A Quantitative Model of Secular Stagnation

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Abstract

Real interests rates in the United States, Europe, and Japan have been negative since 2007, a fact seemingly at odds with economic theory. We show that in a canonical overlapping generations model, calibrated to the US economy using standard parameters from the literature, the equilibrium real interest rate need not be positive; our preferred calibration produces an equilibrium rate of -2.5%. Through the lens of our model, we explore the economic forces which have lead to a secular decline in interest rates over the past thirty years. The main factors which caused rates to decline include lower productivity growth, slower population growth, and an increase in life expectancy and years of retirement. In economies with negative real rates, there can be permanent slumps unless the monetary authority induces sufficient inflation for the zero lower bound not to bind.

Keywords: Natural rate of interest, lifecycle models, secular stagnation
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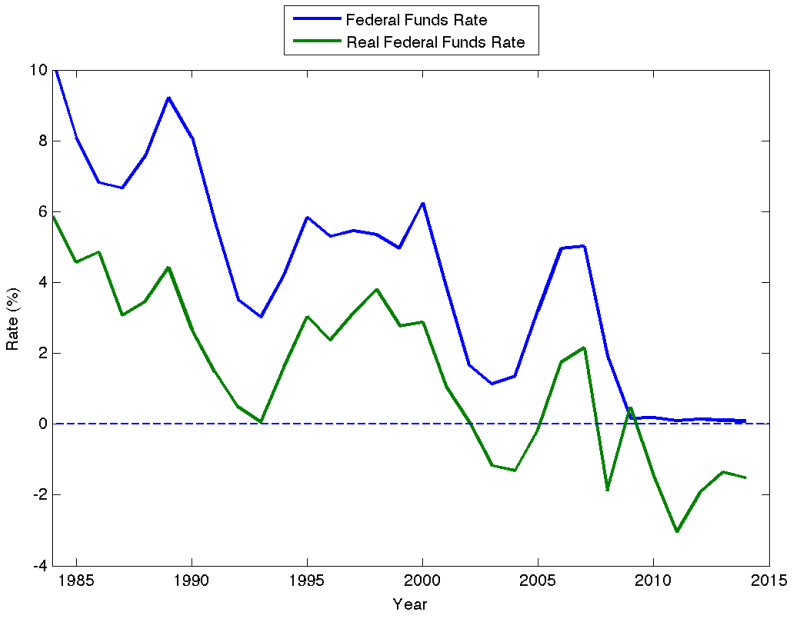


Figure 1: Nominal and real Federal funds yield

1 Introduction

In the United States and across other developed nations, the real interest rate has declined substantially over the past thirty years, with many economies experiences zero or negative real rates over the past decade. Figure 1 shows nominal and real US fed funds rate from 1984 to 2015. The real federal funds rate shows a declining trend over the entire time period, falling below zero in 2007.[1] Figure 2 shows yields of US 10 year Treasury bonds for the same time period. In the span of a single generation, the real rate of return on these bonds declined from nine percent to around 0 percent, with yields dipping briefly negative in 2012.

Data on other OECD countries follows the same general trend as the US. Figure Figure 3 shows real central bank discount rates from 1999-2013 for the US, Eurozone, and Japan. The trend for all three series is similar, and all fall below zero by 2014.[2] Even nominal yields in some countries have entered negative territory. For example, German bonds yield negative returns on maturities lasting until 2022. The Japanese 10 year bond yield is negative, and the 30 year bond yield is only 44 basis points. The ECB?s nominal discount rate is negative at -40 basis points as are the policy rates in Sweden, Denmark, Switzerland, and Japan.

[1]Here and in other figures the real rate is measured as *nominal rate - expected inflation*, where *expected inflation* is estimated using a five year moving average of current and past inflation. This decline in US rates is robust to other measures of expected inflation (see Hamilton et al. (2015))

[2]For additional countries, see Hamilton et al. (2015), Figures 2.3 and 2.4

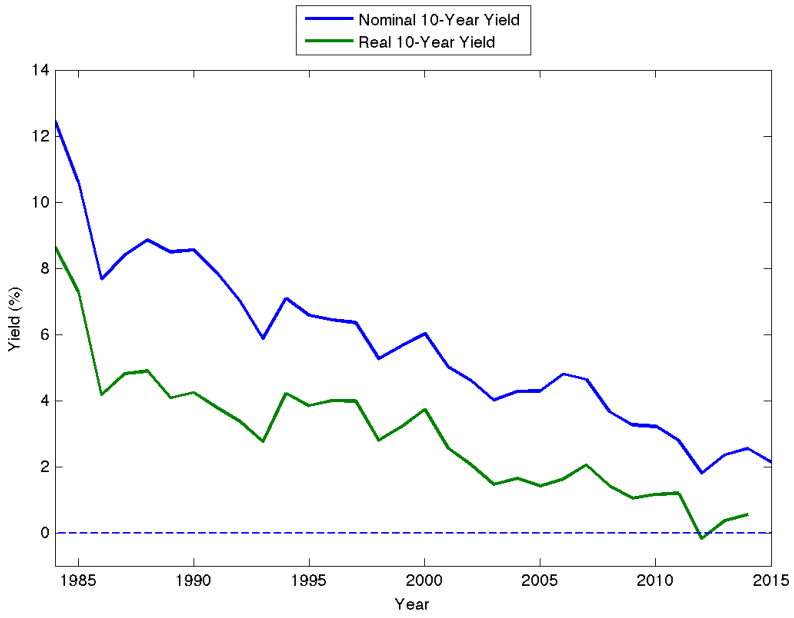


Figure 2: US 10 year treasury yield

The persistent decline in interest rates has sparked concerns over secular stagnation (see Summers (2013) and Eggertsson and Mehrotra (2014)) - that the natural rate of interest is low or negative making the zero lower bound problem a chronic rather than temporary phenomenon. If the natural rate (the interest rate consistent with full employment) is negative, a central bank cannot keep rates low enough to sustain sufficient demand. Moreover, a central bank?s ability to respond to negative demand shocks is impeded since monetary policy cannot be eased in recessions.

In resurrecting the secular stagnation hypothesis, Summers (2013) offers several potential explanations for a low and declining natural rate of interest. He points out that rising income inequality, falling relative prices for capital goods, and declining population growth - all slow moving and persistent trends - may raise desired saving while lower investment demand, reducing the real interest rate. Alternatively, lower productivity growth as suggested by Gordon (2016) or Fernald and Jones (2014) may also account for lower rates. Eggertsson and Mehrotra (2014) show in a three-period OLG model that the forces emphasized by Summers (2013) along with deleveraging shocks can lower the natural rate of interest.

Figure 4 shows the population growth rate for selected developed and developing countries, with historical data until 2010 and projections afterward. Population growth rates have declined significantly in both groups since the 1980s, and are projected to fall even further looking forward to the year 2100. Both Germany and Japan will have declining populations in the near future, and through the miracle of compounding are projected to have huge population losses by 2100. The

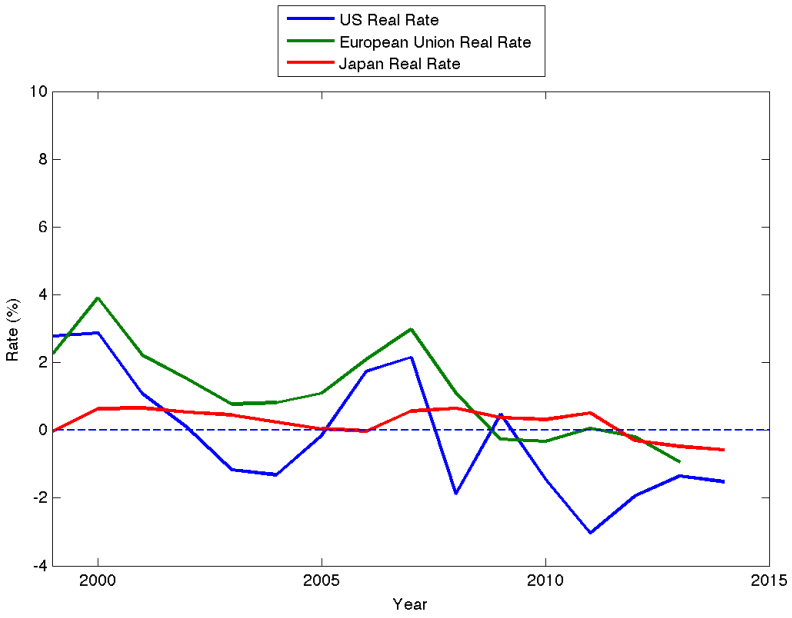


Figure 3: Real central bank discount rates, US, Eurozone, Japan

US and UK's demographics are more favorable, with population growth rate projected to slowly asymptote to zero by 2100. If we look at workforce growth numbers instead of population, the story is the same. Table 1 shows the growth rate of the workforce for selected countries. Japan and Germany's workforce are projected to decline by about 30% over the next 40 years, while the US, UK will grow at a sharply reduced rate from the past.

In this paper, we seek to quantify the forces that account for a lower natural rate of interest. Representative agent models are not well-suited for this task because the steady state real interest rate is determined solely by the household?s discount factor. Therefore, we build a 55 period quantitative lifecycle model to find a model-based estimate of the current US natural rate of interest and to determine which factors are most important in accounting for the fall in the natural rate in the US over the last 30 years. Our lifecycle model will allow us to consider the various factors identified by Summers along with supply side factors like slower growth in the labor force and decline in the growth rate of total factor productivity. We also analyze quantitatively the implications for output gap and inflation if a negative natural rate of interest generates a secular stagnation.

The departure point for our theoretical economy is a version of overlapping generations model with 55 periods, together with updated data on the lifecycle income profile and US demographic statistics. Households make consumption and saving choices, face borrowing constraints, and make bequests to their descendants. Household face mortality risk, and inequality is captured by

Table 1: Growth in the workforce, %, selected countries. Source: Goodhart and Erfurth (2014)

	1990-2000	2000-2013	2013-2050
Japan	3.1	-2.1	-33.7
US	15.2	5.1	18.0
UK	2.3	8.8	1.8
Germany	1.7	6.3	-34.3
China	11.3	6.8	-10.5

permanent differences in income profiles across households.

We show that in a standard calibration of our model with standard preferences and technology, the natural rate of interest in the stationary equilibrium is -2.5% for the US - below the negative inflation target of the Federal Reserve. In other words, we estimate the the interest rate consistent with full employment would not be attained given the Federal Reserve inflation target and a nominal rate at the zero bound. Our calibration strategy follows closely Auerbach and Kotlikoff (1987), with lower population growth and productivity growth accounting for a low natural rate of interest. The negative rates we find are robust to a variety of different specifications.

We also provide a decomposition of the factors that have lead to a decrease in interest rates over the past thirty years. A decline in the rate of population growth, which has indeed slowed from around 1% per year in 1980 to 0.7% in 2015 account for a 70 basis point decline in the nominal interest rate. Lower population growth raises the ratio of savers to borrowers requiring a lower interest rate to fall to clear the asset market. Also, life expectancy at birth has increased from 73.4 years in 1980 to 78.7 year in 2010, which has increased the number of years of retirement and thus the savings motive, leading to a decrease in real interest rate of 2 percentage points.

Beyond demographic factor, a slowdown productivity growth can also account for a substantial decrease in the real interest rate. A decrease from 2% to 1% per year in the rate of productivity growth lowers the real interest rate by 3.7 percentage points. A decrease in productivity growth leads to a decrease in interest rates through two channels. First, with lower productivity growth, individuals receive a greater portion of their income earlier in life, thus they tend to consume less and save more in their youth and middle age. Second, when productivity growth slows, the marginal product of capital grows more slowly, and thus firms demand fewer capital goods. Both channels mean that lower productivity growth is associated with lower interest rates.

While demographic factors and productivity can account for a substantial decline in the nat-

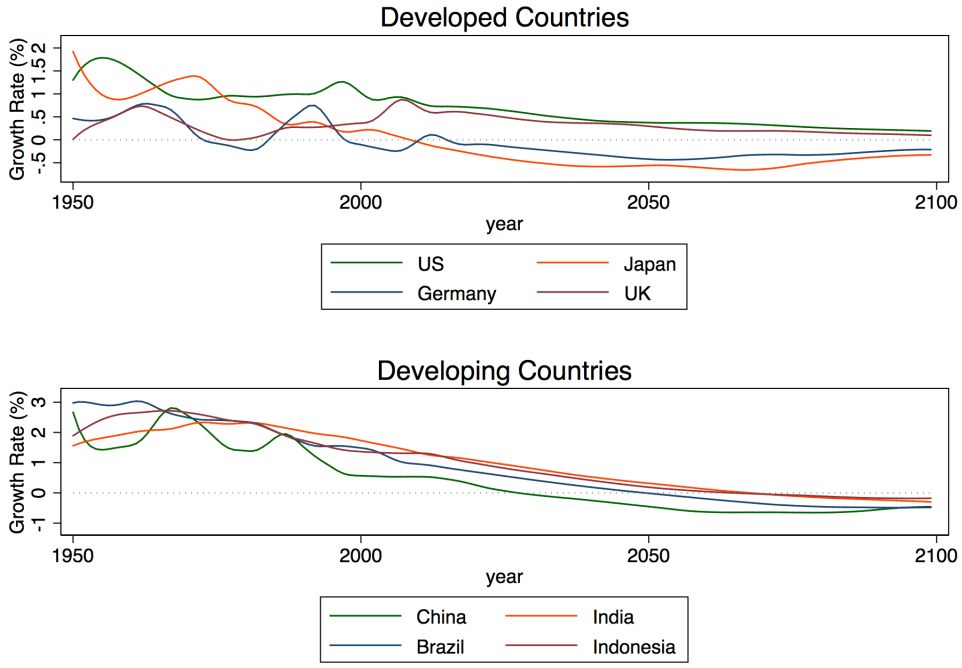


Figure 4: World population growth rate, 1950-2010, and projections, 2010-2100. Source: United Nations Population Projections

ural rate of interest, we also consider how increases in income inequality affect interest rates. As shown in Eggertsson and Mehrotra (2014), rising income inequality could theoretically have an ambiguous effect on the interest rates. However, we find that an increase in income inequality increases the overall propensity to save in the economy. Wealthier individuals are induced to save more through three channels: (1) steeper income profiles (2) lower time preference (3) higher bequest motives. We show that an increase in the gini coefficient of 20% lowers the natural rate of interest by 4 percentage points.

One further channel that may have lead to a decrease in the real interest rate is flows of savings from abroad. In fact, Bernanke (2005) has argued for the primacy of the global savings glut in explaining low interest rates. We model a global savings glut as a foreign country who has a large exogenous demand for US bond holdings in equilibrium. We find that indeed foreign demand for US bonds has had a significant impact on the real interest rate. The US net foreign asset position has decline for 0% of GDP in 1980 to -40% of GDP in 2015. Adding a foreign demand of this size to our model decreases interest rates by 1.3 percentage points. However, this decline in foreign demand has been partly offset by a large increase in the US public debt over this same period.

Finally, we consider the implications of a negative natural rate for secular stagnation as described in Eggertsson and Mehrotra (2014). Following their work, we consider the implications

of downward nominal wage rigidity and show under a standard calibration for the wage rigidity parameter that a negative natural rate of 2.5% implies an output gap (unemployment) of 10% and a shortfall in the inflation rate of 50 basis points from target. To the extent that prolonged high unemployment leads workers to leave the labor force, our model would predict a gradual closing of the output gap and return of the inflation rate to target.

1.1 Related Literature

Our quantitative analysis builds on the existing secular stagnation literature that consider the causes of low natural rates of interest and implications for monetary policy and output in the presence of nominal frictions and the zero lower bound. The basic ingredients needed to model demand-side secular stagnation are discussed in Eggertsson and Mehrotra (2014). Related models by Caballero and Farhi (2014) and Thwaites (2015) also use an overlapping generations structure to explain lower real interest rates. In the former, low interest rates are driven by an increase in uncertainty, while in the latter, low interest rates are driven by a decline in the relative price of investment goods. Carvalho and Ferrero (2014) build a quantitative perpetual youth model to examine the effects of slowing population growth on interest rates and inflation in Japan.

Low real interest rates in a global context have been analyzed in purely real model and models with nominal frictions. Perhaps closest to our approach is the two-country overlapping generations model of Coeurdacier, Guibaud and Jin (2015) who consider global capital market integration and its effect on US interest rates. Caballero, Farhi and Gourinchas (2008) and Mendoza, Quadrini et al. (2009) present real models of low global interest rates based on differences in financial market development. Recent work by Eggertsson et al. (2016) and Caballero, Farhi and Gourinchas (2015) examine the implications of low natural rates at the zero lower bound with nominal frictions. Favilukis, Ludvigson and Van Nieuwerburgh (2015) study the effect of the global savings glut on US interest rates and house prices in the 2000s.

Finally, our analysis contributes to an emerging literature that studies interest rates and deleveraging in incomplete markets models. Some examples include Guerrieri and Lorenzoni (2011), Auclert (2014), and Kaplan, Moll and Violante (2016). In particular, recent work by Wong (2014) present evidence of heterogenous effects of monetary policy by age cohort and builds a OLG model of the transmission mechanism of monetary policy.

2 Model

The starting point of our model is Auerbach and Kotlikoff (1987) multi period OLG model, and our exposition focuses on the elements which determine the real interest rate. The main determinants of savings, and thus the real interest rate, are the desire to save for retirement, to smooth

consumption, and to leave bequests. Individuals also interact with firms, who demand capital, and governments, who borrow to finance spending. As will be seen, the model is able to easily produce negative equilibrium real interest rates, as long as there is one key assumption: output cannot be stored without paying storage costs. Individuals want to smooth consumption, in particular to save for retirement, and thus are willing to bear a tax of negative interest on savings in order to do so. In this equilibrium the marginal product of capital ρ is still positive, however the interest rate $r = \rho - \delta_K$, where δ_K is physical capital depreciation, is negative.

We are interested in long-run negative interest rates, and thus it is natural to focus purely on steady states. This simplifies the exposition as well as the computation of the equilibrium. When there is positive productivity growth, we solve for the balanced growth path, where income, capital, and consumption grow at a constant rate. When there is population growth, we solve for the equilibrium in which capital grows at a constant rate. In both cases, however, the real interest rate is constant.

2.1 Lifecycle Decisions

The economy consists of a large number of identical individuals N, with a constant rate of population growth of $n\%$ per year. Individuals enter economic maturity at age 26, after which they work, consume, have children, participate in markets, and finally pass away at age 81. They may perish stochastically, however, before their maximum life age, and the probability of surviving between age t and $t+1$ is given by s_t[3]. Individuals have children at age 26, thus to achieve a population growth rate of rate n, each agent must have $(1+n)^{25}$ children. Individuals receive utility from two sources: consumption, which is given by time-separable constant elasticity of substitution (CES) utility, and bequests, which are divided equally amongst all descendants. The lifecycle is summarized in figure 5.

Individuals receive bequests from previous generations, and in turn save in order to pass on bequests to the next generation. Agents receive receive bequests ν_r at age a_ν, and give bequests ν_g during the year they pass away. In the utility function, the parameter μ governs an individual's utility from giving a bequest.

Individuals thus maximize the utility function

$$U = \sum_{t=1}^{L} s^t (\beta^{t-1} \frac{C_t^{1-\sigma}}{1-\sigma}) + s^L \beta^{L-1} \mu \frac{\nu_g^{1-\sigma}}{1-\sigma} \tag{1}$$

Where C_t is consumption of an individual at time t, β is the time discount rate, and σ is the coefficient of relative risk aversion. We define s^t as the unconditional probability of being alive

[3]Following Ríos-Rull (1996) we assume all individuals participate in annuity markets, and thus all interest rates will be indexed by probability of death s_t

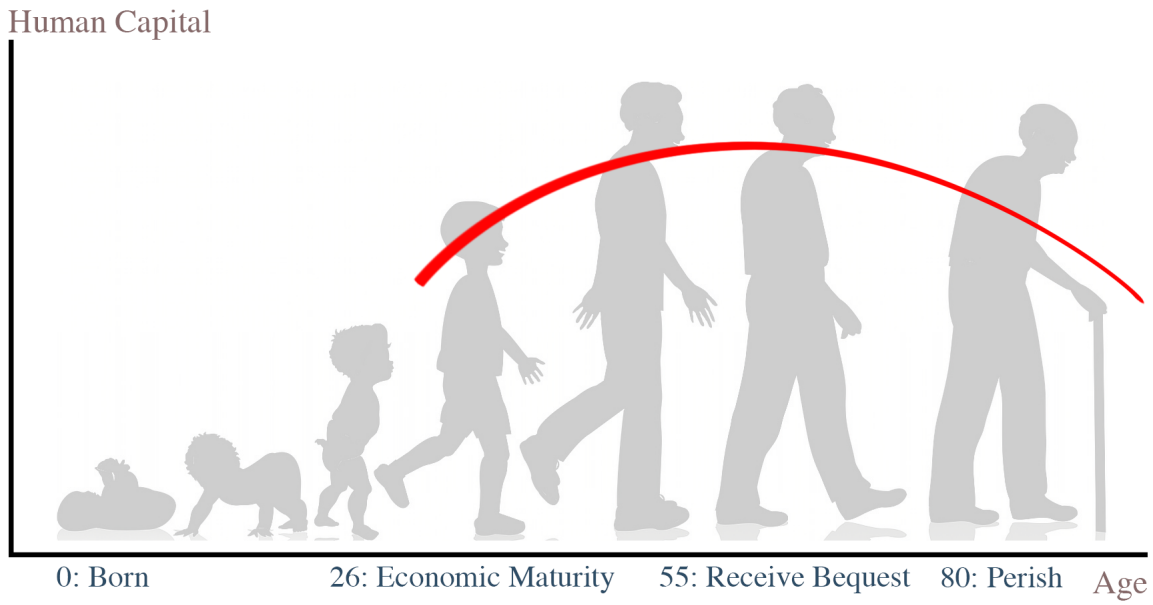


Figure 5: The lifecycle of the model

at age t[4]. In the above notation, $t = 1$ corresponds to lifetime age 26 and L corresponds to the maximum life age of 81.

Individuals purchase consumption and bequests using labor income and bequests received from their parents. Each agent is given an identical schedule of lifetime exogenous labor productivity, denoted hc_a, which varies based on age. Figure 6 shows a standard profile of human capital, with low productivity when first entering the workforce, reaching a maximum in middle age, and then declining in old age before dropping to zero in retirement.

Individuals supply labor inelastically, and thus wage income is equal to the wage multiplied by an individual's age specific human capital. Agents retire at age 65, after which they receive no labor income for the rest of their lives. Agents must pay average labor income taxes $\overline{\tau_w}$ and average capital income taxes $\overline{\tau_r}$. The inelasticity labor supply means that the earnings profile of an individual exactly follows the human capital profile, as seen in Figure 6.

We assume that technological progress is labor enhancing, and labor productivity grows at a rate g. In a steady state, wages will grow at a constant rate g, and thus the earnings profile will become steeper with positive economic growth. Figure 6 shows the updated earnings profile with $g = 2\%$.

Optimal consumption and bequest decisions are constrained by a series of flow budget constraints

$$b_t = w_t hc_t(1 - \overline{\tau_w}) - C_t + (1 + r(1 - \overline{\tau_r}))b_{t-1} + \nu_{r,t} - \nu_{g,t} \tag{2}$$

[4]This can be calculated by the multiplicative series $s^t = \Pi_{m=1}^{t-1} s_m$

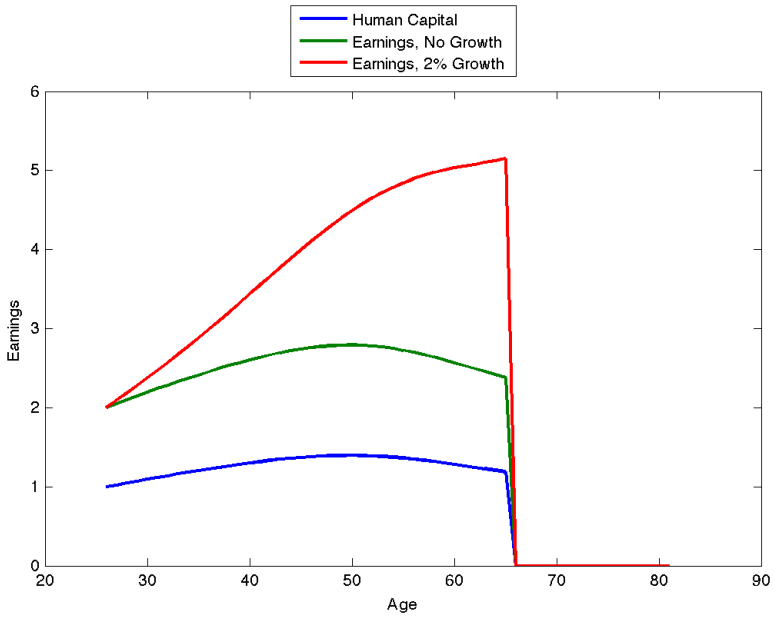


Figure 6: Human Capital Profile, as estimated by Gourinchas and Parker (2002)

Where b_t are one period bonds(a negative bond denotes lending). Bequests received $\nu_{r,t}$ are zero except at age a_ν, and bequests given $\nu_{g,t}$ are zero except at age L. At their maximum age of L, individuals' bond holdings must be equal zero.

It may be the case that individuals would like to borrow in order to smooth consumption, but are unable to do so because of credit constraints. We specify that individuals are subject to the debt limit $\bar{D}$, thus bond holdings must satisfy the inequality

$$b_t \geq -\bar{D} \tag{3}$$

The flow budgets can be combined into an intertemporal budget constraint:

$$\begin{aligned} &\sum_{s=1}^{L} \frac{C_s}{(1+r(1-\overline{\tau_r}))^{s-1}} + \frac{\nu_g}{(1+r(1-\overline{\tau_r})))^{L-1}} = \\ &\sum_{s=1}^{L} \frac{w * hc_s(1-\overline{\tau_w})}{(1+r(1-\overline{\tau_r})))^{s-1}} + \frac{\nu_r}{(1+r(1-\overline{\tau_r})))^{a_\nu - 1}} \end{aligned} \tag{4}$$

2.2 Production

Output is produced by perfectly competitive firms with Cobb Douglas production function

$$Y_t = z_t \cdot K_t^{\alpha}((1+g)^t \mathcal{L}_t)^{1-\alpha} \tag{5}$$

In this equation the total human capital adjusted labor supply is denoted $\mathcal{L}_t$, which sums labor supplies across generations and takes into account human capital. Every individual of age a supplies 1 unit of labor inelastically, which is adjusted by the individual's human capital hc_a. Summing across generations, and keeping in mind the relative size of the generations, the total human capital adjusted labor supply is

$$\sum_{a=1}^{L}(1+n)^{1-a}N * hc_a = \mathcal{L}_t \tag{6}$$

With perfect competition, the return on capital is equal to the marginal product of capital

$$\rho_s = z_s \cdot \alpha \cdot \left(\frac{K_s}{(1+g)^s \mathcal{L}_s}\right)^{\alpha-1} \tag{7}$$

The wage is equal to the marginal product of labor:

$$w_s = z_s \cdot (1-\alpha) \cdot (1+g)^s \cdot \left(\frac{K_s}{(1+g)^s \mathcal{L}_s}\right)^{\alpha} \tag{8}$$

And the interest rate is equal to the marginal product of capital minus the depreciation rate, δ:

$$r_s = \rho_s - \delta \tag{9}$$

Capital evolves according to the law of motion

$$K_{t+1} = (1-\delta_K)K_t + I_t \tag{10}$$

Where δ_K is the rate of depreciation, and I_t is investment.

2.3 Government Behavior

The government spends a pre specified G_s each period, and may accumulate debt. The budget constraint for the government is given by:

$$D_s = G_s + (1+r) * D_{s-1} - T_s \tag{11}$$

For simplicity, we assume government spending does not appear in the utility function of individuals.

2.4 Investment Good Production

Investment goods are produced by perfectly competitive firms, who turn savings into investment goods using final output. These firms maximize their profit function

$$\Pi^I = P_I * I_t - Y_t^I \tag{12}$$

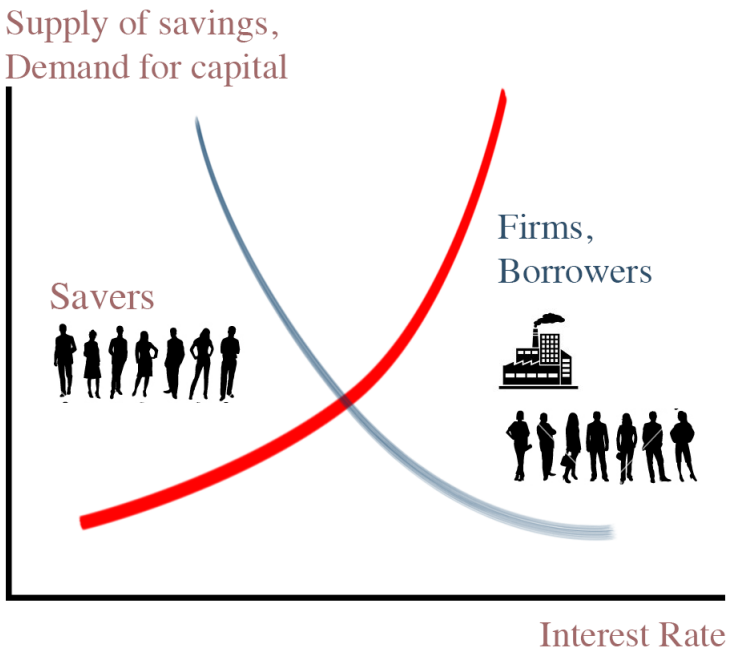


Figure 7: The loanable funds market

Subject to the technological constraint

$$I_t = \Upsilon Y_t^I \tag{13}$$

The zero profit conditions means that in equilibrium $P_I = \frac{1}{\Upsilon}$. Thus the more productive is the investment goods sector, the lower the price of investment goods.In our standard calibration the price of capital goods will be equal to 1, and for simplicity we will omit discussion of investment goods in our discussion of equilibrium[5].

2.5 Equilibrium

The major market in this economy is the market for bonds, and the interaction of individuals, firms, and the government determines the equilibrium real interest rate. The demand for bonds consists of all of the individuals who want to lend at a given interest rate. The supply of bonds consists of individuals who want to borrow, firms who want capital, and governments who want to finance spending. The loanable funds market equilibrium is shown in figure 7.

At any period of time there are many generations alive, and all of these demands must be accounted for, as well as the relative population sizes of the generations in the computation of the equilibrium. Let $\mathcal{F}_t$ be the set of generations who are lenders for a given interest rate r_{t+1}, and

[5]For full details, see the appendix.

$\mathcal{H}_t$ be the set of generations who are borrowers. Furthermore, denote $b_{\tau,t}$ as the bond demand for generation τ in time t. The equilibrium condition for interest rates can then be written as:

$$\sum_{\tau \in \mathcal{F}_t} N \cdot (1+n)^{\tau - t} \cdot b_{\tau,t}(r_{t+1}) = \sum_{\tau \in \mathcal{H}_t} N \cdot (1+n)^{\tau - t} \cdot b_{\tau,t}(r_{t+1}) + K_{t+1}(r_{t+1}) + D_t \tag{14}$$

We define a new term, net bond demand (NBD), which summarizes all extant generations' demand for bonds:

$$NBD(r_{t+1}) = \sum_{\tau \in \mathcal{F}_t} N \cdot (1+n)^{\tau - t} \cdot b_{\tau,t}(r_{t+1}) - \sum_{\tau \in \mathcal{H}_t} N \cdot (1+n)^{\tau - t} \cdot b_{\tau,t}(r_{t+1}) \tag{15}$$

We now formally define a balanced growth equilibrium.

Definition 1. *A balanced growth equilibrium is an interest rate r, a sequence of wage rates* $\{w_t\}_{t=1}^{\infty}$*, consumption paths* $\{\{C_{a,t}\}_{a=1}^{L}\}_{t=1}^{\infty}$*, bond demands* $\{\{b_{a,t}\}_{a=1}^{L}\}_{t=1}^{\infty}$*, capital supply* $\{K_t\}_{t=1}^{\infty}$*, adjusted labor supply* $\{\mathcal{L}_t\}_{t=1}^{\infty}$*, and production* $\{Y_t\}_{t=1}^{\infty}$ *such that:*

1. *Labor productivity grows at rate g. Consumption, bond profiles, bequests, and wages grow at rate g. Labor supply, capital, and government spending grow at rate g.*
2. *Consumption and bequets maximizes (1) subject to (4) and (3)*
3. *Bond holdings satisfy (2). In addition, individuals must have zero bond holdings when they die:* b^L*=0.*
4. *Bequests received equal bequests given by the parents divided by the total fertility rate*
5. *Bond markets clear and (15) is satisfied.*
6. *The human capital adjusted labor supply is given by (6)*
7. *Capital evolves according to the law of motion (10).*
8. *There are perfect factor markets, and thus (7), (8), and (9) hold.*
9. *Production satisfies (5).*
10. *The government satisfies budget equation (11)*

3 Calibration and Quantitative Results

3.1 Baseline

Our parameters come from three main sources, and are chosen to represent the current state of the US economy. The source of many parameters is from Auerbach and Kotlikoff (1987), who

pioneered multi period OLG models in order to explore the effects of tax reform. The source of the human capital earnings profile comes from Gourinchas and Parker (2002), who estimate average income profiles by age[6]. Finally, many of the basic facts from the lifecycle model come from statistics about the United States, such as the mortality tables, retirement age, and population growth. Table 2 gives the parameters we chose for the baseline calibration.

The rate of productivity growth is a key determinant of the real interest rate. Our baseline model gives productivity growth of 1% per year, generally below the 2% per year growth of GDP noted by Jones (2014). In this sense it is a version of the American economy in a mild stagnation. The depreciation rate is also important in determining the equilibrium interest rate. A rate of 12% is in line with BEA estimates of 14% depreciation for private and governmental nonresidential equipment [7]. Our model, however, does not include housing capital, which has considerably slower depreciation. Presumably, however, housing capital has a lower marginal product to go along with its lower depreciation, which would tend to depress real interest rates.

In our baseline calibration individuals retire at age 65, which gives more years of retirement than many other OLG lifecycle models. The average age of retirement in the US is 62, and the average lifespan is 79 years, so in this sense the the number of years of retirement is conservative.

The borrowing limit and the bequest utility μ are difficult to calibrate. Our initial calibration does not allow any borrowing. In other specifications, we select the debt limit to match observed US moments of consumer debt as a percentage of GDP. We choose parameter μ to match ratio of bequests given to permanent income from Fullerton and Rogers (1993).

3.2 Results

Table 3 gives the results from our main specification. Of particular interest is the real interest rate, which has an equilibrium value of -2.5%. The savings rate, capital output ratio, and tax revenue as a percentage of GDP are relatively similar to current US values of 4.5%, 2.4, and 17.4%, respectively [8]. With no heterogeneity other than age, the income and wealth Gini coefficient are small compared to US values.

Figure 8 shows optimal consumption decisions over the lifecycle. Consumption tracks income over the early part of the lifecycle due to borrowing constraints, then declines gradually in order to save for retirement. For comparison purposes, Figure 8 also includes average expenditure profiles from the Consumer Expenditure Survey, as estimated by Gourinchas and Parker (2002).

[6] In our model, the human capital profile and income profiles have the same shape

[7] BEA reports average depreciation rate for equipment of 14% (BEA (2016*a*)). Jorgenson (1996) and Fraumeni (1997) report average depreciation rates of 15% for durable equipment in US manufacturing; Kollintzas and Choi (1985) and Bischoff and Kokkelenberg (1987) report values of 12.5% of 10.6%, respectively; Epstein and Denny (1980) account for endogenous utilization and find an average depreciation rate of 12.6% 10 % for the period 1947-1971.

[8] See Piketty (2014) and National Income Accounts (BEA (2016*b*))

Table 2: Calibration Parameters

Variable	Value	Source
Consumer Parameters		
Rate of Time Preference	.99	Auerbach & Kotlikoff (1987)
Elasticity of Intertemporal Substitution	.25	Auerbach & Kotlikoff (1987)
Human Capital profile	Lifecycle	Gourinchas & Parker (2000)
Survival	Lifecycle	US Survival Tables
Borrowing Limit, % of Perm. Income	0%	Gourinchas & Parker (2000)
Years of Retirement	16	Calibrated
Bequests, % of Lifetime Income	5%	Fullerton & Rogers
Production Parameter		
Capital Share	.25	Auerbach & Kotlikoff
Population Growth	.007	US Population Growth Rate
Productivity Growth	.01	
Depreciation	.12	BEA, Jorgensen
Government Parameters		
Wage Income Tax	.20	
Capital Income Tax	0	

Table 3: Model Results

Variable	Model Value	US Value
Real Interest Rate	-2.5%	-1.5%
Capital Output Ratio	2.55	2.4
Personal Savings Rate	10%	5.4%
Income Gini	.32	.41
Wealth Gini	.45	.8

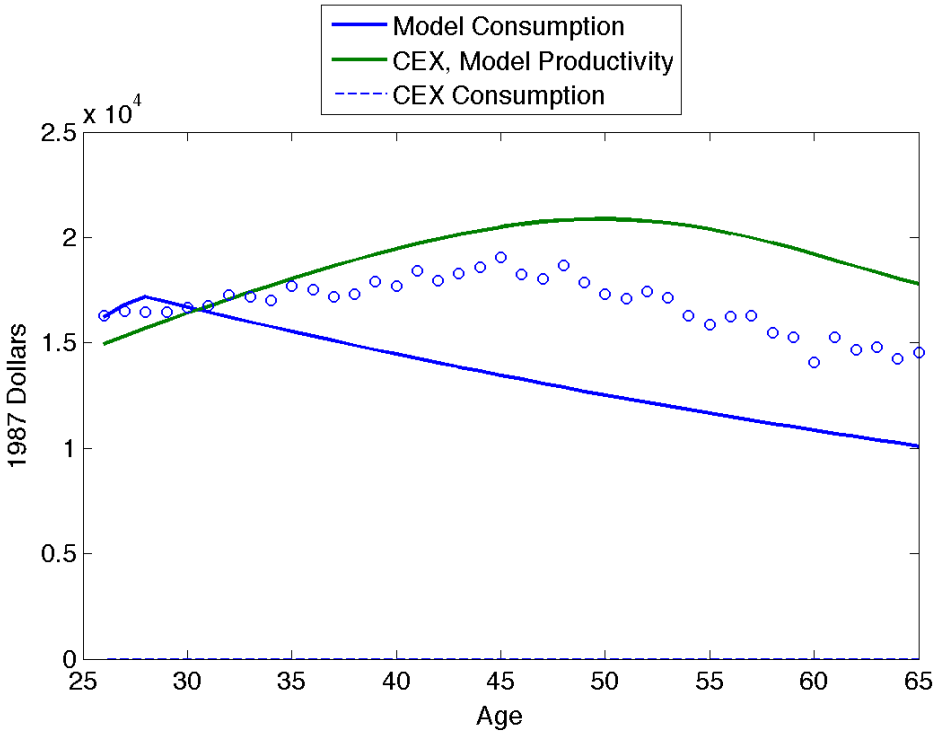


Figure 8: Baseline consumption and income profile

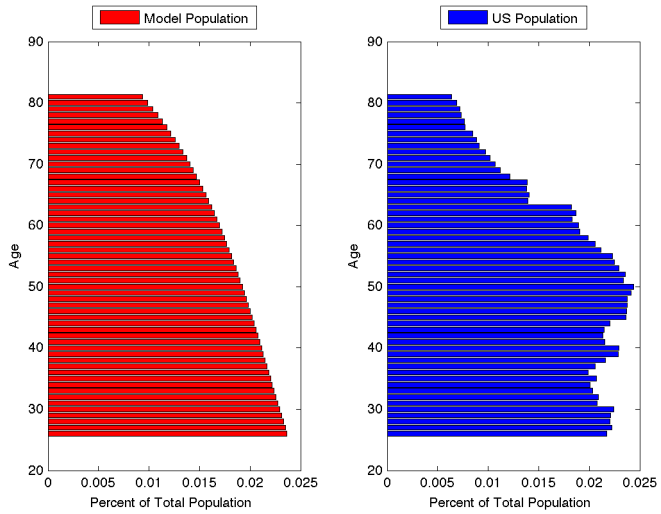


Figure 9: Model and US Population Pyramids

Table 4: Robustness

σ	β	α	δ	**r**	σ	β	α	δ	**r**	σ	β	α	δ	**r**
4.00	0.99	0.25	0.12	-0.03	10.00	0.99	0.25	0.12	-0.04	4.00	1.01	0.25	0.12	-0.03
4.00	0.99	0.25	0.10	-0.00	10.00	0.99	0.25	0.10	0.01	4.00	1.01	0.25	0.10	-0.02
4.00	0.99	0.25	0.14	-0.04	10.00	0.99	0.25	0.14	-0.05	4.00	1.01	0.25	0.14	-0.05
4.00	0.99	0.23	0.12	-0.04	10.00	0.99	0.23	0.12	-0.05	4.00	1.01	0.25	0.14	-0.03

Figure 9 show the model's population pyramid as compared to the current US distribution. One aspect which the model is not matching is the aging of the baby boom population, who are just now coming into retirement and creating a hump in the US pyramid.

A natural question is whether the results are robust to changes in parameters. Table 4 shows the model is able to generate negative interest rates with a variety of utility and production parameters. Higher σ tends to lower real interest rates, higher β tends to lower real interest rates, higher δ tends to decrease rates, and higher α tends to increase rates. While these are simulation results, the appendix has algebraic comparative static results for many parameters in the model.

The borrowing limit has a moderate effect on the real interest rate in this model. There is a strong desire of individuals to save for retirement and bequests, and the borrowing limit only effects them in the first years of their economic lives. Completely eliminating the borrowing limit raises the real interest rate to -1.5% in the standard calibration. Bequests have a moderate to large effect on the interest rate. Cutting the bequest motive in half raises the real rate to -1.7%, and

Table 5: Comparison of Auerbach & Kotlikoff (1981) Model

Model Change	Real Interest Rate
Original A&K Parameters	6.3 %
Add Depreciation	4.2%
Decrease economic growth	3.2%
Add retirement years	2.1%
Change Taxes	1.9 %
Change σ and β	-.7 %

eliminating the bequest motive raises the rate to -.5%. Productivity and population growth have strong effects on the real interest rate. Raising productivity growth to 2% per year raises real rates to .4%, while eliminating all productivity growth lowers rates to -4.8%. Raising population growth to 1% per year raises rates to -2%, while lowering population growth to 0 lowers rates to -3.5%.

To further understand how these results differ from other OLG models in the literature, we perform an exercise comparing our results with the canonical Auerbach, Kotlikoff and Skinner (1981) model. Beginning with the exact parameters of A&K, we change them one by one to match our baseline model and examine the effect on the real interest rate. Here it is seen the combined effects of higher depreciation, lower economic growth, and increase in retirement years on the real interest rate.

3.3 Recapitulation and Discussion

The hallmark of secular stagnation, in the words of Lawrence Summers, is a "chronic excess of savings over investment" (Summers (2015)), and this is exactly how the above model produces negative interest rates. There is a large supply of savings, due to the need to save for retirement as well as to leave bequests. Investment opportunities are limited by the curvature of the production function. If there is not sufficient population growth or productivity growth to sufficiently increase the effective worker population, the marginal product of capital will decrease. Using our baseline parameters, calibrated to that of the US economy, the lines for supply and demand for bonds cross at a negative interest rate. Even with low interest rates, individuals still want to save for retirement and leave bequests. Individuals are able to store goods and suffer depreciation(or storage costs) δ_s, which we assume is greater than the equilibrium interest rate $r^\star$, ensuring that individuals prefer to use the capital markets to save.

The model economy of this paper invests heavily in capital for very little marginal return. Us-

ing the criteria from Abel et al. (1989), it can be shown that the economy is dynamically inefficient. Since the growth rate of the value of the capital stock is positive, and the return on safe assets is less than zero, by their proposition two the economy is dynamically inefficient.

4 US Secular Changes Since the 1980s

In the early 1980s the real federal funds rate was around 6%, and the comparable number in 2014 was around -2%. Can the model explain this change? Unfortunately from a measurement perspective, far from being in a steady state, the US has been in a state of constant flux for at least the past century. Population growth, life expectancy, retirement, productivity growth, and inequality have changed drastically and are projected to continue to change.

With these limitations in mind, we present comparative static results, starting from a 1980s calibrated US economy, that show changes in the real interest rate as the parameters are changed to those of the 2015 calibration. In 1980 the population growth rate was .96% per year. We set the probability of death to match the US life tables from 1980 [9]. We set productivity growth to 2% per year. Individuals retired a year earlier in 1980, at an average age of 62.5 compared to 63.9 in 2015 (Munnell et al. (2011)). However, because individuals are living longer, the total number of years of retirement is actually increasing. To take into account these two changes, we change two factors at once: individuals work one fewer year in 1980, but the maximum lifespan is two years shorter.

Table 6 gives the results of the 1980 calibration. In the 1980 economy, the equilibrium real interest rate is 3.3%. The rest of the columns of Table 6 show the effect on the real interest rate of changing various parameters to their 2015 values. The decrease in productivity growth since 1980 had the largest effect, lowering the real rate 3.7 percentage points. Population growth and survival changes also tended to lower the real rate. The changes in retirement and survival also lead to a decrease in the real interest rate. The takeaways from this exercise are twofold. First, the importance of productivity growth in determining the real interest rate. Second, it gives an estimate of the magnitudes of the effect of demographic changes on the real interest rate.

5 Model Extensions

5.1 Inequality and Interest Rates

As is now widely recognized, income inequality has increased drastically since 1980. Piketty and Saez (2003) data shows that the top 1%'s share of income in the US has grown from 8% to 19% of total income during this time period, while the top 10%'s share of income has grown from 38% to 48%. The "superstars" of the top .1% of the income distribution have done especially well, with

[9] Life expectancy at birth was 73.4 years in 1980, compared with 78.7 in 2010

Table 6: Decomposition of Interest Rate Changes

Variable	Effect of change	Decomposition (%)
Δ Population Growth	-.7	12
Δ Productivity Growth	-3.7	64
Δ Retirement & Survival	-2.2	38

income shares rising from 2.2% in 1980 to 7.5% in 2014. Along with income inequality, wealth inequality has also been increasing. Saez and Zucman (2015) show that the top 0.1% wealth share has grown from 7% in 1980 to 22% in 2013.

In theory, inequality could affect interest rates through several channels. The two most direct are the savings and bequest channel. It is well known that the rich save a greater proportion of their income and give larger bequests (see for example Dynan, Skinner and Zeldes (2004)). Higher savings and bequests translates into greater demand for bonds, and lower interest rates. Inequality can also affect interest rates through the age income profile channel. Age income profiles which have a greater proportion of income at earlier ages tend to increase demand for bonds over the lifecycle, and thus lower the interest rate.

We model inequality in the manner of Fullerton & Rogers (1993) and Altig et al. (2001). There are J types of agents, each with their own separate productivities, utility parameters, and bequest parameters. There are agents with high incomes, who will have higher savings rates and bequest motives, and lower income agents, who will save and bequest less. To simplify the calibration, we make the assumption that agents only differ on three metrics: their human capital profiles, bequest motives, and time discount rates. We include 12 different types of agents, with agent type 1 having the lowest income, and agent type 12 having the highest. The 12 types are not equally sized: type 1 and 12 agents are each 2% of the population, type 2 and 11 agents are each 8% of the population, and the rest are 10% of the population.

Income profiles are calibrated using estimates from Altig et al. (2001), who fit income age profiles using a cubic function using the Panel Survey of Income Data from 1976 to 1988. We calibrate bequests to match the observed ratio of bequests to lifetime income, using data from Fullerton and Rogers (1993). The time discount rates for all agent types we set to .99, as in the baseline model, with two exceptions. We set $\beta = 1$ for agent type 11, and $\beta = 1.01$ for agent type 12. In this way, we are making sure that the rich save more as a percentage of their income. Finally, all other parameters we set as in the baseline model.

Figure 10 gives the income profiles for type 1, 6, and 10 type agents, as well as the income

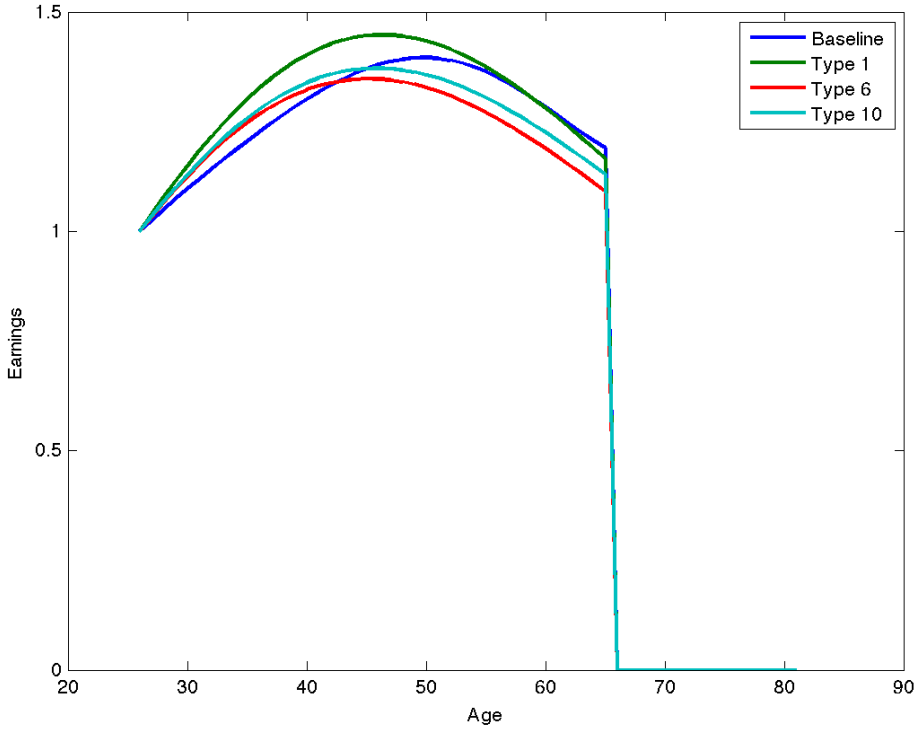


Figure 10: Human Capital Age Profiles, Scaled

Table 7: Inequality Model Results

Variable	Value
Real Interest Rate	-4.3%
Capital Output Ratio	3.1
Savings Rate	11.3%
Income Gini	.52
Wealth Gini	.52

profile for the baseline calibration of section 2, all scaled so the initial income is at 1 in order to better compare the shape. Income profiles tend to peak earlier for all income groups compared to the baseline calibration. With more lifetime income in the earlier portion of the lifecycle we would expected to see lower real interest rates, and indeed we see that in the results, shown in Table 7. The equilibrium real interest rate is -4.3%, and the capital output ratio higher, at 3.1. The wealth gini coefficient is 0.52, higher than in the base calibration, but still lower than the current US value of 0.80. The income Gini index, however, is .52, higher than the actual US value of 0.40 (Bank (2016)).

What is the effect on the real interest rate if we increase income and wealth inequality? As an exercise, we increase income and wealth inequality by multiplying the human capital profile of each type j agent by $(1.1)^{j-1}$, so that the rich receive a greater share of the income and the poor a lower share. In the new equilibrium, the Gini wealth coefficient increases to .61, and the real interest rate decreases to -4.6%.

5.2 Foreign Savings Gluts and Sovereign Debt

Bernanke (2005) argues that the main driver of lower interest rates has been a global savings glut. This hypothesis states that there is a global excess of savings over investment, driven largely by emerging market countries, oil producers, and a few developed countries such as Germany and Japan. These surplus funds flow into the US because of it's developed financial markets and the status of the US dollar as a reserve currency, reducing US interest rates. The data indeed shows a large foreign influx of capital. The US current account deficit grew from $75 billion in 1990 to $802 billion in 2006. Recently, as China has cut back its lending and the price of oil has fallen, the current account deficit has widened, reaching $400 billion in 2014.

A simple way to include an international aspect in our model is to assume the existence of a

foreign nation, the "rest of the world", who wants to hold US bonds. Since we are concerned with steady states, the foreign nation does not demand repayment of the bonds, but simply holds them and consumes the interest indefinitely. This extra demand for bonds will tend to push down the real interest rate.

A natural question is how to specify the demand function of foreigners for US bonds. If Bernanke is correct in his hypothesis, much of the demand for US securities has been driven by emerging market countries such as China to build up war chests against potential capital outflows, as well as oil producing governments who hold US assets because of it's financial markets and status as a reserve currency. If this theory of foreign asset demand is true, then it is likely that the demand for US bonds would be relatively inelastic with respect to the real interest rate, and indeed the data seems to bear this out. Even as US treasury yields reached record lows in the 2000s, foreigners continue to buy the bonds at record rates. Their holdings seem to be concentrated in securities with relatively low yields. While foreigners own nearly 60% of outstanding treasuries as of 2016, they hold relatively little equity, only about 15% of total market value. They do have significant holdings in the corporate bond market, around 40% of the market. (Walker (2014)).

For simplicity, we will model foreign demand for US bonds as completely inelastic. All other things equal, this will tend to increase the effect of foreign demand for bonds on the real interest rate. To calibrate foreign demand for bonds, we use the US Net International Investment Position, which is defined as US ownership of foreign assets minus US liabilities. At the end of the second quarter 2015, the US net investment deficit stood at 40% of GDP. This is a significant deterioration of the US International position, which stood at around 0 in the early 1980s. Adding the foreign demand for US bonds to the baseline calibration decreases the real interest rate to -3.8%.

Counteracting the large foreign demand for bonds has been the supply of bonds from the US government in order to finance its growing debt. The debt has grown from 25% of GDP in 1980, to 74% of GDP in 2015, thus the increase in debt has been larger than the foreign inflow of savings.

6 Risk of Secular Stagnation

In the above sections we have shown that it is quantitatively possible for an economy to have long run negative real interest rates in a purely "real" model of the economy. But what happens if we introduce money into the model? If the real rate is -2.5% and the central bank is targeting 2% inflation, this would mean nominal rates of -.5%, which cannot be reached if there is a "zero lower bound". But unlike in typical New Keynesian models, the natural rate is not temporarily negative, it is *permanently* negative. If we further add nominal frictions to the model, we can generate an equilibrium with a permanent stagnation.

We now extend the model in line with Eggertsson & Mehrotra (2015) to include nominal price level determination. We introduce a nominal price level by assuming that one period nominal

debt denominated in money is traded, and that that the government controls the rate of return of this asset (the nominal interest rate). Individuals in the economy now have access to risk-free nominal debt which is indexed in dollars in addition to one period risk-free real debt[10].The flow budget constraint is now given as

$$P_t b_t = P_t W_t h c_t (1 - \overline{\tau_w}) - P_t C_t + P_t (1 + r_t (1 - \overline{\tau_r})) b_{t-1} + P_t \nu_{r,t} - P_t \nu_{g,t} \tag{16}$$

Where P_t is the price level. We follow the literature and impose a non-negativity constraint on nominal interest rates. Implicitly, we assume that the existence of money precludes the possibility of negative nominal rates, thus at all times we must have

$$i_t \geq 0 \tag{17}$$

The existence of both nominal and real debt implies the Fisher equation:

$$1 + r_t = (1 + i_t) \frac{P_t}{P_{t+1}} \tag{18}$$

Finally, we denote the rate of inflation by $\Pi_t = \frac{P_{t+1}}{P_t}$.

6.1 Aggregate Supply

We depart from the frictionless world of section 2 and add nominal rigidities on wages in line with Eggertsson & Mehrohtra (2015). In particular, we assume that nominal wages are downwardly rigid, but are perfectly flexible in adjusting upward. To formalize this, we assume that households never accept working for wages that fall below their wage in the previous period by a certain amount. The minimum required wage is denoted the household's "wage norm", which is indexed to expected inflation, and is given by the equation

$$\tilde{W}_t = \gamma \Pi^{\star} W_{t-1} + (1 - \gamma) P_t w_t^{flex} \tag{19}$$

where $w_t^{flex} = (1 - \alpha) \bar{K}^{\alpha} \bar{L}^{-\alpha}$. The parameter γ determines the degree of wage rigidity in the economy. If $\gamma = 1$, wages are perfectly downwardly rigid, and they cannot decrease. If $\gamma = 0$, wages are perfectly flexible as in our baseline model.

The actual equilibrium wage in the economy will then be taken to be the maximum of the flexible wage and the norm:

$$W_t = max \left\{ \tilde{W}_t, P_t w_t^{flex} \right\} \tag{20}$$

[10]For simplicity, this asset trades in zero net supply

Whether or not the wage norm will influence the equilibrium wage will depend upon the inflation rate, which is set by monetary policy. In our model, the central bank sets the nominal rate according to a standard Taylor rule:

$$1 + i_t = max\left(1, (1 + i^\star)\left(\frac{\Pi_t}{\Pi^\star}\right)^{\phi_\Pi}\right) \tag{21}$$

where $\Pi^\star$ and $i^\star$ are parameters of the policy rule which are held constant. This rule states that the central bank tries to stabilize inflation around a target $\Pi^\star$, unless it is constrained by the zero lower bound.

6.2 Steady State

Now we analyze the steady state of the model. We do so in two steps: first by tracing out the aggregate supply curve of the economy, and then the aggregate demand.

There are two possible aggregate supply regimes. In the first regime, inflation exceeds target inflation, the wage norm does not bind, and the equilibrium wage rate is equal to $P_t w^{flex}$. In this equilibrium wages behave as if they are flexible, and there is full employment:

$$\frac{\mathcal{L}}{\bar{\mathcal{L}}} = 1 \; for \; \Pi \geq \Pi^\star \tag{22}$$

This is seen in figure 11 as the horizontal line at the top of the aggregate supply curve.

Now, let us analyze the equilibrium in which inflation is below the targeted rate: $\Pi < \Pi^\star$, and thus $W = \tilde{W}$. With inflation less than the target, the wage norm binds, and the real wage is greater than the market clearing wage rate. In the steady state, the real wage takes the value of

$$w = \frac{(1-\gamma)w^{flex}}{1 - \gamma\Pi^\star\Pi^{-1}} \tag{23}$$

Substituting this into the equation for labor market equilibrium (8) yields an equation for the employment rate as a function of inflation:

$$\frac{\gamma\Pi^\star}{\Pi} = 1 - (1-\gamma)\left(\frac{\mathcal{L}}{\bar{\mathcal{L}}}\right)^{\alpha} \tag{24}$$

This equation is a non-linear Phillips curve, and is shown in figure 11 as the lower part of the aggregate supply curve. In this regime, higher inflation means lower real wages, which are in turn closer to the market clearing level and thus a higher level of employment.

Turning now to aggregate demand, once again there are two regimes. In the first regime, monetary policy is not constrained by the zero lower bound, and the aggregate demand curve is download sloping. The logic for this relationship is familiar: if inflation increases, the central bank raises the nominal interest rate by more than one for one, which in turn increases the real interest

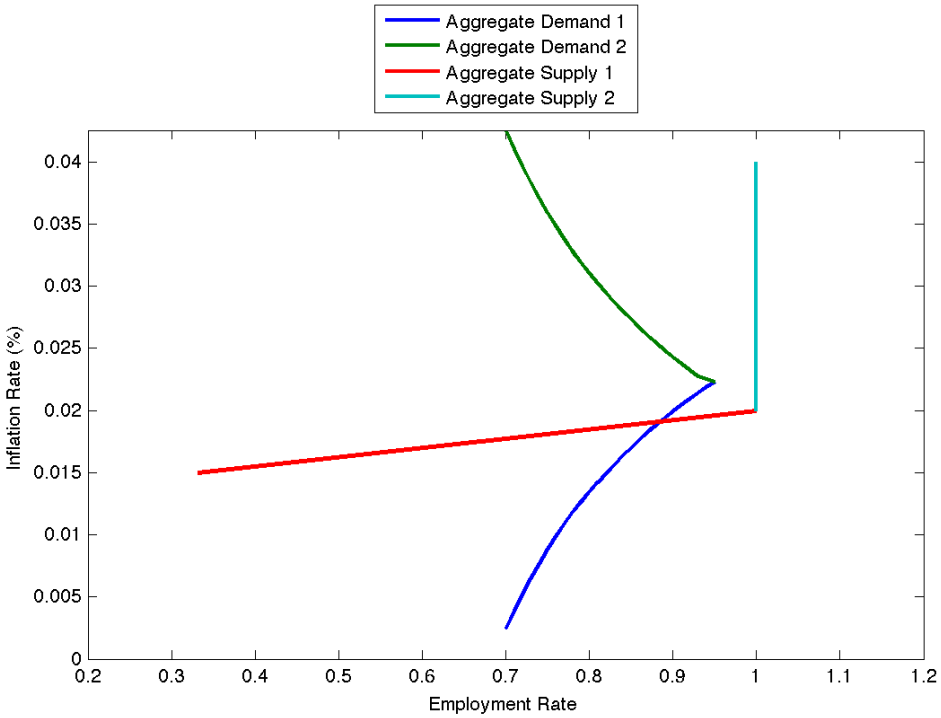


Figure 11: Secular Stagnation Equilibrium

rate, and reduces demand. This is shown by the top portion of the aggregate demand curve in figure 11.

The kink in the aggregate demand curve occurs at the inflation rate at which monetary policy is constrained by the zero lower bound. At this point, the zero lower bound is binding and aggregate demand curve becomes upward sloping. In this regime, higher inflation means lower real interest rates, and thus a higher level of output.

In figure 11, the target inflation rate $\Pi^\star$ is 2%, while the equilibrium real interest rate is -2.5%. Since at the inflation target the nominal interest rate is negative, we are in the "secular stagnation" equilibrium, which in this calibration leads to an unemployment rate of 10%.

7 Conclusion

Are long term negative real interest rates possible in the real world? Now we are able to speak to the classical argument against zero and negative interest rates. As stated by Bernanke:

> As ... Paul Samuelson taught me in graduate school at MIT, if the real interest rate were expected to be negative indefinitely, almost any investment is profitable. For example, at a negative (or even zero) interest rate, it would pay to level the Rocky Mountains

> to save even the small amount of fuel expended by trains and cars that currently must climb steep grades. It's therefore questionable that the economy's equilibrium real rate can really be negative for an extended period

In the model economy of this paper, there are no "Rocky Mountain" type of investments available, which pay out positive sums indefinitely with no depreciation. There is also no land, which also pays out a positive sum over an infinite time horizon. Both of these pose a problem in that with zero or negative interest rates their present value is infinite.

There are several potential ways to answer this criticism. The most straightforward way is to concede the point that perhaps rates cannot be negative indefinitely, but they instead can have a finite but long time horizon, say 20 to 50 years.

As many commentators have noted, another way around this criticism is to suggest that in an economy with stochastic risk, the risk free interest rate could be negative, while the expected return on capital would be positive. In this way indefinitely negative interest rates could be sustained. Also, if there are financial frictions in intermediation, there could be a sufficient wedge between the borrowing and lending rate such that the borrowing rate is positive and the lending rate negative.

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