Portfolio Choice under the APT with Model Misspecification*

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Abstract

In this paper, we study portfolio choice when asset returns satisfy the arbitrage pricing theory (APT), which allows for the possibility of model misspecification and mispricing. We show that, under the APT, the mean-variance portfolio can be decomposed into two components: a portfolio that depends on only pricing errors and a portfolio that depends on only factor risk premia (and their loadings). We demonstrate that these two portfolio components are orthogonal to each other and the squares of their Sharpe ratios sum to the square of the Sharpe ratio of the mean-variance portfolio. When number of assets is asymptotically large, the portfolio weights in the component that depends on pricing errors typically dominate the weights in the component that depends on the risk premia. However, the asymptotic return on the mean-variance portfolio is not dominated by either component. Analogously, we show that the global-minimum-variance portfolio can be decomposed into two components: a portfolio that depends on only the residual covariance matrix and a portfolio that depends on only the factor covariance matrix. We show that these two components are orthogonal to each other and that the sum of the squares of the Sharpe ratios of the two component portfolios is less than or equal to the square of the Sharpe ratio of the global-minimum-variance portfolio. When the number of assets is asymptotically large, then the portfolio weights associated with the residual covariance matrix dominate the weights associated with the factor covariance matrix. In the remainder of the paper, we (will) investigate a variety of theoretical and empirical applications of these properties of the mean-variance and global-minimum-variance portfolios.

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1 Introduction

How should one choose optimal portfolios when asset returns have pricing errors? How does the impact of pricing errors on portfolio weights vary depending on whether the pricing errors are a consequence of missing or latent factors versus pricing errors that are unrelated to factors. What is the relative contribution of the pricing error and the risk premia to portfolio weights, portfolio returns, and their Sharpe ratios? How can one improve the estimation of mean-variance and global-minimum-variance portfolios in the presence of pricing errors. How do the answers to the above questions change as we increase the number of assets in the portfolio?

To address these questions, we study portfolio choice when asset returns satisfy the Arbitrage Pricing Theory (APT), while allowing for the possibility of model misspecification and mispricing. The APT is a very general model that can accommodate a variety of factors. The factors could be statistical, for example, based on a principal-component decomposition of returns; macroeconomic, for example, shocks to inflation, interest rates, and exchange rates; and, characteristic-based, for instance, industry, country, size, value, return momentum, and liquidity.[1] The factor model for returns that we consider is general enough also to allow for correlated residual errors, missing and latent factors, and pure pricing errors that are unrelated to factors. We also show how to extend the traditional APT to the case of unbounded residual variation; that is, we do not limit ourselves to the traditional case of residual errors that are mildly correlated across assets.

In our analysis, we study both the mean-variance and the global-minimum-variance portfolios. Our main results are the following. We show that, under the APT, the mean-variance portfolio can be decomposed into two components: a portfolio that depends only on pricing errors and a portfolio that depends only on factor risk premia (and their loadings). We demonstrate that these two portfolio components are orthogonal to each other and that the squares of the Sharpe ratios of the returns on these two component portfolios sum to the square of the Sharpe ratio of the mean-variance portfolio. For the case in which the number of assets is asymptotically large,

[1] For further details of the variety of applications of the APT, see Connor, Goldberg, and Korajczyk (2010, Ch. 4–6).

the portfolio weights in the component that depends on pricing errors typically dominate the portfolio weights in the component that depends on the factor risk premia. However, for the asymptotic return on the mean-variance portfolio neither component dominates. We also relate our results to the "orthogonal portfolios" studied in Roll (1980), MacKinlay (1995), and MacKinlay and Pástor (2000).

Similarly, we show that the global-minimum-variance portfolio can be decomposed into two components: a portfolio that depends on only the residual covariance matrix and a portfolio that depends on only the factor covariance matrix. We show that these two components are orthogonal to each other and that the sum of the squares of the Sharpe ratios of the returns on these two component portfolios is less than or equal to the square of the Sharpe ratio of the global-minimum-variance portfolio. If the number of assets is asymptotically large, then the portfolio weights associated with the residual covariance matrix dominate the weights associated with the factor covariance matrix.

After developing these theoretical results, in the second part of the paper we (will) investigate a variety of theoretical and empirical applications of these properties of the mean-variance and global-minimum-variance portfolios.

We conclude this introduction by providing a short summary of the literature on the APT. Ross (1976, 1977) shows that if asset returns have a strict factor structure, then the mean returns are approximately linear functions of factor loadings. He obtains this result by showing that in a market with an infinite number of assets, there are sufficiently many riskless portfolios such that prices of assets are determined by a no-arbitrage condition. Huberman (1982) formalizes the argument proposed for linear factor pricing in Ross. He also shows that the arbitrage portfolios used to prove the APT need not be well diversified (that is, these portfolios could have idiosyncratic risk); instead, they need to satisfy only two conditions: they are zero-cost portfolios and their weights are orthogonal to factor loadings (which implies that they have zero factor risk).

Chamberlain (1983), Chamberlain and Rothschild (1983), and Ingersoll (1984) extend the results of Ross (1976, 1977) and Huberman (1982) to a setting where returns

need to satisfy only an approximate factor structure; that is, the idiosyncratic components of returns are allowed to be correlated across assets. Chamberlain also shows that arbitrage pricing is exact if and only if there is a risky, well-diversified portfolio on the mean-variance frontier. Moreover, if all mean-variance efficient portfolios are well-diversified, then the well-diversified portfolios provide mutual-fund separation. Chamberlain and Rothschild (1983) also show that if the covariance matrix of the asset returns has only K unbounded eigenvalues, then there is an approximate factor structure and it is unique.

The rest of the paper is organized as follows. In Section 2, we specify the linear factor model for asset returns, summarize the APT, and describe the various portfolio strategies we will study. In Section 3, we derive the properties of mean-variance and global-minimum-variance portfolios when the number of assets is finite; Section 4 extends this analysis to the setting where the number of assets is asymptotically large. Applications of our these results are (to be) reported in Section 5. We conclude in Section 6. Proofs for all our results are reported in the appendix.

2 Notation, Factor Model for Returns, and APT

2.1 Notation

Throughout the paper, we will refer to returns in *excess* of the risk-free return. Let the N-dimensional vector $\mathbf{r}_t = (r_{1t}, r_{2t}, ..., r_{Nt})'$ denote the vector of asset excess returns.

Given an arbitrary portfolio strategy a with weights $\mathbf{w}^a$ of N assets, we define the associated portfolio excess return as

$$r_t^a = \mathbf{r}_t' \mathbf{w}^a,$$

with conditional mean, standard deviation, and Sharpe ratio indicated by

$$\begin{aligned} \mu^a &= E(r_t^a), \\ \sigma^a &= \sqrt{\text{var}(r_t^a)}, \\ SR^a &= \mu^a / \sigma^a. \end{aligned}$$

2.2 Linear factor model for asset returns

Assumption 2.1 (Linear factor model)**.** *We assume the N-dimensional vector $\mathbf{r}_t$ of asset returns can be characterized by the following linear factor model*

$$\mathbf{r}_t = \boldsymbol{\mu} + \mathbf{B}\,\mathbf{z}_t + \boldsymbol{\varepsilon}_t,$$

where $\mathbf{z}_t = (z_{1t}, z_{2t}, ..., z_{Kt})'$ is the $K \times 1$ vector of common factors, assumed to be with zero (conditional) mean, without loss of generality, as explained below; $\mathbf{B} = (\beta_1, \beta_2, \ldots, \beta_N)'$ is an $N \times K$ full-rank matrix of factor loadings with ith row β_i'; $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \varepsilon_{2t}, ..., \varepsilon_{Nt})'$ is an $N \times 1$ vector of residuals, and the $N \times 1$ vector $\boldsymbol{\mu}$ represents the mean of the vector of excess returns, $\mathbf{r}_t$.[2]

At any time t

$$\mathbf{z}_t \sim (0, \boldsymbol{\Omega}),$$
$$\boldsymbol{\varepsilon}_t \sim (0, \boldsymbol{\Sigma}),$$

where the $K \times K$ covariance matrix $\boldsymbol{\Omega}$ and $N \times N$ covariance matrix $\boldsymbol{\Sigma}$ are positive definite; that is, $\boldsymbol{\Omega} > \mathbf{0}$ and $\boldsymbol{\Sigma} > \mathbf{0}$. Moreover, $\boldsymbol{\varepsilon}_t$ and $\mathbf{z}_t$ are uncorrelated; that is, $E(\boldsymbol{\varepsilon}_t \mathbf{z}_t') = \mathbf{0}$.

Assumption 2.1 implies that the conditional variance-covariance matrix for asset-returns is:

$$E\left[(\mathbf{r}_t - \boldsymbol{\mu})(\mathbf{r}_t - \boldsymbol{\mu})'\right] = \mathbf{V} = \mathbf{B}\boldsymbol{\Omega}\mathbf{B}' + \boldsymbol{\Sigma}. \tag{1}$$

Note that, just as in Chamberlain (1983), Chamberlain and Rothschild (1983), and Ingersoll (1984), we too do not restrict $\boldsymbol{\Sigma}$ to be diagonal; that is, we allow for correlated error terms. Moreover, do not restrict the maximum eigenvalues of $\boldsymbol{\Sigma}$ to be bounded, in contrast to Chamberlain (1983) and Chamberlain and Rothschild (1983).

[2]Throughout the paper, it will be assumed that $K < N$.

2.3 Arbitrage Pricing Theory (APT)

Definition 2.1 (Asymptotic arbitrage)**.** *A sequence of portfolios is said to generate an* asymptotic arbitrage opportunity *if along some subsequence N':*

$$var(\mathbf{r}_t'\mathbf{w}^a) \to 0 \quad as \quad N' \to \infty$$
$$\boldsymbol{\mu}'\mathbf{w}^a \geq \delta > 0 \quad for\ all \quad N',$$

where δ represents some positive scalar.

Assumption 2.2 (No asymptotic arbitrage)**.** *There are no asymptotic arbitrage opportunities.*

Theorem 2.1 (APT (Huberman, 1982; Ingersoll, 1984))**.** *Assumptions 2.1 and 2.2 imply that for all N there exists a positive number δ such that the weighted sum of the squared pricing errors is uniformly bounded:*

$$\hat{\boldsymbol{\alpha}}'\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\alpha}} \leq \delta < \infty,$$

where,

$$\hat{\boldsymbol{\alpha}} = \boldsymbol{\mu} - \mathbf{B}\hat{\boldsymbol{\lambda}}, \tag{2}$$

and

$$\hat{\boldsymbol{\lambda}} = \left(\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B}\right)^{-1}\mathbf{B}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}. \tag{3}$$

Remark 2.1.1. Observe that $\hat{\boldsymbol{\lambda}}$ is the projection coefficient when projecting $\boldsymbol{\mu}$ on $\mathbf{B}$ and $\hat{\boldsymbol{\alpha}}$ is the projection residual that satisfies

$$\mathbf{B}'\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\alpha}} = \mathbf{0}. \tag{4}$$

Remark 2.1.2. Observe that the projection residual $\hat{\boldsymbol{\alpha}}$ can be interpreted as a "pricing error." For instance, Roll (1980) points out that such a pricing error could arise in the Black (1972) linear model of returns if the index with respect to which risk (beta) is measured is not mean-variance efficient. This could happen if the theory was correct but the true value-weighted market portfolio was measured with error. But, the pricing error could arise also if the index was measured without error but the theory was incorrect. These pricing errors would also be present if the asset-pricing model had missing factors (MacKinlay and Pástor, 2000) or if some of the factors were latent (Connor and Korajczyk, 1986; Lehmann and Modest, 1988).

2.4 Portfolio strategies

We now consider several portfolio strategies and establish a useful correspondence between their weights and associated portfolio returns.

The first portfolio strategy we investigate is the *mean-variance portfolio*, defined by the solution to the following optimization problem:

$$\mathbf{w}^{mv} = \text{argmax}_{\mathbf{w}} \left(\mathbf{w}'\boldsymbol{\mu} - \frac{\gamma}{2}\mathbf{w}'\mathbf{V}\mathbf{w} \right),$$

where $0 < \gamma < \infty$ is the coefficient of risk aversion; $\mathbf{w}^{mv} = (w_1^{mv}, \ldots, w_N^{mv})'$ is the vector of portfolio weights in the N risky assets; and, the investment in the risk-free asset is given by $1 - \mathbf{e}'\mathbf{w}^{mv}$.

The solution to the above optimization problem is

$$\mathbf{w}^{mv} = \frac{1}{\gamma}\mathbf{V}^{-1}\boldsymbol{\mu}.$$

By standard arguments, its associated portfolio return has conditional mean, standard deviation, and Sharpe ratio given by the following three expressions:

$$\mu^{mv} = \gamma^{-1} \left(\boldsymbol{\mu}'\mathbf{V}^{-1}\boldsymbol{\mu} \right), \tag{5}$$

$$\sigma^{mv} = \gamma^{-1} \left(\boldsymbol{\mu}'\mathbf{V}^{-1}\boldsymbol{\mu} \right)^{1/2}, \tag{6}$$

$$SR^{mv} = \left(\boldsymbol{\mu}'\mathbf{V}^{-1}\boldsymbol{\mu} \right)^{1/2}. \tag{7}$$

The next trading strategy we consider is the *global minimum-variance portfolio*. This is a special case of the mean-variance portfolio, where the expected returns for all the assets are identical and all wealth is invested in only risky assets.[3] It is given by:

$$\mathbf{w}^{gmv} = \frac{\mathbf{V}^{-1}\mathbf{e}}{\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}}.$$

[3]Observe that the mean-variance portfolio with investments in only risky assets, can then be written as:

$$\frac{\mathbf{w}^{mv}}{\mathbf{e}'\mathbf{w}^{mv}} = \frac{\mathbf{V}^{-1}\boldsymbol{\mu}}{\mathbf{e}'\mathbf{V}^{-1}\boldsymbol{\mu}}.$$

In addition to the two standard portfolio strategies considered above, we also consider two new portfolios. The first is a portfolio that is often used to prove the APT (see, for example, Ingersoll (1984, Theorem 1)). We label this portfolio $\mathbf{w}^{\hat{\alpha}}$ because it depends on the pricing errors but not the factor premia. This portfolio can be expressed as

$$\mathbf{w}^{\hat{\alpha}} = \frac{1}{\gamma}\mathbf{V}^{-1}\hat{\boldsymbol{\alpha}}, \tag{8}$$

or, in terms of normalized portfolio weights:

$$\frac{\mathbf{V}^{-1}\hat{\boldsymbol{\alpha}}}{\mathbf{e}'\mathbf{V}^{-1}\hat{\boldsymbol{\alpha}}},$$

where $\hat{\boldsymbol{\alpha}}$ is defined in equation (2). The portfolio $\mathbf{w}^{\hat{\alpha}}$ can be interpreted as one that is "factor neutral;" that is, it is the portfolio whose return is *not* exposed to the risks arising from the common factors:

$$\begin{aligned}
\mathbf{r}_t'\mathbf{w}^{\hat{\alpha}} &= (\boldsymbol{\mu} + \mathbf{B}\mathbf{z}_t + \boldsymbol{\varepsilon}_t)'\frac{1}{\gamma}\mathbf{V}^{-1}\hat{\boldsymbol{\alpha}} \\
&= (\hat{\boldsymbol{\alpha}} + \mathbf{B}\hat{\boldsymbol{\lambda}} + \mathbf{B}\mathbf{z}_t + \boldsymbol{\varepsilon}_t)'\frac{1}{\gamma}\mathbf{V}^{-1}\hat{\boldsymbol{\alpha}} \\
&= (\hat{\boldsymbol{\alpha}} + \boldsymbol{\varepsilon}_t)'\frac{1}{\gamma}\mathbf{V}^{-1}\hat{\boldsymbol{\alpha}} \\
&= (\hat{\boldsymbol{\alpha}} + \boldsymbol{\varepsilon}_t)'\frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\alpha}},
\end{aligned}$$

where the last two equalities are obtained using the result in (4).

The portfolio $\mathbf{w}^{\hat{\alpha}}$ in (8) has four equivalent representations, which we will use in the next section to gain insights for the portfolio weights when the number of assets is large. These four representations are:

$$\begin{aligned}
\mathbf{w}^{\hat{\alpha}} &= \frac{1}{\gamma}\mathbf{V}^{-1}\hat{\boldsymbol{\alpha}} \\
&= \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\alpha}} \\
&= \frac{1}{\gamma}\breve{\mathbf{V}}\hat{\boldsymbol{\alpha}} \\
&= \frac{1}{\gamma}\breve{\mathbf{V}}\boldsymbol{\mu}.
\end{aligned} \tag{9}$$

The first equality is the definition of the $\mathbf{w}^{\hat{\alpha}}$ portfolio in (8) ; the second equality follows from the orthogonality of the projection in (4); the third and fourth equalities follow from the definition of $\mathbf{V}$ in (1), the definition of $\breve{\mathbf{V}}$ below,

$$\breve{\mathbf{V}} = \left[\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{B}(\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}^{-1}\right]. \tag{10}$$

and the fact that

$$\breve{\mathbf{V}}\mathbf{B} = \mathbf{0}. \tag{11}$$

To see the relation between $\mathbf{V}^{-1}$ and $\breve{\mathbf{V}}$, note that the Sherman-Morrison-Woodbury formula implies:

$$\mathbf{V}^{-1} = \left[\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{B}(\boldsymbol{\Omega}^{-1} + \mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}^{-1}\right].$$

Setting $\boldsymbol{\Omega}^{-1} = \mathbf{0}$ in the expression above then leads to the expression for $\breve{\mathbf{V}}$ in (10). To understand the intuition for this, observe that mean-variance optimization penalizes assets with large variances. Therefore, as the elements of $\boldsymbol{\Omega}$ become arbitrarily large, the only way to obtain a mean-variance portfolio, which is the portfolio maximizing the Sharpe ratio, is to set $\boldsymbol{\Omega}^{-1} = \mathbf{0}$ in the expression of $\mathbf{V}^{-1}$, which leads to $\breve{\mathbf{V}}$, which then guarantees that the condition in (11) is satisfied.

Finally, we define the portfolio strategy, $\mathbf{w}^{\hat{\lambda}}$, which in contrast to $\mathbf{w}^{\hat{\alpha}}$, depends on the risk premia but not on the pricing errors:

$$\mathbf{w}^{\hat{\lambda}} = \frac{1}{\gamma}\mathbf{V}^{-1}(\mathbf{B}\hat{\boldsymbol{\lambda}}).$$

The scaling factor, γ, does not affect our main results and without loss of generality will be set as $\gamma = 1$ in the rest of the paper.

3 Results for finite number of assets

3.1 Results for mean-variance portfolios

The following theorem, which is valid for any finite $N > K$, establishes the relations that exist across the mean-variance portfolio, $\mathbf{w}^{mv}$, and the two portfolios that depend on only the pricing error, $\mathbf{w}^{\hat{\alpha}}$, and the portfolio that depends only on factor risk, $\mathbf{w}^{\hat{\lambda}}$.

Theorem 3.1 (Weights of mean-variance portfolios)**.** *Suppose that the vector of asset returns, $\mathbf{r}_t$, satisfies Assumptions 2.1 and 2.2. Then for any finite $N > K$, the mean-variance portfolio weights satisfies the following decomposition:*

$$\mathbf{w}^{mv} = \mathbf{w}^{\hat{\alpha}} + \mathbf{w}^{\hat{\lambda}},$$

and portfolios $\mathbf{w}^{\hat{\alpha}}$ and $\mathbf{w}^{\hat{\lambda}}$ satisfy the orthogonality condition:

$$\mathbf{w}^{\hat{\alpha}\prime}\mathbf{V}\mathbf{w}^{\hat{\lambda}} = \mathbf{w}^{\hat{\alpha}\prime}\mathbf{\Sigma}\mathbf{w}^{\hat{\lambda}} = 0. \tag{12}$$

Remark 3.1.1. Note that the portfolio $\mathbf{w}^{\hat{\alpha}}$, defined in (9), does not depend on either the risk premia or the factor-covariance matrix.

Remark 3.1.2. The orthogonality condition in (12) above says that the two portfolios $\mathbf{w}^{\hat{\alpha}}$ and $\mathbf{w}^{\hat{\lambda}}$ are uncorrelated to the factors, both conditional on the factors and also unconditionally.

Theorem 3.2 (Returns of mean-variance portfolios)**.** *Suppose that the vector of asset returns, $\mathbf{r}_t$, satisfies Assumptions 2.1 and 2.2. Then for any finite $N > K$, the returns on the portfolios $\mathbf{w}^{\hat{\alpha}}$ and $\mathbf{w}^{\hat{\lambda}}$ have mean, volatility (standard deviation), and Sharpe ratio that have the same functional form as the corresponding expressions for $\mathbf{w}^{mv}$ given in equations* (5)*,* (6)*, and* (7)*. Moreover, these three quantities are functions of the same quadratic form, and satisfy, respectively:*

$$\mu^{\hat{\alpha}} = \hat{\boldsymbol{\alpha}}'\mathbf{\Sigma}^{-1}\hat{\boldsymbol{\alpha}}; \qquad \mu^{\hat{\lambda}} = \hat{\boldsymbol{\lambda}}'\mathbf{B}'\mathbf{V}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}}; \tag{13}$$

$$\sigma^{\hat{\alpha}} = (\hat{\boldsymbol{\alpha}}'\mathbf{\Sigma}^{-1}\hat{\boldsymbol{\alpha}})^{\frac{1}{2}}; \qquad \sigma^{\hat{\lambda}} = (\hat{\boldsymbol{\lambda}}'\mathbf{B}'\mathbf{V}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}})^{\frac{1}{2}}; \tag{14}$$

$$SR^{\hat{\alpha}} = (\hat{\boldsymbol{\alpha}}'\mathbf{\Sigma}^{-1}\hat{\boldsymbol{\alpha}})^{\frac{1}{2}}; \qquad SR^{\hat{\lambda}} = (\hat{\boldsymbol{\lambda}}'\mathbf{B}'\mathbf{V}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}})^{\frac{1}{2}}.$$

Moreover, the levels of the Sharpe ratios satisfy the following bounds:

$$0 \leq SR^{\hat{\alpha}} < \infty \quad \textit{and} \quad 0 \leq SR^{\hat{\lambda}} < \infty,$$

and their squares satisfy

$$(SR^{mv})^2 = (SR^{\hat{\alpha}})^2 + (SR^{\hat{\lambda}})^2.$$

Remark 3.2.1. Observe that the return on the portfolio $\mathbf{w}^{\hat{\alpha}}$ is independent of the distribution of the factors.

Note that the Sharpe ratios for the $\mathbf{w}^{\hat{\alpha}}$ and $\mathbf{w}^{\hat{\lambda}}$ portfolios are positive because both portfolios earn a positive expected return, which follows from quadratic forms for the expressions for the respective means, $\mu^{\hat{\alpha}}$ and $\mu^{\hat{\lambda}}$.

3.2 Results for global-minimum-variance portfolios

Theorem 3.3 (Weights of global minimum-variance portfolios)**.** *Suppose that the vector of asset returns, $\mathbf{r}_t$, satisfies Assumptions 2.1 and 2.2. Then for any finite $N > K$, the minimum-variance portfolio weights satisfies the following decomposition:*

$$\mathbf{w}^{gmv} = \mathbf{w}^{\Sigma} + \mathbf{w}^{\Omega},$$

where

$$\mathbf{w}^{\Sigma} = \frac{\breve{\mathbf{V}}\mathbf{e}}{\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}},$$
$$\mathbf{w}^{\Omega} = \frac{(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{e}}{\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}},$$

and portfolios $\mathbf{w}^{\Sigma}$ and $\mathbf{w}^{\Omega}$ satisfy the orthogonality condition:

$$\mathbf{w}^{\Sigma\prime}\mathbf{V}\mathbf{w}^{\Omega} = \mathbf{w}^{\Sigma\prime}\mathbf{\Sigma}\mathbf{w}^{\Omega} = 0.$$

Theorem 3.4 (Returns of global minimum-variance portfolios)**.** *Suppose that the vector of asset returns, $\mathbf{r}_t$, satisfies Assumptions 2.1 and 2.2. Then for any finite*

$N > K$: the returns on the portfolios $\mathbf{w}^{\Sigma}$ and $\mathbf{w}^{\Omega}$ have mean, volatility (standard deviation), and Sharpe ratio given by the following expressions,

$$\mu^{\Sigma} = \frac{\mathbf{e}'\breve{\mathbf{V}}\hat{\boldsymbol{\alpha}}}{\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}}; \qquad \mu^{\Omega} = \frac{\mathbf{e}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{B}\hat{\boldsymbol{\lambda}}}{\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}};$$

$$\sigma^{\Sigma} = \frac{(\mathbf{e}'\breve{\mathbf{V}}\mathbf{e})^{\frac{1}{2}}}{\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}}; \qquad \sigma^{\Omega} = \frac{\left(\mathbf{e}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{e}\right)^{\frac{1}{2}}}{\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}};$$

$$SR^{\Sigma} = \frac{\mathbf{e}'\breve{\mathbf{V}}\hat{\boldsymbol{\alpha}}}{(\mathbf{e}'\breve{\mathbf{V}}\mathbf{e})^{\frac{1}{2}}}; \qquad SR^{\Omega} = \frac{\mathbf{e}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{B}\hat{\boldsymbol{\lambda}}}{(\mathbf{e}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{e})^{\frac{1}{2}}}.$$

And, these Sharpe ratios satisfy the following bounds:

$$\left|SR^{\Sigma}\right| < \infty \quad \textit{and} \quad \left|SR^{\Omega}\right| < \infty, \quad \textit{and}$$

$$(SR^{gmv})^2 \leq (SR^{\Sigma})^2 + (SR^{\Omega})^2.$$

Remark 3.4.1. Observe that the return on the portfolio $\mathbf{w}^{\Sigma}$ depends on factor distribution only through $\boldsymbol{\Omega}$. As we will see below, this dependence on $\boldsymbol{\Omega}$ becomes weaker as the number of assets increases. However, even for a finite number of assets, the Sharpe ratio of the portfolio $\mathbf{w}^{\Sigma}$ is independent of $\boldsymbol{\Omega}$.

Remark 3.4.2. Unlike the case for the mean-variance portfolio $\mathbf{w}^{mv}$ and its components, for the global minimum-variance portfolio and its components there is no guarantee that their Sharpe ratios will be positive.

4 Results for asymptotically large number of assets

In this section, we study the case where the number of assets is asymptotically large, that is, $N \to \infty$. For this reason, we index all quantities that depend on the number of assets by the subscript N. As we did in the section above, we will analyze the behavior of the two portfolios, $\mathbf{w}_N^{\hat{\alpha}}$ and the $\mathbf{w}_N^{\hat{\lambda}}$ separately.

Just like in Chamberlain and Rothschild (1983) and Ingersoll (1984), we study a market with an infinite number of assets. That is, instead of considering a sequence of

distinct economies, we consider a *fixed* infinite economy in which we study a sequence of nested subsets of assets. That is, for the N^{th} step, one new asset is added to the first $N-1$ assets whose parameters do not change with the addition of the new assets. These unchanging parameters can be interpreted as the parameters one would get in the limit as the number of assets becomes asymptotically large.

4.1 Assumptions for case of asymptotically large number of assets

We adopt the following notation. Consider a symmetric $M \times M$ matrix $\mathbf{A}$. Let $g_{iM}(\mathbf{A})$ denote the i-th eigenvalue of $\mathbf{A}$ in decreasing order for $1 \leq i \leq M$. Thus, the maximum eigenvalue is $g_{1M}(\mathbf{A})$ and the minimum eigenvalue is $g_{MM}(\mathbf{A})$.

Recall that in the section above, where we study the case of a finite number of risky assets, we had started out by defining $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{\alpha}}$. We now study the limiting behavior of these two quantities as the number of assets increases.

We start with the definition of a "regular" factor economy.

Definition 4.1 (Regular factor economy (Ingersoll, 1984, p. 1028 and footnote 10))**.** *A factor economy is* regular *if the maximum eigenvalue* $g_{1K}(\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N)^{-1} \to 0$ *as* $N \to \infty$*. If the limit is positive instead of zero, then the factor representation is* irregular*.*

We now introduce the limit of $\hat{\boldsymbol{\lambda}}$ as $N \to \infty$, which we label $\boldsymbol{\lambda}_\infty = \boldsymbol{\lambda}$, and the associated vector of pricing errors, $\boldsymbol{\alpha}_N = (\boldsymbol{\mu}_N - \mathbf{B}_N\boldsymbol{\lambda})$. Under Assumptions 2.1 and 2.2, and the regularity assumption above, Ingersoll (1984, Theorem 3 and ftn. 10) shows that $\boldsymbol{\lambda}$ is unique and prices assets with bounded squared error:

$$\boldsymbol{\alpha}_N'\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\alpha}_N \leq \delta < \infty, \tag{15}$$

where δ represents some positive scalar. Observe that, as discussed in Ingersoll (1984, footnote 10), the weighting of the squared pricing error uses $\boldsymbol{\Sigma}_N^{-1}$. The intuition for why the squared error bound in (15) is weighted by the (inverse of the) the matrix of residual variances, $\boldsymbol{\Sigma}$, is that an asset's usefulness in arbitrages is limited by its

residual variance. As Ingersoll (1984) explains: "Roughly speaking, the smaller an asset's residual variance, the more extreme its weighting can be in a portfolio, while maintaining diversification, and the more extreme is its weighting, the bigger is its effect on the portfolio's expected return. Thus, an asset with a small residual variance can have a major impact on a diversified portfolio's expected return, and to prevent arbitrage its own expected return must be more nearly in line with the prediction of the APT. Assets with larger residual variances can only have small effects on a diversified portfolio's expected return, so their own expected returns need not be as closely in line with the prediction."

To understand the condition in (15), note that from (2) and (3) we have:

$$\begin{aligned}\hat{\boldsymbol{\alpha}}_N &= \boldsymbol{\mu}_N - \mathbf{B}_N\hat{\boldsymbol{\lambda}} \\ &= (\mathbf{I}_N - \mathbf{B}_N(\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N)^{-1}\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1})(\boldsymbol{\alpha}_N + \mathbf{B}_N\boldsymbol{\lambda}) \\ &= (\mathbf{I}_N - \mathbf{B}_N(\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N)^{-1}\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1})\boldsymbol{\alpha}_N.\end{aligned}$$

Because

$$\boldsymbol{\Sigma}_N^{-1}\hat{\boldsymbol{\alpha}}_N = \boldsymbol{\Sigma}_N^{-1}(\mathbf{I}_N - \mathbf{B}_N(\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N)^{-1}\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1})\boldsymbol{\alpha}_N = \breve{\mathbf{V}}_N\boldsymbol{\alpha}_N,$$

it follows that

$$\begin{aligned}\hat{\boldsymbol{\alpha}}_N'\boldsymbol{\Sigma}_N^{-1}\hat{\boldsymbol{\alpha}}_N &= \hat{\boldsymbol{\alpha}}_N'\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\Sigma}_N\boldsymbol{\Sigma}_N^{-1}\hat{\boldsymbol{\alpha}}_N \\ &= \boldsymbol{\alpha}_N'\breve{\mathbf{V}}_N\boldsymbol{\Sigma}_N\breve{\mathbf{V}}_N\boldsymbol{\alpha}_N \\ &= \boldsymbol{\alpha}_N'\breve{\mathbf{V}}_N\boldsymbol{\alpha}_N. \qquad (16)\end{aligned}$$

Therefore, the condition in (17) below,

$$\hat{\boldsymbol{\alpha}}_N'\boldsymbol{\Sigma}_N^{-1}\hat{\boldsymbol{\alpha}}_N \leq \delta < \infty, \qquad (17)$$

which is the central result of the APT (see, for instance, Ingersoll (1984, Theorem 1), Huberman (1982, Theorem 1), and Chamberlain and Rothschild (1983, Theorem 3')), implies from (16) that

$$\boldsymbol{\alpha}_N'\breve{\mathbf{V}}_N\boldsymbol{\alpha}_N = \boldsymbol{\alpha}_N'\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\alpha}_N - \boldsymbol{\alpha}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N(\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N)^{-1}\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\alpha}_N \leq \delta < \infty. \qquad (18)$$

Furthermore, because $\boldsymbol{\alpha}_N'\breve{\mathbf{V}}_N\boldsymbol{\alpha}_N \geq 0$, therefore (18) implies that

$$0 \leq \boldsymbol{\alpha}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N(\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N)^{-1}\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\alpha}_N \leq \boldsymbol{\alpha}_N'\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\alpha}_N \leq \delta < \infty,$$

implying that (15), as well as the equation below, hold by no-arbitrage.

$$\boldsymbol{\alpha}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N(\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N)^{-1}\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\alpha}_N \leq \delta < \infty$$

4.2 Conditions imposed on pricing errors and residual covariance matrix by no arbitrage

The condition in (15), reproduced below,

$$\boldsymbol{\alpha}_N' \boldsymbol{\Sigma}_N^{-1} \boldsymbol{\alpha}_N \leq \delta < \infty,$$

which is a consequence of asymptotic no arbitrage, links the pricing error $\boldsymbol{\alpha}_N$ and the idiosyncratic covariance matrix, $\boldsymbol{\Sigma}_N$. There are two possible, complementary, cases for $\boldsymbol{\Sigma}_N$ as $N \to \infty$. In the first case, for large N all the eigenvalues of $\boldsymbol{\Sigma}_N$ are bounded, and in the second case, at least one of the eigenvalues is unbounded. The existing literature has focused on studying the case in which all the eigenvalues of $\boldsymbol{\Sigma}_N$ are bounded, which is summarized in the theorem below, the proof for which is provided in Huberman (1982) and Ingersoll (1984).

Theorem 4.1 (Constraint imposed on $\boldsymbol{\alpha}_N$ by no arbitrage; Huberman (1982); Ingersoll (1984))**.** *If $\boldsymbol{\Sigma}_N$ has bounded eigenvalues for large N, then the condition in* (15) *requires the elements of the pricing-error vector $\boldsymbol{\alpha}_N$ to become small for large N in the following sense:*

$$\boldsymbol{\alpha}_N' \boldsymbol{\alpha}_N \leq g_{1N}(\boldsymbol{\Sigma}_N)(\boldsymbol{\alpha}_N' \boldsymbol{\Sigma}_N^{-1} \boldsymbol{\alpha}_N) \leq \delta < \infty,$$

but without $\boldsymbol{\alpha}_N$ being tied down to $\boldsymbol{\Sigma}_N$.

The assumption that the covariance matrix for the residuals of returns, $\boldsymbol{\Sigma}_N$, has bounded eigenvalues for large N implies that the return of any well-diversified portfolio (sometimes referred to a "well-spread" portfolio) has zero residual risk in the limit. The above theorem then says that the squared sum of the pricing errors is bounded, which is equivalent to saying that the squared pricing errors, $\boldsymbol{\alpha}_N$, get smaller as the number of assets increases.

In contrast to the existing literature, we also consider the case where, for large N, all the eigenvalues of $\boldsymbol{\Sigma}_N$ are *not* bounded.

Theorem 4.2 (Constraint imposed on $\boldsymbol{\Sigma}_N$ and $\boldsymbol{\alpha}_N$ by no arbitrage)**.** *Suppose that the vector of asset returns, $\mathbf{r}_t$, satisfies Assumptions 2.1 and 2.2. Suppose that for*

some finite $1 \leq p < N$ *the following three conditions hold: (i)* $\sup_N g_{pN}(\boldsymbol{\Sigma}_N) = \infty$*; (ii)* $\sup_N g_{p+1\,N}(\boldsymbol{\Sigma}_N) \leq \delta < \infty$*; and, (iii)* $\inf_N g_{NN}(\boldsymbol{\Sigma}_N) \geq \delta > 0$*. Then, the APT condition in equation* (15) *is satisfied by the pricing error* $\boldsymbol{\alpha}_N$*, represented as*

$$\boldsymbol{\alpha}_N = \mathbf{A}_N \mathbf{m} + \mathbf{a}_N, \tag{19}$$

where $\mathbf{A}_N$ *is a* $N \times p$ *matrix whose* j*th column equals* $g_{jN}^{\frac{1}{2}}(\boldsymbol{\Sigma}_N)\mathbf{v}_{jN}(\boldsymbol{\Sigma}_N)$*, where* $1 \leq j \leq p$*,* $\mathbf{v}_{jN}(\boldsymbol{\Sigma}_N)$ *is the eigenvector of* $\boldsymbol{\Sigma}_N$ *associated with the eigenvalue* $g_{jN}(\boldsymbol{\Sigma}_N)$*,* $\mathbf{m}$ *is some* $p \times 1$ *vector, and* $\mathbf{a}_N$ *is some non-zero* $N \times 1$ *vector that satisfies*

$$\mathbf{a}_N' \mathbf{a}_N \leq \delta < \infty.$$

As we will explain in greater detail below, one can interpret the two components of $\boldsymbol{\alpha}_N$ in (19) in a variety of ways. The first term, $\mathbf{A}_N\mathbf{m}$, could be associated with p latent or missing pervasive factors, where $\mathbf{A}_N$ are the factor loadings and $\mathbf{m}$ are the risk premia for the missing factors. The second term, $\mathbf{a}_N$, is the idiosyncratic part of the pricing error $\boldsymbol{\alpha}_N$; for instance, $\mathbf{a}_N$ could be interpreted as representing managerial skill.

More importantly, Theorem 4.2 shows that the common perception that the pricing errors in $\boldsymbol{\alpha}_N$ need to be small in the APT is not completely accurate. In particular, if the maximum eigenvalue of the residual covariance matrix is not bounded, then the pricing errors can also be large without violating the no-arbitrage condition given in (15). Recall our earlier comment that, if the covariance matrix for the residuals of returns, $\boldsymbol{\Sigma}_N$, has bounded eigenvalues for large N, then the return of any well-diversified portfolio has zero residual risk in the limit. What Theorem 4.2 states is that if the maximum eigenvalue of $\boldsymbol{\Sigma}_N$ is unbounded, then the contribution of the pricing error to the portfolio return could be large, but for this to satisfy the no-arbitrage condition, any portfolio earning this high return would not be well diversified. To see this, observe that by Chamberlain and Rothschild (1983, Theorem 4), under the assumptions made for deriving Theorem 4.2, the covariance matrix of residuals, $\boldsymbol{\Sigma}_N$, has the following approximate p-factor structure:

$$\boldsymbol{\Sigma}_N = \mathbf{A}_N \mathbf{A}_N' + \mathbf{C}_N,$$

where $\mathbf{C}_N$ is a $N \times N$ positive semi-definite matrix with bounded eigenvalues. Therefore, the residual variance of the return on any portfolio weights $\mathbf{w}_N$ is given by

$$\mathbf{w}_N' \mathbf{\Sigma}_N \mathbf{w}_N = \mathbf{w}_N' \mathbf{A}_N \mathbf{A}_N' \mathbf{w}_N + \mathbf{w}_N' \mathbf{C}_N \mathbf{w}_N.$$

Whereas the second term, $\mathbf{w}_N' \mathbf{C}_N \mathbf{w}_N$, goes to zero for well-spread portfolios (that is, for portfolios with $\mathbf{w}_N' \mathbf{w}_N \to 0$), there is no guarantee that the same occurs for the first term, $\mathbf{w}_N' \mathbf{A}_N \mathbf{A}_N' \mathbf{w}_N$.

4.3 Results for mean-variance portfolios

In contrast to the definition of "regularity" given above (see Definition 4.1), an alternative is to consider "pervasive factors." Whereas regularity implies only that all the eigenvalues of $\mathbf{B}_N' \mathbf{\Sigma}_N^{-1} \mathbf{B}_N$ diverge as $N \to \infty$, we use a stronger definition of pervasiveness, in which all the rates of divergence of the eigenvalues are identical.

We wish to extend the definition of a regular economy in two different dimensions. The earlier definition of regularity was applied to $\mathbf{B}_N$ and $\mathbf{\Sigma}_N$. We extend the definition to any arbitrary matrix of dimension $N \times p$ and an arbitrary positive definite matrix $N \times N$. Second, we impose that the eigenvalues are diverging at precisely the same rate.

Definition 4.2 ($\mathbf{C_N}$-Regularity)**.** *The definition of* regularity *requires that there exists an increasing function of N, $f(N)$, such that for any $1 \leq j \leq K$, the eigenvalues $g_{jK}(\frac{1}{f(N)} \mathbf{D}_N' \mathbf{C}_N^{-1} \mathbf{D}_N) \to \delta_j > 0$, where δ_j is some finite positive constant.*

A special case of the above definition is the notion of pervasive factors defined in Connor and Korajczyk (1986, Assumption 6) and Connor, Goldberg, and Korajczyk (2010, p. 85). Both papers use $f(N) = N$, but Connor and Korajczyk require all the eigenvalues to diverge at least at that rate, whereas Connor, Goldberg, and Korajczyk is similar to our definition in the sense that it requires all eigenvalues to diverge precisely at the same rate.

The above definition also nests as a special case the notion of a well-spread portfolio, where a well-spread portfolio $\mathbf{w}$ is defined to be one where $\mathbf{w}' \mathbf{\Sigma}_N \mathbf{w} \to 0$ as $N \to \infty$, implying that $\mathbf{w}$ is $\mathbf{\Sigma}_N^{-1}$-regular.

Below, we will present results for the two cases considered in Theorems 4.1 and 4.2. Ingersoll (1984, p. 1026) defines the first setting as one with "bounded residual variation;" we label the second setting as one with "unbounded residual variation."

Theorem 4.3 (Weights of mean-variance portfolio for large N with bounded residual variation)**.** *Suppose that the vector of asset returns, $\mathbf{r}_t$, satisfies Assumptions 2.1 and 2.2, and the loadings $\mathbf{B}_N$ are $\boldsymbol{\Sigma}_N$-regular. Then, as $N \to \infty$, the absolute value of the components of the mean-variance portfolio vectors, $\mathbf{w}_N^{\hat{\alpha}}$ and $\mathbf{w}_N^{\hat{\lambda}}$, decrease at most at the rate:*

$$|w_i^{\hat{\alpha}}| = \mathcal{O}\left(|\mathbf{e}_i'\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\alpha}_N| + \frac{\|\mathbf{e}_i'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N\|}{g_{KK}^{\frac{1}{2}}(\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N)}\right); \tag{20}$$

$$|w_i^{\hat{\lambda}}| = \mathcal{O}\left(\frac{\|\mathbf{e}_i'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N\|}{g_{KK}(\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N)}\right). \tag{21}$$

From equations (20) and (21), we see that $w_i^{\hat{\alpha}}$ can dominate $w_i^{\hat{\lambda}}$ as the number of assets increases. In particular, $w_i^{\hat{\alpha}}$ dominates $w_i^{\hat{\lambda}}$ when the pricing-error term, $|\mathbf{e}_i'\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\alpha}_N|$, goes to zero slowly as the number of assets increases. Recall that the APT bounds the pricing error from above; that is, $\boldsymbol{\alpha}_N'\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\alpha}_N \leq \delta < \infty$. But, the APT is silent about whether $\boldsymbol{\alpha}_N'\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\alpha}_N$ is bounded away from zero. When this expression is bounded away zero, then one can show that the ratio $w_i^{\hat{\lambda}}/w_i^{\hat{\alpha}}$ always decreases at the rate that is equal or faster than $1/g_{KK}^{\frac{1}{2}}(\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N)$. In the corollary below, we look at a special case to get a better understanding of this result.

Corollary 4.1 (Weights of mean-variance portfolios for large N with bounded residual variation and $f(N) = N$)**.** *Suppose that the vector of asset returns, $\mathbf{r}_t$, satisfies Assumptions 2.1 and 2.2, and the loadings $\mathbf{B}_N$ are $\boldsymbol{\Sigma}_N$-regular with $f(N) = N$. Then, as $N \to \infty$, the absolute value of the components of the mean-variance portfolio vectors, $\mathbf{w}_N^{\hat{\alpha}}$ and $\mathbf{w}_N^{\hat{\lambda}}$, decrease at the rate:*

$$|w_i^{\hat{\alpha}}| = \mathcal{O}\left(|\mathbf{e}_i'\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\alpha}_N| + \frac{\|\mathbf{e}_i'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N\|}{N^{\frac{1}{2}}}\right); \tag{22}$$

$$|w_i^{\hat{\lambda}}| = \mathcal{O}\left(\frac{\|\mathbf{e}_i'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N\|}{N}\right). \tag{23}$$

Moreover, if in addition, the rows of $\mathbf{B}_N$ *and* $\mathbf{\Sigma}_N^{-1}$*, are uniformly bounded:*

$$\sup_i \|\mathbf{e}_i'\mathbf{B}_N\| \leq \delta < \infty, \qquad \sup_i \|\mathbf{e}_i'\mathbf{\Sigma}_N^{-1}\| \leq \delta < \infty,$$

then

$$|w_i^{\hat{\alpha}}| = \mathcal{O}\left(\max\left\{|\mathbf{e}_i'\mathbf{\Sigma}_N^{-1}\boldsymbol{\alpha}_N|, \frac{1}{N^{\frac{1}{2}}}\right\}\right); \tag{24}$$

$$|w_i^{\hat{\lambda}}| = \mathcal{O}\left(\frac{1}{N}\right). \tag{25}$$

Comparing (24) and (25), we see that $w_i^{\hat{\lambda}}$ will always go to zero at a slower rate than $w_i^{\hat{\alpha}}$. The expression in (25) represents the rate at which the mean-variance portfolio weights decay to zero as we increase the number of assets under the traditional setting of exact pricing where the pricing error $\boldsymbol{\alpha}_N$ is zero. The expression in (26) for $w_i^{\hat{\alpha}}$ shows how the pricing error impacts portfolio weights for large N. The asymptotic no-arbitrage condition of the APT in (15) allows us to say only that $\mathbf{e}_i'\mathbf{\Sigma}_N^{-1}\boldsymbol{\alpha}_N$ decays to zero, without restricting the rate. On the other hand, mathematically speaking, $\mathbf{e}_i'\mathbf{\Sigma}_N^{-1}\boldsymbol{\alpha}_N$ cannot decay slower than $\frac{1}{N^{\frac{1}{2}-\epsilon}}$, for any small $\epsilon > 0$.

Comparing the set of equations in (26) and (25) with those in equations (22) and (23), we see that the only additional term is $\|\mathbf{e}_i'\mathbf{\Sigma}_N^{-1}\mathbf{B}_N\|$. If this term were not restricted to be uniformly bounded, then one can show that it is not increasing at a rate $g_{KK}^{\frac{1}{2}}(\mathbf{B}_N'\mathbf{\Sigma}_N^{-1}\mathbf{B}_N)$ or faster.

We now study the case of unbounded residual variation, that is, the case where $\mathbf{\Sigma}_N$ satisfies the conditions listed in Theorem 4.2, which implies that

$$\mathbf{\Sigma}_N = \mathbf{A}_N\mathbf{A}_N' + \mathbf{C}_N.$$

As discussed before, the following specification of $\boldsymbol{\alpha}_N$ is consistent with no arbitrage:

$$\boldsymbol{\alpha}_N = \mathbf{A}_N\mathbf{m} + \mathbf{a}_N.$$

Theorem 4.4 (Weights of mean-variance portfolios for large N with unbounded residual variation)**.** *Suppose that the assumptions of Theorem 4.2 are satisfied. Suppose also that* $\mathbf{A}_N$ *and* $\mathbf{B}_N$ *are* $\mathbf{C}_N$*-regular with the same scaling factor* $f(N)$ *and* $\mathbf{A}_N$

and $\mathbf{B}_N$ *are not asymptotically collinear. Then, as* $N \to \infty$*, the absolute value of the components of the mean-variance portfolio vectors,* $\mathbf{w}_N^{\hat{\alpha}}$ *and* $\mathbf{w}_N^{\hat{\lambda}}$*, decrease at most at the rate:*

$$|w_i^{\hat{\alpha}}| = \mathcal{O}\left(|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{a}_N| + \|\mathbf{a}_N\|\frac{\|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{B}_N\|}{g_{KK}^{\frac{1}{2}}(\mathbf{B}_N'\mathbf{C}_N^{-1}\mathbf{B}_N)} + \|\mathbf{m}\|\frac{\|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{A}_N\|}{g_{pp}(\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)}\right);$$
$$|w_i^{\hat{\lambda}}| = \mathcal{O}\left(\frac{\|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{B}_N\|}{g_{KK}(\mathbf{B}_N'\mathbf{C}_N^{-1}\mathbf{B}_N)}\right).$$

Corollary 4.2 (Weights of mean-variance portfolios for large N with unbounded residual variation and $f(N) = N$)**.** *Suppose that the assumptions of Theorem 4.2 are satisfied. Suppose also that* $\mathbf{A}_N$ *and* $\mathbf{B}_N$ *are* $\mathbf{C}_N$*-regular with the same scaling factor* $f(N) = N$*. Then, as* $N \to \infty$*, the absolute value of the components of the mean-variance portfolio vectors,* $\mathbf{w}_N^{\hat{\alpha}}$ *and* $\mathbf{w}_N^{\hat{\lambda}}$*, decrease at most at the rate:*

$$|w_i^{\hat{\alpha}}| = \mathcal{O}\left(|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{a}_N| + \|\mathbf{a}_N\|\frac{\|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{B}_N\|}{N^{\frac{1}{2}}} + \|\mathbf{m}\|\frac{\|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{A}_N\|}{N}\right);$$
$$|w_i^{\hat{\lambda}}| = \mathcal{O}\left(\frac{\|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{B}_N\|}{N}\right).$$

Moreover, if in addition, the rows of $\mathbf{A}_N, \mathbf{B}_N$ *and* $\mathbf{C}_N^{-1}$ *are uniformly bounded:*

$$\sup_i \|\mathbf{e}_i'\mathbf{A}_N\| \leq \delta < \infty, \qquad \sup_i \|\mathbf{e}_i'\mathbf{B}_N\| \leq \delta < \infty, \qquad \sup_i \|\mathbf{e}_i'\mathbf{C}_N^{-1}\| \leq \delta < \infty,$$

then

$$|w_i^{\hat{\alpha}}| = \mathcal{O}\left(\max\left\{|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{a}_N|, \frac{\|\mathbf{a}_N\|}{N^{\frac{1}{2}}} + \frac{\|\mathbf{m}\|}{N}\right\}\right); \tag{26}$$
$$|w_i^{\hat{\lambda}}| = \mathcal{O}\left(\frac{1}{N}\right).$$

We now study the properties of the returns of the mean-variance portfolio for the case in which the number of assets is large.

Theorem 4.5 (Returns of mean-variance portfolio for large N)**.** *Suppose that the vector of asset returns, $\mathbf{r}_t$, satisfies Assumptions 2.1 and 2.2 but ruling out exact pricing. Then, for portfolio weights $\mathbf{w}^{\hat{\alpha}}$ and $\mathbf{w}^{\hat{\lambda}}$ that satisfy either the assumptions of Theorem 4.3 for the case of bounded residual variation, or the assumptions of Theorem 4.4 for the case of unbounded residual variation, we have the following results. For $N \to \infty$, the asymptotic means of the excess returns on the portfolios $\mathbf{w}^{\hat{\alpha}}$ and $\mathbf{w}^{\hat{\lambda}}$ defined by the expressions below:*

$$\mu_\infty^{\hat{\alpha}} = \lim_{N\to\infty} \mu_N^{\hat{\alpha}} = \lim_{N\to\infty} \boldsymbol{\alpha}_N' \breve{\mathbf{V}}_N \boldsymbol{\alpha}_N, \qquad \mu_\infty^{\hat{\lambda}} = \lim_{N\to\infty} \mu_N^{\hat{\lambda}} = \boldsymbol{\lambda}' \boldsymbol{\Omega}^{-1} \boldsymbol{\lambda},$$

satisfy the following bounds:

$$0 < \mu_\infty^{\hat{\alpha}} < \infty \quad \textit{and} \quad 0 < \mu_\infty^{\hat{\lambda}} < \infty.$$

Furthermore, the asymptotic variances and squared Sharpe ratios of the returns on the portfolios $\mathbf{w}^{\hat{\alpha}}$ and $\mathbf{w}^{\hat{\lambda}}$ are given by the same expressions as the means and satisfy the same properties.

Remark 4.5.1. The above theorem states that $\mu_\infty^{\hat{\alpha}} > 0$. We now interpret this inequality in terms of the asymptotic variance of the portfolio. Recall that $\mu_\infty^{\hat{\alpha}}$ represents not just the asymptotic mean return but also the asymptotic variance of the portfolio, which is equal to the limit of the idiosyncratic variance of the portfolio because for any N

$$\text{var}(\mathbf{r}_t' \mathbf{w}_N^{\hat{\alpha}}) = \text{var}(\boldsymbol{\varepsilon}_t' \mathbf{w}_N^{\hat{\alpha}}) = \boldsymbol{\alpha}_N' \breve{\mathbf{V}}_N \boldsymbol{\alpha}_N.$$

We will have exact equality, $\mu_\infty^{\hat{\alpha}} = 0$ in two cases. One, where each element of $\boldsymbol{\alpha}_N$ is zero; that is, the case of exact pricing (see Chamberlain (1983, Corollary 1)). Two, where $\boldsymbol{\alpha}_N$ is asymptotically collinear with $\mathbf{B}_N$; that is, $\boldsymbol{\alpha}_N = B_N \boldsymbol{\delta}_K + \boldsymbol{\delta}_N$ for some nonzero constant vector $\boldsymbol{\delta}_K$ and $\boldsymbol{\delta}_N$ so that the squared sum of the elements of the residual vector, $\boldsymbol{\delta}_N' \boldsymbol{\delta}_N$, decays to zero.

Observe that the two portfolios, $\mathbf{w}^{\hat{\alpha}}$ and $\mathbf{w}^{\hat{\lambda}}$, generate portfolio returns with bounded Sharpe ratios. These Sharpe ratios are, in general, of the same order of magnitude even for large N. This is in sharp contrast to the portfolio weights themselves, where $\mathbf{w}^{\hat{\alpha}}$ asymptotically dominates the $\mathbf{w}^{\hat{\lambda}}$, in general. For instance, focusing on the case of bounded residual variation, the $\mathbf{w}^{\hat{\lambda}}$ portfolio decays to zero at a fast

rate equal to $1/N$ (see Corollary 4.2) whereas $\mathbf{w}^{\hat{\alpha}}$ portfolio decays to zero at a rate to $1/N^{\frac{1}{2}}$ at most. The reason for this contrast is that, for finite N, the portfolio return associated with $\mathbf{w}^{\hat{\alpha}}$ is

$$r_t^{\hat{\alpha}} = \mathbf{r}_t' \mathbf{w}^{\hat{\alpha}} = (\hat{\boldsymbol{\alpha}}_N' + \boldsymbol{\varepsilon}_t')\mathbf{w}^{\hat{\alpha}} = (\boldsymbol{\alpha}_N' + \boldsymbol{\varepsilon}_t')\mathbf{w}^{\hat{\alpha}}$$

and whose mean and variance are given in equations (13) and (14), which are bounded by no arbitrage asymptotically. Note that the $\mathbf{B}_N$ is absent in the expression for the portfolio return because of orthogonality of the $\boldsymbol{\alpha}_N$ and $\mathbf{B}_N$; if they were not orthogonal, then their product would have generated an explosive term in the expression for the return on portfolio $\mathbf{w}^{\hat{\alpha}}$, in which case the return of portfolio $\mathbf{w}^{\hat{\alpha}}$ would dominate the return on the mean-variance portfolio. Note that the contribution of the idiosyncratic component to the return on the $\mathbf{w}^{\hat{\alpha}}$ portfolio, $\boldsymbol{\varepsilon}_t' \mathbf{w}^{\hat{\alpha}}$, will not necessarily be diversified away by arbitrage. The reason for this is that the mean of the return on the $\mathbf{w}^{\hat{\alpha}}$ portfolio, which from Theorem 3.2 is $\boldsymbol{\alpha}_N' \mathbf{w}^{\hat{\alpha}} = \boldsymbol{\alpha}_N' \breve{\mathbf{V}}_N \boldsymbol{\alpha}_N$, cannot be zero unless the variance is also zero because the variance of $\mathbf{r}_t' \mathbf{w}^{\hat{\alpha}}$ is also $\boldsymbol{\alpha}_N' \breve{\mathbf{V}}_N \boldsymbol{\alpha}_N$.

In contrast, the return on the portfolio $\mathbf{w}^{\hat{\lambda}}$ is

$$\begin{aligned} r_t^{\hat{\lambda}} &= \mathbf{r}_t' \mathbf{w}^{\hat{\lambda}} \\ &= (\hat{\boldsymbol{\lambda}}' + \mathbf{z}_t')\mathbf{B}_N' \mathbf{w}^{\hat{\lambda}} + (\hat{\boldsymbol{\alpha}}_N' + \boldsymbol{\varepsilon}_t')\mathbf{w}^{\hat{\lambda}} \\ &= (\hat{\boldsymbol{\lambda}}' + \mathbf{z}_t')\mathbf{B}_N' \mathbf{w}^{\hat{\lambda}} + \boldsymbol{\varepsilon}_t' \mathbf{w}^{\hat{\lambda}}, \end{aligned}$$

and whose mean and variance are given in equations (13) and (14), which are bounded by no arbitrage asymptotically. Again by orthogonality, $\hat{\boldsymbol{\alpha}}_N$ is absent in the last expression.[4] Moreover, the contribution of $\boldsymbol{\varepsilon}_t$ in the expression for $r_t^{\hat{\lambda}}$ will vanish for well-spread (diversified) portfolios, in contrast to its contribution to $r_t^{\hat{\alpha}}$, where it does not vanish in general, as explained in Remark 4.5.1. Observe that, because $\mathbf{z}_t$ is random, the return $r_t^{\hat{\lambda}}$ is random even if the variance of the idiosyncratic component of the return, $\boldsymbol{\varepsilon}_t' \mathbf{w}^{\hat{\lambda}}$, is diversified away, and hence, the mean of $r_t^{\hat{\lambda}}$ is not constrained to go to zero with diversification.

For well-spread portfolios, under the conditions of Theorem 4.1 the variance of the term $\boldsymbol{\varepsilon}_t' \mathbf{w}_N^{\hat{\lambda}}$ is of order $\mathcal{O}(g_{KK}^{-1}(\mathbf{B}_N' \boldsymbol{\Sigma}_N^{-1} \mathbf{B}_N))$, which decays to zero if $\mathbf{B}_N$ is $\boldsymbol{\Sigma}_N$-regular.

[4] Observe that if we had expressed the return $r_t^{\hat{\lambda}}$ in terms of $\boldsymbol{\alpha}$ instead of $\hat{\boldsymbol{\alpha}}_N$, then $\boldsymbol{\alpha}$ would be eliminated by no arbitrage only asymptotically.

4.4 Results for global-minimum-variance portfolios

Just as in the case of the mean-variance portfolio, for the global-minimum-variance portfolio of the two components, $\mathbf{w}_N^{\Omega}$ and $\mathbf{w}_N^{\Sigma}$, the latter dominates. However, in contrast to the case of the mean-variance portfolio, here the $\mathbf{w}_N^{\Sigma}$ component dominates for any form of the pricing error, $\boldsymbol{\alpha}_N$. Below, we show that $\mathbf{w}_N^{\Sigma}$ decays to zero at the rate $\mathcal{O}(1/N)$, whereas $\mathbf{w}_N^{\Omega}$ decays at the rate $\mathcal{O}(1/N^2)$.

Theorem 4.6 (Weights of global-minimum-variance portfolio for large N with bounded residual variation)**.** *Suppose that the vector of asset returns, $\mathbf{r}_t$, satisfies Assumptions 2.1 and 2.2, and the loadings $\mathbf{B}_N$ and $\mathbf{e}_N$ are $\boldsymbol{\Sigma}_N$-regular. Moreover, $\mathbf{B}_N$ and $\mathbf{e}_N$ are not asymptotically collinear. Then, as $N \to \infty$, the absolute value of the components of the global-minimum-variance portfolio vectors, $\mathbf{w}_N^{\Sigma}$ and $\mathbf{w}_N^{\Omega}$, decrease at most at the rate:*

$$|w_i^{\Sigma}| = \mathcal{O}\Big(\frac{|\mathbf{e}_i'\boldsymbol{\Sigma}_N^{-1}\mathbf{e}_N|}{N} + \frac{\|\mathbf{e}_i'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N\|}{N^{\frac{1}{2}} g_{KK}^{\frac{1}{2}}(\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N)}\Big),$$

$$|w_i^{\Omega}| = \mathcal{O}\Big(\frac{\|\mathbf{e}_i'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N\|}{N^{\frac{1}{2}} g_{KK}^{\frac{3}{2}}(\mathbf{B}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N)}\Big).$$

Corollary 4.3 (Weights of global-minimum-variance portfolio for large N with bounded residual variation and $f(N) = N$)**.** *Suppose that the vector of asset returns, $\mathbf{r}_t$, satisfies Assumptions 2.1 and 2.2, and the loadings $\mathbf{B}_N$ and $\mathbf{e}_N$ are $\boldsymbol{\Sigma}_N$-regular with $f(N) = N$. Moreover, $\mathbf{B}_N$ and $\mathbf{e}_N$ are not asymptotically collinear. Then, as $N \to \infty$, the absolute value of the components of the global-minimum-variance portfolio vectors, $\mathbf{w}_N^{\Sigma}$ and $\mathbf{w}_N^{\Omega}$, decrease at most at the rate:*

$$|w_i^{\Sigma}| = \mathcal{O}\Big(\frac{|\mathbf{e}_i'\boldsymbol{\Sigma}_N^{-1}\mathbf{e}_N|}{N} + \frac{\|\mathbf{e}_i'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N\|}{N}\Big),$$

$$|w_i^{\Omega}| = \mathcal{O}\Big(\frac{\|\mathbf{e}_i'\boldsymbol{\Sigma}_N^{-1}\mathbf{B}_N\|}{N^2}\Big).$$

Moreover, if in addition, the rows of $\mathbf{B}_N$ and $\boldsymbol{\Sigma}_N^{-1}$, are uniformly bounded:

$$\sup_i \|\mathbf{e}_i'\mathbf{B}_N\| \leq \delta < \infty, \qquad \sup_i \|\mathbf{e}_i'\boldsymbol{\Sigma}_N^{-1}\| \leq \delta < \infty,$$

then

$$|w_i^{\Sigma}| = \mathcal{O}\left(\frac{1}{N}\right),$$

$$|w_i^{\Omega}| = \mathcal{O}\left(\frac{1}{N^2}\right).$$

Theorem 4.7 (Weights of global-minimum-variance portfolio for large N with unbounded residual variation)**.** *Suppose that the assumptions of Theorem 4.2 are satisfied. Suppose also that* $\mathbf{A}_N$, $\mathbf{B}_N$, $\mathbf{e}_N$ *are* $\mathbf{C}_N$*-regular with the same scaling factor* $f(N)$*. Moreover, neither* $\mathbf{B}_N$ *nor* $\mathbf{e}_N$ *are asymptotically collinear with* $\mathbf{A}_N$*. Then, as* $N \to \infty$*, the absolute value of the components of the global-minimum-variance portfolio vectors,* $\mathbf{w}_N^{\Sigma}$ *and* $\mathbf{w}_N^{\Omega}$*, decrease at most at the rate:*

$$|w_i^{\Sigma}| = \mathcal{O}\Big(\frac{|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{e}_N|}{N} + \frac{\|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{A}_N\|}{N^{\frac{1}{2}} g_{KK}^{\frac{1}{2}}(\mathbf{B}_N'\mathbf{C}_N^{-1}\mathbf{B}_N)}\Big),$$

$$|w_i^{\Omega}| = \mathcal{O}\Big(\frac{\|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{B}_N\| + \|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{A}_N\|}{N^{\frac{1}{2}} g_{KK}^{\frac{3}{2}}(\mathbf{B}_N'\mathbf{C}_N^{-1}\mathbf{B}_N)}\Big).$$

Corollary 4.4 (Weights of global-minimum-variance portfolio for large N with unbounded residual variation and $f(N) = N$)**.** *Suppose that the assumptions of Theorem 4.2 are satisfied. Suppose also that* $\mathbf{A}_N$, $\mathbf{B}_N$, $\mathbf{e}_N$ *are* $\mathbf{C}_N$*-regular with the same scaling factor* $f(N) = N$*. Moreover, neither* $\mathbf{B}_N$ *nor* $\mathbf{e}_N$ *are asymptotically collinear with* $\mathbf{A}_N$*. Then, as* $N \to \infty$*, the absolute value of the components of the global-minimum-variance portfolio vectors,* $\mathbf{w}_N^{\Sigma}$ *and* $\mathbf{w}_N^{\Omega}$*, decrease at most at the rate:*

$$|w_i^{\Sigma}| = \mathcal{O}\Big(\frac{|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{e}_N|}{N} + \frac{\|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{A}_N\|}{N}\Big),$$

$$|w_i^{\Omega}| = \mathcal{O}\Big(\frac{\|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{B}_N\| + \|\mathbf{e}_i'\mathbf{C}_N^{-1}\mathbf{A}_N\|}{N^2}\Big).$$

Moreover, if in addition, the rows of $\mathbf{A}_N$, $\mathbf{B}_N$, *and* $\mathbf{\Sigma}_N^{-1}$*, are uniformly bounded:*

$$\sup_i \|\mathbf{e}_i'\mathbf{A}_N\| \leq \delta < \infty, \qquad \sup_i \|\mathbf{e}_i'\mathbf{B}_N\| \leq \delta < \infty, \qquad \sup_i \|\mathbf{e}_i'\mathbf{\Sigma}_N^{-1}\| \leq \delta < \infty,$$

then

$$|w_i^{\Sigma}| = \mathcal{O}\left(\frac{1}{N}\right),$$

$$|w_i^{\Omega}| = \mathcal{O}\left(\frac{1}{N^2}\right).$$

Theorem 4.8 (Returns of minimum-variance portfolio for large N)**.** *Suppose that the vector of asset returns, $\mathbf{r}_t$, satisfies Assumptions 2.1 and 2.2 but ruling out exact pricing. Then, for portfolio weights $\mathbf{w}^{\Sigma}$ and $\mathbf{w}^{\Omega}$ that satisfy either the assumptions of Theorem 4.3 for the case of bounded residual variation, or the assumptions of Theorem 4.4 for the case of unbounded residual variation, we have the following results. For $N \to \infty$, the excess returns on the portfolios $\mathbf{w}^{\Sigma}$ and $\mathbf{w}^{\Omega}$ have asymptotic means given by the expressions below:*

$$\mu_{\infty}^{\Sigma} = \lim_{N\to\infty} \mu_N^{\Sigma} = 0; \qquad \mu_{\infty}^{\Omega} = \lim_{N\to\infty} \mu_N^{\Omega} = 0.$$

Furthermore, the asymptotic variances of the excess returns on the portfolios $\mathbf{w}^{\Sigma}$ and $\mathbf{w}^{\Omega}$ also converge to zero. The Sharpe ratios

$$SR_{\infty}^{\Sigma} = \lim_{N\to\infty} SR_N^{\Sigma} = \lim_{N\to\infty} \frac{\mathbf{e}'\breve{\mathbf{V}}\hat{\boldsymbol{\alpha}}}{(\mathbf{e}'\mathbf{V}^{-1}\mathbf{e})^{\frac{1}{2}}} \quad \textit{and} \quad SR_{\infty}^{\Omega} = \lim_{N\to\infty} SR_N^{\Omega} = \lim_{N\to\infty} \frac{\mathbf{e}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{B}\hat{\boldsymbol{\lambda}}}{(\mathbf{e}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{e})^{\frac{1}{2}}}$$

satisfy the following bounds:

$$\left|SR_{\infty}^{\Sigma}\right| < \infty \quad \textit{and} \quad \left|SR_{\infty}^{\Omega}\right| < \infty.$$

Remark 4.8.1. As the number of assets increases, the expected excess return and the volatility of return on the global-minimum-variance portfolio go to zero. This implies that the return on the global-minimum-variance portfolio will converge to the risk-free rate of return. However, the excess return and the volatility of return on the global-minimum-variance portfolio decay at the same rate, $\mathcal{O}(1/(\mathbf{e}'\mathbf{V}^{-1}\mathbf{e})^{\frac{1}{2}})$, implying that the absolute Sharpe ratio is bounded. Observe that we have stated the above boundedness condition in terms of the absolute value of the Sharpe ratio because the expression for the mean excess return is not a quadratic form, and hence, it is not guaranteed to be positive.

5 Applications of the Theoretical Results

To be added.

6 Conclusion

In this paper, we have investigated the properties of mean-variance portfolios and global-minimum-variance portfolios when asset returns satisfy the Arbitrage Pricing Theory, while allowing for the possibility of model misspecification and mispricing. We have shown that the mean-variance portfolio can be decomposed into two components: a portfolio that depends on only pricing errors and a portfolio that depends on only factor risk premia (and their loadings). We demonstrated that these two portfolio components are orthogonal to each other and the squares of their Sharpe ratios sum to the square of the Sharpe ratio of the mean-variance portfolio. When number of assets is asymptotically large, the portfolio weights in the component that depends on pricing errors typically dominate the weights in the component that depends on the risk premia. However, the asymptotic return on the mean-variance portfolio is not dominated by either component.

Analogously, we have shown that the global-minimum-variance portfolio can be decomposed into two components: a portfolio that depends on only the residual covariance matrix and a portfolio that depends on only the factor covariance matrix. We have shown that these two components are also orthogonal to each other but the sum of the squares of the Sharpe ratios of the two component portfolios is less than or equal to the square of the Sharpe ratio of the global-minimum-variance portfolio. When the number of assets is asymptotically large, then the portfolio weights associated with the residual covariance matrix dominate the weights associated with the factor covariance matrix.

A Appendix with proofs

Proof of Theorem 3.1

Given that $\boldsymbol{\mu} = \hat{\boldsymbol{\alpha}} + \mathbf{B}\hat{\boldsymbol{\lambda}}$

$$\begin{aligned}\mathbf{w}^{mv} &= \mathbf{V}^{-1}\boldsymbol{\mu} \\ &= \mathbf{V}^{-1}\hat{\boldsymbol{\alpha}} + \mathbf{V}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}} \\ &= \breve{\mathbf{V}}\hat{\boldsymbol{\alpha}} + \mathbf{V}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}} \\ &= \mathbf{w}^{\hat{\alpha}} + \mathbf{w}^{\hat{\lambda}}.\end{aligned}$$

Moreover

$$\begin{aligned}\mathbf{w}^{\hat{\alpha}\prime}\mathbf{V}\mathbf{w}^{\hat{\lambda}} &= \mathbf{w}^{\hat{\alpha}\prime}\mathbf{V}\mathbf{V}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}} \\ &= \mathbf{w}^{\hat{\alpha}\prime}\mathbf{B}\hat{\boldsymbol{\lambda}} \\ &= \hat{\boldsymbol{\alpha}}'\breve{\mathbf{V}}\mathbf{B}\hat{\boldsymbol{\lambda}} \\ &= 0,\end{aligned}$$

because $\breve{\mathbf{V}}$ is orthogonal to $\mathbf{B}$. Similarly,

$$\begin{aligned}\mathbf{w}^{\hat{\alpha}\prime}\boldsymbol{\Sigma}\mathbf{w}^{\hat{\lambda}} &= \mathbf{w}^{\hat{\alpha}\prime}\boldsymbol{\Sigma}\mathbf{V}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}} \\ &= \mathbf{w}^{\hat{\alpha}\prime}\mathbf{B}(\mathbf{I}_K - (\boldsymbol{\Omega}^{-1} + \mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B})^{-1}\mathbf{B}`\boldsymbol{\Sigma}^{-1}\mathbf{B})\hat{\boldsymbol{\lambda}} \\ &= \hat{\boldsymbol{\alpha}}'\breve{\mathbf{V}}\mathbf{B}(\mathbf{I}_K - (\boldsymbol{\Omega}^{-1} + \mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B})^{-1}\mathbf{B}`\boldsymbol{\Sigma}^{-1}\mathbf{B})\hat{\boldsymbol{\lambda}} \\ &= 0.\end{aligned}$$

Proof of Theorem 3.2

Because $\breve{\mathbf{V}}\mathbf{V}\breve{\mathbf{V}} = \breve{\mathbf{V}}\boldsymbol{\Sigma}\breve{\mathbf{V}} = \breve{\mathbf{V}}$, it follows that

$$\begin{aligned}\boldsymbol{\mu}^{\hat{\alpha}} = \mathbf{w}^{\hat{\alpha}\prime}\boldsymbol{\mu} = \hat{\boldsymbol{\alpha}}'\breve{\mathbf{V}}(\hat{\boldsymbol{\alpha}} + \mathbf{B}\hat{\boldsymbol{\lambda}}) = \hat{\boldsymbol{\alpha}}'\breve{\mathbf{V}}\hat{\boldsymbol{\alpha}}, \\ (\boldsymbol{\sigma}^{\hat{\alpha}})^2 = \mathbf{w}^{\hat{\alpha}\prime}\mathbf{V}\mathbf{w}^{\hat{\alpha}} = \hat{\boldsymbol{\alpha}}'\breve{\mathbf{V}}\mathbf{V}\breve{\mathbf{V}}\hat{\boldsymbol{\alpha}} = \hat{\boldsymbol{\alpha}}'\breve{\mathbf{V}}\hat{\boldsymbol{\alpha}}.\end{aligned}$$

Then use $\hat{\boldsymbol{\alpha}}'\breve{\mathbf{V}}\hat{\boldsymbol{\alpha}} = \hat{\boldsymbol{\alpha}}'\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\alpha}}$.

Because $\mathbf{B}'\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\alpha}} = \mathbf{0}$ implies $\mathbf{B}'\mathbf{V}^{-1}\hat{\boldsymbol{\alpha}} = \mathbf{0}$, one gets

$$\boldsymbol{\mu}^{\hat{\lambda}} = \mathbf{w}^{\hat{\lambda}'}\boldsymbol{\mu} = \hat{\boldsymbol{\lambda}}'\mathbf{B}'\mathbf{V}^{-1}(\hat{\boldsymbol{\alpha}} + \mathbf{B}\hat{\boldsymbol{\lambda}}) = \hat{\boldsymbol{\lambda}}'\mathbf{B}'\mathbf{V}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}},$$
$$(\boldsymbol{\sigma}^{\hat{\lambda}})^2 = \mathbf{w}^{\hat{\lambda}'}\mathbf{V}\mathbf{w}^{\hat{\lambda}} = \hat{\boldsymbol{\lambda}}'\mathbf{B}'\mathbf{V}^{-1}\mathbf{V}\mathbf{V}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}} = \hat{\boldsymbol{\lambda}}'\mathbf{B}'\mathbf{V}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}}.$$

The boundedness of $\hat{\boldsymbol{\alpha}}'\breve{\mathbf{V}}\hat{\boldsymbol{\alpha}} = \hat{\boldsymbol{\alpha}}'\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\alpha}}$ follows from By Theorem 2.1 (APT). Next,

$$\begin{aligned}
\mathbf{B}'\mathbf{V}^{-1}\mathbf{B} &= \mathbf{B}'(\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{B}(\boldsymbol{\Omega}^{-1} + \mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}^{-1})\mathbf{B} \\
&= \mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B}(\boldsymbol{\Omega}^{-1} + \mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B})^{-1}\boldsymbol{\Omega}^{-1}, \\
&= (\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B} + \boldsymbol{\Omega}^{-1} - \boldsymbol{\Omega}^{-1})(\boldsymbol{\Omega}^{-1} + \mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B})^{-1}\boldsymbol{\Omega}^{-1} \\
&= \boldsymbol{\Omega}^{-1} - \boldsymbol{\Omega}^{-1}(\boldsymbol{\Omega}^{-1} + \mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B})^{-1}\boldsymbol{\Omega}^{-1}.
\end{aligned}$$

Therefore the positive definite matrix $\mathbf{B}'\mathbf{V}^{-1}\mathbf{B}$ is bounded above by the constant matrix $\boldsymbol{\Omega}^{-1}$, implying boundedness of the former. Premultiplying and postmultiplying the above expression by $\hat{\boldsymbol{\lambda}}$ yields (REF).

Finally,

$$\begin{aligned}
(SR^{mv})^2 &= \boldsymbol{\mu}'\mathbf{V}^{-1}\boldsymbol{\mu} \\
&= (\hat{\boldsymbol{\alpha}} + \mathbf{B}\hat{\boldsymbol{\lambda}})'\mathbf{V}^{-1}(\hat{\boldsymbol{\alpha}} + \mathbf{B}\hat{\boldsymbol{\lambda}}) \\
&= \hat{\boldsymbol{\alpha}}'\mathbf{V}^{-1}\hat{\boldsymbol{\alpha}} + \hat{\boldsymbol{\lambda}}'\mathbf{B}'\mathbf{V}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}} \\
&= \hat{\boldsymbol{\alpha}}'\mathbf{V}^{-1}\hat{\boldsymbol{\alpha}} + \hat{\boldsymbol{\lambda}}'\mathbf{B}'\mathbf{V}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}} \\
&= \hat{\boldsymbol{\alpha}}'\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\alpha}} + \hat{\boldsymbol{\lambda}}'\mathbf{B}'\mathbf{V}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}} \\
&= (SR^{\hat{\alpha}})^2 + (SR^{\hat{\lambda}})^2,
\end{aligned}$$

where the third equality follows by the orthogonality.

Proof of Theorem 3.3

$$\mathbf{w}^{gmv} = \frac{\mathbf{V}^{-1}\mathbf{e}}{\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}} = \frac{\breve{\mathbf{V}}\mathbf{e}}{\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}} + \frac{(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{e}}{\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}} = \mathbf{w}^{\Sigma} + \mathbf{w}^{\Omega}.$$

Moreover

$$\mathbf{w}^{\Sigma\prime}\mathbf{V}\mathbf{w}^{\Omega} = \frac{1}{(\mathbf{e}'\mathbf{V}^{-1}\mathbf{e})^2}\mathbf{e}'\breve{\mathbf{V}}\mathbf{V}(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{e}$$

$$= \frac{1}{(\mathbf{e}'\mathbf{V}^{-1}\mathbf{e})^2}\mathbf{e}'(\breve{\mathbf{V}} - \breve{\mathbf{V}}\mathbf{V}\breve{\mathbf{V}})\mathbf{e}$$
$$= 0,$$

because of the identity $\breve{\mathbf{V}}\mathbf{V}\breve{\mathbf{V}} = \breve{\mathbf{V}}$.

A.1 Lemma on Sharpe-ratio Decomposition

Lemma A.1. *Consider the portfolio weights $w = w_1 + w_2$ such that*

$$w_1' V w_2 = 0.$$

Then, setting $SR_i = w_i'\mu/(w_i'\mathbf{V}w_i)^{1/2}$,

$$SR^2 = \frac{(w'\mu)^2}{w'\mathbf{V}w} \leq (SR_1)^2 + (SR_2)^2.$$

Finally, equality holds if and only if:

$$\frac{w_1'\mu}{w_1' V w_1} = \frac{w_2'\mu}{w_2' V w_2}$$

Proof. Setting for simplicity

$$\mu_i = w_i'\mu, \sigma_i^2 = w_i'\mathbf{V}w_i,$$

$$\begin{aligned} SR^2 &= \frac{(w_1'\mu)^2}{w'\mathbf{V}w} + \frac{(w_2'\mu)^2}{w'\mathbf{V}w} + 2\frac{(w_1'\mu)(w_2'\mu)}{w'\mathbf{V}w} \\ &= \frac{\mu_1^2}{\sigma_1^2}\frac{\sigma_1^2}{w'\mathbf{V}w} + \frac{\mu_2^2}{\sigma_2^2}\frac{\sigma_2^2}{w'\mathbf{V}w} + 2\frac{\mu_1\mu_2}{w'\mathbf{V}w} \\ &= \frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} + \left[\frac{\mu_1^2}{\sigma_1^2}\Big(-1 + \frac{\sigma_1^2}{w'\mathbf{V}w}\Big) + \frac{\mu_2^2}{\sigma_2^2}\Big(-1 + \frac{\sigma_2^2}{w'\mathbf{V}w}\Big) + 2\frac{\mu_1\mu_2}{w'\mathbf{V}w}\right]. \end{aligned}$$

Recalling that $w'\mathbf{V}w = w_1'\mathbf{V}w_1 + w_2'\mathbf{V}w_2$, the term in square-brackets can we rewritten as

$$\begin{aligned} -\frac{\mu_1^2}{\sigma_1^2}\frac{\sigma_2^2}{w'\mathbf{V}w} - \frac{\mu_2^2}{\sigma_2^2}\frac{\sigma_1^2}{w'\mathbf{V}w} + 2\frac{\mu_1\mu_2}{w'\mathbf{V}w} &= \frac{1}{w'\mathbf{V}w}\Big(-\mu_1^2\frac{\sigma_2^2}{\sigma_1^2} - \mu_2^2\frac{\sigma_1^2}{\sigma_2^2} + 2\mu_1\frac{\sigma_2}{\sigma_1}\mu_2\frac{\sigma_1}{\sigma_2}\Big) \\ &= -\frac{1}{w'\mathbf{V}w}\Big(\mu_1\frac{\sigma_2}{\sigma_1} - \mu_2\frac{\sigma_1}{\sigma_2}\Big)^2. \end{aligned}$$

Hence,

$$SR^2 = \frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} - \frac{1}{w'\mathbf{V}w}(\mu_1\frac{\sigma_2}{\sigma_1} - \mu_2\frac{\sigma_1}{\sigma_2})^2 \leq \frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} = (SR_1)^2 + (SR_2)^2.$$

Equality holds if and only if

$$\left(\mu_1\frac{\sigma_2}{\sigma_1} - \mu_2\frac{\sigma_1}{\sigma_2}\right)^2 = 0,$$

which, in turn, can be rearranged as

$$\frac{\mu_1}{\sigma_1^2} = \frac{\mu_2}{\sigma_2^2}.$$

Proof of Theorem 3.4

$$\boldsymbol{\mu}^{\Sigma} = \mathbf{w}^{\Sigma\prime}\boldsymbol{\mu} = \frac{1}{\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}}\mathbf{e}'\breve{\mathbf{V}}(\hat{\boldsymbol{\alpha}} + \mathbf{B}\hat{\boldsymbol{\lambda}}) = \frac{\mathbf{e}'\breve{\mathbf{V}}\hat{\boldsymbol{\alpha}}}{\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}},$$
$$(\boldsymbol{\sigma}^{\Sigma})^2 = \mathbf{w}^{\Sigma\prime}\mathbf{V}\mathbf{w}^{\Sigma} = \frac{\mathbf{e}'\breve{\mathbf{V}}\mathbf{V}\breve{\mathbf{V}}\mathbf{e}}{(\mathbf{e}'\mathbf{V}^{-1}\mathbf{e})^2} = \frac{\mathbf{e}'\breve{\mathbf{V}}\mathbf{e}}{(\mathbf{e}'\mathbf{V}^{-1}\mathbf{e})^2}.$$

and

$$\boldsymbol{\mu}^{\Omega} = \mathbf{w}^{\Omega\prime}\boldsymbol{\mu} = \frac{1}{\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}}\mathbf{e}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})(\hat{\boldsymbol{\alpha}} + \mathbf{B}\hat{\boldsymbol{\lambda}}) = \frac{\mathbf{e}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{B}\hat{\boldsymbol{\lambda}}}{\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}},$$
$$(\boldsymbol{\sigma}^{\Omega})^2 = \mathbf{w}^{\Omega\prime}\mathbf{V}\mathbf{w}^{\Omega} = \frac{\mathbf{e}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{V}(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{e}}{(\mathbf{e}'\mathbf{V}^{-1}\mathbf{e})^2} = \frac{\mathbf{e}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{e}}{(\mathbf{e}'\mathbf{V}^{-1}\mathbf{e})^2},$$

because $\mathbf{V}^{-1}\hat{\boldsymbol{\alpha}} = \breve{\mathbf{V}}\hat{\boldsymbol{\alpha}} = \boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\alpha}}$ and $(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{V}(\mathbf{V}^{-1} - \breve{\mathbf{V}}) = \mathbf{V}^{-1} - \breve{\mathbf{V}}$.

Next, because

$$|SR^{\Sigma}| = \frac{|\mathbf{e}'\breve{\mathbf{V}}\hat{\boldsymbol{\alpha}}|}{(\mathbf{e}'\breve{\mathbf{V}}\mathbf{e})^{\frac{1}{2}}} \leq \frac{(\mathbf{e}'\breve{\mathbf{V}}\mathbf{e})^{\frac{1}{2}}(\hat{\boldsymbol{\alpha}}\breve{\mathbf{V}}\hat{\boldsymbol{\alpha}})^{\frac{1}{2}}}{(\mathbf{e}'\breve{\mathbf{V}}\mathbf{e})^{\frac{1}{2}}} = (\hat{\boldsymbol{\alpha}}\breve{\mathbf{V}}\hat{\boldsymbol{\alpha}})^{\frac{1}{2}}$$

boundedness follow from Theorem 2.1 (APT). Regarding boundeness of $|SR^{\Omega}|$, because the matrix $(\mathbf{V}^{-1} - \breve{\mathbf{V}})$ is positive semi-definite, one gets

$$|SR^{\Omega}| = \frac{|\mathbf{e}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{B}\hat{\boldsymbol{\lambda}}|}{(\mathbf{e}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{e})^{\frac{1}{2}}} \leq \frac{(\mathbf{e}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{e})^{\frac{1}{2}}(\hat{\boldsymbol{\lambda}}'\mathbf{B}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{B}\hat{\boldsymbol{\lambda}})^{\frac{1}{2}}}{(\mathbf{e}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{e})^{\frac{1}{2}}}$$

$$= (\hat{\boldsymbol{\lambda}}'\mathbf{B}'(\mathbf{V}^{-1} - \breve{\mathbf{V}})\mathbf{B}\hat{\boldsymbol{\lambda}})^{\frac{1}{2}} = (\hat{\boldsymbol{\lambda}}'\mathbf{B}'\mathbf{V}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}})^{\frac{1}{2}}$$

and boundedness of $\mathbf{B}'\mathbf{V}^{-1}\mathbf{B}$ follows directly from the proof of Theorem 3.2.

Finally, by Lemma A.1,

$$(SR^{gmv})^2 \leq (SR^{\Sigma})^2 + (SR^{\Omega})^2.$$

Proof of Theorem 4.2

By Chamberlain and Rothschild (1983, Theorem 4) the residual covariance matrix satisfies

$$\boldsymbol{\Sigma}_N = \mathbf{A}_N\mathbf{A}_N' + \mathbf{C}_N$$

where $\mathbf{C}_N$ is a positive definite matrix with eigenvalues uniformly bounded by $g_{p+1\,N}(\boldsymbol{\Sigma}_N)$. By the Sherman decomposition

$$\boldsymbol{\Sigma}_N^{-1} = \mathbf{C}_N^{-1} - \mathbf{C}_N^{-1}\mathbf{A}_N(\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1}\mathbf{A}_N'\mathbf{C}_N^{-1}.$$

Therefore, by substitution,

$$\begin{aligned}
\boldsymbol{\alpha}_N'\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\alpha}_N &= \boldsymbol{\alpha}_N'\mathbf{C}_N^{-1}\boldsymbol{\alpha}_N - \boldsymbol{\alpha}_N'\mathbf{C}_N^{-1}\mathbf{A}_N(\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1}\mathbf{A}_N'\mathbf{C}_N^{-1}\boldsymbol{\alpha}_N \\
&= (\mathbf{A}_N\mathbf{m} + \mathbf{a}_N)'\mathbf{C}_N^{-1}(\mathbf{A}_N\mathbf{m} + \mathbf{a}_N) \\
&- (\mathbf{A}_N\mathbf{m} + \mathbf{a}_N)'\mathbf{C}_N^{-1}\mathbf{A}_N(\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1}\mathbf{A}_N'\mathbf{C}_N^{-1}(\mathbf{A}_N\mathbf{m} + \mathbf{a}_N) \\
&= \mathbf{m}'\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N\mathbf{m} - \mathbf{m}'\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N(\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1}\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N\mathbf{m} \\
&\mathbf{a}_N'\mathbf{C}_N^{-1}\mathbf{a}_N - \mathbf{a}_N'\mathbf{C}_N^{-1}\mathbf{A}_N(\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1}\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{a}_N \\
&+ 2\mathbf{a}_N'\mathbf{C}_N^{-1}\mathbf{A}_N\mathbf{m} - \mathbf{a}_N'\mathbf{C}_N^{-1}\mathbf{A}_N(\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1}\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N\mathbf{m}.
\end{aligned}$$

We now show that $\boldsymbol{\alpha}_N'\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\alpha}_N$ is bounded even as N diverges. We look each of the term on the right hand side of the last equality sign, one by one. Thus:

$$\begin{aligned}
&\mathbf{m}'\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N\mathbf{m} - \mathbf{m}'\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N(\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1}\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N\mathbf{m} \\
&= \mathbf{m}'(\mathbf{I}_N - \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N(\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1})\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N\mathbf{m} \\
&= \mathbf{m}'(\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1}\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N\mathbf{m} \leq \mathbf{m}'\mathbf{m},
\end{aligned}$$

since $\mathbf{I}_p - (\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1}\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N$ is positive semi definite. Next, for the second term

$$\mathbf{a}_N'\mathbf{C}_N^{-1}\mathbf{a}_N \leq \mathbf{a}_N'\mathbf{a}_N g_{NN}^{-1}(\mathbf{C}_N).$$

Now, the jth element of $\mathbf{a}_N'\mathbf{C}_N^{-1}\mathbf{A}_N$, obtained by considering the jth column of $\mathbf{A}_N$, for every $1 \leq j \leq p$, satisfies

$$|\mathbf{a}_N'\mathbf{C}_N^{-1}g_{jN}^{\frac{1}{2}}\mathbf{v}_{jN}| \leq g_{jN}^{\frac{1}{2}}(\mathbf{a}_N'\mathbf{C}_N^{-1}\mathbf{a}_N)^{\frac{1}{2}}(\mathbf{v}_{jN}'\mathbf{C}_N^{-1}\mathbf{v}_{jN})^{\frac{1}{2}},$$

where for simplicity we set $\mathbf{v}_{jN} = \mathbf{v}_{jN}(\mathbf{\Sigma}_N), g_{jN} = g_{jN}(\mathbf{\Sigma}_N)$. Since

$$\begin{aligned}
&\mathbf{a}_N'\mathbf{C}_N^{-1}\mathbf{A}_N(\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1}\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{a}_N \\
&\leq g_{1p}((\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1})(\mathbf{a}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)(\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{a}_N) \\
&\leq g_{1p}((\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1})(\mathbf{a}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)(\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{a}_N) \\
&= g_{pp}^{-1}(\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)(\mathbf{a}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)(\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{a}_N) \leq \delta < \infty.
\end{aligned}$$

since the (i,j)th element, for every $1 \leq i,j \leq p$, of $(\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)$ is equal to $g_{iN}^{\frac{1}{2}}g_{jN}^{\frac{1}{2}}\mathbf{v}_{iN}'\mathbf{C}_N^{-1}\mathbf{v}_{jN}$. Concerning the third term, it turns out that it will converges to zero. In fact:

$$\begin{aligned}
&|2\mathbf{a}_N'\mathbf{C}_N^{-1}\mathbf{A}_N\mathbf{m} - 2\mathbf{a}_N'\mathbf{C}_N^{-1}\mathbf{A}_N(\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1}\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N\mathbf{m}| \\
&= 2|\mathbf{a}_N'\mathbf{C}_N^{-1}\mathbf{A}_N(\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1}\mathbf{m}| \\
&\leq (\mathbf{a}_N'\mathbf{C}_N^{-1}\mathbf{A}_N(\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1}\mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{a}_N)^{\frac{1}{2}}(\mathbf{m}'(\mathbf{I}_p + \mathbf{A}_N'\mathbf{C}_N^{-1}\mathbf{A}_N)^{-1}\mathbf{m})^{\frac{1}{2}} \leq \delta g_{pN}^{-\frac{1}{2}} \to 0.
\end{aligned}$$

Proof of Theorem 4.3

Proof of Theorem 4.4

Proof of Theorem 4.5

Proof of Theorem 4.6

Proof of Theorem 4.7

Proof of Theorem 4.8

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