Timing Decisions in Organizations: Communication and Authority in a Dynamic Environment*

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Abstract

We consider a problem in which an uninformed principal repeatedly solicits advice from an informed but biased agent on when to exercise an option. This problem is common in firms: examples include headquarters deciding when to shut down an underperforming division, drill an oil well, or launch a new product. We show that equilibria are different from those in the static "cheap talk" setting. When the agent has a bias for late exercise, full communication of information often occurs, but communication and option exercise are inefficiently delayed. In contrast, when the agent is biased towards early exercise, communication is partial, while exercise is either unbiased or delayed. Given the same absolute bias, the principal is better off when the agent has a delay bias. Next, we consider delegation as an alternative to centralized decision-making with communication. If the agent favors late exercise, delegation is always weakly inferior. In contrast, if the agent is biased towards early exercise, delegation is optimal if the bias is low. Thus, it is not optimal to delegate decisions with a late exercise bias, such as plant closures, but may be optimal to delegate decisions such as product launches.

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1 Introduction

In organizations, it is common for a decision-maker to seek advice from an agent on when to take a certain action. It is also not uncommon for the agent to be better informed but biased in terms of timing. Consider the following three examples of such settings. 1) In a typical hierarchical firm, top executives may rely on the advice of a product manager to determine the timing of the launch of a new generation of a product. It would not be surprising for an empire-building product manager to be biased in favor of an earlier launch. 2) The CEO of a multinational corporation seeks advice from a local plant manager about when to shut down a plant in a struggling economic region. The plant manager is better informed about the prospects of the plant, but is biased towards a later shutdown due to his personal costs of relocation. 3) Emerging companies seek advice from investment bankers as to when to take their firms public. While the banker is better informed about the prospects for an IPO, he is also likely to be biased towards an earlier IPO due to fees.

All three of these examples share a common theme. An uninformed decision-maker faces an optimal stopping-time problem (when to exercise a real option) and gets advice from an informed but biased expert. Since contracts that specify payments for advice are often infeasible, the decision-maker must rely on informal communication with the agent ("cheap talk"). In this paper, we provide a theoretical analysis of such a setting. We show that the economics underlying this problem are quite different from those when the decision is static rather than dynamic, and the decision variable is scale of the action rather than a stopping time. In particular, there is a large asymmetry in the equilibrium properties of communication and decision-making depending on the direction of the agent's bias. For example, in the first and third examples above, the agent is biased in favor of early exercise, while in the second example above the agent is biased in the direction of later exercise. Unlike in the static problem (e.g., Crawford and Sobel, 1982), the results for these two cases are not mirror images of each other. As we discuss below, this has implications for the choice between centralized and decentralized decision-making and for the value of commitment in organizations based on the direction of the agent's bias. For example, within our framework, there is no benefit from delegating decisions for which the agent has a preference for later exercise, such as plant closures, as opposed to decisions for which the agent has a preference for earlier exercise. Since most decisions that firms make have option-like features as they can be delayed, our results are important for understanding the economics of firms.

Our setting combines the framework of optimal stopping time problems under continuous-time uncertainty with the framework of cheap talk communication between an agent and a principal. The principal must choose when to exercise an option whose payoff depends on an unknown parameter. The agent knows the parameter, but the agent's payoff from exercise differs from

the principal's due to a bias. We first consider the problem in which the principal keeps formal authority to exercise the option. At any point in time, the agent communicates with the principal about whether or not to exercise the option. Conditional on the received message and the history of the game, the principal chooses whether to follow the advice or not. Importantly, not exercising today provides an option to get advice in the future. In equilibrium, the agent's communication strategy and the principal's exercise decisions are mutually optimal, and the principal rationally updates his beliefs about the agent's private information. In most of the paper, we look for stationary equilibria in this setting. After analyzing the case where the principal has formal authority, we consider the problem in which the principal delegates the option exercise decision to the agent and study under what conditions delegation helps.

When the agent is biased towards later exercise and the bias is not too high, the equilibrium in the communication game is often characterized by full revelation of information. However, the equilibrium stopping time will always involve delay relative to the principal's preferences. This is different from the static cheap talk setting of Crawford and Sobel (1982), where information is only partially revealed but the decision is conditionally optimal from the decision-maker's standpoint. In contrast, when the agent is biased towards earlier exercise, the equilibrium of our model features incomplete revelation of information with an infinite number of partitions. However, conditional on this incomplete information, the equilibrium exercise times are often unbiased from the decision-maker's standpoint.

The intuition for these strikingly different results for the two directions of the agent's bias lies in the nature of time as a decision variable. While the decision-maker always has the choice to get advice and exercise at a point later than the present, he cannot do the reverse, i.e., get advice and exercise at a point earlier than the present. If the agent is biased towards later exercise, she can withhold information and reveal it later, when the agent's and the decision-maker's interests will be aligned on immediate exercise at precisely the agent's first-best stopping time. Hence, using the terminology of Aghion and Tirole (1997), the agent has full real authority over the exercise decision, even though the principal has formal authority. Conversely, if the agent is biased towards earlier exercise, she does not benefit from withholding information, but when she discloses it, the decision-maker can always postpone exercise if it is not in his interests.

These results have implications for the informativeness and timeliness of option exercise decisions in organizations. First, other things equal, the agent's information is likely to explain more variation in the timing of option exercise for decisions with a late exercise bias (e.g., shutting down a plant) than for decisions with an early exercise bias (e.g., making an acquisition). Second, decisions with a late exercise bias are always delayed relative to the optimal exercise time from the decision-maker's perspective. In contrast, the timing of decisions with an early exercise bias

is on average unbiased.

We next show that the asymmetric nature of time has important implications for the principal's delegation decisions. We show that if the agent is biased towards late exercise, as in the case of a plant closure, the principal is always weakly better off keeping formal authority and communicating with the agent, rather than delegating the decision to the agent. Intuitively, when the agent with a late exercise bias makes the exercise recommendation, the principal knows that it is too late and is tempted to go back in time and exercise the option in the past. This, however, is not feasible since time only moves forward. This inability to revise past decisions allows the principal to commit to follow the recommendation of the agent, i.e., to exercise exactly when the agent recommends to exercise. Since the agent knows that the principal will follow her recommendation, the agent communicates honestly, which increases the principal's value of retaining authority.

In contrast, if the agent is biased towards early exercise, as in the case of a product launch, delegation is optimal for the principal if the agent's bias is not too high. Intuitively, in this case, when the agent recommends to exercise the option, the principal is tempted to delay the decision. Unlike changing past decisions, changing future decisions is possible, and hence time does not have valuable built-in commitment. Thus, communication is not as efficient as in the case when the agent is biased towards late exercise. As a consequence, delegation can now be optimal because it allows for more effective use of the agent's private information. The trade-off between information and bias suggests that delegation is superior when the agent's bias is not too high, similar to the argument for static decisions (Dessein, 2002).

We also study the comparative statics of the communication equilibrium with respect to the parameters of the stochastic environment. We show that in settings in which the agent is biased towards early exercise, an increase in volatility or the growth rate of the option payoff, as well as a decrease in the discount rate, lead to less information being revealed in equilibrium. Intuitively, these changes increase the value of the option to delay exercise and thereby effectively increase the conflict of interest between the agent and the decision-maker.

Since the communication framework is one in which the decision-maker cannot commit to future actions, it is interesting to consider the potential benefits to the decision-maker from the ability to commit. We thus compare the equilibrium of the communication game to the case where the decision-maker can commit to any decision-making mechanism. Importantly, when the agent is biased towards later exercise, the advising equilibrium coincides with the solution under the optimal contract with commitment, and hence the ability to commit does not improve the decision-maker's payoff. Intuitively, the decision-maker's inability to go back in time and act on the information before it is received creates an implicit commitment device for the principal to follow the agent's advice. In contrast, when the agent is biased towards earlier exercise, the advising

equilibrium differs significantly from the solution under the optimal contract with commitment. From the organizational design perspective, these results imply that investing in commitment power is not important for decisions in which the agent wishes to delay exercise, as in the case of headquarters seeking a local plant manager's advice on closing the plant. In contrast, investing in commitment power is important for decisions in which the agent is biased towards early exercise, such as making an acquisition or launching a new product line. We also show that given the same absolute bias, the principal is better off with an agent who is biased towards late exercise.

The paper proceeds as follows. The remainder of this section discusses the related literature. Section 2 describes the setup of the model and solves for the benchmark case of full information. Section 3 provides the analysis of the main model of advising under asymmetric information. Section 4 examines the delegation problem. Section 5 considers comparative statics and other implications. Finally, Section 6 concludes.

Related literature

Most importantly, our paper is related to models that study decision-making in the presence of an informed but biased expert. The seminal paper in this literature is Crawford and Sobel (1982), who consider a setting where the advisor sends a message to the decision-maker and the decision-maker cannot commit to the way he reacts to the messages of the advisor. Melumad and Shibano (1991) and Goltsman et al. (2009) consider settings similar to Crawford and Sobel (1982) but allow for commitment and more general decision-making procedures. Our base model is similar to Crawford and Sobel's in that the decision-maker has no commitment power, but the important difference is that the decision problem is dynamic and concerns the timing of option exercise, rather than the scale of a project. To our knowledge, ours is the first paper that studies option exercise problem in a "cheap talk" setting. Surprisingly, even though there is no flow of additional private information to the agent, equilibria differ conceptually from the ones in Crawford and Sobel (1982).

Our paper also contributes to the literature on authority in organizations (e.g., Aghion and Tirole, 1997), surveyed in Bolton and Dewatripont (2013). It is most closely related to Dessein (2002), who studies the decision-maker's choice between delegating a decision to an expert and communicating with the expert to make the decision himself, as in Crawford and Sobel (1982). Dessein (2002) shows that delegation dominates communication provided that the expert's bias is not too large. Relatedly, Harris and Raviv (2005, 2008) and Chakraborty and Yilmaz (2011) analyze the optimality of delegation in settings with two-sided private information.[1] Our paper

[1] For a broader review of the literature on decisions in organizations, see Gibbons, Matouschek, and Roberts (2013).

contributes to this literature by studying delegation of option exercise decisions and showing that unlike in static settings, the optimality of delegation crucially depends on the direction of the agent's bias.

Dynamic extensions of Crawford and Sobel (1982) are very difficult. Because of multiplicity of equilibria in the static model, existing models that study repeated versions of a cheap talk game usually restrict attention to binary signals and types (Sobel, 1985; Benabou and Laroque, 1992; Morris, 2001). Ottaviani and Sorensen (2006a,b) study the advisor's reputation-building incentives but model them in reduced-form, and have static decision-making. Our model differs from this literature in the nature of the decision problem. Even though the decision whether to exercise or not is made repeatedly, the game ends when the option is exercised and the agent's private information is persistent. These features, as well as stationarity of the problem, make the analysis tractable.

Finally, our paper is related to the literature on option exercise in the presence of agency problems. Grenadier and Wang (2005), Gryglewicz and Hartman-Glaser (2013), and Kruse and Strack (2013) study such settings but assume that the principal can commit to contracts and make contingent transfers to the agent, which makes the problem conceptually different from ours. Several papers study signaling through option exercise.[2] They assume that the decision-maker is informed, while in our setting the decision-maker is uninformed.

2 Model setup

A firm (or an organization, more generally) has a project and needs to decide on the optimal time to implement it. The organization consists of two players, the uninformed party (principal, P) and the informed party (the agent, A). Both parties are risk-neutral and share the same discount rate $r > 0$. Time is continuous and indexed by $t \in [0, \infty)$. The persistent type $\theta \in \Theta$ is drawn and learned by the agent at the initial date $t = 0$. The principal does not know θ. It is common knowledge that θ is a random draw from the uniform distribution over $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$, where $0 \leq \underline{\theta} < \overline{\theta}$. Without loss of generality, we normalize $\overline{\theta} = 1$. For much of the paper, we also assume $\underline{\theta} = 0$. We start by considering the exercise of a call option. We will refer to it as the option to invest, but it can capture any perpetual American call option, such as the option to go public or the option to launch a new generation of the product. In unreported results, we also extended the analysis to a put option (e.g., if the decision is about shutting down a poorly performing division) and show that the main results continue to hold.

[2] Grenadier and Malenko (2011), Morellec and Schuerhoff (2011), Bustamante (2012), Grenadier, Malenko, and Strebulaev (2013).

The exercise at time t generates the payoff to the principal of $\theta X(t) - I$, where $I > 0$ is the exercise price (the investment cost), and $X(t)$ follows geometric Brownian motion with drift μ and volatility σ:

$$dX(t) = \mu X(t)\, dt + \sigma X(t)\, dB(t), \tag{1}$$

where $\sigma > 0$, $r > \mu$, and $dB(t)$ is the increment of a standard Wiener process. The starting point $X(0)$ is low enough. Process $X(t)$, $t \geq 0$ is observable by both the principal and the agent. While the agent knows θ, she is biased. Specifically, upon exercise, the agent receives the payoff of $\theta X(t) - I + b$, where $b \in (-\infty, I)$ is the commonly known bias of the agent. Positive bias $b > 0$ means that the agent is biased in the direction of early exercise: His personal exercise price $(I - b)$ is lower than the principal's (I), so his most preferred timing of exercise is earlier than the principal's for any θ. In contrast, negative bias $b < 0$ means that the agent is biased in the direction of late exercise. These preferences can be viewed as reduced-form implications of an existing revenue-sharing agreement.[3]

The principal has formal authority on deciding when to exercise the option. We adopt an incomplete contracting approach by assuming that the timing of the exercise cannot be contracted upon. Furthermore, the organization is assumed to have a resource, controlled by the principal, which is critical for the implementation of the project. This resource is the reason why the agent cannot implement the project without the principal's approval. Some examples include rights to contract with suppliers and human capital of the managerial team. We initially make an extreme assumption that nothing is contractible, so the principal can only rely on informal "cheap talk" communication with the agent. This problem is the option exercise analogue of Crawford and Sobel's (1982) "cheap talk" model. Then, we relax this assumption by allowing the principal to grant the agent authority over the exercise of the option. This problem is the option exercise analogue of Dessein's (2002) analysis on authority and communication.

As an example, consider an oil-producing firm that owns an oil well and needs to decide on the optimal time to drill it. The publicly observable oil price process is represented by $X(t)$. The top management of the firm has formal authority over the decision to drill. The regional manager has private information about how much oil the well contains (θ), which stems from his local knowledge and prior experience with neighborhood wells. The firm cannot simply sell the oil well to the regional manager, because of its resources, such as human capital and existing relationships with suppliers. Depending on its ability and willingness to delegate, the top management may

[3] For example, suppose that the principal supplies financial capital $\hat{I}$, the agent supplies human capital ("effort") valued at $\hat{e}$, and the principal and the agent hold fractions α_P and α_A of equity of the realized value from the project. Then, at exercise, the principal's (agent's) expected payoff is $\alpha_P \theta X(t) - \hat{I}$ $(\alpha_A \theta X(t) - \hat{e})$. This is analogous to the specification in the model with $I = \frac{\hat{I}}{\alpha_P}$ and $b = \frac{\hat{I}}{\alpha_P} - \frac{\hat{e}}{\alpha_A}$.

assign the right to decide on the timing of drilling to the regional manager. In contrast, if the top management is not willing or unable to commit to delegate, the top management is the party that decides on the timing of drilling.

For now, assume that authority is not contractible. The timing is as follows. At each time t, knowing the state of nature $\theta \in \Theta$ and the history of the game $\mathcal{H}_t$, the agent decides on a message $m(t) \in M$ to send to the principal, where M is a set of messages. At each t, the principal decides whether to exercise the option or not, given $\mathcal{H}_t$ and the current message $m(t)$. If the principal exercises the option, the game ends. If the principal does not exercise the option, the game continues. Because the game ends when the principal exercises the option, we can only consider histories such that the option has not been exercised yet. Then, the history of the game at time t has two components: the sample path of the public state $X(t)$ and the history of messages of the agent. Formally, it is represented by $(\mathcal{H}_t)_{t\geq 0}$, where $\mathcal{H}_t = \{X(s), s \leq t, m(s), s < t\}$. Thus, the strategy m of the agent is a family of functions $(m_t)_{t\geq 0}$ such that for any t function m_t maps the agent's information set at time t into the message she sends to the principal: $m_t : \Theta \times \mathcal{H}_t \to M$. The strategy e of the principal is a family of functions $(e_t)_{t\geq 0}$ such that for any t function e_t maps the principal's information set at time t into the binary exercise decision: $e_t : \mathcal{H}_t \times M \to \{0, 1\}$. Here, $e_t = 1$ stands for "exercise" and $e_t = 0$ stands for "wait." Let $\tau(e) \equiv \inf\{t : e_t = 1\}$ denote the stopping time implied by strategy e of the principal. Finally, let $\mu(\theta|\mathcal{H}_t)$ denote the updated probability that the principal assigns to the type of the agent being θ given that she observed history $\mathcal{H}_t$.

Heuristically, the timing of events over an infinitesimal time interval $[t, t + dt]$ prior to option exercise can be described as follows:

1. The nature determines the realization of X_t.
2. The agent sends message $m_t \in \mathcal{M}$ to the principal.
3. The principal decides whether to exercise the option or not. If the option is exercised, the principal obtains the payoff of $\theta X_t - I$, the agent obtains the payoff of $\theta X_t - I + b$, and the game ends. Otherwise, the game continues, and the nature draws $X_{t+dt} = X_t + dX_t$.

This is a dynamic game with observed actions (messages and the exercise decision) and incomplete information (type θ of the agent). We focus on equilibria in pure strategies. The equilibrium concept is Perfect Bayesian Equilibrium in Markov strategies, defined as:

Definition 1. Strategies $m^* = \{m_t^*, t \geq 0\}$ and $e^* = \{e_t^*, t \geq 0\}$, beliefs μ^*, and a message spaces M constitute a *Perfect Bayesian equilibrium in Markov strategies (PBEM)* if and only if:

1. For every t, $\mathcal{H}_t$, $\theta \in \Theta$, and strategy m,

$$\begin{aligned} & \mathbb{E}\left[e^{-r\tau(e^*)}\left(\theta X\left(\tau\left(e^*\right)\right)-I+b\right)|\mathcal{H}_t,\theta,\mu^*\left(\cdot|\mathcal{H}_t\right),m^*,e^*\right] \\ \geq\ & \mathbb{E}\left[e^{-r\tau(e^*)}\left(\theta X\left(\tau\left(e^*\right)\right)-I+b\right)|\mathcal{H}_t,\theta,\mu^*\left(\cdot|\mathcal{H}_t\right),m,e^*\right]. \end{aligned} \tag{2}$$

2. For every t, $\mathcal{H}_t$, $m\left(t\right) \in M$, and strategy e,

$$\begin{aligned} & \mathbb{E}\left[e^{-r\tau(e^*)}\left(\theta X\left(\tau\left(e^*\right)\right)-I\right)|\mathcal{H}_t,\mu^*\left(\cdot|\mathcal{H}_t,m\left(t\right)\right),e^*,m^*\right] \\ \geq\ & \mathbb{E}\left[e^{-r\tau(e)}\left(\theta X\left(\tau\left(e\right)\right)-I\right)|\mathcal{H}_t,\mu^*\left(\cdot|\mathcal{H}_t,m\left(t\right)\right),e,m^*\right]. \end{aligned} \tag{3}$$

3. Bayes' rule is used to update beliefs $\mu^*\left(\theta|\mathcal{H}_t\right)$ to $\mu^*\left(\theta|\mathcal{H}_t,m\left(t\right)\right)$ whenever possible: For every $\mathcal{H}_t$ and $m\left(t\right) \in M$, if there exists θ such that $m_t^*\left(\theta,\mathcal{H}_t\right) = m\left(t\right)$, then for all θ

$$\mu^*\left(\theta|\mathcal{H}_t,m\left(t\right)\right) = \frac{\mu^*(\theta|\mathcal{H}_t)\mathbf{1}\{m_t^*(\theta,\mathcal{H}_t)=m\left(t\right)\}}{\int_{\underline{\theta}}^{1}\mu^*(\tilde{\theta}|\mathcal{H}_t)\mathbf{1}\{m_t^*(\tilde{\theta},\mathcal{H}_t)=m\left(t\right)\}\frac{1}{1-\underline{\theta}}d\tilde{\theta}}, \tag{4}$$

with $\mu^*\left(\theta|\mathcal{H}_0\right) = 1$ for $\theta \in \Theta$ and $\mu^*\left(\theta|\mathcal{H}_0\right) = 0$, otherwise.

4. For every t, $\mathcal{H}_t$, $\theta \in \Theta$, and $m\left(t\right) \in M$,

$$m_t^*\left(\theta,\mathcal{H}_t\right) = m^*\left(\theta,X\left(t\right),\mu^*\left(\cdot|\mathcal{H}_t\right)\right); \tag{5}$$

$$e_t^*\left(\mathcal{H}_t,m\left(t\right)\right) = e^*\left(X\left(t\right),\mu^*\left(\cdot|\mathcal{H}_t,m\left(t\right)\right)\right). \tag{6}$$

Conditions (2)–(4) are requirements of the Perfect Bayesian equilibrium. Inequalities (2) require the equilibrium strategy m^* to be sequentially optimal for the agent for any possible history $\mathcal{H}_t$ and type realization θ. Similarly, inequalities (3) require equilibrium strategy e^* to be sequentially optimal for the principal for any possible history. Equation (4) requires beliefs to be updated according to Bayes' rule. Finally, conditions (5)–(6) are requirements that the equilibrium strategies and the message space are Markov.

Bayes' rule does not apply if the principal observes a message that should be sent by no type. To restrict beliefs following such off-equilibrium actions, we impose another constraint:

Assumption 1. If, at any point t, the principal's belief $\mu\left(\theta|\mathcal{H}_t\right)$ and the observed message $m\left(t\right)$ are such that no type that could exist (according to the principal's belief) could possibly send message $m\left(t\right)$, then the principal's belief is unchanged: If $\{\theta : m_t^*(\theta,\mathcal{H}_t) = m\left(t\right),\mu^*\left(\theta|\mathcal{H}_t\right) > 0\} = \emptyset$, then $\mu^*\left(\theta|m\left(t\right),\mathcal{H}_t\right) = \mu^*\left(\theta|\mathcal{H}_t\right)$.

This assumption is related to a frequently imposed restriction in models with two types that if, at any point, the posterior assigns probability one to a given type, then this belief persists no matter what happens (e.g., Rubinstein, 1985; Halac, 2012). Because our model features a continuum of types, an action that no one was supposed to take may occur off equilibrium even if the belief is not degenerate. As a consequence, we impose a stronger restriction.

Let stopping time $\tau^*(\theta)$ denote the equilibrium exercise time of the option if the type is θ. In almost all standard option exercise models, the optimal exercise strategy for a perpetual American call option is a threshold: It is optimal to exercise the option at the first instant the state process $X(t)$ exceeds some critical level, which depends on the parameters of the environment. It is thus natural to look for equilibria that exhibit a similar property, formally defined as:

Definition 2. An equilibrium is a *threshold-exercise* PBEM if $\tau^*(\theta) = \inf\left\{t \geq 0 | X(t) \geq \bar{X}(\theta)\right\}$ for some $\bar{X}(\theta)$ (possibly infinite), $\theta \in \Theta$.

For any threshold-exercise equilibrium, let $\mathcal{X}$ denote the set of equilibrium exercise thresholds: $\mathcal{X} \equiv \left\{X : \exists \theta \in \Theta \text{ such that } \bar{X}(\theta) = X\right\}$. We next prove two useful auxiliary results that hold in any threshold-exercise PBEM. The next lemma shows that in any threshold-exercise PBEM, the option is exercised weakly later if the agent has less favorable information:

Lemma 1. *Let $\tau^*(\theta) = \inf\left\{t \geq 0 | X(t) \geq \bar{X}(\theta)\right\}$ be the equilibrium exercise time in a threshold-exercise PBEM. Then, $\bar{X}(\theta_1) \geq \bar{X}(\theta_2)$ for any $\theta_1, \theta_2 \in \Theta$ such that $\theta_2 \geq \theta_1$.*

Intuitively, because talk is "cheap," the agent with information θ_1 can adopt the message strategy of the agent with information $\theta_2 > \theta_1$ (and the other way around) at no cost. Thus, between choosing dynamic communication strategies that induce exercise at thresholds $\bar{X}(\theta_1)$ and $\bar{X}(\theta_2)$, the type-θ_1 agent must prefer the former, while the type-θ_2 agent must prefer the latter. This is simultaneously possible only if $\bar{X}(\theta_1) \geq \bar{X}(\theta_2)$.

The second auxiliary result is that it is without loss of generality to reduce the message space significantly. Specifically, the next lemma shows that for any threshold-exercise equilibrium, it is possible to construct an equilibrium with a binary message space $M = \{0, 1\}$ and simple equilibrium strategies that implements the same exercise time:

Lemma 2. *If there exists a threshold-exercise PBEM with some threshold $\bar{X}(\theta)$, then there exists an equivalent threshold-exercise PBEM with the binary message space $M = \{0, 1\}$ and the*

following strategies of the agent and the principal:

1. *The agent with type θ sends message $m(t)=1$ if and only if $X(t)$ is greater or equal than threshold $\bar{X}(\theta)$:*

$$\bar{m}_t\left(\theta, X(t), \bar{\mu}\left(\cdot|\mathcal{H}_t\right)\right)=\begin{cases} 1, & \text{if } X(t)\geq \bar{X}(\theta), \\ 0, & \text{otherwise.}\end{cases} \tag{7}$$

   Given Lemma 1 and the fact that the agent plays (γ), the posterior belief of the principal at any time t is that θ is distributed uniformly over $\left[\check{\theta}_t, \hat{\theta}_t\right]$ for some $\check{\theta}_t$ and $\hat{\theta}_t$ (possibly, equal).

2. *The option exercise strategy of the principal is*

$$\bar{e}_t\left(X(t), \check{\theta}_t, \hat{\theta}_t\right)=\begin{cases} 1, & \text{if } X(t)\geq \check{X}\left(\check{\theta}_t, \hat{\theta}_t\right) \\ 0, & \text{otherwise,}\end{cases} \tag{8}$$

   for some threshold $\check{X}\left(\check{\theta}_t, \hat{\theta}_t\right)$.

   Function $\check{X}\left(\check{\theta}_t, \hat{\theta}_t\right)$ is such that on equilibrium path the option is exercised on the first instant when the agent sends message $m_t=1$, which happens when $X(t)$ hits threshold $\bar{X}(\theta)$ for the first time.

Lemma 2 implies that it is without loss of generality to focus on equilibria of the following simple form. At any time t, the agent can send one of the two messages, 1 or 0. Message $m=1$ can be interpreted as a recommendation of exercise, while message $m=0$ can be interpreted as a recommendation of waiting. The agent plays a threshold strategy, recommending exercise if and only if the public state $X(t)$ is above threshold $\bar{X}(\theta)$, which depends on private information θ of the agent. The principal also plays a threshold strategy: the principal that believes that $\theta\in\left[\check{\theta}_t, \hat{\theta}_t\right]$ exercises the option if and only if $X(t)$ exceeds some threshold $\check{X}\left(\check{\theta}_t, \hat{\theta}_t\right)$. As a consequence of the agent's strategy, there is a set $\mathcal{T}$ of "informative" times, when the agent's message has information content, i.e., it affects the belief of the principal and, in turn, her exercise decision. These are instances when the state process $X(t)$ passes a new threshold from the set of possible exercise thresholds $\mathcal{X}$. At all other times, the agent's message has no information content, as it does not lead the principal to update his belief. In equilibrium, each type θ of the agent recommends exercise (sends $m=1$) at the first time when the state process $X(t)$ passes some threshold $\bar{X}(\theta)$ for the first time, and the principal exercises the option immediately.

Lemma 2 states that if there is exists *some* equilibrium with the set of thresholds $\left\{\bar{X}(\theta), \theta \in \Theta\right\}$, then there exists an equilibrium of the above form with the same set of exercise thresholds. The intuition behind this result is that at each time the principal faces a binary decision: to exercise or to wait. Because the information of the agent is important only for the timing of the exercise, one can achieve the same efficiency by choosing the timing of communicating a binary message as through the richness of the message space. Therefore, richer than binary message spaces cannot improve the efficiency of decision making. And because the relevant set of actions is only the set of equilibrium thresholds, there is no benefit from communication at times other than when $X(t)$ passes one of the potential exercise thresholds.

In what follows, we focus on threshold-exercise PBEM of the form in Lemma 2 and refer to them as simply "equilibria." When $\underline{\theta} = 0$, the problem exhibits stationarity in the following sense. Because the prior distribution of types is uniform over $[0,1]$ and the payoff structure is multiplicative, a time-t sub-game in which the posterior belief of the principal is uniform over $\left[0, \hat{\theta}\right]$ is equivalent to the game with the belief is that θ is uniform over $[0,1]$, the true type is $\frac{\theta}{\hat{\theta}}$, and the modified state process $\tilde{X}(t) = \hat{\theta} X(t)$. Because of this scalability of the game, it is natural to restrict attention to stationary equilibria, which are formally defined as follows:

Definition 2. Suppose that $\underline{\theta} = 0$. A threshold-exercise PBEM (m^*, e^*, μ^*, M) is *stationary* if whenever posterior belief $\mu^*(\cdot|\mathcal{H}_t)$ is uniform over $[0, \hat{\theta}]$ for some $\hat{\theta} \in (0,1)$:

$$m^*(\theta, X(t), \mu^*(\cdot|\mathcal{H}_t)) = m^*\left(\frac{\theta}{\hat{\theta}}, \hat{\theta} X(t), \mu^*(\cdot|\mathcal{H}_0)\right), \tag{9}$$

$$e^*(X(t), \mu^*(\cdot|\mathcal{H}_t, m(t))) = e^*\left(\hat{\theta} X(t), \mu^*(\cdot|\mathcal{H}_0, m(t))\right), \tag{10}$$

for all $\theta \in \left[0, \hat{\theta}\right]$.

Condition (9) means that every type $\theta \in \left[0, \hat{\theta}\right]$ sends the same message when the public state is $X(t)$ and the posterior is uniform over $\left[0, \hat{\theta}\right]$ as type $\frac{\theta}{\hat{\theta}}$ when the public state is $\hat{\theta} X(t)$ and the posterior is uniform over $[0,1]$. Condition (10) means that the exercise strategy of the principal is the same when the public state is $X(t)$ and his belief is that θ is uniform over $\left[0, \hat{\theta}\right]$ as when the public state is $\hat{\theta} X(t)$ and his belief is that θ is uniform over $[0,1]$.

Motivated by the result of Lemma 2, from now on we focus on threshold-exercise PBEM in the form stated in Lemma 2. We refer to these equilibria simply as *equilibria*. In the model with $\underline{\theta} = 0$, we focus on threshold-exercise PBEM in the form stated in Lemma 2 that are stationary. We refer to these equilibria as *stationary equilibria*.

2.1 Benchmark cases

As benchmarks, we consider two simple settings: one in which the principal knows θ and the other in which the agent has formal authority to exercise the option.

2.1.1 Optimal exercise for the principal

Suppose that the principal knows θ, so communication with the agent is irrelevant. Let $V_P^*(X,\theta)$ denote the value of the option to the principal in this case, if the project's type is θ and the current value of $X(t)$ is X. According to the standard argument (e.g., Dixit and Pindyck, 1994), in the range prior to exercise, $V_P^*(X,\theta)$ solves

$$rV_P^*(X,\theta) = \mu X \frac{\partial V_P^*(X,\theta)}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 V_P^*(X,\theta)}{\partial X^2}. \tag{11}$$

Suppose that type θ exercises the option when $X(t)$ reaches threshold $X_P^*(\theta)$. Then,

$$V_P^*(X_P^*(\theta),\theta) = \theta X_P^*(\theta) - I. \tag{12}$$

Solving (11) subject to this boundary condition and condition $V_P^*(0,\theta) = 0$, we obtain[4]

$$V_P^*(X,\theta) = \begin{cases} \left(\frac{X}{X_P^*(\theta)}\right)^{\beta} (\theta X_P^*(\theta) - I), & \text{if } X \le X_P^*(\theta) \\ \theta X - I, & \text{if } X > X_P^*(\theta), \end{cases} \tag{13}$$

where

$$\beta = \frac{1}{\sigma^2}\left[-\left(\mu - \frac{\sigma^2}{2}\right) + \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}\right] > 1. \tag{14}$$

is the positive root of the fundamental quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0$.

The optimal exercise trigger $X_P^*(\theta)$ maximizes the value of the option (13), and is given by

$$X_P^*(\theta) = \frac{\beta}{\beta - 1}\frac{I}{\theta}. \tag{15}$$

2.1.2 Optimal exercise for the agent

Suppose that the agent has complete formal authority over when to exercise the option. Substituting $I - b$ for I in (11)–(15), we obtain that the optimal exercise strategy for the agent is to

[4] $V_P^*(0,\theta) = 0$, because $X = 0$ is an absorbing barrier.

exercise the option when $X(t)$ reaches threshold

$$X_A^*(\theta) = \frac{\beta}{\beta - 1}\frac{I-b}{\theta}. \tag{16}$$

The value of the option to the agent in this case is

$$V_A^*(X,\theta) = \begin{cases} \left(\frac{X}{X_A^*(\theta)}\right)^{\beta}(\theta X_A^*(\theta) - I + b), & \text{if } X \leq X_A^*(\theta), \\ \theta X - I + b, & \text{if } X > X_A^*(\theta). \end{cases} \tag{17}$$

3 Communication game

By Lemmas 1 and 2, the history of the game at time t on equilibrium path can be summarized by two cut-offs, $\check{\theta}_t$ and $\hat{\theta}_t$. Moreover, prior to recommending to exercise ($m = 1$), the history of the game can be summarized by a single cut-off $\hat{\theta}_t$, where $\hat{\theta}_t \equiv \sup\left\{\theta : \bar{X}(\theta) > \max_{s \leq t} X(s)\right\}$. Indeed, by Lemma 2, on equilibrium path, the principal exercises the option at the first time t with $X(t) \in \mathcal{X}$ at which the agent sends $m_t = 1$. If the agent has not recommended exercise by time t, the principal infers that the agent's type does not exceed $\bar{X}\left(\hat{\theta}_t\right)$. Therefore, process $\hat{\theta}_t$ summarizes the belief of the principal at time t, provided that he has not deviated from his equilibrium strategy of exercising the option at first instant $X(s) \in \mathcal{X}$ at which the agent recommends exercise.

Consider the case $\underline{\theta} = 0$, in which the problem becomes stationary. Using Lemma 1 and the stationarity condition, we conclude that any stationary equilibrium must either have continuous exercise or partitioned exercise. If the equilibrium exercise has a partition structure, such that the set of types Θ is partitioned into intervals with each interval inducing the exercise at a given threshold, then stationarity implies that the set of partitions must take the form $[\omega, 1]$, $\left[\omega^2, \omega\right]$, ..., $\left[\omega^n, \omega^{n-1}\right]$, $n \in \mathbb{N}$, for some $\omega \in [0,1)$, where $\mathbb{N}$ is the set of natural numbers. This implies that the set of exercise thresholds $\mathcal{X}$ is given by $\left\{\bar{X}, \frac{\bar{X}}{\omega}, \frac{\bar{X}}{\omega^2}, ..., \frac{\bar{X}}{\omega^n}, ...\right\}$, $n \in \mathbb{N}$, such that if $\theta \in \left(\omega^n, \omega^{n-1}\right)$, the option is exercised at threshold $\frac{\bar{X}}{\omega^{n-1}}$. We refer to an equilibrium of this form as the *ω-equilibrium*.

For ω and $\bar{X}$ to constitute an equilibrium, incentive compatibility conditions for the principal and the agent must hold. Because the problem is stationary, it is enough to consider only incentive compatibility conditions for the play up to reaching the first threshold $\bar{X}$. First, consider the agent's problem. From the agent's point of view, the set of possible exercise thresholds is given by $\mathcal{X}$. The agent can induce exercise at any threshold in $\mathcal{X}$ by recommending exercise at the first instant $X(t)$ reaches a desired point in $\mathcal{X}$. At the same time, the agent cannot induce exercise at any point not in $\mathcal{X}$. The reason is simple: Once the agent sends $m_t = 0$ when $X(t)$ reaches a threshold in $\mathcal{X}$, the principal updates her belief that the agent's type is not in the partition that

recommends exercise at that threshold, and by Assumption 1, the agent is unable to convince the principal in the opposite going forward.

Pair $\left(\omega, \bar{X}\right)$ satisfies the agent's incentive compatibility if and only if types above ω have incentives to recommend exercise ($m = 1$) at threshold $\bar{X}$ rather than to wait, whereas types below ω have incentives to recommend delay ($m = 0$). This holds if and only if type ω is exactly indifferent between exercising the option at threshold $\bar{X}$ and at threshold $\frac{\bar{X}}{\omega}$. This yields the following equation:

$$\left(\frac{X(t)}{\bar{X}}\right)^{\beta}\left(\omega\bar{X} + b - I\right) = \left(\frac{X(t)}{\bar{X}/\omega}\right)^{\beta}\left(\omega\frac{\bar{X}}{\omega} + b - I\right), \tag{18}$$

which can be simplified to

$$\omega\bar{X} + b - I = \omega^{\beta}\left(\bar{X} + b - I\right). \tag{19}$$

Indeed, if (18) holds, then $\left(\frac{X(t)}{\bar{X}}\right)^{\beta}\left(\theta\bar{X} + b - I\right) \gtrless \left(\frac{X(t)}{\bar{X}/\omega}\right)^{\beta}\left(\theta\frac{\bar{X}}{\omega} + b - I\right)$ if $\theta \gtrless \omega$, because the left-hand side is more sensitive to θ than the right-hand side. Hence, if type ω is indifferent between recommending exercise at threshold $\bar{X}$ and recommending delay, then any higher type strictly prefers recommending exercise, while any lower type strictly prefers recommending delay. By stationarity, if (18) holds, then type ω^2 is indifferent between recommending exercise and delay at threshold $\frac{\bar{X}}{\omega}$, so types $\left(\omega^2, \omega\right)$ recommend $m = 1$ at threshold $\frac{\bar{X}}{\omega}$, and so on. Equation (19) implies the following relation between the first possible exercise threshold $\bar{X}$ and ω:

$$\bar{X} = \frac{\left(1 - \omega^{\beta}\right)(I - b)}{\omega\left(1 - \omega^{\beta-1}\right)}. \tag{20}$$

Denote the right-hand side of (20) by $Y(\omega)$.

Next, consider the principal's problem. For ω and $\bar{X}$ to constitute an equilibrium, the principal must have incentives (1) to exercise the option immediately when she gets recommendation $m = 1$ from the agent at threshold in $\mathcal{X}$; and (2) not to exercise the option otherwise. We refer to the former (latter) incentive compatibility condition as the *ex-post (ex-ante) incentive compatibility* constraint. Suppose that $X(t)$ reaches threshold $\bar{X}$ for the first time, and the principal receives recommendation $m = 1$ at that instant. By Bayes' rule, the principal updates his beliefs about θ to θ being uniform over $[\omega, 1]$. If the principal exercises immediately, he obtains the expected payoff of $\frac{\omega+1}{2}\bar{X} - I$. If the principal delays, he expects that there will be no further informative communication in the continuation game, given the conjectured equilibrium strategy of the agent. Therefore, upon receiving recommendation $m = 1$ at threshold $\bar{X}$, the principal faces the standard perpetual call option exercise problem (e.g., Dixit and Pindyck, 1994) as if the type of the project

were $\frac{\omega+1}{2}$. The solution to this problem is immediate exercise if and only if exercising at threshold $\bar{X}$ dominates waiting until $X(t)$ reaches a higher threshold $\hat{X}$ and exercising the option there for any possible $\hat{X} > \bar{X}$:

$$\bar{X} \in \arg\max_{\hat{X} \geq \bar{X}} \left(\frac{\bar{X}}{\hat{X}}\right)^{\beta} \left(\frac{\omega+1}{2}\hat{X} - I\right). \tag{21}$$

Using the fact that the unconditional maximizer of the right-hand side is $\hat{X} = \frac{\beta}{\beta-1}\frac{2I}{\omega+1}$ and that the right-hand side is an inverted U-shaped function of $\hat{X}$, the ex-post incentive compatibility condition for the principal can be equivalently written as

$$Y(\omega) \geq \frac{\beta}{\beta-1}\frac{2I}{\omega+1}. \tag{22}$$

This condition has a clear intuition. It means that at the moment when the agent recommends the principal to exercise the option, it must be "too late" to delay exercise. If (22) is violated, the principal delays exercise, so the recommendation loses its responsiveness, as the principal does not follow it. In contrast, if (22) holds, the principal's optimal response to getting the recommendation to exercise is to exercise the option immediately. As with the incentive compatibility condition of the agent, stationarity implies that if (22) holds, then a similar condition holds for all higher thresholds in $\mathcal{X}$. The fact that constraint (22) is an inequality rather than an equality highlights the built-in asymmetric nature of time. When the agent recommends exercise to the principal, the principal can either exercise immediately or can delay, but cannot go back in time and exercise in the past, even though it is tempting to do so, if (22) holds as a strict inequality.

The ex-ante incentive compatibility constraint is that the principal is better off waiting at any time prior to receiving $m = 1$ at $X(t) \in \mathcal{X}$. Let $V_P\left(X(t), \hat{\theta}_t; \bar{X}, \omega\right)$ denote the expected value to the principal, given that the public state is $X(t)$, his belief is that θ is uniform over $\left[0, \hat{\theta}_t\right]$, and he expects types $\left(\omega^n, \omega^{n-1}\right)$, $n \in \mathbb{N}$ to recommend exercise at threshold $\bar{X}\omega^{1-n}$. Then, the ex-ante incentive-compatibility constraint is

$$V_P\left(X(t), \hat{\theta}_t; \bar{X}, \omega\right) \geq \frac{\hat{\theta}_t}{2}X(t) - I \tag{23}$$

for any $X(t)$ and $\hat{\theta}_t = \sup\left\{\theta : \bar{X}(\theta) > \max_{s \leq t} X(s)\right\}$. By stationarity, it is sufficient to verify the ex-ante incentive-compatibility constraint of the principal for $X(t) \leq \bar{X}$ and beliefs equal to the prior. Then, the ex-ante incentive-compatibility constraint becomes

$$V_P\left(X(t), 1; \bar{X}, \omega\right) \geq \frac{1}{2}X(t) - I \ \forall X(t) \leq \bar{X}. \tag{24}$$

This inequality states that at any time up to threshold $\bar{X}$, the principal is better off waiting than exercising the option. If (24) does not hold, then the principal is better off exercising the option rather than waiting for informative recommendations from the agent. If (24) holds, then the principal does not exercise the option prior to reaching threshold $\bar{X}$. By stationarity, if (24) holds, then a similar condition holds for the n^{th} partition for any $n \in \mathbb{N}$, which implies that (24) and (23) are equivalent.

Given $Y(\omega)$, we can solve for the principal's value $V_P\left(X(t), \hat{\theta}_t; \omega\right)$ in the ω-equilibrium in closed form (see the appendix for the derivation):

$$V_P(X, 1; \omega) = \frac{1-\omega}{1-\omega^{\beta+1}} \left(\frac{X}{Y(\omega)}\right)^{\beta} \left(\frac{1}{2}(1+\omega) Y(\omega) - I\right), \tag{25}$$

for any $X \leq Y(\omega)$, where $Y(\omega)$ is given by (20). Using stationarity, (25) can be generalized to

$$V_P\left(X, \hat{\theta}; \omega\right) = V_P\left(\hat{\theta} X, 1; \omega\right) = \frac{1-\omega}{1-\omega^{\beta+1}} \left(\frac{X\hat{\theta}}{Y(\omega)}\right)^{\beta} \left(\frac{1}{2}(1+\omega) Y(\omega) - I\right). \tag{26}$$

Then, the ex-ante incentive compatibility conditions of the principal are equivalent to:

$$V_P(X(0), 1; \omega) \geq \left(\frac{X(0)}{\bar{X}_u}\right)^{\beta} \left(\frac{1}{2}\bar{X}_u - I\right), \tag{27}$$

where $\bar{X}_u = \frac{\beta}{\beta-1} 2I$ is the optimal uniformed exercise strategy of the principal.

The analysis above considered only partitioned equilibria, i.e., $\bar{X}(\theta) = \bar{X}(1)$ for any $\theta \in (\omega, 1]$. In contrast, if $\bar{X}(\theta) \neq \bar{X}(1)$ for all $\theta < 1$, then by stationarity of the problem, $\bar{X}(\theta) = \bar{X}(1)/\theta$ for any θ. We refer to such equilibria, if they exist, as *equilibria with continuous exercise*.

3.1 Preference for later exercise

Suppose that the agent is biased in the direction of later exercise. Formally, $b < 0$. First, consider the potential equilibrium with continuous exercise. By stationarity, $\mathcal{X} = \{X : X \geq \underline{X}\}$ for some $\underline{X}$. Incentive compatibility of the agent can be written as

$$\bar{X}(\theta) \in \arg\max_{\hat{X} \geq \underline{X}} \left(\frac{X(t)}{\hat{X}}\right)^{\beta} \left(\theta \hat{X} - I + b\right).$$

It implies that exercise occurs at the agent's most preferred threshold as long as it is above $\underline{X}$:

$$\bar{X}(\theta) = X_A^*(\theta) = \frac{\beta}{\beta-1} \frac{I-b}{\theta}. \tag{28}$$

Stationarity implies that separation must hold for all types, including $\theta = 1$, which implies that (28) holds for any $\theta \in \Theta$. Hence, $\mathcal{X} = \{X : X \geq X_A^*(1)\}$. This exercise schedule satisfies the ex-post incentive compatibility of the principal. Since the agent has a delay bias and follows the strategy of recommending exercise at her most preferred threshold, when the agent recommends to exercise, the principal infers that it is already too late and thus does not benefit from delaying exercise even further. Formally, $X_A^*(\theta) > X_P^*(\theta)$.

Consider the ex-ante incentive compatibility condition for the principal. Let $V_P^c\left(X(t), \hat{\theta}\right)$ denote the expected value to the principal, given that the public state is $X(t)$, his belief is that θ is uniform over $\left[0, \hat{\theta}_t\right]$, and type θ recommends exercise at threshold $X_A^*(\theta)$, under the assumption that the principal does not exercise the option prior to getting $m = 1$. By stationarity of the problem, it is sufficient to verify the ex-ante incentive compatibility for $\hat{\theta} = 1$, which yields inequality

$$V_P^c(X(t), 1) \geq \left(\frac{X(t)}{\bar{X}_u}\right)^{\beta} \left(\frac{1}{2}\bar{X}_u - I\right), \ \forall X(t) \leq X_A^*(1). \tag{29}$$

It can be verified that this constraint holds if and only if $b \geq -I$.[5]

Second, consider the case of partitioned exercise for a fixed ω. To be an equilibrium, the implied exercise thresholds must satisfy the incentive-compatibility conditions of the principal (22)–(27). The following proposition summarizes the set of all stationary equilibria:[6]

Proposition 1. *Suppose that $b \in (-I, 0)$. The set of non-babbling stationary equilibria is given by:*

1. *Equilibrium with continuous exercise. The principal exercises at the first time t at which the agent sends $m = 1$, provided that $X(t) \geq X_A^*(1)$ and $X(t) = \max_{s \leq t} X(s)$. The agent of type θ sends message $m = 1$ at the first moment when $X(t)$ crosses her most-preferred threshold $X_A^*(\theta)$.*

2. *Equilibria with partitioned exercise (ω-equilibria). The principal exercises at time t at which $X(t)$ crosses threshold $Y(\omega)$, $\frac{1}{\omega}Y(\omega)$, ... for the first time, where $Y(\omega)$ is given by (20), provided that the agent sends message $m = 1$ at that point. The principal does not exercise the option at any other time. The agent of type θ sends message $m = 1$ the first moment $X(t)$ crosses threshold $Y(\omega)\frac{1}{\omega^n}$, where $n \geq 0$ is such that $\theta \in \left[\omega^{n+1}, \omega^n\right]$. There exists a unique equilibrium for each ω that satisfies (27).*

[5] See the proof of Proposition 1 in the appendix.

[6] As always in cheap talk games, there always exists a "babbling" equilibrium, in which the agent's recommendations are uninformative, and the principal exercises at her optimal uninformed threshold, $\frac{\beta}{\beta-1}2I$. We do not consider this equilibrium, unless it is the unique equilibrium of the game.

If $b < -I$, the unique stationary equilibrium has no information revelation. The principal exercises this option at threshold $\frac{\beta}{\beta-1}2I$.

Thus, as long as $b > -I$, there exist an infinite number of stationary equilibria: one equilibrium with continuous exercise and infinitely many equilibria with partitioned exercise. Both the equilibrium with continuous exercise and the equilibria with partitioned exercise feature delay relative to the principal's optimal timing given the information available to him at the time of exercise.

Clearly, not all of these equilibria are equally "reasonable." It is common in cheap talk games to focus on the equilibrium with the most information revelation, which here corresponds to the equilibrium with continuous exercise. It turns out, as the next proposition shows, that the equilibrium with continuous exercise dominates all equilibria with partitioned exercise in the Pareto sense: It leads to a weakly higher expected payoff for both the principal and all types of the agent.

Proposition 2. *The equilibrium with continuous exercise from Proposition 1 dominates all equilibria with partitioned exercise in the Pareto sense.*

Using Pareto dominance as a selection criterion, we conclude that there is full revelation of information if the agent's bias is not very large, $b > -I$. However, although information is communicated fully in equilibrium, communication and exercise are inefficiently (from the principal's point of view) delayed. Using the terminology of Aghion and Tirole (1997), the equilibrium is characterized by unlimited real authority of the agent, even though the principal has unlimited formal authority. The left panel of Figure 1 illustrates how the equilibrium exercise thresholds depend on the bias and type. If the bias is not too big, there is full revelation of information but procrastination in the action. If the bias is very big, no information is revealed at all, and the principal exercises according to his prior.

Now, consider the case of $\underline{\theta} > 0$. In this case, we show that the equilibrium with continuous exercise from the stationary case of $\underline{\theta} = 0$ takes the form of the equilibrium with continuous exercise up to a cut-off:

Proposition 3. *Suppose that $\underline{\theta} > 0$ and $b \in \left(-\frac{1-\underline{\theta}}{1+\underline{\theta}}I, 0\right]$. The equilibrium with continuous exercise from Proposition 2 no longer exists. However, the equilibrium with continuous exercise up to a cut-off exists. In this equilibrium, the principal's exercise strategy is: (1) to exercise at the first time t at which the agent sends $m = 1$, provided that $\hat{X} \geq X(t) \geq X_A^*(1)$ and*

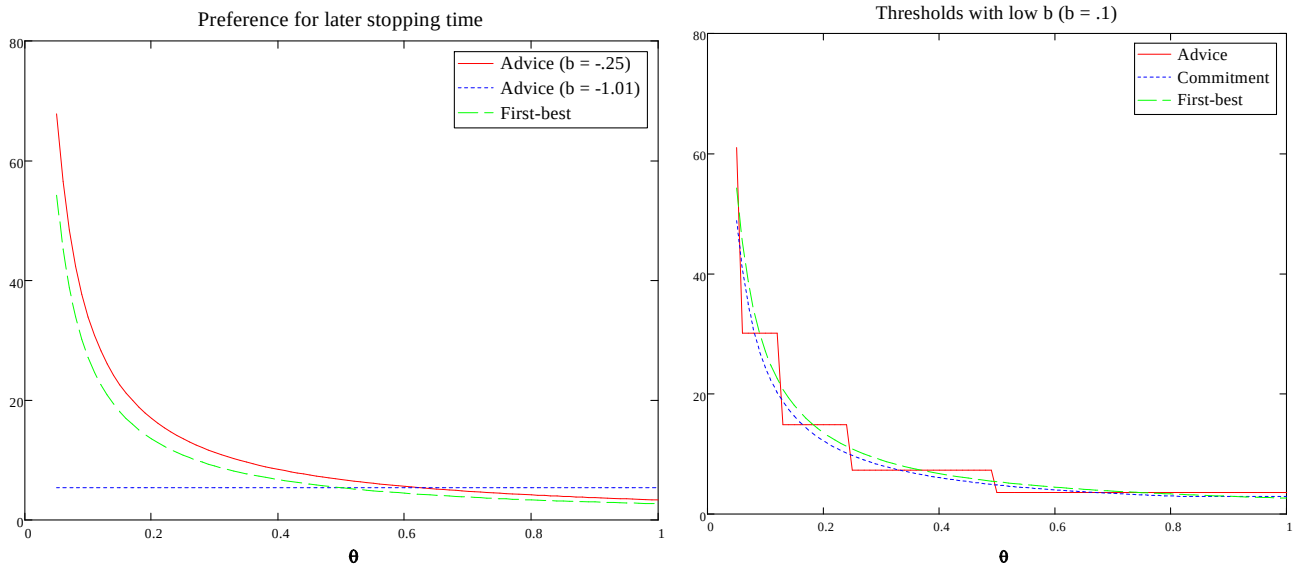


Figure 1. Equilibrium exercise threshold for b<0 and b>0 cases. The left panel illustrates the equilibrium the case of b<0. The right panel illustrates the case of b>0.

$X(t) = \max_{s \leq t} X(s)$; *(2) to exercise at the first time* t *at which* $X(t) \geq \hat{X}$, *regardless of the agent's recommendation. The agent of type* θ *sends message* $m = 1$ *at the first moment when* $X(t)$ *crosses the minimum between her most-preferred threshold* $X_A^*(\theta)$ *and* $\hat{X}$. *Threshold* $\hat{X}$ *is given by*

$$\hat{X} = \frac{\beta}{\beta - 1} \frac{I + b}{\underline{\theta}} = X_A^* \left(\hat{\theta}^* \right), \text{ where } \hat{\theta}^* = \left(\frac{I - b}{I + b} \right) \underline{\theta}.$$

The intuition is as follows. At any time, the principal, who obtains a recommendation against exercise, faces the following trade-off. On the one hand, she can wait and see what the agent will recommend in the future. This option leads to informative exercise, because the agent communicates his information to the principal, but has a drawback in that communication and exercise will be excessively delayed. On the other hand, the principal can overrule the agent's recommendation and exercise immediately. This option results in less informative exercise, but not in excessive delay. Thus, the principal's trade-off is between the value of information and the cost of excessive delay. When $\underline{\theta} = 0$, the problem is stationary and the trade-off persists over time: If the agent's bias is not too high ($b > -I$), waiting for his recommendation is strictly better, while if the agent's bias is too high ($b < -I$), waiting for the agent's recommendation is too costly and communication never happens. However, if $\underline{\theta} > 0$, the problem is non-stationary, and the trade-off between information and delay changes over time. Specifically, as time goes by and the agent recommends against exercise, the principal learns that the agent's type is not too high. This results in the shrinkage of the principal's belief about where θ is: The interval $\left[\underline{\theta}, \hat{\theta}_t\right]$ shrinks over time. As

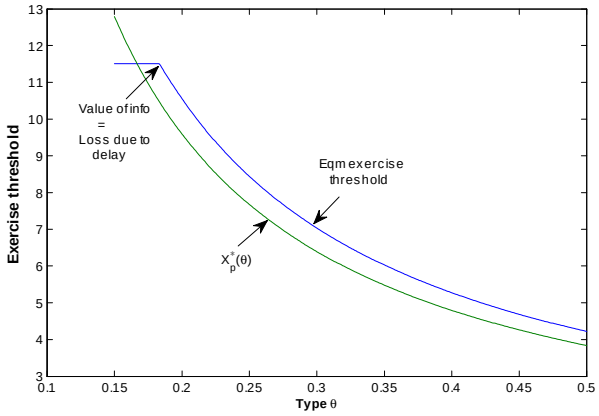


Figure 2. Equilibrium exercise threshold for the case of b<0 and $\underline{\theta} > 0$.

a consequence, the remaining value of private information of the agent declines over time. At the same time, the cost of waiting for information persists. Once the interval shrinks to $\left[\underline{\theta}, \hat{\theta}^*\right]$, which happens at threshold $\hat{X}$, the remaining value of private information that the agent has is not worth it for the principal to wait any longer, and the principal exercises regardless of the recommendation then. Figure 2 illustrates this logic.

The comparative statics of the cut-off type $\hat{\theta}^*$ are intuitive. As b decreases, i.e., the conflict of interest gets bigger, $\hat{\theta}^*$ increases and $\hat{X}$ decreases, implying that the principal waits less for the agent's recommendation. Going back to the terminology of Aghion and Tirole (1997), the equilibrium features limited real authority of the agent: The principal has unlimited formal authority, but the agent has limited real authority, which is limited by an endogenous cut-off $\hat{X}$.

3.2 Preference for earlier exercise

Suppose that $b > 0$, i.e., the agent is biased in the direction of earlier exercise, and focus again on the stationary case $\underline{\theta} = 0$. Because the principal prefers delay over immediate exercise whenever the agent sends message $m = 1$ at his most-preferred threshold, there is no equilibrium with continuous exercise. Indeed, if the agent follows the strategy of recommending exercise at his most-preferred threshold $X_A^*(\theta)$, the principal does not respond by exercising immediately upon getting the recommendation to exercise. Knowing this, the agent is tempted to change his recommendation strategy, mimicking a lower type. Thus, no equilibrium with continuous exercise exists in this case.

For $\omega-$equilibrium with partitioned exercise to exist, the expected value that the principal gets from waiting for recommendations of the agent, $V_P(X, 1; \omega)$, and threshold $Y(\omega)$ must satisfy the

ex-post and ex-ante incentive compatibility conditions (22) and (27). First, suppose that (22) holds as an equality:

$$Y(\omega) = \frac{\beta}{\beta - 1} \frac{2I}{1 + \omega}.$$

Then, using the agent's indifference condition (19) for $Y(\omega)$, we can express ω as the solution to:

$$\omega = \frac{1}{\frac{\beta}{\beta-1} \frac{1-\omega^{\beta-1}}{1-\omega^{\beta}} \frac{2I}{I-b} - 1}. \tag{30}$$

The next lemma shows that when $b > 0$, equation (30) has a unique solution, denoted ω^*:

Lemma 3. *In the range* $[0, 1]$*, equation (30) has a unique solution* $\omega^* \in (0, 1)$*, where* ω^* *decreases in* b *and* $\lim_{b \to 0} \omega^* = 1$*.*

Second, suppose that (22) holds as a strict inequality. Since $Y(\omega)$ increases in ω, then $Y(\omega) > \frac{\beta}{\beta-1} \frac{2I}{1+\omega}$ is equivalent to $\omega < \omega^*$. The ω-equilibrium will exist as long as the ex ante incentive compatibility condition (27) holds as well. Since the left-hand side of (27) increases in ω and the right-hand side does not depend on ω, then (27) is satisfied if ω is high enough. This puts a lower bound on ω, denoted $\underline{\omega}$. The next proposition summarizes the set of equilibria:

Proposition 4. *Suppose that* $b > 0$*. The set of non-babbling equilibria is given by equilibria with partitioned exercise, indexed by* $\omega \in [\underline{\omega}, \omega^*]$*, where* $0 < \underline{\omega} < \omega^* < 1$*,* $\underline{\omega}$ *is implicitly given by (27) binding and* ω^* *is given by the unique solution to (30). The decision-maker exercises at any time* t *at which* $X(t)$ *equals* $Y(\omega)$*,* $\frac{1}{\omega} Y(\omega)$*, ..., where* $Y(\omega)$ *is given by (20), provided that the agent sends message* $m = 1$ *at that point. The decision-maker does not exercise the option at any other time. The agent of type* θ *sends message* $m = 1$ *the first moment* $X(t)$ *crosses threshold* $\frac{1}{\omega^n} Y(\omega)$*, where* $n \geq 0$ *is such that* $\theta \in \left(\omega^{n+1}, \omega^n\right)$*. There exists a unique equilibrium for each* $\omega \in [\underline{\omega}, \omega^*]$*.*

As in the case of the delay preference, the equilibria can also be ranked in informativeness. The most informative equilibrium is the one with the smallest partitions, i.e., ω^*. In this equilibrium, exercise is unbiased: the exercise rule maximizes the payoff to the principal given that the type of the agent lies in a given partition. In all other equilibria, there is both loss of information and procrastination in option exercise. The latter occurs despite the fact that the agent is biased in the direction of exercising too early. Because the ω^*-equilibrium features both the highest degree of communication and a more optimal exercise conditional on the communicated information, it

is more appealing than all equilibria with $\omega \in [\underline{\omega}, \omega^*)$, so going forward we focus on it.[7]

3.3 The role of dynamic communication

In this section, we highlight the role of dynamic communication by comparing our communication model to a model where communication is restricted to a one-shot interaction at the beginning of the game. Specifically, we analyze the setting of the basic model where instead of communicating with the principal continuously, the agent sends a single message m_0 at time $t = 0$, and there is no subsequent communication. After receiving the message, the principal updates his beliefs about the agent's type and then exercises the option at the optimal threshold given these beliefs.

First, note that for any equilibrium of the static communication game, there exists an equivalent equilibrium of the dynamic communication game where all communication after time 0 is uninformative (babbling), and the principal's beliefs are that the agent's messages are uninformative. However, as we show next, the opposite is not true: many equilibria of the dynamic communication game described by Propositions 1 and 4 do not exist in the static communication game. The following result summarizes our findings.

Proposition 5. *Suppose $\underline{\theta} = 0$. If $b < 0$, there is no non-babbling stationary equilibrium of the dynamic communication game that is also an equilibrium of the static communication game. If $b > 0$, the only non-babbling stationary equilibrium of the dynamic communication game that is also an equilibrium of the static communication game is the one with $\omega = \omega^*$, where ω^* is given by the unique solution to (30).*

The intuition behind this result is the following. All non-babbling stationary equilibria of the dynamic communication game for $b < 0$ feature delay relative to what the principal's optimal timing of exercise would have been ex ante, given the information he learns in equilibrium. In a dynamic communication game, this delay is feasible because the principal learns information with delay, after his optimal exercise time has passed. However, in a static communication game, this delay cannot be sustained in equilibrium because it would violate the principal's IC constraint: since the principal learns all the information at time 0, his exercise decision is always optimal given the available information.[8] By the same argument, the only sustainable equilibrium of the

[7] In general, equilibrium selection in cheap-talk games is a delicate issue. Unfortunately, most equilibrium refinements that reduce the set of equilibria in costly signaling games, do not work well in the games of costless signaling (i.e., cheap talk). Here, we follow the standard of the literature to simply assume that the most informative equilibrium is selected. Some formal approaches to equilibrium selection in cheap-talk games are provided by Farrell (1993) and Chen, Kartik, and Sobel (2008).

[8] Similarly, if $\underline{\theta} > 0$, the equilibrium with continuous exercise up to a cut-off, described in Proposition 3, does not exist in the static communication game either.

dynamic communication game for $b > 0$ is the one that features no delay relative to the principal's optimal threshold, i.e., the equilibrium with $\omega = \omega^*$.

Thus, even though information of the agent is persistent, the ability to communicate dynamically makes the analysis rather different from the static "cheap talk" problem. When the agent has a bias for late exercise, timing the recommendation strategically is helpful for both the principal and the agent, because it ensures that the principal does not overrule the agent's recommendation and thereby makes communication effective.

4 Delegation versus communication

So far we made an extreme assumption that the principal has no commitment power at all. In this section, we relax this assumption by allowing the principal to choose between delegating formal authority to exercise the option to the agent and keeping the formal authority but playing the communication game, analyzed in the previous section. The question we are interested in is under what conditions the principal benefits from delegating the decision to the agent.

Formally, we consider the problem studied by Dessein (2002) but for stopping time decisions. In an insightful paper, Dessein (2002) shows that delegating the decision to the informed-but-biased agent often dominates keeping the decision right but communicating with the agent. Delegation always dominates communication in the quadratic-uniform case of Crawford and Sobel (1982), provided that an informative equilibrium exists in the cheap-talk game. For general payoff functions, delegation always dominates if the bias is small enough. Importantly, Dessein (2002) considers static decisions. In this section, we show that the choice between delegation and centralization can be quite different for decisions about the timing of doing something. In particular, we show that if the agent is biased towards late exercise, then delegation is always weakly inferior to communication, and is strictly inferior if $\underline{\theta} > 0$. In contrast, if the agent is biased towards early exercise, we obtain a Dessein-like result that delegation dominates communication for small biases and communication dominates delegation for large biases. Because most decisions that organizations can be delayed and thus involve the stopping time component, these results are important for organizational design.

4.1 Optimal mechanism with commitment

To help us analyze the choice between delegation and not, we first derive an auxiliary result. Suppose that the decision-maker can commit to any decision-making mechanism. By the revelation principle, we can restrict attention to direct revelation mechanisms, i.e., mechanisms in which the message space is $M = \Theta$ and that provide the agent with incentives to report state θ truthfully.

Furthermore, it is easy to show that no mechanism in which exercise occurs not at the first passage time is optimal.

Hence, we can restrict attention to mechanisms in the form $\left\{\hat{X}(\theta) \geq X_0, \theta \in \Theta\right\}$. If the agent reports θ, the decision-maker exercises when $X(t)$ passes threshold $\hat{X}(\theta)$ for the first time. Let $\tilde{U}_A\left(\hat{X}, \theta\right)$ and $\tilde{U}_D\left(\hat{X}, \theta\right)$ denote the expected payoffs as of the initial date to the agent and the decision-maker, respectively, when the true state is θ and the exercise occurs at threshold $\hat{X}$:

$$\tilde{U}_A\left(\hat{X}, \theta\right) \equiv \left(\frac{X_0}{\hat{X}}\right)^{\beta}\left(\theta\hat{X} - I + b\right) \tag{31}$$

$$\tilde{U}_D\left(\hat{X}, \theta\right) \equiv \left(\frac{X_0}{\hat{X}}\right)^{\beta}\left(\theta\hat{X} - I\right) \tag{32}$$

The optimal mechanism maximizes the ex-ante expected payoff to the decision-maker:

$$\max_{\left\{\hat{X}(\theta), \theta \in \Theta\right\}} \int_{\underline{\theta}}^{\overline{\theta}} \tilde{U}_D\left(\hat{X}(\theta), \theta\right) \frac{1}{\overline{\theta} - \underline{\theta}} d\theta, \tag{33}$$

subject to the incentive-compatibility constraint of the agent:

$$\tilde{U}_A\left(\hat{X}(\theta), \theta\right) \geq \tilde{U}_A\left(\hat{X}\left(\hat{\theta}\right), \theta\right) \ \forall \theta, \hat{\theta} \in \Theta. \tag{34}$$

The next proposition uses this lemma to characterize the optimal decision-making rule under commitment:

Proposition 6. *The optimal incentive-compatible threshold schedule $\hat{X}(\theta)$, $\theta \in \Theta$ is given by:*

- *If* $b \in \left(-\infty, -\frac{1-\underline{\theta}}{1+\underline{\theta}} I\right) \cup (I, \infty)$, *then*

$$\hat{X}(\theta) = \frac{\beta}{\beta - 1} \frac{2I}{\underline{\theta} + 1} \ \forall \theta \in \Theta\text{`}. \tag{35}$$

- *If* $b \in \left(-\frac{1-\underline{\theta}}{1+\underline{\theta}} I, 0\right]$, *then*

$$\hat{X}(\theta) = \begin{cases} \frac{\beta}{\beta-1} \frac{I+b}{\underline{\theta}}, & \text{if } \theta \leq \left(\frac{I-b}{I+b}\right) \underline{\theta}; \\ \frac{\beta}{\beta-1} \frac{I-b}{\theta}, & \text{if } \theta \geq \left(\frac{I-b}{I+b}\right) \underline{\theta}. \end{cases} \tag{36}$$

- *If* $b \in (0, I)$*, then*

$$\hat{X}(\theta) = \begin{cases} \frac{\beta}{\beta-1}\frac{I-b}{\theta}, & \text{if } \theta \leq \frac{I-b}{I+b}; \\ \frac{\beta}{\beta-1}(I+b), & \text{if } \theta \geq \frac{I-b}{I+b}. \end{cases} \tag{37}$$

The reasoning behind Proposition 6 is similar to that in the optimal decision-making rule in the static linear-quadratic model (Melumad and Shibano, 1991; Goltsman et al., 2009). Intuitively, because the agent does not receive additional private information over time and the optimal stopping rule can be summarized by a threshold, the optimal dynamic contract is no different from the optimal contract in a static game with equivalent payoff functions.

By comparing previous propositions with Proposition 6, it is easy to see that if the agent is biased towards later exercise, the solutions to the communication problem under no commitment and under commitment coincide. This result does not hold when $b > 0$, i.e., when the agent is biased in the direction of earlier exercise. In this case, the principal could benefit from commitment power. This asymmetry occurs because of the asymmetric nature of time: one cannot go back in time. Therefore, as time passes, even without formal commitment power, the principal effectively commits not to exercise earlier because she cannot go back in time. This is not true when the decision is not a stopping time, but rather a point in the continuum, such as the scale of the project. The idea that one cannot go back in time highlights the unique characteristic of stopping time decisions.

The next proposition summarizes these results:

Proposition 7. *If* $b < 0$*, then* $\bar{X}(\theta) = \hat{X}(\theta)$ *for all* $\theta \in \Theta$*. In particular, the equilibrium payoffs of both parties in the advising game coincide with their payoffs under the optimal commitment mechanism. If* $b > 0$*, then the payoff of the decision-maker in the advising game is lower than his payoff under the optimal commitment mechanism.*

From the organizational design point of view, this result implies that investing in commitment power is not important for decisions in which the agency problem is that the agent wants to delay exercise, such as the decision when to close a plant. In contrast, investing in the commitment power is important for decisions in which the agency problem is that the agent is biased towards early exercise, such as the decision when to do an IPO or when to launch a new product.

4.2 Delegation when the agent has a preference for later exercise

It follows from Proposition 7 and the asymmetric nature of the equilibrium in the communication game that implications for delegations are quite different between the "delay bias" case and the "early exercise bias" case. First, consider $b < 0$, i.e., the case when the agent is biased in the direction of later exercise. If the principal does not delegate the decision to the agent, the principal and the agent will play a communication game. The outcome of this game is either that the option is exercised at the agent's most preferred threshold (if $\underline{\theta} = 0$) or that the option is exercised at the agent's most preferred threshold up to a cut-off (if $\underline{\theta} > 0$). In contrast, if the principal delegates formal authority to exercise the option to the agent, the agent will exercise the option at his most preferred threshold $X_A^*(\theta)$. Clearly, if $\underline{\theta} = 0$, delegation and communication are equivalent. However, if $\underline{\theta} > 0$, they are not equivalent: Not delegating the decision and playing the communication game implements *conditional* delegation (delegation up to a cut-off $\hat{X}$), while delegation implements *unconditional* delegation. By Proposition 7, the principal is strictly better off with the former rather than the latter. This result is summarized in the following proposition.

Proposition 8. *If $b < 0$, i.e., the agent is biased in the direction of late exercise, then the principal prefers retaining control and getting advice from the agent to delegating the exercise decision. The preference is strict if $\underline{\theta} > 0$. If $\underline{\theta} = 0$, then retaining control and delegation are equivalent.*

This result contrasts with the opposite implications for static decisions, such as deciding on the scale of the project that cannot be delayed (Dessein, 2002). Dessein (2002) shows that in the leading quadratic-uniform example of Crawford and Sobel (1982) delegation always dominates communication, as long as the agent's bias is not too high so that at least some informative communication is possible. While this strong result is based on the quadratic-uniform setup, Dessein also shows the general result that delegation dominates communication if the bias is small enough. In contrast, we obtain the opposite result: Regardless of the magnitude of the bias, if the agent is biased towards late exercise, the principal never wants to delegate decision making. Intuitively, time has asymmetric nature: It moves forward, but not backward. Inability to go back in time is a useful feature when the agent has a preference for later exercise, because it allows the principal to commit to follow the recommendation of the agent: Even though ex-post the principal is tempted to revise history and exercise in the past, it is not feasible. This built-in commitment role of time helps informative communication to the extent that delegation has no further benefit.

4.3 Delegation when the agent has a preference for earlier exercise

In contrast to the case when the agent has a preference for later exercise, we show that delegation helps if the agent has a preference for earlier exercise and the bias is low enough. This result is similar to that of Dessein (2002) for static decisions and shows that Dessein's argument extends to stopping time decisions when the agent has a preference for earlier exercise.

For a given b, the expected value to the decision-maker under delegation is:

$$\begin{aligned} VD(b) &= \int_0^1 \left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b)-I\right)d\theta \\ &= \frac{1}{\beta+1}\left(\frac{\beta}{\beta-1}(I-b)\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b)-I\right). \end{aligned}$$

Intuitively, if the decision is completely delegated to the agent, then exercise occurs at threshold $\frac{\beta}{\beta-1}\frac{I-b}{\theta}$, where θ is the private information of the agent, and the resulting payoff upon exercise is $\frac{\beta}{\beta-1}(I-b)-I$.

Consider the most informative equilibrium of the advising game. For a given b, the implied value to the decision-maker is:

$$VA(b) = \frac{1-\omega(b)}{1-\omega(b)^{\beta+1}}\left(\frac{\beta}{\beta-1}\frac{2I}{1+\omega(b)}\right)^{-\beta}\frac{I}{\beta-1} \tag{38}$$

where $\omega(b)$ is the unique solution to (30), given b. First, consider the behavior of $VA(b)$ and $VD(b)$ around $b = I$:

We have the properties:

$$VD(I) = -\infty \tag{39}$$

$$VA(I) = \left(\frac{\beta}{\beta-1}2I\right)^{-\beta}\frac{I}{\beta-1} \tag{40}$$

By continuity of $VD(I)$ and $VA(I)$, (39) and (40) imply that there exists $b_h \in (0, I)$, such that for any $b > b_h$, $VA(I) > VD(I)$. In other words, advising dominates delegation if the conflict of interest between the agent and the decision-maker is big enough.

Second, consider the behavior of $VA(b)$ and $VD(b)$ for small but positive b. We have:

$$VD(0) = VA(0) = \frac{1}{\beta+1}\left(\frac{\beta}{\beta-1}I\right)^{-\beta}\frac{I}{\beta-1}.$$

The derivative of $VD(b)$ is

$$VD'(b) = -\frac{\beta b}{(I-b)(\beta+1)\left(\frac{\beta(I-b)}{\beta-1}\right)^{\beta}}.$$

In particular, $VD'(b) = 0$. The derivative of $VA(\omega)$ is

$$VA'(b) = \left(\frac{\beta}{\beta-1}2I\right)^{-\beta} \frac{I(1+\omega)^{\beta-1}\left(\beta-1-(\beta+1)\left(\omega-\omega^{\beta}\right)\right)\omega^{\beta-2}}{(\beta-1)(I-b)} \frac{\left(1-\omega^{\beta-1}\right)\left(1-\omega^{\beta}\right)}{\left(1-\omega^{\beta+1}\right)^2\left(-\beta+1-\omega^{\beta}+\beta\omega\right)},$$

where ω denotes $\omega(b)$. Applying the product rule of limits and using the fact that $\lim_{b\downarrow 0}\omega(b) = 1$, we have:

$$\lim_{b\downarrow 0} VA'(b) = \frac{1}{2}\left(\frac{\beta}{\beta-1}I\right)^{-\beta} \lim_{\omega\uparrow 1} \frac{\left(1-\omega^{\beta-1}\right)\left(1-\omega^{\beta}\right)}{\left(1-\omega^{\beta+1}\right)^2\left(-\beta+1-\omega^{\beta}+\beta\omega\right)}. \tag{41}$$

Let $n(\omega)$ and $d(\omega)$ denote the numerator and the denominator of the limit term on the right-hand side of (41). Manually verify that $n(1) = n'(1) = 0$ and $n''(1) = 2\beta(\beta-1) > 0$. Similarly, manually verify that $d(1) = d'(1) = d''(1) = 0$. Therefore, by l'Hopital's rule,

$$\lim_{b\downarrow 0} VA'(b) = \frac{1}{2}\left(\frac{\beta}{\beta-1}I\right)^{-\beta} \lim_{\omega\uparrow 1} \frac{n''(\omega)}{d''(\omega)} = -\infty, \tag{42}$$

as $d''(\omega) < 0$ for ω close enough but less than one. By continuity of $VA'(b)$ and $VD'(b)$ for $b > 0$, there exists $b_l > 0$ such that $VA'(b) < VD'(b)$ for any $b < b_l$. Because $VA(0) = VD(0) = 0$,

$$\begin{aligned} VD(b) - VA(b) &= \int_0^b \left(VD'(x) - VA'(x)\right)dx \\ &> 0 \end{aligned}$$

for any $b \in (0, b_l]$. In other words, delegation dominates advising when the agent is biased but the bias is low enough.

5 Implications

5.1 Comparative statics

The model delivers interesting comparative statics with respect to the underlying parameters. First, consider the case of an agent with a late exercise bias, $b < 0$. There is full revelation of information, independent of the agent's bias and the underlying parameters of the model μ, σ, and r. The decision is delayed with the factor of $\frac{I-b}{I}$, which is also independent of these parameters.

Second, consider the case of an agent biased towards early exercise, $b > 0$. In this case, equilibrium exercise is unbiased, but there is loss of information since $\omega^* < 1$. The comparative statics results are presented in the next proposition and illustrated in Figure 3.

Proposition 9. *Consider the case of an agent biased towards early exercise, $b > 0$. Then, ω^* decreases in b and increases in β and hence decreases in σ and μ, and increases in r.*

The equilibrium partitioning multiple ω^* is decreasing in the bias of the agent, in line with Crawford and Sobel's (1982) result that less information is revealed if the misalignment of preferences is bigger. More interesting are the comparative statics results in parameters μ, σ, and r. We find that that the parameters that drive the magnitude of the option to wait reduce information revelation in equilibrium. For example, there is less information revelation in equilibrium (ω^* is lower) if the underlying environment is more uncertain (σ is higher). Intuitively, unlike in a standard investment-under-uncertainty model, volatility plays two roles in our setting. First, its standard role is that it increases the value of the option to wait for the decision maker, resulting in a higher exercise threshold. Its second role, however, is that higher uncertainty also increases the value of the option to wait for the agent. This effect makes it more difficult to extract information from the agent, leading to less efficient communication in equilibrium. This is so despite the fact that the informational advantage of the agent is unchanged. Similarly, communication is less efficient in the lower-interest and higher-growth environments.

5.2 Strategic choice of the agent

We next show that asymmetry has also important implications for the strategic choice of an agent. Specifically, the next proposition shows that if the decision-maker needs to choose between an agent biased towards early exercise and an agent biased towards late exercise with the same (in absolute value) bias, the decision-maker is better off choosing the agent with a late exercise bias:

Proposition 10. *Let $V_0(b)$ be the expected payoff of the decision-maker at the initial date $t = 0$ in the most informative equilibrium, given that the agent's bias is b. Then, $V_0(-b) \geq V_0(b)$ for any $b \geq 0$ and $V_0(-b) > V_0(b)$ for any $b \in (0, I)$.*

Proposition 10 implies that between the two problems, poor communication but unbiased timing and full communication but a procrastinated timing, the former is a bigger problem. In-

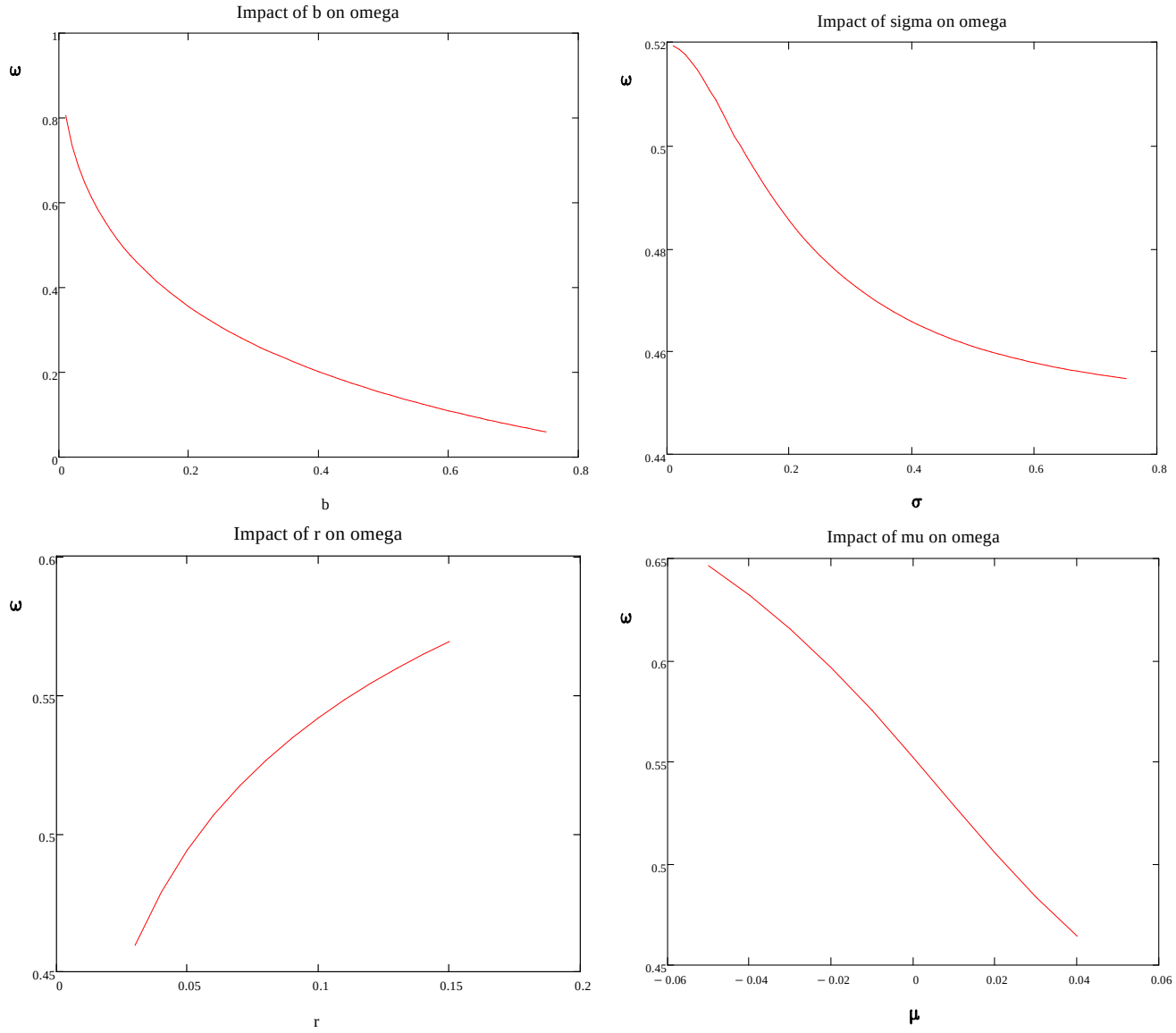


Figure 3. Comparative statics of ω^* in parameters of the model.

tuitively, when dealing with an agent biased towards late exercise, the problem features a built-in commitment power of the decision-maker. Because revising past decisions is not feasible, the decision-maker has effectively no conflict of interest when the agent with a late exercise bias recommends exercise. This makes the promise to exercise credible and communication effective. This is not true if the decision-maker deals with an agent biased towards early exercise. Even though, the decision-maker would have liked to promise the agent to speed up exercise in order to elicit his private information, such promise is not credible. Because of the built-in commitment power when dealing with an agent biased towards late exercise but not when dealing with an agent biased towards early exercise, the decision-maker is better off dealing with an agent biased towards late exercise.

6 Conclusion

This paper considers a problem in which an uninformed decision-maker solicits advice from an informed but biased agent on when to exercise an option. Depending on the application, the agent may be biased in favor of late or early exercise. In contrast to the static cheap talk setting, where the decision variable is scale rather than stopping time, the equilibrium is asymmetric in the direction of the agent's bias. When the agent is biased towards late exercise, there is full revelation of information but suboptimal delay. Conversely, when the agent is biased towards early exercise, there is partial revelation of information but unbiased exercise. The reason for this asymmetry lies in the asymmetric nature of time as a decision variable. While the decision-maker can always get advice and exercise the option at a later point in time, he cannot go back and get advice and exercise the option at an earlier point in time.

The analysis has implications for the informativeness and timeliness of option exercise decisions, depending on the direction of the agent's bias and on the parameters of the stochastic environment, such as volatility, growth rate, and discount rate.

We next analyze the case of delegation, in which the decision-maker can delegate authority over option exercise to the agent. We show that the optimal choice between delegation and centralization is also asymmetric in the direction of the agent's bias. Delegation is always inferior when the agent is biased towards late exercise, but is optimal when the agent is biased towards early exercise and his bias is not very large. The decision-maker's benefit from the ability to commit also depends on the direction of the agent's bias. When the agent is biased towards late exercise, the ability to commit to any decision-making mechanism does not improve the decision-maker's payoff relative to the advisory setting without commitment. In contrast, when the agent is biased towards early exercise, the decision-maker would benefit from an ability to commit.

In unreported results, we consider the setting of put options rather than call options and find that the important conclusions of our model continue to hold. A potentially interesting avenue to explore would be more general settings of decisions under uncertainty beyond these simple option payoffs. Other extensions of possible interest would be to consider more general stochastic processes as well as non-stationary or non-Markov equilibria.

Appendix: Proofs

Proof of Lemma 1. By contradiction, suppose that $\bar{X}(\theta_1) < \bar{X}(\theta_2)$ for some $\theta_2 > \theta_1$. Because the message strategy of type θ_1 is feasible for type θ_2, the incentive compatibility of type θ_2 implies:

$$\left(\frac{X(t)}{\bar{X}(\theta_2)}\right)^{\beta}\left(\theta_2\bar{X}(\theta_2) - I + b\right) \geq \left(\frac{X(t)}{\bar{X}(\theta_1)}\right)^{\beta}\left(\theta_2\bar{X}(\theta_1) - I + b\right). \tag{43}$$

Similarly, because the message strategy of type θ_2 is feasible for type θ_1,

$$\left(\frac{X(t)}{\bar{X}(\theta_1)}\right)^{\beta}\left(\theta_1\bar{X}(\theta_1) - I + b\right) \geq \left(\frac{X(t)}{\bar{X}(\theta_2)}\right)^{\beta}\left(\theta_1\bar{X}(\theta_2) - I + b\right). \tag{44}$$

These inequalities imply

$$\begin{aligned}\theta_2\bar{X}(\theta_1)\left(1 - \left(\frac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)}\right)^{\beta-1}\right) &\leq (I-b)\left(1 - \left(\frac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)}\right)^{\beta}\right) \\ &\leq \theta_1\bar{X}(\theta_1)\left(1 - \left(\frac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)}\right)^{\beta-1}\right),\end{aligned} \tag{45}$$

which is a contradiction, because $\theta_2 > \theta_1$ and $\frac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)} < 1$. Therefore, $\bar{X}(\theta_1) \geq \bar{X}(\theta_2)$ for any $\theta_1, \theta_2 \in \Theta$ such that $\theta_2 \geq \theta_1$. ■

Proof of Lemma 2. Consider a threshold exercise equilibrium E with an arbitrary message space M^* and equilibrium message strategy $m^*(\theta) = \{m_t^*(\theta), t \geq 0\}$, in which exercise occurs at stopping time $\tau^*(\theta) = \inf\left\{t \geq 0 | X(t) \geq \bar{X}(\theta)\right\}$ for some set of thresholds $\bar{X}(\theta)$, $\theta \in \Theta$. By Lemma 1, $\bar{X}(\theta)$ is weakly decreasing. Define $\theta_l(X) \equiv \inf\left\{\theta : \bar{X}(\theta) = X\right\}$ and $\theta_h(X) \equiv \sup\left\{\theta : \bar{X}(\theta) = X\right\}$ for any $X \in \mathcal{X}$. We will construct a different equilibrium, denoted by $\bar{E}$, which implements the same equilibrium exercise time $\tau^*(\theta)$ and has the structure specified in the formulation of the lemma. As we shall see, it will imply that on the equilibrium path, the principal exercises the option at the first informative time $t \in \mathcal{T}$ at which he receives message $m(t) = 1$, where the set $\mathcal{T}$ of informative times is defined as

$$\mathcal{T} \equiv \left\{t : X(t) = \bar{X} \text{ for some } \bar{X} \in \mathcal{X} \text{ and } X(t) > X(s) \ \ \forall s < t \ \right\}. \tag{46}$$

For the collection of strategies (7) and (8) and the corresponding beliefs to be an equilibrium, we need to verify incentive compatibility of the agent and the principal.

1 - Incentive compatibility of the agent. The condition is that any type θ is better off when exercise occurs at threshold $\bar{X}(\theta)$. By Assumption 1, a deviation to sending $m_t = 1$ at any $t \notin \mathcal{T}$ does not lead to the principal changing her beliefs, and hence, her behavior. Thus, it is without loss of generality to only consider deviations at $t \in \mathcal{T}$. If type θ deviates to sending message $m_t = 1$ when $X(t)$ hits threshold $\bar{X}\left(\hat{\theta}\right)$, $\hat{\theta} > \theta_h\left(\bar{X}(\theta)\right)$ for the first time, the principal exercises immediately. If type θ deviates to sending message $m_t = 0$ when $X(t)$ hits threshold $\bar{X}(\theta)$, then the principal would exercise the option at one of thresholds $\hat{X} \in \mathcal{X}$, $\hat{X} > \bar{X}(\theta)$. Furthermore, type θ can always ensure exercise at any threshold $\hat{X} \in \mathcal{X}$ by adopting the equilibrium message strategy of type $\tilde{\theta}$ at which $\bar{X}\left(\tilde{\theta}\right) = \hat{X}$. Thus, the incentive compatibility condition of the agent is:

$$\left(\frac{X(t)}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta) - I + b\right) \geq \max_{\hat{X}\in\mathcal{X},\hat{X}>X(t)}\left(\frac{X(t)}{\hat{X}}\right)^{\beta}\left(\theta\hat{X} - I + b\right) \tag{47}$$

Let us argue that it holds. Suppose otherwise. Then, there exists a pair $\left(\theta, \hat{X}\right)$ with $\hat{X} \in \mathcal{X}$ such that

$$\frac{\theta\bar{X}(\theta) - I + b}{\bar{X}(\theta)^{\beta}} < \frac{\theta\hat{X} - I + b}{\hat{X}^{\beta}} \tag{48}$$

However, (48) implies that in equilibrium E type θ is better off deviating from message strategy $m^*(\theta)$ to the message strategy $m^*\left(\tilde{\theta}\right)$, where $\tilde{\theta}$ is any type satisfying $\bar{X}\left(\tilde{\theta}\right) = \hat{X}$ (since $\hat{X} \in X$, at least one such $\tilde{\theta}$ exists). Thus, (47) holds. Hence, if the principal plays strategy (8), the agent finds it optimal to play strategy (7).

2 - "Ex-post" incentive compatibility of the principal. Second, consider incentive compatibility of the principal. It is comprised of two parts, as evident from (8): we refer to the top line of (8) (not waiting when the principal "should" exercise) as the ex-post incentive compatibility condition, and to the bottom line of (8) (not exercising when the principal "should" wait) as the ex-ante incentive compatibility condition. First, consider the ex-post incentive compatibility condition. Suppose that the principal believes that $\theta \sim Uni\left[\theta_l\left(\hat{X}\right), \theta_h\left(\hat{X}\right)\right]$ for some $\hat{X} \in X$. Because the principal expects the agent to play (7), the principal expects the agent to send $m_t = 1$, if $X(t) \geq \hat{X}$, and $m_t = 0$, otherwise, regardless of $\theta \in \left[\theta_l\left(\hat{X}\right), \theta_h\left(\hat{X}\right)\right]$. Therefore, the principal's problem is equivalent to the standard option exercise problem with the option paying off $\frac{\theta_l\left(\hat{X}\right)+\theta_h\left(\hat{X}\right)}{2}X(t)$ upon exercise at time t. The solution (e.g., Dixit and Pindyck, 1994) is to exercise the option if and only if

$$X(t) \geq \frac{\beta}{\beta - 1}\frac{2I}{\theta_l\left(\hat{X}\right) + \theta_h\left(\hat{X}\right)}, \tag{49}$$

where β is given by (14). Let us show that any threshold $\hat{X} \in \mathcal{X}$ and the corresponding type cut-offs $\theta_l\left(\hat{X}\right)$ and $\theta_h\left(\hat{X}\right)$ in equilibrium E satisfy (49). Consider equilibrium E. For the principal to exercise at threshold $\bar{X}(\theta)$, the value that the principal gets upon exercise must be greater or equal than what she gets from delaying the exercise. The former equals $\mathbb{E}\left[\theta|\mathcal{H}_t, m_t = m_t^*(\theta)\right]\bar{X}(\theta) - I$, where $(\mathcal{H}_t, m_t)$ is any history of the sample path $\{X(s), s \leq t\}$ and equilibrium messages that leads to exercise at time t at threshold $\bar{X}(\theta)$. Suppose that the principal delays exercise. Because waiting until $X(t)$ hits a threshold $\tilde{X} > \bar{X}(\theta)$ and exercising then is a feasible strategy, the value from delay cannot be lower than the value from such a deviation, which equals

$$\left(\frac{\bar{X}(\theta)}{\tilde{X}}\right)^{\beta}\left(\mathbb{E}\left[\theta|\mathcal{H}_t, m_t\right]\tilde{X} - I\right). \tag{50}$$

Hence, $\bar{X}(\theta)$ must satisfy

$$\bar{X}(\theta) \in \arg\max_{\tilde{X} \geq \bar{X}(\theta)}\left(\frac{\bar{X}(\theta)}{\tilde{X}}\right)^{\beta}\left(\mathbb{E}\left[\theta|\mathcal{H}_t, m_t\right]\tilde{X} - I\right). \tag{51}$$

Using the fact that the unconditional maximizer of the right-hand side is $\tilde{X} = \frac{\beta}{\beta-1}\frac{I}{\mathbb{E}[\theta|\mathcal{H}_t, m_t]}$ and that function $\left(\frac{\bar{X}(\theta)}{\tilde{X}}\right)^{\beta}\left(\mathbb{E}\left[\theta|\mathcal{H}_t, m_t\right]\tilde{X} - I\right)$ is inverted U-shaped in $\tilde{X}$, this condition can be equivalently re-written as

$$\bar{X}(\theta) \geq \frac{\beta}{\beta - 1}\frac{I}{\mathbb{E}\left[\theta|\mathcal{H}_t, m_t\right]}, \tag{52}$$

for any history $(\mathcal{H}_t, m_t)$ with $X(t) = \bar{X}(\theta)$ and $m_s = m_s^*(\mathcal{H}_s, \theta)$ for some $\theta \in \left[\theta_l\left(\hat{X}\right), \theta_h\left(\hat{X}\right)\right]$. Let $\mathbb{H}_t^*$ denote the set of such histories. Therefore,

$$\bar{X}(\theta) \geq \frac{\beta}{\beta - 1}\max_{(\mathcal{H}_t, m_t) \in \mathbb{H}_t^*}\frac{I}{\mathbb{E}\left[\theta|\mathcal{H}_t, m_t\right]}, \tag{53}$$

or, equivalently,

$$\begin{aligned}
\frac{\beta}{\beta - 1}\frac{I}{\bar{X}(\theta)} &\leq \min_{(\mathcal{H}_t, m_t) \in \mathbb{H}_t^*}\mathbb{E}\left[\theta|\mathcal{H}_t, m_t\right] \\
&\leq \mathbb{E}\left[\mathbb{E}\left[\theta|\mathcal{H}_t, m_t\right]|\theta \in \left[\theta_l\left(\hat{X}\right), \theta_h\left(\hat{X}\right)\right], \mathcal{H}_0\right] \\
&= \mathbb{E}\left[\theta|\theta \in \left[\theta_l\left(\hat{X}\right), \theta_h\left(\hat{X}\right)\right]\right] = \frac{\theta_l\left(\hat{X}\right) + \theta_h\left(\hat{X}\right)}{2},
\end{aligned} \tag{54}$$

where the inequality follows from the fact that the minimum of a random variable cannot exceed its mean, and the equality follows from the law of iterated expectations. Therefore, when the principal obtains

message $m = 1$ at threshold $\hat{X} \in X$, her optimal reaction is to exercise immediately. Thus, the ex-post incentive compatibility condition of the principal is satisfied.

3 - "Ex-ante" incentive compatibility of the principal. Finally, consider the ex-ante incentive compatibility condition of the principal stating that the principal is better off waiting following a history $\mathcal{H}_t$ with $m_s = 0$, $s \leq t$, and $\max_{s \leq t} X(s) < \bar{X}(\underline{\theta})$. Given that the agent follows (8), at any such history $\mathcal{H}_t$, the principal's belief is that $\theta \sim Uni\left[0, \theta_l\left(\hat{X}\right)\right]$ for some $\hat{X} \in \mathcal{X}$. If the principal exercises immediately, her expected payoff is $\frac{\theta_l(\hat{X})}{2} X(t) - I$. If the principal waits, her expected payoff is

$$\int_{\underline{\theta}}^{\theta_l(\hat{X})} \left(\frac{X(t)}{\bar{X}(\theta)}\right)^{\beta} \left(\mathbb{E}\left[\theta | \theta \in \left[\theta_l\left(\bar{X}(\theta)\right), \theta_h\left(\bar{X}(\theta)\right)\right]\right] \bar{X}(\theta) - I\right) \frac{1}{\theta_l\left(\hat{X}\right)} d\theta. \tag{55}$$

Suppose that there exists a pair $\hat{X}$ and $X(t) < \lim_{\theta \uparrow \theta_l(\hat{X})} \bar{X}(\theta)$ such that immediate exercise is optimal:

$$\frac{\theta_l\left(\hat{X}\right)}{2} X(t) - I > \int_0^{\theta_l(\hat{X})} \left(\frac{X(t)}{\bar{X}(\theta)}\right)^{\beta} \left(\mathbb{E}\left[\theta | \theta \in \left[\theta_l\left(\bar{X}(\theta)\right), \theta_h\left(\bar{X}(\theta)\right)\right]\right] \bar{X}(\theta) - I\right) \frac{1}{\theta_l\left(\hat{X}\right)} d\theta. \tag{56}$$

Denote the pair of such values by $\left(\hat{X}, \tilde{X}\right)$. Then, we can re-write (56) as

$$\mathbb{E}_\theta\left[\left(\frac{1}{\tilde{X}}\right)^{\beta} \left(\theta \tilde{X} - I\right) | \theta < \theta_l\left(\hat{X}\right)\right] > \mathbb{E}_\theta\left[\left(\frac{1}{\bar{X}(\theta)}\right)^{\beta} \left(\theta \bar{X}(\theta) - I\right) | \theta < \theta_l\left(\hat{X}\right)\right]. \tag{57}$$

Let us show that if equilibrium E exists, then (57) must be satisfied. Consider equilibrium E, any type $\tilde{\theta} < \theta_l\left(\hat{X}\right)$, time $t < \tau^*\left(\tilde{\theta}\right)$, and any history $(\mathcal{H}_t, m_t)$ such that $X(t) = \tilde{X}$, $\max_{s \leq t, s \in \mathcal{T}} X(s) = \hat{X}$, which is consistent with the equilibrium play of type $\tilde{\theta}$, i.e., with $m_s = m_s^*\left(\tilde{\theta}, \mathcal{H}_s\right)$ $\forall s \leq t$. Let $\mathbb{H}_t^{**}\left(\tilde{\theta}, \tilde{X}, \hat{X}\right)$ denote the set of such histories. Because the principal prefers waiting in equilibrium E, the payoff from immediate exercise cannot exceed the payoff from waiting:

$$\mathbb{E}\left[\theta \tilde{X} - I | \mathcal{H}_t, m_t\right] \leq \mathbb{E}\left[\left(\frac{\tilde{X}}{\bar{X}(\theta)}\right)^{\beta} \left(\theta \bar{X}(\theta) - I\right) | \mathcal{H}_t, m_t\right],$$

or, equivalently,

$$\mathbb{E}\left[\left(\frac{1}{\bar{X}(\theta)}\right)^{\beta} \left(\theta \bar{X}(\theta) - I\right) - \left(\frac{1}{\tilde{X}}\right)^{\beta} \left(\theta \tilde{X} - I\right) | \mathcal{H}_t, m_t\right] \geq 0. \tag{58}$$

This inequality must hold for all histories $(\mathcal{H}_t, m_t) \in \mathbb{H}_t^{**}\left(\tilde{\theta}, \tilde{X}, \hat{X}\right)$. In any history $(\mathcal{H}_t, m_t) \in$

$\mathbb{H}_t^{**}\left(\tilde{\theta},\tilde{X},\hat{X}\right)$, the option is never exercised by time t if $\tilde{\theta}<\theta_l\left(\hat{X}\right)$ and is exercised before time t if $\tilde{\theta}>\theta_l\left(\hat{X}\right)$. Therefore, conditional on $\tilde{X}$, $\hat{X}$, and $\tilde{\theta}<\theta_l\left(\hat{X}\right)$, the distribution of $\tilde{\theta}$ is independent of the sample path of $X(s), s\leq t$. Fixing $\tilde{X}$ and $\hat{X}$, and integrating over histories $(\mathcal{H}_t,m_t)\in\mathbb{H}_t^{**}\left(\tilde{\theta},\tilde{X},\hat{X}\right)$ and then over $\tilde{\theta}\in\left[0,\theta_l\left(\hat{X}\right)\right)$, we obtain that

$$\mathbb{E}_\theta\left[\mathbb{E}_{(\mathcal{H}_t,m_t)}\left[\left(\frac{1}{\bar{X}(\theta)}\right)^\beta\left(\theta\bar{X}(\theta)-I\right)-\left(\frac{1}{\hat{X}}\right)^\beta\left(\theta\hat{X}-I\right)|\,(\mathcal{H}_t,m_t)\in\mathbb{H}_t^{**}\left(\tilde{\theta},\tilde{X},\hat{X}\right)\right]|\tilde{\theta}\in\left[\underline{\theta},\theta_l\left(\hat{X}\right)\right),\hat{X},\tilde{X}\right] \tag{59}$$

must be non-negative. Equivalently,

$$\mathbb{E}\left[\left(\frac{1}{\bar{X}(\theta)}\right)^\beta\left(\theta\bar{X}(\theta)-I\right)-\left(\frac{1}{\tilde{X}}\right)^\beta\left(\theta\tilde{X}-I\right)|\theta<\theta_l\left(\hat{X}\right)\right]\geq 0, \tag{60}$$

where we applied the law of iterated expectations and the conditional independence of the sample path of $X(t)$ and the distribution of $\tilde{\theta}$ (conditional on $\tilde{X}$, $\hat{X}$, and $\tilde{\theta}<\theta_l\left(\hat{X}\right)$). Therefore, (57) cannot hold. Hence, the ex-ante incentive compatibility condition of the principal is also satisfied.

We conclude that if there exists a threshold exercise equilibrium E in which $\tau^*(\theta)=\inf\left\{t\geq 0|X(t)\geq\bar{X}(\theta)\right\}$ for some threshold $\bar{X}(\theta)$, then there exists a threshold exercise equilibrium $\bar{E}$ of the form specified in the lemma, in which the option is exercised at the same time. Finally, let us see that on the equilibrium path, the option is indeed exercised at the first informative time t at which the principal receives message $m(t)=1$. Because any message sent at $t\notin\mathcal{T}$ does not lead to updating of the principal's beliefs and because of the second part of (8), the principal never exercises the option prior to the first informative time $t\in\mathcal{T}$ at which she receives message $m(t)=1$. Consider the first informative time $t\in\mathcal{T}$ at which the principal receives $m(t)=1$. By the Bayes' rule, the principal believes that θ is distributed uniformly over $(\theta_l(X(t)),\theta_h(X(t)))$. Equilibrium strategy of the agent (7) implies $X(t)=\bar{X}(\theta)$ $\forall\theta\in(\theta_l(X(t)),\theta_h(X(t)))$. Therefore, in equilibrium the principal exercises the option immediately. ■

Derivation of the principal's value function in the ω-equilibrium, $V_P\left(X(t),\hat{\theta}_t;\omega\right)$. It satisfies

$$rV_P(X,1;\omega)=\mu XV_{P,X}(X,1;\omega)+\frac{1}{2}\sigma^2X^2V_{P,XX}(X,1;\omega). \tag{61}$$

The value matching condition is:

$$V_P(Y(\omega),1;\omega)=\int_\omega^1(\theta Y(\omega)-I)\,d\theta+\omega V_P(Y(\omega),\omega;\omega). \tag{62}$$

The intuition behind (62) is as follows. With probability $1-\omega$, θ is above ω. In this case, the agent

recommends exercise, and the principal follows the recommendation. The payoff of the principal, given θ, is $\theta Y(\omega) - I$. With probability ω, θ is below ω, so the agent recommends against exercise, and the option is not exercised. The continuation payoff of the principal in this case is $V_P(Y(\omega), \omega; \omega)$. Solving (61) subject to (62), we obtain

$$V_P(X, 1; \omega) = \left(\frac{X}{Y(\omega)}\right)^{\beta} \left(\int_{\omega}^{1} (\theta Y(\omega) - I)\, d\theta + \omega V_P(Y(\omega), \omega; \omega)\right). \tag{63}$$

By stationarity,

$$V_P(Y(\omega), \omega; \omega) = V_P(\omega Y(\omega), 1; \omega). \tag{64}$$

Evaluating (63) at $X = \omega Y(\omega)$ and using the stationarity condition (64), we obtain:

$$V_P(\omega Y(\omega), 1; \omega) = \omega^{\beta}\left[\frac{1}{2}\left(1 - \omega^2\right) Y(\omega) - (1 - \omega) I\right] + \omega^{\beta+1} V_P(\omega Y(\omega), 1; \omega). \tag{65}$$

Therefore,

$$V_P(\omega Y(\omega), 1; \omega) = \frac{\omega^{\beta}(1 - \omega)}{1 - \omega^{\beta+1}}\left[\frac{1}{2}(1 + \omega) Y(\omega) - I\right]. \tag{66}$$

Plugging (66) into (63), we obtain the principal's value function. ■

Proof of Proposition 1. First, we demonstrate that $\bar{X}(1, \omega) \geq \frac{\beta}{\beta - 1}\frac{2I}{1+\omega}$ for any positive $\omega < 1$. Define:

$$G(\omega) = \frac{\left(1 - \omega^{\beta}\right)(I - b)}{\omega\left(1 - \omega^{\beta-1}\right)} - \frac{\beta}{\beta - 1}\frac{2(I - b)}{1 + \omega}.$$

Rewriting:

$$\begin{aligned} G(\omega) &= \frac{2(I - b)}{1 + \omega} g(\omega), \\ g(\omega) &= \frac{\left(1 - \omega^{\beta}\right)(1 + \omega)}{2\left(\omega - \omega^{\beta}\right)} - \frac{\beta}{\beta - 1}. \end{aligned}$$

We have:

$$\begin{aligned} \lim_{\omega \to 1} g(\omega) &= \lim_{\omega \to 1} \frac{1 - \omega^{\beta} - \beta\omega^{\beta-1}(1 + \omega)}{2\left(1 - \beta\omega^{\beta-1}\right)} - \frac{\beta}{\beta - 1} = 0, \\ g'(\omega) &= \frac{\beta\left(\omega^{\beta-1} - \omega^{\beta+1}\right) + \omega^{2\beta} - 1}{2\left(\omega - \omega^{\beta}\right)^2}, \end{aligned}$$

where the first limit holds by l'Hopitale's rule. Let $h(\omega) \equiv \omega^{2\beta} - \beta\omega^{\beta+1} + \beta\omega^{\beta-1} - 1$. Function $h(\omega)$ is a generalized polynomial. By an extension of Descartes' Rule of Signs to generalized polynomials (Laguerre,

1883),[9] the number of positive roots of $h(\omega)$, counted with their orders, does not exceed three. Because $\omega = 1$ is the root of $h(\omega)$ of order three and $h(0) < 0$, we can write

$$h(\omega) = (1-\omega)^3 \eta(\omega),$$

where $\eta(\omega) < 0$ for all $\omega \geq 0$. Therefore, $g'(\omega) < 0$ for all $\omega \in [0,1)$. Combining with $\lim_{\omega \to 1} g(\omega) = 0$, this implies $g(\omega) > 0$ and $G(\omega) > 0$ for all $\omega \in [0,1)$. Thus,

$$\bar{X}\left(\hat{\theta}\right) \geq \frac{\beta}{\beta - 1} \frac{2(I-b)}{1+\omega} \geq \frac{\beta}{\beta - 1} \frac{2I}{1+\omega},$$

where the second inequality follows from the fact that $b < 0$.

Now we consider constraint (27). If it does not hold, the $\omega-$equilibrium does not exist, because the decision-maker benefits from deviating to exercising at $\bar{X}_u$ instead of waiting to get a message at $\bar{X}(1,\omega) > \bar{X}_u$. If it holds, then the decision-maker does not exercise the option before the agent sends message $m = 1$, and when the agent sends $m = 1$ when $X(t) \in X$, the decision-maker is better off exercising the option immediately. Hence, the ω-equilibrium exists in this case. ■

Proof of Proposition 2. Note that in equilibrium with continuous exercise, exercise occurs at the unconstrained optimal time of any type θ of the agent. Therefore, the payoff of any type θ of the agent is higher in the equilibrium with continuous exercise than in any other equilibrium. As Section 4 shows, if the decision maker could commit to any mechanism, the exercise strategy implied by the optimal mechanism coincides with the exercise strategy in the equilibrium with continuous exercise. Therefore, the decision-maker's expected payoff in it exceeds the decision-maker's expected payoff under any other exercise rule, in particular, under the exercise rules implied by equilibria with partitioned exercise. ■

Proof of Proposition 3. Given that the principal plays the strategy stated in Proposition 3, it is clear that the strategy of any type θ of the agent is incentive compatible. Indeed, for any type $\theta \geq \left(\frac{I-b}{I+b}\right)\underline{\theta}$, exercise occurs at the unconstrained optimal time. Therefore, no type $\theta \geq \left(\frac{I-b}{I+b}\right)\underline{\theta}$ can benefit from a deviation. Any type $\theta < \left(\frac{I-b}{I+b}\right)\underline{\theta}$ cannot benefit from a deviation either, since inducing the principal to delay even further is not possible, given that the principal exercises at threshold $\hat{X}$ regardless of the recommendation, and the agent only loses from inducing the principal to exercise earlier. ■

[9] See Theorem 3.1 in Jameson (2006).

Proof of Lemma 3. We can rewrite this equation as

$$\omega = \frac{(\beta-1)\left(1-\omega^{\beta}\right)(I-b)}{\beta\left(1-\omega^{\beta-1}\right)2I-(\beta-1)\left(1-\omega^{\beta}\right)(I-b)}, \tag{67}$$

or, equivalently,

$$\frac{2\beta I\left(\omega-\omega^{\beta}\right)+(\beta-1)(I-b)\left(\omega^{\beta+1}-\omega-1+\omega^{\beta}\right)}{\beta\left(1-\omega^{\beta-1}\right)2I-(\beta-1)\left(1-\omega^{\beta}\right)(I-b)}=0. \tag{68}$$

Denote the left-hand side as a function of ω by $l(\omega)$. The denominator of $l(\omega)$, $l_d(\omega)$, is nonnegative on $\omega\in[0,1]$ and equals zero only at $\omega=1$. This follows from:

$$\begin{aligned} l_d(0) &= 2\beta I-(\beta-1)(I-b)>0, \\ l_d(1) &= 0, \\ l_d'(\omega) &= -2\beta(\beta-1)I\omega^{\beta-2}+\beta(\beta-1)(I-b)\omega^{\beta-1} \\ &= \beta(\beta-1)\omega^{\beta-2}\left(-2I+\omega(I-b)\right)<0. \end{aligned} \tag{69}$$

Therefore, $l(\omega)=0$ if and only if the numerator of $l(\omega)$, $l_n(\omega)$, equals zero at $\omega\in(0,1)$. Note that

$$\begin{aligned} l_n'(\omega) &= 2\beta I\left(1-\beta\omega^{\beta-1}\right)+(\beta-1)(I-b)\left((\beta+1)\omega^{\beta}-1+\beta\omega^{\beta-1}\right) \\ l_n''(\omega) &= -2\beta^2(\beta-1)I\omega^{\beta-2}+(\beta-1)(I-b)\left(\beta(\beta+1)\omega^{\beta-1}+\beta(\beta-1)\omega^{\beta-2}\right) \end{aligned}$$

and

$$l_n''(\omega)<0\Leftrightarrow(I-b)\left((\beta+1)\omega+\beta-1\right)<2\beta I\Leftrightarrow\omega<\frac{(\beta+1)I+(\beta-1)b}{(\beta+1)(I-b)}$$

Since $\frac{(\beta+1)I+(\beta-1)b}{(\beta+1)(I-b)}>1$, $l_n''(\omega)<0$ for any $\omega\in[0,1]$. Since $l_n'(0)=2\beta I-(\beta-1)(I-b)>0$ and $l_n'(1)=-2\beta(\beta-1)b<0$, there exists $\hat{\omega}\in(0,1)$ such that $l_n(\omega)$ increases to the left of $\hat{\omega}$ and decreases to the right. Since $\lim_{\omega\to1}l_n(\omega)=0$, then $l_n(\hat{\omega})>0$, and hence $l_n(\omega)$ has a unique root ω^* on $(0,\hat{\omega})$.

Since the function $l_n(\omega)$ increases in b and is strictly increasing at the point ω^*, then ω^* decreases in b. To prove that $\lim_{b\to0}\omega^*=1$, it is sufficient to prove that for any small $\varepsilon>0$, there exists $b(\varepsilon)>0$ such that $l_n(1-\varepsilon)<0$ for $b<b(\varepsilon)$. Since $l_n(\omega)>0$ on $(\omega^*,1)$, this would imply that $\omega^*\in(1-\varepsilon,1)$, i.e., that ω^* is infinitely close to 1 when b is close to zero.

Using the expression for $l_n(\omega)$,

$$l_n(\omega)<0\Leftrightarrow\frac{2\beta}{\beta-1}\frac{\omega}{\omega+1}\frac{1-\omega^{\beta-1}}{1-\omega^{\beta}}<1-\frac{b}{I}$$

Denote the left-hand side of the inequality by $L(\omega)$. Note that $L(\omega)$ is increasing on $(0,1)$. Indeed, $L'(\omega)>0\Leftrightarrow g(\omega)\equiv1-\omega^{2\beta}-\beta\omega^{\beta-1}+\beta\omega^{\beta+1}>0$. The function $g(\omega)$ is decreasing on $(0,1)$ because $g'(\omega)<0\Leftrightarrow h(\omega)\equiv-2\omega^{\beta+1}-(\beta-1)+(\beta+1)\omega^2<0$, where $h'(\omega)>0$ and $h(1)=0$. Since $g(\omega)$ is decreasing and $g(1)=0$, then, indeed, $g(\omega)>0$ and hence $L'(\omega)>0$ for all $\omega\in(0,1)$.

In addition, by l'Hôpital's rule, $\lim_{\omega\to 1} L(\omega) = 1$. Hence, $L(1-\varepsilon) < 1$ for any $\varepsilon > 0$, and thus $l_n(1-\varepsilon) < 0$ for $b \in [0, I(1-L(1-\varepsilon))]$. ■

Proof of Proposition 5. First, consider the case $b < 0$. Proposition 1 shows that in the dynamic communication game, there exists an equilibrium with continuous exercise, where for each type θ, the option is exercised at the threshold $X_A^*(\theta)$. No such equilibrium exists in the static communication game. Indeed, continuous exercise requires that the principal perfectly infers the agent's type. However, since the principal gets this information at time 0, he will exercise the option at $X_P^*(\theta) \neq X_A^*(\theta)$.

We next show that no stationary equilibrium with partitioned exercise exists in the static communication game either. To see this, note that for such an equilibrium to exist, the following conditions must hold. First, the boundary type ω must be indifferent between exercise at $\bar{X}$ and at $\frac{\bar{X}}{\omega}$. Repeating the derivations in Section 3, this requires that (20) holds: $\bar{X} = \frac{\left(1-\omega^{\beta}\right)(I-b)}{\omega\left(1-\omega^{\beta-1}\right)}$. Second, given that the exercise threshold $\bar{X}$ is optimally chosen by the principal given beliefs $\theta \in [\omega, 1]$, it must satisfy $\bar{X} = \frac{\beta}{\beta-1}\frac{2I}{\omega+1}$. Combining these two equations, ω must be the solution to (30), which can be rewritten as

$$2\beta I\left(\omega - \omega^{\beta}\right) - (\beta-1)(I-b)(1+\omega)\left(1-\omega^{\beta}\right) = 0. \tag{70}$$

We next show that the left-hand side of (70) is negative for any $b < 0$ and $\omega < 1$. Since $b < 0$, it is sufficient to prove that

$$\begin{aligned} 2\beta\left(\omega - \omega^{\beta}\right) &< (\beta-1)(1+\omega)\left(1-\omega^{\beta}\right) \Leftrightarrow \\ l_n(\omega) &\equiv 2\beta\left(\omega - \omega^{\beta}\right) + (\beta-1)\left(\omega^{\beta+1} - \omega - 1 + \omega^{\beta}\right) < 0 \end{aligned}$$

It is easy to show that $l_n'(1) = 0$ and that $l_n''(\omega) < 0 \Leftrightarrow \omega < 1$, and hence $l_n'(\omega) > 0$ for any $\omega < 1$. Since $l_n(1) = 0$, then, indeed, $l_n(\omega) < 0$ for all $\omega < 1$.

Next, consider the case $b > 0$. As argued above, for ω-equilibrium to exist in the static communication game, ω must satisfy (30). According to Lemma 3, for $b > 0$, this equation has a unique solution, denoted by ω^*. Thus, among equilibria with $\omega \in [\underline{\omega}, \omega^*]$, which exist in the dynamic communication game, only equilibrium with $\omega = \omega^*$ is an equilibrium of the static communication game. ■

For Proposition 6, we prove the following lemma, which is an analogue of Proposition 1 in Melumad and Shibano (1991) for the payoff specification in our model, characterizes the structure of any incentive-compatible decision-making rule:[10]

[10] We cannot simply use Proposition 1 in Melumad and Shibano (1991), because $\frac{\partial^2 \tilde{U}_A\left(\hat{X},\theta\right)}{\partial \hat{X}^2}$ is not everywhere negative.

Lemma 4. An incentive-compatible threshold schedule $\hat{\mathrm{X}}(\theta)$, $\theta \in \Theta$ must satisfy the following conditions:

1. $\hat{\mathrm{X}}(\theta)$ is weakly decreasing in θ.

2. If $\hat{\mathrm{X}}(\theta)$ is strictly decreasing on (θ_1, θ_2), then

$$\hat{X}(\theta) = \frac{\beta}{\beta - 1} \frac{I - b}{\theta}. \tag{71}$$

3. If $\hat{\mathrm{X}}(\theta)$ is discontinuous at $\hat{\theta}$, then the discontinuity satisfies

$$\tilde{U}_A\left(\hat{X}^-\left(\hat{\theta}\right), \theta\right) = \tilde{U}_A\left(\hat{X}^+\left(\hat{\theta}\right), \theta\right), \tag{72}$$

$$\hat{X}(\theta) = \begin{cases} \hat{X}^-\left(\hat{\theta}\right), & \forall \theta \in \left[\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^-\left(\hat{\theta}\right)}, \hat{\theta}\right), \\ \hat{X}^+\left(\hat{\theta}\right), & \forall \theta \in \left(\hat{\theta}, \frac{I-b}{\hat{X}^+\left(\hat{\theta}\right)}\right], \end{cases} \tag{73}$$

$$\hat{X}\left(\hat{\theta}\right) \in \left\{\hat{X}^-\left(\hat{\theta}\right), \hat{X}^+\left(\hat{\theta}\right)\right\}, \tag{74}$$

where $\hat{\mathrm{X}}^-\left(\hat{\theta}\right) \equiv \lim_{\theta \uparrow \hat{\theta}} \hat{\mathrm{X}}(\theta)$ and $\hat{\mathrm{X}}^+\left(\hat{\theta}\right) \equiv \lim_{\theta \downarrow \hat{\theta}} \hat{\mathrm{X}}(\theta)$.

Proof of Lemma 4. The first part of the lemma can be proven by contradiction. Suppose there exist $\theta_1, \theta_2 \in \Theta$, $\theta_2 > \theta_1$, such that $\hat{X}(\theta_2) > \hat{X}(\theta_1)$. Inequality (34) for $\theta = \theta_1$ and $\hat{\theta} = \theta_2$, $\tilde{U}_A\left(\hat{X}(\theta_1), \theta_1\right) \geq \tilde{U}_A\left(\hat{X}(\theta_2), \theta_1\right)$, can be written in the integral form:

$$\int_{\hat{X}(\theta_1)}^{\hat{X}(\theta_2)} \left(\frac{X_0}{\hat{X}}\right)^{\beta} \frac{-(\beta - 1)\theta_1 \hat{X} + \beta(I - b)}{\hat{X}} d\hat{X} \leq 0. \tag{75}$$

Because $\theta_2 > \theta_1$ and $\beta > 1$, (75) implies

$$\int_{\hat{X}(\theta_1)}^{\hat{X}(\theta_2)} \left(\frac{X_0}{\hat{X}}\right)^{\beta} \frac{-(\beta - 1)\theta_2 \hat{X} + \beta(I - b)}{\hat{X}} d\hat{X} < 0, \tag{76}$$

or, equivalently, $\tilde{U}_A\left(\hat{X}(\theta_1), \theta_2\right) > \tilde{U}_A\left(\hat{X}(\theta_2), \theta_2\right)$. However, this inequality violates (34) for $\theta = \theta_2$ and $\hat{\theta} = \theta_1$: $\tilde{U}_A\left(\hat{X}(\theta_2), \theta_2\right) \geq \tilde{U}_A\left(\hat{X}(\theta_1), \theta_2\right)$. Therefore, $\hat{X}(\theta)$ is weakly decreasing in θ.

To prove the second part of the lemma, note that $\tilde{U}_A\left(\hat{X}, \theta\right)$ is differentiable and absolutely continuous in θ for all $\hat{X} \in (X(0), \infty)$. Also,

$$\sup_{\hat{X} \in \mathbf{X}} \left|\frac{\partial \tilde{U}_A\left(\hat{X}, \theta\right)}{\partial \theta}\right| = \sup_{\hat{X} \in \mathbf{X}} \left|\left(\frac{X(0)}{\hat{X}}\right)^{\beta} \hat{X}\right|. \tag{77}$$

Hence, $\sup_{\hat{X}\in\mathbf{X}}\left|\frac{\partial\tilde{U}_A\left(\hat{X},\theta\right)}{\partial\theta}\right|$ is integrable on $\theta\in\Theta$. By the generalized envelope theorem (see Corollary 1 in Milgrom and Segal, 2002), the equilibrium utility of the informed party in any mechanism implementing exercise at thresholds $X^*(\theta)$, $\theta\in\Theta$, denoted $V_A(\theta;X^*(\cdot))$ satisfies the integral condition,

$$V_A(\theta;X^*(\cdot))=V_A(\underline{\theta};X^*(\cdot))+\int_{\underline{\theta}}^{\theta}\left(\frac{X(0)}{X^*(s)}\right)^{\beta}X^*(s)\,ds. \tag{78}$$

On the other hand, $V_A(\theta;X^*(\cdot))=\tilde{U}_A(X^*(\theta),\theta)$. At any point θ, at which $X^*(\theta)$ is strictly decreasing, we have

$$\begin{aligned}\frac{\partial V_A(\theta;X^*(\cdot))}{\partial\theta} &= \left(\frac{X(0)}{X^*(\theta)}\right)^{\beta}X^*(\theta)\\ &= \left(\frac{X(0)}{X^*(\theta)}\right)^{\beta}X^*(\theta)-\left(\frac{X(0)}{X^*(\theta)}\right)^{\beta}\frac{(\beta-1)X^*(\theta)-\beta(I-b)}{X^*(\theta)}X^{*\prime}(\theta).\end{aligned}$$

Because $X^{*\prime}(\theta)<0$, it must be that $X^*(\theta)$ satisfies (71). This proves the second part of the lemma.

Finally, consider the third part of the lemma. Eq. (72) follows from continuity of $\tilde{U}_A(\cdot)$ and incentive compatibility of $\hat{X}(\theta)$. Otherwise, for example, if $\tilde{U}_A\left(\hat{X}^+\left(\hat{\theta}\right),\hat{\theta}\right)>\tilde{U}_A\left(\hat{X}^-\left(\hat{\theta}\right),\hat{\theta}\right)$, then types close enough to $\hat{\theta}$ from below would benefit from a deviation to $\hat{X}^+\left(\hat{\theta}\right)$. Because $\hat{X}^-\left(\hat{\theta}\right)=\arg\max_{\hat{X}}\tilde{U}_A\left(\hat{X},\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^-\left(\hat{\theta}\right)}\right)$ and $\hat{X}^+\left(\hat{\theta}\right)=\arg\max_{\hat{X}}\tilde{U}_A\left(\hat{X},\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^+\left(\hat{\theta}\right)}\right)$, (73) is satisfied at the boundaries. First, suppose that $\hat{X}(\theta)\neq\hat{X}^-\left(\hat{\theta}\right)$ for some $\theta\in\left(\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^-\left(\hat{\theta}\right)},\hat{\theta}\right)$. By part 1 of the lemma, $\hat{X}(\theta)>\hat{X}^-\left(\hat{\theta}\right)$. By incentive compatibility, $\tilde{U}_A\left(\hat{X}(\theta),\theta\right)\geq\tilde{U}_A\left(\hat{X}^-\left(\hat{\theta}\right),\theta\right)$, which can be written in the integral form as:

$$\int_{\hat{X}^-\left(\hat{\theta}\right)}^{\hat{X}(\theta)}\left(\frac{X_0}{\hat{X}}\right)^{\beta}\frac{-(\beta-1)\theta\hat{X}+\beta(I-b)}{\hat{X}}d\hat{X}\geq 0. \tag{79}$$

The left-hand side is strictly decreasing in θ. Therefore, $\tilde{U}_A\left(\hat{X}(\theta),\tilde{\theta}\right)>\tilde{U}_A\left(\hat{X}^-\left(\hat{\theta}\right),\tilde{\theta}\right)$ for every $\tilde{\theta}\in\left[\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^-\left(\hat{\theta}\right)},\theta\right)$. However, this contradicts $\hat{X}^-\left(\hat{\theta}\right)=\arg\max_{\hat{X}}\tilde{U}_A\left(\hat{X},\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^-\left(\hat{\theta}\right)}\right)$. Second, suppose that $\hat{X}(\theta)\neq\hat{X}^+\left(\hat{\theta}\right)$ for some $\theta\in\left(\hat{\theta},\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^+\left(\hat{\theta}\right)}\right)$. By part 1 of the lemma, $\hat{X}(\theta)<\hat{X}^+\left(\hat{\theta}\right)$. By incentive compatibility, $\tilde{U}_A\left(\hat{X}(\theta),\theta\right)\geq\tilde{U}_A\left(\hat{X}^+(\theta),\theta\right)$, which can be written as

$$\int_{\hat{X}(\theta)}^{\hat{X}^+(\theta)}\left(\frac{X_0}{\hat{X}}\right)^{\beta}\frac{-(\beta-1)\theta\hat{X}+\beta(I-b)}{\hat{X}}d\hat{X}\leq 0. \tag{80}$$

Because the left-hand side is strictly decreasing in θ, $\tilde{U}_A\left(\hat{X}(\theta),\tilde{\theta}\right)>\tilde{U}_A\left(\hat{X}^+(\theta),\tilde{\theta}\right)$ for every $\tilde{\theta}\in\left(\theta,\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^+\left(\hat{\theta}\right)}\right]$. However, this contradicts $\hat{X}^+\left(\hat{\theta}\right)=\arg\max_{\hat{X}}\tilde{U}_A\left(\hat{X},\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^+\left(\hat{\theta}\right)}\right)$. Lastly, (74) follows from continuity of $\tilde{U}(\cdot)$ and incentive compatibility of $\hat{X}(\theta)$. Because $\hat{\theta}\in\left(\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^-\left(\hat{\theta}\right)},\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^+\left(\hat{\theta}\right)}\right)$, every

policy with thresholds below $\hat{X}^{-}(\theta)$ or above $\hat{X}^{+}(\theta)$ is dominated by $\hat{X}^{-}(\theta)$ and $\hat{X}^{+}(\theta)$, respectively, and thus cannot be incentive compatible. Suppose that $\hat{X}\left(\hat{\theta}\right) \in \left(\hat{X}^{-}\left(\hat{\theta}\right), \hat{X}^{+}(\theta)\right)$. Incentive compatibility and part one of the lemma imply $\tilde{U}_A\left(\hat{X}\left(\hat{\theta}\right), \hat{\theta}\right) \geq \tilde{U}_A\left(\hat{X}^{-}\left(\hat{\theta}\right), \hat{\theta}\right) = \tilde{U}_A\left(\hat{X}^{+}(\theta), \hat{\theta}\right)$. Because $\tilde{U}\left(\hat{X}, \hat{\theta}\right)$ is strictly increasing for $\hat{X} < \frac{\beta}{\beta-1}\frac{I-b}{\hat{\theta}}$ and strictly decreasing for $\hat{X} > \frac{\beta}{\beta-1}\frac{I-b}{\hat{\theta}}$, the inequality is strict: $\tilde{U}_A\left(\hat{X}\left(\hat{\theta}\right), \hat{\theta}\right) > \tilde{U}_A\left(\hat{X}^{-}\left(\hat{\theta}\right), \hat{\theta}\right) = \tilde{U}_A\left(\hat{X}^{+}(\theta), \hat{\theta}\right)$. However, this together with continuity of $\tilde{U}_A(\cdot)$ imply that types close enough to $\hat{\theta}$ benefit from a deviation to threshold $\hat{X}\left(\hat{\theta}\right)$. Hence, it must be that $\hat{X}\left(\hat{\theta}\right) \in \left\{\hat{X}^{-}\left(\hat{\theta}\right), \hat{X}^{+}(\theta)\right\}$. ■

Proof of Proposition 6. The proof proceeds in two steps. First, we show that the optimal contract cannot have discontinuities. Second, we show that the optimal continuous contract is as specified in the proposition.

By contradiction, suppose that the optimal contract $C = \left\{\hat{X}(\theta), \theta \in \Theta\right\}$ has a discontinuity at some point $\hat{\theta} \in (\underline{\theta}, 1)$. By Lemma 2, the discontinuity must satisfy (72)–(74). Let $\theta_1 < \hat{\theta}$ and $\theta_2 > \hat{\theta}$ be points such that types $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta_1}$ for $\theta \in \left[\theta_1, \hat{\theta}\right)$ and $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta_2}$ for $\theta_2 \in \left(\hat{\theta}, \theta_2\right]$. By (72) in Lemma 2,

$$\frac{\hat{\theta}\hat{X}(\theta_1) - I + b}{\hat{X}(\theta_1)^{\beta}} = \frac{\hat{\theta}\hat{X}(\theta_2) - I + b}{\hat{X}(\theta_2)^{\beta}}. \tag{81}$$

Consider a contract $C_2 = \left\{\hat{X}_2(\theta), \theta \in \Theta\right\}$, defined as

$$\hat{X}_2(\theta) = \begin{cases} \hat{X}(\theta), & \text{if } \theta \in (-\infty, \theta_1] \cup [\theta_2, 1], \\ \frac{\beta}{\beta-1}\frac{I-b}{\theta_1}, & \text{if } \theta \in \left[\theta_1, \tilde{\theta}\right), \\ \frac{\beta}{\beta-1}\frac{I-b}{\tilde{\theta}_2}, & \text{if } \theta \in \left(\tilde{\theta}, \tilde{\theta}_2\right], \\ \frac{\beta}{\beta-1}\frac{I-b}{\theta}, & \text{if } \theta \in \left(\tilde{\theta}_2, \theta_2\right). \end{cases} \tag{82}$$

where $\tilde{\theta}_2 \in \left(\hat{\theta}, \theta_2\right]$ and $\tilde{\theta}$ satisfies

$$\frac{\tilde{\theta}\hat{X}(\theta_1) - I + b}{\hat{X}(\theta_1)^{\beta}} = \frac{\tilde{\theta}\hat{X}\left(\tilde{\theta}_2\right) - I + b}{\hat{X}\left(\tilde{\theta}_2\right)^{\beta}}. \tag{83}$$

Intuitively, contract C_2 is the same as contract C, except that it substitutes a part of a flat region, $\left[\tilde{\theta}_2, \theta_2\right]$, with a continuous region $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$. Because contract C is incentive-compatible and $\tilde{\theta}$ satisfies (83), contract C_2 is incentive-compatible too. Below we show that the payoff to the principal from contract C_2 exceeds the payoff to the principal from contract C for $\tilde{\theta}_2$ very close to θ_2. Because $\hat{X}_2(\theta) = \hat{X}(\theta)$ for

$\theta \leq \theta_1$ and $\theta \geq \theta_2$, it is enough to restrict attention to the payoff in the range $\theta \in (\theta_1, \theta_2)$. The payoff to the principal from contract C_2 in this range is

$$\int_{\theta_1}^{\tilde{\theta}} \frac{\theta \hat{X}(\theta_1) - I}{\hat{X}(\theta_1)^{\beta}} d\theta + \int_{\tilde{\theta}}^{\tilde{\theta}_2} \frac{\theta \hat{X}\left(\tilde{\theta}_2\right) - I}{\hat{X}\left(\tilde{\theta}_2\right)^{\beta}} d\theta + \int_{\tilde{\theta}_2}^{\theta_2} \frac{\theta \hat{X}(\theta) - I}{\hat{X}(\theta)^{\beta}} d\theta. \tag{84}$$

The derivative with respect to $\tilde{\theta}_2$, after the application of (83) and Leibniz's integral rule, is

$$\int_{\tilde{\theta}}^{\tilde{\theta}_2} \frac{\beta I - (\beta - 1)\theta \hat{X}\left(\tilde{\theta}_2\right)}{\hat{X}\left(\tilde{\theta}_2\right)^{\beta+1}} \hat{X}'\left(\tilde{\theta}_2\right) d\theta + b\left(\frac{1}{\hat{X}\left(\tilde{\theta}_2\right)^{\beta}} - \frac{1}{\hat{X}(\theta_1)^{\beta}}\right) \frac{d\tilde{\theta}}{d\tilde{\theta}_2}. \tag{85}$$

The first term of (85) can be simplified to

$$\begin{aligned} & \frac{(\beta - 1)\hat{X}\left(\tilde{\theta}_2\right) \frac{\tilde{\theta}_2^2 - \tilde{\theta}^2}{2} - \beta I\left(\tilde{\theta}_2 - \tilde{\theta}\right)}{\hat{X}\left(\tilde{\theta}_2\right)^{\beta+1}} \frac{\hat{X}\left(\tilde{\theta}_2\right)}{\tilde{\theta}_2} \\ = \quad & (\beta - 1)\frac{\tilde{\theta}_2 - \tilde{\theta}}{\tilde{\theta}_2} \frac{\hat{X}\left(\tilde{\theta}_2\right) \frac{\tilde{\theta}_2 + \tilde{\theta}}{2} - \frac{\beta}{\beta - 1} I}{\hat{X}\left(\tilde{\theta}_2\right)^{\beta}} = \beta \frac{\tilde{\theta}_2 - \tilde{\theta}}{\tilde{\theta}_2} \hat{X}\left(\tilde{\theta}_2\right)^{-\beta} \left[\frac{I - b}{\tilde{\theta}_2} \frac{\tilde{\theta}_2 + \tilde{\theta}}{2} - I\right]. \end{aligned} \tag{86}$$

The second term of (83) can be simplified to

$$\frac{b}{\hat{X}\left(\tilde{\theta}_2\right)^{\beta}} \left(1 - \left(\frac{\tilde{\theta}_2}{\theta_1}\right)^{-\beta}\right) \frac{d\tilde{\theta}}{d\tilde{\theta}_2} = \beta \frac{\tilde{\theta}_2 - \tilde{\theta}}{\tilde{\theta}_2} \hat{X}\left(\tilde{\theta}_2\right)^{-\beta} \left(\frac{\tilde{\theta}}{\tilde{\theta}_2}\right) b, \tag{87}$$

where we have used (83) and

$$\frac{d\tilde{\theta}}{d\tilde{\theta}_2} = \frac{(\beta - 1)\tilde{\theta}_2^{\beta-2}\left(\tilde{\theta} - \tilde{\theta}_2\right)}{\left(\frac{\theta_1^{\beta}}{\theta_1} - \frac{\tilde{\theta}_2^{\beta}}{\tilde{\theta}_2}\right)}. \tag{88}$$

Adding up (86) and (87), we obtain

$$-\beta \frac{(\tilde{\theta}_2 - \tilde{\theta})^2}{2\tilde{\theta}_2^2} \hat{X}\left(\tilde{\theta}_2\right)^{-\beta} (I + b) < 0, \tag{89}$$

if $b \geq 0$. Therefore, a deviation from contract C to contract C_2 with $\tilde{\theta}_2 = \theta_2 - \varepsilon$ for an infinitesimal positive ε is beneficial for the principal. Hence, contract C cannot be optimal for $b \geq 0$. If $b < 0$, an

equivalent deviation to contract $C_1 = \left\{\hat{X}_1(\theta), \theta \in \Theta\right\}$, defined as

$$\hat{X}_2(\theta) = \begin{cases} \hat{X}(\theta), & \text{if } \theta \in (-\infty, \theta_1] \cup [\theta_2, 1], \\ \frac{\beta}{\beta-1}\frac{I-b}{\tilde{\theta}_1}, & \text{if } \theta \in \left[\tilde{\theta}_1, \tilde{\theta}\right), \\ \frac{\beta}{\beta-1}\frac{I-b}{\theta_2}, & \text{if } \theta \in \left(\tilde{\theta}, \theta_2\right], \\ \frac{\beta}{\beta-1}\frac{I-b}{\theta}, & \text{if } \theta \in \left(\theta_1, \tilde{\theta}_1\right), \end{cases} \tag{90}$$

with $\tilde{\theta}$ satisfying

$$\frac{\tilde{\theta}\hat{X}\left(\tilde{\theta}_1\right) - I + b}{\hat{X}\left(\tilde{\theta}_1\right)^{\beta}} = \frac{\tilde{\theta}\hat{X}(\theta_2) - I + b}{\hat{X}(\theta_2)^{\beta}}, \tag{91}$$

is profitable for $\tilde{\theta}_1 = \theta_1 + \varepsilon$ for an infinitesimal positive ε. In this case, following similar calculations, the derivative of the principal's payoff with respect to $\tilde{\theta}_1$ is

$$\beta\frac{\tilde{\theta} - \tilde{\theta}_1}{2\tilde{\theta}_1^2}\hat{X}\left(\tilde{\theta}_1\right)^{-\beta}\left[(I-3b)\tilde{\theta} - (I+b)\tilde{\theta}_1\right] > 0. \tag{92}$$

Therefore, contract C cannot be optimal for $b < 0$. Hence, no contract with a discontinuity can be optimal.

Second, we prove that among continuous contracts satisfying Lemma 2, the one specified in the proposition maximizes the payoff to the principal. By Lemma 2 and continuity, it is sufficient to restrict attention to contracts that are combinations of, at most, one downward sloping part $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$ and two flat parts. Otherwise, Lemma 2 implies a discontinuity, which cannot occur in the optimal contract, as shown above. Consider a contract such that $\hat{X}(\theta)$ is flat for $\theta \in [0, \theta_1]$, is downward-sloping for $\theta \in [\theta_1, \theta_2]$, and is again flat for $\theta \in [\theta_2, 1]$, for some $\theta_1 \in [0, \theta_2]$ and $\theta_2 \in [\theta_1, 1]$.[11] The payoff to the principal is

$$\int_{\underline{\theta}}^{\theta_1} \frac{\theta X_a^*(\theta_1) - I}{X_a^*(\theta_1)^{\beta}} d\theta + \int_{\theta_1}^{\theta_2} \frac{\theta X_a^*(\theta) - I}{X_a^*(\theta)^{\beta}} d\theta + \int_{\theta_2}^{1} \frac{\theta X_a^*(\theta_2) - I}{X_a^*(\theta_2)^{\beta}} d\theta, \tag{93}$$

where $X_a^*(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$ is the exercise threshold most preferred by type θ the agent. The derivative with

[11] This consideration is always feasible, because each of the three parts need not exist (i.e., it can be that $\theta_1 = 0$ and/or $\theta_2 = 1$, or $\theta_1 = \theta_2$).

respect to θ_1 is

$$\begin{aligned}
\int_{\underline{\theta}}^{\theta_1} \frac{\beta I - (\beta-1)\,\theta X_a^*(\theta_1)}{X_a^*(\theta_1)^{\beta+1}} X_a^{*\prime}(\theta_1)\,d\theta &= -\frac{\beta}{\theta_1 X_a^*(\theta_1)^{\beta}} \int_{\underline{\theta}}^{\theta_1} \left(I - \theta \frac{I-b}{\theta_1}\right) d\theta \\
&= -\frac{\beta}{\theta_1 \hat{X}(\theta_1)^{\beta}} \left[I\theta - \theta^2 \frac{I-b}{2\theta_1}\right]_{\underline{\theta}}^{\theta_1} \qquad (94) \\
&= -\frac{\beta}{\theta_1 \hat{X}(\theta_1)^{\beta}} \left[\theta\left(I - \frac{\theta}{\theta_1}\frac{I-b}{2}\right)\right]_{\underline{\theta}}^{\theta_1} \qquad (95) \\
&= -\frac{\beta}{\theta_1 \hat{X}(\theta_1)^{\beta}} \left[\theta_1 \frac{I+b}{2} - \theta_1\left(I\frac{\underline{\theta}}{\theta_1} - \left(\frac{\underline{\theta}}{\theta_1}\right)^2 \frac{I-b}{2}\right)\right] \qquad (96) \\
&= -\frac{\beta}{\hat{X}(\theta_1)^{\beta}} \left[\frac{I+b}{2} - I\frac{\underline{\theta}}{\theta_1} + \left(\frac{\underline{\theta}}{\theta_1}\right)^2 \frac{I-b}{2}\right] \qquad (97)
\end{aligned}$$

Note that $I - \frac{\theta}{\theta_1}\frac{I-b}{2} \geq 0$ if $b > -I$. Quadratic function $x^2\frac{I-b}{2} - Ix + \frac{I+b}{2}$ has two roots, one and $\frac{I+b}{I-b}$. If $b \in (0, I)$, then this function is positive for $x < 1$. If $b > I$, then $\frac{I+b}{I-b} < 0$ and the quadratic function is inversely U-shaped. Therefore, it is positive for $x \in (0,1)$. Therefore, if $b > 0$, then (93) is strictly decreasing in θ_1. Consequently, (93) is maximized at $\theta_1 = \underline{\theta}$. If $b < 0$, then $\frac{I+b}{I-b} < 1$ and the quadratic function is U-shaped. Therefore, (93) is decreasing in θ_1 when $\frac{\underline{\theta}}{\theta_1} < \frac{I+b}{I-b}$ or $\frac{\underline{\theta}}{\theta_1} > 1$, and increasing in θ_1 when $\frac{\underline{\theta}}{\theta_1} \in \left(\frac{I+b}{I-b}, 1\right)$. Because $\theta_1 \geq 1$, we have that (93) is increasing in θ_1 in the range $\theta_1 < \frac{I-b}{I+b}\underline{\theta}$ and decreasing in θ_1 in the range $\theta_1 > \frac{I-b}{I+b}\underline{\theta}$. Therefore, if $b < 0$, (93) reaches its maximum at $\theta_1 = \min\left\{\frac{I-b}{I+b}\underline{\theta}, 1\right\}$. In particular, $\theta_1 = \frac{I-b}{I+b}\underline{\theta}$, if $b \in \left(-\frac{1-\underline{\theta}}{1+\underline{\theta}}I, 0\right)$, and $\theta_1 = 1$, if $b \leq -\frac{1-\underline{\theta}}{1+\underline{\theta}}I$.

Similarly, the derivative of (93) with respect to θ_2 is

$$\begin{aligned}
\int_{\theta_2}^{1} \frac{\beta I - (\beta-1)\,\theta \hat{X}(\theta_2)}{\hat{X}(\theta_2)^{\beta+1}} \hat{X}'(\theta_2)\,d\theta &= -\frac{\beta}{\theta_2 \hat{X}(\theta_2)^{\beta}} \int_{\theta_2}^{1} \left(I - \theta\frac{I-b}{\theta_2}\right) d\theta \qquad (98) \\
&= \frac{\beta(1-\theta_2)}{2\theta_2^2 \hat{X}(\theta_2)^{\beta}} \left(I - b - (I+b)\,\theta_2\right).
\end{aligned}$$

If $b \in (-\infty, -I]$, then (93) is strictly increasing in θ_2. In this case, (93) is maximized at $\theta_2 = 1$. Similarly, if $b \in \left(-I, -\frac{1-\underline{\theta}}{1+\underline{\theta}}I\right)$, then (98) is positive for any $\theta_2 \in [\underline{\theta}, 1]$. Therefore, (93) is again maximized at $\theta_2 = 1$. Combining with $\theta_1 = \theta_2$, this implies that $\theta_1 = \theta_2 = 1$, i.e., the decision-maker always prefers to "flatten" the contract. If $b > I$, then (93) is strictly decreasing in θ_2. Hence, (93) is maximized at $\theta_2 = \theta_1$. Combining with $\theta_1 = \underline{\theta}$, this implies that $\theta_1 = \theta_2 = \underline{\theta}$, i.e., the decision-maker always prefers to "flatten" the contract. Among flat contracts, the one that maximizes the payoff to the principal solves

$$\arg\max_{\bar{X}} \int_0^1 \frac{\theta\bar{X} - I}{\bar{X}^{\beta}} d\theta = \frac{2\beta}{\beta - 1} I.$$

This proves the first part of the proposition. If $b \in \left(-\frac{1-\underline{\theta}}{1+\underline{\theta}}I, 0\right]$, then (93) is increasing in θ_2 up to $\frac{I-b}{I+b}$. Because $\frac{I-b}{I+b} \geq 1$ in this case, (93) is maximized at $\theta_2 = 1$. Combining with $\theta_1 = \frac{I-b}{I+b}\underline{\theta}$, this proves the second part of the proposition. Finally, if $b \in (0, I)$, then (93) is increasing in θ_2 up to $\frac{I-b}{I+b}$ and decreasing after that. Hence, (93) is maximized at $\theta_2 = \frac{I-b}{I+b}$. Combining with $\theta_1 = 0$, this proves the final part of the proposition. ■

Proof of Proposition 9. From (30), ω^* solves $F(\omega, \beta) = 0$, where

$$F(\omega, \beta) = \frac{\beta}{\beta - 1}\frac{1 - \omega^{\beta-1}}{1 - \omega^{\beta}}\frac{2I}{I - b} - 1 - \frac{1}{\omega}. \tag{99}$$

Denote the unique solution by $\omega^*(\beta)$. Function $F(\omega, \beta)$ is continuously differentiable in both arguments on $\omega \in (0, 1)$, $\beta > 1$. Differentiating $F(\omega^*(\beta), \beta)$ in β:

$$\frac{\partial \omega^*}{\partial \beta} = -\frac{F_\beta(\omega^*(\beta), \beta)}{F_\omega(\omega^*(\beta), \beta)}. \tag{100}$$

Since $F_\omega(\omega, \beta) > 0$, it is sufficient to prove that $F_\beta(\omega, \beta) < 0$. Differentiating $F(\omega, \beta)$ with respect to β and reorganizing the terms, we obtain that $F_\beta(\omega, \beta) < 0$ is equivalent to

$$\frac{\left(1 - \omega^{\beta-1}\right)\left(1 - \omega^{\beta}\right)}{\omega^{\beta-1}(1 - \omega)} + \beta(\beta - 1)\ln\omega > 0. \tag{101}$$

Denote the left-hand side as a function of β by $L(\beta)$. Because $L(1) = 0$, a sufficient condition for $L(\beta) > 0$ for any $\beta > 1$ is that $L'(\beta) > 0$ for $\beta > 1$. Differentiating $L(\beta)$:

$$L'(\beta) = \ln\omega\left[-\frac{\omega^{1-\beta} - \omega^{\beta}}{1 - \omega} + 2\beta - 1\right]. \tag{102}$$

Because $\ln\omega < 0$ for any $\omega \in (0, 1)$, condition $L'(\beta) > 0$ is equivalent to

$$d(\beta) = \frac{\omega^{1-\beta} - \omega^{\beta}}{1 - \omega} - 2\beta + 1 > 0. \tag{103}$$

Note that $\lim_{\beta \to 1} d(\beta) = 0$ and

$$d'(\beta) = -\left(\omega^{1-\beta} + \omega^{\beta}\right)\frac{\ln\omega}{1 - \omega} - 2 \equiv g(\beta). \tag{104}$$

Note that

$$\begin{aligned} g(\beta) &= g(1) + \int_1^{\beta} g'(x)\,dx \\ &= -\frac{(1+\omega)\ln\omega}{1-\omega} - 2 + \frac{(\ln\omega)^2}{1-\omega}\int_1^{\beta}\left(\left(\frac{1}{\omega}\right)^{2x-1} - 1\right)\omega^x dx. \end{aligned} \tag{105}$$

The second term of (105) is positive, because $\left(\frac{1}{\omega}\right)^{2x-1} - 1 > 0$, since $\frac{1}{\omega} > 1$ and $2x - 1 > 1$ for any $x > 1$. The first term of (105) is positive, because

$$\lim_{\omega\to 1}\left(-\frac{(1+\omega)\ln\omega}{1-\omega} - 2\right) = \lim_{\omega\to 1}\left(\ln\omega + \frac{1+\omega}{\omega}\right) - 2 = 0, \tag{106}$$

$$\begin{aligned} \frac{\partial\left(-\frac{(1+\omega)\ln\omega}{1-\omega} - 2\right)}{\partial\omega} &= \frac{-2\ln\omega - \frac{1}{\omega} + \omega}{(1-\omega)^2} < \frac{-3 + 2\omega + \omega^2}{\omega(1-\omega)^2} \\ &= -\frac{\omega+3}{\omega(1-\omega)} < 0 \end{aligned} \tag{107}$$

where (106) is by l'Hopitale's rule, and (107) is because $\ln\omega > 1 - \frac{1}{\omega}$. Therefore, $g(\beta) > 0$ for any $\beta > 1$, which implies $d(\beta) > 0$, which in turn implies that $L(\beta) > 0$ for any $\beta > 1$. Hence, $F_\beta(\omega, \beta) < 0$. Therefore, ω^* is strictly increasing in $\beta > 1$. Finally, a standard calculation shows that $\frac{\partial\beta}{\partial\sigma} < 0$, $\frac{\partial\beta}{\partial\mu} < 0$, and $\frac{\partial\beta}{\partial r} > 0$. Therefore, ω^* is decreasing in β and μ and increasing in r. ■

Proof of Proposition 10. To prove this proposition, we use the solution for the optimal contract offered by the decision-maker assuming full commitment power (Proposition 6). In the case of commitment, we show that the direction of the bias is irrelevant to the decision-maker. Specifically, the effect of $+b$ on the decision-maker's expected payoff is the same as the effect of $-b$. Let $VC(b)$ denote the ex-ante utility of the decision-maker under commitment as a function of b. We now show that $VC(b)$ is an even function in that $VC(b) = VC(-b)$ for any $b > 0$.

First, consider $b \notin (-I, I)$. In this case, the exercise trigger equals $\frac{2\beta}{\beta-1}I$, regardless of θ and b. Hence, $VC(b) = VC(b)$ for any $b \geq I$. Second, consider $b \in (-I, I)$. For $b \in (-I, 0]$, the exercise trigger for type θ is $\frac{\beta}{\beta-1}\frac{I-b}{\theta}$. Thus, the expected payoff of the decision-maker is:

$$\begin{aligned} VC(b) &= \int_0^1 \left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b) - I\right)d\theta \\ &= \frac{(\beta-1)^{\beta-1}}{(\beta(I+b))^{\beta}}\frac{I+\beta b}{\beta+1}. \end{aligned}$$

Now, consider $b \in [0, I)$. The exercise trigger for type θ is $\frac{\beta}{\beta-1}\frac{I-b}{\theta}$ if $\theta \leq \frac{I-b}{I+b}$ and $\frac{\beta}{\beta-1}(I+b)$ otherwise.

Thus, the utility of the decision-maker is:

$$\begin{aligned} VC(b) &= \int_0^{\frac{I-b}{I+b}} \left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta} \left(\frac{\beta}{\beta-1}(I-b)-I\right) d\theta \\ &\quad + \int_{\frac{I-b}{I+b}}^{1} \left(\frac{\beta}{\beta-1}(I+b)\right)^{-\beta} \left(\frac{\beta}{\beta-1}(I+b)\,\theta - I\right) d\theta \\ &= \frac{(\beta-1)^{\beta-1}}{(\beta(I+b))^{\beta}} \frac{I+\beta b}{\beta+1}. \end{aligned}$$

Therefore, $VC(b) = VC(-b)$ for any $b \in [0, I)$. Combining the two cases, we conclude that $VC(b) = VC(-b)$ for any $b \geq 0$. ■

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