Asset Management Contracts and Equilibrium Prices

ANDREA M. BUFFA
Boston University

DIMITRI VAYANOS
London School of Economics, CEPR and NBER

PAUL WOOLLEY
London School of Economics

February 4, 2015*

Abstract

We study the joint determination of fund managers' contracts and equilibrium asset prices. Because of agency frictions, investors make managers' fees more sensitive to performance and benchmark performance against a market index. This makes managers unwilling to deviate from the index and exacerbates price distortions. Because trading against overvaluation exposes managers to greater risk of deviating from the index than trading against undervaluation, agency frictions bias the aggregate market upwards. They can also generate a negative relationship between risk and return because they raise the volatility of overvalued assets. Socially optimal contracts provide steeper performance incentives and cause larger pricing distortions than privately optimal contracts.

*buffa@bu.edu, d.vayanos@lse.ac.uk, p.k.woolley@lse.ac.uk. We thank Sergey Chernenko, Chris Darnell, Peter DeMarzo, Ken French, Jeremy Grantham, Zhiguo He, Ron Kaniel, seminar participants at Bocconi, Boston University, CEU, Cheung Kong, Dartmouth, LSE, Maryland, Stanford, and conference participants at AEA, BIS, CRETE, ESSFM Gerzenzee, FRIC, Jackson Hole, and LSE PWC for helpful comments.

1 Introduction

Asset management is a large and growing industry. For example, individual investors held directly 47.9% of U.S. stocks in 1980 and 21.5% in 2007, with the remainder held by financial institutions of various types, run by professional managers (French (2008)). Asset managers' risk and return is measured against benchmarks, and performance relative to the benchmarks determines the managers' compensation and the funds they get to manage. In this paper we study how the delegation of asset management from investors to professional managers affects equilibrium asset prices. Unlike most prior literature, we endogenize both equilibrium prices and managers' contracts, including the extent of benchmarking. We also perform a normative analysis, comparing privately optimal contracts to socially optimal ones.

We show that when agency frictions between investors and managers are more severe, managers' compensation is more sensitive to performance, and performance is tied more closely to a benchmark. As a consequence, managers become less willing to deviate from the benchmark, and the price distortions that they are hired to exploit become more severe. While distortions are exacerbated in both directions, i.e., undervalued assets become cheaper and overvalued assets become more expensive, the positive distortions dominate, biasing the aggregate market upwards and its expected return downwards. Indeed, overvalued assets account for an increasingly large fraction of market movements relative to undervalued assets. Therefore, trading against overvaluation, by underweighting the overvalued assets, exposes managers to greater risk of under-performing their benchmark than trading against undervaluation.

In addition to exacerbating price distortions, agency frictions can generate a negative relationship between risk and expected return in the cross-section. Such a negative relationship has been documented empirically, with risk being measured by return volatility or CAPM beta, and contradicts basic predictions of standard theories.[1] Agency frictions can generate a negative risk-return relationship because they raise the volatility of overvalued assets, through an amplification mechanism. Consider a positive shock to the expected cashflows of an overvalued asset. Because the asset then accounts for a larger fraction of market movements, managers become less willing to trade against overvaluation, and prefer instead to buy the asset. Their buying pressure amplifies the price increase caused by the higher cashflows.

[1] Haugen and Baker (1996) and Ang, Hodrick, Xing, and Zhang (2006) document that expected return is negatively related to volatility in the cross-section of U.S. stocks. The latter paper also documents a negative relationship between expected return and the idiosyncratic component of volatility. Since volatility is negatively related to expected return, it is also negatively related to CAPM alpha, which is expected return adjusted for beta, i.e., for systematic risk. Black (1972), Black, Jensen, and Scholes (1972), and Frazzini and Pedersen (2014) document that alpha is negatively related to beta in the cross-section of U.S. stocks. The relationship between expected return and beta is almost flat during 1926-2012 (Frazzini and Pedersen (2014)), and turns negative during the second half of the sample (Baker, Bradley, and Wurgler (2011)).

Our model, presented in Section 2, is as follows. We assume a continuous-time infinite-horizon economy with multiple risky assets and an exogenous riskless rate. An investor can invest in the risky assets directly by holding a market index that includes all assets according to their supplies, or indirectly by holding a fund run by a manager. Both investor and manager are price-takers, and can be interpreted as a continuum of identical investors and managers. The manager's contract consists of a fee, paid by the investor. We assume that the fee must be an affine function of the fund's performance and the index performance, and we optimize over the coefficients of that function. The manager chooses the fund's portfolio. We model agency frictions by assuming that the manager can additionally undertake a "shirking" action that lowers the fund's return but delivers a private benefit to him.

If the investor and the manager were the only agents in the model, then they would hold the index because of market clearing. Equilibrium prices would adjust to make the index an optimal portfolio, and the investor would not employ the manager because she can hold the index directly. To ensure that the manager can add value over the index, we introduce a third set of agents, buy-and-hold investors, who hold a portfolio that differs from the index. The portfolio choice of these agents could be driven, for example, by corporate-control or hedging considerations.[2] Assets that are in low demand by the buy-and-hold investors must earn high expected returns in equilibrium, so that the manager is induced to give them a weight larger than the index weight. Conversely, assets in high demand must earn low expected returns so that the manager underweights them. The former assets are undervalued, when measuring risk by the covariance with the market index, while the latter assets are overvalued. By overweighting the undervalued assets and underweighting the overvalued assets, the manager adds value over the index.

In Section 3 we solve the model in the case where there are no agency frictions. We show that the manager's fee does not depend on the index performance, and hence there is no benchmarking. The fee depends only on the fund's performance, in a way that implements optimal risk-sharing between the investor and the manager, who are both risk-averse. A negative relationship between risk and return in the cross-section of assets can arise even in the absence of agency frictions (but is stronger when the frictions are present). Consider an asset that is in high demand by buy-and-hold investors. This asset earns low expected return and is underweighted by the manager. The reason why its return can be highly volatile is as follows. Following a positive shock to an asset's expected cashflows, the asset accounts for a larger fraction of the manager's portfolio volatility. The increase in volatility makes the manager less willing to hold the asset, and attenuates the price increase caused by the improved fundamentals. The attenuation effect is weak, however, for an asset that the manager underweights because the asset's contribution to his portfolio volatility

[2] Fama and French (2007) perform a similar construction in a static setting and show how the presence of investors not holding the market portfolio generates superior opportunities for other investors.

is small. Therefore, the asset's price is highly sensitive to the cashflow shock, resulting in high volatility.

In Section 4 we solve the model in the case where there are agency frictions. We show that the investor makes the manager's fee more sensitive to the fund's performance than in the frictions' absence. This reduces the manager's incentive to undertake the shirking action. It also exposes him to more risk, but the manager can offset the increase in his personal exposure by choosing a less risky portfolio for the fund. The investor restores the manager's incentives to take risk by making the fee sensitive to the index performance; this encourages risk-taking because the manager's personal exposure to market drops is reduced. Benchmarking, however, only incentivizes the manager to take risk that correlates closely with the index, and discourages deviations from that benchmark. Thus, the manager becomes less willing to overweight assets in low demand by buy-and-hold investors, and to underweight assets in high demand. The former assets become more undervalued in equilibrium, and the latter assets become more overvalued.

Agency frictions exacerbate not only cross-sectional price distortions but also the negative relationship between risk and return. This is because they raise the volatility of overvalued assets. Recall that in the absence of frictions, a positive shock to an asset's expected cashflows is attenuated by an increase in risk premium because the asset accounts for a larger fraction of the manager's portfolio volatility. In the presence of frictions, the risk premium instead *decreases* for overvalued assets, and hence the shock is amplified. Indeed, the manager underweights overvalued assets, but becomes less willing to do so when these assets account for a larger fraction of market movements. Benchmarking amounts to a short position in the underweighted assets, which the manager seeks to reduce when volatility increases.

The cross-sectional price distortions that agency frictions introduce do not cancel out in the aggregate. We show that the positive distortions are more severe than the negative ones, biasing the aggregate market upwards. This is because overvalued assets account, through the amplification effect, for an increasingly large fraction of market movements relative to undervalued assets. Therefore, trading against overvaluation exposes the manager to greater risk of under-performing the index than trading against undervaluation.

Endogenizing fund managers' contracts allows us to perform a normative analysis. In Section 5 we show that the contract chosen by a social planner provides the manager with steeper incentives than the contract chosen by private agents. The price distortions under the socially optimal contract are also larger. The inefficiency of private contracts can be viewed as a free-rider problem, by interpreting our price-taking investor and manager as a continuum of identical such agents. When one investor in the continuum gives steeper performance incentives to her manager, this induces less shirking. At the same time, the manager offsets the increase in his personal risk exposure by

choosing a less risky portfolio, hence exploiting mispricings to a lesser extent. Other managers, however, remain equally willing to exploit mispricings, benefiting their investors. When all investors give steeper incentives to their managers, mispricings become more severe in equilibrium, and all managers remain equally willing to exploit them despite being exposed to more risk.

Throughout our analysis, we assume constant absolute risk aversion (CARA) utility for the investor and the manager, and square-root processes for asset cashflows. Square-root processes have the property that the volatility of an asset's cashflows *per share* increases with the cashflow level. This property is realistic (e.g., the risk of a firm in absolute terms, i.e., not relative to the firm's size, increases with size) and is key for our results. We underscore its importance in Section 6, where we consider a familiar CARA-normal setting, where the volatility of cashflows per share is constant. We show that the risk-return relationship is then always positive and agency frictions do not affect the aggregate market. The combination of CARA utility and square-root processes for cashflows is to our knowledge new to the literature, including in a frictionless setting. We show that it yields closed-form solutions for asset prices and can accommodate any number of risky assets.

The effects of asset management on equilibrium prices are the subject of a growing literature. Our paper is closest to the strand of that literature that focuses on managers' contractual incentives.[3] Brennan (1993) assumes a static setting where some investors have preferences over the return relative to a benchmark. Equilibrium expected returns are given by a two-factor model, with the factors being the market portfolio and the benchmark. Basak and Pavlova (2013) assume a dynamic setting where some investors have preferences over wealth and a benchmark. Demand by these investors raises the prices of the assets included in the benchmark and makes them more volatile and more correlated with each other. In both papers benchmarks are introduced directly into investors' utility functions.

Cuoco and Kaniel (2011) model delegation and contracts explicitly, in a dynamic setting with two risky assets. Investors delegate the management of the risky part of their portfolio to managers, whose fee is a piece-wise affine function of absolute return and of the return relative to a benchmark. Managers' demand raises the prices of the assets included in the benchmark, but the effect on volatility depends on the convexity of the managers' fee. Affine fees are not optimal because investors cannot commit to an allocation in the fund when choosing the fee. Garcia and Vanden (2009), Malamud and Petrov (2014), and Qiu (2014) explore static settings where managers observe private signals about the payoff of a single risky asset. They show, respectively, that competition between fund managers makes prices more informative; convexity of the managers' fee reduces

[3]Other papers in the literature focus on fund flows, e.g., Shleifer and Vishny (1997), Berk and Green (2004), Vayanos (2004), He and Krishnamurthy (2012, 2013), Kaniel and Kondor (2013), and Vayanos and Woolley (2013), and on managers' reputation concerns, e.g., Dasgupta and Prat (2008), Dasgupta, Prat, and Verardo (2011), and Guerrieri and Kondor (2012).

volatility, and the socially optimal fee provides weaker incentives than the privately optimal fee; and relative-performance evaluation can make prices less informative. Relative to these papers, we assume symmetrically informed managers and linear contracts, but we allow for a general number of risky assets. This sharpens the analysis of cross-sectional asset pricing, and indeed we derive new implications for the risk-return relationship and the pricing of the aggregate market.[4]

One explanation of the negative relationship between risk and return is based on leverage (Black (1972), Frazzini and Pedersen (2014)). Stocks with high CAPM beta have the same systematic risk as a suitably levered portfolio of low-beta assets, but the latter portfolio is not available to leverage-constrained investors. The demand by these investors pushes up the prices of high-beta assets, and lowers their expected return. Another explanation is based on disagreement (Hong and Sraer (2013)). Investors' disagreement about the future return of the aggregate market is larger for high-beta assets because they are more sensitive to market movements. Moreover, assets for which disagreement is larger are priced only by optimists and hence offer low expected returns because short-sale constraints drive pessimists out of the market.[5]

Karceski (2002) explains the negative risk-return relationship based on fund managers' incentives. If fund flows are more sensitive to performance when the market goes up, then managers prefer high-beta assets because they outperform the market during good times. Baker, Bradley, and Wurgler (2011) suggest an explanation that is based on benchmarking. Fund managers view high- and low-beta assets as equally risky because they care about deviations from a benchmark and not about absolute returns. Therefore, the expected return of high-beta assets does not reflect their underlying risk and is too low. Leverage constraints are implicit in both explanations because managers cannot replace high-beta assets by a suitably levered portfolio of low-beta assets.

Our explanation assumes no leverage constraints or disagreement. Moreover, unlike the previous explanations, we do not show that exogenous differences in betas yield overpricing, but rather that high investor demand yields both overpricing and high beta. We also can generate a negative relationship not only between beta and CAPM alpha (expected return adjusted for beta), but also between beta and expected return. By contrast, the negative relationship that the leverage explanation generates is only between beta and alpha: for leverage-constrained investors to prefer high-beta assets, the relationship between beta and expected return must be positive.

[4] A number of papers study the choice of contracts taking prices as given. Stoughton (1993) shows that when faced with steeper performance incentives, fund managers choose less risky portfolios, and their incentives to collect information on asset payoffs remain unchanged. Admati and Pfleiderer (1997) rely on this observation to show that benchmarking distorts managers' portfolio choice without encouraging them to collect more information. Both results are shown for affine contracts. Ou-Yang (2003) shows that affine contracts and benchmarking can be optimal when moral hazard pertains to other activities than information collection. Bhattacharya and Pfleiderer (1985), Starks (1987), Stoughton (1993), Das and Sundaram (2002), Palomino and Prat (2003), Li and Tiwari (2009), and Dybvig, Farnsworth, and Carpenter (2010) study non-affine contracts and whether they can dominate affine ones.

[5] Cohen, Polk, and Vuolteenaho (2005) find a negative relationship between risk and return during times of high inflation, and relate this result to money illusion.

2 Model

2.1 Assets

Time t is continuous and goes from zero to infinity. There is an exogenous riskless rate r, and N risky assets. The price S_{it} of asset $i = 1, .., N$ is determined endogenously in equilibrium. The dividend flow D_{it} of asset i is given by

$$D_{it} = b_i s_t + e_{it}, \tag{2.1}$$

where s_t is a component common to all assets and e_{it} is a component specific to asset i. The variables $(s_t, e_{1t}, .., e_{Nt})$ are positive and mutually independent, and we specify their stochastic evolution below. The constant $b_i \geq 0$ measures the exposure of asset i to the common component s_t. We set $D_t \equiv (D_{1t}, .., D_{Nt})'$, $S_t \equiv (S_{1t}, .., S_{Nt})'$, and $b \equiv (b_1, .., b_N)'$. We denote by $dR_t \equiv (dR_{1t}, .., dR_{Nt})'$ the vector of assets' returns per share in excess of the riskless rate:

$$dR_t \equiv D_t dt + dS_t - rS_t dt. \tag{2.2}$$

Dividing dR_{it} by the price S_{it} of asset i yields asset i's return per dollar in excess of the riskless rate. For simplicity, we refer to dR_{it} and $dR^{\$}_{it} \equiv \frac{dR_{it}}{S_{it}}$ as share return and dollar return, respectively, omitting that they are in excess of the riskless rate. Asset i is in supply of $\eta_i > 0$ shares. We denote the market portfolio by $\eta \equiv (\eta_1, .., \eta_N)$, and refer to it as the index.

The variables $(s_t, e_{1t}, .., e_{Nt})$ evolve according to square-root processes:

$$ds_t = \kappa\left(\bar{s} - s_t\right) dt + \sigma_s \sqrt{s_t} dw_{st}, \tag{2.3}$$

$$de_{it} = \kappa\left(\bar{e}_i - e_{it}\right) dt + \sigma_i \sqrt{e_{it}} dw_{it}, \tag{2.4}$$

where $(\kappa, \bar{s}, \bar{e}_1, .., \bar{e}_N, \sigma_s, \sigma_1, .., \sigma_N)$ are positive constants, and the Brownian motions $(w_{st}, w_{1t}, .., w_{Nt})$ are mutually independent. The square-root specification (2.3) and (2.4) allows for closed-form solutions, while also ensuring that dividends remain positive. An additional property of this specification is that the volatility of dividends per share (i.e., of D_{it}) increases with the dividend level. This property is realistic and key for our results.

The constants $(\bar{s}, \bar{e}_1, .., \bar{e}_N)$ are the unconditional (long-term) means of the variables $(s_t, e_{1t}, .., e_{Nt})$. The increments $(ds_t, de_{1t}, .., de_{Nt})$ of these variables have variance rates $(\sigma_s^2 s_t, \sigma_1^2 e_{1t}, .., \sigma_N^2 e_{Nt})$ conditionally and $(\sigma_s^2 \bar{s}, \sigma_1^2 \bar{e}_1, .., \sigma_N^2 \bar{e}_N)$ unconditionally. We occasionally consider the special case of "scale invariance," where the ratio of unconditional standard deviation to unconditional mean is identical across the $N + 1$ processes. This occurs when the vector $(\sigma_s^2, \sigma_1^2, .., \sigma_N^2)$ is collinear with

$(\bar{s}, \bar{e}_1, .., \bar{e}_N)$.

2.2 Agents

The main agents in our model are an investor and a fund manager. Both agents are price-takers and can be interpreted as a continuum of identical investors and managers. The investor can invest in the risky assets directly by holding the index, or indirectly by holding a fund run by a manager. Employing the manager is the only way for the investor to hold a portfolio that differs from the index, and hence to "participate" in the markets for the individual risky assets. One interpretation of this participation friction is that the investor cannot identify assets that offer higher returns than the index, and hence must employ the manager for non-index investing.

If the investor and the manager were the only agents in the model, then the participation friction would not matter. This is because the index is the market portfolio, so equilibrium prices would adjust to make that portfolio optimal for the investor. For the participation friction to matter, the manager must add value over the index. To ensure that this can happen, we introduce a third set of agents, buy-and-hold investors, who do not hold the index. These agents could be holding assets for hedging purposes, or could be additional unmodeled fund managers. We denote their aggregate portfolio by $\eta - \theta$, and assume that $\theta \equiv (\theta_1, .., \theta_N)$ is constant over time and not proportional to η. The number of shares of asset i available to the investor and the manager is thus θ_i, and represents the residual supply of asset i to them. Assets in large residual supply (large θ_i) must earn high expected returns in equilibrium, so that the manager is willing to give them weight larger than the index weight. Conversely, assets in small residual supply must earn low expected returns so that the manager is willing to underweight them. By overweighting high-expected-return assets and underweighting low-expected-return ones, the manager adds value over the index. We assume that the residual supply of each asset is positive ($\theta_i > 0$ for all i). We refer to residual supply simply as supply from now on.[6]

The investor chooses an investment x in the index η, i.e., holds $x\eta_i$ shares of asset i. She also decides whether or not to employ the manager. Both decisions are made once and for all at $t = 0$. If the manager is employed by the investor, then he chooses the fund's portfolio $z_t \equiv (z_{1t}, .., z_{Nt})$ at each time t, where z_{it} denotes the number of shares of asset i held by the fund. The manager can also undertake a "shirking" action $m_t \geq 0$ that delivers to him a private benefit $\left(Am_t - \frac{B}{2}m_t^2\right) dt$, where $1 \geq A \geq 0$ and $B \geq 0$, and reduces the fund's return by $m_t dt$. A literal interpretation of m_t is as cash diverted from the fund, with diversion involving a deadweight cost except when $A = 1$ and $B = 0$. Alternatively, m_t could be interpreted in reduced form as insufficient effort to lower

[6] An alternative interpretation of our setting is that there are no buy-and-hold investors, θ is the market portfolio, and η is an index that differs from the market portfolio, e.g., does not include private equity.

operating costs or to identify a more efficient portfolio. When $A = 0$, the private benefit is non-positive for all values of m_t and there are no agency frictions. The investor can influence the choices of z_t and m_t through a compensation contract that she offers to the manager at $t = 0$. (Assuming that the manager offers the contract to the investor would not change our analysis, provided that competition drives the manager's utility to his outside option of not being employed.) The contract specifies a fee that the investor pays to the manager over time. It is chosen optimally within a parametrized class, described as follows. The fee is paid as a flow, and the flow fee df_t is an affine function of the fund's return $z_t dR_t - m_t dt$ and the index return ηdR_t. Moreover, the coefficients of this affine function are chosen at $t = 0$ and remain constant over time. Thus, the flow fee df_t is given by

$$df_t = \phi\left(z_t dR_t - m_t dt\right) - \chi\eta dR_t + \psi dt, \tag{2.5}$$

where (ϕ, χ, ψ) are constants. The constant ϕ is the fee's sensitivity to the fund's performance, and the constant χ is the sensitivity to the index performance. We assume that the manager invests his personal wealth in the riskless rate. This is without loss of generality: since the manager is exposed to the risky assets through the fee, and can adjust this exposure by changing the fund's portfolio, a personal investment in those assets is redundant. If the manager is not employed by the investor, then he chooses a personal portfolio $\bar{z}_t$ in the risky assets, receives no fee, and has no shirking action available.[7]

Our setting, in which one investor contracts with one manager, fits best institutional asset management, whereby large institutions such as pension funds or sovereign-wealth funds contract with asset management firms on a target return relative to a benchmark. Yet, we abstract away from a number of real-world features. For example, fees typically depend on assets under management, but we assume that they can only depend on the return achieved by the manager and on the return of the benchmark. Moreover, fees in some cases are convex, but we restrict them to be linear. We also abstract away from implicit incentives generated by fund flows that depend on past returns. Our intention is to capture in a simple manner two key features of asset management contracts: managers' fees depend on their performance, and performance is evaluated relative to a benchmark. These features are present not only in institutional asset management, but in other forms of asset management as well, such as mutual funds offered to retail investors.

[7] Ruling out the shirking action for an unemployed manager is without loss of generality: since the manager invests his personal wealth, he would not undertake the shirking action even if that action were available.

2.2.1 Manager's Optimization Problem

The manager derives utility over intertemporal consumption. Utility is exponential:

$$\mathbb{E}\left[\int_0^\infty -\exp(-\bar{\rho}\bar{c}_t - \bar{\delta}t)dt\right], \tag{2.6}$$

where $\bar{\rho}$ is the coefficient of absolute risk aversion, $\bar{c}_t$ is consumption, and $\bar{\delta}$ is the discount rate. We denote the manager's wealth by $\bar{W}_t$.

The manager decides at $t = 0$ whether or not to accept the contract offered by the investor. If the manager accepts the contract and is hence employed by the investor, then he chooses at each time t the fund's portfolio z_t and the shirking action m_t. His budget constraint is

$$d\bar{W}_t = r\bar{W}_t dt + df_t + \left(Am_t - \frac{B}{2}m_t^2\right)dt - \bar{c}_t dt, \tag{2.7}$$

where the first term is the riskless return, the second term is the fee paid by the investor, the third term is the private benefit from shirking, and the fourth term is consumption. The manager's optimization problem is to choose controls $(\bar{c}_t, z_t, m_t)$ to maximize the expected utility (2.6) subject to the budget constraint (2.7) and the fee (2.5). We denote by $z_t(\phi, \chi, \psi)$ and $m_t(\phi, \chi, \psi)$ the manager's optimal choices of z_t and m_t, and by $\bar{V}(\bar{W}_t, s_t, e_t)$ his value function, where $e_t \equiv (e_{1t}, .., e_{Nt})'$.

If the manager is not employed by the investor, then he chooses his personal portfolio $\bar{z}_t$. His budget constraint is

$$d\bar{W}_t = r\bar{W}_t dt + \bar{z}_t dR_t - \bar{c}_t dt. \tag{2.8}$$

The manager's optimization problem is to choose controls $(\bar{c}_t, \bar{z}_t)$ to maximize (2.6) subject to (2.8). We denote by $\bar{V}_u(\bar{W}_t, s_t, e_t)$ his value function. The manager accepts the contract offered by the investor if

$$\bar{V}(\bar{W}_0, s_0, e_0) \geqslant \bar{V}_u(\bar{W}_0, s_0, e_0). \tag{2.9}$$

2.2.2 Investor's Optimization Problem

The investor derives utility over intertemporal consumption. Utility is exponential:

$$\mathbb{E}\left[\int_0^\infty -\exp(-\rho c_t - \delta t)dt\right], \tag{2.10}$$

where ρ is the coefficient of absolute risk aversion, c_t is consumption, and δ is the discount rate.

The investor chooses an investment x in the index η, and whether or not to employ the manager. Both decisions are made at $t = 0$. If the investor employs the manager, then she offers him a contract (ϕ, χ, ψ). We denote the investor's wealth by W_t. The investor's budget constraint is

$$dW_t = rW_t dt + x\eta dR_t + z_t dR_t - m_t dt - df_t - c_t dt, \tag{2.11}$$

where the first term is the riskless return, the second term is the return from the investment in the index, the third and fourth term are the return from the fund, the fifth term is the fee paid to the manager, and the sixth term is consumption. The investor's optimization problem is to choose controls $(c_t, x, \phi, \chi, \psi)$ to maximize the expected utility (2.10) subject to the budget constraint (2.11), the fee (2.5), the manager's incentive compatibility constraint

$$\begin{aligned} z_t &= z_t(\phi, \chi, \psi), \\ m_t &= m_t(\phi, \chi, \psi), \end{aligned}$$

and the manager's individual rationality constraint (2.9). We denote by $V(W_t, s_t, e_t)$ the investor's value function.

If the investor does not employ the manager, then her budget constraint is

$$dW_t = rW_t dt + x\eta dR_t - c_t dt. \tag{2.12}$$

The investor's optimization problem is to choose controls (c_t, x) to maximize (2.10) subject to (2.12). We denote by $V_u(W_t, s_t, e_t)$ her value function. The investor employs the manager if

$$V(W_0, s_0, e_0) \geqslant V_u(W_0, s_0, e_0). \tag{2.13}$$

2.3 Equilibrium Concept

We look for equilibria in which the investor employs the manager, i.e., offers a contract that the manager accepts. These equilibria are described by a price process S_t, a compensation contract (ϕ, χ, ψ) that the investor offers to the manager, and a direct investment x in the index by the investor.

Definition 1 (**Equilibrium prices and contract**). *A price process S_t, a contract (ϕ, χ, ψ), and an index investment x, form an equilibrium if:*

(i) Given S_t and (ϕ, χ, ψ), $z_t = \theta - x\eta$ solves the manager's optimization problem.

(ii) Given S_t, the investor chooses to employ the manager, and (x, ϕ, χ, ψ) solve the investor's optimization problem.

The equilibrium in Definition 1 involves a two-way feedback between prices and contracts. A contract offered by the investor affects the manager's portfolio choice, and hence equilibrium prices. Equilibrium prices are determined by the market-clearing condition that the fund's portfolio z_t plus the portfolio $x\eta$ that the investor holds directly add up to the supply portfolio θ. Conversely, the contract that the investor offers to the manager depends on the equilibrium prices. We conjecture that the equilibrium price of asset i is an affine function of s_t and e_{it}:

$$S_{it} = a_{0i} + a_{1i} s_t + a_{2i} e_{it}, \tag{2.14}$$

where (a_{0i}, a_{1i}, a_{2i}) are constants.

3 Equilibrium without Agency Frictions

In this section we solve for equilibrium in the absence of agency frictions. We eliminate agency frictions by setting the parameter A in the manager's private-benefit function $Am_t - \frac{B}{2}m_t^2$ to zero. This ensures that the private benefit is non-positive for all values of m_t. When $A = 0$, the investor and the manager share risk optimally, through the contract. The equilibrium becomes one with a representative agent, whose risk tolerance is the sum of the investor's and the manager's. We compute prices in that equilibrium in closed form. We show that the combination of exponential utility and square-root dividend processes—which to our knowledge is new to the literature—yields a framework that is not only tractable but can also help address empirical puzzles about the risk-return relationship.

Theorem 3.1 (**Equilibrium Prices and Contract without Agency Frictions**). *When $A = 0$, the following form an equilibrium: the price process S_t given by (2.14) with*

$$a_{0i} = \frac{\kappa}{r}\left(a_{1i}\bar{s} + a_{2i}\bar{e}_i\right), \tag{3.1}$$

$$a_{1i} = \frac{b_i}{\sqrt{(r+\kappa)^2 + 2r\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}\theta b \sigma_s^2}} \equiv a_1 b_i, \tag{3.2}$$

$$a_{2i} = \frac{1}{\sqrt{(r+\kappa)^2 + 2r\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}\theta_i \sigma_i^2}}; \tag{3.3}$$

the contract $(\phi, \chi, \psi) = \left(\frac{\rho}{\rho+\bar{\rho}}, 0, 0\right)$; and the index investment $x = 0$.

Since $x = 0$, the investor does not invest directly in the index. Market-clearing hence implies that the fund holds the supply portfolio θ. Since, in addition, $\chi = 0$, the manager is compensated based only on absolute performance and not on performance relative to the index. Therefore, the manager receives a fraction $\phi = \frac{\rho}{\rho+\bar{\rho}}$ of the return of the supply portfolio, and the investor receives the complementary fraction $1-\phi = \frac{\bar{\rho}}{\rho+\bar{\rho}}$. This coincides with the standard rule for optimal risk-sharing under exponential utility.

The coefficient a_{2i} measures the sensitivity of asset i's price to changes in the asset-specific component e_{it} of dividends. A unit increase in e_{it} causes asset i's dividend flow at time t to increase by one. In the absence of risk aversion ($\rho\bar{\rho} = 0$), (3.3) implies that the price of asset i would increase by $a_{2i} = \frac{1}{r+\kappa}$. This is the present value, discounted at the riskless rate r, of the increase in asset i's expected dividends from time t onwards: the dividend flow at time t increases by one, and the effect decays over time at rate κ.

The coefficient a_{1i} measures the sensitivity of asset i's price to changes in the common component s_t of dividends. We normalize a_{1i} by b_i, the sensitivity of asset i's dividend flow to changes in s_t. This yields a coefficient a_1 that is common to all assets, and that measures the sensitivity of any given asset's price to a unit increase in the asset's dividend flow at time t caused by an increase in s_t. In the absence of risk aversion, (3.2) implies that the price of asset i would increase by $a_1 = \frac{1}{r+\kappa}$. Hence, a_1 and a_{2i} would be equal: an increase in an asset's dividend flow would have the same effect on the asset's price regardless of whether it comes from the common or from the asset-specific component.

Risk aversion lowers a_1 and a_{2i}. This is because increases in s_t or e_{it} not only raise expected dividends but also make them riskier, and risk has a negative effect on prices when agents are risk averse. The effect of increased risk attenuates that of higher expected dividends. One would expect the attenuation to be larger when the increased risk comes from increases in s_t rather than in e_{it}. This is because agents are more averse to risk that affects all assets rather than a specific asset. Equations (3.2) and (3.3) imply that $a_1 < a_{2i}$ if

$$\theta b \sigma_s^2 = \left(\sum_{i=1}^{N} \theta_i b_i \right) \sigma_s^2 > \theta_i \sigma_i^2. \tag{3.4}$$

Equation (3.4) evaluates how a unit increase in asset i's dividend flow at time t affects the covariance between the dividend flow of asset i and of the supply portfolio. This covariance captures the relevant risk in our model. The left-hand side of the inequality in (3.4) is the increase in the covariance when the increase in dividend flow is caused by an increase in s_t. The right-hand side is the increase in the same covariance when the increase in dividend flow is caused by an increase in e_{it}. When (3.4) holds, the change in s_t has a larger effect on the covariance compared to the

change in e_{it}. Therefore, it has a larger attenuating effect on the price.

Equation (3.4) holds when the number N of assets exceeds a threshold, which can be zero. This is because the left-hand side increases when assets are added, while the right-hand side remains constant. In the special case of scale invariance, (3.4) takes the intuitive form

$$\theta b\bar{s} > \theta_i \bar{e}_i. \tag{3.5}$$

The left-hand side is the dividend flow of the supply portfolio that is derived from the common component. The right-hand side is the dividend flow of the same portfolio that is derived from the component specific to asset i. Equation (3.5) obviously holds when N is large enough.

3.1 Supply Effects

We next examine how differences in supply in the cross-section of assets are reflected into prices and return moments. We compare two assets i and i' that differ only in supply ($\theta_i \neq \theta_{i'}$), but have otherwise identical characteristics ($(b_i, \bar{e}_i, \sigma_i, \eta_i) = (b_{i'}, \bar{e}_{i'}, \sigma_{i'}, \eta_{i'})$). When comparing the prices of the assets at a time t, we assume that the asset-specific components of dividends at t are also identical ($e_{it} = e_{i't}$). We take asset i to be the one in smaller supply ($\theta_i < \theta_{i'}$).

We compute unconditional (long-term) moments of returns, and consider both returns per share and returns per dollar invested. Moments of share returns can be computed in closed form. To compute closed-form solutions for moments of dollar returns, we approximate from now on the dollar return of an asset by its share return divided by the unconditional mean of the share price, i.e., modify the definition of $dR^{\$}_{it}$ to $dR^{\$}_{it} \equiv \frac{dR_{it}}{\mathbb{E}(S_{it})}$. For example, expected dollar return is the expected ratio of share return to share price but we approximate it by the ratio of expected share return to expected share price.

Proposition 3.1 (**Price and Expected Return**). *Suppose that $A = 0$. An asset i in smaller supply than an otherwise identical asset i' has higher price at time t ($S_{it} > S_{i't}$), higher expected price ($\mathbb{E}(S_{it}) > \mathbb{E}(S_{i't})$), lower expected share return ($\mathbb{E}(dR_{it}) < \mathbb{E}(dR_{i't})$), and lower expected dollar return ($\mathbb{E}(dR^{\$}_{it}) < \mathbb{E}(dR^{\$}_{i't})$).*

An asset i in small supply θ_i must offer low expected share return, so that the manager is induced to hold a small number of shares of the asset. Therefore, the asset's price must be high. The asset's expected dollar return is low because of two effects working in the same direction: low expected share return in the numerator, and high price in the denominator.

The effect of θ_i on the asset price is through the coefficient a_{2i}, which measures the price sensitivity to changes in the asset-specific component e_{it} of dividends. When θ_i is small, an increase in e_{it} is accompanied by a small increase in the covariance between the dividend flow of the asset and of the supply portfolio θ. Therefore, the positive effect that the increase in e_{it} has on the price through higher expected dividends is attenuated by a small negative effect due to the increase in risk. As a consequence, a_{2i} is large. Since an increase in e_{it} away from its lower bound of zero has a large effect on the price, the price is high.

Note that θ_i does not have an effect through the coefficient a_{1i}, which measures the price sensitivity to changes in the common component s_t of dividends. This coefficient depends on θ_i only through the aggregate quantity θb, which is constant in cross-sectional comparisons.

Proposition 3.2 (**Return Volatility**). *Suppose that $A = 0$. An asset i in smaller supply than an otherwise identical asset i' has higher share return variance ($\mathbb{V}\mathrm{ar}(dR_{it}) > \mathbb{V}\mathrm{ar}(dR_{i't})$). It has higher dollar return variance ($\mathbb{V}\mathrm{ar}(dR^{\$}_{it}) > \mathbb{V}\mathrm{ar}(dR^{\$}_{i't})$) if and only if*

$$\mathcal{D}_1 \equiv a_1 b_i (a_{2i} + a_{2i'})(\sigma_i^2 \bar{s} - \sigma_s^2 \bar{e}_i) + 2(a_{2i} a_{2i'} \sigma_i^2 \bar{e}_i - a_1^2 b_i^2 \sigma_s^2 \bar{s}) > 0. \tag{3.6}$$

Since dividend changes have a large effect on the price of an asset that is in small supply, such an asset has high share return volatility (square root of variance). This effect is concentrated on the part of volatility that is driven by the asset-specific component, while there is no effect on the part that is driven by the common component. Whether small supply is associated with high or low dollar return volatility depends on two effects working in opposite directions: high share return volatility in the numerator, and high price in the denominator. The first effect dominates when (3.6) holds.

Since the effect of supply on volatility is concentrated on the part that is driven by the asset-specific component, (3.6) should hold if that part is large enough. This can be confirmed, for example, in the case of scale invariance. Equation (3.6) becomes

$$\sqrt{a_{2i} a_{2i'}}\, \bar{e}_i > a_1 b_i \bar{s}, \tag{3.7}$$

and has the simple interpretation that the volatility driven by the common component is smaller than the geometric average, across assets i and i', of the volatilities driven by the asset-specific components. Indeed, the conditional variance rate driven by the common component is $a_1^2 b_i^2 \sigma_s^2 s_t$ for both assets. The conditional variance rate driven by the asset-specific component is $a_{2i}^2 \sigma_i^2 e_{it}$ for asset i and $a_{2i'}^2 \sigma_i^2 e_{it}$ for asset i'. Taking expectations, we find the unconditional variance rates $a_1^2 b_i^2 \sigma_s^2 \bar{s}$, $a_{2i}^2 \sigma_i^2 \bar{e}_i$, and $a_{2i'}^2 \sigma_i^2 \bar{e}_i$, respectively. Under scale invariance, the former is smaller than the

geometric average of the latter if (3.7) holds.

We next examine how supply affects the systematic and idiosyncratic parts of volatility. When the number N of assets is large, these coincide, respectively, with the parts driven by the common and the asset-specific component. For small N, however, the systematic part includes volatility driven by the asset-specific component. To compute the systematic and idiosyncratic parts, we regress the return dR_{it} of asset i on the return $dR_{\eta t} \equiv \eta dR_t$ of the index:

$$dR_{it} = \beta_i dR_{\eta t} + d\epsilon_{it}. \tag{3.8}$$

The CAPM beta of asset i is

$$\beta_i = \frac{\mathbb{C}\text{ov}(dR_{it}, dR_{\eta t})}{\mathbb{V}\text{ar}(dR_{\eta t})}, \tag{3.9}$$

and measures the systematic part of volatility. The variance $\mathbb{V}\text{ar}(d\epsilon_{it})$ of the regression residual measures the idiosyncratic part. These quantities are defined in per-share terms. Their per-dollar counterparts are

$$\beta_i^{\$} = \frac{\mathbb{C}\text{ov}(dR_{it}^{\$}, dR_{\eta t}^{\$})}{\mathbb{V}\text{ar}(dR_{\eta t}^{\$})}, \tag{3.10}$$

and $\mathbb{V}\text{ar}(d\epsilon_{it}^{\$})$, where $dR_{\eta t}^{\$} \equiv \frac{dR_{\eta t}}{\mathbb{E}(S_{\eta t})}$, $S_{\eta t} \equiv \eta S_t$, and $d\epsilon_{it}^{\$} \equiv \frac{d\epsilon_{it}}{\mathbb{E}(S_{it})}$.

Proposition 3.3 (**Beta and Idiosyncratic Volatility**). *Suppose that $A = 0$. An asset i in smaller supply than an otherwise identical asset i' has higher share beta ($\beta_i > \beta_{i'}$) and idiosyncratic share return variance ($\mathbb{V}\text{ar}(d\epsilon_{it}) > \mathbb{V}\text{ar}(d\epsilon_{i't})$). It has higher dollar beta ($\beta_i^{\$} > \beta_{i'}^{\$}$) if and only if*

$$\mathcal{D}_2 \equiv a_1 b_i (a_{2i} + a_{2i'}) \eta_i \sigma_i^2 \bar{s} + a_{2i} a_{2i'} \eta_i \sigma_i^2 \bar{e}_i - a_1^2 b_i \eta b \sigma_s^2 \bar{s} > 0, \tag{3.11}$$

and higher idiosyncratic dollar return variance ($\mathbb{V}\text{ar}(d\epsilon_{it}^{\$}) > \mathbb{V}\text{ar}(d\epsilon_{i't}^{\$})$) if and only if

$$\left(a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^{N} a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j \right) a_1 b_i \bar{s} \mathcal{D}_1$$
$$- \left[\left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i \right) (a_1 b_i \bar{s} + a_{2i'} \bar{e}_i) + \left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i'}^2 \eta_i \sigma_i^2 \bar{e}_i \right) (a_1 b_i \bar{s} + a_{2i} \bar{e}_i) \right] \mathcal{D}_2 > 0. \tag{3.12}$$

The share beta and idiosyncratic volatility are large for an asset that is in small supply because of the effect identified in Propositions 3.1 and 3.2: changes to the asset-specific component of dividends

have a large effect on the price of such an asset. This yields high idiosyncratic volatility. It also yields large beta because asset-specific shocks have a large contribution to the asset's covariance with the index. Whether small supply is associated with high or low dollar beta and idiosyncratic volatility depends on two effects working in opposite directions: high share beta and idiosyncratic volatility in the numerator, and high price in the denominator. To build intuition on which effect dominates, we consider the case where the number N of assets is large.

For large N, an asset's covariance with the index is driven mainly by the common shocks, whose effect on price does not depend on supply. Since supply affects only a small fraction of the covariance, the effect of supply on price should dominate that on share beta. Hence, dollar beta should be small for an asset that is in small supply. This can be confirmed, for example, in the case of scale invariance and symmetric assets with identical characteristics $(b_i, \bar{e}_i, \eta_i, \theta_i)$. (To ensure that assets i and i' differ in their supply, we assume that $\theta_i - \theta_{i'}$ is close but not equal to zero.) We denote by $(b_c, \bar{e}_c, \eta_c, \theta_c)$ the common values of $(b_i, \bar{e}_i, \eta_i, \theta_i)$ across all assets, by a_{2c} the common value of a_{2i}, and by $y \equiv \frac{a_{2c}\bar{e}_c}{a_1 b_c \bar{s}}$ the ratio of volatility driven by the asset-specific component to the volatility driven by the common component. We can write (3.11) as

$$2y + y^2 - N > 0. \tag{3.13}$$

As N increases, (3.13) is satisfied for values of y that exceed an increasingly large threshold.

An asset's idiosyncratic volatility, for large N, is driven mainly by the shocks specific to that asset. Since the effect of supply is only through those shocks, while common shocks account for a potentially large fraction of the price, the effect of supply on idiosyncratic share volatility should dominate the effect of supply on price. Hence, idiosyncratic dollar return volatility should be large for an asset that is in small supply. For example, in the case of scale invariance and symmetric assets, we can write (3.12) as

$$N - 2 - y > 0. \tag{3.14}$$

As N increases, (3.14) is satisfied for values of y that are below an increasingly large threshold.

We finally determine how supply affects assets' CAPM alphas, i.e., the expected returns that assets are earning in excess of the CAPM. The CAPM alpha of asset i is

$$\alpha_i = \mathbb{E}(dR_{it}) - \beta_i \mathbb{E}(dR_{\eta t}), \tag{3.15}$$

in per-share terms. Its per-dollar counterpart is

$$\alpha_i^{\$} = \mathbb{E}(dR_{it}^{\$}) - \beta_i^{\$}\mathbb{E}(dR_{\eta t}^{\$}) = \frac{\alpha_i}{\mathbb{E}(S_{it})}. \tag{3.16}$$

Proposition 3.4 (Alpha). *Suppose that $A = 0$. An asset i in smaller supply than an otherwise identical asset i' has lower share alpha ($\alpha_i < \alpha_{i'}$) and lower dollar alpha ($\alpha_i^{\$} < \alpha_{i'}^{\$}$).*

An asset in small supply has low share alpha because it has low expected share return (Proposition 3.1) and high share beta (Proposition 3.3). The effect of supply on share alpha carries through to dollar alpha, so an asset in small supply has low dollar alpha as well.

Our results have implications for the relationship between risk and expected return in the cross-section of assets. Standard theories predict that this relationship should be positive: riskier assets should compensate investors with higher expected return. Empirically, however, a negative relationship has often been documented. Moreover, in those instances where a positive relationship has been documented, it has been found to be weaker than the theoretical one. The weakness of the relationship has been shown using alpha, which is expected return in excess of its theoretical value. Alpha has been shown to be negative for riskier assets and positive for less risky ones. This means that the expected returns of the former assets are not as high as predicted by theory, while the expected returns of the latter assets are not as low. The empirical findings concern both CAPM alpha, as well as alphas computed using other risk-adjustment methods such as the Fama-French three-factor model.

Haugen and Baker (1996) and Ang, Hodrick, Xing, and Zhang (2006) document that U.S. stocks with high return volatility earn lower returns on average than stocks with low volatility. The latter paper also shows that the negative relationship holds not only for return volatility but also for the idiosyncratic component of that volatility. Since a negative relationship holds between volatility and expected return, it also holds between volatility and alpha: adjusting for risk can only make the negative relationship stronger. Since alpha averages to zero across stocks, high-volatility stocks earn negative alpha and low-volatility stocks earn positive alpha. The negative relationship between volatility on one hand and expected return or alpha on the other is known as the *volatility anomaly*.

Black (1972), Black, Jensen, and Scholes (1972), and Frazzini and Pedersen (2014) document that U.S. stocks with high CAPM beta earn negative alpha while stocks with low beta earn positive alpha. The relationship between expected return and beta is almost flat during 1926-2012 (Frazzini and Pedersen (2014)), and turns negative during the second half of the sample (Baker, Bradley, and Wurgler (2011)). The negative relationship between beta and alpha, as well as the weak or negative relationship between beta and expected return, is known as the *beta anomaly*.

The results in this section suggest a mechanism that could help explain the volatility and beta anomalies, even in the absence of agency frictions. A negative relationship between volatility or beta on one hand, and expected return or alpha on the other, can be generated by the way that these variables depend on supply. Assets in small supply earn low expected dollar return (Proposition 3.1) and negative alpha (Proposition 3.4). Under some conditions, they also have high dollar return volatility (Proposition 3.2), high idiosyncratic dollar return volatility (Proposition 3.3), and high dollar beta (Proposition 3.3). Under these conditions our model can generate a negative relationship both between risk and alpha, as well as between risk and expected return. We further explore the relationship between risk and return implied by our model in the next section, where we add agency frictions and quantify the effects in the context of a numerical example.

4 Equilibrium with Agency Frictions

In this section we solve for equilibrium in the presence of agency frictions. We introduce agency frictions by setting the parameter A in the manager's private-benefit function $Am_t - \frac{B}{2}m_t^2$ to a positive value. For simplicity, we set the parameter B to zero. This pins down immediately the coefficient ϕ that characterizes how sensitive the manager's fee is to the fund's performance. Indeed, if $\phi < A$, then the manager will choose an arbitrarily large shirking action m_t. This forces the investor to offer $\phi \geq A$, in which case there is no shirking, i.e., $m_t = 0$.[8] When $A \leq \frac{\rho}{\rho+\bar{\rho}}$, the constraint $\phi \geq A$ is not binding, since in the equilibrium without agency frictions the investor offers $\phi = \frac{\rho}{\rho+\bar{\rho}}$. When instead $A > \frac{\rho}{\rho+\bar{\rho}}$, the constraint is binding, and the investor offers $\phi = A$.

Allowing B to be positive yields a richer theory of contract determination, both on the positive and on the normative front. The asset pricing results, however, remain essentially the same. For this reason we defer the case $B > 0$ to Section 5, where we perform a normative analysis of contracts.

Theorem 4.1 (**Equilibrium Prices and Contract with Agency Frictions**). *Suppose that $B = 0$. When $\frac{\rho}{\rho+\bar{\rho}} \geq A > 0$, the equilibrium in Theorem 3.1 remains an equilibrium. When $A > \frac{\rho}{\rho+\bar{\rho}}$, the following form an equilibrium: the price process S_t given by (2.14) with a_{0i} given by (3.1),*

$$a_{1i} = \frac{b_i}{\sqrt{(r+\kappa)^2 + 2r\bar{\rho}(\phi\theta - \chi\eta)b\sigma_s^2}} \equiv a_1 b_i, \tag{4.1}$$

$$a_{2i} = \frac{1}{\sqrt{(r+\kappa)^2 + 2r\bar{\rho}(\phi\theta_i - \chi\eta_i)\sigma_i^2}}; \tag{4.2}$$

[8] For $\phi = A$, the manager is indifferent between all values of m_t. We assume that he chooses $m_t = 0$, as would be the case for any positive value of B, even arbitrarily small.

the contract (ϕ, χ, ψ) *with* $\phi = A$, $\psi = 0$, *and* $\chi > 0$ *being the unique solution to*

$$(a_1 - \check{a}_1)\,\eta b \hat{s}_0 + \sum_{i=1}^{N} (a_{2i} - \check{a}_{2i})\,\eta_i \hat{e}_{i0} = 0, \tag{4.3}$$

where

$$\check{a}_1 \equiv \frac{1}{\sqrt{(r+\kappa)^2 + 2r\rho[(1-\phi)\theta + \chi\eta]b\sigma_s^2}}, \tag{4.4}$$

$$\check{a}_{2i} \equiv \frac{1}{\sqrt{(r+\kappa)^2 + 2r\rho[(1-\phi)\theta_i + \chi\eta_i]\sigma_i^2}}, \tag{4.5}$$

$\hat{s}_t \equiv s_t + \frac{\kappa}{r}\bar{s}$, *and* $\hat{e}_{it} \equiv e_{it} + \frac{\kappa}{r}\bar{e}_i$; *and the index investment* $x = 0$.

When $A > \frac{\rho}{\rho+\bar{\rho}}$, the investor renders the manager's fee more sensitive to the fund's performance compared to the equilibrium without agency frictions ($\phi = A > \frac{\rho}{\rho+\bar{\rho}}$). This exposes the manager to more risk, but eliminates his incentive to undertake the shirking action m_t. If the increase in ϕ were the only change in the contract, then the manager would respond by scaling down the fund's holdings of the risky assets and investing more in the riskless rate. This would offset the increase in his personal risk exposure caused by the larger ϕ. The investor restores the manager's incentives to take risk by making the fee sensitive to the index performance ($\chi > 0$). This induces the manager to scale up the fund's holdings of the risky assets because his personal exposure to market drops becomes smaller. The increase in the risky-asset holdings, however, is according to the weights in the index η and not those in the supply portfolio θ. The fund's portfolio thus changes in response to the increases in ϕ and χ, and becomes closer to the index. This causes equilibrium prices to change, as we show in Section 4.1. The investor does not invest directly in the index ($x = 0$) because she can control the fund's index exposure by changing χ.

The compensation that the manager receives for performance relative to the index is analogous to relative-performance evaluation in models of optimal contracting under moral hazard (e.g., Holmstrom 1979). The mechanism is somewhat different, however. In typical moral-hazard models, relative-performance evaluation is used to insulate the agent from risk that he cannot control. In our model, instead, the agent can control his risk exposure through his choice of the fund's portfolio. Compensation based on relative performance is instead used to induce the agent to take risk.

Equations (4.1) and (4.2) show how the contract parameters (ϕ, χ) affect equilibrium prices. Prices are determined by the covariance with the portfolio $\phi\theta - \chi\eta$. This is the portfolio that describes the manager's personal risk exposure: the fee is ϕ times the fund's return, which in equilibrium is the return of the supply portfolio θ, minus χ times the return of the index portfolio

η. The covariance is multiplied by the manager's risk aversion coefficient $\bar{\rho}$. Prices are determined by the manager's risk aversion and risk exposure because the manager is marginal in pricing the assets. We examine the properties of prices in Sections 4.1 and 4.2.

Equation (4.3), which characterizes the the contract parameter χ, can be given an intuitive interpretation. The quantity $S_{i0} = a_1 b_i \hat{s}_0 + a_{2i}\hat{e}_{i0}$ is the price of asset i at time zero. We can also construct the counterpart $\check{S}_{i0} \equiv \check{a}_1 b_i \hat{s}_0 + \check{a}_{2i}\hat{e}_{i0}$ of this expression for the coefficients $\check{a}_1$ and $\check{a}_{2i}$ defined in (4.4) and (4.5). This is the hypothetical price of asset i at time zero under the assumption that the asset is priced from the investor instead of the manager. The price $\check{S}_{i0}$ can be derived from S_{i0} by replacing the manager's risk exposure $\phi\theta - \chi\eta$ by the investor's exposure $(1-\phi)\theta + \chi\eta$, and the manager's risk-aversion coefficient $\bar{\rho}$ by the investor's coefficient ρ. Equation (4.3) states that the investor and the manager agree on their valuation of the index: $\eta\check{S}_0 = \eta S_0$. This is because the investor can invest directly in the index, and hence is marginal in pricing the index. The investor and the manager can disagree on their valuation of other portfolios. In particular, and as we show in the proof of Theorem 4.1, the investor values the supply portfolio more than the manager: $\theta\check{S}_0 > \theta S_0$. The investor could acquire more of the supply portfolio by lowering ϕ, but this would incentivize the manager to undertake the shirking action m_t. Proposition 4.1 summarizes how the contract parameters (ϕ, χ) depend on agency frictions, as measured by the private-benefit parameter A.

Proposition 4.1 (**Effect of Agency Frictions on Manager's Contract**). *Suppose that* $A > \frac{\rho}{\rho+\bar{\rho}}$ *and* $B = 0$. *Following an increase in the private-benefit parameter* A, *the manager's fee becomes more sensitive to the fund's performance* ($\frac{\partial\phi}{\partial A} > 0$) *and to the index performance* ($\frac{\partial\chi}{\partial A} > 0$).

4.1 Cross-Sectional Pricing and Amplification

We next examine how agency frictions, as measured by the private-benefit parameter A, affect the cross-section of asset prices and of return moments. Following an increase in A, the manager's fee becomes more sensitive to the fund's performance, and performance becomes benchmarked to the index to a larger extent. This renders the manager less willing to deviate from the index. Recall that in equilibrium the manager deviates from the index by overweighting assets in large supply and underweighting assets in small supply. Hence, when A increases, the prices of large-supply assets must decrease so that the manager remains equally willing to overweight them, and the prices of small-supply assets must increase so that the manager remains equally willing to underweight them. Proposition 4.2 confirms these results in two simple cases of the model. First, when dividends vary over time only because of the asset-specific component. This case can be derived by setting the volatility parameter σ_s of the common component to zero. Second, when supply is the only driver

of cross-sectional variation, i.e., the remaining characteristics $(b_i, \bar{e}_i, \sigma_i, \eta_i)$ are common to all assets. We denote the common values of these characteristics by $(b_c, \bar{e}_c, \sigma_c, \eta_c)$. For simplicity, we assume that in both cases and for the rest of Section 4 the time-zero values of the processes $(s_t, e_{1t}, .., e_{Nt})$ are equal to the processes' unconditional means, i.e., $(s_0, e_{10}, .., e_{N0}) = (\bar{s}, \bar{e}_1, .., \bar{e}_N)$.

Proposition 4.2 (**Effect of Agency Frictions on Price and Expected Return**). *Suppose that* $A > \frac{\rho}{\rho+\bar{\rho}}$ *and* $B = 0$. *Following an increase in the private-benefit parameter* A, *the following results hold:*

(i) When $\sigma_s = 0$, *there exists a threshold* $\gamma > 0$ *such that the prices of assets* i *for which* $\frac{\theta_i}{\eta_i} > \gamma$ *decrease* $(\frac{\partial S_{it}}{\partial A} < 0)$, *and the prices of assets* i *for which* $\frac{\theta_i}{\eta_i} < \gamma$ *increase* $(\frac{\partial S_{it}}{\partial A} > 0)$. *Both sets of assets are non-empty.*

(ii) When $(b_i, \bar{e}_i, \sigma_i, \eta_i) = (b_c, \bar{e}_c, \sigma_c, \eta_c)$ *for all* i, *the price of asset* $i = \arg\max_{j\in\{1,..,N\}} \theta_j$ *decreases* $(\frac{\partial S_{it}}{\partial A} < 0)$, *and the expected price of asset* $i = \arg\min_{j\in\{1,..,N\}} \theta_j$ *increases* $(\frac{\partial \mathbb{E}(S_{it})}{\partial A} > 0)$.

For assets whose prices decrease, expected returns increase, both in share $(\frac{\partial \mathbb{E}(dR_{it})}{\partial A} > 0)$ *and dollar* $(\frac{\partial \mathbb{E}(dR^{\$}_{it})}{\partial A} > 0)$ *terms. Conversely, for assets whose prices increase, expected returns decrease, both in share* $(\frac{\partial \mathbb{E}(dR_{it})}{\partial A} < 0)$ *and dollar* $(\frac{\partial \mathbb{E}(dR^{\$}_{it})}{\partial A} < 0)$ *terms.*

When the time-variation of dividends is only asset-specific, the effect of agency frictions on prices takes a simple form. Assets are ordered according to the ratio $\frac{\theta_i}{\eta_i}$ of the weight in the supply portfolio θ relative to the index portfolio η. Assets for which the ratio exceeds a threshold γ, and are hence overweighted by the manager, drop in price when A increases. Conversely, underweighted assets, for which the ratio is below γ, rise. These results can also be stated in terms of the risk premium associated to the asset-specific component, i.e., the compensation that the manager requires for bearing the asset-specific risk. Agency frictions raise the risk premium for the assets in large supply, and lower it for the assets in small supply.

When the time-variation of dividends has a common component in addition to the asset-specific one, the analysis becomes more complicated. This is because agency frictions affect the risk premium associated to the common component and this effect is the same for all assets regardless of their supply. When supply is the only driver of cross-sectional variation, agency frictions raise the risk premium associated to the common component. Since they also raise the risk premium associated to the asset-specific component for the large-supply assets, the prices of these assets decrease. For the small-supply assets instead, the two effects go in opposite directions because the risk premium associated to the asset-specific component decreases. The prices of these assets

increase in expectation, i.e., in terms of their unconditional (long-term) means.

Proposition 4.2 implies that agency frictions exacerbate price distortions caused by supply. Indeed, assets in large supply, which the manager overweights in equilibrium, trade at low prices holding else equal. Agency frictions cause their prices to become even lower. Conversely, assets in small supply, which the manager underweights, trade at high prices, and agency frictions cause their prices to rise further. Agency frictions effectively raise the supply of assets whose supply is already large, and lower the supply of assets whose supply is already small. Formally, in the presence of frictions, prices are determined by the covariance with the portfolio $\phi\theta - \chi\eta$ that describes the manager's personal risk exposure. Frictions raise ϕ and χ in such a way that $\phi\theta_i - \chi\eta_i$ increases for large-θ_i assets and decreases for small-θ_i assets.

Agency frictions affect not only prices and expected returns, but also the volatility of returns. Indeed, because they magnify differences in supply, they also magnify the relationship between supply and volatility shown in Section 3.1. As shown in that section, supply is related to volatility through an attenuation effect. Following a positive shock to an asset's expected cashflows, the asset accounts for a larger fraction of the manager's portfolio volatility. The increase in volatility makes the manager less willing to hold the asset, and attenuates the price increase caused by the improved fundamentals. The extent of attenuation depends on the asset's supply. If supply is large, then attenuation is strong because the asset's contribution to portfolio volatility is large. Therefore, the cashflow shock has a weak effect on the asset's price, resulting in low share return volatility. Conversely, if supply is small, then attenuation is weak, and share return volatility is high.

The effect of agency frictions on return volatility is most striking for assets in small supply. For these assets, the attenuation effect described in the previous paragraph can reverse sign and become an amplification effect. Consider again a positive shock to an asset's expected cashflows. Following the shock, the asset accounts for a larger fraction of the manager's portfolio volatility. The manager, however, cares not only about the volatility of his portfolio, as is the case in the absence of agency frictions, but also about the volatility of his deviation from the index. When the shock concerns an asset in small supply, which the manager underweights, the latter volatility increases and it can be reduced by buying the asset, i.e., reducing the underweight. Buying pressure by the manager to reduce the underweight amplifies the price increase caused by the improved fundamentals and results in high share return volatility. This is the amplification effect. Conversely, for assets in large supply, which the manager overweights, the volatility of the manager's deviation from the index can be reduced by selling the asset, i.e., reducing the overweight. Hence, attenuation is stronger, resulting in lower share return volatility. Proposition 4.3 confirms these results in the two special cases of the model, and examines the behavior of dollar return volatility as well.

Proposition 4.3 (**Effect of Agency Frictions on Return Volatility**). *Suppose that* $A > \frac{\rho}{\rho+\bar{\rho}}$ *and* $B = 0$. *Following an increase in the private-benefit parameter* A, *the following results hold:*

(i) When $\sigma_s = 0$, *the return volatility of assets* i *for which* $\frac{\theta_i}{\eta_i} > \gamma$ *decreases, both in share* ($\frac{\partial \mathbb{V}\text{ar}(dR_{it})}{\partial A} < 0$) *and dollar* ($\frac{\partial \mathbb{V}\text{ar}(dR^{\$}_{it})}{\partial A} < 0$) *terms, and the return volatility of assets* i *for which* $\frac{\theta_i}{\eta_i} < \gamma$ *increases, both in share* ($\frac{\partial \mathbb{V}\text{ar}(dR_{it})}{\partial A} > 0$) *and dollar* ($\frac{\partial \mathbb{V}\text{ar}(dR^{\$}_{it})}{\partial A} > 0$) *terms, where the threshold* $\gamma > 0$ *is as in Proposition 4.2. Both sets of assets are non-empty.*

(ii) When $(b_i, \bar{e}_i, \sigma_i, \eta_i) = (b_c, \bar{e}_c, \sigma_c, \eta_c)$ *for all* i, *the share return volatility of asset* $i = \arg\max_{j\in\{1,..,N\}} \theta_j$ *decreases* ($\frac{\partial \mathbb{V}\text{ar}(dR_{it})}{\partial A} < 0$). *The return volatility of asset* $i = \arg\min_{j\in\{1,..,N\}} \theta_j$ *increases, both in share* ($\frac{\partial \mathbb{V}\text{ar}(dR_{it})}{\partial A} > 0$) *and dollar* ($\frac{\partial \mathbb{V}\text{ar}(dR^{\$}_{it})}{\partial A} > 0$) *terms, provided that*

$$a_{2i}\sigma_i^2 > a_1 b_i \sigma_s^2. \tag{4.6}$$

Condition (4.6) is sufficient for the share return volatility, and necessary and sufficient for the dollar return volatility.

When the time-variation of dividends is only asset-specific, the effect of agency frictions takes the same form as in Proposition 4.2: assets whose weight in the supply portfolio relative to the index portfolio exceeds a threshold γ become less volatile, and assets below the threshold become more volatile. When a time-varying common component is added, the analysis becomes more complicated but the results have a similar flavor. Agency frictions lower the share return volatility of assets in large supply. They also raise the share return volatility of assets in small supply, under the sufficient condition (4.6). Moreover, (4.6) is necessary and sufficient for the dollar return volatility of small-supply assets to increase with agency frictions. Condition (4.6) parallels (3.6), which is necessary and sufficient for dollar return volatility to be higher for assets in small supply: under scale invariance and $i = i'$, (3.6) and (4.6) become identical, and require that the volatility driven by the asset-specific component exceeds that driven by the common component.

4.2 Aggregate Market

We next examine how agency frictions affect the valuation of the aggregate market, i.e., the index η. Recall from Proposition 4.2 that frictions cause the prices of assets in large supply to drop and the prices of assets in small supply to rise. The effect on the aggregate market is thus ambiguous a priori. We show, however, that the cross-sectional differences do not cancel out, and the aggregate market goes up. Therefore, agency frictions distort the prices of small-supply assets upwards more than they distort the prices of large-supply assets downwards.

The intuition for the asymmetry lies in the interaction between the manager's risk-taking incentives and the amplification effect described in Section 4.1. When agency frictions are more severe, the manager becomes less willing to deviate from the index because he is benchmarked more tightly on it. The manager's deviations are to overweight assets in large supply, which earn high expected return, and underweight assets in small supply, which earn low expected return. The latter deviation becomes increasingly costly relative to the former because of the amplification effect: since the share return volatility of small-supply assets increases, the manager is exposed to an increased risk of deviating from the index by underweighting these assets. Therefore, when the manager is benchmarked more tightly on the index, he becomes particularly keen to reduce the underweights. As a consequence, the price of the underweights goes up more than the price of the overweights goes down.

Proposition 4.4 confirms these results in two simple cases of the model. These cases parallel those in Propositions 4.2 and 4.3, but are somewhat more restrictive. When dividends vary over time only because of the asset-specific component, assets are assumed symmetric in terms of some of their characteristics. And when the time-variation of dividends has also a common component, supply is assumed to take only two values.

Proposition 4.4 (**Effect of Agency Frictions on Aggregate Market**). *Suppose that* $A > \frac{\rho}{\rho+\bar{\rho}}$, $B = 0$, *and that one of the following conditions holds:*

(i) $\sigma_s = 0$, *and* $(\sigma_i, \eta_i) = (\sigma_c, \eta_c)$ *for all* i *or* $(\sigma_i, \theta_i) = (\sigma_c, \theta_c)$ *for all* i.

(ii) $(b_i, \bar{e}_i, \sigma_i, \eta_i) = (b_c, \bar{e}_c, \sigma_c, \eta_c)$ *for all* i, *and* θ_i *can take only two values.*

Following an increase in the private-benefit parameter A, *the expected price of the aggregate market increases* $(\frac{\partial \mathbb{E}(S_{\eta t})}{\partial A} > 0)$, *and the expected return decreases both in share* $(\frac{\partial \mathbb{E}(dR_{\eta t})}{\partial A} < 0)$ *and dollar* $(\frac{\partial \mathbb{E}(dR^{\$}_{\eta t})}{\partial A} < 0)$ *terms.*

4.3 Numerical Example

We illustrate our results with a numerical example. We set the investor's risk-aversion coefficient ρ to one. This is a normalization because we can redefine the units of the consumption good. We set the manager's risk-aversion coefficient $\bar{\rho}$ to 50, meaning that the manager accounts for $\frac{\rho}{\rho+\bar{\rho}} = 2\%$ of aggregate risk tolerance. The ratio $\frac{\rho}{\rho+\bar{\rho}}$ reflects the "size" of fund managers relative to fund investors, and can be related to the size of the financial sector. Philippon (2008) reports that the GDP share of the Finance and Insurance industry was 5.5% on average during 1960-2007 in the US. Since only part of that industry concerns asset management, 5.5% can be viewed as an upper

bound for $\frac{\rho}{\rho+\bar{\rho}}$. We set the riskless rate r to 4%. We set the mean-reversion parameter κ to 10%, meaning that the half-life of dividend shocks is $\frac{\log(2)}{0.1} = 6.93$ years.

If the assets in our model are interpreted as individual stocks, then supply effects will be small. Indeed, since there is a large number of stocks, the specific risk associated to each stock is small relative to the aggregate risk tolerance. We interpret instead our assets as segments of the stock market, e.g., style portfolios such as value and growth, or industry-sector portfolios. Under this interpretation, asset-specific risk concerns market segments, and so does the demand by the buy-and-hold investors. The volatility and beta anomalies that we show in this section are also at the segment level. We set the number N of assets to six.

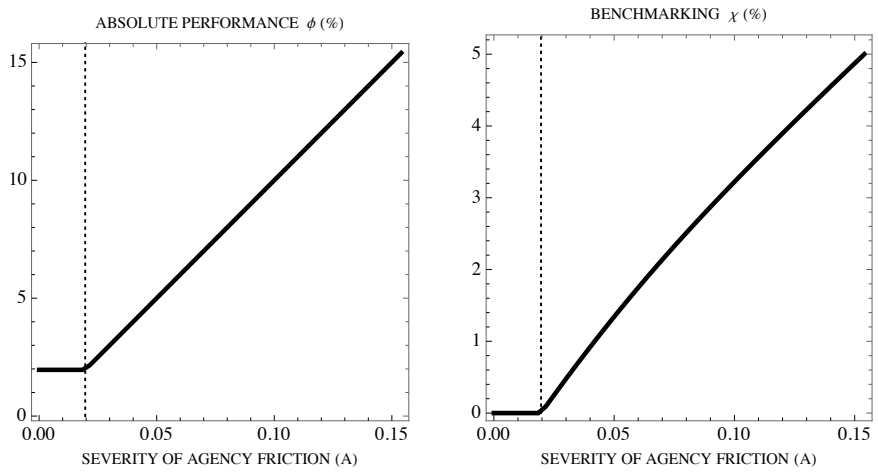
We assume that the six assets are divided into two groups, with three assets in each group. Assets in each group have identical characteristics, except for supply. Thus, assets are identical in terms of number of shares η_i included in the index, sensitivity b_i of dividends to the dividends' common component s_t, long-run mean $\bar{e}_i$ of the asset-specific component, and volatility parameter σ_i of the asset-specific component. We assume that the index η includes one share of each asset ($\eta_i = 1$ for $i = 1, .., 6$). This is a normalization because we can redefine one share of each asset. We set the supply of assets 1, 2, and 3, left over by the buy-and-hold investors, to 0.7 share, and the corresponding supply of assets 4, 5, and 6 to 0.3 share ($\theta_1 = \theta_2 = \theta_3 = 0.7$ and $\theta_4 = \theta_5 = \theta_6 = 0.3$). We set the dividend sensitivities to one ($b_i = 1$ for $i = 1, .., 6$). This is a normalization because we can redefine s_t.

The remaining parameters are the long-run mean $\bar{s}$ of the common component of dividends, the long-run mean $\bar{e}_i$ of the asset-specific component, the volatility parameter σ_s of the common component, and the volatility parameter σ_i of the asset-specific component. We determine σ_i as function of $(\bar{s}, \bar{e}_i, \sigma_s)$ by imposing scale invariance, i.e., $\frac{\sigma_s^2}{\bar{s}} = \frac{\sigma_i^2}{\bar{e}_i}$. We determine the ratio $\frac{\bar{e}_i}{\bar{s}}$ based on the fraction of assets' return variance that is idiosyncratic. Finally, we determine $(\bar{s}, \sigma_s)$ based on the mean and the variance of the return of the aggregate market η. We set $(\bar{s}, \sigma_s) = (0.65, 1)$; under these choices the market's expected return (in excess of the riskless rate) is 5.27% in the absence of agency frictions ($A \leq \frac{\rho}{\rho+\bar{\rho}}$), and the market's return volatility is 16.6%. We also set $\bar{e}_i = 0.4$; under this choice idiosyncratic risk accounts for 60% of assets' return variance in the absence of frictions. Our results are sensitive to the size of idiosyncratic risk, as we show in Propositions 3.2, 3.3, and 4.3, and emphasize again later in this section.

Figure 1 plots the sensitivity ϕ of the manager's fee to the fund's performance and the sensitivity χ to the index performance as a function of the private-benefit parameter A. We express the sensitivities as percentages, and allow A to vary from zero to 0.15. Thus, under the maximum value of A, the manager's fee increases by fifteen cents when the fund's assets increase by one dollar. A sensitivity of that magnitude is typical for hedge funds; under the 2/20 contracts that

are common in the industry, hedge-fund managers receive 20% of profits. For mutual funds, typical fees are 0.5-2% of assets under management, but the fees' sensitivity to returns may be significantly higher because high returns attract inflows (an effect which is absent from our model). Figure 1 confirms the result of Proposition 4.1 that ϕ and χ increase in A for $A > \frac{\rho}{\rho+\bar{\rho}}$.

Figure 1: Optimal Contract

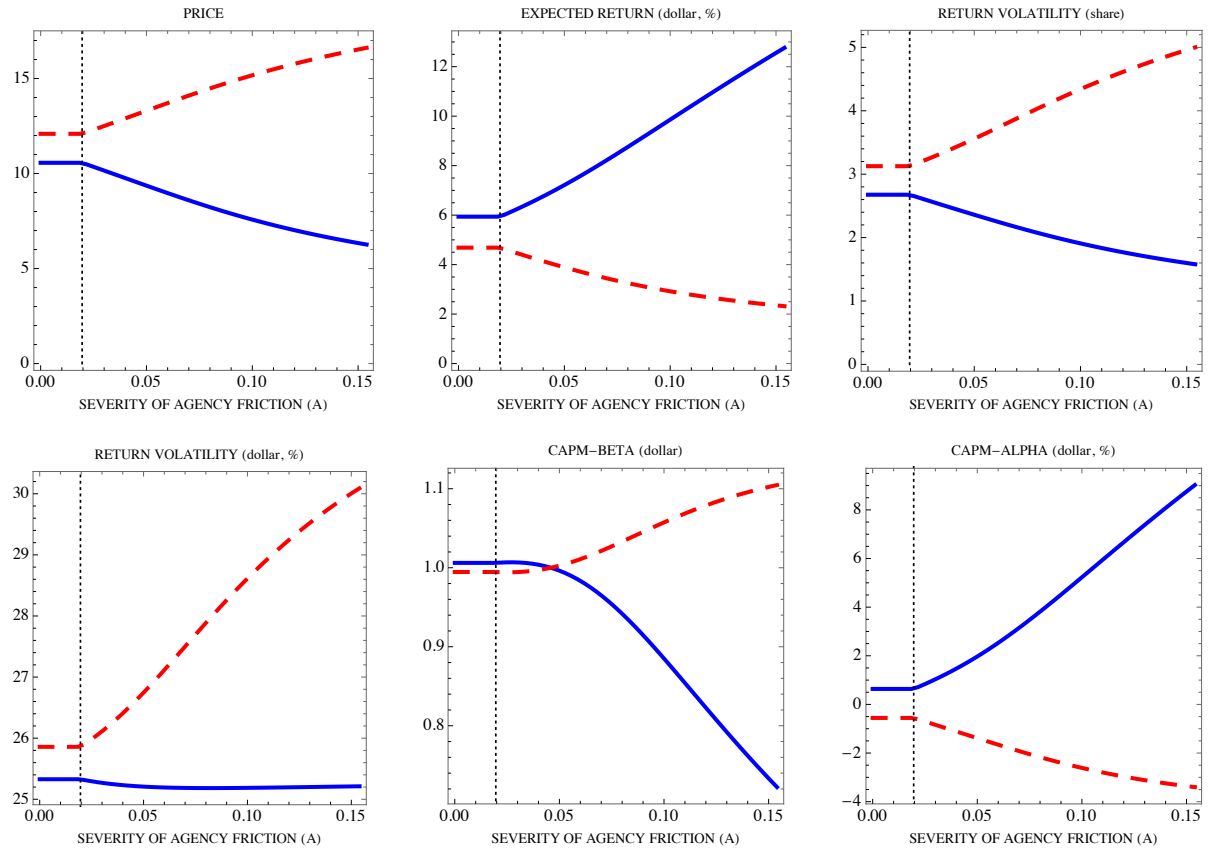


The sensitivity ϕ of the manager's fee to the fund's performance and the sensitivity χ to the index performance as a function of the private-benefit parameter A. There are two groups of assets, with three assets in each group. Assets in each group have identical characteristics, except for supply. Parameter values are: $\rho = 1$, $\bar{\rho} = 50$, $r = 4\%$, $\kappa = 10\%$, $N = 6$, $\eta_i = 1$, $\theta_1 = \theta_2 = \theta_3 = 0.7$, $\theta_4 = \theta_5 = \theta_6 = 0.3$, $b_i = 1$, $\bar{s} = 0.65$, $\bar{e}_i = 0.4$, $\sigma_s = 1$, $\frac{\sigma_{\bar{s}}^2}{\bar{s}} = \frac{\sigma_i^2}{\bar{e}_i}$, for $i = 1, .., 6$.

Figure 2 plots price and return moments for individual assets as a function of the private-benefit parameter A. The blue solid line represents assets in large supply and the red dashed line assets in small supply. Consistent with Proposition 3.1, assets in small supply are more expensive than assets in large supply and earn lower expected return. Moreover, consistent with Proposition 4.2, the supply-driven distortions are exacerbated by agency frictions. These effects are significant quantitatively. In the absence of frictions, small-supply assets earn an expected return of 4.5% and large-supply assets earn 6%. When $A = 0.15$, the expected return of small-supply assets drops to 2.5% and that of large-supply assets rises to 12.5%. Thus, the expected-return differential increases from 1.5% to 10%.

The results on return volatility parallel those on expected return. Assets in small supply are more volatile than assets in small supply, and the supply-driven distortions are exacerbated by agency frictions. In the absence of frictions, small-supply assets have return volatility equal to 25.8% and small-supply assets 25.4%. When $A = 0.15$, the volatility of small-supply assets rises to 30% and that of large-supply assets drops to 25.2%. Thus, the volatility differential increases from

Figure 2: Agency Frictions and Cross-Sectional Pricing



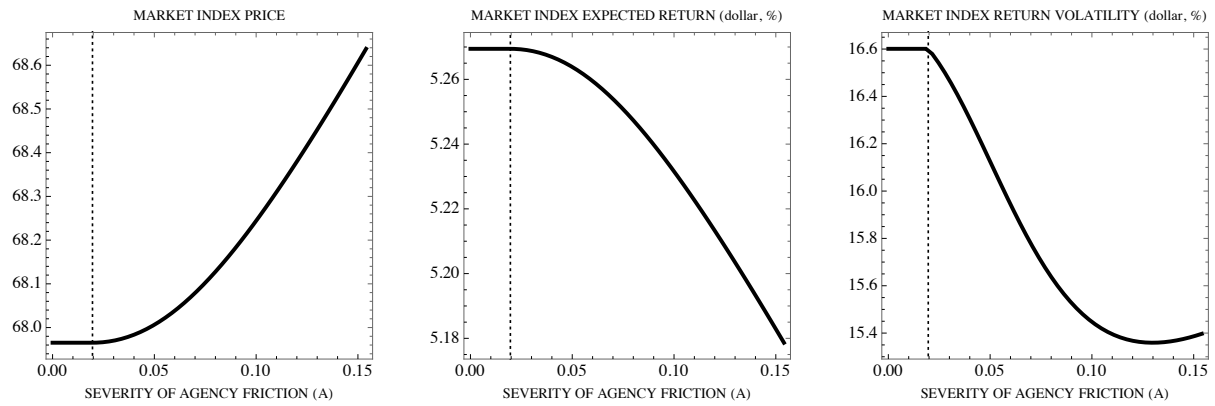
Expected price $\mathbb{E}(S_{it})$, expected dollar return $\frac{1}{dt}\mathbb{E}(dR^{\$}_{it})$, share return volatility $\sqrt{\frac{1}{dt}\mathbb{V}\text{ar}(dR_{it})}$, dollar return volatility $\sqrt{\frac{1}{dt}\mathbb{V}\text{ar}(dR^{\$}_{it})}$, dollar beta $\beta^{\$}_i$, and dollar alpha $\frac{1}{dt}\alpha^{\$}_i$, as a function of the private-benefit parameter A. There are two groups of assets, with three assets in each group. Assets in each group have identical characteristics, except for supply. Parameter values are as in Figure 1. Assets 1, 2, and 3, in the high-supply group are represented by the blue solid line, and assets 4, 5, and 6, in the low-supply group are represented by the red dashed line.

0.4% to 4.8%, with most of the increase being driven by the assets in small supply.

The results on CAPM beta parallel those on volatility except when agency frictions are small. In the absence of frictions, small-supply assets have beta equal to 0.99 and large-supply assets have a slightly larger beta equal to 1.01. The difference in beta reverses when A exceeds 0.05. When $A = 0.15$, the beta of small-supply assets rises to 1.1 and that of large-supply assets drops to 0.72.

The results in Figure 2 are consistent with the volatility and beta anomalies. Assets in small supply have high return volatility and low expected return. Hence, volatility is negatively related to expected return in the cross-section. The negative relationship continues to hold when volatility

Figure 3: Agency Frictions and the Aggregate Market



Expected price $\mathbb{E}(S_{\eta t})$, expected dollar return $\frac{1}{dt}\mathbb{E}(dR^{\$}_{\eta t})$, and dollar return volatility $\sqrt{\frac{1}{dt}\mathbb{V}\text{ar}(dR^{\$}_{\eta t})}$ of the aggregate market, as a function of the private-benefit parameter A. There are two groups of assets, with three assets in each group. Assets in each group have identical characteristics, except for supply. Parameter values are as in Figure 1.

is replaced by its idiosyncratic component (not plotted in Figure 2). It also continues to hold when volatility is replaced by beta, provided that agency frictions are large enough ($A > 0.05$). Moreover, the negative relationship is reinforced when expected return is replaced by alpha. Indeed, alpha is expected return adjusted for beta, and assets in small supply both earn low expected return, and have high beta when frictions are large enough. For example, when $A = 0.15$, the alpha differential between large- and small-supply assets is 13%, which is larger than the expected-return differential of 10.5%.

Propositions 3.2, 3.3, and 4.3 show that volatility or beta are not always negatively related to supply; the relationship is negative when idiosyncratic risk accounts for a large enough fraction of return variance. The same applies to the relationship between risk measures and expected return since supply and expected return are positively related. For example, under scale invariance and no agency frictions, Proposition 3.2 implies that volatility is negatively related to expected return if the asset-specific component of dividends accounts for more than half of return variance. Our numerical example meets this condition. The extent of idiosyncratic risk that is required to generate a negative risk-return relationship can be significantly lower, however, if the assumption of scale invariance is dropped.

Figure 3 plots price and return moments for the aggregate market as a function of the private-benefit parameter A. Consistent with Proposition 4.4, agency frictions cause the price of the aggregate market to increase and its expected return to drop. The effect is quantitatively small,

however: the expected return in the absence of agency frictions is 5.27%, and it drops to 5.18% when $A = 0.15$. One reason why the effect is small is that fund investors can react to market overvaluation by lowering their direct investment in the index or by changing the manager's contract to induce him to hold a smaller position in the index. If investors lacked the ability or understanding to undertake these actions, the effects of agency frictions on the aggregate market might be significantly larger.

Figure 3 shows additionally that agency frictions can have a non-monotonic effect on the volatility of the aggregate market: volatility decreases for small frictions, but increases when frictions become large enough. This result stands in-between that in Cuoco and Kaniel (2011), who find that market volatility decreases with the extent of benchmarking, and that in Basak and Pavlova (2013), who find that volatility increases.

5 Social Optimality

In this section we examine whether the privately optimal contract, determined in Section 4, is socially optimal. We assume that a social planner chooses contract parameters (ϕ, χ, ψ) at time zero. This is the social planner's only intervention: given the contract, the manager is free to choose the fund's portfolio z_t and the shirking action m_t, and prices S_t must clear markets. Without loss of generality, we restrict the investor's direct investment in the index to be zero ($x = 0$).[9]

The social planner maximizes the investor's value function at time zero, subject to the manager's incentive compatibility and individual rationality constraints. This optimization problem is the same as the investor's but the social planner internalizes that a change in the contract parameters affects equilibrium prices. Formally, the value functions of the investor and the manager at time zero can be written as $V(W_0, s_0, e_0, \phi, \chi, \psi, \mathcal{S})$ and $\bar{V}(\bar{W}_0, s_0, e_0, \phi, \chi, \psi, \mathcal{S})$, respectively, where $\mathcal{S}$ consists of the parameters $(a_{01}, .., a_{0N}, a_{11}, .., a_{1N}, a_{21}, .., a_{2N})$ that describe the price process. The investor chooses (ϕ, χ, ψ) taking $\mathcal{S}$ as given. The social planner instead internalizes the dependence of $\mathcal{S}$ on (ϕ, χ).

The social planner's optimization problem involves the utility of the investor and the manager, but not of the buy-and-hold investors. These investors, however, are neutral for our normative analysis, in the sense that the contract choice does not affect their asset holdings and dividend stream. Indeed, buy-and-hold investors are endowed with the portfolio $\eta - \theta$ at time zero and do not trade. Therefore, the dividend stream that they receive from their portfolio does not depend

[9]The set $\mathcal{A}$ of allocations that the social planner can achieve when restricting x to be zero includes the set $\mathcal{A}'$ of allocations when the investor can choose any value of x. This is because the social planner can induce the investor to choose $x = 0$ through an appropriate choice of χ, without affecting the allocation. Moreover, the social planner's optimal allocation in $\mathcal{A}$ also belongs to $\mathcal{A}'$. Indeed, Proposition 5.1 shows that under the optimal allocation the investor and the manager agree on their valuation of the index ((4.3) holds). Therefore, the investor would choose $x = 0$ even if the restriction $x = 0$ were lifted.

on prices and on the contract choice.

When the parameter B in the manager's private-benefit function $Am_t - \frac{B}{2}m_t^2$ is equal to zero, as assumed in Section 4, the social planner's problem yields the same solution as the investor's. Indeed, the coefficient ϕ that characterizes the fee's sensitivity to the fund's performance must satisfy $\phi \geq A$, so that the manager does not choose an arbitrarily large shirking action m_t. Moreover, any $\phi \geq A$ yields no shirking, i.e., $m_t = 0$. When A exceeds the value $\frac{\rho}{\rho+\bar{\rho}}$ that ϕ takes in the absence of agency frictions, the constraint $\phi \geq A$ is binding. Hence, the social planner sets $\phi = A$, as does the investor.

The social planner chooses the same contract as the investor because $\phi = A$ is a corner solution. The differences in marginal trade-offs between the social planner and the investor become apparent when instead ϕ is an interior solution. Interior solutions are possible when the parameter B is positive. Theorem 5.1 generalizes the equilibrium derived in the previous section to $B > 0$. Proposition 5.1 solves the social planner's problem and shows that solutions for the investor and the social planner differ.

Theorem 5.1 (**Equilibrium Prices and Contract with General Agency Frictions**). *When* $\frac{\rho}{\rho+\bar{\rho}} \geq A > 0$*, the equilibrium in Theorem 3.1 remains an equilibrium. When* $A > \frac{\rho}{\rho+\bar{\rho}}$ *and* $B \in [0, \underline{B}] \cup [\bar{B}, \infty)$ *for two constants* $\bar{B} > \underline{B}$*, the following form an equilibrium: the price process* S_t *given by (2.14), (3.1), (4.1), and (4.2); the contract* (ϕ, χ, ψ) *with* $A \geq \phi > \frac{\rho}{\rho+\bar{\rho}}$ *and* $\chi > 0$ *solving the system of equations*

$$\left(\frac{\phi(1-\phi)}{B} + r\left(a_1 - \check{a}_1\right)\theta b \hat{s}_0 + r\sum_{i=1}^{N}\left(a_{2i} - \check{a}_{2i}\right)\theta_i \hat{e}_{i0} = 0 \quad \textit{and} \quad \phi < A \right) \quad \textit{or} \tag{5.1}$$

$$\left(\frac{\phi(1-\phi)}{B} + r\left(a_1 - \check{a}_1\right)\theta b \hat{s}_0 + r\sum_{i=1}^{N}\left(a_{2i} - \check{a}_{2i}\right)\theta_i \hat{e}_{i0} \geq 0 \quad \textit{and} \quad \phi = A \right), \tag{5.2}$$

and (4.3), where $\check{a}_1$*,* $\check{a}_{2i}$*,* $\hat{s}_0$*, and* $\hat{e}_{i0}$ *are as in Theorem 4.1, and* $\psi = -\frac{(A-\phi)^2}{2B}$*; and the index investment* $x = 0$*.*

The behavior of ϕ for $A > \frac{\rho}{\rho+\bar{\rho}}$ is as follows. When B is positive but close to zero, $\phi = A$ is a corner solution, as in the case $B = 0$. When B exceeds a threshold, $\phi = A$ ceases to be a corner solution, and the solution becomes interior to the interval $(\frac{\rho}{\rho+\bar{\rho}}, A)$. Intuitively, the investor's benefit from raising ϕ is that the manager has a smaller incentive to undertake the shirking action. At the same time, larger ϕ involves a cost to the investor because the manager becomes less willing to take risk and hence to exploit price differentials driven by supply, i.e., invest relatively more in

high-θ assets and less in low-θ assets. When B increases, the manager derives a smaller benefit from shirking. Hence the investor's benefit from raising ϕ is smaller, which is why ϕ decreases below A when B exceeds a threshold. When B becomes large, and so the manager's benefit from shirking converges to zero, ϕ converges to its value $\frac{\rho}{\rho+\bar{\rho}}$ under no agency frictions.

The system of equations (4.3), (5.2), and (5.1) that determines (ϕ, χ, ψ) can have multiple solutions when $B > 0$. This means that multiple equilibria can exist. The comparison between socially and privately optimal contract shown in Proposition 5.1 applies to the privately optimal contract in any of these equilibria.

Equilibrium may fail to exist for intermediate values of B. This is because the investor may not be willing to employ the manager. Note that the investor is willing to employ the manager not only when the benefit of shirking is small ($B \geq \bar{B}$) but also—and more surprisingly—when it is large ($B \leq \underline{B}$). This is because of a general-equilibrium effect: when the benefit from shirking is large, equilibrium prices are more distorted, making the supply portfolio an even better investment than the index portfolio.

Proposition 5.1 (**Socially Optimal Contract**). *When $\frac{\rho}{\rho+\bar{\rho}} \geq A \geq 0$, the socially optimal contract (ϕ^*, χ^*, ψ^*) is as in Theorem 3.1. When $A > \frac{\rho}{\rho+\bar{\rho}}$, the socially optimal contract is as follows: $A \geq \phi^* > \frac{\rho}{\rho+\bar{\rho}}$ and $\chi^* > 0$ are the unique solution to the system of equations*

$$\left(\frac{1-\phi}{B} + r\left(a_1 - \check{a}_1\right) \theta b \hat{s}_0 + r \sum_{i=1}^{N} \left(a_{2i} - \check{a}_{2i}\right) \theta_i \hat{e}_{i0} = 0 \quad \textit{and} \quad \phi < A \right) \quad \textit{or} \tag{5.3}$$

$$\left(\frac{1-\phi}{B} + r\left(a_1 - \check{a}_1\right) \theta b \hat{s}_0 + r \sum_{i=1}^{N} \left(a_{2i} - \check{a}_{2i}\right) \theta_i \hat{e}_{i0} \geq 0 \quad \textit{and} \quad \phi = A \right), \tag{5.4}$$

and (4.3), where $\check{a}_1$, $\check{a}_{2i}$, $\hat{s}_0$, and $\hat{e}_{i0}$ are as in Theorem 4.1, and $\psi^* = -\frac{(A-\phi^*)^2}{2B}$. The manager's fee under the socially optimal contract is more sensitive to the fund's performance and to the index performance than under the privately optimal contract ($\phi^* \geq \phi$, $\chi^* \geq \chi$), with the inequalities being strict when $\phi < A$.

Under the socially optimal contract, the manager has steeper incentives than under the privately optimal contract, with the inequalities being strict when the privately optimal ϕ is an interior solution. The intuition is that because the social planner internalizes price effects, he is more effective than private agents in providing incentives. Indeed, recall that the investor's benefit from raising ϕ is that the manager shirks less, and her cost is that the manager becomes less willing to take risk and exploit price differentials driven by supply. The cost is lower for the social planner.

This is because when the social planner raises ϕ, equilibrium prices become more distorted, to the point where the manager remains equally willing to exploit supply-driven price differentials. Because cost and benefit are equalized at an interior solution, such a solution for the investor must be strictly smaller than for the social planner.

The inefficiency can be viewed as a free-rider problem. Interpret the investor and the manager as a continuum of identical investors and managers. When one investor in the continuum gives steeper incentives to her manager, this makes the manager less willing to exploit mispricings. Other managers, however, remain equally willing to do so, benefiting their investors. When all investors give steeper incentives to their managers, mispricings become more severe, and all managers remain equally willing to exploit them despite being exposed to more risk.

Because the social planner chooses steeper incentives than private agents, supply effects are stronger under the socially optimal contract. Thus, the volatility and beta anomalies are stronger. Agency frictions also have a larger positive effect on the price of the aggregate market.

6 Normally Distributed Cashflows

Our analysis in the previous sections assumes that asset cashflows evolve according to square-root (SR) processes. While we use this particular specification for tractability, the key property that we want to capture is that the volatility of cashflows per share increases with the cashflow level. To show how this property matters for our results, we consider in this section a modification of our model where the volatility of cashflows per share is constant. We replace the SR processes (2.3) and (2.4) by the Ornstein-Uhlenbeck (OU) processes:

$$ds_t = \kappa\left(\bar{s} - s_t\right) dt + \sigma_s dw_{st}, \tag{6.1}$$

$$de_{it} = \kappa\left(\bar{e}_i - e_{it}\right) dt + \sigma_i dw_{it}. \tag{6.2}$$

With OU processes, the diffusion coefficients $(\sigma_s, \sigma_1, .., \sigma_N)$ are constant, and so is the volatility of cashflows per share. Moreover, future cashflows per share are normally distributed conditional on current information. Theorem 6.1 derives the equilibrium with OU processes and with or without agency frictions.

Theorem 6.1 (**Equilibrium Prices and Contract with OU Cashflow Processes**). *Suppose that $B = 0$ and $(s_t, e_{1t}, .., e_{Nt})$ evolve according to (6.1) and (6.2). The following form an*

equilibrium: the price process S_t given by (2.14) with

$$a_{0i} = \frac{\kappa(b_i\bar{s} + \bar{e}_i)}{r(r+\kappa)} - \frac{\bar{\rho}\left[b_i(\phi\theta - \chi\eta)b\sigma_s^2 + (\phi\theta_i - \chi\eta_i)\sigma_i^2\right]}{(r+\kappa)^2}, \tag{6.3}$$

$$a_{1i} = \frac{b_i}{r+\kappa} \equiv a_1 b_i, \tag{6.4}$$

$$a_{2i} = \frac{1}{r+\kappa}; \tag{6.5}$$

the contract (ϕ, χ, ψ) with

$$\phi = \max\{A, \frac{\rho}{\rho+\bar{\rho}}\}, \tag{6.6}$$

$$\chi = \left(\phi - \frac{\rho}{\rho+\bar{\rho}}\right)\frac{\eta b\theta b\sigma_s^2 + \sum_{i=1}^N \eta_i\theta_i\sigma_i^2}{(\eta b)^2\sigma_s^2 + \sum_{i=1}^N \eta_i^2\sigma_i^2}, \tag{6.7}$$

$$\psi = 0; \tag{6.8}$$

and the index investment $x = 0$.

The key difference between Theorem 6.1 and its counterparts under SR processes (Theorems 3.1 and 4.1) is that the coefficients (a_{1i}, a_{2i}) of (s_t, e_{it}) in the price function do not depend on supply. Since the price can vary over time only because of (s_t, e_{it}), supply has no effect on share return volatility. Supply affects the price only through the term a_{0i}, and this relationship is negative, as is the case under SR processes. Thus, when comparing two assets that differ only in supply but have otherwise identical characteristics, the asset in smaller supply is more expensive, has lower expected share return, and has the same share return volatility. If share returns were to be converted into dollar returns, the small-supply asset would have lower expected return and lower return volatility because it is more expensive.[10] Hence, supply would always induce a positive relationship between risk and expected return, while the relationship can be negative under SR processes.

Proposition 6.1 (**Supply Effects with OU Cashflow Processes**). *Suppose that $A = 0$ and $(s_t, e_{1t}, .., e_{Nt})$ evolve according to (6.1) and (6.2). An asset i in smaller supply than an otherwise identical asset i' has higher price at time t ($S_{it} > S_{i't}$), lower expected share return ($\mathbb{E}(dR_{it}) < \mathbb{E}(dR_{i't})$), and same share return volatility ($\mathbb{V}\text{ar}(dR_{it}) = \mathbb{V}\text{ar}(dR_{i't})$).*

The effects of agency frictions have a similar flavor to those of supply. When frictions are more

[10] Computing dollar returns under OU processes is made complicated by the fact that prices can be zero or even negative. We use expected prices, as under SR processes, and assume that these are positive as would be the case if the unconditional means $(\bar{s}, \bar{e}_1, .., \bar{e}_N)$ are large enough.

severe, prices become more distorted but share return volatilities are unaffected. The aggregate market is also unaffected because a change in share return volatilities is key for such an effect.

Proposition 6.2 (**Effects of Agency Frictions with OU Cashflow Processes**). *Suppose that* $A > \frac{\rho}{\rho+\bar{\rho}}$, $B = 0$, *and* $(s_t, e_{1t}, .., e_{Nt})$ *evolve according to (6.1) and (6.2). Following an increase in the private-benefit parameter A, prices and expected share returns are affected as described in Proposition 4.2, while share return volatilities and the price of the aggregate market are unaffected.*

7 Conclusion

In this paper we study how the delegation of asset management affects equilibrium prices. Unlike most prior literature, we endogenize both equilibrium prices and fund managers' contracts. We show that because of agency frictions, managers are compensated based on their performance relative to a benchmark. As a consequence, they become less willing to deviate from the benchmark, and the price distortions that they are hired to exploit become more severe. While distortions are exacerbated in both directions, i.e., undervalued assets become cheaper and overvalued assets become more expensive, the positive distortions dominate, biasing the aggregate market upwards. This is because overvalued assets account for an increasingly large fraction of market movements relative to undervalued assets, and hence trading against overvaluation is riskier for managers than trading against undervaluation. Agency frictions can also generate a negative relationship between risk and expected return in the cross-section, in line with empirical evidence but in contrast to standard theories. This is because they raise the volatility of overvalued assets, through an amplification mechanism. Following a positive shock to the expected cash flows of an overvalued asset, the asset accounts for a larger fraction of market movements. This makes it riskier for managers to underweight the asset, and the resulting buying pressure amplifies the price increase caused by the higher cash flows.

Our model combines square-root processes for asset cashflows with constant absolute risk aversion utility. This combination is to our knowledge new to the literature, including in a frictionless setting. We show that it yields closed-form solutions for asset prices and can accommodate any number of risky assets. It also generates more realistic properties for asset cashflows and prices than the tractable CARA-normal alternative.

The effects of agency frictions in our model are reflected only on prices and not on fund managers' portfolios. This is because we are assuming a representative manager, who must hold in equilibrium a fixed portfolio supplied by other agents. A natural extension of our analysis is to introduce multiple managers with different levels of agency frictions, and examine how their port-

folios differ. Such an extension could provide additional points of contact with the data. For example, Christoffersen and Simutin (2014) find that fund managers who face greater pressure to meet benchmarks hold a larger fraction of their portfolios in high-beta stocks and achieve lower alphas. Their findings are in the spirit of our theoretical results.

Our analysis can also be extended on the normative front. One extension is to examine whether the result that socially optimal contracts cause larger pricing distortions than privately optimal contracts continues to hold when distortions affect the real economy. Another extension is to allow for more degrees of freedom in contract design, e.g., the choice of benchmark which in our model is assumed to be the index, and study how privately optimal choices compare with socially optimal ones.

Appendix

Proof of Theorem 3.1. The theorem follows from the proof of Theorem 4.1, which covers the case $A = 0$. □

Proof of Proposition 3.1. Substituting a_{0i} from (3.1), we can write the price (2.14) of asset i as

$$S_{it} = a_1 b_i \hat{s}_t + a_{2i} \hat{e}_{it}, \tag{A.1}$$

and asset i's expected price as

$$\mathbb{E}(S_{it}) = \frac{r+\kappa}{r} \left(a_1 b_i \bar{s} + a_{2i} \bar{e}_i\right), \tag{A.2}$$

where $\hat{s}_t \equiv s_t + \frac{\kappa}{r}\bar{s}$ and $\hat{e}_{it} \equiv e_{it} + \frac{\kappa}{r}\bar{e}_i$. Since asset i' differs from asset i only in its supply ($\theta_i \neq \theta_{i'}$), its price differs from the price of asset i only because $a_{2i} \neq a_{2i'}$. Since, in addition, $\theta_i < \theta_{i'}$, (3.3) implies that $a_{2i} > a_{2i'}$. Therefore, (A.1) implies that $S_{it} > S_{i't}$, and (A.2) implies that $\mathbb{E}(S_{it}) > \mathbb{E}(S_{i't})$.

Substituting a_{0i} from (3.1), we can write the share return (A.23) of asset i as

$$dR_{it} = \left\{[1-(r+\kappa)a_1]b_i s_t + [1-(r+\kappa)a_{2i}]e_{it}\right\}dt + a_1 b_i \sigma_s \sqrt{s_t} dw_{st} + a_{2i}\sigma_i \sqrt{e_{it}} dw_{it}. \tag{A.3}$$

The expected share return is

$$\begin{aligned}\mathbb{E}(dR_{it}) &= \left\{[1-(r+\kappa)a_1]b_i\bar{s} + [1-(r+\kappa)a_{2i}]\bar{e}_i\right\}dt \\ &= \left[b_i\bar{s} + \bar{e}_i - r\mathbb{E}(S_{it})\right]dt,\end{aligned} \tag{A.4}$$

where the second step follows from (A.2). Since $\mathbb{E}(S_{it}) > \mathbb{E}(S_{i't})$, (A.4) implies that $\mathbb{E}(dR_{it}) < \mathbb{E}(dR_{i't})$. The expected dollar return is

$$\mathbb{E}(dR^{\$}_{it}) = \mathbb{E}\left(\frac{dR_{it}}{\mathbb{E}(S_{it})}\right) = \frac{\mathbb{E}(dR_{it})}{\mathbb{E}(S_{it})} = \left[\frac{b_i\bar{s}+\bar{e}_i}{\mathbb{E}(S_{it})} - r\right]dt, \tag{A.5}$$

where the third step follows from (A.4). Since $b_i\bar{s} + \bar{e}_i > 0$ and $\mathbb{E}(S_{it}) > \mathbb{E}(S_{i't})$, (A.5) implies that $\mathbb{E}(dR^{\$}_{it}) < \mathbb{E}(dR^{\$}_{i't})$. □

Proof of Proposition 3.2. Equation (A.3) implies that the share return variance of asset i is

$$\begin{aligned}\mathbb{V}\text{ar}(dR_{it}) &= \mathbb{E}\left[(dR_{it})^2\right] - \left[\mathbb{E}(dR_{it})\right]^2 \\ &= \mathbb{E}\left[(dR_{it})^2\right] \\ &= \mathbb{E}\left[\left(a_1^2 b_i^2 \sigma_s^2 s_t + a_{2i}^2 \sigma_i^2 e_{it}\right)dt\right] \\ &= \left(a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i}^2 \sigma_i^2 \bar{e}_i\right)dt,\end{aligned} \tag{A.6}$$

where the second step follows because the term $\mathbb{E}\left[(dR_{it})^2\right]$ is of order dt and the term $\left[\mathbb{E}(dR_{it})\right]^2$ is of order $(dt)^2$. Since $a_{2i} > a_{2i'}$, (A.6) implies that $\mathbb{V}\text{ar}(dR_{it}) > \mathbb{V}\text{ar}(dR_{i't})$.

Equations (A.2) and (A.6) imply that the dollar return variance of asset i is

$$\mathbb{V}\text{ar}(dR_{it}^{\$}) = \mathbb{V}\text{ar}\left(\frac{dR_{it}}{\mathbb{E}(S_{it})}\right) = \frac{a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i}^2 \sigma_i^2 \bar{e}_i}{\frac{(r+\kappa)^2}{r^2}\left(a_1 b_i \bar{s} + a_{2i}\bar{e}_i\right)^2} dt. \tag{A.7}$$

Since asset i' differs from asset i only in its supply, (A.7) implies that $\mathbb{V}\text{ar}(dR_{it}^{\$}) > \mathbb{V}\text{ar}(dR_{i't}^{\$})$ if and only if

$$\begin{aligned}
&\frac{a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i}^2 \sigma_i^2 \bar{e}_i}{\left(a_1 b_i \bar{s} + a_{2i}\bar{e}_i\right)^2} > \frac{a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i'}^2 \sigma_i^2 \bar{e}_i}{\left(a_1 b_i \bar{s} + a_{2i'}\bar{e}_i\right)^2} \\
&\Leftrightarrow \left(a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i}^2 \sigma_i^2 \bar{e}_i\right)\left(a_1 b_i \bar{s} + a_{2i'}\bar{e}_i\right)^2 - \left(a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i'}^2 \sigma_i^2 \bar{e}_i\right)\left(a_1 b_i \bar{s} + a_{2i}\bar{e}_i\right)^2 > 0 \qquad \text{(A.8)} \\
&\Leftrightarrow (a_{2i} - a_{2i'})\left[a_1 b_i (a_{2i} + a_{2i'})(\sigma_i^2 \bar{s} - \sigma_s^2 \bar{e}_i) + 2(a_{2i} a_{2i'} \sigma_i^2 \bar{e}_i - a_1^2 b_i^2 \sigma_s^2 \bar{s})\right] > 0. \qquad \text{(A.9)}
\end{aligned}$$

Since $a_{2i} > a_{2i'}$, (A.9) is equivalent to (3.6). □

Proof of Proposition 3.3. Equation (A.3) implies that the share return of the index is

$$dR_{\eta t} = \left\{[1-(r+\kappa)a_1]\eta b s_t + \sum_{j=1}^{N}[1-(r+\kappa)a_{2i}]\eta_j e_{jt}\right\}dt + a_1 \eta b \sigma_s \sqrt{s_t} dw_{st} + \sum_{j=1}^{N} a_{2j}\eta_j \sigma_j \sqrt{e_{jt}} dw_{jt}. \tag{A.10}$$

Equations (3.9), (A.3) and (A.10) imply that the share beta of asset i is

$$\beta_i = \frac{\mathbb{C}\text{ov}(dR_{it}, dR_{\eta t})}{\mathbb{V}\text{ar}(dR_{\eta t})} = \frac{\mathbb{E}\left(dR_{it} dR_{\eta t}\right)}{\mathbb{E}\left[(dR_{\eta t})^2\right]} = \frac{a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i}{a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^{N} a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j}. \tag{A.11}$$

Since $a_{2i} > a_{2i'}$, (A.11) implies that $\beta_i > \beta_{i'}$.

Equation (A.2) implies that the expected share price of the index is

$$\mathbb{E}(\eta S_t) = \frac{r+\kappa}{r}\left(a_1 \eta b \bar{s} + \sum_{j=1}^{N} a_{2j}\eta_j \bar{e}_j\right). \tag{A.12}$$

Equations (3.10), (A.2), (A.11), and (A.12) imply that the dollar beta of asset i is

$$\beta_i^{\$} = \beta_i \frac{\mathbb{E}(S_{\eta t})}{\mathbb{E}(S_{it})} = \frac{a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i}{a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^{N} a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j} \frac{a_1 \eta b \bar{s} + \sum_{j=1}^{N} a_{2j}\eta_j \bar{e}_j}{a_1 b_i \bar{s} + a_{2i}\bar{e}_i}. \tag{A.13}$$

Equation (A.13) implies that $\beta_i^{\$} > \beta_{i'}^{\$}$ if and only if

$$\begin{aligned}
&\frac{a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i}{a_1 b_i \bar{s} + a_{2i}\bar{e}_i} > \frac{a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i'}^2 \eta_i \sigma_i^2 \bar{e}_i}{a_1 b_i \bar{s} + a_{2i'}\bar{e}_i} \\
&\Leftrightarrow \left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i\right)\left(a_1 b_i \bar{s} + a_{2i'}\bar{e}_i\right) - \left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i'}^2 \eta_i \sigma_i^2 \bar{e}_i\right)\left(a_1 b_i \bar{s} + a_{2i}\bar{e}_i\right) > 0 \qquad \text{(A.14)} \\
&\Leftrightarrow (a_{2i} - a_{2i'})\left[a_1 b_i (a_{2i} + a_{2i'})\eta_i \sigma_i^2 \bar{s} + a_{2i} a_{2i'} \eta_i \sigma_i^2 \bar{e}_i - a_1^2 b_i \eta b \sigma_s^2 \bar{s}\right] > 0. \qquad \text{(A.15)}
\end{aligned}$$

Since $a_{2i} > a_{2i'}$, (A.15) is equivalent to (3.11).

Equations (A.6), (A.10), and (A.11) imply that the idiosyncratic share return variance of asset i is

$$\begin{aligned}\mathbb{V}\text{ar}(d\epsilon_{it}) &= \mathbb{V}\text{ar}(dR_{it}) - \beta_i^2 \mathbb{V}\text{ar}(dR_{\eta t}) \\ &= \left[a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i}^2 \sigma_i^2 \bar{e}_i - \frac{\left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i\right)^2}{a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j} \right] dt. \end{aligned} \tag{A.16}$$

Equation (A.16) implies that $\mathbb{V}\text{ar}(d\epsilon_{it}) > \mathbb{V}\text{ar}(d\epsilon_{i't})$ if and only if

$$\begin{aligned} & a_{2i}^2 \sigma_i^2 \bar{e}_i - \frac{\left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i\right)^2}{a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j} > a_{2i'}^2 \sigma_i^2 \bar{e}_i - \frac{\left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i'}^2 \eta_i \sigma_i^2 \bar{e}_i\right)^2}{a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j} \\ \Leftrightarrow\ & (a_{2i}^2 - a_{2i'}^2) \sigma_i^2 \bar{e}_i \left(a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j \right) - \left[\left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i\right)^2 - \left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i'}^2 \eta_i \sigma_i^2 \bar{e}_i\right)^2 \right] > 0 \\ \Leftrightarrow\ & (a_{2i}^2 - a_{2i'}^2) \left[a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j - \eta_i \left(2 a_1^2 b_i \eta b \sigma_s^2 \bar{s} + (a_{2i}^2 + a_{2i'}^2) \eta_i \sigma_i^2 \bar{e}_i \right) \right] > 0. \end{aligned} \tag{A.17}$$

Eq. (A.17) holds because $a_{2i} > a_{2i'}$,

$$\begin{aligned} & (\eta b)^2 \geq (\eta_i b_i + \eta_{i'} b_{i'})(\eta b) = 2 \eta_i b_i \eta b, \\ & \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j \geq a_{2i}^2 \eta_i^2 \sigma_i^2 \bar{e}_i + a_{2i'}^2 \eta_{i'}^2 \sigma_{i'}^2 \bar{e}_{i'} = (a_{2i}^2 + a_{2i'}^2) \eta_i^2 \sigma_i^2 \bar{e}_i. \end{aligned}$$

Equations (A.2) and (A.16) imply that the idiosyncratic dollar return variance of asset i is

$$\mathbb{V}\text{ar}(d\epsilon_{it}^{\$}) = \mathbb{V}\text{ar}\left(\frac{d\epsilon_{it}}{\mathbb{E}(S_{it})} \right) = \frac{a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i}^2 \sigma_i^2 \bar{e}_i - \frac{\left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i\right)^2}{a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j}}{\frac{(r+\kappa)^2}{r^2} \left(a_1 b_i \bar{s} + a_{2i} \bar{e}_i\right)^2} dt. \tag{A.18}$$

Equation (A.18) implies that $\mathbb{V}\text{ar}(d\epsilon_{it}^{\$}) > \mathbb{V}\text{ar}(d\epsilon_{i't}^{\$})$ if and only if

$$\begin{aligned} & \frac{a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i}^2 \sigma_i^2 \bar{e}_i - \frac{\left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i\right)^2}{a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j}}{\left(a_1 b_i \bar{s} + a_{2i} \bar{e}_i\right)^2} > \frac{a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i'}^2 \sigma_i^2 \bar{e}_i - \frac{\left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i'}^2 \eta_i \sigma_i^2 \bar{e}_i\right)^2}{a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j}}{\left(a_1 b_i \bar{s} + a_{2i'} \bar{e}_i\right)^2} \\ \Leftrightarrow\ & \left(a_1^2 (\eta b)^2 \sigma_s^2 \bar{s} + \sum_{j=1}^N a_{2j}^2 \eta_j^2 \sigma_j^2 \bar{e}_j \right) \left[\left(a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i}^2 \sigma_i^2 \bar{e}_i\right) \left(a_1 b_i \bar{s} + a_{2i'} \bar{e}_i\right)^2 - \left(a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i'}^2 \sigma_i^2 \bar{e}_i\right) \left(a_1 b_i \bar{s} + a_{2i} \bar{e}_i\right)^2 \right] \\ & - \left[\left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i\right)^2 \left(a_1 b_i \bar{s} + a_{2i'} \bar{e}_i\right)^2 - \left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i'}^2 \eta_i \sigma_i^2 \bar{e}_i\right)^2 \left(a_1 b_i \bar{s} + a_{2i} \bar{e}_i\right)^2 \right] > 0. \end{aligned} \tag{A.19}$$

Using the same calculations as when deriving (A.9) from (A.8) and (A.15) from (A.14), we can write the right-hand side of (A.19) as $(a_{2i} - a_{2i'})\bar{e}_i$ times the right-hand side of (3.12). Since $a_{2i} > a_{2i'}$, (A.19) is equivalent to (3.12). □

Proof of Proposition 3.4. Since $\mathbb{E}(dR_{it}) < \mathbb{E}(dR_{i't})$ and $\beta_i > \beta_{i'}$, (3.15) implies that $\alpha_i < \alpha_{i'}$. Equations

(3.16), (A.2), (A.4), (A.12), and (A.13) imply that the dollar alpha of asset i is

$$\alpha_i^{\$} = \frac{r(b_i\bar{s}+\bar{e}_i)}{(r+\kappa)(a_1b_i\bar{s}+a_{2i}\bar{e}_i)} - r - \frac{a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i}{a_1^2(\eta b)^2\sigma_s^2\bar{s} + \sum_{j=1}^N a_{2j}^2\eta_j^2\sigma_j^2\bar{e}_j} \frac{r[1-(r+\kappa)a_1]\eta b\bar{s} + r\sum_{j=1}^N[1-(r+\kappa)a_{2j}]\eta_j\bar{e}_j}{(r+\kappa)(a_1b_i\bar{s}+a_{2i}\bar{e}_i)}. \tag{A.20}$$

Equation (A.20) implies that $\alpha_i^{\$} < \alpha_{i'}^{\$}$ if and only if

$$\begin{aligned}
&\frac{b_i\bar{s}+\bar{e}_i}{a_1b_i\bar{s}+a_{2i}\bar{e}_i} - \frac{a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i}{a_1^2(\eta b)^2\sigma_s^2\bar{s} + \sum_{j=1}^N a_{2j}^2\eta_j^2\sigma_j^2\bar{e}_j} \frac{[1-(r+\kappa)a_1]\eta b\bar{s} + \sum_{j=1}^N[1-(r+\kappa)a_{2j}]\eta_j\bar{e}_j}{a_1b_i\bar{s}+a_{2i}\bar{e}_i} \\
&< \frac{b_i\bar{s}+\bar{e}_i}{a_1b_i\bar{s}+a_{2i'}\bar{e}_i} - \frac{a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i'}^2 \eta_i \sigma_i^2 \bar{e}_i}{a_1^2(\eta b)^2\sigma_s^2\bar{s} + \sum_{j=1}^N a_{2j}^2\eta_j^2\sigma_j^2\bar{e}_j} \frac{[1-(r+\kappa)a_1]\eta b\bar{s} + \sum_{j=1}^N[1-(r+\kappa)a_{2j}]\eta_j\bar{e}_j}{a_1b_i\bar{s}+a_{2i'}\bar{e}_i} \\
&\Leftrightarrow (a_{2i}-a_{2i'})\bar{e}_i(b_i\bar{s}+\bar{e}_i) + \frac{[1-(r+\kappa)a_1]\eta b\bar{s} + \sum_{j=1}^N[1-(r+\kappa)a_{2j}]\eta_j\bar{e}_j}{a_1^2(\eta b)^2\sigma_s^2\bar{s} + \sum_{j=1}^N a_{2j}^2\eta_j^2\sigma_j^2\bar{e}_j} \\
&\times \left[\left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i}^2 \eta_i \sigma_i^2 \bar{e}_i\right)(a_1b_i\bar{s}+a_{2i'}\bar{e}_i) - \left(a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i'}^2 \eta_i \sigma_i^2 \bar{e}_i\right)(a_1b_i\bar{s}+a_{2i}\bar{e}_i)\right] > 0 \\
&\Leftrightarrow (a_{2i}-a_{2i'})\bar{e}_i \left[b_i\bar{s}+\bar{e}_i + \frac{[1-(r+\kappa)a_1]\,\eta b\bar{s} + \sum_{j=1}^N[1-(r+\kappa)a_{2j}]\,\eta_j\bar{e}_j}{a_1^2(\eta b)^2\sigma_s^2\bar{s} + \sum_{j=1}^N a_{2j}^2\eta_j^2\sigma_j^2\bar{e}_j}\right. \\
&\left.\times \left(a_1b_i(a_{2i}+a_{2i'})\eta_i\sigma_i^2\bar{s} + a_{2i}a_{2i'}\eta_i\sigma_i^2\bar{e}_i - a_1^2 b_i\eta b\sigma_s^2\bar{s}\right)\right] > 0.
\end{aligned} \tag{A.21}$$

Since $a_{2i} > a_{2i'}$, (A.21) is equivalent to

$$b_i\bar{s}+\bar{e}_i + \frac{[1-(r+\kappa)a_1]\,\eta b\bar{s} + \sum_{j=1}^N[1-(r+\kappa)a_{2j}]\,\eta_j\bar{e}_j}{a_1^2(\eta b)^2\sigma_s^2\bar{s} + \sum_{j=1}^N a_{2j}^2\eta_j^2\sigma_j^2\bar{e}_j}\left(a_1b_i(a_{2i}+a_{2i'})\eta_i\sigma_i^2\bar{s} + a_{2i}a_{2i'}\eta_i\sigma_i^2\bar{e}_i - a_1^2 b_i\eta b\sigma_s^2\bar{s}\right) > 0. \tag{A.22}$$

Since $\sum_{j=1}^N \eta_j\alpha_j = 0$, there exists i^* such that $\alpha_{i^*} \geq 0$ and $\alpha_{i^*}^{\$} \geq 0$. Equation (A.20) written for i^* implies that the left-hand side of (A.22) is larger than

$$\begin{aligned}
&(r+\kappa)(a_1b_i\bar{s}+a_{2i^*}\bar{e}_i) + \frac{[1-(r+\kappa)a_1]\,\eta b\bar{s} + \sum_{j=1}^N[1-(r+\kappa)a_{2j}]\,\eta_j\bar{e}_j}{a_1^2(\eta b)^2\sigma_s^2\bar{s} + \sum_{j=1}^N a_{2j}^2\eta_j^2\sigma_i^2\bar{e}_i} \\
&\times \left[a_1^2 b_i \eta b \sigma_s^2 \bar{s} + a_{2i^*}^2 \eta_i \sigma_i^2 \bar{e}_i + \left(a_1b_i(a_{2i}+a_{2i'})\eta_i\sigma_i^2\bar{s} + a_{2i}a_{2i'}\eta_i\sigma_i^2\bar{e}_i - a_1^2 b_i\eta b\sigma_s^2\bar{s}\right)\right] \\
&= (r+\kappa)(a_1b_i\bar{s}+a_{2i^*}\bar{e}_i) + \frac{[1-(r+\kappa)a_1]\,\eta b\bar{s} + \sum_{j=1}^N[1-(r+\kappa)a_{2j}]\,\eta_j\bar{e}_j}{a_1^2(\eta b)^2\sigma_s^2\bar{s} + \sum_{j=1}^N a_{2j}^2\eta_j^2\sigma_i^2\bar{e}_i} \\
&\times \left[a_1b_i(a_{2i}+a_{2i'})\eta_i\sigma_i^2\bar{s} + (a_{2i^*}^2 + a_{2i}a_{2i'})\eta_i\sigma_i^2\bar{e}_i\right] > 0.
\end{aligned}$$

Therefore, (A.22) holds, and so does (A.21). □

Proof of Theorem 4.1. We allow A to be zero so that the proof can also cover Theorem 3.1. The proof assumes $B = 0$, but when $A = 0$ the proof carries through unchanged to $B > 0$. We proceed in two steps:

- *Step 1:* We fix a contract (ϕ, χ, ψ) with $\phi \geq A$, and show that for the price function (2.14) and the coefficients (a_{0i}, a_{1i}, a_{2i}) given by (3.1), (4.1), and (4.2), $z_t = \theta$ solves the optimization problem of an employed manager. Hence, markets clear provided that the manager accepts the contract (ϕ, χ, ψ) and the investor invests $x = 0$ in the index.
- *Step 2:* We fix prices given by (2.14), (3.1), (4.1), and (4.2), and show that the investor decides to employ the manager, i.e., offer a contract that the manager accepts, and that the contract (ϕ, χ, ψ) given in Theorems 3.1 and 4.1, and the index investment $x = 0$, solve the investor's optimization problem. Hence, an equilibrium exists, and is as in the theorems.

Step 1. Substituting S_{it} from (2.14) into (2.2), and using (2.1), (2.3), and (2.4), we can write the excess return dR_{it} of asset i as

$$dR_{it} = \mu_{it}dt + a_{1i}\sigma_s\sqrt{s_t}dw_{st} + a_{2i}\sigma_i\sqrt{e_{it}}dw_{it}, \tag{A.23}$$

where

$$\mu_{it} \equiv (\kappa a_{1i}\bar{s} + \kappa a_{2i}\bar{e}_i - ra_{0i}) + [b_i - (r+\kappa)a_{1i}]s_t + [1 - (r+\kappa)a_{2i}]e_{it}. \tag{A.24}$$

We set $\mu_t \equiv (\mu_{1t}, .., \mu_{Nt})'$.

We conjecture that the value function of an employed manager takes the form

$$\bar{V}(\bar{W}_t, s_t, e_t) = -\exp\left[-\left(r\bar{\rho}\bar{W}_t + \bar{q}_0 + \bar{q}_1 s_t + \sum_{i=1}^{N}\bar{q}_{2i}e_{it}\right)\right], \tag{A.25}$$

where $(\bar{q}_0, \bar{q}_1, \bar{q}_{21}, .., \bar{q}_{2N})$ are constants. The manager's Bellman equation is

$$\max_{\bar{c}_t, z_t, m_t}\left[-\exp(-\bar{\rho}\bar{c}_t) + \mathcal{D}\bar{V}_t - \bar{\delta}\bar{V}_t\right] = 0, \tag{A.26}$$

where $\mathcal{D}\bar{V}_t$ is the drift of $\bar{V}_t$.

Using (2.3), (2.4), (2.5), (2.7), and (A.23), we find that the dynamics of $\bar{J}_t \equiv r\bar{\rho}\bar{W}_t + \bar{q}_0 + \bar{q}_1 s_t + \sum_{i=1}^{N}\bar{q}_{2i}e_{it}$ are

$$d\bar{J}_t = \bar{G}_t dt + \bar{H}_t dw_{st} + \sum_{i=1}^{N}\bar{K}_{it}dw_{it}, \tag{A.27}$$

where

$$\begin{aligned}
\bar{G}_t &\equiv r\bar{\rho}\left[r\bar{W}_t + (\phi z_t - \chi\eta)\mu_t + \psi + (A-\phi)m_t - \bar{c}_t\right] + \kappa\left[\bar{q}_1(\bar{s} - s_t) + \sum_{i=1}^{N}\bar{q}_{2i}(\bar{e}_i - e_{it})\right], \\
\bar{H}_t &\equiv \left[r\bar{\rho}\sum_{i=1}^{N}(\phi z_{it} - \chi\eta_i)a_{1i} + \bar{q}_1\right]\sigma_s\sqrt{s_t}, \\
\bar{K}_{it} &\equiv \left[r\bar{\rho}(\phi z_{it} - \chi\eta_i)a_{2i} + \bar{q}_{2i}\right]\sigma_i\sqrt{e_{it}}.
\end{aligned}$$

Using $\bar{V}(\bar{W}_t, s_t, e_t) = -\exp(\bar{J}_t)$, (A.27), and Ito's lemma, we find that the drift $\mathcal{D}\bar{V}_t$ of $\bar{V}_t$ is

$$\mathcal{D}\bar{V}_t = -\bar{V}_t \left(\bar{G}_t - \frac{1}{2}\bar{H}_t^2 - \frac{1}{2}\sum_{i=1}^{N} \bar{K}_{it}^2 \right). \tag{A.28}$$

Substituting into (A.26), we can write the manager's Bellman equation as

$$\max_{\bar{c}_t, z_t, m_t} \left[-\exp(-\bar{\rho}\bar{c}_t) - \bar{V}_t \left(\bar{G}_t - \frac{1}{2}\bar{H}_t^2 - \frac{1}{2}\sum_{i=1}^{N} \bar{K}_{it}^2 \right) - \bar{\delta}\bar{V}_t \right] = 0. \tag{A.29}$$

The first-order condition with respect to $\bar{c}_t$ is

$$\bar{\rho}\exp(-\bar{\rho}\bar{c}_t) + r\bar{\rho}\bar{V}_t = 0.$$

Using (A.25) to substitute for $\bar{V}_t$, and solving for $\bar{c}_t$, we find

$$\bar{c}_t = r\bar{W}_t + \frac{1}{\bar{\rho}} \left(\bar{q}_0 - \log(r) + \bar{q}_1 s_t + \sum_{i=1}^{N} \bar{q}_{2i} e_{it} \right). \tag{A.30}$$

The first-order condition with respect to m_t is

$$m_t = 0 \tag{A.31}$$

because $\phi \geq A$. (For $\phi > A$, the manager has a strict preference for $m_t = 0$. For $\phi = A$, the manager is indifferent between all values of m_t, and we assume that he chooses $m_t = 0$.) The first-order condition with respect to z_{it} is

$$r\bar{\rho}\phi\mu_{it} - r\bar{\rho}\phi a_{1i} \left[r\bar{\rho} \sum_{i=1}^{N} (\phi z_{it} - \chi\eta_i) a_{1i} + \bar{q}_1 \right] \sigma_s^2 s_t - r\bar{\rho}\phi a_{2i} \left[r\bar{\rho}(\phi z_{it} - \chi\eta_i) a_{2i} + \bar{q}_{2i} \right] \sigma_i^2 e_{it} = 0. \tag{A.32}$$

The portfolio $z_t = \theta$ solves the manager's optimization problem if (A.32) holds for $z_t = \theta$ and for all values of $(s_t, e_{1t}, .., e_{Nt})$. Substituting μ_{it} from (A.24), and dividing by $r\bar{\rho}\phi$ throughout, we can write (A.32) for $z_t = \theta$ as

$$A_{0i} + A_{1i} s_t + A_{2i} e_{it} = 0, \tag{A.33}$$

where

$$\begin{aligned}
A_{0i} &\equiv \kappa(a_{1i}\bar{s} + a_{2i}\bar{e}_i) - r a_{0i}, \\
A_{1i} &\equiv b_i - (r+\kappa) a_{1i} - a_{1i} \left[r\bar{\rho} \sum_{i=1}^{N} (\phi\theta_i - \chi\eta_i) a_{1i} + \bar{q}_1 \right] \sigma_s^2, \\
A_{2i} &\equiv 1 - (r+\kappa) a_{2i} - a_{2i} \left[r\bar{\rho}(\phi\theta_i - \chi\eta_i) a_{2i} + \bar{q}_{2i} \right] \sigma_i^2.
\end{aligned}$$

The left-hand side of (A.33) is an affine function of (s_t, e_{it}). Therefore, (A.32) holds for $z_t = \theta$ and for all

values of $(s_t, e_{1t}, .., e_{Nt})$ if $A_{0i} = A_{1i} = A_{2i} = 0$. Before linking these equations to the coefficients (a_{0i}, a_{1i}, a_{2i}) given in the proposition, we determine a set of additional equations that follow from the requirement that the manager's Bellman equation (A.29) holds. Using (A.30), (A.31), and (A.32) to substitute $\bar{c}_t$, m_t, and μ_{it}, we can write (A.29) for $z_t = \theta$ as

$$\bar{Q}_0 + \bar{Q}_1 s_t + \sum_{i=1}^{N} \bar{Q}_{2i} e_{it} = 0, \tag{A.34}$$

where

$$\begin{aligned}
\bar{Q}_0 &\equiv r\bar{q}_0 - r\bar{\rho}\psi - \kappa\left(\bar{q}_1\bar{s} + \sum_{i=1}^{N}\bar{q}_{2i}\bar{e}_i\right) + r - \bar{\delta} - r\log(r),\\
\bar{Q}_1 &\equiv (r+\kappa)\bar{q}_1 + \frac{1}{2}\bar{q}_1^2\sigma_s^2 - \frac{1}{2}\left[r\bar{\rho}\sum_{i=1}^{N}(\phi\theta_i - \chi\eta_i)a_{1i}\right]^2\sigma_s^2,\\
\bar{Q}_{2i} &\equiv (r+\kappa)\bar{q}_{2i} + \frac{1}{2}\bar{q}_{2i}^2\sigma_i^2 - \frac{1}{2}\left[r\bar{\rho}(\phi\theta_i - \chi\eta_i)a_{2i}\right]^2\sigma_i^2.
\end{aligned}$$

The left-hand side of (A.34) is an affine function of $(s_t, e_{1t}, .., e_{Nt})$. Therefore, (A.34) holds for $z_t = \theta$ and for all values of $(s_t, e_{1t}, .., e_{Nt})$ if $\bar{Q}_0 = \bar{Q}_1 = \bar{Q}_{21} = .. = \bar{Q}_{2N} = 0$.

We next show that equations $A_{0i} = A_{1i} = A_{2i} = 0$ and $\bar{Q}_0 = \bar{Q}_1 = \bar{Q}_{2i} = 0$ determine the coefficients $(a_{0i}, a_{1i}, a_{2i}, \bar{q}_0, \bar{q}_1, \bar{q}_{2i})$ uniquely, with (a_{0i}, a_{1i}, a_{2i}) being as in the proposition. This will imply that $z_t = \theta$ solves the manager's optimization problem given the prices in the proposition. Equation $A_{1i} = 0$ implies that $a_{1i} = a_1 b_i$, with a_1 being independent of i. Hence, $A_{1i} = 0$ can be replaced by $A_1 = 0$ with

$$A_1 \equiv 1 - (r+\kappa)a_1 - a_1\left[r\bar{\rho}(\phi\theta - \chi\eta)ba_1 + \bar{q}_1\right]\sigma_s^2.$$

Moreover, $\bar{Q}_1$ can be written as

$$\bar{Q}_1 = (r+\kappa)\bar{q}_1 + \frac{1}{2}\bar{q}_1^2\sigma_s^2 - \frac{1}{2}\left[r\bar{\rho}(\phi\theta - \chi\eta)ba_1\right]^2\sigma_s^2.$$

The quadratic equation $\bar{Q}_1 = 0$ has the unique positive root[11]

$$\bar{q}_1 = \frac{\sqrt{(r+\kappa)^2 + \left[r\bar{\rho}(\phi\theta - \chi\eta)ba_1\right]^2\sigma_s^4} - (r+\kappa)}{\sigma_s^2}. \tag{A.35}$$

[11]Holding wealth $\bar{W}_t$ constant, the manager is better off the larger s_t is. This is because with larger s_t, dividends are more volatile, and the manager must earn higher compensation in equilibrium for investing in the risky assets. (In the extreme case where volatility is zero, risky assets earn the riskless return r, and the manager derives no benefit from investing in them.) Because the manager's utility increases in s_t, the coefficient $\bar{q}_1$ must be positive. The coefficients $(\bar{q}_{21}, .., \bar{q}_{2N})$, and the counterparts of $(\bar{q}_1, \bar{q}_{21}, .., \bar{q}_{2N})$ in the investor's value function, must be positive for the same reason.

Substituting (A.35) into $A_1 = 0$, we find

$$\begin{aligned}
&1 - a_1^2 r\bar{\rho}(\phi\theta - \chi\eta)b\sigma_s^2 = a_1\sqrt{(r+\kappa)^2 + \left[r\bar{\rho}(\phi\theta - \chi\eta)ba_1\right]^2\sigma_s^4} \\
&\Rightarrow 1 - a_1^2\left[(r+\kappa)^2 + 2r\bar{\rho}(\phi\theta - \chi\eta)b\sigma_s^2\right] = 0 \\
&\Rightarrow a_1 = \frac{1}{\sqrt{(r+\kappa)^2 + 2r\bar{\rho}(\phi\theta - \chi\eta)b\sigma_s^2}}
\end{aligned} \tag{A.36}$$

where the second equation follows from the first by squaring both sides and simplifying. Eqs. $a_{1i} = b_i a_1$ and (A.36) coincide with (4.1). Substituting (A.36) into (A.35) we can determine $\bar{q}_1$:

$$\bar{q}_1 = \frac{(r+\kappa)^2 + r\bar{\rho}(\phi\theta - \chi\eta)b\sigma_s^2}{\sigma_s^2\sqrt{(r+\kappa)^2 + 2r\bar{\rho}(\phi\theta - \chi\eta)b\sigma_s^2}} - \frac{r+\kappa}{\sigma_s^2}. \tag{A.37}$$

Following the same procedure to solve the system of $A_{2i} = \bar{Q}_{2i} = 0$, we find (4.2) and

$$\bar{q}_{2i} = \frac{(r+\kappa)^2 + r\bar{\rho}(\phi\theta_i - \chi\eta_i)\sigma_i^2}{\sigma_i^2\sqrt{(r+\kappa)^2 + 2r\bar{\rho}(\phi\theta_i - \chi\eta_i)b\sigma_i^2}} - \frac{r+\kappa}{\sigma_i^2}. \tag{A.38}$$

Finally, $A_{0i} = 0$ implies (3.1), and $\bar{Q}_0 = 0$ implies

$$\bar{q}_0 = \bar{\rho}\psi + \frac{\kappa}{r}\left(\bar{q}_1\bar{s} + \sum_{i=1}^{N}\bar{q}_{2i}\bar{e}_i\right) - 1 + \frac{\bar{\delta}}{r} + \log(r). \tag{A.39}$$

Step 2. We conjecture that the value function of the investor when he employs the manager, offers contract $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$ that satisfies $\tilde{\phi} \geq A$ and can differ from the equilibrium contract (ϕ, χ, ψ), and invests x in the index, takes the form

$$V(W_t, s_t, e_t) = -\exp\left[-\left(r\rho W_t + q_0 + q_1 s_t + \sum_{i=1}^{N} q_{2i}e_{it}\right)\right], \tag{A.40}$$

where $(q_0, q_1, q_{21}, .., q_{2N})$ are constants. The investor's Bellman equation is

$$\max_{c_t}\left[-\exp(-\rho c_t) + \mathcal{D}V_t - \delta V_t\right] = 0, \tag{A.41}$$

where $\mathcal{D}V_t$ is the drift of V_t.

When the investor offers the equilibrium contract (ϕ, χ, ψ), the manager's first-order condition (A.32) is satisfied for $z_t = \theta$, as shown in Step 1. When the investor offers contract $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$ with $\tilde{\phi} \geq A$, (A.32) is satisfied for z_t given by

$$\begin{aligned}
&\tilde{\phi}z_{it} - \tilde{\chi}\eta_i = \phi\theta_i - \chi\eta_i \\
&\Rightarrow z_{it} = \frac{\phi\theta_i + (\tilde{\chi} - \chi)\eta_i}{\tilde{\phi}}.
\end{aligned} \tag{A.42}$$

This is because (A.32) depends on $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$ only through the quantity $\tilde{\phi}z_{it} - \tilde{\chi}\eta_i$: if $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$ changes, then z_{it} also changes in a way that $\tilde{\phi}z_{it} - \tilde{\chi}\eta_i$ is kept constant. The economic intuition is that the manager chooses

the fund's portfolio z_t to "undo" a change in contract: his personal risk exposure, through the fee, is the same under $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$ and (ϕ, χ, ψ). The manager's personal risk exposure arises through the fee's variable component, which is $(\tilde{\phi} z_t - \tilde{\chi}\eta) dR_t$ under $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$, and $(\phi\theta - \chi\eta) dR_t$ under (ϕ, χ, ψ). Eq. (A.42) relies on the assumption that the investor and the manager take asset prices as given and independent of the contract. Formally, the contract $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$ in (A.32) does not affect the price coefficients (a_{i0}, a_{1i}, a_{2i}). We drop the time subscript from the portfolio z_t in (A.42) because that portfolio is constant over time.

Using (2.3), (2.4), (2.5), (2.11), (A.23), $m_t = 0$ (which holds because $\tilde{\phi} \geq A$), and (A.42), we find that the dynamics of $J_t \equiv r\rho W_t + q_0 + q_1 s_t + \sum_{i=1}^{N} q_{2i} e_{it}$ are

$$dJ_t = G_t dt + H_t dw_{st} + \sum_{i=1}^{N} K_{it} dw_{it}, \tag{A.43}$$

where

$$\begin{aligned}
G_t &\equiv r\rho \left[rW_t + (x\eta + z - \phi\theta + \chi\eta)\mu_t - \tilde{\psi} - c_t \right] + \kappa \left[q_1(\bar{s} - s_t) + \sum_{i=1}^{N} q_{2i}(\bar{e}_i - e_{it}) \right], \\
H_t &\equiv \left[r\rho(x\eta + z - \phi\theta + \chi\eta) b a_1 + q_1 \right] \sigma_s \sqrt{s_t}, \\
K_{it} &\equiv \left[r\rho(x\eta_i + z - \phi\theta_i + \chi\eta_i) a_{2i} + q_{2i} \right] \sigma_i \sqrt{e_{it}}.
\end{aligned}$$

Proceeding as in Step 1, we can write the investor's Bellman equation (A.41) as

$$\max_{cb_t} \left[-\exp(-\rho c_t) - V_t \left(G_t - \frac{1}{2} H_t^2 - \frac{1}{2} \sum_{i=1}^{N} K_{it}^2 \right) - \delta V_t \right] = 0. \tag{A.44}$$

The first-order condition with respect to c_t is

$$\rho \exp(-\rho c_t) + r\rho V_t = 0,$$

and yields

$$c_t = rW_t + \frac{1}{\rho} \left(q_0 - \log(r) + q_1 s_t + \sum_{i=1}^{N} q_{2i} e_{it} \right). \tag{A.45}$$

Using (A.45) to substitute c_t, we can write (A.44) as

$$Q_0 + Q_1 s_t + \sum_{i=1}^{N} Q_{2i} e_{it} = 0, \tag{A.46}$$

where

$$
\begin{aligned}
Q_0 &\equiv rq_0 + r\rho\tilde{\psi} - \kappa\left(q_1\bar{s} + \sum_{i=1}^{N} q_{2i}\bar{e}_i\right) + r - \bar{\delta} - r\log(r),\\
Q_1 &\equiv (r+\kappa)q_1 + \frac{1}{2}q_1^2\sigma_s^2 - r\rho(x\eta + z - \phi\theta + \chi\eta)ba_1\\
&\quad\times\left[r\bar{\rho}(\phi\theta - \chi\eta)ba_1 - \frac{1}{2}r\rho(x\eta + z - \phi\theta + \chi\eta)ba_1 + \bar{q}_1 - q_1\right]\sigma_s^2,\\
Q_{2i} &\equiv (r+\kappa)q_{2i} + \frac{1}{2}q_{2i}^2\sigma_i^2 - r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)a_{2i}\\
&\quad\times\left[r\bar{\rho}(\phi\theta_i - \chi\eta_i)a_{2i} - \frac{1}{2}r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)a_{2i} + \bar{q}_{2i} - q_{2i}\right]\sigma_i^2.
\end{aligned}
$$

The left-hand side of (A.34) is an affine function of $(s_t, e_{1t}, .., e_{Nt})$. Therefore, (A.34) holds for all values of $(s_t, e_{1t}, .., e_{Nt})$ if $Q_0 = Q_1 = Q_{21} = .. = Q_{2N} = 0$. Using $A_1 = 0$ we can simplify Q_1 to

$$
\begin{aligned}
Q_1 &= (r+\kappa)q_1 + \frac{1}{2}q_1^2\sigma_s^2 + r\rho(x\eta + z - \phi\theta + \chi\eta)ba_1\\
&\quad\times\left[\frac{1}{2}r\rho(x\eta + z - \phi\theta + \chi\eta)ba_1\sigma_s^2 - q_1\sigma_s^2 + r + \kappa - \frac{1}{a_1}\right],
\end{aligned}
$$

and using $A_{2i} = 0$ we can simplify Q_{2i} to

$$
\begin{aligned}
Q_{2i} &= (r+\kappa)q_{2i} + \frac{1}{2}q_{2i}^2\sigma_i^2 + r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)a_{2i}\\
&\quad\times\left[\frac{1}{2}r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)a_{2i}\sigma_i^2 - q_{2i}\sigma_i^2 + r + \kappa - \frac{1}{a_{2i}}\right].
\end{aligned}
$$

Using the simplified expressions, we find that the positive root of $Q_1 = 0$ is

$$
q_1 = \frac{\sqrt{(r+\kappa)^2 + 2r\rho(x\eta + z - \phi\theta + \chi\eta)b\sigma_s^2} - (r+\kappa)}{\sigma_s^2} - r\rho(x\eta + z - \phi\theta + \chi\eta)ba_1, \tag{A.47}
$$

and the positive root of $Q_{2i} = 0$ is

$$
q_{2i} = \frac{\sqrt{(r+\kappa)^2 + 2r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)\sigma_i^2} - (r+\kappa)}{\sigma_i^2} - r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)a_{2i}. \tag{A.48}
$$

Moreover, $Q_0 = 0$ implies

$$
q_0 = -\rho\tilde{\psi} + \frac{\kappa}{r}\left(q_1\bar{s} + \sum_{i=1}^{N} q_{2i}\bar{e}_i\right) - 1 + \frac{\delta}{r} + \log(r). \tag{A.49}
$$

If the investor decides to employ the manager, then she chooses a contract $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$ and index investment x to maximize her time-zero value function $V(W_0, s_0, e_0)$. This objective is equivalent to $q_0 + q_1 s_0 +$

$\sum_{i=1}^{N} q_{2i} e_{i0}$ because of (A.40), and the latter objective is equivalent to

$$-\rho\tilde{\psi} + q_1 \hat{s}_0 + \sum_{i=1}^{N} q_{2i} \hat{e}_{i0} \tag{A.50}$$

because of (A.49). The maximization is subject to the manager's individual rationality (IR) constraint (2.9). To derive the time-zero value function $\bar{V}(\bar{W}_0, s_0, e_0)$ of an employed manager under a contract $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$, we recall from (A.42) that the contract does not affect the manager's personal risk exposure ($\tilde{\phi} z_t - \tilde{\chi}\eta = \phi\theta - \chi\eta$). The contract also does not affect the manager's shirking action m_t, which is zero because $\tilde{\phi} \geq A$. Hence, the value function is as in Step 1, i.e., as under the equilibrium contract (ϕ, χ, ψ), with $(\bar{q}_1, \bar{q}_{21}, .., \bar{q}_{2N})$ given by (A.37) and (A.38), and $\bar{q}_0$ given by

$$\bar{q}_0 = \bar{\rho}\tilde{\psi} + \frac{\kappa}{r}\left(\bar{q}_1 \bar{s} + \sum_{i=1}^{N} \bar{q}_{2i} \bar{e}_i\right) - 1 + \frac{\bar{\delta}}{r} + \log(r)$$

instead of (A.39). The time-zero value function $\bar{V}_u(\bar{W}_0, s_0, e_0)$ of an unemployed manager follows by the same argument. An unemployed manager can be viewed as an employed one with contract $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi}) = (1, 0, 0)$ and shirking action $m_t = 0$. Hence, the value function is as in Step 1, with $(\bar{q}_1, \bar{q}_{21}, .., \bar{q}_{2N})$ given by (A.37) and (A.38), and $\bar{q}_0$ given by

$$\bar{q}_0 = \frac{\kappa}{r}\left(\bar{q}_1 \bar{s} + \sum_{i=1}^{N} \bar{q}_{2i} \bar{e}_i\right) - 1 + \frac{\bar{\delta}}{r} + \log(r).$$

The manager's IR constraint (2.9) thus reduces to

$$\tilde{\psi} \geq 0.$$

The investor chooses $\tilde{\psi}$ that meets this constraint with equality: $\tilde{\psi} = 0$. Substituting into (A.50), we can write the investor's optimization problem as

$$\max_{\tilde{\phi}, \tilde{\chi}, x} \left(q_1 \hat{s}_0 + \sum_{i=1}^{N} q_{2i} \hat{e}_{i0}\right),$$

subject to the constraint $\tilde{\phi} \geq A$. Because this problem is concave, the first-order conditions characterize an optimum. To confirm that $(\tilde{\phi}, \tilde{\chi}, x) = (\phi, \chi, 0)$ is an optimum, we thus need to check that the first-order conditions are satisfied for $(\phi, \chi, 0)$. Equation (A.42) implies that

$$\left|\frac{\partial z}{\partial \tilde{\phi}}\right|_{(\tilde{\phi}, \tilde{\chi}) = (\phi, \chi)} = -\frac{\theta}{\phi}, \tag{A.51}$$

$$\left|\frac{\partial z}{\partial \tilde{\chi}}\right|_{(\tilde{\phi}, \tilde{\chi}) = (\phi, \chi)} = \frac{\eta}{\phi}. \tag{A.52}$$

Using (A.47), (A.48), (A.51), and (A.52), we find

$$\frac{\partial}{\partial\tilde{\phi}}\left(q_1\hat{s}_0+\sum_{i=1}^{N}q_{2i}\hat{e}_{i0}\right)\Bigg|_{(\tilde{\phi},\tilde{\chi},x)=(\phi,\chi,0)}=\frac{r\rho}{\phi}\left[\left(a_1-\check{a}_1\right)\theta b\hat{s}_0+\sum_{i=1}^{N}\left(a_{2i}-\check{a}_{2i}\right)\theta_i\hat{e}_{i0}\right]\equiv\frac{r\rho}{\phi}\Phi, \tag{A.53}$$

$$\frac{\partial}{\partial\tilde{\phi}}\left(q_1\hat{s}_0+\sum_{i=1}^{N}q_{2i}\hat{e}_{i0}\right)\Bigg|_{(\tilde{\phi},\tilde{\chi},x)=(\phi,\chi,0)}=-\frac{r\rho}{\phi}\left[\left(a_1-\check{a}_1\right)\eta b\hat{s}_0+\sum_{i=1}^{N}\left(a_{2i}-\check{a}_{2i}\right)\eta_i\hat{e}_{i0}\right]\equiv-\frac{r\rho}{\phi}X, \tag{A.54}$$

$$\frac{\partial}{\partial x}\left(q_1\hat{s}_0+\sum_{i=1}^{N}q_{2i}\hat{e}_{i0}\right)\Bigg|_{(\tilde{\phi},\tilde{\chi},x)=(\phi,\chi,0)}=-r\rho X. \tag{A.55}$$

The first-order conditions with respect to $\tilde{\chi}$ and x require that $X=0$, which is equivalent to (4.3). The first-order condition with respect to $\tilde{\phi}$ requires that Φ is non-positive if $\phi=A$ and is equal to zero if $\phi>A$. To show that the values of (ϕ,χ) implied by these conditions are as in Theorems 3.1 and 4.1, we first characterize the solution χ of (4.3) and then determine the sign of Φ.

Given $\phi\in[0,1]$, X is increasing in χ because a_1 is increasing in χ from (4.1), a_{2i} is increasing in χ from (4.2), $\check{a}_1$ is decreasing in χ from (4.4), and $\check{a}_{2i}$ is decreasing in χ from (4.5). It converges to ∞ when χ goes to

$$\bar{\chi}\equiv\min\left\{\left[\frac{(r+\kappa)^2}{2r\bar{\rho}\sigma_s^2}+\phi\theta b\right]\frac{1}{\eta b},\min_{i=1..,N}\left[\frac{(r+\kappa)^2}{2r\bar{\rho}\sigma_i^2}+\phi\theta_i\right]\frac{1}{\eta_i}\right\},$$

and to $-\infty$ when χ goes to

$$\underline{\chi}\equiv-\min\left\{\left[\frac{(r+\kappa)^2}{2r\bar{\rho}\sigma_s^2}+(1-\phi)\theta b\right]\frac{1}{\eta b},\min_{i=1..,N}\left[\frac{(r+\kappa)^2}{2r\bar{\rho}\sigma_i^2}+(1-\phi)\theta_i\right]\frac{1}{\eta_i}\right\}.$$

Therefore, (4.3) has a unique solution $\chi(\phi)$. Moreover, X is decreasing in ϕ because a_1 is decreasing in ϕ from (4.1), a_{2i} is decreasing in ϕ from (4.2), $\check{a}_1$ is increasing in ϕ from (4.4), and $\check{a}_{2i}$ is increasing in ϕ from (4.5). Therefore, $\chi(\phi)$ is increasing in ϕ. Since $X=0$ for $(\phi,\chi)=(\frac{\rho}{\rho+\bar{\rho}},0)$, $\chi(\phi)$ has the same sign as $\phi-\frac{\rho}{\rho+\bar{\rho}}$.

We next substitute $\chi(\phi)$ into Φ, and show property $(\mathcal{P})$: Φ has the same sign as $\frac{\rho}{\rho+\bar{\rho}}-\phi$. Property $(\mathcal{P})$ will imply that the values of (ϕ,χ) are as in Theorems 3.1 and 4.1. Indeed, when $A\leq\frac{\rho}{\rho+\bar{\rho}}$, Φ cannot be negative: the first-order condition with respect to $\tilde{\phi}$ would then imply that $\phi=A\leq\frac{\rho}{\rho+\bar{\rho}}$, and property $(\mathcal{P})$ would imply that Φ has to be non-negative. Therefore, $\Phi=0$, which implies $\phi=\frac{\rho}{\rho+\bar{\rho}}$ and $\chi(\phi)=0$. When instead $A>\frac{\rho}{\rho+\bar{\rho}}$, ϕ cannot be strictly larger than A: the first-order condition with respect to $\tilde{\phi}$ would then imply that $\Phi=0$, and property $(\mathcal{P})$ would imply that $\phi=\frac{\rho}{\rho+\bar{\rho}}<A$. Therefore, $\phi=A>\frac{\rho}{\rho+\bar{\rho}}$, which implies $\chi(\phi)>0$.

Setting $\Delta \equiv a_1 - \check{a}_1$ and $\Delta_i \equiv a_{2i} - \check{a}_{2i}$, we can write Φ and (4.3) as

$$\Phi = \Delta\theta b\hat{s}_0 + \sum_{i=1}^{N} \Delta_i \theta_i \hat{e}_{i0}, \tag{A.56}$$

$$\Delta\eta b\hat{s}_0 + \sum_{i=1}^{N} \Delta_i \eta_i \hat{e}_{i0} = 0, \tag{A.57}$$

Eqs. (4.1) and (4.4) imply that Δ has the same sign as

$$[\rho - (\rho + \bar{\rho})\phi]\theta b + (\rho + \bar{\rho})\chi\eta b.$$

Likewise, (4.2) and (4.5) imply that Δ_i has the same sign as

$$[\rho - (\rho + \bar{\rho})\phi]\theta_i + (\rho + \bar{\rho})\chi\eta_i.$$

For $\phi = \frac{\rho}{\rho+\bar{\rho}}$, $\chi(\phi) = 0$, and hence $\Delta = \Delta_i = 0$ and $\Phi = 0$. For $\phi < \frac{\rho}{\rho+\bar{\rho}}$,

$$\begin{aligned}
\Phi &= \Delta\theta b\hat{s}_0 + \sum_{i=1}^{N} \Delta_i \theta_i \hat{e}_{i0} \\
&> -\frac{(\rho + \bar{\rho})\chi(\phi)}{\rho - (\rho + \bar{\rho})\phi} \left(\Delta\eta b\hat{s}_0 + \sum_{i=1}^{N} \Delta_i \eta_i \hat{e}_{i0} \right) \\
&= 0,
\end{aligned}$$

where the second step follows by distinguishing cases according to the signs of Δ and Δ_i, and the third step follows from (A.57). The inequality in the second step is strict. This is because θ is not proportional to η, and hence the components of the vector $[\rho - (\rho + \bar{\rho})\phi]\theta + (\rho + \bar{\rho})\chi\eta$ cannot all be zero. For $\phi > \frac{\rho}{\rho+\bar{\rho}}$, the same reasoning implies that $\Phi < 0$. Therefore, property $(\mathcal{P})$ holds. Note that property $(\mathcal{P})$ implies that when $A > \frac{\rho}{\rho+\bar{\rho}}$, the investor values the supply portfolio more than the manager: $\theta\check{S}_0 > \theta S_0$. This is because $\theta\check{S}_0 - \theta S_0$ has the same sign as $-\Phi$, which is positive when $\phi > \frac{\rho}{\rho+\bar{\rho}}$.

Setting $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi}) = (\phi, \chi, \psi)$ in (A.47), (A.48), and (A.49), and using (A.42), we find that the coefficients q_1, q_{2i}, and q_0 when the investor offers the equilibrium contract (ϕ, χ, ψ) are

$$q_1 = \frac{\sqrt{(r+\kappa)^2 + 2r\rho((1-\phi)\theta + \chi\eta)b\sigma_s^2} - (r+\kappa)}{\sigma_s^2} - r\rho((1-\phi)\theta + \chi\eta)ba_1, \tag{A.58}$$

$$q_{2i} = \frac{\sqrt{(r+\kappa)^2 + 2r\rho((1-\phi)\theta_i + \chi\eta_i)\sigma_i^2} - (r+\kappa)}{\sigma_i^2} - r\rho((1-\phi)\theta_i + \chi\eta_i)a_{2i}, \tag{A.59}$$

$$q_0 = -\rho\psi + \frac{\kappa}{r}\left(q_1\bar{s} + \sum_{i=1}^{N} q_{2i}\bar{e}_i \right) - 1 + \frac{\delta}{r} + \log(r). \tag{A.60}$$

The investor decides to employ the manager if (2.13) is satisfied. To derive the time-zero value function $V_u(W_0, s_0, e_0)$ of the investor when he does not employ the manager, we can follow the same steps as when she employs the manager, but with two modifications. First, we replace $x\eta + z - \phi\theta + \chi\eta$ by $x\eta$ since the

investor's only exposure to the risky assets when he does not employ the manager is through the investment x in the index. Second, we replace $\tilde{\psi}$ by zero because the investor does not offer a contract. The value function is given by (A.40), with

$$q_{1u} \equiv \frac{\sqrt{(r+\kappa)^2 + 2r\rho x\eta b\sigma_s^2} - (r+\kappa)}{\sigma_s^2} - r\rho x\eta b a_1, \tag{A.61}$$

$$q_{2iu} = \frac{\sqrt{(r+\kappa)^2 + 2r\rho x\eta_i\sigma_i^2} - (r+\kappa)}{\sigma_i^2} - r\rho x\eta_i a_{2i}, \tag{A.62}$$

$$q_{0u} = \frac{\kappa}{r}\left(q_{1u}\bar{s} + \sum_{i=1}^{N} q_{2iu}\bar{e}_i\right) - 1 + \frac{\delta}{r} + \log(r), \tag{A.63}$$

instead of q_1, q_{2i}, and q_0, respectively. The investor's optimization problem is

$$\max_x \left[q_{1u}\hat{s}_0 + \sum_{i=1}^{N} q_{2iu}\hat{e}_{i0}\right].$$

The investor decides to employ the manager if

$$\max_{\tilde{\phi},\tilde{\chi},x}\left(q_1\hat{s}_0 + \sum_{i=1}^{N} q_{2i}\hat{e}_{i0}\right) > \max_x\left(q_{1u}\hat{s}_0 + \sum_{i=1}^{N} q_{2iu}\hat{e}_{i0}\right). \tag{A.64}$$

To show that (A.64) holds, we show that it holds when setting $(\tilde{\phi}, x) = (\phi, 0)$ in the left-hand side. Using (A.58), (A.59), (A.61), (A.62), and setting

$$f_1(y) \equiv \frac{\sqrt{(r+\kappa)^2 + 2r\rho y\sigma_s^2} - (r+\kappa)}{\sigma_s^2} - r\rho y a_1,$$
$$f_{2i}(y) \equiv \frac{\sqrt{(r+\kappa)^2 + 2r\rho y\sigma_i^2} - (r+\kappa)}{\sigma_i^2} - r\rho y a_{2i},$$

for a scalar y, we can write the latter condition as

$$\max_{\tilde{\chi}}\left[f_1\left((1-\phi)\theta b + \left(\frac{\tilde{\chi}-\chi}{\phi} + \chi\right)\eta b\right)\hat{s}_0 + \sum_{i=1}^{N} f_{2i}\left((1-\phi)\theta_i + \left(\frac{\tilde{\chi}-\chi}{\phi} + \chi\right)\eta_i\right)\hat{e}_{i0}\right]$$
$$> \max_x\left[f_1(x\eta b)\hat{s}_0 + \sum_{i=1}^{N} f_{2i}(x\eta_i)\hat{e}_{i0}\right]. \tag{A.65}$$

The function $f_1(y)$ is concave and maximized for y given by

$$\frac{1}{\sqrt{(r+\kappa)^2 + 2r\rho y\sigma_s^2}} - a_1 = 0$$
$$\Leftrightarrow y = \frac{\bar{\rho}}{\rho + \bar{\rho}}\theta b,$$

where the second step follows from (3.2). Likewise, the function $f_2(y)$ is concave and maximized for y given

by

$$\frac{1}{\sqrt{(r+\kappa)^2+2r\rho y\sigma_s^2}}-a_{2i}=0$$
$$\Leftrightarrow y=\frac{\bar{\rho}}{\rho+\bar{\rho}}\theta_i,$$

where the second step follows from (3.3). For any given x, we can write

$$(1-\phi)\theta+\left(\frac{\tilde{\chi}-\chi}{\phi}+\chi\right)\eta=\lambda\frac{\bar{\rho}}{\rho+\bar{\rho}}\theta+(1-\lambda)x\eta$$

by defining $(\lambda,\tilde{\chi})$ by

$$\lambda\frac{\bar{\rho}}{\rho+\bar{\rho}}\equiv 1-\phi,$$
$$(1-\lambda)x\equiv\frac{\tilde{\chi}-\chi}{\phi}+\chi.$$

Since $1\geq\phi\geq\frac{\rho}{\rho+\bar{\rho}}$, $\lambda\in[0,1]$. Therefore, the arguments of $f_1(y)$ and $f_2(y)$ in the left-hand side of (A.65) are convex combinations of the corresponding arguments in the right-hand side and of the maximands of $f_1(y)$ and $f_2(y)$. Concavity of $f_1(y)$ and $f_2(y)$ then implies that the values of $f_1(y)$ and $f_2(y)$ in the left-hand side of (A.65) exceed the corresponding values in the right-hand side. Moreover, at least one of the inequalities is strict. This is because θ is not proportional to η and hence the arguments of $f_1(y)$ and $f_2(y)$ in the right-hand side of (A.65) cannot all coincide with the maximands of $f_1(y)$ and $f_2(y)$. Therefore, (A.65) holds. □

Proof of Proposition 4.1. When $A>\frac{\rho}{\rho+\bar{\rho}}$, ϕ is equal to A and hence is increasing in A. Since $\chi(\phi)$ is increasing in ϕ, χ is also increasing in A. □

Proof of Proposition 4.2. We first compute the derivatives of $(a_1,a_{2i},\check{a}_1,\check{a}_{2i},S_{it},\mathbb{E}(S_{it}))$ with respect to A. Differentiating (4.1), (4.2), (4.4), and (4.5), and using $\phi=A$, we find

$$\frac{\partial a_1}{\partial A}=-r\bar{\rho}a_1^3\left(\theta-\frac{\partial\chi}{\partial A}\eta\right)b\sigma_s^2, \tag{A.66}$$
$$\frac{\partial a_{2i}}{\partial A}=-r\bar{\rho}a_{2i}^3\left(\theta_i-\frac{\partial\chi}{\partial A}\eta_i\right)\sigma_i^2, \tag{A.67}$$
$$\frac{\partial\check{a}_1}{\partial A}=r\rho\check{a}_1^3\left(\theta-\frac{\partial\chi}{\partial A}\eta\right)b\sigma_s^2, \tag{A.68}$$
$$\frac{\partial\check{a}_{2i}}{\partial A}=r\rho\check{a}_{2i}^3\left(\theta_i-\frac{\partial\chi}{\partial A}\eta_i\right)\sigma_i^2. \tag{A.69}$$

Differentiating (4.3) with respect to A, and using (A.66)-(A.69) and $(s_0,e_{10},..,e_{N0})=(\bar{s},\bar{e}_1,..,\bar{e}_N)$, we find

$$-(r+\kappa)\left(\bar{\rho}a_1^3+\rho\check{a}_1^3\right)\left(\theta-\frac{\partial\chi}{\partial A}\eta\right)b\sigma_s^2\eta b\bar{s}-(r+\kappa)\sum_{i=1}^{N}\left(\bar{\rho}a_{2i}^3+\rho\check{a}_{2i}^3\right)\left(\theta_i-\frac{\partial\chi}{\partial A}\eta_i\right)\sigma_i^2\eta_i\bar{e}_i=0$$
$$\Rightarrow\frac{\partial\chi}{\partial A}=\frac{\left(\bar{\rho}a_1^3+\rho\check{a}_1^3\right)\eta b\theta b\sigma_s^2\bar{s}+\sum_{i=1}^{N}\left(\bar{\rho}a_{2i}^3+\rho\check{a}_{2i}^3\right)\eta_i\theta_i\sigma_i^2\bar{e}_i}{\left(\bar{\rho}a_1^3+\rho\check{a}_1^3\right)(\eta b)^2\sigma_s^2\bar{s}+\sum_{i=1}^{N}\left(\bar{\rho}a_{2i}^3+\rho\check{a}_{2i}^3\right)\eta_i^2\sigma_i^2\bar{e}_i}. \tag{A.70}$$

Substituting $\frac{\partial \chi}{\partial A}$ from (A.70) into (A.66) and (A.67), we find

$$\frac{\partial a_1}{\partial A} = \frac{r\bar{\rho}a_1^3 \sum_{j=1}^N \left(\bar{\rho}a_{2j}^3 + \rho\check{a}_{2j}^3\right)\eta_j(\eta b\theta_j - \theta b\eta_j)\sigma_s^2\sigma_j^2\bar{e}_j}{\left(\bar{\rho}a_1^3 + \rho\check{a}_1^3\right)(\eta b)^2\sigma_s^2\bar{s} + \sum_{j=1}^N \left(\bar{\rho}a_{2j}^3 + \rho\check{a}_{2j}^3\right)\eta_j^2\sigma_j^2\bar{e}_j}, \tag{A.71}$$

$$\frac{\partial a_{2i}}{\partial A} = \frac{r\bar{\rho}\left(\bar{\rho}a_1^3 + \rho\check{a}_1^3\right)a_{2i}^3\eta b(\eta_i\theta b - \theta_i\eta b)\sigma_s^2\sigma_i^2\bar{s} + r\bar{\rho}a_{2i}^3\sum_{j=1}^N \left(\bar{\rho}a_{2j}^3 + \rho\check{a}_{2j}^3\right)\eta_j(\eta_i\theta_j - \theta_i\eta_j)\sigma_i^2\sigma_j^2\bar{e}_j}{\left(\bar{\rho}a_1^3 + \rho\check{a}_1^3\right)(\eta b)^2\sigma_s^2\bar{s} + \sum_{j=1}^N \left(\bar{\rho}a_{2j}^3 + \rho\check{a}_{2j}^3\right)\eta_j^2\sigma_j^2\bar{e}_j}. \tag{A.72}$$

Differentiating (A.1) with respect to A, we find that $\frac{\partial S_{it}}{\partial A}$ has the same sign as

$$\frac{\partial a_1}{\partial A}b_i\hat{s}_t + \frac{\partial a_{2i}}{\partial A}\hat{e}_{it}, \tag{A.73}$$

and $\frac{\partial \mathbb{E}(S_{it})}{\partial A}$ has the same sign as

$$\frac{\partial a_1}{\partial A}b_i\bar{s} + \frac{\partial a_{2i}}{\partial A}\bar{e}_i. \tag{A.74}$$

When $\sigma_s = 0$, (A.71)-(A.73) imply that $\frac{\partial S_{it}}{\partial A}$ is negative if

$$\frac{\theta_i}{\eta_i} > \frac{\sum_{j=1}^N \left(\bar{\rho}a_{2j}^3 + \rho\check{a}_{2j}^3\right)\eta_j\theta_j\sigma_j^2\bar{e}_j}{\sum_{j=1}^N \left(\bar{\rho}a_{2j}^3 + \rho\check{a}_{2j}^3\right)\eta_j^2\sigma_j^2\bar{e}_j}, \tag{A.75}$$

and is positive if (A.75) holds in the opposite direction. This establishes the threshold result in the proposition, with γ equal to the right-hand side of (A.75). Since γ is a weighted average of $\frac{\theta_j}{\eta_j}$ over $j \in \{1, .., N\}$ with the weights

$$\frac{\left(\bar{\rho}a_{2j}^3 + \rho\check{a}_{2j}^3\right)\eta_j^2\sigma_j^2\bar{e}_j}{\sum_{j'=1}^N \left(\bar{\rho}a_{2j'}^3 + \rho\check{a}_{2j'}^3\right)\eta_{j'}^2\sigma_{j'}^2\bar{e}_{j'}}$$

that are positive and sum to one, (A.75) holds for $i = \arg\max_{j\in\{1,..,N\}} \frac{\theta_j}{\eta_j}$, and the opposite inequality holds for $i = \arg\min_{j\in\{1,..,N\}} \frac{\theta_j}{\eta_j}$. Therefore, each inequality holds for a non-empty set of assets.

When $(b_i, \bar{e}_i, \sigma_i, \eta_i) = (b_c, \bar{e}_c, \sigma_c, \eta_c)$ for all i, we can write (A.71) and (A.72) as

$$\frac{\partial a_1}{\partial A} = \frac{r\bar{\rho}a_1^3 b_c\eta_c^2\sigma_s^2\sigma_c^2\bar{e}_c\sum_{j=1}^N \left(\bar{\rho}a_{2j}^3 + \rho\check{a}_{2j}^3\right)\left(N\theta_j - \sum_{j'=1}^N \theta_{j'}\right)}{\left(\bar{\rho}a_1^3 + \rho\check{a}_1^3\right)b_c^2\eta_c^2N^2\sigma_s^2\bar{s} + \sum_{j=1}^N \left(\bar{\rho}a_{2j}^3 + \rho\check{a}_{2j}^3\right)\eta_c^2\sigma_c^2\bar{e}_c}, \tag{A.76}$$

$$\frac{\partial a_{2i}}{\partial A} = \frac{r\bar{\rho}\left(\bar{\rho}a_1^3 + \rho\check{a}_1^3\right)a_{2i}^3 b_c^2\eta_c^2N\sigma_s^2\sigma_c^2\bar{s}\left(\sum_{j=1}^N \theta_j - N\theta_i\right) + r\bar{\rho}a_{2i}^3\eta_c^2\sigma_c^4\bar{e}_c\sum_{j=1}^N \left(\bar{\rho}a_{2j}^3 + \rho\check{a}_{2j}^3\right)(\theta_j - \theta_i)}{\left(\bar{\rho}a_1^3 + \rho\check{a}_1^3\right)b_c^2\eta_c^2N^2\sigma_s^2\bar{s} + \sum_{j=1}^N \left(\bar{\rho}a_{2j}^3 + \rho\check{a}_{2j}^3\right)\eta_c^2\sigma_c^2\bar{e}_c}, \tag{A.77}$$

respectively. We next show some properties of $(a_1, \check{a}_1)$. The expression $\bar{\rho}a_{2j}^3 + \rho\check{a}_{2j}^3$ decreases in θ_j because of (4.2), (4.5), and $\phi = A \in (\frac{\rho}{\rho+\bar{\rho}}, 1]$. Using this observation and denoting by $\bar{a}_2$ the value of $\bar{\rho}a_{2j}^3 + \rho\check{a}_{2j}^3$ for

$\bar{\theta} \equiv \frac{\sum_{j'=1}^{N} \theta_{j'}}{N}$, we find

$$\begin{aligned}
&\sum_{j=1}^{N} \left(\bar{\rho} a_{2j}^3 + \rho \check{a}_{2j}^3\right) \left(N\theta_j - \sum_{j'=1}^{N} \theta_{j'}\right) \\
&= \sum_{\theta_j \leq \bar{\theta}} \left(\bar{\rho} a_{2j}^3 + \rho \check{a}_{2j}^3\right) N(\theta_j - \bar{\theta}) + \sum_{\theta_j > \bar{\theta}} \left(\bar{\rho} a_{2j}^3 + \rho \check{a}_{2j}^3\right) N(\theta_j - \bar{\theta}) \\
&< \sum_{\theta_j \leq \bar{\theta}} \bar{a}_2 N(\theta_j - \bar{\theta}) + \sum_{\theta_j > \bar{\theta}} \bar{a}_2 N(\theta_j - \bar{\theta}) \\
&= \sum_{j=1}^{N} \bar{a}_2 N(\theta_j - \bar{\theta}) = 0. \qquad \text{(A.78)}
\end{aligned}$$

Equations (A.76) and (A.78) imply that $\frac{\partial a_1}{\partial A} < 0$. Since (A.66) and (A.68) imply that $\frac{\partial a_1}{\partial A}$ and $\frac{\partial \check{a}_1}{\partial A}$ have opposite signs, $\frac{\partial \check{a}_1}{\partial A} > 0$. Since, in addition, $a_1 = \check{a}_1$ for $A = \frac{\rho}{\rho+\bar{\rho}}$ (no frictions), $a_1 < \check{a}_1$ for $A \in (\frac{\rho}{\rho+\bar{\rho}}, 1]$. We next show analogous properties of $(a_{2i}, \check{a}_{2i})$ for $i = \arg\max_{j \in \{1,..,N\}} \theta_j$ and $i = \arg\min_{j \in \{1,..,N\}} \theta_j$. For $i = \arg\max_{j \in \{1,..,N\}} \theta_j$, (A.77) implies that $\frac{\partial a_{2i}}{\partial A} < 0$. Since (A.67) and (A.69) imply that $\frac{\partial a_{2i}}{\partial A}$ and $\frac{\partial \check{a}_{2i}}{\partial A}$ have opposite signs, $\frac{\partial \check{a}_{2i}}{\partial A} > 0$. Since, in addition, $a_{2i} = \check{a}_{2i}$ for $A = \frac{\rho}{\rho+\bar{\rho}}$, $a_{2i} < \check{a}_{2i}$ for $A \in (\frac{\rho}{\rho+\bar{\rho}}, 1]$. For $i = \arg\min_{j \in \{1,..,N\}} \theta_j$, (A.77) implies that $\frac{\partial a_{2i}}{\partial A} > 0$. Therefore, repeating the previous argument, we find $a_{2i} > \check{a}_{2i}$ for $A \in (\frac{\rho}{\rho+\bar{\rho}}, 1]$. Since, in addition, a_{2j} and $\check{a}_{2j}$ decrease in θ_j, $a_{2i} = \max\{a_{2j}, \check{a}_{2j}\}_{j=1,..,N}$.

For $i = \arg\max_{j \in \{1,..,N\}} \theta_j$, $\frac{\partial a_1}{\partial A} < 0$, $\frac{\partial a_{2i}}{\partial A} < 0$, and (A.73) imply that $\frac{\partial S_{it}}{\partial A} < 0$. For $i = \arg\min_{j \in \{1,..,N\}} \theta_j$, the second term in the numerator in (A.77) is positive. Denoting the numerator in (A.76) by N_1, and the first term in the numerator in (A.77) by N_2, the term $b_c \bar{s} N_1 + \bar{e}_c N_2$ is positive because

$$\begin{aligned}
&a_1^3 b_c^2 \eta_c^2 \sigma_s^2 \sigma_c^2 \bar{s} \bar{e}_c \sum_{j=1}^{N} \left(\bar{\rho} a_{2j}^3 + \rho \check{a}_{2j}^3\right) \left(N\theta_j - \sum_{j'=1}^{N} \theta_{j'}\right) + \left(\bar{\rho} a_1^3 + \rho \check{a}_1^3\right) a_{2i}^3 b_c^2 \eta_c^2 N \sigma_s^2 \sigma_c^2 \bar{s} \bar{e}_c \left(\sum_{j=1}^{N} \theta_j - N\theta_i\right) \\
&> a_1^3 b_c^2 \eta_c^2 \sigma_s^2 \sigma_c^2 \bar{s} \bar{e}_c \sum_{j=1}^{N} \left(\bar{\rho} a_{2j}^3 + \rho \check{a}_{2j}^3\right) \left(N\theta_j - \sum_{j'=1}^{N} \theta_{j'}\right) + (\bar{\rho} + \rho) a_1^3 a_{2i}^3 b_c^2 \eta_c^2 N \sigma_s^2 \sigma_c^2 \bar{s} \bar{e}_c \left(\sum_{j=1}^{N} \theta_j - N\theta_i\right) \\
&> a_1^3 b_c^2 \eta_c^2 \sigma_s^2 \sigma_c^2 \bar{s} \bar{e}_c \sum_{j=1}^{N} \left(\bar{\rho} a_{2j}^3 + \rho \check{a}_{2j}^3\right) \left(N\theta_j - \sum_{j'=1}^{N} \theta_{j'}\right) + a_1^3 b_c^2 \eta_c^2 \sigma_s^2 \sigma_c^2 \bar{s} \bar{e}_c \sum_{j=1}^{N} \left(\bar{\rho} a_{2j}^3 + \rho \check{a}_{2j}^3\right) \left(\sum_{j'=1}^{N} \theta_{j'} - N\theta_i\right) \\
&= a_1^3 b_c^2 \eta_c^2 \sigma_s^2 \sigma_c^2 \bar{s} \bar{e}_c \sum_{j=1}^{N} \left(\bar{\rho} a_{2j}^3 + \rho \check{a}_{2j}^3\right) \left(N\theta_j - \sum_{j'=1}^{N} \theta_{j'} + \sum_{j'=1}^{N} \theta_{j'} - N\theta_i\right) \\
&= a_1^3 b_c^2 \eta_c^2 \sigma_s^2 \sigma_c^2 \bar{s} \bar{e}_c N \sum_{j=1}^{N} \left(\bar{\rho} a_{2j}^3 + \rho \check{a}_{2j}^3\right) (\theta_j - \theta_i) > 0,
\end{aligned}$$

where the first step follows from $a_1 < \check{a}_1$ and the second from $a_{2i} = \max\{a_{2j}, \check{a}_{2j}\}_{j=1,..,N}$. Therefore, (A.74) implies that $\frac{\partial \mathbb{E}(S_{it})}{\partial A} > 0$.

The results on expected returns in both cases of the proposition follow by combining the results on prices (which translate to expected prices) with (A.4) and (A.5). □

Proof of Proposition 4.3. Differentiating (A.6) with respect to A, we find that $\frac{\partial \mathbb{V}\text{ar}(dR_{it})}{\partial A}$ has the same

sign as

$$a_1 \frac{\partial a_1}{\partial A} b_i^2 \sigma_s^2 \bar{s} + a_{2i} \frac{\partial a_{2i}}{\partial A} \sigma_i^2 \bar{e}_i. \tag{A.79}$$

Likewise, differentiating (A.7), we find that $\frac{\partial \mathbb{V}\mathrm{ar}(dR_{it}^{\$})}{\partial A}$ has the same sign as

$$\begin{aligned}
&\left(a_1 \frac{\partial a_1}{\partial A} b_i^2 \sigma_s^2 \bar{s} + a_{2i} \frac{\partial a_{2i}}{\partial A} \sigma_i^2 \bar{e}_i \right) (a_1 b_i \bar{s} + a_{2i} \bar{e}_i) - \left(a_1^2 b_i^2 \sigma_s^2 \bar{s} + a_{2i}^2 \sigma_i^2 \bar{e}_i \right) \left(\frac{\partial a_1}{\partial A} b_i \bar{s} + \frac{\partial a_{2i}}{\partial A} \bar{e}_i \right) \\
&= \left(a_1 \frac{\partial a_{2i}}{\partial A} - a_{2i} \frac{\partial a_1}{\partial A} \right) \left(a_{2i} \sigma_i^2 - a_1 b_i \sigma_s^2 \right) b_i \bar{s} \bar{e}_i.
\end{aligned} \tag{A.80}$$

When $\sigma_s = 0$, (A.72) and (A.79) imply that $\frac{\partial \mathbb{V}\mathrm{ar}(dR_{it})}{\partial A}$ is negative if (A.75) holds, and is positive if (A.75) holds in the opposite direction. Likewise, (A.71), (A.72), and (A.80) imply that $\frac{\partial \mathbb{V}\mathrm{ar}(dR_{it}^{\$})}{\partial A}$ is negative if (A.75) holds, and is positive if (A.75) holds in the opposite direction. These observations establish the threshold results in the proposition. The same argument as in the proof of Proposition 4.2 implies that each inequality holds for a non-empty set of assets.

Consider next the case $(b_i, \bar{e}_i, \sigma_i, \eta_i) = (b_c, \bar{e}_c, \sigma_c, \eta_c)$ for all i. For $i = \arg\max_{j \in \{1,..,N\}} \theta_j$, $\frac{\partial a_1}{\partial A} < 0$, $\frac{\partial a_{2i}}{\partial A} < 0$, and (A.79) imply that $\frac{\partial \mathbb{V}\mathrm{ar}(dR_{it})}{\partial A} < 0$. For $i = \arg\min_{j \in \{1,..,N\}} \theta_j$, the second term in the numerator in (A.77) is positive. The term $a_1 b_c^2 \sigma_s^2 \bar{s} N_1 + a_{2i} \sigma_i^2 \bar{e}_c N_2$ (where N_1 denotes the numerator in (A.76), and N_2 the first term in the numerator in (A.77)) is positive if (4.6) holds. This follows from $N_2 > 0$ and because $b_c \bar{s} N_1 + \bar{e}_c N_2$ is positive as shown in the proof of Proposition 4.2. Therefore, (A.79) implies that $\frac{\partial \mathbb{V}\mathrm{ar}(dR_{it})}{\partial A} > 0$ if (4.6) holds. Moreover, (A.80) implies that $\frac{\partial \mathbb{V}\mathrm{ar}(dR_{it}^{\$})}{\partial A} > 0$ if and only if (4.6) holds, because of $\frac{\partial a_1}{\partial A} < 0$ and $\frac{\partial a_{2i}}{\partial A} > 0$. □

Proof of Proposition 4.4. Differentiating (A.12) with respect to A, we find that $\frac{\partial \mathbb{E}(S_{\eta t})}{\partial A}$ has the same sign as

$$\frac{\partial a_1}{\partial A} \eta b \bar{s} + \sum_{i=1}^{N} \frac{\partial a_{2i}}{\partial A} \eta_i \bar{e}_i. \tag{A.81}$$

Equations (A.71) and (A.72) imply that (A.81) has the same sign as

$$\begin{aligned}
&a_1^3 \eta b \sigma_s^2 \bar{s} \sum_{i=1}^{N} \left(\bar{\rho} a_{2i}^3 + \rho \check{a}_{2i}^3 \right) \eta_i (\eta b \theta_i - \theta b \eta_i) \sigma_i^2 \bar{e}_i + \left(\bar{\rho} a_1^3 + \rho \check{a}_1^3 \right) \eta b \sigma_s^2 \bar{s} \sum_{i=1}^{N} a_{2j}^3 \eta_i (\eta_i \theta b - \theta_i \eta b) \sigma_i^2 \bar{e}_i \\
&+ \sum_{i=1}^{N} \sum_{j=1}^{N} a_{2i}^3 \left(\bar{\rho} a_{2j}^3 + \rho \check{a}_{2j}^3 \right) \eta_i \eta_j (\eta_i \theta_j - \theta_i \eta_j) \sigma_i^2 \sigma_j^2 \bar{e}_i \bar{e}_j \\
&= \rho \eta b \sigma_s^2 \bar{s} \sum_{i=1}^{N} \left(a_1^3 \check{a}_{2i}^3 - \check{a}_1^3 a_{2i}^3 \right) \eta_i (\eta b \theta_i - \theta b \eta_i) \sigma_i^2 \bar{e}_i + \rho \sum_{i=1}^{N} \sum_{j=1}^{N} a_{2i}^3 \check{a}_{2j}^3 \eta_i \eta_j (\eta_i \theta_j - \theta_i \eta_j) \sigma_i^2 \sigma_j^2 \bar{e}_i \bar{e}_j.
\end{aligned} \tag{A.82}$$

When $\sigma_s = 0$, the first term in (A.82) is zero. The second term can be written as

$$\rho\left(\sum_{i=1}^{N} a_{2i}^3 \eta_i^2 \sigma_i^2 \bar{e}_i\right)\left(\sum_{i=1}^{N} \check{a}_{2i}^3 \eta_i \theta_i \sigma_i^2 \bar{e}_i\right) - \rho\left(\sum_{i=1}^{N} a_{2i}^3 \eta_i \theta_i \sigma_i^2 \bar{e}_i\right)\left(\sum_{i=1}^{N} \check{a}_{2i}^3 \eta_i^2 \sigma_i^2 \bar{e}_i\right),$$

and has the same sign as

$$\frac{\sum_{i=1}^{N} \check{a}_{2i}^3 \eta_i \theta_i \sigma_i^2 \bar{e}_i}{\sum_{i=1}^{N} \check{a}_{2i}^3 \eta_i^2 \sigma_i^2 \bar{e}_i} - \frac{\sum_{i=1}^{N} a_{2i}^3 \eta_i \theta_i \sigma_i^2 \bar{e}_i}{\sum_{i=1}^{N} a_{2i}^3 \eta_i^2 \sigma_i^2 \bar{e}_i}. \tag{A.83}$$

Both fractions in (A.83) are weighted averages of $\frac{\theta_i}{\eta_i}$ over $i \in \{1, .., N\}$ with weights that are positive and sum to one. The weights are

$$\check{w}_i \equiv \frac{\check{a}_{2i}^3 \eta_i^2 \sigma_i^2 \bar{e}_i}{\sum_{j=1}^{N} \check{a}_{2j}^3 \eta_j^2 \sigma_j^2 \bar{e}_j}$$

for the first fraction, and

$$w_i \equiv \frac{a_{2i}^3 \eta_i^2 \sigma_i^2 \bar{e}_i}{\sum_{j=1}^{N} a_{2j}^3 \eta_j^2 \sigma_j^2 \bar{e}_j}$$

for the second fraction. When $(\sigma_i, \eta_i) = (\sigma_c, \eta_c)$ for all i, the ratio

$$\frac{w_i}{\check{w}_i} = \frac{a_{2i}^3}{\check{a}_{2i}^3} \frac{\sum_{j=1}^{N} a_{2j}^3 \eta_j^2 \sigma_j^2 \bar{e}_j}{\sum_{j=1}^{N} \check{a}_{2j}^3 \eta_j^2 \sigma_j^2 \bar{e}_j}$$

of weights depends on i only through θ_i. It also decreases in θ_i because (4.2), (4.5), and $\phi \in (\frac{\rho}{\rho+\bar{\rho}}, 1]$ imply that $\frac{a_{2i}^3}{\check{a}_{2i}^3}$ decreases in θ_i. Denote by i^* the asset that maximizes θ_i within the set of assets for which $w_{i^*} \geq \check{w}_{i^*}$. (That set is non-empty: $w_i < \check{w}_i$ for all $i \in \{1, .., N\}$ is not possible since $\sum_{i=1}^{N} w_i = \sum_{i=1}^{N} \check{w}_i = 1$.) Since $\frac{w_i}{\check{w}_i}$ decreases in θ_i, it is larger than one for $\theta_i < \theta_{i^*}$, and smaller than one for $\theta_i > \theta_{i^*}$. Using this property, we find

$$\begin{aligned}
\sum_{i=1}^{N} (\check{w}_i - w_i)\frac{\theta_i}{\eta_c} &= \sum_{\theta_i \leq \theta_{i^*}} (\check{w}_i - w_i)\frac{\theta_i}{\eta_c} + \sum_{\theta_i > \theta_{i^*}} (\check{w}_i - w_i)\frac{\theta_i}{\eta_c} \\
&= \sum_{\theta_i \leq \theta_{i^*}} \check{w}_i \left(1 - \frac{w_i}{\check{w}_i}\right)\frac{\theta_i}{\eta_c} + \sum_{\theta_i > \theta_{i^*}} \check{w}_i \left(1 - \frac{w_i}{\check{w}_i}\right)\frac{\theta_i}{\eta_c} \\
&> \sum_{\theta_i \leq \theta_{i^*}} \check{w}_i \left(1 - \frac{w_i}{\check{w}_i}\right)\frac{\theta_{i^*}}{\eta_c} + \sum_{\theta_i > \theta_{i^*}} \check{w}_i \left(1 - \frac{w_i}{\check{w}_i}\right)\frac{\theta_{i^*}}{\eta_c} \\
&= \sum_{\theta_i \leq \theta_{i^*}} (\check{w}_i - w_i)\frac{\theta_{i^*}}{\eta_c} + \sum_{\theta_i > \theta_{i^*}} (\check{w}_i - w_i)\frac{\theta_{i^*}}{\eta_c} \\
&= \sum_{i=1}^{N} (\check{w}_i - w_i)\frac{\theta_{i^*}}{\eta_c} = 0.
\end{aligned} \tag{A.84}$$

Therefore, (A.83) is positive and so is (A.82). When $(\sigma_i, \theta_i) = (\sigma_c, \theta_c)$ for all i, we can follow the same steps to show that (A.83) is again positive. The modifications are that $\frac{w_i}{\check{w}_i}$ and $\frac{\theta_i}{\eta_i}$ depend on i only through η_i

rather than only through θ_i, $\frac{w_i}{\bar{w}_i}$ increases in η_i rather than decreases in θ_i, and $\frac{\theta_i}{\eta_i}$ decreases in η_i rather than increases in θ_i.

When $(b_i, \bar{e}_i, \sigma_i, \eta_i) = (b_c, \bar{e}_c, \sigma_c, \eta_c)$ for all i, the second term in (A.82) is positive because of the previous argument. The first term can be written as

$$\begin{aligned}
&\rho b_c^2 \eta_c^3 N \sigma_s^2 \sigma_c^2 \bar{s}\bar{e}_c \sum_{i=1}^{N} \left(a_1^3 \check{a}_{2i}^3 - \check{a}_1^3 a_{2i}^3\right) \left(N\theta_i - \sum_{j=1}^{N} \theta_j\right), \\
&= \rho b_c^2 \eta_c^3 N \sigma_s^2 \sigma_c^2 \bar{s}\bar{e}_c \left[a_1^3 \sum_{i=1}^{N} \left(\check{a}_{2i}^3 - a_{2i}^3\right) \left(N\theta_i - \sum_{j=1}^{N} \theta_j\right) + (a_1^3 - \check{a}_1^3) \sum_{i=1}^{N} a_{2i}^3 \left(N\theta_i - \sum_{j=1}^{N} \theta_j\right)\right]. \quad \text{(A.85)}
\end{aligned}$$

The same argument as for equation (A.78) establishes that

$$\sum_{i=1}^{N} a_{2i}^3 \left(N\theta_i - \sum_{j=1}^{N} \theta_j\right) < 0.$$

Since, in addition, $a_1 < \check{a}_1$, the second term in the bracket in (A.85) is positive. When θ_i can take only two values, $a_{2i} > \check{a}_{2i}$ for the smaller value, and $a_{2i} < \check{a}_{2i}$ for the larger value, as shown in the proof of Proposition 4.2. Therefore, the first term in the bracket in (A.85) is positive, and so (A.82) is positive.

The results on expected returns in both cases of the proposition follow by combining the result on expected prices with (A.4) and (A.5). □

Proof of Theorem 5.1. We proceed in two steps, as in the proof of Theorem 4.1.

Step 1. Same as for Theorem 4.1, except that we do not impose the restriction $\phi \geq A$, and we replace $\bar{G}_t$, (A.31), and (A.39) by

$$\begin{aligned}
\bar{G}_t &\equiv r\bar{\rho}\left[r\bar{W}_t + (\phi z_t - \chi\eta)\mu_t + \psi + (A-\phi)m_t - \frac{B}{2}m_t^2 - \bar{c}_t\right] + \kappa\left[\bar{q}_1(\bar{s} - s_t) + \sum_{i=1}^{N} \bar{q}_{2i}(\bar{e}_i - e_{it})\right], \\
m_t &= \frac{A-\phi}{B} 1_{\{\phi \leq A\}}, \\
\bar{q}_0 &= \bar{\rho}\left(\psi + \frac{(A-\phi)^2}{2B} 1_{\{\phi \leq A\}}\right) + \frac{\kappa}{r}\left(\bar{q}_1 \bar{s} + \sum_{i=1}^{N} \bar{q}_{2i}\bar{e}_i\right) - 1 + \frac{\bar{\delta}}{r} + \log(r), \quad \text{(A.86)}
\end{aligned}$$

respectively, where $1_{\mathcal{S}}$ is the indicator function of the set $\mathcal{S}$.

Step 2. Same as for Theorem 4.1, with the following changes. We replace G_t, (A.49), and (A.60) by

$$G_t \equiv r\rho\left[rW_t + (x\eta + z - \phi\theta + \chi\eta)\mu_t - \tilde{\psi} - (1-\tilde{\phi})m_t - c_t\right] + \kappa\left[q_1(\bar{s} - s_t) + \sum_{i=1}^{N} q_{2i}(\bar{e}_i - e_{it})\right],$$

$$q_0 = -\rho\left(\tilde{\psi} + \frac{(1-\tilde{\phi})(A-\tilde{\phi})}{B}1_{\{\tilde{\phi}\leq A\}}\right) + \frac{\kappa}{r}\left(q_1\bar{s} + \sum_{i=1}^{N} q_{2i}\bar{e}_i\right) - 1 + \frac{\delta}{r} + \log(r), \tag{A.87}$$

$$q_0 = -\rho\left(\psi + \frac{(1-\phi)(A-\phi)}{B}1_{\{\phi\leq A\}}\right) + \frac{\kappa}{r}\left(q_1\bar{s} + \sum_{i=1}^{N} q_{2i}\bar{e}_i\right) - 1 + \frac{\delta}{r} + \log(r), \tag{A.88}$$

respectively. The manager's individual rationality constraint becomes

$$\tilde{\psi} + \frac{(A-\tilde{\phi})^2}{2B}1_{\{\tilde{\phi}\leq A\}} \geq 0,$$

and the investor chooses $\tilde{\psi} = -\frac{(A-\tilde{\phi})^2}{2B}1_{\{\tilde{\phi}\leq A\}}$. The investor's optimization problem becomes

$$\max_{\tilde{\phi},\tilde{\chi},x}\left(-\rho\frac{(A-\tilde{\phi})(2-A-\tilde{\phi})}{2B}1_{\{\tilde{\phi}\leq A\}} + q_1\hat{s}_0 + \sum_{i=1}^{N} q_{2i}\hat{e}_{i0}\right),$$

without the constraint $\tilde{\phi} \geq A$. The first-order conditions with respect to $\tilde{\chi}$ and x are equivalent to (4.3). Using (A.53), we find that the first-order condition with respect to $\tilde{\phi}$ is

$$\frac{1-\phi}{B} + \frac{r}{\phi}\Phi = 0 \quad \text{if} \quad \phi < A, \tag{A.89}$$

$$\frac{1-\phi}{B} + \frac{r}{\phi}\Phi \geq 0 \quad \text{and} \quad \Phi \leq 0 \quad \text{if} \quad \phi = A, \tag{A.90}$$

$$\Phi = 0 \quad \text{if} \quad \phi > A. \tag{A.91}$$

Equations (A.89)-(A.91) rule out that Φ is positive. When $A \leq \frac{\rho}{\rho+\bar{\rho}}$, Φ cannot be negative: (A.91) would then imply that $\phi \leq A \leq \frac{\rho}{\rho+\bar{\rho}}$, and property $(\mathcal{P})$ would imply that Φ has to be non-negative. Therefore, $\Phi = 0$, which implies $\phi = \frac{\rho}{\rho+\bar{\rho}}$ and $\chi(\phi) = 0$. When instead $A > \frac{\rho}{\rho+\bar{\rho}}$, ϕ cannot be strictly larger than A: (A.91) would then imply that $\Phi = 0$, and property $(\mathcal{P})$ would imply that $\phi = \frac{\rho}{\rho+\bar{\rho}} < A$. Moreover, ϕ cannot be smaller than $\frac{\rho}{\rho+\bar{\rho}}$: (A.89) would then imply that $\Phi < 0$, and property $(\mathcal{P})$ would imply that $\phi > \frac{\rho}{\rho+\bar{\rho}}$. Therefore, $\phi \in (\frac{\rho}{\rho+\bar{\rho}}, A]$, which implies $\chi(\phi) > 0$. Equations (A.89) and (A.90) yield (5.1) and (5.2). The condition that the investor decides to employ the manager becomes

$$\max_{\tilde{\phi},\tilde{\chi},x}\left(-\rho\frac{(A-\tilde{\phi})(2-A-\tilde{\phi})}{2B}1_{\{\tilde{\phi}\leq A\}} + q_1\hat{s}_0 + \sum_{i=1}^{N} q_{2i}\hat{e}_{i0}\right) > \max_{x}\left(q_{1u}\hat{s}_0 + \sum_{i=1}^{N} q_{2iu}\hat{e}_{i0}\right) \tag{A.92}$$

instead of (A.64). Equation (A.92) is satisfied for $B = 0$, as shown in Theorem 4.1. It is also satisfied for $B = \infty$ because the left-hand side of (A.92) becomes identical to that of (A.64). By continuity, it is satisfied for B close to zero and to infinity. □

Proof of Proposition 5.1. The social planner maximizes the investor's value function $V(W_0, s_0, e_0)$ at

time zero, subject to the manager's incentive compatibility (IC) and individual rationality (IR) constraints. The IC constraint is that the manager's choices of (z_t, m_t) are optimal given the contract. The IR constraint is that the manager's value function $\bar{V}(\bar{W}_0, s_0, e_0)$ exceeds the value function $\bar{V}_u(\bar{W}_0, s_0, e_0)$ from being unemployed. From Steps 1 and 2 of the proof of Theorem 5.1, the social planner's problem reduces to maximizing

$$r\rho W_0 + q_0 + q_1 s_0 + \sum_{i=1}^{N} q_{2i} e_{i0}$$

subject to

$$r\bar{\rho}\bar{W}_0 + \bar{q}_0 + \bar{q}_1 s_0 + \sum_{i=1}^{N} \bar{q}_{2i} e_{i0} \geq \bar{q}_{0u} + \bar{q}_{1u} s_0 + \sum_{i=1}^{N} \bar{q}_{2iu} e_{i0},$$

where $(q_0, q_1, q_{21}, .., q_{2N})$ are given by (A.58), (A.59), and (A.88), $(\bar{q}_0, \bar{q}_1, \bar{q}_{21}, .., \bar{q}_{2N})$ are given by (A.37), (A.38), and (A.86), and $(\bar{q}_{0u}, \bar{q}_{1u}, \bar{q}_{21u}, .., \bar{q}_{2Nu})$ are the counterparts of $(\bar{q}_0, \bar{q}_1, \bar{q}_{21}, .., \bar{q}_{2N})$ for an unemployed manager. The values of $(\bar{q}_{0u}, \bar{q}_{1u}, \bar{q}_{21u}, .., \bar{q}_{2Nu})$ computed in Theorem 5.1 depend on (ϕ, χ). (In particular, $(\bar{q}_{1u}, \bar{q}_{21u}, .., \bar{q}_{2Nu}) = (\bar{q}_1, \bar{q}_{21}, .., \bar{q}_{2N})$.) This is because the manager computes his value function when unemployed under the equilibrium prices, which depend on the contract. The values of $(\bar{q}_{0u}, \bar{q}_{1u}, \bar{q}_{21u}, .., \bar{q}_{2Nu})$ computed by the social planner, however, do not depend on (ϕ, χ). This is because the social planner internalizes that when the manager is unemployed, prices change and do not depend on the contract.

Using (A.86) and (A.88), we can write the social planner's problem as

$$\max_{\phi,\chi} \left[r(W_0 + \bar{W}_0) - \frac{(A-\phi)(2-A-\phi)}{2B} 1_{\{\phi \leq A\}} + \frac{1}{\rho}\left(q_1 \hat{s}_0 + \sum_{i=1}^{N} q_{2i} \hat{e}_{i0} \right) + \frac{1}{\bar{\rho}}\left(\bar{q}_1 \hat{s}_0 + \sum_{i=1}^{N} \bar{q}_{2i} \hat{e}_{i0} \right) \right]. \quad \text{(A.93)}$$

Since the investor and the manager are endowed collectively with the portfolio θ at time zero, the problem (A.93) is equivalent to

$$\max_{\phi,\chi} \left[r\theta S_0 - \frac{(A-\phi)(2-A-\phi)}{2B} 1_{\{\phi \leq A\}} + \frac{1}{\rho}\left(q_1 \hat{s}_0 + \sum_{i=1}^{N} q_{2i} \hat{e}_{i0} \right) + \frac{1}{\bar{\rho}}\left(\bar{q}_1 \hat{s}_0 + \sum_{i=1}^{N} \bar{q}_{2i} \hat{e}_{i0} \right) \right]. \quad \text{(A.94)}$$

Using (A.58), (A.59),

$$\bar{q}_1 = \frac{\sqrt{(r+\kappa)^2 + 2r\bar{\rho}(\phi\theta - \chi\eta) b \sigma_s^2} - (r+\kappa)}{\sigma_s^2} - r\bar{\rho}(\phi\theta - \chi\eta) b a_1, \quad \text{(A.95)}$$

which follows from (4.1) and (A.37),

$$\bar{q}_{2i} = \frac{\sqrt{(r+\kappa)^2 + 2r\bar{\rho}(\phi\theta_i - \chi\eta_i) \sigma_i^2} - (r+\kappa)}{\sigma_i^2} - r\bar{\rho}(\phi\theta_i - \chi\eta_i) a_{2i}, \quad \text{(A.96)}$$

which follows from (4.2) and (A.38), and

$$S_{i0} = a_1 b_i \hat{s}_0 + a_{2i} \hat{e}_{i0},$$

which follows from (A.1), we can write (A.94) as

$$\max_{\phi,\chi}\left[-\frac{(A-\phi)(2-A-\phi)}{2B}1_{\{\phi\le A\}}\right.$$
$$+\frac{1}{\rho}\left(\frac{\sqrt{(r+\kappa)^2+2r\rho((1-\phi)\theta+\chi\eta)b\sigma_s^2}}{\sigma_s^2}\hat{s}_0+\sum_{i=1}^{N}\frac{\sqrt{(r+\kappa)^2+2r\rho((1-\phi)\theta_i+\chi\eta_i)\sigma_i^2}}{\sigma_i^2}\hat{e}_{i0}\right)$$
$$\left.+\frac{1}{\bar{\rho}}\left(\frac{\sqrt{(r+\kappa)^2+2r\bar{\rho}(\phi\theta-\chi\eta)b\sigma_s^2}}{\sigma_s^2}\hat{s}_0+\sum_{i=1}^{N}\frac{\sqrt{(r+\kappa)^2+2r\bar{\rho}(\phi\theta_i-\chi\eta_i)\sigma_i^2}}{\sigma_i^2}\hat{e}_{i0}\right)\right]. \tag{A.97}$$

The first-order condition with respect to χ is (4.3). The first-order condition with respect to ϕ is

$$\frac{1-\phi}{B}+r\Phi=0 \quad \text{if} \quad \phi<A, \tag{A.98}$$
$$\frac{1-\phi}{B}+r\Phi\ge 0 \quad \text{and} \quad \Phi\le 0 \quad \text{if} \quad \phi=A, \tag{A.99}$$
$$\Phi=0 \quad \text{if} \quad \phi>A. \tag{A.100}$$

Since (A.97) is strictly concave, the first-order conditions characterize a unique maximum (ϕ^*,χ^*). Using the same arguments as in the proof of Theorem 5.1, we find that (ϕ^*,χ^*,ψ^*) are as in the theorem.

We finally show that $\phi^*\ge\phi$ and $\chi^*\ge\chi$ for the privately optimal (ϕ,χ), with the inequalities being strict when $\phi<A$. Since χ^* solves (4.3), it is equal to $\chi(\phi^*)$ for the function $\chi(\phi)$ defined in the proof of Theorem 4.1. Since $\chi(\phi)$ is increasing in ϕ, it suffices to show the inequalities for ϕ^*. When $\phi<A$, ϕ is determined by (A.89). Using (A.89), we can write the left-hand side of (A.98) as

$$\frac{1-\phi}{B}-\frac{\phi(1-\phi)}{B}=\frac{(1-\phi)^2}{B}>0.$$

Since the derivative of the social planner's objective with respect to ϕ is positive at the privately optimal ϕ, $\phi^*>\phi$. When $\phi=A$, ϕ satisfies (A.90). Using (A.90), we find that the left-hand side of (A.98) is larger than $\frac{(1-\phi)^2}{B}>0$. Since the derivative of the social planner's objective with respect to ϕ is positive at the privately optimal $\phi=A$, $\phi^*=\phi=A$. □

Proof of Theorem 6.1. We follow the same steps as in the proof of Theorem 4.1.

Step 1. Equation (A.23) is replaced by

$$dR_{it}=\mu_{it}dt+a_{1i}\sigma_s dw_{st}+a_{2i}\sigma_i dw_{it}, \tag{A.101}$$

which can be derived by substituting S_{it} from (2.14) into (2.2), and using (2.1), (6.1), and (6.2). The manager's Bellman equation (A.29) remains the same, except that $(\bar{H}_t,\bar{K}_{it})$ are replaced by

$$\bar{H}_t\equiv\left[r\bar{\rho}\sum_{i=1}^{N}(\phi z_{it}-\chi\eta_i)a_{1i}+\bar{q}_1\right]\sigma_s,$$
$$\bar{K}_{it}\equiv\left[r\bar{\rho}(\phi z_{it}-\chi\eta_i)a_{2i}+\bar{q}_{2i}\right]\sigma_i.$$

The first-order condition (A.32) is replaced by

$$r\bar{\rho}\phi\mu_{it} - r\bar{\rho}\phi a_{1i}\left[r\bar{\rho}\sum_{i=1}^{N}(\phi z_{it} - \chi\eta_i)a_{1i} + \bar{q}_1\right]\sigma_s^2 - r\bar{\rho}\phi a_{2i}\left[r\bar{\rho}(\phi z_{it} - \chi\eta_i)a_{2i} + \bar{q}_{2i}\right]\sigma_i^2 = 0. \tag{A.102}$$

The terms $(A_{0i}, A_{1i}, A_{2i}, \bar{Q}_0, \bar{Q}_1, \bar{Q}_{2i})$ are replaced by

$$\begin{aligned}
A_{0i} &\equiv \kappa(a_{1i}\bar{s} + a_{2i}\bar{e}_i) - ra_{0i} - a_{1i}\left[r\bar{\rho}\sum_{i=1}^{N}(\phi\theta_i - \chi\eta_i)a_{1i} + \bar{q}_1\right]\sigma_s^2 - \sum_{i=1}^{N} a_{2i}\left[r\bar{\rho}(\phi\theta_i - \chi\eta_i)a_{2i} + \bar{q}_{2i}\right]\sigma_i^2,\\
A_{1i} &\equiv b_i - (r+\kappa)a_{1i},\\
A_{2i} &\equiv 1 - (r+\kappa)a_{2i},\\
\bar{Q}_0 &\equiv r\bar{q}_0 - r\bar{\rho}\psi - \kappa\left(\bar{q}_1\bar{s} + \sum_{i=1}^{N}\bar{q}_{2i}\bar{e}_i\right) + r - \bar{\delta} - r\log(r)\\
&\quad - \frac{1}{2}\left[r\bar{\rho}\sum_{i=1}^{N}(\phi\theta_i - \chi\eta_i)a_{1i}\right]^2\sigma_s^2 - \sum_{i=1}^{N}\frac{1}{2}\left[r\bar{\rho}(\phi\theta_i - \chi\eta_i)a_{2i}\right]^2\sigma_i^2,\\
\bar{Q}_1 &\equiv (r+\kappa)\bar{q}_1 + \frac{1}{2}\bar{q}_1^2\sigma_s^2,\\
\bar{Q}_{2i} &\equiv (r+\kappa)\bar{q}_{2i} + \frac{1}{2}\bar{q}_{2i}^2\sigma_i^2.
\end{aligned}$$

Setting $\bar{Q}_1 = \bar{Q}_{2i} = 0$ yields $\bar{q}_1 = \bar{q}_{2i} = 0$. Setting $A_{0i} = A_{1i} = A_{2i} = 0$ yields (6.3)-(6.5). Setting $\bar{Q}_0 = 0$, and using (6.4) and (6.5), yields

$$\bar{q}_0 = \bar{\rho}\psi - 1 + \frac{\bar{\delta}}{r} + \log(r) + \frac{r\bar{\rho}^2\left[\left[(\phi\theta - \chi\eta)b\right]^2\sigma_s^2 + \sum_{i=1}^{N}(\phi\theta_i - \chi\eta_i)\sigma_i^2\right]}{2(r+\kappa)^2}. \tag{A.103}$$

Step 2. The investor's Bellman equation (A.44) remains the same, except that (H_t, K_{it}) are replaced by

$$\begin{aligned}
H_t &\equiv \left[r\rho(x\eta + z - \phi\theta + \chi\eta)ba_1 + q_1\right]\sigma_s,\\
K_{it} &\equiv \left[r\rho(x\eta_i + z - \phi\theta_i + \chi\eta_i)a_{2i} + q_{2i}\right]\sigma_i.
\end{aligned}$$

The terms (Q_0, Q_1, Q_{2i}) are replaced by

$$\begin{aligned}
Q_0 &\equiv rq_0 + r\rho\tilde{\psi} - \kappa\left(q_1\bar{s} + \sum_{i=1}^{N}q_{2i}\bar{e}_i\right) + r - \delta - r\log(r)\\
&\quad - r\rho(x\eta + z - \phi\theta + \chi\eta)ba_1\left[r\bar{\rho}(\phi\theta - \chi\eta)ba_1 - \frac{1}{2}r\rho(x\eta + z - \phi\theta + \chi\eta)ba_1 + \bar{q}_1 - q_1\right]\sigma_s^2\\
&\quad - \sum_{i=1}^{N} r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)a_{2i}\left[r\bar{\rho}(\phi\theta_i - \chi\eta_i)a_{2i} - \frac{1}{2}r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)a_{2i} + \bar{q}_{2i} - q_{2i}\right]\sigma_i^2,\\
Q_1 &\equiv (r+\kappa)q_1 + \frac{1}{2}q_1^2\sigma_s^2,\\
Q_{2i} &\equiv (r+\kappa)q_{2i} + \frac{1}{2}q_{2i}^2\sigma_i^2.
\end{aligned}$$

Setting $Q_1 = Q_{2i} = 0$ yields $q_1 = q_{2i} = 0$. Setting $Q_0 = 0$, and using (6.4) and (6.5), yields

$$\begin{aligned} q_0 = &-\rho\tilde{\psi} - 1 + \frac{\delta}{r} + \log(r) \\ &+ \frac{r\rho(x\eta + z - \phi\theta + \chi\eta)b\left[\bar{\rho}(\phi\theta - \chi\eta)b - \frac{1}{2}\rho(x\eta + z - \phi\theta + \chi\eta)b\right]\sigma_s^2}{(r+\kappa)^2} \\ &+ \frac{\sum_{i=1}^N r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)\left[\bar{\rho}(\phi\theta_i - \chi\eta_i) - \frac{1}{2}\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)\right]\sigma_i^2}{(r+\kappa)^2}. \end{aligned} \tag{A.104}$$

If the investor decides to employ the manager, then she chooses a contract $(\tilde{\phi}, \tilde{\chi}, \tilde{\psi})$ and index investment x to maximize $q_0 + q_1 s_0 + \sum_{i=1}^N q_{2i} e_{i0}$. Using $q_1 = q_{2i} = 0$ and (A.104), and noting that the investor chooses $\tilde{\psi} = 0$ to meet the manager's IR constraint $\tilde{\psi} \geq 0$ with equality, we can write the investor's optimization problem as

$$\begin{aligned} \max_{\tilde{\phi},\tilde{\chi},x} \Bigg\{ &(x\eta + z - \phi\theta + \chi\eta)b\left[\bar{\rho}(\phi\theta - \chi\eta)b - \frac{1}{2}\rho(x\eta + z - \phi\theta + \chi\eta)b\right]\sigma_s^2 \\ &+ \sum_{i=1}^N r\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)\left[\bar{\rho}(\phi\theta_i - \chi\eta_i) - \frac{1}{2}\rho(x\eta_i + z_i - \phi\theta_i + \chi\eta_i)\right]\sigma_i^2 \Bigg\}, \end{aligned}$$

subject to the constraint $\tilde{\phi} \geq A$. The partial derivatives of the investor's objective with respect to $(\tilde{\phi}, \tilde{\chi}, x)$ at $(\tilde{\phi}, \tilde{\chi}, x) = (\phi, \chi, 0)$ are

$$\frac{[\rho - (\rho+\bar{\rho})\phi]\left[(\theta b)^2\sigma_s^2 + \sum_{i=1}^N \theta_i^2\sigma_i^2\right] + (\rho+\bar{\rho})\chi\left[\eta b\theta b\sigma_s^2 + \sum_{i=1}^N \eta_i\theta_i\sigma_i^2\right]}{\phi}, \tag{A.105}$$

$$-\frac{[\rho - (\rho+\bar{\rho})\phi]\left[\eta b\theta b\sigma_s^2 + \sum_{i=1}^N \eta_i\theta_i\sigma_i^2\right] + (\rho+\bar{\rho})\chi\left[(\eta b)^2\sigma_s^2 + \sum_{i=1}^N \eta_i^2\sigma_i^2\right]}{\phi}, \tag{A.106}$$

$$-[\rho - (\rho+\bar{\rho})\phi]\left[\eta b\theta b\sigma_s^2 + \sum_{i=1}^N \eta_i\theta_i\sigma_i^2\right] - (\rho+\bar{\rho})\chi\left[(\eta b)^2\sigma_s^2 + \sum_{i=1}^N \eta_i^2\sigma_i^2\right], \tag{A.107}$$

respectively. Because the investor's optimization problem is concave, $(\tilde{\phi}, \tilde{\chi}, x) = (\phi, \chi, 0)$ is an optimum if (A.106) and (A.107) are equal to zero, and if (A.105) is non-positive when $\phi = A$ and is equal to zero when $\phi > A$. Setting (A.106) and (A.107) to zero yields (6.7). Using (6.7), we can write (A.105) as

$$[\rho - (\rho+\bar{\rho})\phi]\frac{\left[(\eta b)^2\sigma_s^2 + \sum_{i=1}^N \eta_i^2\sigma_i^2\right]\left[(\theta b)^2\sigma_s^2 + \sum_{i=1}^N \theta_i^2\sigma_i^2\right] - \left[\eta b\theta b\sigma_s^2 + \sum_{i=1}^N \eta_i\theta_i\sigma_i^2\right]^2}{\phi\left[(\eta b)^2\sigma_s^2 + \sum_{i=1}^N \eta_i^2\sigma_i^2\right]}. \tag{A.108}$$

Since θ is not proportional to η, the Cauchy-Schwarz inequality implies that the denominator in (A.108) is positive. Therefore, (A.108) has the same sign as $\rho - (\rho+\bar{\rho})\phi$. Property $(\mathcal{P})$ in the proof of Theorem 4.1 is thus satisfied, and the same argument as in that proof implies that ϕ is given by (6.6). □

Proof of Proposition 6.1. Equation (6.3) implies that $a_{0i} > a_{0i'}$. Since, in addition, (6.4) and (6.5) imply that $(a_{1i}, a_{2i}) = (a_{1i'}, a_{2i'})$, $S_{it} > S_{i't}$. Equations (A.23), which remains valid under OU processes,

and $\mathbb{E}(S_{it}) > \mathbb{E}(S_{i't})$ imply that $\mathbb{E}(dR_{it}) < \mathbb{E}(dR_{i't})$. Proceeding as in the derivation of (A.6), we find

$$\mathbb{V}\text{ar}(dR_{it}) = \left(a_1^2 b_i^2 \sigma_s^2 + a_{2i}^2 \sigma_i^2\right) dt = \frac{b_i^2 \sigma_s^2 + \sigma_i^2}{(r+\kappa)^2} dt, \tag{A.109}$$

where the second step follows from (6.4) and (6.5). Therefore, $\mathbb{V}\text{ar}(dR_{it}) = \mathbb{V}\text{ar}(dR_{i't})$. □

Proof of Proposition 6.2. Equations (6.3)-(6.5), (6.7), and $\phi = A$ imply that

$$\frac{\partial S_{it}}{\partial A} = \frac{\bar{\rho}\left[b_i \sum_{j=1}^N \eta_j (\eta b \theta_j - \theta b \eta_j)\sigma_s^2 \sigma_j^2 + \eta b(\eta_i \theta b - \theta_i \eta b)\sigma_s^2 \sigma_i^2 + \sum_{j=1}^N \eta_j (\eta_i \theta_j - \eta_j \theta_i)\sigma_i^2 \sigma_j^2\right]}{(r+\kappa)^2\left[(\eta b)^2 \sigma_s^2 + \sum_{i=1}^N \eta_i^2 \sigma_i^2\right]}. \tag{A.110}$$

When $\sigma_s = 0$, the result follows as in the proof of Proposition 4.2. When $(b_i, \bar{e}_i, \sigma_i, \eta_i) = (b_c, \bar{e}_c, \sigma_c, \eta_c)$ for all i, the first term in the numerator in (A.110) is zero. The second and third terms are negative for $i = \arg\max_{j \in \{1,..,N\}} \theta_j$, and positive for $i = \arg\min_{j \in \{1,..,N\}} \theta_j$. The results on expected returns in both cases follow by combining the results on prices with (A.4). The share return variance of asset i is given by (A.109) and hence does not depend on A. Multiplying (A.110) by η_i and summing across i, we find

$$\frac{\partial S_{\eta t}}{\partial A} = \frac{\bar{\rho}\left[\eta b \sum_{i=1}^N \eta_i (\eta b \theta_i - \theta b \eta_i)\sigma_s^2 \sigma_i^2 + \eta b \sum_{i=1}^N \eta_i (\eta_i \theta b - \theta_i \eta b)\sigma_s^2 \sigma_i^2 + \sum_{i=1}^N \sum_{j=1}^N \eta_i \eta_j (\eta_i \theta_j - \eta_j \theta_i)\sigma_i^2 \sigma_j^2\right]}{(r+\kappa)^2\left[(\eta b)^2 \sigma_s^2 + \sum_{i=1}^N \eta_i^2 \sigma_i^2\right]}. \tag{A.111}$$

Since the first two terms in (A.111) cancel and the third term is zero, $\frac{\partial S_{\eta t}}{\partial A} = 0$. □

References

Admati, Anat, and Paul Pfleiderer, 1997, Does it all add up? Benchmarks and the compensation of active portfolio managers, *Journal of Business* 70, 323–350.

Ang, Andrew, Robert Hodrick, Yuhang Xing, and Xiaoyan Zhang, 2006, The cross-section of volatility and expected returns, *Journal of Finance* 61, 259–299.

Baker, Malcolm, Brendan Bradley, and Jeffrey Wurgler, 2011, Benchmarks as limits to arbitrage: Understanding the low-volatility anomaly, *Financial Analysts Journal* 67.

Basak, Suleyman, and Anna Pavlova, 2013, Asset prices and institutional investors, *American Economic Review* 103, 1728–1758.

Berk, Jonathan, and Richard Green, 2004, Mutual fund flows and performance in rational markets, *Journal of Political Economy* 112, 1269–1295.

Bhattacharya, Sudipto, and Paul Pfleiderer, 1985, Delegated portfolio management, *Journal of Economic Theory* 36, 1–25.

Black, Fischer, 1972, Capital market equilibrium with restricted borrowing, *Journal of Business* 45, 444–55.

——— , Michael Jensen, and Myron Scholes, 1972, The capital asset pricing model: Some empirical tests, in Michael Jensen, ed.: *Studies in the Theory of Capital Markets* . pp. 79–121 (New York: Praeger).

Brennan, Michael, 1993, Agency and asset pricing, Working paper 1147 UCLA.

Christoffersen, Susan E.K., and Mikhail Simutin, 2014, On the demand for high-beta stocks: Evidence from mutual funds, Working paper University of Toronto.

Cohen, Randolph, Christopher Polk, and Tuomo Vuolteenaho, 2005, Money illusion in the stock market: The Modigliani-Cohn hypothesis, *Quarterly Journal of Economics* 120, 639–668.

Cuoco, Domenico, and Ron Kaniel, 2011, Equilibrium prices in the presence of delegated portfolio management, *Journal of Financial Economics* 101, 264–296.

Das, Sanjiv Ranjan, and Rangarajan K. Sundaram, 2002, Fee speech: Signaling, risk-sharing, and the impact of fee structures on investor welfare, *Review of Financial Studies* 15, 1465–1497.

Dasgupta, Amil, and Andrea Prat, 2008, Information aggregation in financial markets with career concerns, *Journal of Economic Theory* 143, 83–113.

——— , and Michela Verardo, 2011, The price impact of institutional herding, *Review of Financial Studies* 24, 892–925.

Dybvig, Philip H., Heber K. Farnsworth, and Jennifer N. Carpenter, 2010, Portfolio performance and agency, *Review of Financial Studies* 23, 1–23.

Fama, Eugene, and Kenneth French, 2007, Disagreement, tastes, and asset prices, *Journal of Financial Economics* 83, 667–689.

Frazzini, Andrea, and Lasse Heje Pedersen, 2014, Betting against beta, *Journal of Financial Economics* 111, 1–25.

French, Kenneth R, 2008, Presidential address: The cost of active investing, *Journal of Finance* 63, 1537–1573.

Garcia, Diego, and Joel M. Vanden, 2009, Information acquisition and mutual funds, *Journal of Economic Theory* 144, 1965–1995.

Guerrieri, Veronica, and Peter Kondor, 2012, Fund managers, career concerns, and asset price volatility, *American Economic Review* 102, 1986–2017.

Haugen, Robert, and Nardin Baker, 1996, Commonality in the determinant of expected stock returns, *Journal of Financial Economics* 41, 401–439.

He, Zhiguo, and Arvind Krishnamurthy, 2012, A model of capital and crises, *Review of Economic Studies* 79, 735–777.

——— , 2013, Intermediary asset pricing, *American Economic Review* 103, 1–43.

Hong, Harrison, and David A. Sraer, 2013, Speculative betas, Working paper Princeton University.

Kaniel, Ron, and Peter Kondor, 2013, The delegated Lucas tree, *Review of Financial Studies* 26, 929–984.

Karceski, Jason, 2002, Returns-chasing behavior, mutual funds, and beta's death, *Journal of Financial and Quantitative Analysis* 37, 559–594.

Li, Wei, and Ashish Tiwari, 2009, Incentive contracts in delegated portfolio management, *Review of Financial Studies* 22, 4681–4714.

Malamud, Semyon, and Evgeny Petrov, 2014, Portfolio delegation and market efficiency, Working paper Swiss Finance Institute.

Ou-Yang, Hui, 2003, Optimal contracts in a continuous-time delegated portfolio management problem, *Review of Financial Studies* 16, 173–208.

Palomino, Frederic, and Andrea Prat, 2003, Risk taking and optimal contracts for money managers, *RAND Journal of Economics* 34, 113–137.

Philippon, Thomas, 2008, The evolution of the US financial industry from 1860 to 2007: Theory and evidence, Working paper New York University.

Qiu, Zhigang, 2014, An institutional REE model with relative performance, Working paper Renmin University.

Shleifer, Andrei, and Robert Vishny, 1997, The limits of arbitrage, *Journal of Finance* 52, 35–55.

Starks, Laura, 1987, Performance incentive fees: An agency theoretic approach, *Journal of Financial and Quantitative Analysis* 22, 17–32.

Stoughton, Neal M., 1993, Moral hazard and the portfolio management problem, *Journal of Finance* 48, 2009–2028.

Vayanos, Dimitri, 2004, Flight to quality, flight to liquidity, and the pricing of risk, Working paper London School of Economics.

——— , and Paul Woolley, 2013, An institutional theory of momentum and reversal, *Review of Financial Studies* 26, 1087–1145.