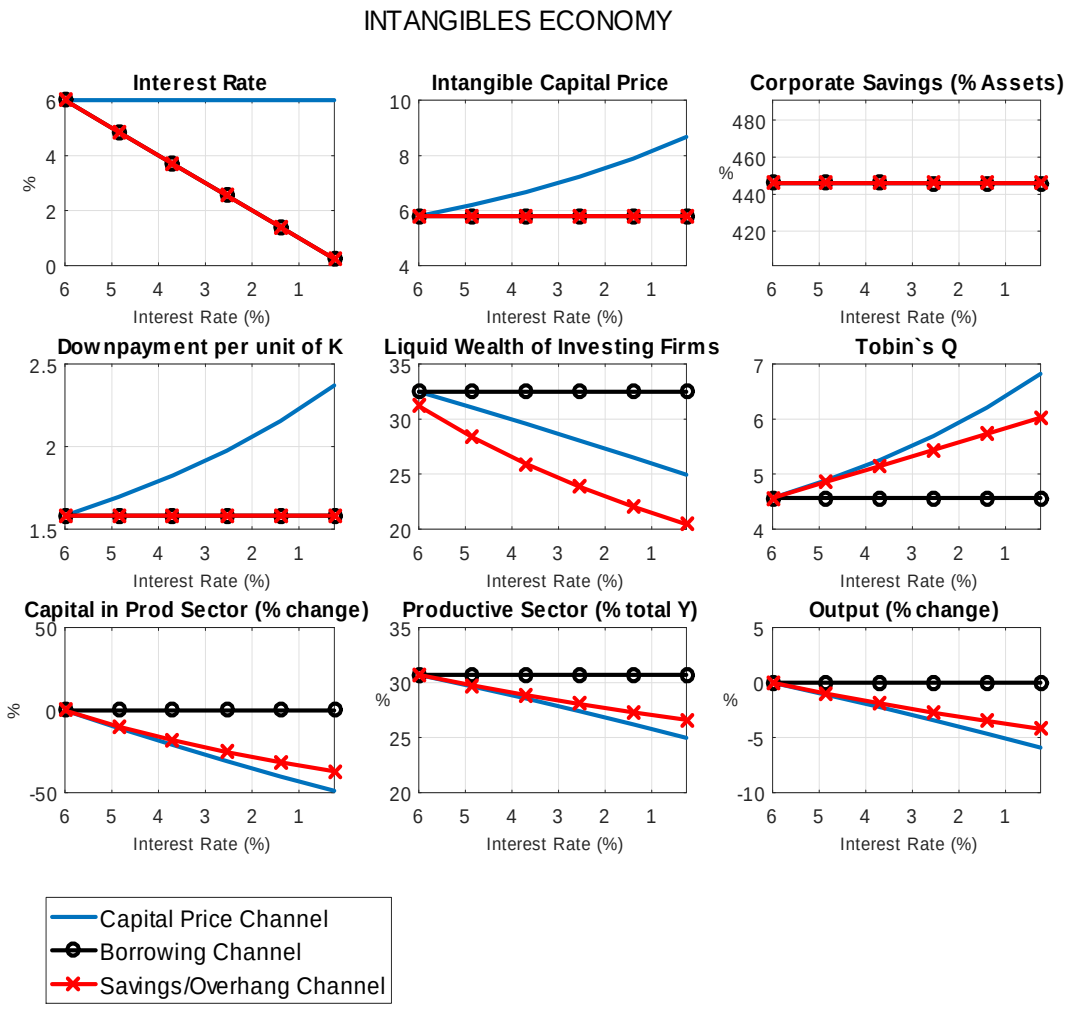


Figure 5: Equilibrium response of aggregate variables to an exogenous decrease in interest rates - intangibles economy.

the tightness of financial constraints. The amount of capital allocated to the productive sector increases and the share of output produced by the productive sector increases. Second, the lower rates enable firms to pay down their debt burden more quickly in non-investing periods (the debt overhang channel). We capture this by fixing capital prices, not allowing interest rates to affect the borrowing constraint, but allowing interest rate variations to affect the interest expense of indebted firms, and the return on savings of saving firms. The amount of capital allocated to the productive sector increases and the share of output produced by the productive sector increases. Third, the lower interest rate increases asset prices and increases the downpayment required to purchase capital, reducing the equilibrium amount of capital the productive sector can absorb. We capture this by allowing interest rate changes to affect capital prices, but not allowing them to influence equilibrium quantities in any other way. The first two channels are expansionary, while the third channel is contractionary, and the former two dominate. The bottom right panel of Figure 2 shows that output increases by close to 0.6% when interest rates decrease from 6% to 0%.

We turn now to the analysis of an intangibles economy, in Figures 4 and 5, in which $\mu = 0.999$ and firms rely almost entirely on intangible capital. The fall in interest rates is strongly contractionary in this economy, for two reasons. First, the decrease in interest rates increases asset prices and makes the purchase of intangible capital, all of which needs to be financed with retained earnings, more expensive (the capital purchase price channel). Second, the lower interest rates decrease the ability of productive firms to accumulate retained earnings (the firm savings channel). The capital allocated to the productive sector falls by more than 60%, and the share of output produced by the productive firms falls from 32% to 24%. Tobin's Q increases sharply, indicating a significant tightening of borrowing constraints, and output falls by 9.4%. Figure 5 shows that both the capital purchase price channel and the savings channel have individually large negative effects on the allocation of capital to the productive sector and on aggregate output.

7 Endogenous Evolution of Interest Rates

In this section, we introduce comparative static exercises that capture how parallel developments in the household and the corporate sector have interacted to generate aggregate patterns

consistent with the secular stagnation hypothesis. In the household sector, we model a progressive increase in individuals' life expectancy, and a progressive decrease in their rate of time preference. Both developments are meant to capture demographic forces that have acted to increase household sector net savings between the 1970s and the present, which puts downward pressure on the equilibrium interest rate. We interpret our exercise as a shortcut for a collection of different factors, such as population aging, wealth and income inequality, and foreign sector developments. In the corporate sector, we introduce a gradual shift in the reliance on intangible capital of firms, from the pre-1980 economy, in which intangible capital accounted only for 20% of aggregate capital ($\mu = 0.2$), to the post-2010 value of 60% of total capital ($\mu = 0.6$) (Corrado and Hulten (2010a), Falato et al (2013), Döttling and Perotti (2015)).

We first introduce the household sector developments in isolation, in an economy in which intangible capital intensity is as in the 1970s ($\mu = 0.2$). The results are in Figure 6. The increase in household net savings puts downward pressure in interest rates and encourages capital creation. Even though it increases productive firms' ability to borrow and pay down their debt, capital allocation is negatively affected by an increase in capital prices through the capital purchase price channel. The net effect is expansionary, and output increases about 0.7%.

In an economy in which we implement a shift to intangible capital, but do not consider any developments in the household sector (Figure 7), firms' increased reliance in a type of capital which attracts less external finance, decreases corporate leverage and tightens firms' borrowing constraints significantly, as evidence by a large increase in Tobin's Q. In fact, firms switch from being net borrowers to being net lenders, consistent with evidence in the U.S. for publicly listed firms (Bates, Khale and Stulz (2009)). The share of output produced by the productive sector drops significantly. The lower corporate borrowing puts downward pressure on interest rates, which amplifies the misallocation of capital through a capital purchase price channel and a savings channel. These negative effects are initially dominated by the higher productivity of intangible capital, but as firms' switch to being net savers intensifies, the negative effects become increasingly important. On net, the effects collectively are mildly expansionary.

When we analyze the household and corporate sector developments jointly, we find that the combined effect of both events is contractionary, despite them being expansionary in isolation. The results are in Figure 8. The decrease in rates caused by household developments has

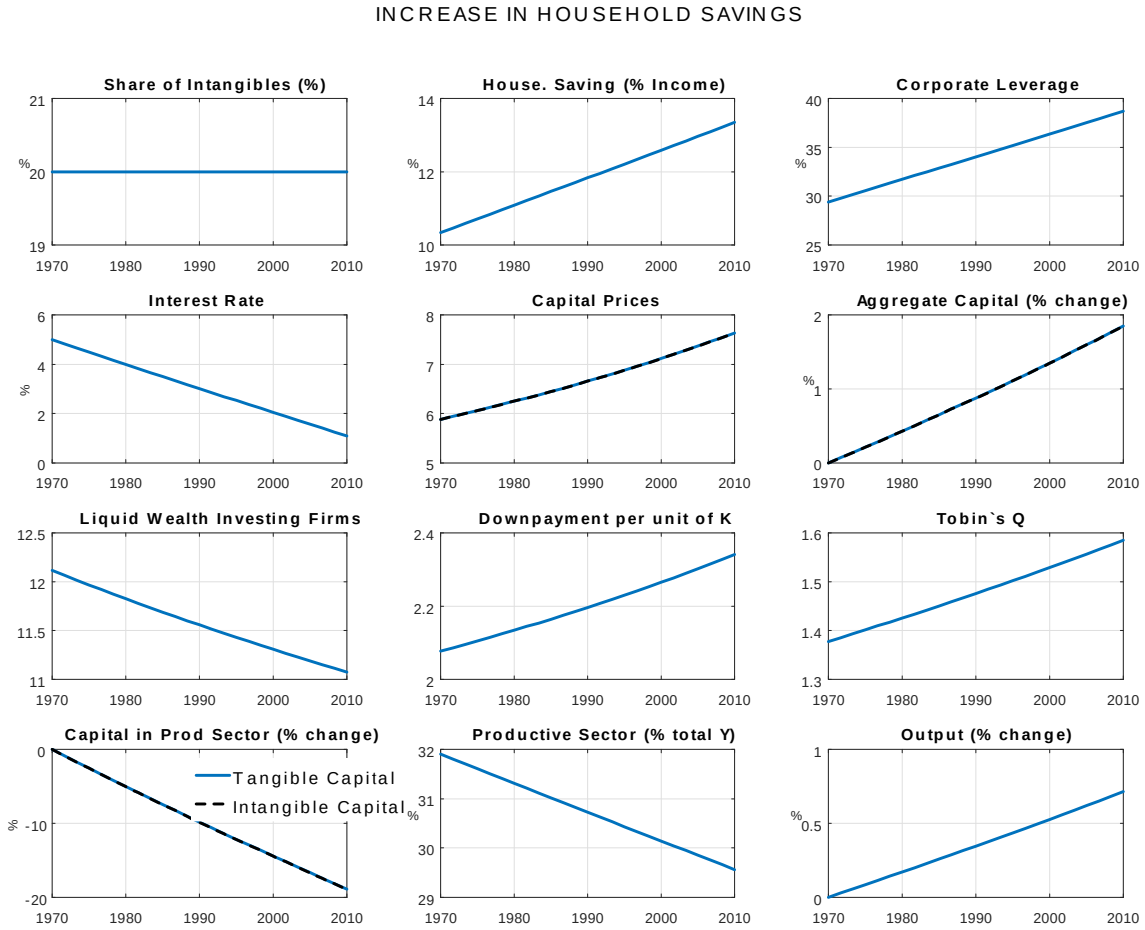


Figure 6: Equilibrium response of aggregate variables to household sector developments that increase net household savings.

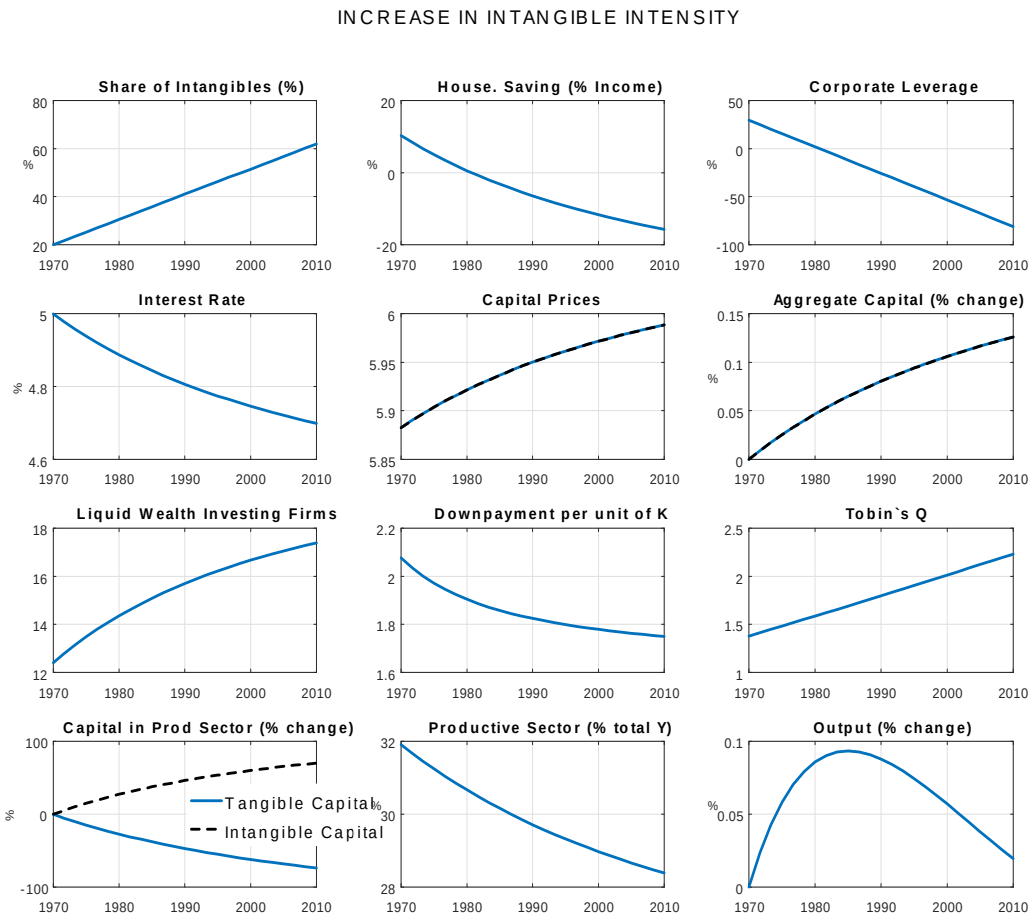


Figure 7: Equilibrium response of aggregate variables to corporate sector developments that increase the reliance on intangible capital.

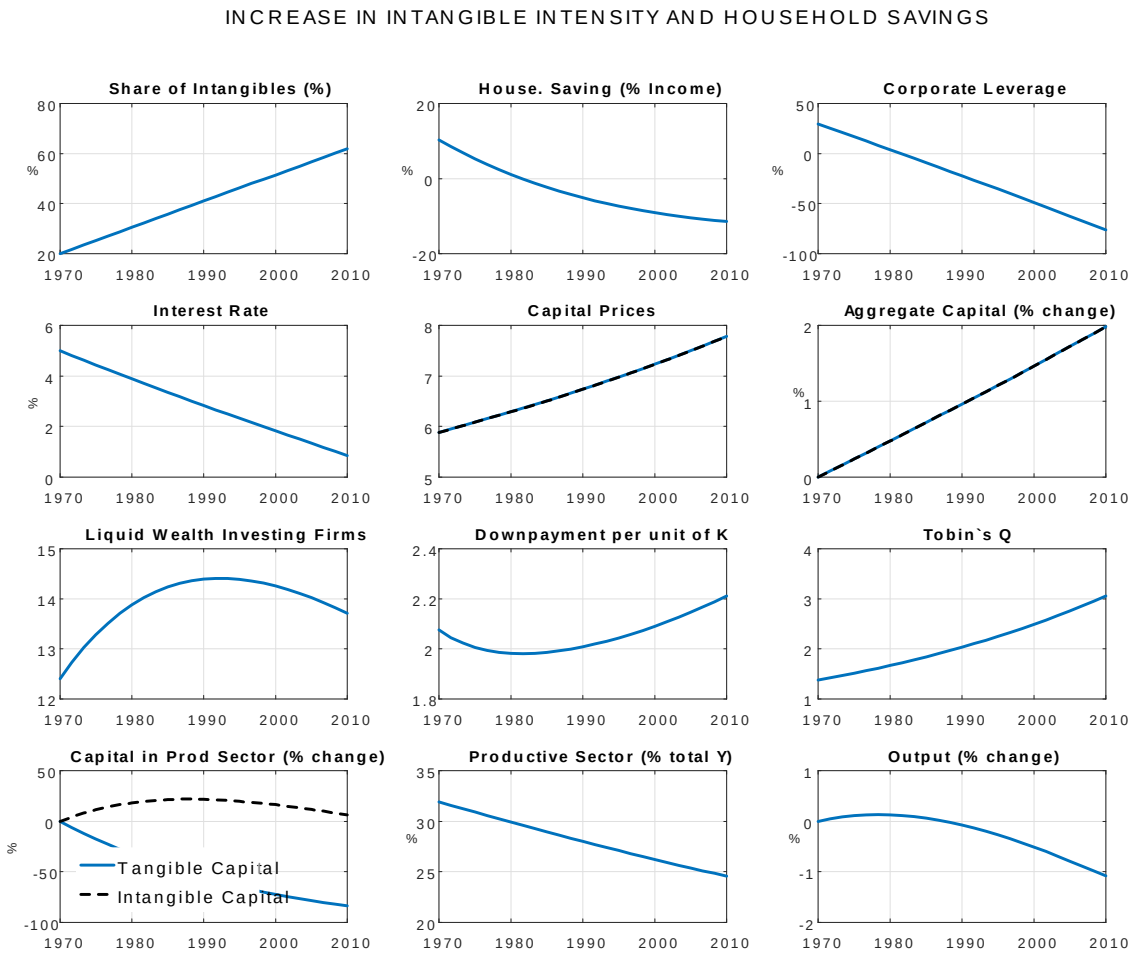


Figure 8: Equilibrium response of aggregate variables to corporate sector developments that increase the reliance on intangible capital, and household sector developments that increase net household savings.

increasingly negative effects on capital allocation as the economy shifts into a higher intangible intensity regime. Productive firms shift from being net borrowers to net savers, so lower rates benefit them initially by decreasing their debt overhang, but hurt them as they become savers and are less able to accumulate savings quickly. In addition, increasing capital prices decrease their ability to purchase capital from exiting firms, worsening capital reallocation. Despite the fact that capital creation increases strongly, and that the economy is shifting to a higher reliance on a type of capital which is significantly more productive, the output drop caused by the combination of both developments is in excess of 1%. We believe that these results are both qualitatively and quantitatively interesting, and that they identify a novel misallocation effect of endogenously low interest rates which has important policy implications. From the qualitative point of view, the model shows that even though the shift to intangible technologies was already taking place in the 1970s, its net negative effects on reallocation, productivity, and output only started to gather pace from 1990 onwards. This is consistent with studies that show a decline in dynamism of U.S. businesses starting in the mid 1980s and gathering speed especially from 2000 onwards (Haltiwanger (2015)). From a quantitative point of view, because we assume the productive sector to be relatively small, and intangible capital to be much more productive than tangible capital, it is likely that the estimated drop in output shown in Figure 8 is a conservative estimate of the actual value.

8 Conclusion

This paper suggests that the changes in firms' financing behavior brought about by technological evolution might help explain the subpar growth experienced in recent years, because they have occurred during a period of low interest rates. Our insights could be extended to develop interesting policy implications. On one hand, the mechanisms described in this paper, operating mostly through the endogenous reaction of interest rates, suggest that the rise in intangibles might have important implications for monetary policy. On the other hand, the negative externality in households' and firms' excessive saving decisions might introduce a role for a fiscal policy that discourages such saving.

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9 Appendix A - Households

9.1 Individual Problem of Old Households

Households have log utility, and a representative old household still living at time t maximizes the following objective function:

$$V_t^o\left(b_t^o\right) = \max_{c_t^o, b_{t+1}^o} \sum_{j=0}^{\infty} (1-\varrho)^j \beta^j \log\left(c_{t+j}\right) \tag{60}$$

subject to

$$c_t^o = b_{t+1}^o + (1-\gamma)d - \frac{(1+r)}{(1-\varrho)} b_t^o. \tag{61}$$

The first order condition with respect to b_{t+1}^o is

$$c_{t+1}^o = \beta(1+r)c_t^o \tag{62}$$

We guess a consumption policy rule:

$$c_t^o = \Delta d + \Theta b_t^o,$$

and by using a guess and verify method on (62) we determine the value of the coefficients Δ and Θ and obtain:

$$c_t^o = (1-(1-\varrho)\beta)\left[\frac{(1-\gamma)(1+r)}{(\varrho+r)} d - \frac{(1+r)}{(1-\varrho)} b_t^o\right]. \tag{63}$$

Moreover, we can use the policy function to derive the evolution of the wealth of old households (*calculations in the Appendix!*):

$$b_{t+1}^o = \frac{(1-\varrho)\left[1-\beta(1+r)\right]}{(\varrho+r)}(1-\gamma)d + (1+r)\beta b_t^o \tag{64}$$

which says that old households slowly consume savings if $\beta(1+r) < 1$, and do so at a faster rate the higher the dividends. In our simulations typically $\beta(1+r) < 1$.[3]

[3] To see this more clearly, denote $a = -b$ as savings, and write

$$a_{t+1}^o = (1+r)\beta a_t^o - \frac{(1-\varrho)\left[1-\beta(1+r)\right]}{(\varrho+r)}(1-\gamma)d$$

9.2 Old Households' Consumption Rule

We guess a consumption policy rule:

$$c_t^o = \Delta d + \Theta b_t^o,$$

and plug it into the FOC

$$\begin{aligned}
\Delta d + \Theta \left[c_t^o - (1-\gamma)\, d + \frac{(1+r)}{(1-\varrho)} b_t^o \right] &= \beta(1+r)\left(\Delta d + \Theta b_t^o\right) \\
c_t^o &= \left[\frac{\beta(1+r)\Delta}{\Theta} + (1-\gamma) - \frac{\Delta}{\Theta} \right] d + (1+r)\left[\beta - \frac{1}{(1-\varrho)} \right] b_t^o,
\end{aligned}$$

and then solve for the unknown coefficients

$$\begin{aligned}
\Delta &= \frac{\beta(1+r)\Delta}{\Theta} + (1-\gamma) - \frac{\Delta}{\Theta} \\
\Theta &= (1+r)\left[\beta - \frac{1}{(1-\varrho)} \right]
\end{aligned}$$

$$\begin{aligned}
\Delta &= \frac{\beta\Delta}{\left[\beta - \frac{1}{(1-\varrho)} \right]} + (1-\gamma) - \frac{\Delta}{(1+r)\left[\beta - \frac{1}{(1-\varrho)} \right]} \\
\Delta &= \frac{(1-\gamma)(1+r)\left[1 - \beta(1-\varrho) \right]}{\varrho + r}
\end{aligned}$$

The policy rule is:

$$\begin{aligned}
c_t^o &= \Delta d + \Theta b_t^o \\
&= \left[1 - \beta(1-\varrho) \right] \left[\frac{(1-\gamma)(1+r)}{\varrho + r} d - \frac{(1+r)}{(1-\varrho)} b_t^o \right]
\end{aligned}$$

9.3 Evolution of the wealth of old households

$$\begin{aligned}
b_{t+1}^o &= c_t^o - (1-\gamma)d + \frac{(1+r)}{(1-\varrho)} b_t^o \\
&= \frac{(1-\varrho)\left[1 - \beta(1+r) \right]}{(\varrho + r)} (1-\gamma) d + (1+r)\beta b_t^o
\end{aligned}$$

9.4 Individual Problem of Young Households

We first consider the optimization problem of an agent of age N in period t, who will become old in period $t+1$[4]:

$$V_{t,N}^{y}\left(b_{t,N}^{y}\right)=\max_{c_{t,N}^{y},b_{t+1}^{o}} u\left(c_{t,N}^{y}\right)+\beta(1-\varrho)V_{t+1}^{o}\left(b_{t+1}^{o}\right) \tag{65}$$

such that:

$$c_{t,N}^{y}=\gamma d+w-(1+r)b_{t,N}^{y}+b_{t+1}^{o}. \tag{66}$$

The first order condition implies that

$$\frac{1}{c_{t,N}^{y}}+\beta\left(\frac{\partial V_{t+1}^{o}\left(b_{t+1}^{o}\right)}{\partial b_{t+1}^{o}}+\frac{\partial V_{t+1}^{o}\left(b_{t+1}^{o}\right)}{\partial b_{t+2}^{o}}\frac{\partial b_{t+2}^{o}}{\partial b_{t+1}^{o}}\right)=0.$$

And applying the envelope theorem we obtain:

$$c_{t+1}^{o}=\beta\left(1+r\right)c_{t,N}^{y}$$

We substitute c_{t+1}^{o} using (63) and we obtain:

$$c_{t,N}^{y}=\left(\frac{1}{\beta}-(1-\varrho)\right)\left(\frac{(1-\gamma)d}{(\varrho+r)}-\frac{b_{t+1}^{o}}{(1-\varrho)}\right) \tag{67}$$

Then we consider the optimization problem for a young household of age $j<N$:

$$V_{t,j}^{y}\left(b_{t,j}^{y}\right)=\max_{c_{t,j}^{y},b_{t+1,j+1}^{y}} u\left(c_{t,j}^{y}\right)+\beta V_{t+1,j+1}^{y}\left(b_{t+1,j+1}^{y}\right) \tag{68}$$

such that:

$$c_{t,j}^{y}=\gamma d+w-(1+r)b_{t,j}^{y}+b_{t+1,j+1}^{y} \tag{69}$$

Which yields the standard Euler equation:

$$c_{t,j}^{y}=\left[\beta(1+r)\right]^{-(N-j)}c_{t+N-j,N}^{y} \tag{70}$$

Equations (67) and (70) fully characterize the lifecycle path of consumption of an household as a function of its assets when entering old age in period $t+1$, b_{t+1}^{o}.

[4] We assume that an agent can also die with probability ϱ in the transition between young and old.

9.5 Value of Savings of Oldest Young: b^o_{t+1}

We use the above equations, the budget constraint 69, and the assumption that newborn households have no endowment ($b^y_{t,1} = 0$) to determine the value of savings for retirement $b^o_{t+1} \equiv b^y_{t+1,N+1}$

We use the budget constraint for j=1 (a young of age=1), in which the debt brought over, $b^y_{t,1}$, is zero:

$$b^y_{t,1} = \frac{(\gamma d + w) - c^y_{t,1} + b^y_{t+1,2}}{(1+r)} = 0,$$

and we solve forward:

$$\begin{aligned} b^y_{t,1} &= \frac{(\gamma d + w) - c^y_{t,1}}{(1+r)} + \frac{(\gamma d + w) - c^y_{t+1,2} + b^y_{t+2,3}}{(1+r)^2} \\ &= (\gamma d + w)\sum_{j=1}^{N} \frac{1}{(1+r)^j} - \sum_{j=1}^{N} \frac{c^y_{t+j-1,j}}{(1+r)^j} + \frac{b^y_{t+N,N+1}}{(1+r)^N} \end{aligned}$$

Making use of the FOC:

$$c^y_{t,j} = [\beta(1+r)]^{-(N-j)}\, c^y_{t+N-j,N} \tag{71}$$

we get

$$\begin{aligned} c^y_{t,j} &= [\beta(1+r)]^{-(N-j)}\, c^y_{t+N-j,N} \\ c^y_{t,1} &= [\beta(1+r)]^{-(N-1)}\, c^y_{t+N-1,N} \\ c^y_{t+1,2} &= [\beta(1+r)]^{-(N-2)}\, c^y_{t+1+N-2,N} = [\beta(1+r)]^{-(N-2)}\, c^y_{t+N-1,N} \\ c^y_{t+2,3} &= [\beta(1+r)]^{-(N-3)}\, c^y_{t+2+N-3,N} = [\beta(1+r)]^{-(N-3)}\, c^y_{t+N-1,N} \\ c^y_{t+N-1,N} &= [\beta(1+r)]^{-(N-N)}\, c^y_{t+N-1+N-N,N} = c^y_{t+N-1,N} \end{aligned}$$

and plug in and simplify

$$\begin{aligned} \sum_{j=1}^{N} \frac{c^y_{t+j-1,j}}{(1+r)^j} &= \frac{c^y_{t,1}}{(1+r)} + \frac{c^y_{t+1,2}}{(1+r)^2} + \frac{c^y_{t+2,3}}{(1+r)^3} + ... + \frac{c^y_{t+N-1,N}}{(1+r)^N} \\ &= \frac{c^y_{t+N-1,N}}{(1+r)^N}\left[\sum_{j=0}^{N-1} \beta^{-(N-1-j)}\right] = -\frac{c^y_{t+N-1,N}}{(1+r)^N}\frac{\beta^N - 1}{\beta^{N-1}(\beta-1)} \end{aligned}$$

We substitute back in and keep simplifying

$$b_{t,1}^{y} = (\gamma d + w)\frac{1}{r}\left[1 - \frac{1}{(1+r)^{N}}\right] - \frac{c_{t+N-1,N}^{y}}{(1+r)^{N}}\frac{\beta^{N}-1}{\beta^{N-1}(\beta-1)} + \frac{b_{t+N,N+1}^{y}}{(1+r)^{N}}$$

$$\begin{aligned}
0 = (\gamma d + w)\frac{1}{r}\left[(1+r)^{N} - 1\right] - \left(\frac{1}{\beta} - (1-\varrho)\right)\frac{\beta^{N}-1}{\beta^{N-1}(\beta-1)}\frac{(1-\gamma)d}{(\varrho+r)} \\
+ \left[\left(\frac{1}{\beta} - (1-\varrho)\right)\frac{1}{(1-\varrho)}\frac{\beta^{N}-1}{\beta^{N-1}(\beta-1)} + 1\right] b_{t+N,N+1}^{y} \\
\left(\frac{1}{\beta} - (1-\varrho)\right)\frac{\beta^{N}-1}{\beta^{N-1}(\beta-1)}\frac{(1-\gamma)d}{(\varrho+r)} - (\gamma d + w)\frac{1}{r}\left[(1+r)^{N} - 1\right] \\
= \left[\left(\frac{1}{\beta} - (1-\varrho)\right)\frac{1}{(1-\varrho)}\frac{\beta^{N}-1}{\beta^{N-1}(\beta-1)} + 1\right] b_{t+N,N+1}^{y}
\end{aligned}$$

Solve:

$$-b_{t+N,N+1}^{y} = \frac{(\gamma d + w)\frac{1}{r}\left[(1+r)^{N} - 1\right] - \Psi\frac{(1-\gamma)d}{(\varrho+r)}}{\frac{\Psi}{1-\varrho} + 1} \tag{72}$$

$$\Psi \equiv \left(\frac{1}{\beta} - (1-\varrho)\right)\frac{\beta^{N}-1}{\beta^{N-1}(\beta-1)}$$

Equation (72) is very intuitive. Savings for retirement $-b_{t+1}^{o}$ increase in the difference between income before and after retirement. Moreover an increase in life expectancy (a drop in ϱ) reduces the value of the term Ψ and therefore increases $-b_{t+1}^{o}$.

9.6 Aggregate savings of the young

The previous section determines a sequence of optimal consumption at every age, $c_{t,1}^{y}, ..., c_{t,N}^{y}$, and applying the budget constraint (69) we can determine a sequence of assets for every age $b_{t,2}^{y}, ..., b_{t,N}^{y}$, which is constant for every period t. In equilibrium there is a measure 1 of households, a fraction $\frac{\phi}{\phi+\varrho}$ old, and a fraction $\frac{\varrho}{\phi+\varrho}$ young. Moreover there is a measure $\frac{\varrho}{\phi+\varrho}\frac{1}{N}$ of young households for each age. Therefore, after dropping the subscript t, we can define aggregate savings of the young households as:

$$B^{y} = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\sum_{j=1}^{N} b_{j+1}^{y}. \tag{73}$$

Savings of a young household are:

$$b_{j+1}^{y} = c_{j}^{y} - \gamma d - w + (1+r)b_{j}^{y} \tag{74}$$

We solve for b_N^y (from now on for simplicity omit the superscript y) :

$$b_N = \frac{1}{1+r}(\gamma d + w) - \frac{1}{1+r}c_N + \frac{1}{1+r}b_{N+1}, \tag{75}$$

where both b_{N+1} and c_N are determined by (67) and (72) above. At age $N-1$ (we use $c_t = [\beta(1+r)]^{-1} c_{t+1}$):

$$b_{N-1} = \frac{1}{1+r}(\gamma d + w) - \frac{1}{1+r}c_{N-1} + \frac{1}{1+r}\left[\frac{1}{1+r}(\gamma d + w) - \frac{1}{1+r}c_N + \frac{1}{1+r}b_{N+1}\right] \tag{76}$$

$$\begin{aligned}
b_{N-1} &= \frac{1}{1+r}(\gamma d + w) - \frac{1}{1+r}[\beta(1+r)]^{-1} c_N + \frac{1}{1+r}\left[\frac{1}{1+r}(\gamma d + w) - \frac{1}{1+r}c_N + \frac{1}{1+r}b_{N+1}\right] \\
&= \frac{1}{1+r}(\gamma d + w) + \frac{1}{(1+r)^2}(\gamma d + w) - \frac{1}{1+r}[\beta(1+r)]^{-1} c_N - \frac{1}{(1+r)^2}c_N + \frac{1}{(1+r)^2}b_{N+1} \\
&= \left(\frac{1}{1+r} + \frac{1}{(1+r)^2}\right)(\gamma d + w) - \left(\frac{1}{\beta} + 1\right)\frac{c_N}{(1+r)^2} + \frac{1}{(1+r)^2}b_{N+1}
\end{aligned}$$

therefore at age $N-2$:

$$\begin{aligned}
b_{N-2} &= \frac{1}{1+r}(\gamma d + w) - \frac{1}{1+r}c_{N-2} + \frac{1}{1+r}b_{N-1} \\
&= \frac{1}{1+r}(\gamma d + w) - \frac{1}{1+r}[\beta(1+r)]^{-1}[\beta(1+r)]^{-1} c_N \\
&\quad + \frac{1}{1+r}\left(\left(\frac{1}{1+r} + \frac{1}{(1+r)^2}\right)(\gamma d + w) - \left(\frac{1}{\beta(1+r)^2} + \frac{1}{(1+r)^2}\right)c_N + \frac{1}{(1+r)^2}b_{N+1}\right) \\
&= \left(\frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3}\right)(\gamma d + w) - \left(\frac{1}{\beta^2} + \frac{1}{\beta} + 1\right)\frac{c_N}{(1+r)^3} + \frac{1}{(1+r)^3}b_{N+1}
\end{aligned}$$

and at a generic age $N-t$:

$$b_{N-t} = \sum_{j=0}^{t} \frac{\gamma d + w}{(1+r)^{j+1}} - \frac{c_N}{(1+r)^{t+1}} \sum_{j=0}^{t} \frac{1}{\beta^j} + \frac{b_{N+1}}{(1+r)^{t+1}} \tag{77}$$

We use general formulas: $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ and $\sum_{j=0}^{t} x^j = \left(1 - x^{t+1}\right)\frac{1}{1-x}$, or $\sum_{j=0}^{t} \frac{1}{(1+r)^j} =$

$\left(1-\frac{1}{(1+r)^{t+1}}\right)\frac{1+r}{r}$ and $\sum_{j=0}^{t}\frac{1}{(1+r)^{j+1}}=\left(1-\frac{1}{(1+r)^{t+1}}\right)\frac{1}{r}$, so that

$$\begin{aligned}\sum_{j=0}^{t}\frac{\gamma d+w}{(1+r)^{j+1}} &= \left(1-\frac{1}{(1+r)^{t+1}}\right)\frac{\gamma d+w}{r}\\ \frac{c_N}{(1+r)^{t+1}}\sum_{j=0}^{t}\frac{1}{\beta^j} &= \frac{c_N}{(1+r)^{t+1}}\left(1-\frac{1}{\beta^{t+1}}\right)\frac{\beta}{\beta-1}\end{aligned}$$

hence:

$$b_{N-t}=\left(1-\frac{1}{(1+r)^{t+1}}\right)\frac{\gamma d+w}{r}-\frac{c_N}{(1+r)^{t+1}}\left(1-\frac{1}{\beta^{t+1}}\right)\frac{\beta}{\beta-1}+\frac{b_{N+1}}{(1+r)^{t+1}} \tag{78}$$

Now we add up the savings/borrowing over all ages from b_2 to b_N

$$\begin{aligned}\sum_{t=0}^{N-2}b_{N-t} &= \frac{\gamma d+w}{r}\sum_{t=0}^{N-2}\left(1-\frac{1}{(1+r)^{t+1}}\right)-\frac{\beta}{\beta-1}c_N\sum_{t=0}^{N-2}\left(\frac{1}{(1+r)^{t+1}}\left(1-\frac{1}{\beta^{t+1}}\right)\right)\\ &\quad +b_{N+1}\sum_{t=0}^{N-2}\left(\frac{1}{(1+r)^{t+1}}\right)\end{aligned}$$

the value of each summation term is:

$$\begin{aligned}\sum_{t=0}^{N-2}\left(1-\frac{1}{(1+r)^{t+1}}\right) &= -\frac{1}{r}\left(r-(r+1)^{-N+1}-Nr+1\right)\\ &= -\frac{1}{r}\left(1+r(1-N)-\frac{1}{(1+r)^{N-1}}\right)\\ &= \frac{1}{r}\left(\frac{1}{(1+r)^{N-1}}+r(N-1)-1\right)\end{aligned}$$

$$\begin{aligned}\sum_{t=0}^{N-2}\left(\frac{1}{(1+r)^{t+1}}\left(1-\frac{1}{\beta^{t+1}}\right)\right) &= \sum_{t=0}^{N-2}\frac{1}{(1+r)^{t+1}}-\sum_{t=0}^{N-2}\left(\frac{1}{[(1+r)\beta]^{t+1}}\right)\\ &= \frac{1-(1+r)^{-N+1}}{(1+r)-1}-\frac{1-[(1+r)\beta]^{-N+1}}{[(1+r)\beta]-1}\end{aligned}$$

$$\sum_{t=0}^{N-2}\left(\frac{1}{(1+r)^{t+1}}\right)=\frac{1-(1+r)^{-N+1}}{(1+r)-1}$$

Substituting them back:

$$\begin{aligned}\sum_{t=0}^{N-2} b_{N-t} &= \frac{\gamma d+w}{r}\frac{1}{r}\left(\frac{1}{(1+r)^{N-1}}+r(N-1)-1\right)-\frac{\beta}{\beta-1}c_N\left[\frac{1-(1+r)^{-N+1}}{(1+r)-1}-\frac{1-[(1+r)\beta]^{-N+1}}{(1+r)\beta-1}\right] \\ &\quad +b_{N+1}\frac{1-(1+r)^{-N+1}}{(1+r)-1}\end{aligned}$$

we rename terms:

$$\sum_{t=0}^{N-2} b_{N-t} = A_1\frac{\gamma d+w}{r} - A_2 c_N + A_3 b_{N+1}$$

$$\begin{aligned}A_1 &\equiv \frac{1}{r}\left(\frac{1}{(1+r)^{N-1}}+r(N-1)-1\right) \\ A_2 &\equiv \frac{\beta}{\beta-1}\left[\frac{1-(1+r)^{-N+1}}{(1+r)-1}-\frac{1-[(1+r)\beta]^{-N+1}}{(1+r)\beta-1}\right] \\ A_3 &\equiv \frac{1-(1+r)^{-N+1}}{(1+r)-1}\end{aligned}$$

9.7 Aggregate savings of the old

In equilibrium there are $\frac{\varrho}{\phi+\varrho}\frac{1}{N}$ households that become old every period, and $\frac{\varrho}{\phi+\varrho}\frac{1}{N}(1-\varrho)^j$ households that survived for j periods . Therefore aggregate savings of the old households are:

$$B^o = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\sum_{j=1}^{\infty}(1-\varrho)^j b_j^0 \tag{79a}$$

also note that b_1^0 is the initial savings from young age as defined in (72)

Recall that (from (64)):

$$b_{j+1}^o = A+\beta(1+r)b_j^o \tag{80}$$

$$A \equiv \frac{(1-\varrho)\left[1-\beta(1+r)\right]}{(\varrho+r)}(1-\gamma)d$$

hence:

$$b_2^o = A+\beta(1+r)b_1^o \tag{81}$$

$$b_3^o = A+\beta(1+r)b_2^o = A+\beta(1+r)A+\beta^2(1+r)^2 b_1^o \tag{82}$$

$$b_t^o = A+\beta(1+r)A+...+\beta^{t-2}(1+r)^{t-2}A+\beta^{t-1}(1+r)^{t-1}b_1^o \tag{83}$$

$$b_t^o = A\left[\sum_{j=0}^{t-2}\beta^j(1+r)^j\right]+\beta^{t-1}(1+r)^{t-1}b_1^o \tag{84}$$

$$b_t^o = \frac{(\beta\,(1+r))^{t-1}-1}{\beta\,(1+r)-1}A + [\beta(1+r)]^{t-1}\,b_1^o \tag{85}$$

$$\begin{aligned}
\sum_{t=1}^{\infty}(1-\varrho)^t b_t^0 &= \sum_{t=1}^{\infty}(1-\varrho)^t\left[\frac{(\beta\,(r+1))^{t-1}-1}{\beta\,(1+r)-1}A + [\beta(1+r)]^{t-1}\,b_1^o\right] \\
&= \frac{A}{\beta\,(1+r)-1}\sum_{t=1}^{\infty}(1-\varrho)^t\left[(\beta\,(r+1))^{t-1}-1\right] + b_1^o\sum_{t=1}^{\infty}(1-\varrho)^t\,[\beta(1+r)]^{t-1} \\
&= \left[\frac{A}{\beta+r\beta-1}+b_1^o\right]\sum_{t=1}^{\infty}(1-\varrho)^t\,(\beta\,(r+1))^{t-1} - \frac{A}{\beta+r\beta-1}\sum_{t=1}^{\infty}(1-\varrho)^t \\
&= \left[\frac{A}{\beta+r\beta-1}+b_1^o\right]\frac{1}{\beta(1+r)}\sum_{t=1}^{\infty}[(1-\varrho)\beta(1+r)]^t - \frac{A}{\beta+r\beta-1}\sum_{t=1}^{\infty}(1-\varrho)^t
\end{aligned}$$

$$\sum_{t=1}^{\infty}[(1-\varrho)\beta(1+r)]^t = \frac{1}{1-(1-\varrho)\beta(1+r)} - 1 = \frac{(1-\varrho)\beta(1+r)}{1-(1-\varrho)\beta(1+r)}$$

$$\sum_{t=1}^{\infty}(1-\varrho)^t = \frac{1}{1-1+\varrho} - 1 = \frac{1-\varrho}{\varrho}$$

hence:

$$\sum_{t=1}^{\infty}(1-\varrho)^t b_t^0 = \left[\frac{A}{\beta+r\beta-1}+b_1^o\right]\frac{(1-\varrho)}{1-(1-\varrho)\beta(1+r)} - \frac{A}{\beta+r\beta-1}\frac{1-\varrho}{\varrho}$$

9.8 Summing up aggregate household borrowing

aggregate household borrowing is:

$$B = B^o + B^y \tag{86}$$

Where savings of the old is:

$$B^o = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\left[\left(\frac{A}{\beta+r\beta-1}+b^{retirement}\right)\frac{(1-\varrho)}{1-(1-\varrho)\beta(1+r)} - \frac{A}{\beta+r\beta-1}\frac{1-\varrho}{\varrho}\right] \tag{87a}$$

$$A \equiv \left(\frac{(1-\varrho)\,(1-(1+r)\beta)}{(\varrho+r)}\right)(1-\gamma)d$$

$$b^{retirement} \equiv \frac{\Psi\frac{(1-\gamma)d}{(\varrho+r)} - (\gamma d+w)\,\frac{1}{r}\left[(1+r)^N-1\right]}{\frac{\Psi}{1-\varrho}+1} \tag{88}$$

$$\Psi \equiv \left(\frac{1}{\beta} - (1-\varrho)\right) \frac{\beta^N - 1}{\beta^{N-1}(\beta - 1)}$$

And savings of the young is:

$$B^y = \frac{\varrho}{\phi + \varrho} \frac{1}{N} \sum_{t=0}^{N-2} b_{N-t} = \frac{\varrho}{\phi + \varrho} \frac{1}{N} \left[A_1 \frac{\gamma d + w}{r} - A_2 c_N + A_3 b^{retirement} \right]$$

$$\begin{aligned} A_1 &\equiv \frac{1}{r} \left(\frac{1}{(1+r)^{N-1}} + r(N-1) - 1 \right) \\ A_2 &\equiv \frac{\beta}{\beta - 1} \left[\frac{1 - (1+r)^{-N+1}}{(1+r) - 1} - \frac{1 - [(1+r)\beta]^{-N+1}}{(1+r)\beta - 1} \right] \\ A_3 &\equiv \frac{1 - (1+r)^{-N+1}}{(1+r) - 1} \end{aligned}$$

And

$$c_N = \left(\frac{1}{\beta} - (1-\varrho)\right) \left(\frac{(1-\gamma)d}{(\varrho + r)} - \frac{b^{retirement}}{(1-\varrho)} \right)$$