A Model of the Reserve Asset*

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Abstract

A portion of the global wealth portfolio is directed towards a safe and liquid reserve asset, which recently has been the US Treasury bond. Our model links the determination of reserve asset status to relative fundamentals and relative debt sizes, by modeling two countries that issue sovereign bonds to satisfy investors' reserve asset demands. A sovereign's debt is more likely to be the reserve asset if its fundamentals are strong relative to other possible reserve assets, but not necessarily strong on an absolute basis. Debt size can enhance or detract from reserve asset status. If global demand for the reserve asset is high, a large-debt sovereign which offers a savings vehicle with better liquidity is more likely to be the reserve asset. If demand for the reserve asset is low, then large debt size is a negative as it carries more rollover risk, leading to a riskier vehicle for saving. When global demand is high, countries may make fiscal/debt-structuring decisions to enhance their reserve asset status. These actions have a tournament feature, and are self-defeating: countries may over-expand debt size to win the reserve asset tournament. Coordination can generate benefits. We use our model to study the benefits of "Eurobonds" – i.e. a coordinated common Europe-wide sovereign bond design. Eurobonds deliver welfare benefits only when they make up a sufficiently large fraction of countries' debts. Small steps towards Eurobonds may hurt countries and not deliver welfare benefits.

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1 Introduction

US government debt has been the world's reserve asset for over half a century. German government debt is the reserve asset within Europe. US and German debt appear to have high valuations relative to the debt of other countries with similar fundamentals, measured in terms of debt or deficit to income ratios. Moreover, as fundamentals in the US and Germany have deteriorated, these high valuations have persisted. Finally, as evident in the financial crises over the last five years, during times of turmoil, the value of these countries' bonds rise relative to the value of other countries' bonds in a "flight to quality." That is, reserve assets have a .

This paper develops a model that helps understand these facts and think about what drives the value of a reserve asset. We study a model with many investors and two countries, each which issues government bonds. The investors have a pool of savings that they must invest in government bonds – there are no alternative savings vehicles. Thus the bonds of one, or possibly both of the countries, will hold these savings and serve as a reserve asset. However, the debts are subject to rollover risk. The countries differ in their fundamentals, which measures their ability to service their debt and factors into their rollover risk; and debt size, which proxies for the financial depth or liquidity of the country's debt market. Our model links fundamentals and debt size to the valuation and equilibrium determination of the reserve asset.

An important assumption is that there are no savings vehicles other than the countries' sovereign debts. That is, all savings needs are satisfied by sovereign debt that is subject to rollover risk. There is no "gold" in the model, nor are there any corporations/banks that are able to honor commitments of repaying debts. Alternatively, the model can be interpreted as one where such vehicles do exist, but their supplies are small relative to the reserve asset needs of investors. Thus, substantially all of the world's reserve asset needs must be satisfied by debt that is subject to rollover risk.

In the model, an investor's valuation of a bond depends on the number of other investors who purchase that bond. If only a few investors demand a country's bond, the bond auction fails and the country defaults on the bond. For a country's bonds to be safe, the number of investors who invest

in the bond must exceed a threshold, which is decreasing in the country's fundamentals (i.e., its fiscal surplus) and increasing in the size of the debt . Our modeling of rollover risk is similar to Calvo (1988) and Cole and Kehoe (2000). Investor actions are *complements* – as more investors invest in a country's bonds, other investors are incentivized to follow suit. Valuation has a coordination aspect which we consider to be an important feature of the economics of a reserve asset.

Besides the above strategic complementarity, the model also has a strategic substitutability force, as is common in models of competitive financial markets. Once the number of investors who invest in the bonds exceeds the threshold required to rollover debts, then investor actions become *substitutes*. Beyond the threshold, more demand for the bond that is in fixed supply drives up the bond price, leading to lower returns. Our model links the debt size to this strategic substitutability: for the same investor demand, a smaller debt size leads to a smaller return to investors. In other words, countries with lower financial depth are at a disadvantage in accommodating investors' demand for a reserve asset.

The model predicts that relative fundamentals more so than absolute fundamentals are an important component of debt valuation. Relative fundamentals matter because of the coordination aspect of valuation. Investors expect that other investors will invest in the country with better fundamentals, and thus relative valuation determines which country's bonds to be the reserve asset. This prediction helps understanding the observations we have made regarding the valuation of US debt in a time of deteriorating fiscal fundamentals. In short, all countries' fiscal conditions have deteriorated along with the US, so that US debt has maintained and perhaps strengthened its reserve asset status. The same logic can be used to understand the value of the German Bund (as a reserve asset within Europe) despite deteriorating German fiscal conditions. The Bund has retained/enhanced its value because of the deteriorating fiscal conditions of other Euro area countries.

The model also predicts that debt size is an important determinant of reserve asset status. If the global demand for reserve assets is high, then large debt size enhances reserve asset status. Consider an extreme example with a large debt country and a small debt country – in fact, infinitesimally

small. If investors coordinate on this small debt as the reserve asset, then the return on the reserve asset will be infinitesimal. That is the quantity of world demand concentrating on a small float of bonds will drive bond prices up to a point that investors' incentives in equilibrium will be to coordinate investment in the large debt. On the other hand, if global demand for reserve assets is low, then investors will be concerned that the large debt may not attract sufficient demand to rollover the debt. That is, when funding conditions are tight, rollover risk for a large debt size is high. In this case, investors will tend to coordinate on the small debt size as the reserve asset.

Our model offers some guidance on when the US may lose its reserve asset status. Many academics have argued that we are and have been in a global savings glut, which in the model will correspond to a high global demand for reserve assets. In this case, the US is likely to continue its reserve asset status unless US fiscal fundamentals deteriorate significantly relative to other countries, or if another sovereign debt can compete with the US Treasury in terms of size. Eurobonds seem like the only possibility of the latter, although there is considerable uncertainty whether such bonds will exist and will have better fundamentals than the US debt. However, if the savings glut ends and the world moves to a low demand for reserve assets, then our model predicts that an alternative high fundamentals country with a relatively low supply of debt may become the reserve asset. The German Bund is a leading example of such an asset.

The size effect of our model also identifies a novel *contagion* channel. In the high savings regime, a larger debt size of the large debt country adversely affects the small debt country by increasing the small country's probability of default and its expected bond yield. This is because the expansion of the supply of the reserve asset creates safe "parking spots" for funds that may otherwise have been invested in small country debt. From 2007Q4 to 2009Q4, the supply of US Treasury bonds increased by $2.7 trillion (the money stock increased another $1.3 trillion). Our model hence hints at a causal link between the increase in US Treasury bond supply/Fed QE and the eruption of the European sovereign debt crisis in 2010.

We use our model to study incentives to change debt size, when doing so may enhance reserve asset status. We study a case where two countries have a "natural" debt size, determined for example

by their GDP, but can deviate from its natural debt size by some adjustment cost. Two interesting cases emerge. When countries are roughly symmetric – similar natural debt size – and when global demand for reserve assets is high, countries will engage in a rat race to become the reserve asset. Starting from the natural debt sizes, and holding fixed the size decision of one country, the other country will have an incentive to increase its debt size since the larger debt size can confer a reserve status. But then the first country will have an incentive to respond in a similar way, and so on so forth. In equilibrium, both countries will expand in a self-defeating manner beyond their natural debt size. This prediction of the model can help to shed some light on the expansion of relatively safe stocks of debt in the US (GSE debt) and Europe (sovereign debt) in the build-up to the crisis. These expansions have ultimately ended badly.

The model identifies a second case, when countries are asymmetric and one country is the natural "top dog." In this case, the larger debt country will have an incentive to reduce debts to the point that balances rollover risk and retaining reserve asset status, while the smaller country will have an incentive to expand its debt size. The model is suggestive that asymmetry leads to better outcomes than symmetry.

We use our model to investigate the benefits of creating "Eurobonds." We are motivated by the Eurobond proposals that have been floated over the last few years (see Claessens, Mody and Vallee, 2012, for a review of various proposals). A shared feature of the many proposals is to create a common Europe-wide reserve asset. Each country receives proceeds from the issuance of the "common bond" which is meant to serve as the reserve asset, in addition to proceeds from the sale of an individual country-specific bond. By issuing a common Euro-wide reserve asset, all countries benefit from investors' need for a reserve asset, as opposed to just one country (Germany) which is the de-facto reserve asset in the absence of a coordinated security design. Our model, in which the determination of the reserve asset is endogenous, is well-suited to analyze these issues formally. Suppose that countries issue α share of common bonds and $1-\alpha$ share as individual bonds. We ask, how does varying α affect welfare, and the probability of safety for each country? Our main finding is that welfare is only unambiguously increased for α above a certain threshold. Above this threshold,

the common-bond structure enhances the safety of both common bonds and individual bonds, and all these bonds develop some reserve asset status. Thus, a higher α makes both countries safer. However, below the threshold, welfare can be increasing or decreasing, depending on the assumed equilibrium; and one country may be made worse off while another may be made better off by increasing α. Thus we conclude that a successful Eurobond proposal requires a significant amount of coordination and volume / size of said Eurobonds.

LITERATURE REVIEW There is a literature in international finance on the reserve currency through history. Historians identify the UK as the provider of the reserve asset in the pre-World War 1 period, and the US as the provider of the reserve asset post-World War 2. There is some disagreement about the interwar period, with some scholars arguing that there was a joint reserve asset in this period. Eichengreen (1995, 2004, 2008) discusses this history.

A reserve currency fulfills three roles: an international store of value, a unit of account, and a medium of exchange (Krugman, 1984, Frankel, 1992). Our paper concerns the store of value role, which is why we present a model of the "reserve asset" rather than the "reserve currency." There is a broader literature in monetary economics on the different roles of money (e.g., Kiyotaki and Write, 1989, Banerjee and Maskin, 1996, Lagos and Wright, 2005, Freeman and Tabellini, 1998, Doepke and Schneider, 2014), and our analysis is most related to the branch of the literature motiving money as a store of value. Samuelson (1958) presents an overlapping generation model where money serves as a store of value, allowing for intergenerational trade. Diamond (1965) presents a related model but where government debt satisfies the store of value role. In this class of models, there is a need for a store of value, but the models do not offer guidance on which asset will be the store of value. For example, it is money in Samuelson (1958) and government debt in Diamond (1965). In our model, the reserve asset arises endogenously. We are unaware of other work on the endogenous determination of the reserve asset as a store of value.

Our paper also belongs to a growing literature on safe asset shortages. Theory work in this area explores the macroeconomic and asset pricing implications of safe asset shortages (Holmstrom

and Tirole, 1998, Caballero, Farhi, Gourinchas 2008, Caballero and Krishnamurthy 2009, Caballero and Farhi, 2015). There is also an empirical literature documenting safe asset shortages and their consequences (Krishnamurthy and Vissing-Jorgensen 2012, 2015, Greenwood and Vayanos 2010, Bernanke et al. 2011). Our analysis presumes that there is a macroeconomic shortage of safe assets (stores of value) and endogenously determines the characteristics of government debt supply that satisfies the safe asset demand.

The element of rollover risk in our model is in the spirit of Calvo (1988) and Cole and Kehoe (2000). Besides the sovereign debt literature, rollover risk is an active research area in the corporate finance literature as well, including Diamond (1991), He and Xiong (2012a,b), and He and Milbradt (2014). We utilize the global games technique (Carlsson and van Damme, 1993; Morris and Shin, 1998; and others) to link the country's relative fundamental to the determination of reserve asset. Relative to that literature, our setting features strategic substitution (in fact, a large country size alleviates the strategic substitution effect) which makes our analysis non-trivial.

2 Model

2.1 The Setting

Consider a two-period model with two countries, indexed by i, and a continuum of homogeneous risk-neutral investors, indexed by j. At date 0 each investor is endowed with one unit of consumption good, which is the numeraire in this economy. Investors invest in the bonds offered by these two countries to maximize their expected date 1 consumption, and there is no other storage technology available. This latter restriction is important to the analysis as will be clear.

There is a large country, called country 1, and a small country, called country 2. We normalize the size of the large country to be one (i.e., $s_1 = 1$), and denote the size of the small country by $s_2 \equiv s \in (0, 1]$. Each country sells bonds at date 0 promising repayment at date 1. The country size determines the total face value (in terms of promised repayment) of bonds that each country sells: the large (small) country offers 1 (s) units of sovereign bonds. Hence the aggregate bond supply is

$1 + s$. All bonds are zero coupon bonds.

The aggregate measure of investors, which is also the aggregate demand for bonds, is $1 + f$, where $f > 0$ is a constant parameterizing the aggregate savings need. To save, we assume that investors place *market orders* to purchase sovereign bonds. In particular, since purchases are via *market orders*, the aggregate investor demand does not depend on the equilibrium price.[1] Denote by p_i the equilibrium price of the bond issued by country i. Since there is no storage technology available to investors, all savings of investors go to buy these sovereign bonds. This implies via the market clearing condition that

$$s_1 p_1 + s_2 p_2 = p_1 + s p_2 = 1 + f.$$

Country i has fundamentals denoted θ_i. Purely as a matter of notation we write the surplus as proportional to country size, i.e., for country i it is $s_i \theta_i$. We assume that each country has an existing debt obligation, assumed to be equal to the country size s_i.[2] The country has total resources consisting of fundamentals $s_i \theta_i$ and the proceeds from newly issued bonds $s_i p_i$,

$$s_i \theta_i + s_i p_i.$$

We assume that a country defaults if and only if

$$\underbrace{s_i \theta_i + s_i p_i}_{\text{total funds available}} < \underbrace{s_i}_{\text{debt obligations}}.$$

which we rewrite as,

[1] Market orders avoid the thorny theoretical issue of investors using the information aggregated by the market clearing price to decide which country to invest in, a topic extensively studied in the literature of Rational Expectation Equilibrium.

[2] One can think of the timing, as discussed in the text, as s_i is past debts that must be rolled over. This is a rollover risk interpretation, where we take the past debt as given. Here is another interpretation. The bonds are auctioned at date 0 with investors anticipating repayment at date 1. The date 0 proceeds of $s_i p_i$ are used by the country in a manner that will generate $s_i \theta_i + s_i p_i$ at date 1 which is then used to repay the auctioned debt of s_i.

$$s_i p_i < s_i(1 - \theta_i).$$

If the country defaults at date 0, there is zero recovery and any investors who purchased the bonds of that country receive nothing.[3] If the country does not default, then each investor in that country receives one at date 1. For simplicity, there is no default possibility at date 1, e.g., this assumption can be justified by a sufficiently high fundamental in period 1.

We note that our model of sovereign debt features a multiple equilibrium crisis, in the sense of Calvo (1988) and Cole and Kehoe (2000). If investors conjecture that other investors will not invest in the debt of a given country, then p_i is low which means the country is more likely to default, which rationalizes the conjecture that other investors will not invest in the debt of the country.

The "fundamentals" of θ_i increase a country's surplus thus giving the country more cushion against default. For most of our analysis we refer to θ_i as the country's fiscal surplus, which then increases the funds available to the country to rollover its debt. But there are other interpretations which are in keeping with our modeling. For the case of foreign currency denominated debt, θ_i can include both the fiscal surplus and the foreign reserves of the country. For the case where the debt is denominated in domestic currency, θ_i can include resources the central bank may be willing to provide to forestall a rollover crisis. In this case, such resources, provided via monetization of debt, may be limited by central bank concerns over inflation or a devalued exchange rate (and its potential negative effects on the country's real surplus). Finally, θ_i can also be interpreted to include reputational costs associated with defaulting on debts, in which case the default equation, $s_i p_i < s_i(1 - \theta_i)$, can be read as one where default is driven by unwillingness-to-pay.

We follow the global games approach to link equilibrium selection to fundamentals. We assume that there is a publicly observable world-level fundamental index θ lying in the interval $(0, 1)$. Our analysis focuses on a measure of relative strength between country 1 and country 2, which we denote by $\tilde{\delta}$. Specifically, conditional on the relative strength $\tilde{\delta}$, the fundamentals of these two countries

[3]For the situation of positive recovery, see Section 3.6.

satisfy

$$1-\theta_1 = (1-\theta)\exp\left(-\tilde{\delta}\right); \quad (1)$$

$$1-\theta_2 = (1-\theta)\exp\left(+\tilde{\delta}\right). \quad (2)$$

Recall that $1-\theta_i$ is the funding need of a country. Given $\tilde{\delta}$, the higher the θ, the greater the surplus of both countries and therefore the lower their funding need. And, given θ, the higher the $\tilde{\delta}$, the better are country 1 fundamentals relative to country 2, and therefore the lower is country 1's relative funding need.[4] Finally, the above specification implies that the funding need for each country is always positive.

We assume that the relative strength of country 1, has a support

$$\tilde{\delta} \in \left[-\overline{\delta}, \overline{\delta}\right] \quad (3)$$

We do not need to take a stand on the distribution over the interval $\left[-\overline{\delta}, \overline{\delta}\right]$. For much of the analysis we set $\overline{\delta} < \ln \frac{1+f}{s(1-\theta)}$, which ensures that for the worse case scenario, financing need of either country exceeds the total savings $1+f$. This gives us the usual dominance regions when the fundamentals take extreme values.

As we will use the global games technique to pin down the unique threshold strategy equilibrium, we assume that the country 1 relative strength $\tilde{\delta}$ is not publicly observable. Instead, each investor $j \in [0,1]$ receives a private signal

$$\delta_j = \tilde{\delta} + \epsilon_j,$$

where $\epsilon_j \sim \mathbb{U}[-\sigma, \sigma]$ and ϵ_j are independent across all investors $j \in [0,1]$. Following the global games literature a la Morris and Shin (2003) we will focus on the limit case where noise $\sigma \to 0$.

Finally, note that although we do not need to take a stand on the distribution of $\tilde{\delta}$, for much of

[4]The scale of $1-\theta$ and exponential noises $e^{\tilde{\delta}}$ in (1) and (2) help in obtaining a simple close-form solution. The Appendix A.1 considers an additive specification $\theta_i = \theta + (-1)^i \tilde{\delta}$ and solves the case for $\sigma > 0$; we show that the main qualitative results hold in that setting.

the analysis, it will make most sense to think of a distribution that places all of the mass around some point δ_0 and almost no mass on other points. This will correspond to a case where investor-j is almost sure that fundamentals are δ_0, but is unsure about what other investors know, and whether other investors know that investor-j knows fundamentals are δ_0. That is, we will focus on a limiting case where there is no fundamental uncertainty and only strategic uncertainty.

2.2 Equilibrium Characterization and Properties

We focus on symmetric threshold equilibria in this section. More specifically, we assume that all investors adopt the same threshold strategy in which each investor purchases country 1 bonds if and only if his private signal about country 1's relative strength is above a certain threshold, i.e. $\delta_j > \delta^*$; otherwise he purchases country 2 bonds. The corner portfolio decisions are for simplicity and are consistent with risk neutrality. Later on we show that this is indeed the unique equilibrium under a monotone strategy assumption given some parameter restrictions.

Deriving the equilibrium threshold. In equilibrium, the marginal investor who receives the threshold signal $\delta_j = \delta^*$ must be indifferent between investing his money in either country. Based on this signal, the marginal investor forms belief about other investors' signals and hence their strategies. Denote by x the fraction of investors who receive signals that are above his own signal $\delta_j = \delta^*$, and as implied by threshold strategies will invest in country 1. It is well-known (e.g., Morris and Shin, 2003) that in the limit of diminishing noise $\sigma \to 0$, the marginal investor forms a "diffuse" view about other investors' strategies, in that he assigns a uniform distribution for $x \sim \mathbb{U}[0,1]$.

Combined with the cutoff strategy, the fraction of investors who purchase the bonds of country 1 is equal to the fraction of investors deemed more optimistic than the marginal agent, x. Thus, the total funds going to country 1 and 2 are $(1+f)\,x$ and $(1+f)\,(1-x)$, respectively. The resulting bond prices are thus

$$p_1 = (1+f)\,x \text{ and } p_2 = \frac{(1+f)\,x}{s}.$$

We now calculate the expected return from investing in bond i, Π_i.

Expected return from investing in country 1. Given x and its fundamental θ_1, country 1 does not default if and only if

$$p_1 - 1 + \theta_1 \quad = \quad (1+f)\,x - 1 + \theta_1 \geq 0 \Rightarrow x \geq \frac{1-\theta_1}{1+f}. \tag{4}$$

This is intuitive: country 1 does not default only when there are sufficient investors who receive favorable signals about country 1 and place their funds in country 1's bonds accordingly. The survival threshold $\frac{1-\theta_1}{1+f}$ is lower when the country 1 fundamental, θ_1, is higher and when the total funds available for savings, f, are higher.

Of course, the country 1 fundamental $1-\theta_1 = (1-\theta)\,e^{-\tilde{\delta}}$ in (1) is uncertain. We take the limit as $\sigma \to 0$, so that the signal is almost perfect and the threshold investor who receives a signal δ^* will be almost certain that[5]

$$1 - \theta_1 = (1-\theta)\,e^{-\delta^*}. \tag{5}$$

Hence, in the limiting case of $\sigma \to 0$, plugging (5) into (4) we find that the large country 1 survives if and only if

$$x \geq \frac{1-\theta_1}{1+f} = \frac{(1-\theta)\,e^{-\delta^*}}{1+f}. \tag{6}$$

Here, either higher average fundamentals θ or a higher threshold δ^* make country 1 more likely to repay its debts.

Now we calculate the investors' return by investing in country 1. Conditional on survival, the realized return is

$$\frac{1}{p_1} = \frac{1}{(1+f)\,x};$$

while if default occurs the realized return is 0. From the point of view of the threshold investor with

[5] In equilibrium, θ_1 depends on the realization of x, which is the fraction of investors with signals above δ^*. Given that the signal noise ϵ_j is drawn from a uniform distribution over $[-\sigma, \sigma]$, we have

$$x = \Pr\left(\tilde{\delta} + \epsilon_j > \delta^*\right) = \frac{\tilde{\delta} + \sigma - \delta^*}{2\sigma} \Rightarrow \tilde{\delta} = \delta^* + (2x-1)\,\sigma.$$

which implies that $\theta_1 = \theta + (1-\theta)\left(1 - e^{-\delta^* - (2x-1)\sigma}\right)$. Taking $\sigma \to 0$ we get (5).

signal δ^*, the chance that country 1 survives is simply the integral w.r.t. to the uniform density dx from $\frac{(1-\theta)e^{-\delta^*}}{1+f}$ to 1:

$$\Pi_1 = \int_{\frac{(1-\theta)e^{-\delta^*}}{1+f}}^{1} \frac{1}{(1+f)\,x} dx = \frac{1}{1+f}\left(\ln\frac{1+f}{1-\theta} + \delta^*\right).$$

The higher the threshold δ^*, the greater the chance that country 1 survives, and hence the higher the return by investing in country 1 bonds.

Expected return from investing in country 2. Denote the measure of investors that are investing in country 2 by $x' \equiv 1 - x$, that is the fractino of investors that are more pessimistic than the marginal agent, which again follows a uniform distribution over $[0, 1]$. If the investor instead purchases country 2's bonds, he knows that country 2 does not default if and only if

$$sp_2 - s + s\theta_2 = (1+f)\,x' - s + s\theta_2 \geq 0 \Leftrightarrow x' \geq \frac{s\,(1-\theta_2)}{1+f}, \tag{7}$$

Country 2 survives if the fraction of investors investing in country 2, x', is sufficiently high. The threshold is lower if the country is smaller, fundamentals are better, and the total funds available for savings are higher.

Similar to the argument in the previous section, in the limiting case of almost perfect signal $\sigma \to 0$, country 2 fundamental θ_2 in (7) is almost certain from the perspective of the threshold investor with signal δ^* (recall (2)):

$$1 - \theta_2 = (1-\theta)\,e^{\delta^*}. \tag{8}$$

Plugging equation (8) into equation (7), we find that country 2 survives if and only if

$$x' \geq \frac{s\,(1-\theta)\,e^{\delta^*}}{1+f}. \tag{9}$$

Relative to (6), country size s plays a role. All else equal, the lower size s and the smaller country 2, the more likely that the country 2 survives.

Given survival, the investors' return of investing in country 2, conditional on x', is

$$\frac{1}{p_2} = \frac{s}{(1+f)\,x'}; \tag{10}$$

while the return is zero if country 2 defaults. As a result, using (10), the expected return from investing in country 2 is

$$\begin{aligned}
\Pi_2 &= \int_{\frac{s(1-\theta)e^{\delta^*}}{1+f}}^{1} \frac{s}{(1+f)\,x'} dx'. \\
&= \frac{1}{1+f} \cdot \underbrace{s}_{\text{payout}} \left(\underbrace{-\ln s}_{\text{survival prob.}} + \underbrace{\ln \frac{1+f}{1-\theta} - \delta^*}_{\text{country 2 fundamental}} \right)
\end{aligned}$$

Here, "country 2 fundamental" essentially comes from turning off the size effect by setting $s = 1$. Then, we see that this term is the same as for country 1.

We decompose the source of the country 2 return in three parts. The third part "country 2 fundamental" is the most transparent: for the threshold investor who observes aggregate fundamental θ and receives a signal δ^* about the relative strength between country 1 and country 2, $\ln \frac{1+f}{1-\theta} - \delta^*$ represents the country 2 fundamental – that is, when setting $s = 1$, this is what remains, analogous to country 1. The first part, "payout", and the second part, "survival prob", are affected by country size. The first part s indicates that the total bond payment is of size s; all else equal, the smaller the country size the lower the return, as can be seen in (10). The second part $-\ln s$, which is decreasing in s, captures the effect of a country's size on its survival probability. This is because all else equal, a smaller country is more likely to survive for a given absolute amount of investment in its bonds, driving up the total return, as can be seen in (9).

Expected return of investing in country 1 versus country 2. Figure 1 plots the return to investing in each country as a function of x (x') which is the measure of investors investing in country 1 (country 2), from the perspective of the marginal investor with $\delta^* = \tilde{\delta} = 0$ so $\theta_1 = \theta_2 = \theta$.

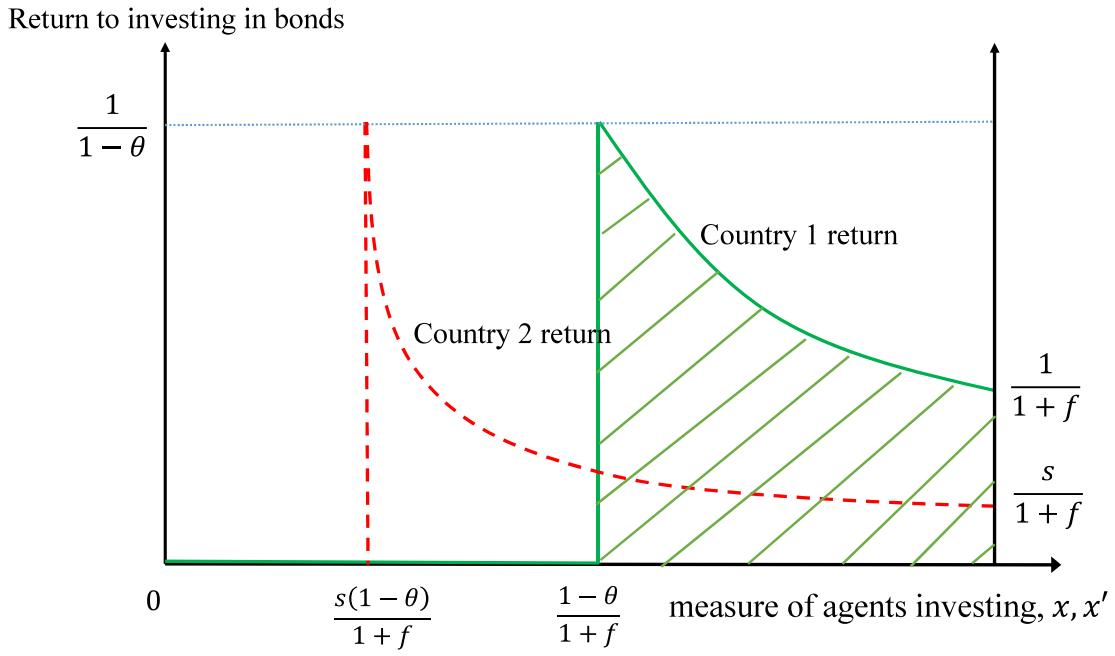


Figure 1: **Returns of the marginal investor when investing in country 1 (country 2) as a function of x (x').** The return to investing in country 1 is in green solid line, while the return to investing in country 2 is in red dashed line. The figure assumes $\delta^* = 0$ so that the marginal investor believes that both countries have the same fundamentals. The bonds issued by the large country 1 only pay when $x > \frac{1-\theta}{1+f}$ while country 2's bonds only pay when $x' > \frac{s(1-\theta)}{1+f}$. The return to country 1's bonds falls to $\frac{1}{1+f}$ when $x = 1$, while for country 2's bonds the return falls more rapidly to $\frac{s}{1+f}$ when $x' = 1$.

Consider the solid green curve first which is the return to investing in country 1. For x below the default threshold $\frac{1-\theta}{1+f}$, the return is zero. This default threshold is relatively high, since country 1 is large and hence it needs a large number of investors to buy bonds to ensure a successful auction. Across the threshold $\frac{1-\theta}{1+f}$, investor actions are strategic complements – i.e., if a given investor knows that other investors are going to invest in country 1, the investor wants to follow suit. Past the threshold, the return falls as the face value of bonds is constant and investors' demand simply bids up the price of the bonds. In this region, investor actions are strategic substitutes. The marginal investor's expected return from investing in country 1 is the integral of shaded area beneath the green solid line.

The dashed red curve plots the return to investing in country 2, as a function of x' which is the

measure of investors investing in country 2. The default threshold for country 2, which is $\frac{s(1-\theta)}{1+f}$, is lower than for country 1 ($\frac{1-\theta}{1+f}$) because country 2 only needs to repay a smaller number of bonds. When $\delta^* = 0$, i.e., the marginal investor with signal $\delta^* = 0$ believes that both countries share the same fundamentals, the threshold return to investing in country 2 is $\frac{1}{1-\theta}$. This is the same as the threshold return to investing in country 1, as shown in Figure 1. While country 2 has a lower default threshold which implies a smaller strategic complementarity effect, past the threshold the return to investing in country 2 falls off quickly. That is, the strategic substitutes effect is more significant for country 2 than country 1. This is because country 2 has a small bond issue and hence an increase in demand for country 2 bonds increases the bond price (decreases return) more than the same increase in demand for country 1 bonds. We see this most clearly at the boundary where $x = x' = 1$, where the return to investing in the large country 1 is $\frac{1}{1+f}$, while the return to investing in country 2 is $\frac{s}{1+f}$.

To sum up, because the large country auctions off more bonds, it needs more investors to participate to ensure no-default. However, the very fact that the large country sells more bonds makes the large country a deeper financial market that can offer a higher return on investment. This tradeoff – size features more rollover risk but provides a more liquid savings vehicle – is at the heart of our analysis.

Equilibrium threshold δ^* The equilibrium threshold δ^* is determined by the indifference condition for the threshold investor between investing in these two countries, i.e.,

$$0 = \Pi_1 - \Pi_2 = \frac{1}{1+f}\left(\ln\frac{1+f}{1-\theta} + \delta^*\right) - \frac{s}{1+f}\left(-\ln s + \ln\frac{1+f}{1-\theta} - \delta^*\right).$$

Solving for the equilibrium threshold signal δ^* yields (recall that $s \in (0,1]$)

$$\delta^* = \underbrace{-\frac{1-s}{1+s}}_{\text{negative, liquidity}} \times z + \underbrace{\frac{-s\ln s}{1+s}}_{\text{positive, rollover risk}}. \tag{11}$$

where we define

$$z \equiv \ln \frac{1+f}{1-\theta} > 0.$$

Here, z measures aggregate funding conditions, which is greater if either more aggregate funds f are available or there is a higher aggregate fundamental θ. The "savings glut" which many have argued to characterize the world economy for the last decade is a case where z is high.

From (11) we see that there are two effects of size. The first term is negative (for $s \in (0,1)$) and reflects the liquidity or market depth benefit that accrues to the larger country, making country 1 safer all else equal. The second term is positive and reflects the rollover risk for country 1, whereby a larger size makes country 1 less safe. The benefit term is modulated by the aggregate funding condition z. We next discuss implications of our model based on the equation (11).

3 Model Implications

3.1 Determination of the reserve asset

The equilibrium threshold δ^* tells us which of country 1 and country 2 will not default and thus which country's debt will serve as the reserve asset. Consider the case where the distribution of $\tilde{\delta}$ places all of the mass around some point δ_0 and almost no mass on other points. This will correspond to a case where investor-j is almost sure that fundamentals are δ_0, but is unsure about what other investors know, and whether other investors know that investor-j knows fundamentals are δ_0. If $\delta_0 > \delta^*$ then country 1 debt is the reserve asset, while if $\delta_0 < \delta^*$ then country 2 debt is the reserve asset. Given that all investors know almost surely the value of δ_0, investors are then almost sure which country is the reserve asset. Mapping this interpretation to thinking about the world, the model says on any given day investors almost sure, for example, that the US Treasury bond is the reserve asset.

The following proposition gives the properties of the equilibrium threshold $\delta^*(s,z)$, as a function of country 2's relative size s and the aggregate funding condition z.

Proposition 1 *We have the following results for the equilibrium threshold* $\delta^*(s,z)$.

1. *The equilibrium threshold* $\delta^*(s,z)$ *is decreasing in the aggregate funding conditions* $z = \ln\frac{1+f}{1-\theta}$, *i.e.,* $\frac{\partial}{\partial z}\delta^*(s,z) < 0$. *Hence, country 1's bonds can be the reserve asset for worse values of country 1 fundamentals* $\tilde{\delta}$, *if the aggregate fundamental* θ *or aggregate saving* f *is higher.*

2. *The equilibrium threshold* $\delta^*(s,z) \leq 0$ *for all* $s \in (0,1]$, *if and only if* $z \geq 1$. *Hence, when the aggregate funding* $z \geq 1$, *the bonds issued by the larger country 1 can be the reserve asset for worse values of country 1 fundamentals* $\tilde{\delta}$.

3. *When* $s \to 0$ *the equilibrium threshold* $\delta^*(s,z)$ *approaches its minimum, i.e.,* $\lim_{s\to 0}\delta^*(s,z) = \inf_{s\in(0,1]}\delta^*(s,z) = -z < 0$. *This implies that all else equal, country 1 is the reserve asset over the widest range of fundamentals when country 2 is smallest.*

Proof. Result (1.) follows because of $\frac{\partial}{\partial z}\delta^*(s,z) = -\frac{1-s}{1+s} < 0$, and $\Pr\left(\tilde{\delta} > \delta^*\right)$ is decreasing in z. To show result (2.), note that when $z = 1$ we have $\delta^*(s, z=1) = \frac{s - s\ln s - 1}{1+s} < 0$ for $s \in (0,1]$. This inequality can be shown by observing 1) $[s - s\ln s - 1]' > 0$ and 2) $[s - s\ln s - 1]_{s=1} = 0$. Result (3.) holds because

$$\delta^*(s,z) = -\frac{1-s}{1+s}z + \frac{-s\ln s}{1+s} > -\frac{1-s}{1+s}z > -z,$$

where the last inequality is due to $-\frac{1-s}{1+s}z$ being increasing in s for $z > 0$. ■

We illustrate these effects in Figure 2. The left panel of Figure 2 plots δ^* as a function of s for the case of $z = 1$, which corresponds to strong aggregate funding conditions with abundant savings and/or good fundamentals. In this case, the equilibrium threshold $\delta^*(s)$ is always negative, and is monotonically increasing in the small country size s. For small s close to zero, the large country is safest for the worst possible values of its fundamental, because in this case country 2 does not exist as an investment alternative. Then because all investors have no choice but to invest in country 1, the bonds issued by country 1 have minimum rollover risk. If we assume that the aggregate savings $1 + f$ are enough to cover country 1's financing shortfall $1 - \theta_1\left(\tilde{\delta}\right)$ even for the worst realization of $\tilde{\delta} = -\overline{\delta}$ then country 1 will always be the reserve asset in this case. This $s = 0$ case offers one

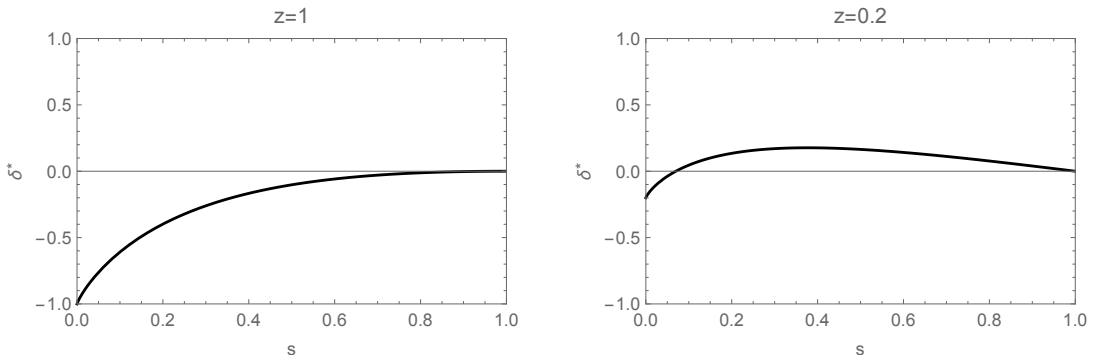


Figure 2: Equilibrium threshold δ^* as a function of country 2 size s. The left panel is for the case of strong aggregate funding conditions with $z = 1$, and the right panel is for the case of low aggregate funding conditions with $z = 0.2$.

perspective on why Japan has been able to sustain a large debt without suffering a rollover crisis. Since many of the investors in Japan are so heavily invested in Japanese government, eschewing foreign alternative investments, Japan's debt is safe.

The right panel in Figure 2 plots δ^* for a case of weak aggregate funding conditions ($z = 0.2$), with insufficient savings and/or low fundamentals. Consistent with the first result in Proposition 1 we see that in this case the large country can be at a disadvantage. For medium levels of s (around 0.4), investors are concerned that there will not be enough demand for the large country bonds, exposing the large country to rollover risk. As a result, investors coordinate on the small country's debt as the reserve asset even if the small country has worse fundamentals. For small s, the size disadvantage of the small country becomes a concern, and large country can be the reserve asset even with poor fundamentals (the third result in Proposition 1). For s large, we are back in the symmetric case. Comparing the right panel with $z = 1$ to the left panel with $z = 0.2$ highlights that the large country's debt size is a clear advantage only when the aggregate funding conditions are strong; as the pool of savings shrink, and in equilibrium interest rates rise, the large debt size triggers rollover risk fears so that investors coordinate on the small country as the reserve asset.

3.2 Relative fundamentals

Our model emphasizes relative fundamentals as being a central ingredient in debt valuation. To clarify this point, consider a standard model without coordination elements and without the reserve asset saving need. In particular, suppose that the world interest rate is R^* and consider any two countries in the world with surpluses given by θ_1 and θ_2. Suppose that investors purchase these countries' bonds for $p_i s_i$ and receive repayment of $s_i \min(\theta_i, 1)$. Then,

$$p_1 = \frac{\mathbb{E}[\min(\theta_1, 1)]}{1 + R^*} \quad \text{and} \quad p_2 = \frac{\mathbb{E}[\min(\theta_2, 1)]}{1 + R^*},$$

so that bond prices depend on fundamentals, but not particularly on relative fundamentals $\theta_1 - \theta_2$. In contrast, in our model when the country 1's relative fundamentals are high $\tilde{\delta} > \delta^*$, bonds issued by country 1 attract all the savings so that

$$p_1 = 1 + f \quad \text{and} \quad p_2 = 0. \tag{12}$$

Similarly, if the country 1's relative fundamentals are low $\tilde{\delta} < \delta^*$, investors only invest in bonds issued by country 2 so that

$$p_1 = 0 \quad \text{and} \quad p_2 = 1 + f. \tag{13}$$

Valuation in our model becomes sensitive to relative fundamentals, as investors are endogenously coordinating to buy bonds issued by relatively stronger country. In Section 3.6 we show that these forces also explain why the reserve asset carries a negative β.

The importance of relative fundamentals helps us to understand why, despite deteriorating US fiscal conditions, US Treasury bond prices have continued to be high: In short, all countries' fiscal conditions have deteriorated along with the US, so that US debt has maintained and perhaps strengthened its reserve asset status. The same logic can be used to understand the value of the German Bund (as a reserve asset within Europe) despite deteriorating German fiscal conditions. The Bund has retained/enhanced its value because of the deteriorating general European fiscal

conditions.

Remark 2 *We have analyzed a case where θ reflects the overall fundamentals of the global economy, and the relative fundamental $\widetilde{\delta}$ is "almost perfectly" observable by investors, thus discussing safety of the two countries in terms of the realization of $\widetilde{\delta}$. In practice, it is common knowledge that the US or Germany have stronger fundamentals than many other countries. We can easily modify our model to capture this effect, by specifying the surpluses of countries to be*

$$1-\theta_1 \;=\; (1-\theta)\exp\left(-\tilde{\delta}-\Delta\right),\ \textit{and}\ 1-\theta_2=(1-\theta)\exp\left(+\tilde{\delta}+\Delta\right),$$

where Δ is a component of the relative fundamental that is common knowledge. The equilibrium threshold now becomes

$$\delta^* = -\frac{1-s}{1+s}z + \frac{-s\ln s}{1+s} - \Delta.$$

Consistent with intuition, common knowledge that a country has better fundamentals makes the country's debt more likely to be the reserve asset.

3.3 Size and aggregate funding conditions

Our model highlights the importance of debt size in determining reserve asset status, and its interactions with the aggregate funding conditions. In the high savings regime, which the literature on the global savings glut has argued to be true of the world in recent history (see, e.g., Bernanke, 2005, Caballero, Farhi, Gourinchas, 2007, and Caballero and Krishnamurthy, 2009), higher debt size increases reserve asset status. The US is the world reserve asset in part because it has maintained large debt issues that can accommodate the world's reserve asset demands.

These predictions of the model also offer some insight into when the US Treasury bond may be displaced as reserve asset. If the world continues in the high savings regime, the US will only be displaced if another country can offer a large debt size and/or good relative fundamentals. This seems unlikely in the foreseeable future. On the other hand, if the world switches to the low savings

regime, it is possible that another country with a smaller debt size and good fundamentals, such as the German Bund, will take on the reserve asset role.

The size effect also offers a perspective on the period prior to World War 1 when the UK consol bond was the world's reserve asset. Despite the fact that the GDP of the US had caught up the GDP of the UK by 1870, the UK offered the reserve currency of the world. Our model provides one explanation for this puzzle. In 1890, the UK Debt/GDP ratio was 0.43, while the US Debt divided by UK GDP was about 0.10. The higher float of UK debt was perhaps one reason that the UK remained the reserve asset during a period when its fundamentals were likely worse. Debt stocks of both US and UK rise quickly in World War 1, with UK Debt/GDP reaching 1.40 by 1920, and US debt/UK GDP reaching 0.80 by 1920. Our model suggests that as the UK debt size grew, size turned from a liquidity advantage to a rollover risk concern. At the same time, the rise in the US debt as a liquid and sizeable alternative allowed the US debt to replace the UK debt as reserve asset to the world.

The size effect of our model also identifies a novel *contagion* channel. In the high savings regime, increasing the debt size of the large debt country reduces δ^* and thus increases the probability of default and expected bond yield of the smaller country.[6] We can see this from Figure 2, left panel with $z = 1$. Suppose that we decrease the relative size of country 2, s, away from 1; it is equivalent to increasing the size of the large country's debt. We see that the default probability of small country, which is just one minus the default probability of the large country, goes up; this comparative static result will be derived formally in equation (44) later. Linking this observation to data, from 2007Q4 to 2009Q4, the supply of US Treasury bonds increased by $2.7 trillion (the money stock increased another $1.3 trillion). Our model suggests that this increase should have increased default probabilities of other country's debts. That is, our model suggests a causal link from the increase in US Treasury bond supply/Fed QE and the eruption of the European sovereign debt crisis in 2010. Intuitively, the expansion of US debt supply created safe "parking spots" for funds that may

[6]The yield on the large country debt is ambiguous because the direct effect of increasing debt size, for fixed f, would cause yields to rise, but since the probability of default falls, there is a countervailing effect that reduces bond yields.

otherwise have been invested in European sovereign debt.

The sovereign debt contagion channel is a novel prediction for which there is no systematic empirical evidence. However, there is a closely related contagion channel for which there is evidence. Let us reinterpret the small country debt of the model as the short-term debt issued by a private bank that can be used to store value, but is subject to a coordination/rollover problem (i.e., bank runs). Then our model suggests that as the supply of country 1 sovereign debt rises, the safety of the private bank debt should fall. Krishnamurthy and Vissing-Jorgensen (2013) provide evidence of this sort of crowding out, documenting a positive relation between the supply of US government debt and the yields on private short-term bank debt.[7]

3.4 Switzerland, Denmark and gold

There are no savings vehicles in the model other than the countries' sovereign debts. That is, all savings needs must be satisfied by sovereign debt that is subject to default risk. There is no "gold" in the model, nor are other companies, banks or other governments that are able to honor commitments to repay debts. In practice such assets do exist. Switzerland and Denmark have been prominent in the news recently because of safe-haven flows into these countries, perhaps because these countries can commit to repay their relatively small outstanding supply of bonds. It is easy to accommodate this possibility into the model.

Suppose that there exists a quantity of full-commitment sovereign bonds. The supply of these bonds is $\underline{s}$, that is, these bonds pay in total $\underline{s}$ at the final date. Investors invest $f - \hat{f}$ in these bonds, with a return of $\frac{\underline{s}}{f-\hat{f}}$. Let us focus on the symmetric case with $s = 1$ and $\delta^* = 0$. Investing in sovereign bonds of country 1 or country 2 depending on the signal $\tilde{\delta}$ gives a return of $\frac{1}{1+\hat{f}}$ as

[7]The model is suggestive of further links between government debt and private bank debt. We would expect that the yields on non-government-backed private bank debt (e.g., large time deposits) would especially rise relative to government-backed bank debt (e.g., checking deposits). Government-backed bank debt would rise in safety with increases in US government debt, if such increases lead to an increase in the reserve status of US government debt. Plausibly, banks would respond by increasing the supply of government-backed debt relative to non-government-backed debt. Krishnamurthy and Vissing-Jorgensen (2013) confirm this prediction in the data. The model also suggests a motive for increasing the backing of bank deposits during periods of rising US government debt, to avoid a contraction of the banking system. To investigate these issues thoroughly our model needs to be extended to accommodate a banking system with many banks, which we intend to do in future work.

the small noise assumption implies that investors are perfectly able to pick the "winner". Thus in equilibrium it must be that the full-commitment bond also offers a return of $\frac{1}{1+\hat{f}}$, which then implies that

$$\frac{\underline{s}}{f-\hat{f}} = \frac{1}{1+\hat{f}} \Rightarrow \hat{f} = \frac{f-\underline{s}}{1+\underline{s}}.$$

We assume that the supply of full-commitment bonds $\underline{s}$ satisfies $\underline{s} < f$ so that $\hat{f} > 0$. We then can solve our model following exactly the same steps, only with f redefined as $\hat{f}$. Thus, the model can be interpreted as one where alternative savings vehicles do exist, but their supplies are such that substantially most of the world's reserve asset needs must still be satisfied by debt that is subject to rollover risk.[8]

Denmark and Switzerland have recently restricted their supplies of safe bonds. The result has been that the prices of their bonds have risen, with interest rates in both countries falling below zero. We can also see this in our model. Reducing $\underline{s}$ causes $\hat{f}$ to rise, and hence the price of safe bonds rises.

3.5 Non-monotone strategies and joint safety equilibria

We have restricted the agents' strategy space to the so-called "cut-off" strategies, i.e., invest in country 1 if δ_i is above certain threshold; otherwise invest in country 2. This section discusses potential equilibria once this strategy space is relaxed.

Denote the probability (or fraction) of investment in country 1 by an agent with signal δ_i by $\phi(\delta_i) \in [0,1]$; the agent's strategy is called monotone if $\phi(\delta_i)$ is monotonically increasing in his signal δ_i of the country 1's fundamental. Within the montone strategy space, we are able to prove, under paremeter restrictions, that the equilibrium with cut-off strategies constructed above will

[8]The total government debt of Switzerland in early 2015 was $127bn. Central bank liabilities were nearer $500bn, having grown significantly with the Europe crisis and the Swiss decision to maintain their exchange rate vis-a-vis the Euro. Total government debt in Denmark was $155bn. Total central bank holdings of gold are approximately $1.2tn, although this amount is largely backing for government liabilities, rather than privately investable gold. It is difficult to get a clear sense of the quantity of gold held privately as an investment, but it is likely not larger than the central bank holdings of $1.2tn. The most liquid gold investment are gold ETFs. Total capitalization of US gold ETFs was $39bn in early 2015. As a comparison, the total supply of Treasury bonds plus central bank liabilities (reserves, cash, repos) in early 2015 was over $16tn.

survive as the unique equilibrium via the iterated deletion of dominated strategies.[9]

Proposition 3 *Under certain parameter restrictions, the equilibrium with cut-off strategies constructed Eq. (11) is the unique equilibrium within the monotone strategy space.*

Once we allow agents to choose among non-monotone strategies, i.e. $\phi(\delta_i)$ is non-monotone, then for a higher z it is possible to construct equilibria where both countries are safe given certain realizations of the relative fundamental signal $\tilde{\delta}$ (while one country fails if $\tilde{\delta}$ is too low or too high). In Appendix A.3 we construct a non-monotone equilibrium in which agent are using "oscillating strategies."

Under this oscillating strategy, agents invest in country 2 for low δ_i, then invest in country 1 given better signals, but go back to investing in country 2 for even higher signals. This strategy profile is driven by the strategic substitutes effect in our model, as it serves to equalize returns from investing when both countries are safe. Indeed, in the constructed equilibria with non-monotone strategies, non-monotonicity occurs only in the region where both countries are safe given the realization of fundamental $\tilde{\delta}$ and equilibrium investment strategies. In this region, knowing that both countries will be safe, investors who are indifferent oscillate between investing in country 1 and country 2 depending on their private signal realizations. Note that the fundamental $\tilde{\delta}$ is no longer payoff relevant for safe countries (recall that without default the payoff of bonds is capped at one). Hence, the private signal is no longer payoff relevant, and this oscillation can be viewed as a randomization scheme to equalize returns. This is because every agent (say with a signal δ_i) in this region knows that other agents whose private signals span an interval of 2σ are investing in both countries in the right (no-arbitrage) proportion, leading to equalized returns in both countries.

Though seemingly exotic, it is interesting that equilibria with non-monotone strategies lead to the economically plausible situation that when funding conditions in the world are plentiful (high z), both countries may be safe and there is a possibility of a joint reserve asset. This possibility cannot emerge in the case we have analyzed where agents play monotone threshold strategies.

[9]The argument is similar to standard global games result, with the caveat that we do not have global strategic complementarity.

We stress that all key qualitative properties in Proposition 1 derived under the threshold strategy equilibria are robust to considering the non-monotone strategy equilibria, with minor modifications. The next proposition summarizes the results parallel to Proposition 1.

Proposition 4 *We have the following results for the equilibrium with oscillating strategies.*

1. *The survival region of the larger country 1 (hence the lowest value of fundamentals such that country 1 bonds are the reserve asset) increases with the aggregate funding conditions z. However, a higher z also increases the survival region of the smaller country 2.*

2. *For sufficiently favorable aggregate funding conditions $z \geq \underline{z}$ where $\underline{z}$ is derived in the appendix, the equilibrium with oscillating strategies exists. In this class of equilibria, the bonds issued by the larger country 1 are the reserve asset for a wider range of fundamentals than the bonds issued by the smaller country 2.*

3. *All else equal, the larger country 1 is the reserve asset for the lowest level of fundamentals when the size of country 2 goes to zero, i.e. $s \to 0$.*

Regarding the first result, recall that in the monotone threshold equilibria studied in the main text, a higher z increases the survival region of the larger country 1 and at the same time *decreases* the survival region of the smaller country 2. This is because only one country survives in the monotone threshold equilibria. In contrast, in the oscillating non-monotone strategy equilibria, both counties may survive, and thus improved aggregate funding conditions makes both countries safer. The second result of Proposition 4 reinforces that in Proposition 1, i.e., under sufficiently favorable aggregate funding conditions so that the non-monotone strategy equilibrium exists, the bonds of the larger country are more likely to be the reserve asset than the bonds of the smaller country. The third result is identical to that of Proposition 1.

3.6 Negative β of the reserve asset

At the height of the US financial crisis, in the aftermath of the Lehman failure, the prices of US Treasury bonds increased dramatically in a flight to quality. Over a period in which the expected liabilities of the US government likely rose by several trillion dollars, the value of US government debt went up. We compute that from September 12, 2008 to the end of trading on September 15, 2008 the value of outstanding US government debt rose by just over $70bn. Over the period from September 1, 2008 to December 31, 2008, the value of US government debt outstanding as of September 1 rose in value by around $210bn. These observations indicate that US Treasury bonds are a "negative β" asset. In this section, we show that the reserve asset is naturally a negative β asset, and this β is closely tied to the strength of an asset's reserve position.

To make these points we introduce a positive recovery value in default, $l_i > 0$. In our baseline model with zero recovery, the price of the reserve asset is equal to $\frac{1+f}{s_i}$ regardless of shocks which does not allow us to derive predictions for the β, which is the sensitivity of price to shocks. Suppose that country i's debt recovery per unit of face value, given default, is $l_i > 0$. This says that the total payouts from the defaulting country 1 or country 2 are l_1 or sl_2, respectively. For simplicity, we do not allow l_i to be dependent on the country's relative fundamental $\tilde{\delta}$. However, l_i may depend on the average fundamental θ, to which we will introduce shocks later when calculating the β of the assets.

When recovery is strictly positive, there is a strong strategic substitution force that pushes investors to buy the defaulting country's debt if nobody else is doing so. This is because an infinitesimal investor would earn an unbounded return if she is the only investor for the defaulting country's bonds, given a strictly positive recovery. But this implies that threshold strategies are no longer optimal in any symmetric equilibrium, especially when the signal noise σ vanishes. To see this, given the conjectured threshold signal δ^*, an investor with a signal $\delta_j = \tilde{\delta} + \varepsilon_j \neq \delta^*$ knows that other investors—who follow the conjectured threshold strategies—are solely investing in the surviving country. But then she will strictly prefer to invest in the defaulting country for an infinite

return.

Similar to the way we deal with strategic substitutability in the joint survival equilibria studied in the previous section, we focus on the strategy space of oscillation strategies to construct an equilibrium for the case of positive recovery. The basic idea, in the spirit of global games, is as follows. Suppose that the relative fundamental of country 1, i.e., $\tilde{\delta}$, is sufficiently high so that country 1 survives for sure, irrespective of investors' strategies. This corresponds to the upper dominance region in global games. Then, investors given their private signals will follow an oscillation strategy so that on average there are $\frac{1}{1+l_2 s}$ ($\frac{l_2 s}{1+l_2 s}$) measure of investors purchasing the bonds issued by country 1 (2). This way, the defaulting country 2 pays out $l_2 s$ while the safe country 1 pays out 1, and each investor receives the same return of

$$\frac{1}{(1+f)\frac{1}{1+l_2 s}} = \frac{l_2 s}{(1+f)\frac{l_2 s}{1+l_2 s}} = \frac{1+l_2 s}{1+f}.$$

When $\tilde{\delta}$ is relatively high but below that upper dominance region, we postulate that this oscillation strategy prevails in equilibrium, so that country 1 is the only safe country. When $\tilde{\delta}$ is relatively low but above the lower dominance region, investors follow an oscillation strategy so that on average there are $\frac{l_1}{l_1+s}$ ($\frac{s}{l_1+s}$) measures of investors purchasing the bonds issued by country 1 (2). This way, defaulting country 1 pays out l_1 while the surviving country 2 pays out s, and each investor receives the same return of

$$\frac{l_1}{(1+f)\frac{l_1}{l_1+s}} = \frac{s}{(1+f)\frac{s}{l_1+s}} = \frac{l_1+s}{1+f}.$$

Again, for $\tilde{\delta}$ being relatively low but above the lower dominance region, we postulate that this oscillation strategy prevails in equilibrium so that country 2 is the safe country.

Similar to how equilibrium is determined in global games, given that these two oscillating strategies prevail for high and low $\tilde{\delta}$, there will be an endogenous switching threshold δ^*, such that it is optimal for investors with private signals above δ^* to follow the oscillation strategies in which country 1 survives, while it is optimal for investors with private signals below δ^* to follow the oscillation strategies in which country 2 survives. In the appendix we show how these two oscillation strategies

paste at the switching threshold δ^*, and prove that the proposed investment strategies as a function of private signals are indeed optimal given the other investors' strategies. The closed-form solution for δ^* is given by

$$\delta^* = \frac{\left[(1-l_2)\,s-(1-l_1)\right]z-(s+l_1)\ln(s+l_1)+(1+sl_2)\ln(1+l_2s)+l_1\ln l_1-sl_2\ln l_2}{(1-l_1)+s\,(1-l_2)}. \quad (14)$$

When setting $l_1 = l_2 = 0$, we recover $\delta^* = \frac{-(1-s)z-s\ln(s)}{1+s}$, which coincides with the equilibrium threshold in the baseline model with zero recovery.

3.6.1 Default Recovery and Equilibrium Prices

We summarize the equilibrium with positive default recovery for the case where signal noise $\sigma \to 0$. For relative fundamental $\tilde{\delta} \in [\underline{\delta}, \delta^*)$ where the endogenous threshold δ^* is given by (14), $\frac{l_1(1+f)}{l_1+s}$ investors purchase bonds issued by country 1 and $\frac{s(1+f)}{l_1+s}$ purchase bonds issued by country 2. As a result, the price of each bond is given by

$$p_1 = \frac{l_1\,(1+f)}{l_1+s}, \text{ and } p_2 = \frac{1+f}{l_1+s}.$$

On the other hand, when the relative fundamental $\tilde{\delta} \in (\delta^*, \underline{\delta}]$, $\frac{1+f}{1+l_2s}$ investors purchase bonds issued by country 1 and $\frac{l_2s(1+f)}{1+l_2s}$ purchase bonds issued by country 2. The resulting prices are

$$p_1 = \frac{1+f}{1+l_2s}, \text{ and } p_2 = \frac{l_2\,(1+f)}{1+l_2s}.$$

Compared to the results with zero default recovery that we have derived in earlier section, this extension with a positive recovery allows us to not only determine the endogenous threshold of the reserve asset status, but also the endognenous bond prices for both countries (in the zero recovery case, the prices are either zero or $(1+f)/s_i$) by equalizing the returns across both countries. As bond prices of the two countries are linked via the cash-in-the-market pricing, the defaulting country's recovery can affect the price of the reserve asset.

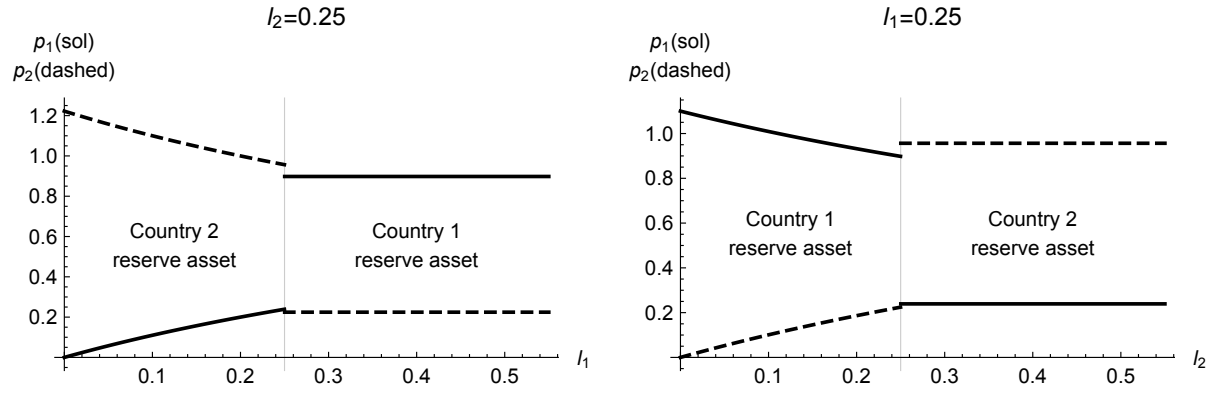


Figure 3: Bond prices p_1 (solid) and p_2 (dashed) when the default recoveries l_i's vary. Here, the parameters are set to $z = 1$, $s = 0.9$, $f = 0.1$ and $\delta = -0.02$.

To illustrate this point, we fix a realization of relative fundamental $\delta > \delta^*$ so that the country 1's bonds are the reserve asset. Figure 3 plots the endogenous bond prices as a function of l_1 (left panel) and l_2 (right panel). Note that both recovery parameters affect the endogenous threshold δ^*, so that δ^* may be below the realized relative fundamental δ for some values of l_i. This explains the discrete jump of bond prices in Figure 3 when $l_1 = 0.25$ and $l_2 = 0.25$, as in these case the jump corresponds to country 1 being the reserve asset. The solid line represents p_1, whereas the dashed-line represents p_2.

As shown in the left panel of Figure 3, both bond prices are unaffected by l_1 in the region that country 1's bonds are the reserve asset. This is intuitive: we are in a situation where country 1 survives, hence its hypothetical recovery value does not matter. However, in the region where l_1 is low and $\delta^* > \delta$, country 1 is not the reserve asset and its bond price is increasing in l_1. In contrast, through the cash-in-the-market pricing effect, the bond's of country 2 are decreasing in l_1 in this region. In the right panel, when the recovery of country 2, i.e., l_2, decreases, then p_2 drops and p_1 rises if country 1 is the reserve asset.

The above comparative statics with respect to l_i suggest that in our model the reserve asset will behave as a negative beta asset. To see this, suppose that aggregate fundamentals deteriorate–say θ falls– leading to lower recovery for both bonds in case of default. However, as shown above, the drop in the reserve asset's recovery, l_1, has no impact on the equilibrium bond prices as long as

country 1 remains the reserve asset. In contrast, the worse recovery of country 2 pushes investors into a flight to quality, and country 1's bond price goes up. Combining both effects, country 1's bonds gain when the aggregate fundamental deterioriates, which makes it a negative beta assset. The next section develops this point fully.

3.6.2 Negative Beta Asset

Suppose that θ, which proxies for the aggregate fundamental for both countries, is subject to shocks. For convenience, suppose that $\tilde{\theta}$ is drawn from the following uniform distribution

$$\tilde{\theta} \sim U\left[\underline{\theta}, \overline{\theta}\right],$$

and recall $z\left(\tilde{\theta}\right) = \ln \frac{1+f}{1-\tilde{\theta}}$. Also, suppose that

$$l_i = l\tilde{\theta}, i \in \{1, 2\}$$

where $l > 0$ is a positive constant, so that recovery is increasing in the funadamental shock. Using (14), we calculate the threshold $\delta^*(\theta)$ as a function of the realization of $\tilde{\theta} = \theta$, to be

$$\delta^*(\theta) = \frac{\left[(1-l\theta)s - (1-l\theta)\right] z(\theta) - (s+l\theta)\ln(s+l\theta) + (1+sl\theta)\ln(1+l\theta s) + l\theta\ln(l\theta) - sl\theta\ln(l\theta)}{(1-l\theta) + s(1-l\theta)}$$

Note that $\frac{d}{d\theta}\delta^*(\theta) < 0$; that is, a higher θ, by reducing rollover risk, makes country 1 safer.

In this exercise we consider a distribution so that the relative fundamental δ is almost surely,

$$\delta > \delta^*\left(\mathbb{E}\left[\theta\right]\right).$$

This implies that ex ante country 1 bonds are more likely to be the reserve asset. Also, define $\hat{\theta}(\delta)$ so that $\delta^*\left(\hat{\theta}\right) = \delta$ holds; this is the critical value of fundamental $\theta = \hat{\theta}$ so that country 1's bonds lose the reserve asset status. We choose δ so that $\hat{\theta} > \underline{\theta}$, which implies that with strictly positive

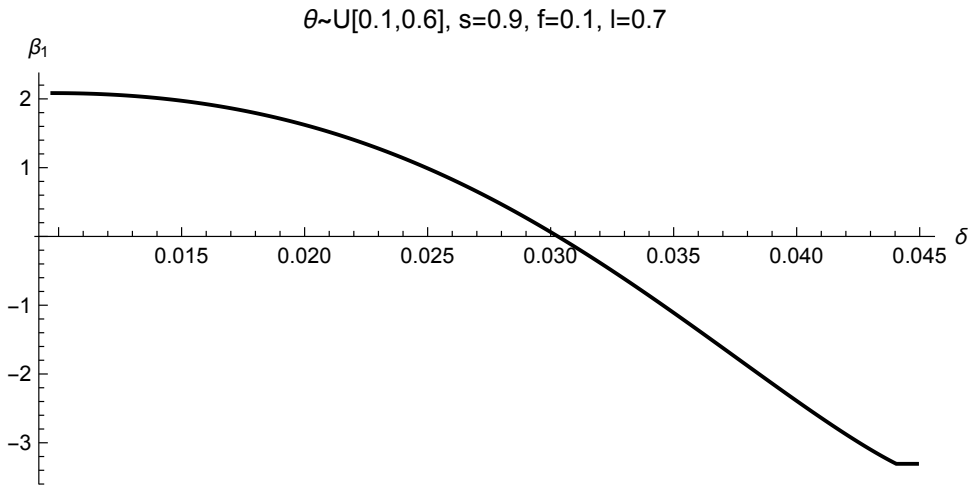


Figure 4: $\beta_1 = \frac{Cov(p_1,\theta_1)}{Var(\theta_1)}$ for the bonds issued by country 1, as function of country 1's relative fundamental δ.

probability, country 1 defaults given a sufficiently low fundamental.

We are interested in the beta of the bond price of each country with respect to the θ shock, i.e.,

$$\beta_i(\delta) = \frac{Cov\left(p_i\left(\tilde{\theta};\delta\right),\tilde{\theta}\right)}{Var\left(\tilde{\theta}\right)} = \frac{\mathbb{E}\left[p_i\left(\tilde{\theta};\delta\right)\cdot\tilde{\theta}\right] - \mathbb{E}\left[\tilde{\theta}\right]\mathbb{E}\left[p_i\left(\tilde{\theta};\delta\right)\right]}{Var\left(\tilde{\theta}\right)}, \tag{15}$$

Using the results in Section 3.6.1, we know that

$$p_1(\theta;\delta) = \begin{cases} \frac{(1+f)l\theta}{s+l\theta} & \text{if } \theta<\hat{\theta}(\delta) \text{ so country 1 defaults;} \\ \frac{1+f}{1+l\theta s} & \text{if } \theta \geq \hat{\theta}(\delta) \text{ so country 1 survives;} \end{cases}$$

and

$$p_2(\theta;\delta) = \begin{cases} \frac{1+f}{s+l\theta} & \text{if } \theta<\hat{\theta}(\delta) \text{ so country 2 survives;} \\ \frac{(1+f)l\theta}{1+l\theta s} & \text{if } \theta \geq \hat{\theta}(\delta) \text{ so country 2 defaults.} \end{cases}$$

Given these pricing functions, it is straightforward to evaluate betas in (15). We vary country 1's relative strength δ and plot the betas for both bonds as a function of δ in Figure 4. We only plot the β for country 1's bonds, because $\beta_2 = -\beta_1/s$ in our model.[10] As suggested by the intuition

[10]This is because cash-in-the-market-pricing implies that $p_1 + sp_2 = 1 + f$.

laid out in the previous section, the beta for the country 1's bonds is negative when the country 1's relative fundamental δ is high, i.e., when country 1 enjoys the reserve asset status. Moreover, the higher country 1's relative fundamental, the more negative the beta of its bonds.

4 Coordination and Security Design

In this section, we characterize the benefits to coordinating through security design. We are motivated by the Eurobond proposals that have been floated over the last few years (see Claessens, Mody and Vallee, 2012, for a review of various proposals). A shared feature of the many proposals is to create a common Europe-wide reserve asset. Each country receives proceeds from the issuance of the "common bond" which is meant to serve as the reserve asset, in addition to proceeds from the sale of an individual country-specific bond. By issuing a common Euro-wide reserve asset, all countries benefit from investors' need for a reserve asset, as opposed to just the one country (Germany) which is the de-facto reserve asset in the absence of a coordinated security design. Our model, in which the determination of the reserve asset is endogenous, seems well-suited to analyze these issues formally. We are unaware of other similar models or formal analysis of this issue.

We return to the model with exogenous sizes, fixing $s_1 = 1$ and $s_2 = s$. The two countries issue a common bond of size $\alpha(1+s)$ as well as individual country bonds of size $(1-\alpha)s_i$, so that total world bond issuance in aggregate face-value is still $(1+s)$. Suppose the equilibrium price for the common bonds is p_c. Let $\frac{s_i}{s_1+s_2}$ be the share of proceeds from the common bond issue that flows to country i, so that country i receives

$$\frac{s_i}{1+s} p_c \alpha (1+s) = s_i \alpha p_c$$

from the common bond auction. Country i issues individual bonds of size $(1-\alpha)s_i$ at endogenous price p_i so that total proceeds from both common and individual bond issuances to country i are,

$$s_i(\alpha p_c + (1-\alpha)p_i).$$

We model the bond auction as a two-stage game. In the first-stage, countries auction the common bonds and investors spend a total of $f - \hat{f}$ to purchase these bonds, so that in equilibrium,

$$f - \hat{f} = (1 + s)\alpha p_c \tag{16}$$

In the second stage, the investors use their remaining funds of $1 + \hat{f}$ to purchase individual country bonds. After both auctions, each country makes its default decision.

Country i avoids default whenever,

$$s_i(\alpha p_c + (1 - \alpha)p_i) + s_i\theta_i > s_i, \tag{17}$$

which is a straightforward extension of the earlier default condition to include the common bond proceeds. We assume that default affects all of the country's obligations. The country defaults fully on its individual bonds and defaults on its portion of common bonds, so that investors in common bonds receive repayments only from any countries that do not default.

We conjecture and verify the following solution.

Proposition 5 *There are two equilibria, a "maximum joint safety" equilibrium and a "minimum joint safety" equilibrium. In both equilibria, the determination of the reserve asset depends on α as follows:*

1. *$\alpha \in [0, \alpha^*]$: There exists a threshold $\delta^*(\alpha)$. If $\tilde{\delta} > \delta^*(\alpha)$, then country 1 is the reserve asset, while if $\tilde{\delta} < \delta^*(\alpha)$ country 2 is the safe asset.*

2. *$\alpha \in [\alpha_{HL}, 1]$: Both countries satisfy (17) for $\tilde{\delta} \in [\delta_L(\alpha), \delta_H(\alpha)]$ so that both debts are safe, whereas for $\tilde{\delta}$ outside this interval, one country violates (17) and defaults and the other country's debt is the reserve asset.*

Furthermore, the two cutoffs $\alpha^* > \alpha_{HL}$ are defined as follows:

1. *In the "maximum joint safety" equilibrium, α_{HL} solves $\delta_L(\alpha_{HL}) = \delta_H(\alpha_{HL})$;*

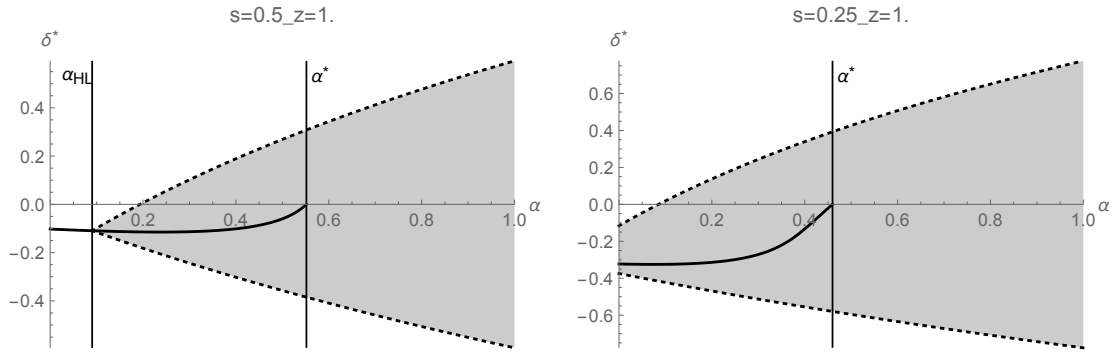


Figure 5: $\delta^*, \delta_H, \delta_L$ for the case of $s = 0.5$ and $s = 0.25$, as a function of α

2. In the "minimum joint safety" equilibrium, α^* solves $\delta^*(\alpha^*) = 0$.

Figure 5 illustrates the statement of the proposition for a case of $s = 0.5$ (left panel) and $s = 0.25$ (right panel), both for $z = 1$. Focusing on the left panel, the black solid line plots δ^* for $\alpha \in [0, \alpha^*]$. We have that δ^* is less than zero for this case because $z = 1$ corresponds to the high savings cases illustrated in earlier graphs. At $\alpha = \alpha^*$ we have that δ^* equals zero. The dashed-lines in the figure indicate the upper/lower bounds of the realizations of $\tilde{\delta}$ in order for the economy to be in the joint safety region where both countries' debts are reserve assets. We see that this joint safety region begins at $\alpha = \alpha_{HL}$, and expands as a function of α. Intuitively, increasing α shares some of the safety benefits of the large country with the small country and hence allows the small country to be safer given realizations of $\tilde{\delta}$. The two possible equilibria overlap between α_{HL} and α^*.

In the right panel we consider a case of $s = 0.25$. We note that the joint safety is a possibility even for $\alpha = 0$, provided that the $\tilde{\delta}$ realizations are sufficiently negative (i.e., country 2 is realized to have sufficiently strong fundamentals than country 1). It is still the case that increasing α makes country 2 safer, expanding the joint safety region and increasing δ^* towards zero. Increasing α makes country 1 less safe in the minimum joint safety region, while it makes country 1 more safe (i.e. δ_L falls) in the maximum joint safety region.

The main result that emerges from our analysis in this section, and which is evident in the figure, is that increases in α only create Pareto gains (when gains are thought of in terms of increasing country safety) when $\alpha > \alpha^*$. In this case, increases in α raise the safety of both country 1 and

country 2. For $\alpha < \alpha^*$ and in the minimum safety equilibrium, increases in α reduce safety of one country while increasing safety of the other country. Common bonds unambiguously increase welfare only when they are large enough in quantity. That is, a small step towards a fiscal union can be worse than no step. The rest of this section derives the equilibrium and results in Proposition 5.

4.1 Case 1: Minimum joint safety

We first solve for an equilibrium where only one country is safe. This equilibrium is a small variant on the equilibrium we have derived so far involving threshold strategies. We will find the largest α so that this threshold equilibrium can exist, which we call α^*.

We work backwards from the second stage. In the second stage, investors have $1+\hat{f}$ to purchase individual country bonds. Consider the marginal investor with signal δ^* who considers that a fraction x of investors have signals exceeding his. Country 1 does not default if,

$$\alpha p_c + (1-\alpha)p_1 + \theta_1 > 1.$$

Since, $f - \hat{f} = (1+s)\alpha p_c$ by (16) and $(1-\alpha)p_1 = x(1+\hat{f})$, we rewrite this condition as,

$$\frac{f-\hat{f}}{(1+s)} + x(1+\hat{f}) + \theta_1 > 1 \Rightarrow x \geq \frac{1-\theta_1 - \frac{1}{1+s}(f-\hat{f})}{1+\hat{f}}.$$

We again take the limit as $\sigma \to 0$ and set $(1-\theta_1) = (1-\theta)e^{-\delta^*}$. Additionally, noting that the return to the marginal investor in investing in country 1 is $\frac{1-\alpha}{(1+\hat{f})x}$ if the country does not default (and zero recovery in default), we integrate over returns when the country does not default to find,

$$\Pi_1 = \frac{1-\alpha}{1+\hat{f}} \ln\left(\frac{1+\hat{f}}{(1-\theta)\,e^{-\delta^*} - \frac{1}{1+s}\left(f-\hat{f}\right)}\right).$$

When $\hat{f} = 0$, this equation is identical to the previously derived profit equation.

We repeat the same steps for the profits to investing in country 2 and find,

$$\Pi_2 = \frac{s\left(1-\alpha\right)}{1+\hat{f}} \ln\left(\frac{1+\hat{f}}{s\left(1-\theta\right)e^{\delta^*} - \frac{s}{1+s}\left(f-\hat{f}\right)}\right).$$

We solve for the threshold in the same way as before:

$$\Pi_1 = \Pi_2 \Rightarrow \delta^*(\hat{f}, \alpha) \tag{18}$$

Next we derive $\hat{f}$ by considering the stage 1 game in which investors make the investment decision on common bonds before $\tilde{\delta}$ realizes. Under the assumed equilibrium where only one country becomes the reserve asset and does not default, the return to investing in the common bond is,

$$\frac{1}{f-\hat{f}}\left[\int_{-\bar{\delta}}^{\delta^*} \alpha s \frac{d\delta}{2\bar{\delta}} + \int_{\delta^*}^{\bar{\delta}} \alpha \frac{d\delta}{2\bar{\delta}}\right].$$

The denominator in the front is the total amount of funds invested in the common bond, while the term in parentheses is the repayment on the common bonds in the cases of repayment only by country 2 and repayment only by country 1, respectively. The returns to keeping one dollar aside and investing in individual country bonds is,

$$\frac{1}{1+\hat{f}}\left[\int_{-\bar{\delta}}^{\delta^*} (1-\alpha) s \frac{d\delta}{2\bar{\delta}} + \int_{\delta^*}^{\bar{\delta}} (1-\alpha) \frac{d\delta}{2\bar{\delta}}\right].$$

Again, the denominator in the front is the total amount of funds invested in individual bonds, while the term in parentheses is the repayment on individual bonds in the cases of repayment only by country 2 and repayment only by country 1. Note the similarity between these last two expressions. The similarity arises because along the nodes of country 2 defaulting or country 1 defaulting, the payoffs, state-by-state, to common bonds and individual bonds are αs_i and $(1-\alpha)s_i$. Since the return from investing in common bonds in stage one of the game and waiting and investing in individual bonds in stage 2 must be equal, we have that:

$$\frac{\alpha}{f-\hat{f}} = \frac{1-\alpha}{1+\hat{f}}$$

which we can solve for $\hat{f}$ to give,

$$f - \hat{f} = \alpha (1 + f) \tag{19}$$

so that

$$p_c = \frac{f - \hat{f}}{\alpha (1 + s)} = \frac{1 + f}{1 + s}. \tag{20}$$

We combine equation (18) and (19) to solve for $\delta^*(\alpha)$.

We next consider the bound α^* We have assumed that only one country becomes the reserve asset, while the other country does not default. However, inspecting (17) we see that as α rises, since $p_c > 0$, it may be that even a country that receives zero proceeds from selling its individual bonds will be able to avoid default. That is, the proceeds from the common bond, which is the reserve asset, provides a default cushion for the weaker country. In this case, it is possible that neither country defaults on its bonds, invalidating our equilibrium assumption that one country defaults for sure.

For each realization of δ, define

$$\theta_{def} (\delta) \equiv \min [\theta_1 (\delta), \theta_2 (\delta)]$$

as the fundamental of the country with the worse realization of fundamentals. Take the best value of this worse realization of fundamentals, which occurs when $\delta = \delta^*$, since even at a slightly higher higher value than this realization one country narrowly beats the other country to become the reserve asset. Now we have constructed an equilibrium in which the beaten country defaults. But is this actually an equilibrium? For this realization, the beaten country defaults if,

$$\theta_{def} (\delta^*) + \alpha p_c < 1 \iff \alpha \frac{1 + f}{1 + s} < 1 - \theta_{def} (\delta^*) \iff \alpha \leq \frac{1 + s}{1 + f} [1 - \theta_{def} (\delta^* (\alpha))].$$

Then α^* solves the equation,

$$\alpha^* = \frac{1+s}{1+f}\left[1 - \theta_{def}\left(\delta^*\left(\alpha^*\right)\right)\right].$$

Let us investigate the slope $\delta^*_\alpha(0)$. We see from the graphs that there are situations in which $\delta^*_\alpha(0) < 0$ so that the large country gains when a small fraction of common bonds are issued. The intuition is the following. Equalizing returns, and realizing that $\hat{f} = f - \alpha(1+f)$, we see that increasing α decreases the available funding for individual bonds. When we look at the returns, we are comparing

$$\ln\left(\frac{1+\hat{f}}{(1-\theta)e^{-\delta^*} - \frac{1}{1+s}\left(f-\hat{f}\right)}\right) \quad \text{vs} \quad s\ln\left(\frac{1+\hat{f}}{s(1-\theta)e^{\delta^*} - \frac{s}{1+s}\left(f-\hat{f}\right)}\right)$$

We see there are two effects. In the denominator, a higher α leads to a lower $\hat{f}$, which means there is less money left over for individual bonds as common bonds divert some of the funding. Size doesn't matter for the numerator as we are in a "winner takes all" equilibrium for individual bonds. This is the *funding effect*. The denominator also changes, however, as $\alpha > 0$ introduces common bond proceeds $\frac{f-\hat{f}}{1+s}$ per common bond sold. We see that these common bonds are scaled by the size of the country as common bonds are issued proportional to country size. Common bond proceeds make a country safer, all else equal. Thus, this is the *safety effect*. For small s, most of the common bond proceeds accrue to the large country. This can lead to a situation in which the large country becomes even safer, i.e., δ^* adjusts downward.

4.2 Case 2: Maximum joint safety

We now construct an equilibrium in which both countries can be safe. Given this equilibrium, we will compute the minimum value of α for which this equilibrium exists. As we show, this minimum value, denoted α_{HL}, will generally be less than α^*. This overlap implies that at least two equilibria exist for some parameters of the model, as described in the Proposition.

The possibility that both countries may be safe means that our equilibrium construction using

threshold strategies is no longer possible. In a region where both countries are known to be safe (recall we consider the limit where $\sigma \to 0$), investors must be indifferent between investing in the two countries, choosing strategies so that in equilibrium the return on both countries' bonds are equal. This "joint safety" region prevails for realizations of $\tilde{\delta} \in [\delta_L, \delta_H]$. Outside this interval we are back to the case where the signals are so strong that only one country survives. We now focus on deriving the boundary between these regions given by functions $\delta_L(\alpha)$ and $\delta_H(\alpha)$. Given these functions we can solve for the boundary α_{HL} , which occurs when both thresholds coincide, i.e., $\delta_L(\alpha_{HL}) = \delta_H(\alpha_{HL})$, and the joint safety region vanishes.

We depart from monotone threshold strategies. The construction of equilibrium follows closely to Appendix A.3. Let us conjecture a non-monotone strategy whereby investment in country 1 and in country 2 alternate on discrete intervals of length $k\sigma$ and $(2-k)\sigma$, with $k \in (0,2)$. We will refer to this strategy as the oscillating strategy. The investor strategy is $\phi(y) \in \{0,1\}$ which is investment in country 1:

$$\phi(y) = \begin{cases} 0, & y < \delta_L \\ 1, & y \in [\delta_L, \delta_L + k\sigma] \cup [\delta_L + 2\sigma, \delta_L + (2+k)\sigma] \cup [\delta_L + 4\sigma, \delta_L + (4+k)\sigma] \cup ... \\ 0, & y \in [\delta_L + k\sigma, \delta_L + 2\sigma] \cup [\delta_L + (2+k)\sigma, \delta_L + 4\sigma] \cup [\delta_L + (4+k)\sigma, \delta_L + 6\sigma] \cup ... \\ 1, & y > \delta_H \end{cases} \tag{21}$$

In the limit as σ approaches zero, these strategies resemble mixed strategies, so that over any interval strictly inside $[\delta_L, \delta_H]$ the probability of investing in each country is such that bond returns are equalized. For more related discussions on this non-monotone strategy, see Section 3.5.

4.2.1 Fraction of agents in investing in country 1

Consider a region on which all investors know that both countries are safe. In this case, the total investment in country 1 and 2 has to be $\left(1+\hat{f}\right)\frac{1}{1+s}$ and $\left(1+\hat{f}\right)\frac{s}{1+s}$, respectively, as otherwise one country will offer strictly higher (expected) returns than the other country. Take an agent with

signal δ. Let us introduce the function $\rho(\delta)$, which is the expected proportion of agents investing in country 1 given (own) signal δ. Then, given the assumed strategy for all agents and given that we are in the region where both countries are safe,

$$\rho(\delta) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{\phi(y)}{2\sigma} dy = \frac{k\sigma}{2\sigma}.$$

We choose k so that,

$$\rho(\delta) = \frac{1}{1+s} \iff k = \frac{2}{1+s} \tag{22}$$

The last equality arises because in equilibrium the proportion must be constant and equal to $\frac{1}{1+s}$ in order to have returns equalized across the two safe bonds.

For any value of δ and x, the proportion of agents investing in country 1 is given by

$$\rho(\delta, x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{\phi(y)}{2\sigma} dy = \begin{cases} 0, & \delta + 2\sigma x < \delta_L \\ \frac{\delta+2\sigma x-\delta_L}{2\sigma} & \delta + 2\sigma x \in (\delta_L, \delta_L + k\sigma) \\ \frac{1}{1+s}, & \delta_H - (2-k)\sigma > \delta > \delta_L + k\sigma \end{cases} \tag{23}$$

We see that $\rho(\delta, x)$ is strictly increasing in x in the vicinity of δ_L, and flat everywhere else. Note further that

$$\rho(\delta_L, x) = \begin{cases} x, & x \in \left[0, \frac{1}{1+s}\right] \\ \frac{1}{1+s}, & x \in \left(\frac{1}{1+s}, 1\right] \end{cases} \tag{24}$$

where we observe that $\rho(\delta_L, x)$ is less than or equal to $\frac{1}{1+s}$.

4.2.2 Lower boundary δ_L

Let $\Pi_i(\delta)$ be the expected payoff of investing in country i for a given agent with signal δ. In the completely safe case, investors were indifferent between both strategies because both paid exactly the same in all states of the world. This is not the case at δ_L, as the countries (at least in the

eyes of the agent) are not always safe. Thus, the country with perceived default risk (at δ_L this is country 1) will need to offer a higher return conditional on survival to be attractive to the agent. To an agent with signal δ_L, the return from investing only in country 2 (i.e. $\phi = 0$) is given by

$$\Pi_2(\delta_L) = \int_0^1 \frac{s}{\left(1+\hat{f}\right)(1-\rho(\delta_L, x))} dx \tag{25}$$

where we integrate over all x as country 2 is safe regardless of x. Thus, substituting in for $\rho(\delta_L, x)$, we have

$$\Pi_2(\delta_L) = \frac{s}{1+\hat{f}}\left[\int_0^{\frac{1}{1+s}} \frac{1}{1-x} dx + \int_{\frac{1}{1+s}}^1 \frac{1}{\frac{s}{1+s}} dx\right] = \frac{s}{1+\hat{f}}\left[\ln\frac{1+s}{s} + 1\right] < \frac{1+s}{1+\hat{f}} \tag{26}$$

where we used $s \ln \frac{1+s}{s} < 1$.

Now consider country 1. The minimum proportion of agents investing in country 1 that are needed to make country 1 safe if the true state of the world is δ solves,

$$\theta_1(\delta) + \alpha p_c + \left(1+\hat{f}\right)\rho_1^{min}(\delta) = 1 \iff \rho_1^{min}(\delta) = \frac{1-\theta_1(\delta)-\alpha p_c}{1+\hat{f}}$$

Define $x_{min}(\delta_L)$ as the solution to $\rho(\delta_L, x) = \rho_1^{min}(\delta_L)$. Given equation (24), we have that,

$$x_{min}(\delta_L) = \frac{1-\theta_1(\delta_L)-\alpha p_c}{1+\hat{f}} \tag{27}$$

The expected return of investing in country 1 given one's own signal δ_L and the strategies $\phi(y)$ of other agents is given by (using (24)),

$$\Pi_1(\delta_L) = \int_{x_{min}(\delta_L)}^1 \frac{1}{\left(1+\hat{f}\right)\rho(\delta_L, x)} dx = \frac{1}{1+\hat{f}}\left[\ln\frac{1}{1+s} - \ln x_{min}(\delta_L) + s\right]$$

Indifference requires that $\Pi_2(\delta_L) = \Pi_1(\delta_L)$, which implies that

$$x_{min}(\delta_L) = \exp\left[s\ln s - (1+s)\ln(1+s)\right] \tag{28}$$

We combine the two equations for $x_{min}(\delta_L)$, (27) and (28), to find an equation that determines δ_L:

$$\exp\{s\ln s-(1+s)\ln(1+s)\}=\frac{1-\theta_1(\delta_L)-\alpha p_c}{1+\hat{f}} \tag{29}$$

Plugging in for $1-\theta_1(\delta)=(1-\theta)\exp(-\delta)$, we finally have

$$\delta_L=-\ln\left\{\frac{1}{1-\theta}\left[\left(1+\hat{f}\right)\frac{s^s}{(1+s)^{(1+s)}}+\alpha p_c\right]\right\}. \tag{30}$$

4.2.3 Upper boundary δ_H

The derivation of the upper boundary δ_H is analogous. We have

$$\rho(\delta,x)=\int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x}\frac{\phi(y)}{2\sigma}dy=\begin{cases}\frac{1}{1+s}, & \delta-2\sigma(1-x)<\delta_H-(2-k)\sigma\\ \frac{\delta+2\sigma x-\delta_H}{2\sigma} & \delta-2\sigma(1-x)\in[\delta_H-(2-k)\sigma,\delta_H]\end{cases} \tag{31}$$

so that

$$\rho(\delta_H,x)=\begin{cases}\frac{1}{1+s}, & x<\frac{1}{1+s}\\ x, & x\in\left[\frac{1}{1+s},1\right]\end{cases} \tag{32}$$

Consider now the agent with signal δ_H who is considering investing in country 1. In his eyes, the return from investing only in country 1 (i.e. $\phi=1$) is given by

$$\Pi_1(\delta_H)=\int_0^1\frac{1}{\left(1+\hat{f}\right)\rho(\delta_H,x)dy}dx=\frac{\ln(1+s)+1}{1+\hat{f}}$$

where we integrated over all x as country 2 is always safe in the vicinity of δ_H.

The default condition for country 2 is

$$s\theta_2(\delta)+s\alpha p_c+\left(1+\hat{f}\right)[1-\rho_2^{max}(\delta)]=s\iff[1-\rho_2^{max}(\delta)]=s\frac{1-\theta_2(\delta)-\alpha p_c}{1+\hat{f}}$$

where $\rho_2^{max}(\delta)$ is the maximum amount of people investing in country 1 so that country 2 does not

default. Let us assume, and later verify, that at δ_H, we have $[1-\rho_2^{max}(\delta_H)] < \frac{s}{1+s}$, that is country 2 would survive even if less than $\frac{s}{1+s}$ of investors invest in it. Define $x_{max}(\delta_H)$ as the solution to $\rho(\delta_H, x_{max}) = \rho_2^{max}(\delta_H)$. Given equation (32), we have that,

$$1 - x_{max}(\delta_H) = s\frac{1-\theta_2(\delta_H) - \alpha p_c}{1+\hat{f}} \tag{33}$$

Then the return to investing in country 2 is,

$$\Pi_2(\delta_H) = \int_0^{x_{max}(\delta_H)} \frac{s}{\left(1+\hat{f}\right)(1-\rho(\delta_H,x))dy}dx = \frac{s}{1+\hat{f}}\left[\frac{1}{s} + \ln\frac{s}{1+s} - \ln\left(1-x_{max}(\delta_H)\right)\right]$$

Indifference requires $\Pi_1(\delta_H) = \Pi_2(\delta_H)$, which implies that

$$1 - x_{max}(\delta_H) = \frac{s}{(1+s)^{\frac{1+s}{s}}} \tag{34}$$

Combining the expressions for $x_{max}(\delta_H)$ from (33) and (34), we solve,

$$\delta_H = \ln\left\{\frac{1}{1-\theta}\left[\frac{1+\hat{f}}{(1+s)^{\frac{1+s}{s}}} + \alpha p_c\right]\right\} \tag{35}$$

We still need to verify that the optimality of proposed investment strategies for investors with signal $\delta_i \in (\delta_L, \delta_H)$. The argument is exactly the same as in Appendix A.3 and hence omitted here.

4.2.4 $\alpha > 0$ and stage 1 asset allocation.

Let us now consider the stage 1 asset allocation, that is, the determination of $\hat{f}$ and $p_c = \frac{f-\hat{f}}{\alpha(1+s)}$ (there are $\alpha(1+s)$ units of common bonds, and there is $f-\hat{f}$ money invested in them). Consider an $\alpha > 0$. Then, we know that the expected returns from investing in common bonds in stage 1 and investing in individual country bonds in stage 2 have to be equalized. The return to investing

in individual bounds is

$$\begin{aligned} R_i &= (\text{only country 2}) + (\text{joint survival}) + (\text{only country 1}) && (36)\\ &= \int_{-\underline{\delta}}^{\delta_L} \frac{(1-\alpha)\, s}{1+\hat{f}} \frac{d\delta}{\overline{\delta}-\underline{\delta}} + \int_{\delta_L}^{\delta_H} \frac{(1-\alpha)\,(1+s)}{1+\hat{f}} \frac{d\delta}{\overline{\delta}-\underline{\delta}} + \int_{\delta_H}^{\overline{\delta}} \frac{(1-\alpha)}{1+\hat{f}} \frac{d\delta}{\overline{\delta}-\underline{\delta}} && (37)\\ &= \frac{1-\alpha}{1+\hat{f}} \left[\int_{-\underline{\delta}}^{\delta_L} s \frac{d\delta}{\overline{\delta}-\underline{\delta}} + \int_{\delta_L}^{\delta_H} (1+s) \frac{d\delta}{\overline{\delta}-\underline{\delta}} + \int_{\delta_H}^{\overline{\delta}} \frac{d\delta}{\overline{\delta}-\underline{\delta}} \right] && (38) \end{aligned}$$

and the expected return for common bonds is given by

$$R_c = \frac{\alpha}{f-\hat{f}} \left[\int_{-\underline{\delta}}^{\delta_L} s \frac{d\delta}{\overline{\delta}-\underline{\delta}} + \int_{\delta_L}^{\delta_H} (1+s) \frac{d\delta}{\overline{\delta}-\underline{\delta}} + \int_{\delta_H}^{\overline{\delta}} \frac{d\delta}{\overline{\delta}-\underline{\delta}} \right] \tag{39}$$

Note the similarity between these last two expressions. The similarity arises because along the states of country 2 defaulting or country 1 defaulting, the payoffs, state-by-state, to common bonds and individual bonds are αs_i and $(1-\alpha)s_i$. Thus,

$$\frac{\alpha}{f-\hat{f}} = \frac{1-\alpha}{1+\hat{f}} \iff f-\hat{f} = \alpha\,(1+f) \iff 1+\hat{f} = (1-\alpha)\,(1+f) \tag{40}$$

and,

$$\alpha p_c = \frac{f-\hat{f}}{1+s} = \alpha \frac{1+f}{1+s} \tag{41}$$

We can now solve out for the thresholds. Plugging the expression for αp_c into (30) and (35), we find,

$$\begin{aligned} \delta_H &= z + \ln\left\{ \frac{1}{1+s} \left[\left(\frac{1}{1+s} \right)^{\frac{1}{s}} (1-\alpha) + \alpha \right] \right\} && (42)\\ \delta_L &= -z - \ln\left\{ \frac{1}{1+s} \left[\left(\frac{s}{1+s} \right)^{s} (1-\alpha) + \alpha \right] \right\} && (43) \end{aligned}$$

The next proposition establishes the existence of the oscillating equilibrium.

Proposition 6 *For a given z, define α_{HL} as the solution $z_{HL}(\alpha_{HL}) = z$ where*

$$z_{HL}(\alpha) = \ln(1+s) - \frac{1}{2}\left\{\ln\left[\left(\frac{s}{1+s}\right)^{s}(1-\alpha)+\alpha\right] + \ln\left[\left(\frac{1}{1+s}\right)^{\frac{1}{s}}(1-\alpha)+\alpha\right]\right\}$$

Then, the oscillating equilibrium exists for $\alpha > \alpha_{HL}$. Furthermore, we have $\delta^*(\alpha_{HL}) = \delta_H(\alpha_{HL}) = \delta_L(\alpha_{HL})$ as well as $\delta^*_\alpha(\alpha_{HL}) < 0$.

There is a smooth transition from δ^* to the oscillating equilibrium as α crosses α_{HL} in that the thresholds $\delta^*, \delta_H, \delta_L$ all coincide at $\alpha = \alpha_{HL}$. Furthermore, this transition occurs at a point at which $\delta^*(\alpha)$ is decreasing in α, implying that it occurs at a point at which the large country gains safety from increasing α.

5 Endogenous Reserve Asset Size

We have thus far taken debt size and country surplus as exogenous, deriving predictions regarding the reserve asset based on these variables. We now endogenize these variables, assuming that each country sees a benefit to becoming the reserve asset, but faces costs in choosing debt size and surplus.

5.1 Incentives and costs of altering debt size and surplus

We suppose that the countries have natural debt sizes of s_1^* and s_2^*, which we will think of as determined by local economic and fiscal conditions. For example, countries with a higher GDP will naturally have a larger stock of debt outstanding. A country can choose to alter the size of its debt, but incurs an adjustment cost. If country 1 increases its debt size to S_1 it must increase the surplus θ proportionately, via increases in the tax base, to support the larger debt issue. We assume that increasing the debt size to S_i requires raising the surplus from θ_i to $\theta_i \frac{S_i}{s_i^*}$ and that countries face an adjustment cost $c(S_i - s_i^*)$ which is increasing and convex in $S_i - s_i^*$, with $c(0) = 0$, $c'(0) = 0$.

Separately, we suppose that countries can change their surplus without altering debt size (i.e.

increasing the ratio of tax revenues to outstanding debt). In particular, a country can choose an increment δ_i, so that the country's fundamental becomes (for country 1):

$$1 - \theta_1 = (1 - \theta)e^{-\tilde{\delta} - \delta_1}$$

That is, for country 1, choosing $\delta_1 > 0$ decreases its funding need $1 - \theta_1$ (we can likewise define a δ_2 that improves country 2's surplus). This action costs $\kappa(\delta_i)$.

We solve for $\delta^*(S_1, S_2, \delta_1, \delta_2)$ following the same analysis as in Section 2.2. The marginal investor with signal δ^* receives the following return from investing in country 1:

$$\Pi_1 = \int_{\frac{S_1(1-\theta)e^{-\delta^*}}{1+f}}^{1} \frac{S_1}{(1+f)\,x} dx = \frac{S_1}{1+f}\left(-\ln S_1 + \ln\frac{1+f}{1-\theta} + \delta^* - \delta_1\right),$$

while the return from investing in country 2 is

$$\Pi_2 = \int_{0}^{\frac{1+f-S_2(1-\theta)e^{\delta^*}}{1+f}} \frac{S_2}{(1+f)\,(1-x)} dx = \frac{S_2}{1+f}\left(-\ln S_2 + \ln\frac{1+f}{1-\theta} - \delta^* + \delta_2\right).$$

Setting $\Pi_1 = \Pi_2$ and solving, we find that the equilibrium threshold is

$$\delta^*\left(S_1, S_2, \delta_1, \delta_2\right) = \frac{S_2 - S_1}{S_1 + S_2} z + \frac{S_1 \ln S_1 - S_2 \ln S_2}{S_1 + S_2} - \delta_1 \frac{S_1}{S_1 + S_2} + \delta_2 \frac{S_2}{S_1 + S_2}. \tag{44}$$

where again $z \equiv \ln\frac{1+f}{1-\theta}$ measures the strength of savings/fundamentals.

By altering S_i and δ_i a country can alter δ^*. Note that a lower δ^* helps country 1 in the sense that country 1 is the reserve asset (and does not default) for a wider range of fiscal fundamentals. Likewise, a higher δ^* helps country 2.

5.2 Endogenous debt size

It is clear from equation (44) that increasing δ_i benefits country-i. Thus, as we would expect a country that increases its fiscal fundamentals improves its reserve asset position. The analysis here

is fairly ordinary: the country chooses δ_i to balance the reserve asset benefit against the cost of increasing its surplus, $\kappa(\delta_i)$.

We instead focus our analysis on endogenous debt size, which yields more nuanced results. Let us set $\delta_i = 0$. Suppose that country 1 chooses S_1 to maximize (focusing again on monotone threshold strategies),

$$\underbrace{-\delta^*(S_1, S_2)}_{\textbf{benefit of reserve asset status}} - \underbrace{c(S_1 - s_1^*)}_{\textbf{adjustment cost}} . \tag{45}$$

This objective can be understood as follows. The second term is the cost of adjustment. The first term captures the benefit of adjustment, i.e., the country is able to lower the minimum level of surplus needed to be the reserve asset, and hence avoid default. For any cdf for $\tilde{\delta}$ and an appropriate renormalization of the cost function, the benefit is linear in $-\delta^*(S_1, S_2)$. Likewise for country 2, the objective is

$$\delta^*(S_1, S_2) - c(S_2 - s_2^*)$$

where country 2 benefits by increasing $\delta^*(S_1, S_2)$.

Introduce the function $h(S_1, S_2; z)$, which is defined as the marginal impact of S_1 on $\delta^*(S_1, S_2)$:

$$h(S_1, S_2; z) \equiv \frac{\partial \delta^*(S_1, S_2)}{\partial S_1} = \frac{1}{(S_1 + S_2)^2}\left[S_1 + S_2(\ln S_1 + \ln S_2 + 1 - 2z)\right] \tag{46}$$

Due to symmetry, the (negative) derivative of δ^* with respect to S_2 is

$$-\frac{\partial \delta^*(S_1, S_2)}{\partial S_2} = h(S_2, S_1; z) = \frac{1}{(S_1 + S_2)^2}\left[S_2 + S_1(\ln S_1 + \ln S_2 + 1 - 2z)\right].$$

As a result, the first order conditions for endogenous debt sizes satisfy,

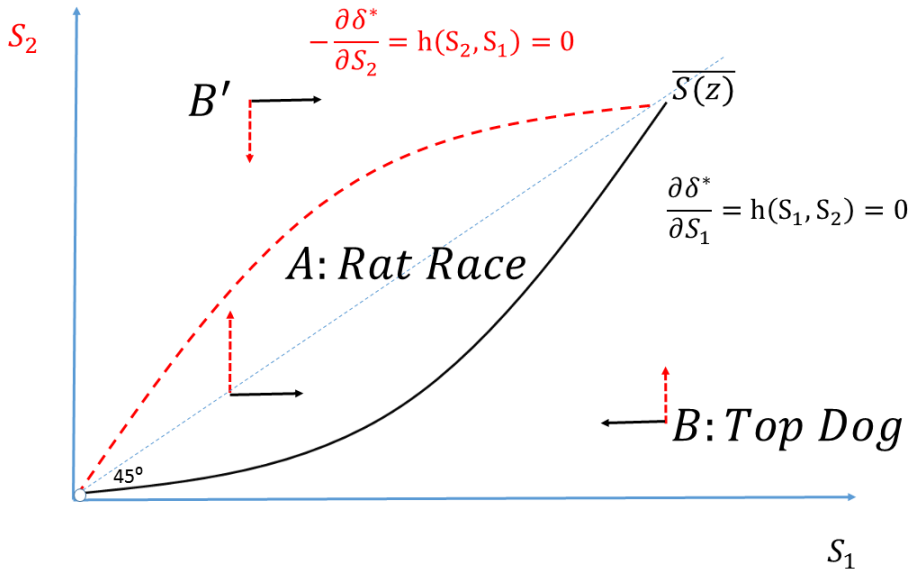


Figure 6: Diagram of incentives to alter size of debt

$$h(S_1, S_2; z) = c'(S_1 - s_1^*) \quad \text{and,} \quad h(S_2, S_1; z) = c'(S_2 - s_2^*).$$

Next we analyze the property of $h(S_1 = s_1^*, S_2 = s_2^*; z)$.

5.3 Equilibrium Characterization and "Phase Diagram"

We illustrate the solution in Figure 6. We first note that if there is no cost of changing size, then the equilibrium solves

$$h(S_1, S_2; z) = h(S_2, S_1; z) = 0.$$

The equilibrium is symmetric. Define the solution in this case as $\overline{S}(z) = S_1 = S_2$, and solving we find that:

$$1 + \ln \overline{S}(z) = z \Rightarrow \overline{S}(z) = e^{z-1}.$$

On Figure 6 we plot the phase diagram where $S_1 = s_1^*$ and $S_2 = s_2^*$ are below $\overline{S}(z)$. The

solid black curve corresponds to the set of points where $h(S_1, S_2; z) = 0$; that is, these are points S_1 and S_2 where changing S_1 has no effect on δ^*. The dashed red curve corresponds to the set of points where $h(S_2, S_1; z) = 0$; that is, these are points where changing S_2 has no effect on δ^*. These two curves are symmetric around the 45-degree line and cross at the point $\left(\overline{S}(z), \overline{S}(z)\right)$. The following lemma shows when country sizes are relatively small with $\max(S_1, S_2) < \overline{S}(z)$, the locus of $h(S_1, S_2; z) = 0$, which is the black curve in the figure, has $S_1 > S_2$. This implies that the black curve lies below the 45-degree line. The result is reversed for the red curve which is the locus of $-\frac{\partial \delta^*(S_1, S_2)}{\partial S_2} = h(S_2, S_1; z) = 0$ and is symmetric around the 45-degree line.

Lemma 1 *When* $\max(S_1, S_2) < \overline{S}(z)$*, the locus of* $h(S_1, S_2; z) = 0$ *has* $S_1 > S_2$..

Proof. We show that for $h(S_1 = S_2 + \epsilon, S_2; z) = 0$ to hold we must have $\epsilon > 0$. Using (46) we have:

$$h(S_1 = S_2 + \epsilon, S_2; z) = \epsilon + S_2(\ln(S_2 + \epsilon) - \ln S_2) + 2S_2(1 + \ln S_2 - z) = 0.$$

When $S_2 < \overline{S}(z)$, the last term $2S_2(1 + \ln S_2 - z)$ on the LHS is negative. To have $h = 0$, we must have that $\epsilon > 0$, since conditional on S_2 the sum of the first two terms is zero when $\epsilon = 0$ and increasing in ϵ. Q.E.D. ■

Intuitively, for any given level of S_2, starting from the 45-degree line country 1 has a strictly positive incentive to expand its debt size to enhance its reserve asset status, since $c'(0) = 0$. This is due to the size benefit, illustrated in the left panel of Figure 2. This explains why along the solid black curve with zero expansion incentive we must have $S_1 > S_2$. However, country 1 does not want its debts to become too large, because as illustrated by the right panel of Figure 2, becoming too large triggers rollover risk fears that may lead investors to coordinate on the smaller country's debt as the reserve asset. The black curve with $h(S_1, S_2; z) = 0$ balances these two effects.

Now suppose that the initial country sizes (s_1^*, s_2^*) lies to the right of the black curve, i.e., $h(S_1 = s_1^*, S_2 = s_2^*; z) > 0$. Then in equilibrium country 1 will choose an $S_1 < s_1^*$ (exactly how much less depends on the cost function). Likewise, for any S_1 if s_2^* is above the red curve, then country 2 will choose an $S_2 < s_2^*$. For points inside the solid black and dashed red curves, both

countries want to expand. The arrows on Figure 6 indicate the direction, as in a phase diagram, in which countries will change debt size given a natural size of (s_1^*, s_2^*).

There are three regions of interest in Figure 6. Region A ("RAT RACE") corresponds to a case where countries' debts are similar in size – so a roughly symmetric case. We see that in this case both countries will increase the size of their debts. This is driven by the reserve asset effect: there is an externality whereby the larger country has a better reserve asset position, and countries compete to become the reserve asset by increasing their debt sizes. Of course, this competition to gain reserve asset status is ultimately self-defeating. Take the fully symmetric case where $s_1^* = s_2^*$ and thus $\delta^* = 0$ in equilibrium. The Nash equilibrium results in countries increasing debt sizes beyond their natural sizes in a rat race to become the reserve asset, while both would be better off and save adjustment costs if they coordinate not to expand. Region B and B' ("TOP DOG") are cases with asymmetric country sizes. Take region B (as B' follows the same logic). Here country 1 is large and country 2 is relatively small. In this case, country 1 is the top dog and not worried about losing its reserve asset status, and is primarily concerned about rollover risk. In this region country 1 chooses to contract its debt size, while country 2 expands, hoping to gain some reserve status.[11]

5.4 Funding Conditions and Incentives for Expansion

By using the fact that $\frac{\partial h(\cdot,\cdot;z)}{\partial z} < 0$ in (46), we can show that the region A—which is the area inside the solid black curve and the dashed green curve—expands unambiguously as the aggregate funding condition z rises. This result has interesting implications, as it suggests that countries, once facing an environment with better aggregate funding conditions, are drawn into the rate race where they compete to offer the reserve asset.

This comparative static offers a unique perspective on the expansion of relatively safe debt supplies in the period preceding the 2007 financial crisis. In the US, the government agencies,

[11] For natural sizes beyond $\overline{S}(z)$, i.e. $\max(S_1, S_2) > \overline{S}(z)$, there is a region (not shown in Figure 6) outside A that covers the 45-degree line, in which both countries have incentives to shrink. There, the debt sizes are relative large compared to the aggregate funding conditions, and the rollover risk concerns drive both countries to issue less debt.

Fannie Mae and Freddie Mac, initiated a program ("Benchmark Notes") in 1998 whose purpose was to offer debt that could compete with US Treasury bonds as a large and liquid savings vehicle. Of course, the expansion of such agency debt stocks ultimately resulted in the bailouts of Fannie Mae and Freddie Mac by the US government, suggesting that welfare would have been improved without these programs, or if the US Treasury had coordinated the debt sizes of the Agencies along with that of the rest of the federal government. In Europe, the expansion of sovereign debt after the formation of the Euro can similarly be seen as a rat race to serve as the reserve asset within Europe. This rat race has also ended badly.

The figure offers a further prediction which may be comforting in today's world of high US government debt. Suppose that the world economy was in region A prior to the financial crisis, but has since transitioned to region B, where US government debt is the top dog. This transition is in keeping with the fact that the amount of US Treasury competitors has fallen since the crisis. The model suggests that the US then has an incentive to shrink its debt.

6 Conclusion

TO BE WRITTEN

References

A Appendix

A.1 Additive Fundamental Structure

We have considered the specification of $1-\theta_i = (1-\theta)\exp\left((-1)^i\tilde{\delta}\right)$ for country i's fundamental. We now show that results are qualitatively similar with the alternative additive specification

$$\theta_1 = \theta + \tilde{\delta}, \text{ and } \theta_2 = \theta - \tilde{\delta}.$$

As $x = \Pr\left(\tilde{\delta} + \epsilon_j > \delta^*\right) = \frac{\tilde{\delta}+\sigma-\delta^*}{2\sigma} \Rightarrow \tilde{\delta} = \delta^* + (2x-1)\sigma$, we know that

$$\begin{aligned}\theta_1 &= \theta + \tilde{\delta} = \theta + \delta^* + (2x-1)\sigma \\ \theta_2 &= \theta - \tilde{\delta} = \theta - \delta^* - (2x-1)\sigma\end{aligned}$$

Given x, the large country 1 survives if and only if

$$p_1 - 1 + \theta_1 = (1+f)x - 1 + \theta + \delta^* + (2x-1)\sigma \geq 0 \Leftrightarrow x \geq \frac{1-\theta-\delta^*+\sigma}{1+f+2\sigma}$$

which implies the expected return from investing in country 1 is

$$\Pi_1 = \int_{\frac{1-\theta-\delta^*+\sigma}{1+f+2\sigma}}^{1} \frac{1}{(1+f)x}dx = \frac{1}{1+f}\ln\frac{1+f+2\sigma}{1-\theta-\delta^*+\sigma}.$$

For country 2, the bond is paid back if

$$\begin{aligned}(1+f)x' - s + s\theta_2 &= (1+f)x' - s + s\left[\theta - \delta^* - (2x-1)\sigma\right] \geq 0 \\ \Leftrightarrow x' &\geq \frac{s(1-\theta+\delta^*-\sigma)}{1+f+2s\sigma}\end{aligned}$$

which implies an expected return of

$$\Pi_2 = \int_{\frac{s(1-\theta+\delta^*-\sigma)}{1+f+2s\sigma}}^{1} \frac{s}{(1+f)x'}dx' = \frac{s}{1+f}\ln\frac{1+f+2s\sigma}{s(1-\theta+\delta^*+\sigma)}$$

As a result, the equilibrium threshold δ^* is pinned by by the indifference condition

$$\ln\frac{1+f+2\sigma}{1-\theta-\delta^*+\sigma} = s\ln\frac{1+f+2s\sigma}{s(1-\theta+\delta^*+\sigma)}.$$

Letting $\sigma \to 0$ we obtain

$$\ln\frac{1+f}{1-\theta-\delta^*} = s\ln\frac{1+f}{s(1-\theta+\delta^*)}. \tag{A.1}$$

We no longer have close-form solution for δ^* in (A.1), as δ^* shows up in both sides. However, the solution is unique because LHS (RHS) is increasing (decreasing) in δ^*. Finally, to ensure $\delta^* < 0$ so that the larger country 1 is relatively safer, we require the same sufficient condition of $z = \ln\frac{1+f}{1-\theta} > 1$ in this alternative specification.

A.2 Uniqueness of the cutoff equilibrium within monotone strategies

Suppose we restrict strategies to be monotone, i.e. $\phi(y)$ is an increasing function of y (of course, not necessarily continuous). Thus, for any x, $\rho(x)$ is increasing and continuous in x, i.e.,

$$\rho(x) = \int_0^1 \phi\left(\delta(x) - (1-2u)\sigma\right)du$$

as $\delta(x)$ is increasing in x. Importantly, we have $\rho'(x) \in [0,1]$, and we restrict $\rho''(x) \geq 0$ by monotonicity. Then, suppose that at x_{flat} we have $\rho'(x_{flat}) = 0$. Then, we must have $\rho(x_{flat}) = \max_{x\in[0,1]}\rho'(x)$. Further, we have $\rho'(0) \geq \rho(0)\frac{2\sigma}{\delta-\underline{\delta}_n}$.

We rule out fundamental uncertainty, and only have strategic uncertainty. That means that each agent who receives a signal $s = \delta$ believes δ to be the true state of the economy, and everyone else being wrong about what they think is the true state of the economy. Let us assume that $\rho_{min}(\delta) > \rho_{max}(\delta)$ for all δ. A sufficient condition would be for example $e^{2z} < 4s$.

What are the strategies the agent might be playing against?

A.2.1 Lower dominance bound $\underline{\delta}_n$

Constant slope strategies are wlog. We will use a sequence of arguments to show that constant slope strategies are optimal. The main argument relies on the fact that without fundamental uncertainty, for any x, x' with $\rho(x) = \rho(x')$ we have the same return as $\rho_{min}(\delta), \rho_{max}(\delta)$ are fixed.

1. Consider two $\rho_0 > \hat{\rho}_0$ with the same slope $\overline{\phi}$. Then we know that $x_{flat} < \hat{x}_{flat}$. As long as $\hat{x}_{flat} < 1$, $\hat{\rho}_0$ cannot be optimal. In other words, amongst strategies with the same slope (and thus the same ultimate flat level), at least under $\rho_{min}(\delta) > \rho_{max}(\delta)$, the strategy that achieves safety of country 1 the fastest dominates any others.
2. Only linear and flat functions can be optimal. Suppose not.

   (a) Consider a function $\rho(x)$ that is strictly convex on $[\rho_0, \rho_{min}(\delta)]$. Let x_1 be defined by $\rho(x_1) = \rho_{min}(x)$ and define x_0 by $\rho_0 + \rho'(x_1) x_0 = \rho_{min}(\delta)$. Then we can find a function

   $$\hat{\rho}(x) = \begin{cases} \rho_0 + \rho'(x_1) x & , x < x_0 \\ \rho(x - x_0 + x_1) & , 1 - x_1 + x_0 > x > x_0 \\ \min[\rho(1) + \rho'(1)(x - x_0 + x_1), 1] & , 1 - x_1 + x_0 < x \end{cases}$$

   that can be formed by monotone strategies. This function achieves safety of country 1 faster than $\rho(x)$, mimics the returns of $\rho(x)$ one for one for $x \in [x_0, 1 - x_1 + x_0]$ and adds some positive returns on $x \in [1 - x_1 + x_0, 1]$. Thus, $\hat{\rho}(x)$ is always better than $\rho(x)$ if $\rho(x)$ is strictly convex.

   (b) Consider any function $\rho(x)$ that ever has an increase in slope on $[\rho_{min}(\delta), 1]$. This simply leads to more investment in country 1, lowering the returns, while not affecting the the safety region of country 1. Let $\rho(x_1) = \rho_{min}(\delta)$. Then, we can always find a function

   $$\hat{\rho}(x) = \begin{cases} \rho(x) & , x < x_1 \\ \min[\rho(x_1) + \rho'(x_1)(x - x_1), \rho(x)] & , x > x_1 \end{cases}$$

   that can be formed by monotone strategies. Intuitively, we always want to have the minimal slope of $\rho(x)$ possible on $[\rho_{min}(\delta), 1]$, which is given by $\rho'(x_1)$.

Constant slope strategies. Let us introduce the following family of strategies $\rho(x)$ defined by the constant slope $\overline{\phi}$:

$$\begin{aligned} \rho_0\left(\overline{\phi}\right) &= \frac{\delta - \underline{\delta}_n}{2\sigma}\overline{\phi} \\ \rho(x) &= \min\left\{\overline{\phi}, \rho_0\left(\overline{\phi}\right) + \overline{\phi} \cdot x\right\} \end{aligned}$$

That is, we assume that everyone that is not dominated plays $\overline{\phi} \in [0, 1]$ (possibly interior). Why this family? It includes $\overline{\phi} = 1$ which basically means that the agents we are playing against are aggressively investing in country 1. But this might not be the best strategy to play against – it might pay to have less people investing in country 1 while it still can survive to boost country 1's return.

What remains is to show that linear and then flat is the best strategy to play against. Let us call $\overline{\phi} = 1$ the max slope strategy. Essentially, a higher slope strategy achieves survival of country 1 faster (i.e. for lower x), but also funnels more investors into country 1, thus lowering the return on country 1, for high x than necessary for country 1 survival. A lower slope strategy may not achieve survival of country 1 as fast (i.e. a higher x), but also funnels less investors into country 1, thus boosting the return on country 1 for high x.

For any strategy constant slope strategy, define x_{flat} as the point at which

$$\overline{\phi} = \rho_0\left(\overline{\phi}\right) + \overline{\phi} x_{flat} \iff x_{flat} = 1 - \frac{\rho_0\left(\overline{\phi}\right)}{\overline{\phi}} = 1 - \frac{\delta - \underline{\delta}_n}{2\sigma}$$

We see that x_{flat} is slope independent, which is intuitive – it occurs when x is such that the expectation is taken exclusively over a non-dominanted area.

Let us note that we will only consider $\overline{\phi} \geq \rho_{min}(\delta)$. If this were not the case, country 1 would never surive, which implies that this strategy can never be incentivizing investment in country 1. Then

$$\begin{aligned}(1+f)\,\Delta(\delta) &= \int_0^1 \left[1_{\{\rho(x)\geq\rho_{min}(\delta)\}}\frac{1}{\rho(x)} - 1_{\{\rho(x)\leq\rho_{max}(\delta)\}}\frac{s}{1-\rho(x)}\right]dx \\ &= \int_{x_{min}}^{x_{flat}} \frac{1}{\rho_0\left(\overline{\phi}\right)+\overline{\phi}x}dx + \int_{x_{flat}}^1 \frac{1}{\overline{\phi}}dx - \int_0^{x_{max}} \frac{s}{1-\rho_0\left(\overline{\phi}\right)-\overline{\phi}x}dx \\ &= \frac{1}{\overline{\phi}}\left[\ln\left(\rho_0\left(\overline{\phi}\right)+\overline{\phi}x_{flat}\right) - \ln\left(\rho_0\left(\overline{\phi}\right)+\overline{\phi}x_{min}\right)\right] + (1-x_{flat})\frac{1}{\overline{\phi}} \\ &\quad -\frac{s}{\overline{\phi}}\left[\ln\left(1-\rho_0\left(\overline{\phi}\right)\right) - \ln\left(1-\rho_0\left(\overline{\phi}\right)-\overline{\phi}x_{max}\right)\right]\end{aligned}$$

Next, we note that

$$\begin{aligned}\rho_0\left(\overline{\phi}\right) &= \frac{\delta-\underline{\delta}_n}{2\sigma}\overline{\phi} \\ \rho_{min}(\delta) &= \rho_0\left(\overline{\phi}\right)+\overline{\phi}x_{min} \\ 1-\rho_{max}(\delta) &= 1-\left(\rho_0\left(\overline{\phi}\right)+\overline{\phi}x_{max}\right)\end{aligned}$$

so that

$$(1+f)\,\Delta(\delta) = \frac{1}{\overline{\phi}}\left\{\ln\left(\overline{\phi}\right) - \ln\left(\rho_{min}(\delta)\right) + \frac{\delta-\underline{\delta}_n}{2\sigma} - s\left[\ln\left(1-\frac{\delta-\underline{\delta}_n}{2\sigma}\overline{\phi}\right) - \ln\left(1-\rho_{max}(\delta)\right)\right]\right\}$$

Let $y \equiv \frac{\delta-\underline{\delta}_n}{2\sigma}$. Then, we have

$$\begin{aligned}(1+f)\,\Delta_{\overline{\phi}}(\delta) &= -\frac{1}{\overline{\phi}}(1+f)\,\Delta(\delta) + \frac{1}{\overline{\phi}}\left\{\frac{1}{\overline{\phi}} + s\frac{y}{1-y\overline{\phi}}\right\} \\ &= \frac{1}{\overline{\phi}}\left\{\frac{1}{\overline{\phi}} + s\frac{y}{1-y\overline{\phi}} - (1+f)\,\Delta(\delta)\right\} \\ &= \frac{1}{\overline{\phi}^2}\left\{1+\frac{y\overline{\phi}}{1-y\overline{\phi}} - \overline{\phi}(1+f)\,\Delta(\delta)\right\} \\ &= \frac{1}{\overline{\phi}^2}\left\{1+\frac{y\overline{\phi}}{1-y\overline{\phi}} - \left\{\ln\left(\frac{\overline{\phi}}{\rho_{min}(\delta)}\right) + y - s\ln\left(\frac{1-y\overline{\phi}}{1-\rho_{max}(\delta)}\right)\right\}\right\}\end{aligned}$$

Note that

$$\begin{aligned}\{\cdot\} &= 1+\frac{y\overline{\phi}}{1-y\overline{\phi}} - \ln\overline{\phi} + \ln\left(\rho_{min}(\delta)\right) - y + s\ln\left(1-y\overline{\phi}\right) - s\ln\left(1-\rho_{max}(\delta)\right) \\ &= 1+\frac{y\overline{\phi}}{1-y\overline{\phi}} - \ln\overline{\phi} - (z+\delta) - y + s\ln\left(1-y\overline{\phi}\right) - s\left(\ln s - z + \delta\right) \\ &= \left\{1+\frac{y\overline{\phi}}{1-y\overline{\phi}} - \ln\overline{\phi} - y + s\ln\left(1-y\overline{\phi}\right)\right\} + (1+s)\left[\frac{-(1-s)z - s\ln s}{1+s}\right] - (1+s)\,\delta \\ &= \left\{1+\frac{y\overline{\phi}}{1-y\overline{\phi}} - \ln\overline{\phi} - y + s\ln\left(1-y\overline{\phi}\right)\right\} + (1+s)\left(\delta^* - \delta\right)\end{aligned}$$

We note that the term $\{\cdot\}$ is strictly positive for $y=0$ and thus will be positive for small $y>0$. The second term is positive as we are approaching δ^* from below via a sequence of δ. Then, we know that for small enough y, for all $\delta<\delta^*$ we have $\overline{\phi}=1$ being the optimal strategy to play against.

Plugging in $\overline{\phi}=1$ and

$$\begin{aligned}\ln\left(\rho_{min}(\delta)\right) &= -z-\delta \\ \ln\left(1-\rho_{max}(\delta)\right) &= \ln s - z + \delta\end{aligned}$$

so that

$$\begin{aligned}(1+f)\,\Delta(\delta;\underline{\delta}_n) &= \ln\left(\overline{\phi}\right) - \ln\left(\rho_{min}(\delta)\right) + \frac{\delta-\underline{\delta}_n}{2\sigma}\frac{1}{\overline{\phi}} - s\left[\ln\left(1-\frac{\delta-\underline{\delta}_n}{2\sigma}\overline{\phi}\right) - \ln\left(1-\rho_{max}(\delta)\right)\right] \\ &= z+\delta+\frac{\delta-\underline{\delta}_n}{2\sigma} - s\left[\ln\left(1-\frac{\delta-\underline{\delta}_n}{2\sigma}\overline{\phi}\right) - \ln s + z - \delta\right]\end{aligned}$$

which is increasing in δ. Let us define

$$\underline{f}(\delta) \equiv (1+f)\,\Delta(\delta;\delta) = (1-s)\,z + (1+s)\,\delta + s\ln s$$

which is increasing in δ. We define $\underline{\delta}_\infty$ by

$$\underline{f}(\underline{\delta}_\infty) = 0 \iff \underline{\delta}_\infty = -\frac{1-s}{1+s}z + \frac{-s}{1+s}\ln s$$

and we see that $\underline{\delta}_\infty = \delta^*$.

A.2.2 Upper dominance bound $\overline{\delta}_n$

Constant slope strategies. Let us introduce the following family of strategies $\rho(x)$ defined by the constant strategy $\overline{\phi}$:

$$\begin{aligned}\rho_1\left(\overline{\phi}\right) &= \frac{\delta+2\sigma-\overline{\delta}_n}{2\sigma} + \frac{\overline{\delta}_n-\delta}{2\sigma}\overline{\phi} = 1 - \frac{\overline{\delta}_n-\delta}{2\sigma}\left(1-\overline{\phi}\right) \\ \rho(x) &= \max\left\{\overline{\phi}, \rho_1\left(\overline{\phi}\right) + \left(1-\overline{\phi}\right)\cdot(x-1)\right\}\end{aligned}$$

Let us for the moment again assume that $e^{2z} < 4s$, so that $\rho_{min}(\delta) > \rho_{max}(\delta)$. Let us note that we will only consider $\overline{\phi} \leq \rho_{max}(\delta)$. If this were not the case, country 2 would never surive, which implies that $\Delta(\delta) \geq 0$ always. Let

$$\rho_1\left(\overline{\phi}\right) + \left(1-\overline{\phi}\right)(x_{flat}-1) = \overline{\phi} \iff x_{flat} = \frac{\overline{\phi}-\rho_1\left(\overline{\phi}\right)}{\left(1-\overline{\phi}\right)} + 1 = \frac{\overline{\delta}_n-\delta}{2\sigma}$$

Then

$$\begin{aligned}(1+f)\,\Delta(\delta) &= \int_0^1 \left[1_{\{\rho(x)\geq\rho_{min}(\delta)\}}\frac{1}{\rho(x)} - 1_{\{\rho(x)\leq\rho_{max}(\delta)\}}\frac{s}{1-\rho(x)}\right]dx \\ &= \int_{x_{min}}^1 \frac{1}{\rho_1\left(\overline{\phi}\right) + \left(1-\overline{\phi}\right)\cdot(x-1)}dx \\ &\quad -s\left[\int_0^{x_{flat}}\frac{1}{1-\overline{\phi}}dx + \int_{x_{flat}}^{x_{max}}\frac{1}{1-\left[\rho_1\left(\overline{\phi}\right)+\left(1-\overline{\phi}\right)\cdot(x-1)\right]}dx\right] \\ &= \frac{1}{1-\overline{\phi}}\left[\ln\left(\rho_1\left(\overline{\phi}\right)\right) - \ln\left(\rho_1\left(\overline{\phi}\right)+\left(1-\overline{\phi}\right)\cdot(x_{min}-1)\right)\right] - s\frac{1}{1-\overline{\phi}}x_{flat} \\ &\quad -s\left\{\frac{1}{1-\overline{\phi}}\left[\ln\left(1-\left[\rho_1\left(\overline{\phi}\right)+\left(1-\overline{\phi}\right)\cdot(x_{flat}-1)\right]\right) - \ln\left(1-\left[\rho_1\left(\overline{\phi}\right)+\left(1-\overline{\phi}\right)\cdot(x_{max}-1)\right]\right)\right]\right\} \\ &= \frac{1}{1-\overline{\phi}}\left[\ln\left(\rho_1\left(\overline{\phi}\right)\right) - \ln\left(\rho_{min}(\delta)\right)\right] - s\left\{\frac{1}{1-\overline{\phi}}x_{flat} + \frac{1}{1-\overline{\phi}}\left[\ln\left(1-\overline{\phi}\right) - \ln\left(1-\rho_{max}(\delta)\right)\right]\right\} \\ &= \frac{1}{1-\overline{\phi}}\left\{\left[\ln\left(1-y\left(1-\overline{\phi}\right)\right) - \ln\left(\rho_{min}(\delta)\right)\right] - s\left\{y + \left[\ln\left(1-\overline{\phi}\right) - \ln\left(1-\rho_{max}(\delta)\right)\right]\right\}\right\}\end{aligned}$$

where $y \equiv \frac{\overline{\delta}_n-\delta}{2\sigma}$. Then, taking the derivative w.r.t. $\left(1-\overline{\phi}\right)$, we have

$$\begin{aligned}(1+f)\,\Delta_{\left(1-\overline{\phi}\right)}\delta &= \frac{1}{1-\overline{\phi}}\left\{-\left(1-\overline{\phi}\right)(1+f)\,\Delta(\delta) + \frac{1}{1-y\left(1-\overline{\phi}\right)}(-y) - s\frac{1}{1-\overline{\phi}}\right\} \\ &= -\frac{1}{1-\overline{\phi}} \times \left\{\begin{array}{c}\left[\ln\left(1-y\left(1-\overline{\phi}\right)\right) - \ln\left(\rho_{min}(\delta)\right)\right] + \frac{y}{1-y\left(1-\overline{\phi}\right)} + \frac{s}{1-\overline{\phi}} \\ -s\left\{y + \left[\ln\left(1-\overline{\phi}\right) - \ln\left(1-\rho_{max}(\delta)\right)\right]\right\}\end{array}\right\}\end{aligned}$$

We want to show that $\{\cdot\}$ is positive. This implies that we want to pick the highest $1-\overline{\phi}=1$ possible, which in turn implies $\overline{\phi}=0$. We have

$$\begin{aligned}\{\cdot\} &= \ln\left(1-y\left(1-\overline{\phi}\right)\right)+z+\delta-sy-s\ln\left(1-\overline{\phi}\right)+s\left[\ln s+\delta-z\right]+\frac{y}{1-y\left(1-\overline{\phi}\right)}+\frac{s}{1-\overline{\phi}}\\ &= \frac{s}{1-\overline{\phi}}+\frac{y}{1-y\left(1-\overline{\phi}\right)}+\ln\left(1-y\left(1-\overline{\phi}\right)\right)-sy-s\ln\left(1-\overline{\phi}\right)+(1+s)\left[\frac{(1-s)z+s\ln s}{1+s}\right]+(1+s)\delta\\ &= \left\{\frac{s}{1-\overline{\phi}}+\frac{y}{1-y\left(1-\overline{\phi}\right)}+\ln\left(1-y\left(1-\overline{\phi}\right)\right)-sy-s\ln\left(1-\overline{\phi}\right)\right\}+(1+s)\left(\delta-\delta^{*}\right)\end{aligned}$$

As we are approaching δ^* from above, we note that $\delta-\delta^*\geq 0$. We note that $\{\cdot\}$ is strictly positive for $y=0$ and thus remains positive for small $y>0$.

Thus, we set $\overline{\phi}=0$ and define

$$\overline{f}\left(\delta\right)=(1+f)\,\Delta\left(\delta;\delta\right)=z+\delta+s\left(\ln s-z+\delta\right)$$

Then, we have

$$\overline{f}\left(\overline{\delta}_{\infty}\right)=0\iff\overline{\delta}_{\infty}=\frac{-(1-s)z-s\ln s}{1+s}=\delta^{*}$$

A.3 Equilibrium with Non-monotone Strategies and zero recovery

We now construct an equilibrium in which 1) agents take non-monotone strategies, and 2) both countries could be safe when the ex post realizations of $\widetilde{\delta}$ fall into an interval $[\delta_L,\delta_H]$ where δ_L and δ_H are endogenously determined. Given this equilibrium, we will compute the minimum value of $z=\underline{z}$ for which this equilibrium exists.

The possibility that both countries may be safe means that our equilibrium construction using threshold strategies is no longer possible. In a region where both countries are known to be safe (recall we consider the limit where $\sigma\to 0$), investors must be indifferent between the two countries, choosing strategies so that in equilibrium the return on both countries' bonds are equal. This "joint safety" region prevails for realizations of $\tilde{\delta}\in[\delta_L,\delta_H]$. Outside this interval, i.e., $\tilde{\delta}\in[-\overline{\delta},\delta_L)\cup(\delta_H,\overline{\delta}]$, we are back in the case where the signal is so strong that only one country is safe.

We will deriving the boundaries δ_L and δ_H as functions of z, i.e., $\delta_L(z)$ and $\delta_H(z)$. Given these functions we can solve for the minimum $\underline{z}$, which occurs when $\delta_L(\underline{z})=\delta_H(\underline{z})$ and the joint safety region vanishes.

We conjecture the following non-monotone strategy whereby investment in country 1 and in country 2 alternates on discrete intervals of length $k\sigma$ and $(2-k)\,\sigma$, with $k\in(0,2)$. The investor i's strategy given his private signal δ_i is $\phi\left(\delta_i\right)\in\{0,1\}$:

$$\phi\left(\delta_i\right)=\begin{cases}0, & \delta_i<\delta_L\\ 1, & \delta_i\in[\delta_L,\delta_L+k\sigma]\cup[\delta_L+2\sigma,\delta_L+(2+k)\,\sigma]\cup[\delta_L+4\sigma,\delta_L+(4+k)\,\sigma]\cup...\\ 0, & \delta_i\in[\delta_L+k\sigma,\delta_L+2\sigma]\cup[\delta_L+(2+k)\,\sigma,\delta_L+4\sigma]\cup[\delta_L+(4+k)\,\sigma,\delta_L+6\sigma]\cup...\\ 1, & \delta_i>\delta_H\end{cases}\tag{A.2}$$

As we will show shortly, the non-monotone oscillation occurs only when both countries are safe, where the equilibrium requires proportional investment in each safe country to equalize returns across two safe bonds. Clearly, k determines the fraction of agents in investing in country 1 when oscillation occurs, to which we turn next.

A.3.1 Fraction of agents in investing in country 1

Consider the region where all investors know that both countries are safe. In this case, the total investment in country 1 and 2 has to be $\frac{1+f}{1+s}$ and $\frac{s(1+f)}{1+s}$, respectively, otherwise one country will offer strictly higher returns than the other. Take an agent with signal δ; introduce the function $\rho\left(\delta\right)$, which is the expected proportion of agents investing in country 1 given (own) signal δ. Then, given the assumed strategy for all agents and given that we are in the region where both countries are safe,

$$\rho(\delta)=\int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x}\frac{\phi\left(y\right)}{2\sigma}dy=\frac{k\sigma}{2\sigma}.$$

We choose k so that,

$$\rho(\delta)=\frac{1}{1+s}\iff k=\frac{2}{1+s}\tag{A.3}$$

The last equality arises because in equilibrium the proportion must be constant and equal to $\frac{1}{1+s}$ in order that returns are equalized across the two safe bonds.

Recall that x denotes the fraction of agents with signal realizations above the agent's private signal δ; and x follows a uniform distribution on $[0,1]$. For any value of δ and x,

$$\rho(\delta, x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{\phi(y)}{2\sigma} dy = \begin{cases} 0, & \delta + 2\sigma x < \delta_L \\ \frac{\delta+2\sigma x-\delta_L}{2\sigma}, & \delta + 2\sigma x \in (\delta_L, \delta_L + k\sigma) \\ \frac{1}{1+s}, & \delta_H - (2-k)\sigma > \delta > \delta_L + k\sigma \end{cases} \tag{A.4}$$

We see that $\rho(\delta, x)$ is strictly increasing in x in the vicinity of δ_L, and flat everywhere else. When we evaluate δ at the marginal agent with signal $\delta = \delta_L$, we have

$$\rho(\delta_L, x) = \begin{cases} 0, & x = 0 \\ x, & x \in \left(0, \frac{1}{1+s}\right) \\ \frac{1}{1+s}, & x > \frac{1}{1+s} \end{cases} \tag{A.5}$$

where we observe that $\rho(\delta_L, x)$ is less than or equal to $\frac{1}{1+s}$.

A.3.2 Lower boundary δ_L

Let $V_\phi(\delta)$ be the expected payoff of strategy $\phi \in \{0,1\}$ when given signal δ. In the completely safe region discussed above (for δ exceeding δ_L sufficiently), investors were indifferent between both strategies because both paid exactly the same in all states of the world. This is not the case for agent with the threshold signal δ_L: as the agent knows investors with signal below is always investing in country 2, country 1 is perceived default risk. We now calculate the return of investing in either country, from the perspective of the boundary agent δ_L.

For the boundary agent δ_L, the return from investing only in country 2 (i.e. $\phi = 0$) is given by

$$\Pi_2(\delta_L) = \int_0^1 \frac{s}{(1+f)(1-\rho(\delta_L, x))} dx \tag{A.6}$$

where we integrate over all x as country 2 is safe regardless of x. Thus, plugging in, we have

$$\Pi_2(\delta_L) = \frac{s}{1+f}\left[\int_0^{\frac{1}{1+s}} \frac{1}{1-x} dx + \int_{\frac{1}{1+s}}^1 \frac{1}{\frac{s}{1+s}} dx\right] = \frac{s}{1+f}\left[\ln\frac{1+s}{s} + 1\right] < \frac{1+s}{1+f}. \tag{A.7}$$

where we used $s \ln \frac{1+s}{s} < 1$. Here, we see that payoff to investing in country 2 is lower than the expected payoff that would have realized if both countries were safe. This reflects the strategic substitution effect: because more people (in expectation) invest in the safe country 2, the return in country 2 is lower.

Now we turn to investment return for country 1. Since country 1 has default risk, we need to calculate the cutoff $x = x_{min}$ so that country 1 becomes safe if there are $x > x_{min}$ measure of agents receiving better signals. To derive x_{min}, we first solve for $\rho_1^{min}(\delta)$, which is the minimum proportion of agents investing in country 1 that are needed to make country 1 safe given fundamental δ. We have

$$\theta_1(\delta) + (1+f)\rho_1^{min}(\delta) = 1 \iff \rho_1^{min}(\delta) = \frac{1-\theta_1(\delta)}{1+f}$$

Define x_{min} as the solution to $\rho(\delta_L, x) = \rho_1^{min}(\delta_L)$. Given equation (A.5), we have that,

$$x_{min} = \frac{1-\theta_1(\delta_L)}{1+f}. \tag{A.8}$$

The expected return of investing in country 1 given one's own signal δ_L and the conjectured strategies $\phi(\cdot)$ of everyone else is given by,

$$\begin{aligned} \Pi_1(\delta_L) &= \int_{x_{min}}^1 \frac{1}{(1+f)\rho(\delta_L, x)} dx = \frac{1}{1+f}\left[\int_{x_{\min}}^{\frac{1}{1+s}} \frac{1}{x} dx + \int_{\frac{1}{1+s}}^1 \frac{1}{1/(1+s)} dx\right] \\ &= \frac{1}{1+f}\left[\ln\frac{1}{1+s} - \ln x_{min} + s\right]. \end{aligned} \tag{A.9}$$

The the boundary agentδ_L must be indifferent between investing in either country, i.e., $\Pi_2(\delta_L) = \Pi_1(\delta_L)$.

Plugging in (A.6) and (A.9), we have

$$\frac{s}{1+f}\left[\ln\frac{1+s}{s}+1\right] = \frac{1}{1+f}\left[\ln\frac{1}{1+s}-\ln x_{min}+s\right] \iff x_{min}=\frac{s^s}{(1+s)^{1+s}}. \tag{A.10}$$

We combine our two equations for x_{min}, (A.8) and (A.10), and use $1-\theta_1(\delta_L)=(1-\theta)\exp(-\delta_L)$, to obtain:

$$\frac{s^s}{(1+s)^{1+s}}=\frac{(1-\theta)\exp(-\delta_L)}{1+f}.$$

Recall $z=\ln\frac{1+f}{1-\theta}$; we have

$$\delta_L(z)=-z+(1+s)\ln(1+s)-s\ln s \tag{A.11}$$

A.3.3 Upper boundary δ_H

The derivation is symmetric to the above. We have

$$\rho(\delta,x)=\int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x}\frac{\phi(y)}{2\sigma}dy=\begin{cases}\frac{1}{1+s}, & \delta-2\sigma(1-x)<\delta_H-(2-k)\sigma\\ \frac{\delta+2\sigma x-\delta_H}{2\sigma} & \delta-2\sigma(1-x)\in(\delta_H-(2-k)\sigma,\delta_H)\\ 1, & \delta-2\sigma(1-x)>\delta_H\end{cases} \tag{A.12}$$

so that

$$\rho(\delta_H,x)=\begin{cases}\frac{1}{1+s}, & x<\frac{1}{1+s}\\ x, & x\in\left(\frac{1}{1+s},1\right)\\ 1, & x=1\end{cases} \tag{A.13}$$

Consider now the agent with signal δ_H who is considering investing in country 1. In his eyes, the return from investing only in country 1 (i.e. $\phi=1$) is given by

$$\Pi_1(\delta_H)=\int_0^1\frac{1}{\left(1+\hat{f}\right)\rho(\delta_H,x)dy}dx=\frac{1}{1+f}\left[\ln(1+s)+1\right]<\frac{1+s}{1+f},$$

where we integrated over all x as country 1 is always safe in the vicinity of δ_H.

The default condition for country 2 is

$$s\theta_2(\delta_H)+(1+f)\left[1-\rho_2^{max}(\delta_H)\right]=s \iff 1-\rho_2^{max}(\delta_H)=s\frac{1-\theta_2(\delta_H)}{1+f}$$

where $\rho_2^{max}(\delta)$ is the maximum amount of agents investing in country 1 so that country 2 does not default. Assume, but later verify, that at δ_H we have $1-\rho_2^{max}(\delta_H)<\frac{s}{1+s}$, that is, country 2 would survive even if less than $\frac{s}{1+s}$ of investors invest in country 2. Define $x_{max}(\delta_H)$ as the solution to $\rho(\delta_H,x_{max})=\rho_2^{max}(\delta_H)$; (A.13) implies that

$$1-x_{max}(\delta_H)=s\frac{1-\theta_2(\delta_H)}{1+f}. \tag{A.14}$$

As a result, the return to country 2 is,

$$\begin{aligned}\Pi_2(\delta_H) &= \int_0^{x_{max}(\delta_H)}\frac{s}{(1+f)(1-\rho(\delta_H,x))dy}dx=\frac{s}{1+\hat{f}}\left[\int_0^{\frac{1}{1+s}}\frac{1}{1-\frac{1}{1+s}}dx+\int_{\frac{1}{1+s}}^{x_{max}(\delta_H)}\frac{1}{1-x}dx\right]\\ &= \frac{s}{1+f}\left[\frac{1}{s}+\ln\frac{s}{1+s}-\ln(1-x_{max}(\delta_H))\right]\end{aligned}$$

Indifference at the boundary agent δ_H requires $\Pi_1(\delta_H)=\Pi_2(\delta_H)$, which yields $1-x_{max}(\delta_H)=\frac{s}{(1+s)^{\frac{1+s}{s}}}$. Combining this result with (A.14) and $1-\theta_2(\delta_H)=(1-\theta)\exp(\delta_H)$, we solve,

$$\delta_H(z) = z-\frac{1+s}{s}\ln(1+s) \tag{A.15}$$

A.3.4 Verifying the equilibrium

We now verify the interior agents $\delta \in (\delta_L, \delta_H)$ have the appropriate incentives to play the conjectured strategy. We just showed that the investor with signal $\delta = \delta_L$ is indifferent; and similar to the argument in the threshold equilibrium, it is easy to show that agents with $\delta < \delta_L$ find it optimal to invest in country 2. Consider an investor with signal $\delta = \delta_L + k\sigma$. Regardless of his relative position (as measured by x) in the signal distribution, this agent knows that a proportion $\frac{1}{1+s}$ of investors invest in country 1, thus making it safe for sure. Further, he knows that a proportion $\frac{s}{1+s}$ of investors invest in country 2, also making it safe. Therefore, this agent knows that (i) both countries are completely safe and that (ii) investment flows give arbitrage free prices. He is thus indifferent, and so is every investor with $\delta_L + k\sigma < \delta < \delta_H - (2-k)\sigma$.

We now consider the investors with $\delta \in (\delta_L, \delta_L + k\sigma)$? We know that country 2 will always survive, and thus we have

$$\Pi_2(\delta) = \int_0^1 \frac{s}{(1+f)\int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{1-\phi(y)}{2\sigma} dy} dx.$$

Note that for any x with $x \geq -\frac{\delta-\delta_L-k\sigma}{2\sigma}$ we are in the oscillating region; for x below we are in the increasing part. Let $\varepsilon \equiv \frac{\delta-\delta_L}{2\sigma} \in \left(0, \frac{1}{1+s}\right)$ so that so that $\delta = \delta_L + 2\sigma\varepsilon$. Thus, we have

$$1 - \rho(\delta, x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{1-\phi(y)}{2\sigma} dy \quad = \quad \begin{cases} 1-\varepsilon-x, & x \in \left(0, \frac{1}{1+s} - \varepsilon\right), \\ \frac{s}{1+s}, & x \in \left(\frac{1}{1+s} - \varepsilon, 1\right). \end{cases} \tag{A.16}$$

Then, we have

$$\Pi_2(\delta) \quad = \quad \frac{s}{1+f}\left[\int_0^{\frac{1}{1+s}-\varepsilon} \frac{1}{1-\varepsilon-x} dx + \int_{\frac{1}{1+s}-\varepsilon}^1 \frac{1}{\frac{s}{1+s}} dx\right] = \Pi_2(\delta_L) + \frac{s\left(\ln(1-\varepsilon) + \frac{1+s}{s}\varepsilon\right)}{1+f}$$

For investment in country 1, we know that, since $\delta > \delta_L$, we have $\rho_1^{min}(\delta) < \rho_1^{min}(\delta_L)$. First, note that

$$\rho(\delta, x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{\phi(y)}{2\sigma} dy = \begin{cases} \varepsilon + x, & x \in \left(0, \frac{1}{1+s} - \varepsilon\right) \\ \frac{1}{1+s}, & x \in \left(\frac{1}{1+s} - \varepsilon, 1\right) \end{cases}$$

Let $x_{min}(\delta)$ be the measure of investors with higher signals than δ so that country 1 is safe. Since $\rho_1^{min}(\delta) = \frac{1-\theta_1(\delta)}{1+f}$, $x_{min}(\delta)$ is the lowest $x \in [0,1]$ such that

$$\rho(\delta, x) = \varepsilon + x \geq \rho_1^{min}(\delta).$$

Thus, we have

$$x_{min}(\delta) = x_{min}(\delta_L + 2\sigma\varepsilon) \quad = \quad \max\left\{\frac{1-\theta_1(\delta_L+2\sigma\varepsilon)}{1+f} - \varepsilon, 0\right\}. \tag{A.17}$$

The expected investment return from country 1 is

$$\begin{aligned} \Pi_1(\delta) \quad &= \quad \int_{x:\rho(\delta,x)\geq\rho_1^{min}(\delta)} \frac{1}{(1+f)\int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{\phi(y)}{2\sigma} dy} dx \\ &= \quad \Pi_1(\delta_L) + \frac{1}{1+f}\left\{\ln x_{min}(\delta_L) - \ln\left[\varepsilon + x_{min}(\delta_L + 2\sigma\varepsilon)\right] + (1+s)\varepsilon\right\} \end{aligned}$$

Thus, to show that $\Pi_1(\delta_L + 2\sigma\varepsilon) \geq \Pi_2(\delta_L + 2\sigma\varepsilon)$, we need to show that the following inequality holds for $\varepsilon \in \left(0, \frac{1}{1+s}\right)$:

$$g_L(\varepsilon) \equiv (1+f)(\Pi_1 - \Pi_2) = \ln x_{min}(\delta_L) - \ln\left[\varepsilon + x_{min}(\delta_L + 2\sigma\varepsilon)\right] - s\ln(1-\varepsilon) \geq 0. \tag{A.18}$$

First, by using $\ln x_{min}(\delta_L) = s\ln s - (1+s)\ln(1+s)$ and $x_{min}\left(\delta_L + 2\sigma\frac{1}{1+s}\right) = 0$, we know the above inequality holds with equality at both end points $\varepsilon = 0$ and $\varepsilon = \frac{1}{1+s}$, i.e., $g_L(0) = g_L\left(\frac{1}{1+s}\right) = 0$. Second, it is easy to show that there exists a unique ε^* such that $\frac{1-\theta_1\left(\delta_L+2\sigma\varepsilon^*\right)}{1+f} = \varepsilon^*$, at which point (A.17) binds at zero. We further note that at $\varepsilon = 0$ we have $\frac{1-\theta_1(\delta_L)}{1+f} > 0$. Thus, in (A.17) we have $\varepsilon^* > 0$ and for $\varepsilon \in (0, \varepsilon^*)$ we have $x_{min}(\delta) = \frac{1-\theta_1(\delta_L+2\sigma\varepsilon)}{1+f} - \varepsilon > 0$,

and for $\varepsilon \in \left[\varepsilon^*, \frac{1}{1+s}\right]$ we have $x_{min}(\delta) = 0$. Plugging in and taking derivative with respect to ε, we have

$$\frac{\partial}{\partial \varepsilon} \ln\left[\varepsilon + x_{min}(\delta_L + 2\sigma\varepsilon)\right] = \begin{cases} \frac{-2\sigma\theta_1'(\delta_L+2\sigma\varepsilon)}{1-\theta_1(\delta_L+2\sigma\varepsilon)} & ,\varepsilon \in (0,\varepsilon^*) \\ \frac{1}{\varepsilon} & ,\varepsilon \in \left[\varepsilon^*, \frac{1}{1+s}\right] \end{cases}$$

Then, for (A.18), we have $g_L(\varepsilon)$ first rises and then drops:

$$g_L'(\varepsilon) = \begin{cases} \frac{2\sigma\theta_1'(\delta_L+2\sigma\varepsilon)}{1-\theta_1(\delta_L+2\sigma\varepsilon)} + \frac{s}{1-\varepsilon} > 0 & ,\varepsilon \in (0,\varepsilon^*), \\ -\frac{1}{\varepsilon} + \frac{s}{1-\varepsilon} = \frac{(1+s)\varepsilon - 1}{1-\varepsilon} < 0 & ,\varepsilon \in \left[\varepsilon^*, \frac{1}{1+s}\right]. \end{cases}$$

Combined with $g_L(0) = g_L\left(\frac{1}{1+s}\right) = 0$ we know that $g_L(\varepsilon) > 0, \forall \varepsilon \in \left(0, \frac{1}{1+s}\right)$, i.e., Thus, on $\varepsilon \in \left(0, \frac{1}{1+s}\right)$ the investors strictly want to invest in country 1.

We now consider the investors with $\delta \in (\delta_H - (2-k)\sigma, \delta_H)$? We know that country 1 will always survive, and thus we have

$$\Pi_1(\delta) = \int_0^1 \frac{1}{(1+f)\int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{\phi(y)}{2\sigma} dy} dx.$$

Let $\varepsilon \equiv \frac{\delta_H - \delta}{2\sigma} \in \left(0, \frac{s}{1+s}\right)$ so that so that $\delta = \delta_H - 2\sigma\varepsilon$. Thus, we have

$$\rho(\delta, x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{\phi(y)}{2\sigma} dy \quad = \quad \begin{cases} \frac{1}{1+s}, & x \in \left(0, \frac{1}{1+s} + \varepsilon\right), \\ x - \varepsilon, & x \in \left(\frac{1}{1+s} + \varepsilon, 1\right). \end{cases} \tag{A.19}$$

Plugging in, we have

$$\Pi_1(\delta) = \frac{1}{1+f}\left[\int_0^{\frac{1}{1+s}+\varepsilon} \frac{1}{\frac{1}{1+s}} dx + \int_{\frac{1}{1+s}+\varepsilon}^1 \frac{1}{x-\varepsilon} dx\right] = \frac{1}{1+f}\left[1 + (1+s)\varepsilon + \ln(1-\varepsilon) + \ln(1+s)\right].$$

For investment in country 2, we know that, since $\delta < \delta_H$, we have $1 - \rho_2^{max}(\delta) < 1 - \rho_2^{max}(\delta_L) \iff \rho_2^{max}(\delta_L) < \rho_2^{max}(\delta)$. First, note that

$$1 - \rho(\delta, x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{1-\phi(y)}{2\sigma} dy = \begin{cases} \frac{s}{1+s}, & x \in \left(0, \frac{1}{1+s} + \varepsilon\right), \\ 1 + \varepsilon - x, & x \in \left(\frac{1}{1+s} + \varepsilon, 1\right). \end{cases}$$

Let $x_{max}(\delta)$ be the measure of investors with higher signals than δ so that country 2 is safe. Since $1 - \rho_2^{max}(\delta) = s\frac{1-\theta_2(\delta)}{1+f}$, $x_{max}(\delta)$ is the highest $x \in [0,1]$ such that

$$1 - \rho(\delta, x) = 1 + \varepsilon - x \leq 1 - \rho_2^{max}(\delta).$$

Thus, we have

$$x_{max}(\delta) = x_{max}(\delta_H - 2\sigma\varepsilon) \quad = \quad \min\left\{1 + \varepsilon - s\frac{1-\theta_2(\delta_H - 2\sigma\varepsilon)}{1+f}, 1\right\}. \tag{A.20}$$

The expected investment return from country 2 is

$$\begin{aligned} \Pi_2(\delta) &= \int_{x:\rho(\delta,x)\leq\rho_2^{max}(\delta)} \frac{s}{(1+f)\int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{1-\phi(y)}{2\sigma} dy} dx \\ &= \frac{s}{1+f}\int_0^{x_{max}(\delta)} \frac{1}{1-\rho(\delta,x)} dx \\ &= \frac{s}{1+f}\left[\int_0^{\frac{1}{1+s}+\varepsilon} \frac{1}{\frac{s}{1+s}} dx + \int_{\frac{1}{1+s}+\varepsilon}^{x_{max}(\delta)} \frac{1}{1+\varepsilon-x} dx\right] \\ &= \frac{s}{1+f}\left[\frac{1+s}{s}\left(\frac{1}{1+s} + \varepsilon\right) - \ln\left[1 + \varepsilon - x_{max}(\delta)\right] + \ln\left(\frac{s}{1+s}\right)\right] \end{aligned}$$

Differencing, we have

$$g_H(\varepsilon) = (1+f)\left[\Pi_1(\varepsilon) - \Pi_2(\varepsilon)\right] = \ln(1-\varepsilon) - s\ln s + (1+s)\ln(1+s) + s\ln\left[1+\varepsilon - x_{max}(\delta)\right]$$

Finally, we need to pick σ appropriately so that there exists some natural number $N > 1$ so that $2N\sigma = \delta_H - \delta_L$. For this particular choice of $\sigma = \hat{\sigma}$, the limiting case of zero signal noise can be achieved when we take the sequence of $\sigma_n = \hat{\sigma}/n$ for $n = 1, 2, ...$

A.3.5 Equilibrium properties

First, with joint safety, the probability of survival for country 1 (or the probability of its bonds being the reserve asset) is not longer one minus the probability of survival of country 2. Using $\tilde{\delta} \sim \mathbb{U}\left(-\overline{\delta}, \overline{\delta}\right)$, the probability of country 1 survival is

$$\Pr(\text{country 1 safe}) = \frac{\overline{\delta} - \delta_L}{2\overline{\delta}} = \frac{\overline{\delta} + z - (1+s)\ln(1+s) + s\ln s}{2\overline{\delta}}, \tag{A.21}$$

and the probability of country 2 survival is

$$\Pr(\text{country 2 safe}) = \frac{\delta_H + \overline{\delta}}{2\overline{\delta}} = \frac{\overline{\delta} + z - \frac{1+s}{s}\ln(1+s)}{2\overline{\delta}}.$$

As a result, the bonds issued by country 1 are more likely to be the reserve assets than that issued by country 2 if the following condition holds:

$$s\ln s - (1+s)\ln(1+s) + \frac{1+s}{s}\ln(1+s) = s\ln s + \left(\frac{1}{s} - s\right)\ln(1+s) > 0. \tag{A.22}$$

In the proof of Proposition 7 we show this condition always holds.[12]

Obviously, the above equilibrium construction requires that $\delta_L(z) < \delta_H(z)$. Since $\delta_L(z)$ in (A.11) is decreasing in z while $\delta_H(z)$ in (A.15) is increasing in z, this condition $\delta_L(z) < \delta_H(z)$ holds if $z > \underline{z}$ so that $\delta_L(\underline{z}) = \delta_H(\underline{z})$ which gives $\underline{z}$:

$$-\underline{z} + (1+s)\ln(1+s) - s\ln s = \underline{z} - \frac{1+s}{s}\ln(1+s) \Rightarrow \underline{z} = \frac{1}{2}\left[\left(2 + s + \frac{1}{s}\right)\ln(1+s) - s\ln s\right]$$

Proposition 7 *We have the following results for the equilibrium with non-monotone strategies.*

1. *The lower equilibrium threshold $\delta_L(s,z)$ is decreasing in z, $\frac{\partial}{\partial z}\delta_L(z) < 0$. Hence, the probability that the larger country 1 is the reserve asset (given in (A.21)) is higher if the aggregate fundamental θ or aggregate saving f is higher. However, the probability of country 2's bonds being the reserve asset is also increasing with z.*
2. *When the aggregate funding conditions z is sufficiently high so that $z > \underline{z}$, the larger country 1 is more likely to be the reserve asset than the smaller country.*
3. *All else equal, country 1 has the highest likelihood of survival when country 2 size goes to zero $s \to 0$.*

Proof. The first result is obvious. For the second result we need to show that $F(s) \equiv s^2\ln s + \left(1-s^2\right)\ln(1+s) > 0$ holds for $s \in (0,1)$. It is clear that $F(0) = 0$ while $F(1) = 0$. Simple algebra shows that

$$F'(s) = 2s\ln s - 2s\ln(1+s) + 1, \frac{1}{2}F''(s) = \ln s - \ln(1+s) + 1 - \frac{s}{1+s} = \ln\left(\frac{s}{1+s}\right) + 1 - \frac{s}{1+s}.$$

[12] We have considered survival, which can be either sole survival or joint survival. If we focus on sole survivals only, i.e., the bonds of country j are the only reserve asset, the condition is exactly the same. This is because

$$\begin{aligned}
\Pr(\text{only country 1 survive}) &= \frac{-\delta_H + \overline{\delta}}{2\overline{\delta}} = \frac{\overline{\delta} - z + \frac{1+s}{s}\ln(1+s)}{2\overline{\delta}}, \\
\Pr(\text{only country 2 survive}) &= \frac{\delta_L + \overline{\delta}}{2\overline{\delta}} = \frac{\overline{\delta} - z + (1+s)\ln(1+s) - s\ln s}{2\overline{\delta}}.
\end{aligned}$$

As a result, $\Pr(\text{only country 1 survive}) > \Pr(\text{only country 2 survive})$ if and only if

$$\frac{1+s}{s}\ln(1+s) - (1+s)\ln(1+s) + s\ln s > 0,$$

which is exactly the same as (A.22).

Let $y = \frac{s}{1+s} <\in (0,1)$; then because it is easy to show $\ln y + 1 - y < 0$ (due to concavity of $\ln y$), we know that $F''(s) < 0$. As a result, $F(s)$ is concave but $F(0) = F(1) = 0$. This immediately implies that $F(s) > 0$, which is our desired result. The third claim follows because $-(1+s)\ln(1+s) + s\ln s$ is decreasing in s.

■

A.4 Equilibrium with non-monotone strategies and positive recovery

Let us say that $s_1 = 1, s_2 = s$ and $l_i s_i$ to be the recovery given default of country i, so that it returns $\frac{l_i s_i}{y_i}$ per unit of dollar invested, where y_i is total investment in country i. Then if country 1 survives, to equalize return, we need

$$\frac{l_2 s}{y_2} = \frac{1}{y_1}, y_1 + y_2 = 1 + f \Rightarrow \frac{y_1}{y_2} = \frac{1}{l_2 s}.$$

This gives prices equal to

$$\begin{aligned} p_1 &= y_1 = \frac{(1+f)}{1+l_2 s} \\ p_2 &= \frac{y_2}{s} = \frac{(1+f)\, l_2}{1+l_2 s} \end{aligned}$$

Similarly, if country 2 survives, then

$$\frac{s}{y_2} = \frac{l_1}{y_1}, y_1 + y_2 = 1 + f \Rightarrow \frac{y_1}{y_2} = \frac{l_1}{s}$$

which results in prices

$$\begin{aligned} p_1 &= y_1 = \frac{(1+f)\, l_1}{l_1 + s} \\ p_2 &= \frac{y_2}{s} = \frac{(1+f)}{l_1 + s} \end{aligned}$$

Let

$$z = \ln \frac{1+f}{1-\theta} > 0$$

and fiscal surplus is given by

$$\begin{aligned} \theta_1 &= 1 - (1-\theta) e^{-\delta} = 1 - (1+f) e^{-z} e^{-\delta} \\ s\theta_2 &= s\left[1 - (1-\theta) e^{\delta}\right] = s\left[1 - (1+f) e^{-z} e^{\delta}\right] \end{aligned}$$

Define two constants $k_1 > 1$ and $k_2 > 1$ (which only occurs if $s < l_1$) so that

$$\begin{aligned} \frac{k_1}{2-k_1} = \frac{1}{l_2 s} &\iff k_1 = \frac{2}{1+l_2 s} > 1 \\ \frac{k_2}{2-k_2} = \frac{s}{l_1} &\iff k_2 = \frac{2s}{s+l_1} > 1 \end{aligned}$$

Then in the country-1-default region, $k_2\sigma$ measure of agents invest in country 2, i.e. play $\phi = 0$, while $(2-k_2)\sigma$ measure of agents play $\phi = 1$. Similarly in the country-2-default region, , $k_1\sigma$ measure of agents play $\phi = 1$ while $(2-k_1)\sigma$ measure of agents play $\phi = 0$.

Conjecture the following equilibrium with cut off δ^*

$$\phi(y) = \begin{cases} .... & \\ 1, & y \in [\delta^* - 2\sigma, \delta^* - k_2\sigma] \\ 0, & y \in [\delta^* - k_2\sigma, \delta^*] \\ 1, & y \in [\delta^*, \delta^* + k_1\sigma] \\ 0, & y \in [\delta^* + k_1\sigma, \delta^* + 2\sigma] \\ 1, & y \in [\delta^* + 2\sigma, \delta^* + 2\sigma + k_1\sigma] \\ .... & \end{cases}$$

In other words, two types of equilibria collide at δ^*. I conjecture that marginal investor at δ^* is indifferent, while the agents between $[\delta^* - k_2\sigma, \delta^*]$ strictly prefer $\phi = 0$, and symmetrically the agents between $[\delta^*, \delta^* + k_1\sigma]$ strictly

prefer $\phi = 1$. Other agents in this economy are indifferent.

Let x denote the fraction of agents with signal realization above the agent's private signal δ_j, so that given x, the true fundamental is

$$\delta(x) = \delta_j - (1 - 2x)\sigma$$

Further, let $\rho(\delta_j, x)$ be the expected proportion agents investing in country 1 given x. Then, we have

$$\rho(\delta_j, x) = \begin{cases} 1 - \frac{k_2}{2}, & \delta + 2\sigma x < \delta^* + (2 - k_2)\sigma \\ x + cst, & else \\ \frac{k_1}{2} & \delta - 2\sigma(1 - x) > \delta^* - (2 - k_1)\sigma \end{cases}$$

where cst is picked so that $\rho(\delta_j, x)$ is continuous in x. We note that the slope is generically x as we are replacing $\phi = 0$ with $\phi = 1$ marginally. At $\delta_j = \delta^*$, we have

$$\rho(\delta^*, x) = \begin{cases} 1 - \frac{k_2}{2}, & x < 1 - \frac{k_2}{2} \\ x, & else \\ \frac{k_1}{2} & x > \frac{k_1}{2} \end{cases}$$

and we need

$$1 - \frac{k_2}{2} < \frac{k_1}{2}$$

Note that if we assume that $\rho_{min}(\delta), 1 - \rho_{max}(\delta) \in \left[1 - \frac{k_2}{2}, \frac{k_1}{2}\right]$ we have a 1-to-1 function between x and ρ that yields

$$x_{min} = \frac{1 - \theta_1(\delta^*)}{1 + f} = \frac{1 - \theta}{1 + f} e^{-\delta^*} \quad \Longleftrightarrow \quad \ln x_{min} = -z - \delta^*$$

$$1 - x_{max} = s\frac{1 - \theta_2(\delta^*)}{1 + f} = s\frac{1 - \theta}{1 + f} e^{\delta^*} \quad \Longleftrightarrow \quad \ln(1 - x_{max}) = \ln s - z + \delta^*$$

Note here that we are ignoring fundmanetal uncertainty. Otherwise, we need to take acocunt of the fact that in the mind of the agent,

$$\rho_{min}(\delta(x)) = e^{-z} e^{-\delta(x)} = e^{-z} e^{-\left[\delta_j - (1-2x)\sigma\right]}$$

is the minimum investment in country 1 needed for it to survive conditional on x. For everything else below, we assume that $\rho_{min}(\delta(x)) = \rho_{min}(\delta_j)$. Next, note that

$$x = \text{Fraction of people with signal above agent}$$

so that $x = 1$ is the most pessimistic agent, and $x = 0$ is the most optimistic. As $\rho(\delta, x)$ is increasing in x, we have

$$\begin{aligned} x < x_{min} \quad &\Longleftrightarrow \quad \text{Country 1 fails} \\ x > x_{min} \quad &\Longleftrightarrow \quad \text{Country 1 survives} \\ x < x_{max} \quad &\Longleftrightarrow \quad \text{Country 2 survives} \\ x > x_{max} \quad &\Longleftrightarrow \quad \text{Country 2 fails} \end{aligned}$$

Then, for the boundary agent, the expected return of investing in country 2 is given by

$$\begin{aligned}
\Pi_2(\delta^*) &= Return_2(survival) + Return_2(default) \\
&= \int_0^{x_{max}} \frac{s}{(1+f)(1-\rho(\delta^*,x))}dx + \int_{x_{max}}^1 \frac{l_2 s}{(1+f)(1-\rho(\delta^*,x))}dx \\
&= \int_0^{1-\frac{k_2}{2}} \frac{s}{(1+f)\left(1-\left(1-\frac{k_2}{2}\right)\right)}dx + \int_{1-\frac{k_2}{2}}^{x_{max}} \frac{s}{(1+f)(1-x)} \\
&\quad + \int_{x_{max}}^{\frac{k_1}{2}} \frac{l_2 s}{(1+f)(1-x)}dx + \int_{\frac{k_1}{2}}^1 \frac{l_2 s}{(1+f)\left(1-\frac{k_1}{2}\right)}dx \\
&= \left(1-\frac{k_2}{2}\right)\frac{s}{(1+f)\frac{k_2}{2}} + \frac{s}{1+f}\left[\ln\left(\frac{k_2}{2}\right) - \ln(1-x_{max})\right] \\
&\quad + \frac{l_2 s}{1+f}\left[\ln(1-x_{max}) - \ln\left(1-\frac{k_1}{2}\right)\right] + \left(1-\frac{k_1}{2}\right)\frac{l_2 s}{(1+f)\left(1-\frac{k_1}{2}\right)} \\
&= \frac{s}{(1+f)}\left\{\left(\frac{1-\frac{k_2}{2}}{\frac{k_2}{2}}\right) + \left[\ln\left(\frac{k_2}{2}\right) - \ln(1-x_{max})\right] + l_2 + l_2\left[\ln(1-x_{max}) - \ln\left(1-\frac{k_1}{2}\right)\right]\right\}
\end{aligned}$$

and the expected return of investing in country 1 is given by

$$\begin{aligned}
\Pi_1(\delta^*) &= \int_0^{x_{min}} \frac{l_1}{(1+f)\rho(\delta^*,x)}dx + \int_{x_{min}}^1 \frac{1}{(1+f)\rho(\delta^*,x)}dx \\
&= \int_0^{1-\frac{k_2}{2}} \frac{l_1}{(1+f)\left(1-\frac{k_2}{2}\right)}dx + \int_{1-\frac{k_2}{2}}^{x_{min}} \frac{l_1}{(1+f)x}dx \\
&\quad + \int_{x_{min}}^{\frac{k_1}{2}} \frac{1}{(1+f)x}dx + \int_{\frac{k_1}{2}}^1 \frac{1}{(1+f)\frac{k_1}{2}}dx \\
&= \left(1-\frac{k_2}{2}\right)\frac{l_1}{(1+f)\left(1-\frac{k_2}{2}\right)} + \frac{l_1}{1+f}\left[\ln(x_{min}) - \ln\left(1-\frac{k_2}{2}\right)\right] \\
&\quad + \frac{1}{1+f}\left[\ln\left(\frac{k_1}{2}\right) - \ln(x_{min})\right] + \left(1-\frac{k_1}{2}\right)\frac{1}{(1+f)\frac{k_1}{2}} \\
&= \frac{1}{1+f}\left\{l_1 + l_1\left[\ln(x_{min}) - \ln\left(1-\frac{k_2}{2}\right)\right] + \left[\ln\left(\frac{k_1}{2}\right) - \ln(x_{min})\right] + \left(\frac{1-\frac{k_1}{2}}{\frac{k_1}{2}}\right)\right\}
\end{aligned}$$

Note that

$$\begin{aligned}
\left(\frac{1-\frac{k_1}{2}}{\frac{k_1}{2}}\right) = \left(\frac{1}{\frac{k_1}{2}} - 1\right) &= 1 + sl_2 - 1 = sl_2 \\
\left(\frac{1-\frac{k_2}{2}}{\frac{k_2}{2}}\right) = \left(\frac{1}{\frac{k_2}{2}} - 1\right) &= \frac{s+l_1}{s} - \frac{s}{s} = \frac{l_1}{s}
\end{aligned}$$

Setting these equal, we have

$$\begin{aligned}
& s\left\{\frac{l_1}{s} + \left[\ln\left(\frac{k_2}{2}\right) - \ln(1-x_{max})\right] + l_2 + l_2\left[\ln(1-x_{max}) - \ln\left(1-\frac{k_1}{2}\right)\right]\right\} \\
= \quad & \left\{l_1 + l_1\left[\ln(x_{min}) - \ln\left(1-\frac{k_2}{2}\right)\right] + \left[\ln\left(\frac{k_1}{2}\right) - \ln(x_{min})\right] + sl_2\right\}
\end{aligned}$$

Plugging in for k_1, k_2 and

$$\begin{aligned}
\frac{k_1}{2} &= \frac{1}{1+l_2 s} \\
\frac{k_2}{2} &= \frac{s}{s+l_1} \\
1-\frac{k_1}{2} &= \frac{l_2 s}{1+l_2 s} \\
1-\frac{k_2}{2} &= \frac{l_1}{s+l_1} \\
\ln(x_{min}) &= -z-\delta^* \\
\ln(1-x_{max}) &= -z+\delta^*+\ln s
\end{aligned}$$

Setting these equal, we have

$$\begin{aligned}
s\left\{\left[\ln\left(\frac{k_2}{2}\right)-\ln(1-x_{max})\right]+l_2\left[\ln(1-x_{max})-\ln\left(1-\frac{k_1}{2}\right)\right]\right\} &= l_1\left[\ln(x_{min})-\ln\left(1-\frac{k_2}{2}\right)\right]+\left[\ln\left(\frac{k_1}{2}\right)-\ln(x_{min})\right] \\
\iff s\left\{-(1-l_2)\ln(1-x_{max})+\left[\ln\left(\frac{k_2}{2}\right)-l_2\ln\left(1-\frac{k_1}{2}\right)\right]\right\} &= -(1-l_1)\ln(x_{min})+\left[\ln\left(\frac{k_1}{2}\right)-l_1\ln\left(1-\frac{k_2}{2}\right)\right] \\
\iff s\left\{(1-l_2)(z-\delta^*-\ln s)+\left[\ln\left(\frac{s}{s+l_1}\right)-l_2\ln\left(\frac{l_2 s}{1+l_2 s}\right)\right]\right\} &= (1-l_1)(z+\delta^*)+\left[\ln\left(\frac{1}{1+l_2 s}\right)-l_1\ln\left(\frac{l_1}{s+l_1}\right)\right]
\end{aligned}$$

Finally, solving for δ^*, we have

$$\begin{aligned}
\delta^* &= \frac{s\left\{(1-l_2)(z-\ln s)+\left[\ln\left(\frac{s}{s+l_1}\right)-l_2\ln\left(\frac{l_2 s}{1+l_2 s}\right)\right]\right\}-(1-l_1)z-\left[\ln\left(\frac{1}{1+l_2 s}\right)-l_1\ln\left(\frac{l_1}{s+l_1}\right)\right]}{(1-l_1)+s(1-l_2)} \\
&= \frac{s\left\{(1-l_2)z-(1-l_2)\ln s+\ln s-\ln(s+l_1)-l_2\ln l_2-l_2\ln s+l_2\ln(1+l_2 s)\right\}}{(1-l_1)+s(1-l_2)} \\
&\quad +\frac{-(1-l_1)z+\ln(1+l_2 s)+l_1\ln(l_1)-l_1\ln(s+l_1)}{(1-l_1)+s(1-l_2)}
\end{aligned}$$

so that finally

$$\delta^* = \frac{\left[(1-l_2)s-(1-l_1)\right]z-(s+l_1)\ln(s+l_1)+(1+sl_2)\ln(1+l_2 s)+l_1\ln l_1-sl_2\ln l_2}{(1-l_1)+s(1-l_2)} \tag{A.23}$$

Plugging in $l_1 = l_2 = 0$, we have

$$\delta^* = \frac{-(1-s)z-s\ln(s)}{1+s}$$

just as we had before.

We want to show that from the perspective of δ^*, for an x small enough so that $\rho(\delta^*, x) = 1-\frac{k_2}{2}$, does country 1 default? We know that $\rho_{min}(\delta^*) = e^{-z}e^{-\delta^*}$, so that

$$\begin{aligned}
\rho_{min}(\delta^*) &> 1-\frac{k_2}{2} \\
\iff \ln(\rho_{min}(\delta^*)) &> \ln\left(1-\frac{k_2}{2}\right) \\
\iff -(\delta^*+z) &> \ln\left(\frac{l_1}{s+l_1}\right) \\
\iff -\left[2(1-l_2)sz-(s+l_1)\ln(s+l_1)+(1+sl_2)\ln(1+l_2 s)+l_1\ln l_1-sl_2\ln l_2\right] &> \left[(1-l_1)+s(1-l_2)\right]\left[\ln l_1-\ln(s+l_1)\right]
\end{aligned}$$

which gives

$$F_1^*(l_1, l_2, s)] \equiv -2(1-l_2)sz-[1+s(1-l_2)]\ln l_1+sl_2\ln l_2+[1+s(2-l_2)]\ln(s+l_1)-(1+l_2 s)\ln(1+l_2 s)$$

and the default condition is given by $F_1^*(l_1, l_2, s) \geq 0$. Assume $l_1 = l_2 = l$. Then, we have

$$F_1^*(l, l, s) = -2(1-l)sz-[1-(1-2l)s]\ln l+[1+s(2-l)]\ln(s+l)-(1+ls)\ln(1+ls)$$

We can show that $F_1^*(l, l, s)$ is always positive for small enough recovery l as the term $-[1-(1-2l)s]\ln l$ explodes,

swamping any negative z effect.[13]

Next, we want to show that from the perspective of δ^*, for an x large enough so that $\rho(\delta^*, x) = \frac{k_1}{2}$, does country 2 default? We know that $1 - \rho_{max}(\delta^*) = se^{-z}e^{\delta^*}$, so that

$$\begin{aligned} 1 - \rho_{max}(\delta^*) &> 1 - \frac{k_1}{2} \\ \iff \ln(1 - \rho_{max}(\delta^*)) &> \ln\left(1 - \frac{k_1}{2}\right) \\ \iff \ln s - z + \delta^* &> \ln\left(\frac{l_2 s}{1 + l_2 s}\right) \end{aligned}$$

so that

$$\begin{aligned} & [(1 - l_1) + s(1 - l_2)] \ln s - 2(1 - l_1) z - (s + l_1) \ln(s + l_1) + (1 + s l_2) \ln(1 + l_2 s) + l_1 \ln l_1 - s l_2 \ln l_2 \\ > \quad & [(1 - l_1) + s(1 - l_2)] [\ln l_2 + \ln s - \ln(1 + l_2 s)] \end{aligned}$$

Define

$$F_2^*(l_1, l_2, s) \equiv -2(1 - l_1) z - (s + l_1) \ln(s + l_1) + (2 - l_1 + s) \ln(1 + l_2 s) + l_1 \ln l_1 - [s + (1 - l_1)] \ln l_2$$

and the default condition is given by $F_2^*(l_1, l_2, s) \geq 0$. Assuming equal recovery $l_1 = l_2 = l$, we have

$$F_2^*(l, l, s) = -2(1 - l) z - (s + l) \ln(s + l) + (2 - l + s) \ln(1 + ls) - [s + (1 - 2l)] \ln l$$

We can show that $F_2^*(l, l, s)$ is always positive for small enough recovery l as the term $-[s + (1 - 2l)] \ln l$ explodes, swamping any negative z effect.[14]

Let us consider an interior agent, i.e., $\delta \in [\delta^* - k_2\sigma, \delta^* + k_1\sigma]$. Let

$$\delta(\varepsilon) = \delta^* + 2\varepsilon\sigma$$

with $\varepsilon \in \left[-\frac{k_2}{2}, \frac{k_1}{2}\right]$. Let us first consider investment in country 1. We have $\rho_{min}(\delta)$ as the default boundary, and actual investment is given by

$$\rho(\delta, x) = \begin{cases} 1 - \frac{k_2}{2}, & \delta^* + \varepsilon 2\sigma + 2\sigma x < \delta^* + (2 - k_2)\sigma \\ x + cst, & else \\ \frac{k_1}{2} & \delta^* + \varepsilon 2\sigma - 2\sigma(1 - x) > \delta^* - (2 - k_1)\sigma \end{cases} = \begin{cases} 1 - \frac{k_2}{2}, & 2\varepsilon\sigma + 2\sigma x < (2 - k_2)\sigma \\ x + cst, & else \\ \frac{k_1}{2} & 2\varepsilon\sigma - 2\sigma(1 - x) > -(2 - k_1)\sigma \end{cases}$$

which gives

$$\rho(\delta, x) = \begin{cases} 1 - \frac{k_2}{2}, & \varepsilon + x < 1 - \frac{k_2}{2} \\ x + \varepsilon, & else \\ \frac{k_1}{2} & \varepsilon + x > \frac{k_1}{2} \end{cases}$$

Note that we have $cst = \varepsilon$ by imposing continuity (which has to follow from $\rho(\delta, x)$ being an integral over strategies

[13] Taking derivatives w.r.t. l and s, we have

$$\begin{aligned} \partial_l F_1^*(l, l, s) &= 2sz + s - \frac{(1 + s)}{l} + \frac{1 + (2 - l) s}{s + l} + 2s \ln l - s \ln(s + l) - s \ln(1 + ls) \\ \partial_s F_1^*(l, l, s) &= \frac{1 + (2 - l) s}{s + l} - l \ln(1 + ls) + (2 - l) \ln(s + l) - 2(1 - l) z - l - (1 - 2l) \ln l \end{aligned}$$

[14] Taking derivatives w.r.t. l and s, we have

$$\begin{aligned} \partial_l F_2^*(l, l, s) &= \\ \partial_s F_2^*(l, l, s) &= \end{aligned}$$

ϕ).[15]

Let $x_{min}(\delta)$ be the lowest $x \in [0,1]$ such that

$$\rho(\delta, x) = \varepsilon + x \geq \rho_{min}(\delta)$$

and we therefore have

$$x_{min}(\delta) = \max\{\rho_{min}(\delta) - \varepsilon, 0\}$$

Similarly, let $x_{max}(\delta)$ be the highest $x \in [0,1]$ such that

$$1 - \rho(\delta, x) = 1 - \varepsilon - x \geq 1 - \rho_{max}(\delta)$$

and thus

$$1 - x_{max}(\delta) = \max\{1 - \rho_{max}(\delta) + \varepsilon, 0\}$$

The expected return of investing in country 1 is then given by

$$\begin{aligned}
\Pi_1(\delta) &= \int_{x:\rho(\delta,x)<\rho_{min}(x)} \frac{l_1}{(1+f)\rho(\delta,x)}dx + \int_{x:\rho(\delta,x)\geq\rho_{min}(x)} \frac{1}{(1+f)\rho(\delta,x)}dx \\
&= \int_0^{x_{min}(\delta)} \frac{l_1}{(1+f)\rho(\delta,x)}dx + \int_{x_{min}(\delta)}^1 \frac{1}{(1+f)\rho(\delta,x)}dx \\
&= \int_0^{1-\frac{k_2}{2}-\varepsilon} \frac{l_1}{(1+f)\left(1-\frac{k_2}{2}\right)}dx + \int_{1-\frac{k_2}{2}-\varepsilon}^{x_{min}(\delta)} \frac{l_1}{(1+f)(x+\varepsilon)}dx \\
&\quad + \int_{x_{min}(\delta)}^{\frac{k_1}{2}-\varepsilon} \frac{1}{(1+f)(x+\varepsilon)}dx + \int_{\frac{k_1}{2}-\varepsilon}^1 \frac{1}{(1+f)\frac{k_1}{2}}dx \\
&= \frac{l_1}{1+f}\left[\frac{1-\frac{k_2}{2}-\varepsilon}{1-\frac{k_2}{2}} + \ln(x_{min}(\delta)+\varepsilon) - \ln\left(1-\frac{k_2}{2}\right)\right] \\
&\quad + \frac{1}{1+f}\left[\ln\left(\frac{k_1}{2}\right) - \ln(x_{min}(\delta)+\varepsilon) + \frac{1-\frac{k_1}{2}+\varepsilon}{\frac{k_1}{2}}\right] \\
&= \frac{l_1}{1+f}\left[1 - \frac{\varepsilon}{1-\frac{k_2}{2}} + \ln(x_{min}(\delta)+\varepsilon) - \ln\left(1-\frac{k_2}{2}\right)\right] \\
&\quad + \frac{1}{1+f}\left[\ln\left(\frac{k_1}{2}\right) - \ln(x_{min}(\delta)+\varepsilon) + \frac{1-\frac{k_1}{2}}{\frac{k_1}{2}} + \frac{\varepsilon}{\frac{k_1}{2}}\right] \\
&= \Pi_1(\delta^*) + \frac{l_1}{1+f}\left[-\frac{\varepsilon}{1-\frac{k_2}{2}} + \ln(x_{min}(\delta)+\varepsilon) - \ln x_{min}(\delta^*)\right] \\
&\quad + \frac{1}{1+f}\left[\ln x_{min}(\delta^*) - \ln(x_{min}(\delta)+\varepsilon) + \frac{\varepsilon}{\frac{k_1}{2}}\right] \\
&= \Pi_1(\delta^*) + \frac{1}{1+f}\left\{\varepsilon\left(\frac{1}{\frac{k_1}{2}} - \frac{l_1}{1-\frac{k_2}{2}}\right) - (1-l_1)\left[\ln(x_{min}(\delta)+\varepsilon) - \ln x_{min}(\delta^*)\right]\right\} \\
&= \Pi_1(\delta^*) + \frac{1}{1+f}\left\{\varepsilon\left[(1-l_1) - s(1-l_2)\right] - (1-l_1)\left[\ln(x_{min}(\delta)+\varepsilon) - \ln x_{min}(\delta^*)\right]\right\}
\end{aligned}$$

[15] and cst is picked so that there is continuity at $x = 1 - \frac{k_2}{2} - \varepsilon$, i.e.

$$\begin{aligned}
\rho\left(\delta, 1-\frac{k_2}{2}-\varepsilon\right) &= 1 - \frac{k_2}{2} = \left(1-\frac{k_2}{2}-\varepsilon\right) + cst \iff cst = \varepsilon \\
\rho\left(\delta, \frac{k_1}{2}-\varepsilon\right) &= \frac{k_1}{2} = \left(\frac{k_1}{2}-\varepsilon\right) + cst \iff cst = \varepsilon
\end{aligned}$$

Similary, investing in country 2 gives

$$\begin{aligned}
\Pi_2(\delta) &= \int_0^{x_{max}(\delta)} \frac{s}{(1+f)(1-\rho(\delta,x))}dx + \int_{x_{max}(\delta)}^1 \frac{l_2 s}{(1+f)(1-\rho(\delta,x))}dx \\
&= \int_0^{1-\frac{k_2}{2}-\varepsilon} \frac{s}{(1+f)\left(1-\left(1-\frac{k_2}{2}\right)\right)}dx + \int_{1-\frac{k_2}{2}-\varepsilon}^{x_{max}(\delta)} \frac{s}{(1+f)(1-x-\varepsilon)} \\
&\quad + \int_{x_{max}(\delta)}^{\frac{k_1}{2}-\varepsilon} \frac{l_2 s}{(1+f)(1-x-\varepsilon)}dx + \int_{\frac{k_1}{2}-\varepsilon}^1 \frac{l_2 s}{(1+f)\left(1-\frac{k_1}{2}\right)}dx \\
&= \frac{s}{1+f}\left[\frac{1-\frac{k_2}{2}-\varepsilon}{\frac{k_2}{2}} + \ln\left(\frac{k_2}{2}\right) - \ln\left(1-x_{max}(\delta)-\varepsilon\right)\right] \\
&\quad + \frac{s l_2}{1+f}\left[\ln\left(1-x_{max}(\delta)-\varepsilon\right) - \ln\left(1-\frac{k_1}{2}\right) + \frac{1-\frac{k_1}{2}+\varepsilon}{1-\frac{k_1}{2}}\right] \\
&= \Pi_2(\delta^*) + \frac{s}{1+f}\left\{\varepsilon\left(l_2\frac{1}{1-\frac{k_1}{2}} - \frac{1}{\frac{k_2}{2}}\right) + (1-l_2)\left[\ln\left(1-x_{max}(\delta^*)\right) - \ln\left(1-x_{max}(\delta)-\varepsilon\right)\right]\right\} \\
&= \Pi_2(\delta^*) + \frac{s}{1+f}\left\{\varepsilon\left[\frac{(1-l_1)-s(1-l_2)}{s}\right] + (1-l_2)\left[\ln\left(1-x_{max}(\delta^*)\right) - \ln\left(1-x_{max}(\delta)-\varepsilon\right)\right]\right\}
\end{aligned}$$

Let us define

$$\begin{aligned}
g(\varepsilon) &\equiv (1+f)\left[\Pi_1(\delta) - \Pi_2(\delta)\right] \\
&= \varepsilon\left[(1-l_1) - s(1-l_2)\right] - (1-l_1)\left[\ln\left(x_{min}(\delta)+\varepsilon\right) - \ln x_{min}(\delta^*)\right] \\
&\quad -s\left\{\varepsilon\left[\frac{(1-l_1)-s(1-l_2)}{s}\right] + (1-l_2)\left[\ln\left(1-x_{max}(\delta^*)\right) - \ln\left(1-x_{max}(\delta)-\varepsilon\right)\right]\right\} \\
&= -(1-l_1)\left[\ln\left(x_{min}(\delta^*+2\sigma\varepsilon)+\varepsilon\right) - \ln x_{min}(\delta^*)\right] + s(1-l_2)\left[\ln\left(1-x_{max}(\delta^*+2\sigma\varepsilon)-\varepsilon\right) - \ln\left(1-x_{max}(\delta^*)\right)\right] \\
&\quad +\varepsilon\left\{\left[(1-l_1)-s(1-l_2)\right] - s\left[\frac{(1-l_1)-s(1-l_2)}{s}\right]\right\} \\
&= -(1-l_1)\left[\ln\left(x_{min}(\delta^*+2\sigma\varepsilon)+\varepsilon\right) - \ln x_{min}(\delta^*)\right] \\
&\quad +s(1-l_2)\left[\ln\left(1-x_{max}(\delta^*+2\sigma\varepsilon)-\varepsilon\right) - \ln\left(1-x_{max}(\delta^*)\right)\right]
\end{aligned}$$

Taking the derivative w.r.t. ε, we have many different cases. The issue is if x_{min} or x_{max} start binding first. Regardless, close to $\varepsilon = 0$ we have neither x_{min} or x_{max} cornered, so that

$$\begin{aligned}
\ln\left(x_{min}(\delta^*+2\sigma\varepsilon)+\varepsilon\right) &= \ln\left(\rho_{min}(\delta(\varepsilon))\right) = -z - \delta(\varepsilon) = -z - (\delta^*+2\sigma\varepsilon) \\
\ln\left(1-x_{max}(\delta^*+2\sigma\varepsilon)-\varepsilon\right) &= \ln\left(1-\rho_{max}(\delta(\varepsilon))\right) = s\ln s - z + \delta(\varepsilon) = s\ln s - z + (\delta^*+2\sigma\varepsilon)
\end{aligned}$$

and thus for ε small we have

$$g'(\varepsilon) = -(1-l_1)(-)2\sigma + s(1-l_2)2\sigma = 2\sigma\left[(1-l_1)+s(1-l_2)\right] > 0$$

and indeed we have the incentives of the agents aligned with the conjectured strategies, at least around δ^*.

Next, we have to account for all the different cases – that is, we know that at some distance ε that x_{min}, x_{max} start binding at $0, 1$, respectively.

Let ε_{min} be the point at which x_{min} becomes cornered, that is

$$\rho_{min}(\delta) = \varepsilon \iff e^{-z}e^{-\left(\delta^*+2\sigma\varepsilon\right)} = \varepsilon \iff 2\sigma\varepsilon + \ln\varepsilon = -z - \delta^*$$

Note that $\rho_{min}(\delta) > 0$ so that there is no solution for $\varepsilon < 0$.

Similarly, let ε_{max} be the point at which x_{max} becomes cornered, that is

$$1 - \rho_{max}(\delta) = -\varepsilon \iff se^{-z}e^{\delta^*+2\sigma\varepsilon} = -\varepsilon \iff 2\sigma(-\varepsilon) + \ln(-\varepsilon) = \ln s - z + \delta^*$$

Note that $1 - \rho_{max}(\delta) \geq 0$ so that there is no solution for $\varepsilon > 0$.

Positive ε. Consider positive ε. Thus, we only have to worry about x_{min} cornered. When x_{min} becomes cornered, then

$$\frac{\partial}{\partial \varepsilon} \ln \left(x_{min} \left(\delta^* + 2\sigma\varepsilon \right) + \varepsilon \right) = \frac{1}{\varepsilon}$$

Then, we have

$$g'(\varepsilon) = -(1 - l_1) \frac{1}{\varepsilon} + s(1 - l_2) 2\sigma$$

The derivative is increasing in ε, and is largest at $\varepsilon = \frac{k_1}{2}$ at a value of

$$g'\left(\frac{k_1}{2}\right) = -(1 - l_1)(1 + l_2 s) + s(1 - l_2) 2\sigma$$

For small enough σ, this is always negative.

Negative ε. Consider negative ε. Thus, we only have to worry about x_{max} cornered. When x_{max} becomes cornered, then

$$\frac{\partial}{\partial \varepsilon} \ln \left(1 - x_{max} \left(\delta^* + 2\sigma\varepsilon \right) - \varepsilon \right) = -\frac{1}{\varepsilon}$$

Then, we have

$$g'(\varepsilon) = (1 - l_1) 2\sigma + s(1 - l_2) \left(-\frac{1}{\varepsilon} \right)$$

The derivative is again increasing in ε, and is largest at $\varepsilon = -\frac{k_2}{2}$ at a value of

$$g'\left(-\frac{k_2}{2}\right) = -(1 - l_2)(s + l_1) + (1 - l_1) 2\sigma$$

For small enough σ, this is always negative.

For $s = 1$ and $l_1 = l_2 = l$, we have symmetric conditions.

The last thing we need to do is to check that

$$g\left(-\frac{k_2}{2}\right) = g(0) = g\left(\frac{k_1}{2}\right) = 0$$

To this end, we can also proof that as $\sigma \to 0$, indeed one country (which one depending on on which side of δ^* the realization of δ falls) will always default. This is equiavlent to the interior assumption for x_{max}, x_{min} we made. For this to hold, we need the following restrictions

$$1 - \frac{k_1}{2} \leq 1 - \rho_{max}(\delta^*) \leq \frac{k_2}{2} \tag{A.24}$$

$$1 - \frac{k_2}{2} \leq \rho_{min}(\delta^*) \leq \frac{k_1}{2} \tag{A.25}$$

The first line says that as $\sigma \to 0$, if $\delta < \delta^*$ then a proportion $\frac{k_2}{2}$ of investors invests in country 2, and it survives. However, if $\delta > \delta^*$, then only a proportion $1 - \frac{k_1}{2}$ of investors invests in country 2, and it defaults. Similar arguments hold for country 1, which is summarized by the second line.

This can be rewritten as

$$\ln\left(1 - \frac{k_1}{2}\right) \leq \ln\left(1 - \rho_{max}(\delta^*)\right) \leq \ln\left(\frac{k_2}{2}\right)$$

$$\ln\left(1 - \frac{k_2}{2}\right) \leq \ln \rho_{min}(\delta^*) \leq \ln\left(\frac{k_1}{2}\right)$$

which gives

$$\ln\left(\frac{l_2 s}{1 + l_2 s}\right) \leq \ln s - z + \delta^* \leq \ln\left(\frac{s}{s + l_1}\right)$$

$$\ln\left(\frac{l_1}{s + l_1}\right) \leq -z - \delta^* \leq \ln\left(\frac{1}{1 + l_2 s}\right)$$

equivalent to

$$\begin{aligned} \ln\left(\frac{l_2}{1+l_2 s}\right)+z \quad &\leq \delta^* \leq \quad \ln\left(\frac{1}{s+l_1}\right)+z \\ \ln\left(\frac{l_1}{s+l_1}\right)+z \quad &\leq -\delta^* \leq \quad \ln\left(\frac{1}{1+l_2 s}\right)+z \end{aligned}$$

equivalent to

$$\begin{aligned} \ln(l_2)-\ln(1+l_2 s)+z \quad &\leq \delta^* \leq \quad -\ln(s+l_1)+z \\ -\ln\left(\frac{1}{1+l_2 s}\right)-z \quad &\leq \delta^* \leq \quad -\ln\left(\frac{l_1}{s+l_1}\right)-z \end{aligned}$$

equivalent to

$$\begin{aligned} \ln(l_2)-\ln(1+l_2 s)+z \quad &\leq \delta^* \leq \quad -\ln(s+l_1)+z \\ \ln(1+l_2 s)-z \quad &\leq \delta^* \leq \quad \ln(s+l_1)-\ln(l_1)-z \end{aligned}$$

so that finally

$$\max\left[\ln(l_2)-\ln(1+l_2 s)+z, \ln(1+l_2 s)-z\right] \leq \delta^* \leq \min\left[-\ln(s+l_1)+z, \ln(s+l_1)-\ln(l_1)-z\right] \tag{A.26}$$

The first term is binding on the RHS for $z > \ln(1+l_2 s) - \frac{1}{2}\ln(l_2)$, and the first term is binding on the left hand side for $z < \ln(s+l_1) - \frac{1}{2}\ln(l_1)$.