A Variational Approach to the Analysis of Tax Systems

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Abstract

We develop a general method to study the effects of non-linear taxation in dynamic settings using variational arguments. When the baseline tax system is sufficiently well behaved, we show how to derive the effect of perturbation of the tax system in terms of intuitive parameters, such as elasticities of relevant factors and hazard ratios. We derive a general theoretical formula that characterizes the welfare effects of local tax reforms and, in particular, the optimal tax system. We then apply this formula to various specific settings. First, we study optimal taxation in the settings that are hard to handle with existing techniques, such as optimal non-linear capital taxation or joint taxation of couples. Second we show how this approach can be used to quantify sources of welfare gains from reforming the existing, potentially suboptimal tax system. The expressions for welfare gains are in terms of parameters that can be easily estimated empirically. Finally, we sequentially decompose the gains arising from each additional element of the reform as the tax system becomes more sophisticated.

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1 Introduction

Many counties use complex system of taxes and transfers. Welfare and social insurance payments depend on individual earnings creating a complicated nonlinear schedule of effective marginal capital and labor tax rates. Figure 1 illustrates this patterns using federal tax programs in the U.S.[1] Moreover, both the eligibility and the amount of payment often depends on past history of labor earnings, assets, marital status, age and the number of children.

It is challenging to develop a theory of taxation that both allows for sufficiently rich tax functions and provides transparent, intuitive insights into effect of taxes. The literature so far have mainly persued on one of the two approaches. The first approach restricts attention to linear taxes and obtains expressions for the optimal taxes in terms of intuitive measures of elasticities. This approach goes all the way back to Ramsey (1927) and the modern application of this technique was started by Diamond and Mirrlees (1971). The other approach imposes explicit informational restrictions on the government and characterizes the informationally-constrained optimum (e.g. Mirrlees (1971) or Golosov, Kocherlakota and Tsyvinski (2003)). Both approaches have limitations. The Ramsey approach does not allow to capture many realistic complexities of tax codes. The mechanism design approach is often sensitive to the assumptions on government's information set. The tax systems that emerge from it are often very unrealistically complex and the intuition for the economic forces that determine the size and the shape of the optimal taxes are not transparent.

In this paper we develop an alternative approach to the analysis of effect of taxation that both preserves the transparency of Ramsey approach and handles more complicated, nonlinear tax systems. Our approach is based on studying perturbations of a given non-linear tax system directly. We show that as long as the baseline tax system is sufficiently well behaved, the effect of perturbations of tax system can be expressed in terms of elasticities and hazard ratios of income distributions that can be easily estimated in the data. Our method is sufficiently flexible both to allow a researcher to restrict attention a-priori to a given class of tax functions (e.g. non-linear labor taxes that do not depend on individual's age) and to study the sources and the magnitude of welfare gains from using more sophisticated taxes (e.g. the sources of welfare gains if labor taxes are allowed to depend on individual's age).

We study a dynamic model, in which individuals' characteristics evolve over their lifetime.[2] The tax system consists of a sequence of tax functions which can be arbitrarily non-linear and joint in the entire history of labor and capital incomes. The generality of the tax functions allows to study the age-dependence and history dependence of taxes, taxation of capital income, and joint conditioning of taxes on labor and capital incomes. The first main contribution of the

[1] This pattern becomes even more complicated once state-level programs are taken into account, see Maag et al (2012).

[2] In most of the paper we focus on a deterministic economy to make our approach transparent. In the last part of the paper and in our working paper Golosov, Tsyvinski and Werquin (2014) we discuss extensions to stochastic environments.

paper is to provide a general formula for the welfare effects of tax reforms in a compact and easily interpretable form despite the complexity of the dynamic problem and the corresponding general tax system and perturbations. Our result is based on deriving the Gateaux differential of individual demands, revenue, and welfare function. Importantly, these formulas are written only as a function of empirically observable and easily interpretable sufficient statistics.

We show several applications of this approach. First, we apply it to optimal taxation and show how it recovers the hallmark results on optimal linear commodity taxation of Diamond (1975) and non-linear labor taxation of static model of Mirrlees (1971), both of which are special cases of our general environment. We then use it to derive several novel predictions such as the optimality conditions for the optimal non-linear capital income tax or for the optimal labor tax on joint income of couples.

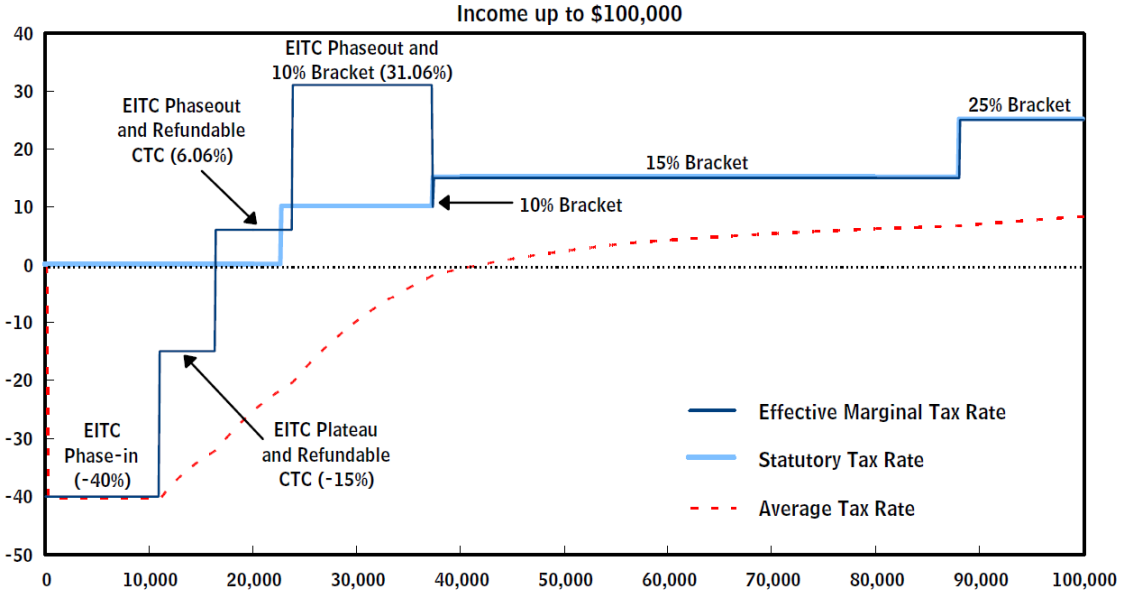
The second application of our approach is to sequentially decompose the gains from reforming existing, non necessarily optimal, tax system as the tax function becomes more sophisticated. We show the effects of taking into account individuals' intertemporal optimization decisions, of allowing for age- and history-dependence, and of joint conditioning of labor and capital income. This sequential decomposition of increasingly sophisticated tax systems shows that the welfare effects of general tax reforms depend on aggregate measures of three key elements: the marginal social welfare weights, which summarize the government's redistributive objective; the labor and capital income elasticities and income effect parameters with respect to the marginal income tax rates, which capture the behavioral effects of taxes; and the properties of the labor and capital income distributions, namely the hazard rates of the marginal and joint distributions.

Finally, we show how one can use easily available empirical moments of income distributions and elasticities to quantify welfare effects of small tax reforms. Unlike the traditional approach to measuring welfare gains, that requires solving often difficult maximization problem of finding the optimum, our method is very transparent and can be done almost "by hand". It does not allow, however to compute gains from the reforms that make big changes in the existing tax system.

Our approach is most closely related to and builds on the work of Piketty (1997) and Saez (2001). Like us, they extend the techniques of Ramsey (1927) and Diamond and Mirrlees (1971) to non-linear taxation and obtain expressions for the optimal labor tax in a uni-dimension static model in terms of elasticities of labor supply and income hazard ratios. Our paper extends their approach to more general dynamic settings and allows the analysis of such questions as capital taxation of joint taxation of several factor incomes. We also show how this approach can be used beyond optimal taxation, to applications of tax reforms and welfare gains from increased sophistication of tax systems.

More broadly our approach is also related to the "sufficient statistics" tax literature (e.g. Chetty (2009), Piketty, Saez, and Stancheva (2013), Rothchild and Scheurer (2013, 2014)) and perturbation methods (e.g. Werning (2007)). Our applications to age-dependence is related to work of Kremer (2002) and Weinzierl (2012), the work on capital taxation also builds on some

Figure 1: Effective Federal Tax Rates (source: CBO 2005)



insights of Piketty and Saez (2013) and Straub and Werning (2014), and joint taxation of factor income is related to work of Kleven, Kreiner and Saez (2009).

The rest of the paper is organized as follows. Section 2 describes our environment. Sections 3 and 4 derive behavioral responses, and revenue and welfare responses to tax perturbations. Section 5 considers the applications of this approach to optimal taxation. Section 6 considers the applications to tax reforms and the decomposition of welfare gains from increased in sophistication of the tax system.

2 Environment

There is a measure one of agents in the economy. An agent lives for $S \leq \infty$ periods, and time is indexed by $s = 1, \ldots, S$. At the beginning of period $s = 1$, there is a draw of an exogenous vector of n characteristics $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^n$ for each individual. These idiosyncratic shocks can be, for instance, the individual's initial level of capital stock k_0, his sequence of productivities, tastes, interest rates (i.e., investment opportunities), etc. over his lifetime. The environment is deterministic: individuals know at the beginning of period $s = 1$ their entire vector of characteristics $\boldsymbol{\theta}$.[3]

Given the draw of vector $\boldsymbol{\theta}$, the individual chooses in each period $s \in \{1, \ldots, S\}$ a level of consumption c_s, labor income y_s, and savings or borrowings k_s which yield capital income z_{s+1} in period $s+1$.[4] The utility function U can be a general, not necessarily time-separable, function

[3]The deterministic environment allows us to show the main insights most transparently. We extend the analysis to the stochastic environment in Golosov, Tsyvinski and Werquin (2014).

[4]The capital income in period $s+1$ can be written as $z_{s+1} = r_{s+1}k_s$, where the interest rate r_{s+1} in each period

of the vector of choices of consumption, labor income and capital income. We assume that the utility function is increasing and concave in each period's consumption (and capital income if it enters explicitly the utility function), decreasing and convex in each period's labor income, and twice differentiable in all of its variables. An example of the utility function which we use in several applications is $U = \sum_{s=1}^{S} \beta^{s-1} u\left(c_s, z_s/\theta_s\right)$. In this case, θ_s is a shock to the productivity of labor supply in period s.

In each period $s \in \{1, \ldots, S\}$, the government levies a tax T_s. The tax liability $T_s(\cdot)$ in period s is a non-linear function of the individual's entire history of labor incomes $\{y_{s'}\}_{s'=1}^{S}$ and capital incomes $\{z_{s'+1}\}_{s'=1}^{S}$.[5,6] The sequence of tax functions $\{T_s(\cdot)\}_{s=1}^{S}$ is known to an individual at the beginning of period $s = 1$, and the government can commit to it. The initial tax system $\mathscr{T}$ thus consists of a set of tax functions $T_s : \mathbb{R}_+^S \times \mathbb{R}^S \to \mathbb{R}$ for each period $s \in \{1, \ldots, S\}$, where each function $T_s(\cdot)$ maps a choice of labor and capital incomes $\mathbf{x} \in \mathbb{R}_+^S \times \mathbb{R}^S$ to a tax liability $T_s(\mathbf{x}) \in \mathbb{R}$. The tax function in period s, $T_s\left(\{y_{s'}\}_{s'=1}^{S}, \{z_{s'+1}\}_{s'=1}^{S}\right)$, is assumed twice continuously differentiable in all of its $2S$ variables, that is $T_s \in \mathcal{C}^2\left(\mathbb{R}_+^S \times \mathbb{R}^S, \mathbb{R}\right)$ for all $s \in \{1, \ldots, S\}$.[7]

The optimization problem of an individual with the vector of types $\boldsymbol{\theta}$ is:

$$\begin{aligned} \mathscr{U}_{\boldsymbol{\theta}}(\mathscr{T}) \equiv \max_{\{c_s, z_s, r_{s+1} k_s\}_{1 \le s \le S}} \quad & U\left(\{c_s\}_{1 \le s \le S}, \{y_s\}_{1 \le s \le S}, \{z_{s+1}\}_{1 \le s \le S}, \boldsymbol{\theta}\right) \\ \text{s.t.} \quad & c_s + k_s = y_s + (k_{s-1} + z_s) - T_s\left(\{y_{s'}\}_{1 \le s' \le S}, \{z_{s'+1}\}_{1 \le s' \le S}\right), \ \forall s. \end{aligned} \tag{1}$$

We denote by $\mathbf{x}_{\boldsymbol{\theta}}(\mathscr{T})$ the argmax of this problem, i.e., the optimal choice of labor and capital incomes of the individual $\boldsymbol{\theta}$ as a function of the tax system $\mathscr{T}$. That is, we define the *individual income functional* as:

$$\mathbf{x}_{\boldsymbol{\theta}}(\mathscr{T}) = \left(y_{\boldsymbol{\theta},1}(\mathscr{T}), \ldots, y_{\boldsymbol{\theta},S}(\mathscr{T}), z_{\boldsymbol{\theta},2}(\mathscr{T}), \ldots, z_{\boldsymbol{\theta},S+1}(\mathscr{T})\right)'.$$

The optimal choices of consumption $\{c_{\boldsymbol{\theta},s}\}_{s=1}^{S}$ are then obtained from the budget constraints. The budget constraint in period s imposes that the sum of consumption c_s and savings k_s is no greater than the sum of labor income y_s and capital income $(k_{s-1} + z_s)$, net of the tax liability T_s.

We denote by $F_{\boldsymbol{\theta}}(\boldsymbol{\theta})$ the c.d.f. of vectors $\boldsymbol{\theta} \in \Theta$, and $f_{\boldsymbol{\theta}}(\boldsymbol{\theta})$ the corresponding density

is exogenous. However, our analysis allows the interest rate to be idiosyncratic, and thus the period-s savings k_s to yield any (deterministic) income z_{s+1} in the next period. The before-tax capital stock at the beginning of period $s+1$ is thus $k_s + z_{s+1}$.

[5] In a given period s, the planner can tax incomes earned in the future periods $s' > s$ because the model is deterministic. We assume here that initial capital k_0 is not taxed, because it is supplied inelastically and hence does not induce any behavioral effects. Our formulas can be trivially extended to the case where it can be taxed.

[6] Throughout the paper we consider only capital income taxes and not wealth taxes. The same approach can be used to analyze wealth taxation.

[7] In the deterministic model, we could without loss of generality write only one tax function, for instance $T_S(\cdot)$. Instead, we choose to define one tax function per period s, at the expense of slightly more cumbersome notation, to make it easier to discuss age-dependent taxes and capital taxes in Sections 6 and 5.

function. We also denote by $F_{\mathbf{x}}(\mathbf{x})$ and $f_{\mathbf{x}}(\mathbf{x})$ the c.d.f. and the p.d.f. of incomes $\mathbf{x} \in \mathrm{X} \subset \mathbb{R}_+^S \times \mathbb{R}^S$, given the tax system $\mathscr{T}$. We assume that the sets Θ and X of vectors of types $\boldsymbol{\theta}$ and incomes $\mathbf{x}$ are compact in $\mathbb{R}^n$ and $\mathbb{R}_+^S \times \mathbb{R}^S$, respectively, and that the densities of types and incomes at the (piecewise smooth) boundaries $\partial\mathrm{X}$ and $\partial\Theta$ of the sets X and Θ are equal to zero.[8] We make the following assumption about the income vectors chosen by individuals with different types $\boldsymbol{\theta}$:

Assumption 1. *The map $\boldsymbol{\theta} \mapsto \mathbf{x}_{\boldsymbol{\theta}}(\mathscr{T})$ between the vector of types $\boldsymbol{\theta}$ and the vector of income choices $\mathbf{x}_{\boldsymbol{\theta}}$ (given the tax system $\mathscr{T}$) is injective. That is, if two individuals have a different vector of types $\boldsymbol{\theta} \neq \boldsymbol{\theta}'$, they choose a different vector of incomes $\mathbf{x}_{\boldsymbol{\theta}}(\mathscr{T}) \neq \mathbf{x}_{\boldsymbol{\theta}'}(\mathscr{T})$.*

We explain below how our main results are affected in the case where Assumption 1 does not hold, e.g., if the space of degrees of heterogeneity has a higher dimension than the space of income choices.

We define the present discounted value of tax revenue as a function of the tax system $\mathscr{T}$, or *tax revenue functional*, as

$$\mathscr{R}(\mathscr{T}) = \int_{\Theta} \left[\sum_{s=1}^{S} \beta^{s-1} T_s(\mathbf{x}_{\boldsymbol{\theta}}(\mathscr{T})) \right] dF_{\boldsymbol{\theta}}(\boldsymbol{\theta}), \tag{2}$$

where β is the marginal rate of transformation of resources across periods for the government, which we assume equal to the individual's discount factor. Tax revenue is thus the sum over time $s \in \{1, \ldots, S\}$ and over individuals $\boldsymbol{\theta} \in \Theta$ of individual tax liabilities, taking into account the agents' optimizing behavior given the tax system $\mathscr{T}$.

We finally define the *social welfare functional* as a weighted average of the indirect utility functions of individual agents and the tax revenue, as a function of the tax system $\mathscr{T}$,

$$\mathscr{W}(\mathscr{T}) = (1-\alpha) \int_{\Theta} \mathcal{G}(\mathscr{U}_{\boldsymbol{\theta}}(\mathscr{T})) \, dF_{\boldsymbol{\theta}}(\boldsymbol{\theta}) + \alpha \mathcal{V}(\mathscr{R}(\mathscr{T})), \tag{3}$$

for some $\alpha \in [0,1]$. Here $\mathcal{V}(\mathscr{R}(\mathscr{T}))$ is a measure of the value of public goods that the government can provide with tax revenues $\mathscr{R}(\mathscr{T})$. The function $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ is defined over lifetime utilities of the individuals, and is assumed continuously differentiable, increasing, and concave. The function $\mathcal{V} : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and increasing.

The tax system $\mathscr{T}$ considered so far is very general and allows for a rich set of non-linearities and non-separabilities in taxing different incomes at different dates. In practice, we are often interested in more restrictive classes of tax systems. For example, the classical Ramsey analysis restricts the tax functions to be separable and linear in each income (e.g., Ramsey 1927, Diamond and Mirrlees 1971, Diamond 1975). Another strand of the literature focuses on the analysis of separable but non-linear tax functions, (e.g., Mirrlees 1971, Diamond 1998, Saez

[8] In some applications of Sections 6 and 5, we let incomes evolve in the whole space $\mathbb{R}_+^S \times \mathbb{R}^S$. Our theory can be generalized to this case by using an increasing sequence of compact sets $X \subset \mathbb{R}_+^S \times \mathbb{R}^S$.

2001, Heathcote, Storesletten and Violante 2014). More generally, the New Dynamic Public Finance literature (e.g., Kocherlakota 2005, Farhi and Werning 2012, Golosov, Troshkin and Tsyvinski 2014) emphasizes the benefits of jointly taxing different incomes, namely labor and capital incomes within periods, or labor incomes across periods (history-dependent taxation), so that the tax rate on income i depends not only on its own level x_i, but also on the levels x_j of other incomes $j \neq i$. When we impose such constraints on the tax system $\mathscr{T}$, we say that $\mathscr{T}$ is "restricted within a class" (e.g., of linear separable, non-linear separable, etc., tax functions).

Our paper focuses on several conceptually distinct, but closely related questions. First, we analyze the revenue or welfare gains and losses of small perturbations of any *baseline tax system* $\mathscr{T}^0$. We refer to such changes as *tax reforms*. Suppose in particular that the tax system $\mathscr{T}^0$ is restricted within a certain class. By deriving the effects of reforms that keep the perturbed tax system within this class, we can shed light on the economic parameters that determine whether the existing tax system is (constrained) optimal, and derive the potential welfare gains obtained by reforming it. Moreover, we can analyze reforms that induce the tax system to leave its restricted class. For instance, we can introduce (a small amount of) non-linearity, age-dependence, history-dependence or joint taxation within a baseline linear, age-independent, or separable tax code. This allows us to sequentially decompose the gains arising from each additional element of reform as the tax system becomes more sophisticated. Second, we derive characterizations of the *optimal tax system*, or the optimum within a certain class. These two questions are closely related, because the characterization of the optimum is obtained by imposing that the net welfare effect of any tax reform is non-positive if the baseline tax system $\mathscr{T}^0$ is optimal.

3 Behavioral Effects of Tax Reforms

In this section, we formally define the admissible perturbations of the initial tax system, i.e., our tax reforms, and study their effect on individual behavior. Starting with a baseline tax system $\mathscr{T}^0$, we analyze the revenue and welfare effects of reforming it to $\tilde{\mathscr{T}} = \mathscr{T}^0 + \mu\mathscr{H}$ where $\mathscr{H}$ is another tax system and $\mu \in \mathbb{R}_+$; we then let $\mu \to 0$. We therefore derive the Gateaux differential of the tax revenue and social welfare functionals, following a local perturbation of the baseline tax system $\mathscr{T}^0$ in the direction $\mathscr{H}$. We can decompose this general perturbation of the tax system into elementary perturbations $h_p : \mathrm{X} \to \mathbb{R}$ of each period-p baseline tax function $T_p : \mathbb{R}_+^S \times \mathbb{R}^S \to \mathbb{R}$. The total effect of the perturbation $\mathscr{H}$ is then equal to the sum of the effects of its components h_p. We thus analyze perturbations of the period-p tax function $T_p(\cdot)$, and we keep the rest of the initial tax system $\mathscr{T}^0$ unchanged. We thus define an *admissible perturbation* of the baseline tax function $T_p(\cdot)$ as a twice continuously differentiable function $h_p \in \mathcal{C}^2(\mathrm{X}, \mathbb{R})$. For any $\mu > 0$, we can then define a new tax function in period p as $\tilde{T}_p = T_p + \mu h_p$. We say that the perturbation h_p is *restricted within a class* if it leaves the perturbed tax system $\tilde{\mathscr{T}}$ in the same class (e.g., of linear, separable, etc., tax systems) as the baseline tax system $\mathscr{T}^0$. In order

to characterize the effects of the perturbation h_p as $\mu \to 0$, we first analyze in this Section the optimal individual behavior given any tax system $\mathscr{T}$, which can be the baseline or the perturbed tax system.

We first characterize the solution to the problem (1) of individual $\boldsymbol{\theta}$, i.e., his choice of incomes $\mathbf{x}_{\boldsymbol{\theta}} \in \mathrm{X}$ given a tax system $\mathscr{T}$. We denote by $\{x_{\boldsymbol{\theta},j}\}_{1\leq j\leq 2S}$ the components of the income vector $\mathbf{x}_{\boldsymbol{\theta}}$, that is labor and capital incomes in each period. The system of first-order conditions that characterizes the individual's income vector given the tax system $\mathscr{T} = \{T_s(\cdot)\}_{1\leq s\leq S}$ can be written formally as

$$F\left(\mathbf{x}_{\boldsymbol{\theta}}, \{\nabla T_s(\mathbf{x}_{\boldsymbol{\theta}})\}_{1\leq s\leq S}, \{T_s(\mathbf{x}_{\boldsymbol{\theta}})\}_{1\leq s\leq S}, \boldsymbol{\theta}\right) = 0, \tag{4}$$

for some function $F : \mathbb{R}^{2S} \times \mathbb{R}^{(2S\times S)} \times \mathbb{R}^S \times \Theta \to \mathbb{R}^{2S}$. The solution to this non-linear system of $2S$ equations is the Marshallian (uncompensated) labor and capital income choices of the individual $\boldsymbol{\theta}$, as a function of the partial derivatives and the absolute height of the tax functions that he faces in every period $1 \leq s \leq S$.

We define the period-s marginal tax rates in each direction j, $\left\{\tau_{s,x_j}\right\}_{1\leq j\leq 2S}$, and the virtual income R_s that an individual with the income vector $\mathbf{x}_{\boldsymbol{\theta}}$ faces, as

$$\tau_{s,x_j}(\mathbf{x}_{\boldsymbol{\theta}}) \equiv \frac{\partial T_s(\mathbf{x}_{\boldsymbol{\theta}})}{\partial x_{\boldsymbol{\theta},j}} \quad \text{and} \quad R_s(\mathbf{x}_{\boldsymbol{\theta}}) \equiv \langle \nabla T_s(\mathbf{x}_{\boldsymbol{\theta}}), \mathbf{x}_{\boldsymbol{\theta}} \rangle - T_s(\mathbf{x}_{\boldsymbol{\theta}}). \tag{5}$$

These definitions allow us to define the *linearized budget constraints* for the individual with labor and capital incomes $\mathbf{x}_{\boldsymbol{\theta}}$, by replacing the tax function $T_s(\cdot)$ with its tangent hyperplane at point $\mathbf{x}_{\boldsymbol{\theta}}$. This tangent hyperplane is defined by the individual's marginal tax rates and virtual income (5) as $\mathcal{H}_s(\mathbf{x}_{\boldsymbol{\theta}}) = -R_s(\mathbf{x}_{\boldsymbol{\theta}}) + \sum_{j=1}^{2S} \tau_{s,x_j}(\mathbf{x}_{\boldsymbol{\theta}})\, x_j$. In particular, the virtual income $R_s(\mathbf{x}_{\boldsymbol{\theta}})$ is the income that the individual would have if he faced this linearized tax function and earned no labor or capital income, i.e., $x_j = 0$ for all $1 \leq j \leq 2S$.

We define the individual elasticities and income effect parameters as the effects on the income vector $\mathbf{x}_{\boldsymbol{\theta}}$ of perturbing each of the $(2S+1)$ coordinates of the linearized budget constraint $\mathcal{H}_s(\mathbf{x}_{\boldsymbol{\theta}})$, keeping the other equal. Specifically, the uncompensated elasticity $\zeta^{u,(\mathbf{x}_{\boldsymbol{\theta}})}_{x_i,\tau_{s,x_j}}$ is the percentage change in the i^{th} income choice $x_{\boldsymbol{\theta},i}$ in response to a percentage change in the marginal tax rate τ_{s,x_j} that the individual faces in the direction of income x_j. More precisely, for each $s \in \{1,\ldots,S\}$, we define these elasticities with respect to the marginal tax rates $\tau_{s,y_j}, \tau_{s,z_j}$ if $j \neq s$, and with respect to the net-of-tax rates $1-\tau_{s,y_s}, 1-\tau_{s,z_s}$ otherwise. Finally, the income effect parameter $\eta^{(\mathbf{x}_{\boldsymbol{\theta}})}_{x_i,R_s}$ is the change in the income $x_{\boldsymbol{\theta},i}$ in response to a uniform (lump-sum) shift of the linearized budget constraint $\mathcal{H}_s(\mathbf{x}_{\boldsymbol{\theta}})$, weighted by the marginal or net-of tax rate on income x_i. We thus define:

$$\zeta^{u,(\mathbf{x}_{\boldsymbol{\theta}})}_{x_i,\hat{\tau}_{s,x_j}} = \frac{\hat{\tau}_{s,x_j}(\mathbf{x}_{\boldsymbol{\theta}})}{x_{\boldsymbol{\theta},i}} \frac{\partial x_{\boldsymbol{\theta},i}}{\partial \hat{\tau}_{s,x_j}} \quad \text{and} \quad \eta^{(\mathbf{x}_{\boldsymbol{\theta}})}_{x_i,R_s} = \hat{\tau}_{s,x_i}(\mathbf{x}_{\boldsymbol{\theta}}) \frac{\partial x_{\boldsymbol{\theta},i}}{\partial R_s}, \tag{6}$$

where in these expressions $\hat{\tau}_{s,x}$ denotes either the marginal tax rate $\tau_{s,x}$, or the net-of-tax rate $1-\tau_{s,x}$ (if $x \in \{y_s, z_s\}$). Finally, we define the compensated elasticities $\zeta^{c,(\mathbf{x}_{\boldsymbol{\theta}})}_{x_i,\hat{\tau}_{s,x_j}}$, i.e., the percentage change in the Hicksian demands in response to percentage changes in the marginal or net-of tax rates, from the Slutsky equations as:

$$\zeta^{c,(\mathbf{x}_{\boldsymbol{\theta}})}_{x_i,\hat{\tau}_{s,x_j}} = \zeta^{u,(\mathbf{x}_{\boldsymbol{\theta}})}_{x_i,\hat{\tau}_{s,x_j}} \pm \frac{\hat{\tau}_{s,x_j}(\mathbf{x}_{\boldsymbol{\theta}})}{\hat{\tau}_{s,x_i}(\mathbf{x}_{\boldsymbol{\theta}})} \frac{x_{\boldsymbol{\theta},j}}{x_{\boldsymbol{\theta},i}} \eta^{(\mathbf{x}_{\boldsymbol{\theta}})}_{x_i,R_s}, \tag{7}$$

where $\pm$ stands for $-$ if $x_j \in \{y_s, z_s\}$, and $+$ otherwise. We derive in the Appendix analytical expressions for all these elasticities and income effect parameters.[9] We finally define the $2S \times 2S$-matrix $\boldsymbol{E}^{c,(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},\tau_s}$ of changes in compensated income x_i with respect to the period-s marginal tax rates τ_{s,x_j}, and the $2S \times 1$-vector $\boldsymbol{I}^{(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},R_s}$ of changes in income x_i with respect to the period-s virtual income R_s, that is,

$$\left[\boldsymbol{E}^{c,(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},\tau_s}\right]_{i,j} = \frac{\partial x^c_{\boldsymbol{\theta},i}}{\partial \tau_{s,x_j}} \quad \text{and} \quad \left[\boldsymbol{I}^{(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},R_s}\right]_i = \frac{\partial x_{\boldsymbol{\theta},i}}{\partial R_s}. \tag{8}$$

The elements of these matrices can be expressed in terms of the compensated elasticities and income effect parameters (6,7) by multiplying by the corresponding marginal or net-of tax rate and dividing by the corresponding income.[10]

We now analyze the change in an individual's income vector $\mathbf{x}_{\boldsymbol{\theta}}$ in response to an admissible perturbation h_p of the tax function T_p in period p. We make the following assumption about the income functional $\mathbf{x}_{\boldsymbol{\theta}}(\cdot)$:[11]

Assumption 2. *The income functional $\mathbf{x}_{\boldsymbol{\theta}}(\cdot)$ is locally Lipschitz continuous in every direction h_p at the initial tax system T_p. That is, for any admissible perturbation $h_p \in \mathcal{C}^2(\mathrm{X}, \mathbb{R})$, there exist $\bar{\mu} > 0$ and $M > 0$ such that $\mu < \bar{\mu}$ implies $\|\mathbf{x}_{\boldsymbol{\theta}}(T_p + \mu h_p) - \mathbf{x}_{\boldsymbol{\theta}}(T_p)\| < M \times \mu$.*

This assumption formalizes the idea that individuals' income choices do not change by a discrete amount in response to infinitesimal admissible perturbations of the initial tax system. Note that this assumption is stronger than the mere continuity of the income correspondence at the initial tax system in the direction h_p. It requires that as the magnitude of the tax reform shrinks to zero, the size of the change in the individual's income vector shrinks to zero at a rate at least as fast as that of the perturbation. This assumption implicitly imposes restrictions on the baseline tax system $\mathscr{T}^0$. Specifically, it requires that every individual's choice of income vector is located at a strict maximum of his choice set, so that no one in the economy is indifferent

[9] In the general model, these expressions are of course complicated. We show in Section 6 that they significantly simplify if the utility function has no income effects on labor supply and the baseline tax system is separable.

[10] Note that in general, the elasticities and income effect parameters depend on the value of the vector of types $\boldsymbol{\theta}$. That is, if two individuals choose the same vector $\mathbf{x}$ of labor and capital incomes but have different vectors of exogenous characteristics $\boldsymbol{\theta}$, their responses to changes in the tax rates, i.e. their elasticities $\zeta^{u,(\mathbf{x}_{\boldsymbol{\theta}})}_{x_i,\tau_{s,x_j}}$, are different. Assumption 1, however, implies that these elasticities depend only on $\mathbf{x}$ and not directly on $\boldsymbol{\theta}$.

[11] With a slight abuse of notation, we denote by $\mathbf{x}_{\boldsymbol{\theta}}(T_p)$ the vector $\mathbf{x}_{\boldsymbol{\theta}}(\mathscr{T}^0)$ of income choices for the individual $\boldsymbol{\theta}$ given the baseline tax system $\mathscr{T}^0$.

between two income vectors $\mathbf{x}_{\boldsymbol{\theta}}$ and $\mathbf{x}'_{\boldsymbol{\theta}}$ under the initial tax system $\mathscr{T}^0$. We also assume that there is no bunching of types at any income level.[12]

In response to a perturbation h_p of the baseline tax system, all the labor and capital incomes chosen by an individual change simultaneously. We derive formally this behavioral change by calculating the Gateaux differential of the individual income function in the direction h_p. We show that despite the apparent complexity of the problem, we can derive, using the matrix notations introduced above, a compact and transparent formula giving the change in individual's behavior following any such perturbation. We express these behavioral responses in terms of: (i) the elasticities and income effect parameters (6,7) of the individual, and (ii) the local characteristics (gradient and Hessian) of the baseline tax function. We view the derivation of the formula in the next proposition and, most importantly, its compact and transparent representation, as one of the main contributions of this paper.

Proposition 1. *Suppose that Assumption 2 is satisfied. Then the income functional* $\mathbf{x}_{\boldsymbol{\theta}}(\cdot)$ *is Gateaux differentiable around the initial tax system. Its Gateaux differential at* T_p *in the direction* h_p *is given by:*

$$\delta \mathbf{x}_{\boldsymbol{\theta}}(T_p, h_p) = \left[\mathrm{i}_{2S} - \sum_{s=1}^{S} \boldsymbol{E}^{c,(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},\tau_s} D^2 T_s(\mathbf{x}_{\boldsymbol{\theta}}) \right]^{-1} \left\{ \boldsymbol{E}^{c,(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},\tau_p} \nabla h_p(\mathbf{x}_{\boldsymbol{\theta}}) - \boldsymbol{I}^{(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},R_p} h_p(\mathbf{x}_{\boldsymbol{\theta}}) \right\} \in \mathbb{R}^{2S}, \quad (9)$$

where i_{2S} *the* $2S \times 2S$ *identity matrix, and* $D^2 T_s(\mathbf{x})$ *the Hessian of the tax function* $T_s(\cdot)$.

Proof. See Appendix. □

We now sketch the main steps of the proof of Proposition 1. The individual's behavior under the baseline tax function T_p (resp., the perturbed tax function $\tilde{T}_p = T_p + \mu h_p$) is described by the system of first-order conditions (4) which lead him to choose the income vector $\mathbf{x}_{\boldsymbol{\theta}}$ (resp., $\tilde{\mathbf{x}}_{\boldsymbol{\theta}}$). Using the assumption that the income functional is locally Lipschitz continuous around the baseline tax system (Assumption 2), we can write a Taylor expansion in the direction h_p of the perturbed set of equations, which yields the first-order (in μ) change in the income vector, $\tilde{\mathbf{x}}_{\boldsymbol{\theta}} - \mathbf{x}_{\boldsymbol{\theta}}$. Using the analytical expressions for the elasticities that we derive in the Appendix, we then show that this Gateaux differential $\delta \mathbf{x}_{\boldsymbol{\theta}}(T_p, h_p)$ can be expressed in terms of the matrices of elasticities and income effect parameters (8), and the gradients and the Hessians of the tax functions.

We finally provide the intuition underlying formula (9). The change $d\mathbf{x}_{\boldsymbol{\theta}} = \tilde{\mathbf{x}}_{\boldsymbol{\theta}} - \mathbf{x}_{\boldsymbol{\theta}}$ in the individual's income vector induced by the perturbation h_p is the consequence of two effects. First, there is a direct elasticity effect due to the exogenous change in the marginal tax rates $\nabla h_p(\mathbf{x}_{\boldsymbol{\theta}})$ that the individual faces, and a direct income effect due to the exogenous change in the

[12] Our analysis relies on the assumption that the first-order conditions characterize the optimal tax system. The literature on multidimensional screening (see, e.g., Rochet and Choné 1998) highlights the issues that bunching creates in the multidimensional case, as non-local incentive constraints may bind.

virtual income dR_p.[13] This direct (elasticity plus income) effect is equal to $\boldsymbol{E}^{c,(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},\tau_p}\nabla h_p(\mathbf{x}_{\boldsymbol{\theta}}) + \boldsymbol{I}^{(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},R_p}(-h_p(\mathbf{x}_{\boldsymbol{\theta}}))$, by definition of the matrix of compensated elasticities $\boldsymbol{E}^{c,(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},\tau_p}$ and the vector of income effect parameters $\boldsymbol{I}^{(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},R_p}$. Second, this shift of the taxpayer $\mathbf{x}_{\boldsymbol{\theta}}$ along the non-linear tax function by $d\mathbf{x}_{\boldsymbol{\theta}}$ produces an additional change in the marginal rates in all periods $s \in \{1,\ldots,S\}$ equal to $d(\nabla T_s)(\mathbf{x}_{\boldsymbol{\theta}}) = \left(D^2 T_s(\mathbf{x}_{\boldsymbol{\theta}})\right) d\mathbf{x}_{\boldsymbol{\theta}}$. This induces indirect elasticity effects, leading to a further change in the income vector $\mathbf{x}_{\boldsymbol{\theta}}$. Note that since the vector $\mathbf{x}_{\boldsymbol{\theta}}$ is taxed in every period, there is such an indirect elasticity effect $\boldsymbol{E}^{c,(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},\tau_s} d(\nabla T_s)(\mathbf{x}_{\boldsymbol{\theta}})$ in every period $s \in \{1,\ldots,S\}$. Therefore, we obtain that the individual changes his income vector $\mathbf{x}_{\boldsymbol{\theta}}$ in response to the perturbation by an amount

$$d\mathbf{x}_{\boldsymbol{\theta}} = \left[\boldsymbol{E}^{c,(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},\tau_p}\nabla h_p(\mathbf{x}_{\boldsymbol{\theta}}) - \boldsymbol{I}^{(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},R_p} h_p(\mathbf{x}_{\boldsymbol{\theta}})\right] + \left[\sum_{s=1}^{S} \boldsymbol{E}^{c,(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},\tau_s} D^2 T_s(\mathbf{x}_{\boldsymbol{\theta}})\right] \times d\mathbf{x}_{\boldsymbol{\theta}},$$

which leads to equation (9).[14]

Proposition 1 is useful, because it allows us to express the response of any individual's choice of labor and capital incomes following any local perturbation of the baseline tax system, without having to solve for the optimization problem (1), and describe this response in terms of empirically observable and easily interpretable parameters. To do so, we simply need to plug the values of the specific perturbation function h_p into equation (9). Importantly, this characterization is valid in very general settings and holds regardless of the specific dimensions of heterogeneity, utility functions, tax systems, etc., as long as the individual's first-order conditions and Assumption 2 are satisfied.

We conclude this section by providing two examples of application of formula (9). First, suppose that the baseline tax system and the perturbation are both separable and linear in incomes. That is, for all $s = 1,\ldots,2S$, an individual who earns income x_s pays the tax liability $\tau_{s,x_s} x_s$ in the corresponding period. Fix $d\tau > 0$ and consider a separable linear perturbation h_p on income x_p in a given period p, defined by $h_p(x_p) = x_p d\tau$, so that the perturbed tax schedule on income x_p is $\tilde{T}_p(x_p) = \left(\tau_{p,x_p} + d\tau\right) x_p$. Applying formula (9), we obtain that as $d\tau \to 0$, an individual with income vector $\mathbf{x}$ under the baseline tax system adjusts his income x_s in every period s by the amount:

$$dx_s = -\frac{x_s}{1-\tau_{p,x_p}} \zeta^{c,(\mathbf{x})}_{x_s,1-\tau_{p,x_p}} d\tau - \frac{\eta^{(\mathbf{x})}_{x_s,R_p}}{1-\tau_{p,x_s}} x_p d\tau = -\frac{x_s}{1-\tau_{p,x_p}} \zeta^{u,(\mathbf{x})}_{x_s,1-\tau_{p,x_p}} d\tau.$$

Second, consider the static Mirrlees model with a non-linear labor income tax schedule

[13] The total income effect of the perturbation is given by the change in the tax liability levied at the income level $\mathbf{x}$, i.e., $(-h_p(\mathbf{x}_{\boldsymbol{\theta}}))$.

[14] If Assumption 1 is not satisfied, then we define $d\mathbf{x}$ as the average of the expression (9) over all individuals $\boldsymbol{\theta}$ who choose the same vector $\mathbf{x}$. If the initial tax system is locally linear around the point $\mathbf{x}$, then this average behavioral response is simply given by $d\bar{\mathbf{x}} = \overline{\boldsymbol{E}}^{c,(\mathbf{x})}_{\mathbf{x},\tau_p}\nabla h(\mathbf{x}) - \overline{\boldsymbol{I}}^{(\mathbf{x})}_{\mathbf{x},R_p} h(\mathbf{x})$, where $\overline{\boldsymbol{E}}$ and $\overline{\boldsymbol{I}}$ are the matrices of average elasticities and income effect parameters among these individuals.

$T(y)$, and let h denote an admissible perturbation of the baseline tax schedule. Suppose that an individual with income y faces a change in his marginal tax rate $h'(y) = d\tau$, and a change in his total tax liability $h(y) = dR$, for some $d\tau, dR > 0$. Applying formula (9), we obtain immediately that the adjustment in the individual y's income as $d\tau, dR \to 0$, given by the Gateaux differential of his income functional, is equal to:

$$dy = -\frac{\frac{y}{1-T'(y)}\zeta^{c,(y)}_{y,1-\tau_y}}{1+\frac{y}{1-T'(y)}\zeta^{c,(y)}_{y,1-\tau_y}T''(y)}d\tau - \frac{\frac{1}{1-T'(y)}\eta^{(y)}_{y,R}}{1+\frac{y}{1-T'(y)}\zeta^{c,(y)}_{y,1-\tau_y}T''(y)}dR.$$

This is the formula obtained by Saez (2001).

4 Welfare Effects of Tax Reforms and Optimal Tax System

Having defined the perturbations and described the effects that they induce on individual behavior, we now derive the revenue and welfare effects of these tax reforms, and characterize the optimal tax system. Specifically, we start from a baseline tax system, which can be suboptimal or optimal. We locally perturb this tax system with tax reform, as defined above. Our first result (Proposition 2) describes the *revenue and welfare effects of these local tax reforms*. Formally, we compute these local effects as the Gateaux differentials of the revenue and social welfare functionals. These give the sign and the magnitude of the potential gains that arise from reforming the current, potentially suboptimal, tax code. If the perturbation yields a strictly positive (revenue or welfare) effect, then the corresponding tax reform is *(revenue or welfare)-improving* and should be implemented. The second result that our theory yields (Proposition 3) is a characterization of the *globally optimal tax function*. Specifically, the baseline tax system is optimal only if there is no local tax reform that yields a strict improvement. Characterizations of the revenue-maximizing or welfare-maximizing tax systems are therefore obtained by setting the Gateaux differentials of the corresponding functionals equal to zero for any admissible perturbation. Note finally that a similar reasoning yields a characterization of the *optimum tax system within a restricted class* (e.g., of linear, separable, etc., tax systems), by restricting the analysis to the perturbations within the corresponding class.

We start by defining the social marginal welfare weights $g_s(\mathbf{x})$ that the planner assigns to agents with various income choices. These weights are defined such that the government is indifferent between having $g_s(\mathbf{x})$ more dollars of public funds in period s and giving one more dollar in period s to the taxpayers with choice vector $\mathbf{x}$. The smaller $g_s(\mathbf{x})$, the less the government values marginal consumption of individuals $\mathbf{x}$. Using the notations of equation (3), let $\lambda \equiv \alpha \mathcal{V}'(\mathscr{R}(\mathscr{T}))$ denote the shadow value of public funds. We formally define the period-s social marginal welfare weight associated with an individual with the choice vector $\mathbf{x}$ (and type

$\boldsymbol{\theta}$ such that $\mathbf{x}_{\boldsymbol{\theta}} = \mathbf{x}$) as[15]

$$g_s(\mathbf{x}) \equiv \frac{1-\alpha}{\lambda}\beta^{-(s-1)}\mathcal{G}'\left(\mathscr{U}_{\boldsymbol{\theta}}\left(\mathscr{T}\right)\right)U_{c_s}(\boldsymbol{\theta}). \tag{10}$$

The social marginal welfare weight $g_s(\mathbf{x})$ represents the value, as of period s, in terms of public funds, of giving one additional dollar in period s uniformly to all individuals whose choice vector is $\mathbf{x}$. This additional dollar increases the individual's indirect utility by $d\mathscr{U} = U_{c_s}$, and using the envelope theorem, social welfare increases by $d\left[\mathcal{G}\left(\mathscr{U}\right)\right] = \mathcal{G}'\left(\mathscr{U}\right)d\mathscr{U} = \mathcal{G}'\left(\mathscr{U}\right)U_{c_t}$. We express this welfare gain in terms of the value of public funds (that is, in monetary units) by dividing this expression by the multiplier λ.

We now characterize the revenue and welfare effects of local tax reforms. We fix a period p and analyze the effects on the tax revenue $\mathscr{R}(\cdot)$ and the social welfare $\mathscr{W}(\cdot)$ of locally perturbing the baseline tax function T_p in the direction $h_p \in \mathcal{C}^2(\mathrm{X},\mathbb{R})$. We show:

Proposition 2. *Suppose that Assumptions 1 and 2 are satisfied. The Gateaux differential of social welfare around the baseline tax system T_p in the direction h_p, is equal to*

$$\begin{aligned}\delta\mathscr{W}(T_p,h_p) = \int_{\mathrm{X}}\Big\{\Big[\beta^{p-1}\left(1-g_p(\mathbf{x})\right) - \boldsymbol{T}'(\mathbf{x})\boldsymbol{D}^{-1}(\mathbf{x})\boldsymbol{I}^{(\mathbf{x})}_{\mathbf{x},R_p}\Big]f_{\mathbf{x}}(\mathbf{x})h_p(\mathbf{x}) \\ + \Big[\boldsymbol{T}'(\mathbf{x})\boldsymbol{D}^{-1}(\mathbf{x})\boldsymbol{E}^{c,(\mathbf{x})}_{\mathbf{x},\tau_p}\Big]f_{\mathbf{x}}(\mathbf{x})\nabla h_p(\mathbf{x})\Big\}d\mathbf{x},\end{aligned} \tag{11}$$

where we let $\boldsymbol{D}(\mathbf{x}) \equiv \mathfrak{i}_{2S} - \sum_s \boldsymbol{E}^{c,(\mathbf{x})}_{\mathbf{x},\tau_s} D^2 T_s(\mathbf{x})$, and $\boldsymbol{T}'(\mathbf{x}) \equiv \sum_s \beta^{s-1}\left(\nabla T_s(\mathbf{x})\right)'$ is the discounted sum of the gradients of the baseline tax functions $T_s(\cdot)$. This expression can be equivalently written as

$$\begin{aligned}\delta\mathscr{W}(T_p,h_p) = \int_{\mathrm{X}}\Big\{\beta^{p-1}\left(1-g_p(\mathbf{x})\right)f_{\mathbf{x}}(\mathbf{x}) - \boldsymbol{T}'(\mathbf{x})\boldsymbol{D}^{-1}(\mathbf{x})\boldsymbol{I}^{(\mathbf{x})}_{\mathbf{x},R_p}f_{\mathbf{x}}(\mathbf{x}) \\ -\nabla\cdot\Big(\boldsymbol{T}'(\mathbf{x})\boldsymbol{D}^{-1}(\mathbf{x})\boldsymbol{E}^{c,(\mathbf{x})}_{\mathbf{x},\tau_p}f_{\mathbf{x}}(\mathbf{x})\Big)\Big\}h_p(\mathbf{x})\,d\mathbf{x}.\end{aligned} \tag{12}$$

The Gateaux differential of the tax revenue functional, $\delta\mathscr{R}(T_p,h)$, is given by equations (11,12) in which $g_p(\mathbf{x})$ is replaced with 0. The perturbation increases (resp., decreases), tax revenue or social welfare if $\delta\mathscr{R}(T_p,h) \geq 0$ or $\delta\mathscr{W}(T_p,h) \geq 0$ (resp., ≤ 0).

Proof. See Appendix. □

Formulas (11) or (12) give the effects on social welfare of any local perturbation of the baseline tax system in the direction h_p. Equation (11) is obtained from the formula

$$\delta\mathscr{W}(T_p,h_p) = \int_{\mathrm{X}}\beta^{p-1}\left(1-g_p(\mathbf{x})\right)h_p(\mathbf{x})f_{\mathbf{x}}(\mathbf{x})\,d\mathbf{x} + \int_{\mathrm{X}}\boldsymbol{T}'(\mathbf{x})\times\delta\mathbf{x}(T_p,h_p)f_{\mathbf{x}}(\mathbf{x})\,d\mathbf{x}, \tag{13}$$

[15] If Assumption 1 is not satisfied, then the social marginal welfare weight $g_s(\mathbf{x})$ should be defined as the average of the expression (10) over all individuals $\boldsymbol{\theta}$ who choose the same vector $\mathbf{x}$.

where $\delta \mathbf{x}(T_p, h)$ denotes the Gateaux differential of the individual income functional derived in (9), and (12) is then obtained by integrating by parts this equation. Intuitively, the first term in equation (13) is the *mechanical effect*, net of the *welfare loss*, of the perturbation h_p, and the second term is the *behavioral effect* of the tax reform. The mechanical effect captures the increase in government revenue due to the tax reform, assuming that individuals do not change their behavior in response to the perturbation. An individual with income $\mathbf{x}$ before the perturbation pays the additional tax liability $h_p(\mathbf{x})$ in period p after the perturbation. By definition of the marginal social welfare weights (10), this induces a loss in social welfare, expressed in units of tax revenue, equal to $g_p(\mathbf{x}) h_p(\mathbf{x})$. Summing over all individuals $\mathbf{x} \in \mathrm{X}$ using the density of incomes $f_{\mathbf{x}}(\mathbf{x})$ yields the first integral in (13). Next, the behavioral effect of the perturbation captures the change in government revenue due to the behavioral response of individuals whose vector of labor and capital incomes $\mathbf{x}$ is affected by changes in the marginal tax rates or the virtual incomes. We derived in Proposition 1 the change $d\mathbf{x} = \delta \mathbf{x}(T_p, h_p)$ in each individual's income vector $\mathbf{x}$ induced by the perturbation h_p. This induces in turn a change in government's revenue in every period s, given by $d[T_s(\mathbf{x})] = (\nabla T_s(\mathbf{x}))' d\mathbf{x}$. The overall behavioral effect of the perturbation is thus equal to the second integral in (13). Finally, the effect on government revenue is identical to the effect on social welfare, except that we do not take into account the welfare loss of the perturbation described above. We call the perturbation h_p *budget-neutral* if $\delta \mathscr{R}(T_p, h_p) = 0$.

The formula (12) (or equivalently (13)) is the most important theoretical result of our paper. It allows to compute in a wide variety of settings the effects on social welfare of any local tax reform h_p of the baseline tax system, by simply plugging the values $h_p(\mathbf{x})$ of the corresponding perturbation in the integral of (12). We analyze several examples of application of this result in Sections 5 and 6.

Moreover, we can use formula (12) to characterize the optimal tax system, or the optimum within a restricted class. Specifically, if the baseline tax system is optimal (possibly within a class), then there is no tax reform (within the corresponding class) that yields a positive effect on social welfare. Thus, by equating the Gateaux differential of social welfare for any such perturbation to zero, we obtain the optimum tax system. We thus obtain the following proposition:

Proposition 3. *Suppose that Assumptions 1 and 2 are satisfied. Then a necessary condition for the baseline tax function T_p to be optimal (resp., optimal within a class) is that, for any perturbation $h_p \in \mathcal{C}^2(\mathrm{X}, \mathbb{R})$ (resp., for any perturbation restricted within this class), we have*

$$\delta \mathscr{W}(T_p, h_p) = 0. \tag{14}$$

In particular, the optimum separable linear tax system is characterized by, for all $p \in \{1, \ldots, 2S\}$:

$$0 = \int_{\mathrm{X}} \left\{ \beta^{p-1} (1 - g_p(\mathbf{x})) x_p + \left[\boldsymbol{T}'(\mathbf{x}) \boldsymbol{E}^{c,(\mathbf{x})}_{\mathbf{x},\tau_{p,x_p}} - x_p \boldsymbol{T}'(\mathbf{x}) \boldsymbol{I}^{(\mathbf{x})}_{\mathbf{x},R_p} \right] \right\} f_{\mathbf{x}}(\mathbf{x}) \, d\mathbf{x}, \tag{15}$$

where $\boldsymbol{E}^{c,(\mathbf{x})}_{\mathbf{x},\tau_{p,x_p}}$ *is the* p^{th} *column of the matrix* $\boldsymbol{E}^{c,(\mathbf{x})}_{\mathbf{x},\tau_p}$.[16] *Moreover, the baseline tax system is the full optimum if, for any compact volume* $\mathrm{V} \subset \mathrm{X}$ *with closed and piecewise smooth boundary* $\mathrm{S} = \partial \mathrm{V}$*, we have, for all* $p \in \{1, \ldots, 2S\}$*:*

$$\begin{aligned} 0 = & \int_{\mathrm{V}} \beta^{p-1} \left(1 - g_p\left(\mathbf{x}\right)\right) f_{\mathbf{x}}\left(\mathbf{x}\right) d\mathbf{x} - \int_{\mathrm{V}} \left(\boldsymbol{T}'\left(\mathbf{x}\right) \boldsymbol{D}^{-1}\left(\mathbf{x}\right) \boldsymbol{I}^{(\mathbf{x})}_{\mathbf{x},R_p}\right) f_{\mathbf{x}}\left(\mathbf{x}\right) d\mathbf{x} \\ & + \int_{\partial \mathrm{V}} \left(\boldsymbol{T}'\left(\mathbf{x}\right) \boldsymbol{D}^{-1}\left(\mathbf{x}\right) \boldsymbol{E}^{c,(\mathbf{x})}_{\mathbf{x},\tau_p} \cdot \overrightarrow{\mathbf{n}}\left(\mathbf{x}\right)\right) f_{\mathbf{x}}\left(\mathbf{x}\right) d\mathrm{S}\left(\mathbf{x}\right), \end{aligned} \tag{16}$$

where $\overrightarrow{\mathbf{n}}\left(\mathbf{x}\right)$ *is the inward-pointing unit normal vector of the closed surface* S *at point* $\mathbf{x}$*.*

Proof. See Appendix. □

Equation (14) formalizes the intuition that the baseline tax system is optimal (resp., optimal within a class) if no tax reform (resp., no tax reform that leaves the tax system within the corresponding class) induces a non-zero welfare gain. In particular, we obtain the characterization of the optimal separable linear tax system given by formula (15), that is the set $\{\tau_{x_s}\}_{1 \leq s \leq 2S}$ of marginal tax rates on each income x_s, by imposing that the welfare effect (11) of separable linear perturbations in each period p, defined by $h_p\left(\mathbf{x}\right) = x_p d\tau$ and $\left[\nabla h_p\left(\mathbf{x}\right)\right]_j = d\tau \mathbf{1}_{\{j=p\}}$,[17] is equal to zero. Note that the optimal linear tax system is such that the total mechanical (net of the welfare loss) and behavioral (elasticitiy and income) effects, averaged over the *whole* population of individuals $\mathbf{x} \in \mathrm{X}$, must sum to zero. This is because in a linear tax system, all the individuals face the same marginal tax rate, so that the tax reforms must increase the tax rates by the same amount for *every* individual in order to leave the perturbed tax system within this class. The total effect of this perturbation is then the sum of these individuals' responses: the linear tax system is optimal if the *average* (over the whole population) behavioral effect due to the uniform increase in the marginal tax rate is equal to the additional revenue generated by this tax reform.

Equation (16) characterizes the fully optimal (in particular, non-linear and non-separable) tax system. We obtain this expression by imposing that every perturbation h_p yields a zero welfare effect, so that the integrand in equation (12) must be equal to zero pointwise:

$$\left[\beta^{p-1}\left(1 - g_p\left(\mathbf{x}\right)\right) - \boldsymbol{T}'\left(\mathbf{x}\right) \boldsymbol{D}^{-1}\left(\mathbf{x}\right) \boldsymbol{I}^{(\mathbf{x})}_{\mathbf{x},R_p}\right] f_{\mathbf{x}}\left(\mathbf{x}\right) - \nabla \cdot \left[\boldsymbol{T}'\left(\mathbf{x}\right) \boldsymbol{D}^{-1}\left(\mathbf{x}\right) \boldsymbol{E}^{c,(\mathbf{x})}_{\mathbf{x},\tau_p} f_{\mathbf{x}}\left(\mathbf{x}\right)\right] = 0. \tag{17}$$

Integrating this equation over any volume $\mathrm{V} \subset \mathrm{X}$ with closed boundary $\mathrm{S} = \partial \mathrm{V}$ must therefore have a zero effect, which implies (16). To understand the intuition underlying formula (16), suppose that the government wants to raise revenue by increasing uniformly and in a lump-sum

[16] The vector $\boldsymbol{E}^{c,(\mathbf{x})}_{\mathbf{x},\tau_{p,x_p}}$ gives the response of the income vector $\mathbf{x}$ to a change in the marginal tax rate in the direction x_p only, i.e., $\partial \mathbf{x} / \partial \tau_{p,x_p}$.

[17] The perturbations h as we defined them in Section 3 give the direction of the perturbation, and we evaluate the effect on social welfare of the function μh as $\mu \to 0$. Here, for concreteness, we write directly $h\left(x\right) = x d\tau$ and let $d\tau \to 0$, so that the perturbed tax schedule is $\tilde{T}\left(x\right) = \left(\tau_x + d\tau\right) x$. We repeatedly use the same slight abuse of notation in the following Sections 5 and 6.

way the tax liability of individuals with income in the region $\mathbf{x} \in \mathrm{V}$ of the space of incomes $\mathrm{X} \subset \mathbb{R}_+^S \times \mathbb{R}^S$. This mechanically increases the government's revenue, since all the individuals in the region V now pay higher taxes; summing the individual mechanical effects (net of the welfare losses) over the region V yields the first integral in equation (16). Moreover, these individuals respond to the lump-sum increase in their tax liability by adjusting their incomes, as captured by the vector of income effect parameters $\boldsymbol{I}_{\mathbf{x},R_p}^{(\mathbf{x})}$; summing these behavioral effects over all the individuals in the region V yields the second integral in (16). Finally, the government can only raise the lump-sum tax liability in the region V by increasing the marginal tax rates of the individuals located on the boundary of V, that is those with income $\mathbf{x}$ on the surface $\mathrm{S} = \partial \mathrm{V}$. These individuals respond to this higher distortion by adjusting their incomes, as captured by the matrix of compensated elasticities $\boldsymbol{E}_{\mathbf{x},\tau_p}^{c,(\mathbf{x})}$; summing these behavioral effects over all the individuals on the boundary $\mathrm{S} = \partial \mathrm{V}$ yields the third integral in (16).

To best understand this last effect, consider the static Mirrlees model, with a single income dimension $y \geq 0$. In order to raise the lump-sum tax liability of individuals with income $y \in [\hat{y}, \infty)$ (the region V), the government must increase the marginal tax rate at the income level $y = \hat{y}$ (the surface $\partial \mathrm{V}$).[18] Similarly, in a model with two dimensions of income (x_1, x_2), a lump-sum tax increase in the region $\mathrm{V} = [\hat{x}_1, \infty) \times [\hat{x}_2, \infty)$ requires an increase in the marginal tax rate τ_{x_1} in the region $\partial \mathrm{V}_1 = \{(x_1, x_2) : x_1 = \hat{x}_1, x_2 \geq \hat{x}_2\}$ and an increase in the marginal tax rate τ_{x_2} in the region $\partial \mathrm{V}_2 = \{(x_1, x_2) : x_1 \geq \hat{x}_1, x_2 = \hat{x}_2\}$; the set $\partial \mathrm{V}_1 \cup \partial \mathrm{V}_2$ forms the boundary of the region V. More generally, formula (16) shows that the optimal tax system is such that, for any region V of the space $\mathrm{X} \subset \mathbb{R}_+^S \times \mathbb{R}^S$, the elasticity effect induced by the additional distortion on the boundary $\partial \mathrm{V}$ of V exactly compensates the mechanical and the income effects due to the lump-sum tax increase inside the region V.

The fundamental difference between the fully optimal tax system (16) and the optimum within the class of separable linear tax systems (15), is that in the latter case the mechanical and behavioral effects of any (restricted) perturbation must cancel out *on average* over the whole population $\mathbf{x} \in \mathrm{X}$, while in the former case these opposing forces must cancel out *pointwise*, that is over every region $\mathbf{x} \in \mathrm{V}$. The fully optimal tax system is thus able to fine-tune optimally the distribution of distortions within the population, whereas a linear tax system is constrained to imposing the same tax rate to every individual, and hence to balance the revenue increase against a measure of the average distortion in the economy. We show in greater detail in Sections 5 and 6 below that the proper averaging of the income and elasticity effects over the various regions determined by any given perturbation (e.g., the boundary and the interior of the set V, where the marginal tax rates and the virtual income are respectively perturbed) is the key to determining the welfare effects of the corresponding tax reform.

Formula (17), along with the individual's first-order conditions (4) which pin down the

[18] Heuristically, this tax reform can be expressed as the limit, as $d\tau, d\hat{y} \to 0$, of a sequence of perturbations that increase the marginal tax rate by $d\tau$ on the interval $y \in [\hat{y}, \hat{y} + d\hat{y}]$, and increase the total tax liability in a lump-sum way by $d\tau d\hat{y}$ on the interval $y \geq \hat{y} + d\hat{y}$. See Saez (2001) and our Section 5.3 below for details.

density function $f_{\mathbf{x}}$ given the exogenous density $f_{\boldsymbol{\theta}}$, is a partial differential equation system which can be integrated, either theoretically or numerically, using standard techniques, to obtain the optimal tax system. Moreover, we can rewrite this formula to characterize the optimal tax system as a function of the exogenous distribution of types $\boldsymbol{\theta} \in \Theta$, rather than the endogenous distribution of incomes $\mathbf{x} \in \mathrm{X}$. For the characterization of the fully optimal tax system, changing variables from $\mathbf{x}$ to $\boldsymbol{\theta}$ in equation (17) is useful because the resulting partial differential equation does not feature the deformation matrix $\boldsymbol{D}(\mathbf{x})$, so that we can solve directly for the marginal tax rates $\boldsymbol{T}'(\mathbf{x}(\boldsymbol{\theta}))$.[19] Assume that individuals have $2S$ dimensions of characteristics, so that their vectors of types and incomes have the same dimension. We show in the Appendix that the optimal tax system is the solution to the partial differential equation:

$$\begin{aligned} 0 = &\left(1 - g_p\left(\mathbf{x}\left(\boldsymbol{\theta}\right)\right)\right) \frac{f_{\boldsymbol{\theta}}\left(\boldsymbol{\theta}\right)}{\det\left(J_{\mathbf{x}}\left(\boldsymbol{\theta}\right)\right)} + \boldsymbol{T}'\left(\mathbf{x}\left(\boldsymbol{\theta}\right)\right) \frac{J_{\mathbf{x}}\left(\boldsymbol{\theta}\right)}{\det\left(J_{\mathbf{x}}\left(\boldsymbol{\theta}\right)\right)} \left[J_{\mathbf{F}}^{-1}\left(\boldsymbol{\theta}\right) J_{\mathbf{F}}\left(T_p\right)\right] f_{\boldsymbol{\theta}}\left(\boldsymbol{\theta}\right) \\ &- \sum_{j=1}^{2S} \sum_{i=1}^{2S} \left[\left(J'_{\mathbf{x}}\left(\boldsymbol{\theta}\right)\right)^{-1}\right]_{i,j} \frac{\partial}{\partial \theta_i} \left\{ \boldsymbol{T}'\left(\mathbf{x}\left(\boldsymbol{\theta}\right)\right) \frac{J_{\mathbf{x}}\left(\boldsymbol{\theta}\right)}{\det\left(J_{\mathbf{x}}\left(\boldsymbol{\theta}\right)\right)} \left[J_{\mathbf{F}}^{-1}\left(\boldsymbol{\theta}\right) J_{\mathbf{F}}\left(\boldsymbol{\tau}_p\right)\right] f_{\boldsymbol{\theta}}\left(\boldsymbol{\theta}\right) \right\}_j, \end{aligned} \tag{18}$$

where $J_{\mathbf{x}}(\boldsymbol{\theta}) = \left[\partial x_{\boldsymbol{\theta},i} / \partial \theta_j\right]_{1 \leq i,j \leq 2S}$ is the Jacobian matrix of the income function $\mathbf{x}(\boldsymbol{\theta})$, $\det\left(J_{\mathbf{x}}(\boldsymbol{\theta})\right)$ is its determinant, and $J_{\mathbf{F}}(\boldsymbol{\tau}_p), J_{\mathbf{F}}(T_p), J_{\mathbf{F}}(\boldsymbol{\theta})$ are defined by $J_{\mathbf{F}}(\boldsymbol{\tau}_p) = \left[\partial F_i / \partial \tau_{p,x_j}\right]_{i,j}$, $J_{\mathbf{F}}(T_p) = \left[\partial F_i / \partial T_p\right]_{i,j}$, and $J_{\mathbf{F}}(\boldsymbol{\theta}) = \left[\partial F_i / \partial \theta_j\right]_{i,j}$, for the function F defined by (4). In practice, we can easily calculate these latter three matrices in specific models, by writing explicitly the system of first-order consitions in the form of (4). In the Appendix, we do so for a dynamic model in which the $2S$ sources of heterogeneity (that is the idiosyncratic vector $\boldsymbol{\theta}$) are the productivity of labor supply and the interest rate on the capital stock in each period. In this model, we obtain the following expressions:

$$\begin{aligned} J_{\mathbf{F}}^{-1}\left(\boldsymbol{\theta}\right) J_{\mathbf{F}}\left(T_p\right) &= -\left[\frac{\mathbf{x}}{\boldsymbol{\theta}} \circ \left(\mathfrak{i}_{2S} + \sum_{s=1}^{S} \zeta_{\mathbf{x},\hat{\tau}_s}^{u,(\mathbf{x})}\right)\right]^{-1} \boldsymbol{I}_{\mathbf{x},R_p}^{(\boldsymbol{\theta})}, \\ J_{\mathbf{F}}^{-1}\left(\boldsymbol{\theta}\right) J_{\mathbf{F}}\left(\boldsymbol{\tau}_p\right) &= \left[\frac{\mathbf{x}}{\boldsymbol{\theta}} \circ \left(\mathfrak{i}_{2S} + \sum_{s=1}^{S} \zeta_{\mathbf{x},\hat{\tau}_s}^{u,(\mathbf{x})}\right)\right]^{-1} \boldsymbol{E}_{\mathbf{x},\tau_p}^{c,(\boldsymbol{\theta})}, \end{aligned} \tag{19}$$

where $(\mathbf{x}/\boldsymbol{\theta})$ denotes the matrix $\left[x_{\boldsymbol{\theta},i} / \theta_j\right]_{1 \leq i,j \leq 2S}$, $\zeta_{\mathbf{x},\hat{\tau}_s}^{u,(\mathbf{x})}$ is the matrix of uncompensated elasticities with respect to the marginal and net-of-tax rates defined in (6), and $\circ$ is the element-wise multiplication of matrices.

[19] On the other hand, it is more useful to work with the distribution of incomes when deriving the welfare effects of local tax reforms of suboptimal tax systems, because it is observed given the current tax code.

5 Applications to Optimal Taxation

In this section we discuss applications of our general analysis to optimal taxation. We first show how our results reproduce two canonical benchmarks in public finance: the optimal Ramsey tax formula of Diamond (1975), and the optimal non-linear income tax formula in a static economy due to Mirrlees (1971). We then apply our analysis to several environments to obtain novel insights about non-linear labor income taxation and capital income taxation in dynamic economies.

In Sections 5.1 to 5.4, we focus on *separable* tax systems. The tax function in period s depends on labor income y_s and capital income z_s as $T_{s,y}(y_s) + T_{s,z}(z_s)$. To simplify the notations, we let $\bar{x}_s \equiv \mathbb{E}[x_s]$ denote the average value of income $x_s \in \{y_s, z_s\}$ in period s in the economy. In latter applications we take expections conditional on vector $\mathbf{x}$ lying in a set V, in which case we denote $\bar{x}_s^V \equiv \mathbb{E}[x_s|\mathbf{x} \in V]$. Similarly, let $\bar{\zeta}^c_{x_s,q_{x_p}}$, $\bar{\zeta}^u_{x_s,q_{x_p}}$, $\bar{\eta}_{x_s,R_p}$ be the average compensated price elasticity, uncompensated price elasticity and income effect parameter of income $x_s \in \{y_s, z_s\}$ in period s with respect to decreases in the prices $q_{x_p} \in \{w_p, r_p\}$ and income R_p in period p, with additional superscript V notation if the elasticities are averages over $\mathbf{x} \in V$. Formally, q_{x_p} denotes the net-of-tax rate $1 - \tau_{p,x_p}$ on income x_p period p, and R_p is the virtual income in period p. We thus define:

$$\begin{aligned}\bar{\zeta}^{(\mathrm{V})}_{x_i,q_{x_j}} &\equiv \int_{\mathbf{x}\in\mathrm{V}} \frac{x_i}{\bar{x}_j^V} \zeta^{(\mathbf{x})}_{x_i,q_{x_j}} f_{\mathbf{x}}(\mathbf{x}\,|\mathbf{x}\in\mathrm{V})\, d\mathbf{x},\\ \bar{\eta}^{(\mathrm{V})}_{x_s,R_p} &\equiv \int_{\mathbf{x}\in\mathrm{V}} \frac{q_{x_s}}{\hat{q}_{x_s}} \eta^{(\mathbf{x})}_{x_s,R_p} f_{\mathbf{x}}(\mathbf{x}\,|\mathbf{x}\in\mathrm{V})\, d\mathbf{x}.\end{aligned} \tag{20}$$

where $\hat{q}_{x_s} = 1 - \tau_{s,x_s}$ if $p = s$, and 1 otherwise.[20] Finally, we define the hazard rate of factor x_s in period s at point $\hat{x}$ as $\mathcal{H}_{x_s}(\hat{x}) = \hat{x} f_{x_s}(\hat{x})/(1 - F_{x_s}(\hat{x}))$.

5.1 Optimal Commodity Taxation

As a first application of our theory, consider the analysis of Ramsey (1927) and Diamond (1975), who restrict the tax system to be separable and *linear* in each income, so that in each period s a consumer pays a proportional tax $T_{s,x}(x) = \tau_{s,x} x$ on income x. Applying formula (15) we obtain that for all p, the optimal tax rates τ_{p,x_p} are determined as functions of the own- and cross-price uncompensated elasticities by:

$$\sum_{s=1}^{S} \sum_{x\in\{y,z\}} \beta^{s-p} \frac{\tau_{s,x_s}}{1-\tau_{p,x_p}} \bar{\zeta}^u_{x_s,q_{x_p}} = 1 - \mathbb{E}\left[\frac{x_p}{\bar{x}_p} g_p\right]. \tag{21}$$

[20] Thus, we have $\hat{q}_{x_s} = \hat{\tau}_{p,x_s}$, where $\hat{\tau}_{p,x_s}$ is defined in (6), so that $\frac{1}{\hat{q}_{x_s}} \eta_{x_s,R_p} = \frac{\partial x_s}{\partial R_p}$.

Define the net social marginal utility of income for individual $\mathbf{x}$ as

$$b_p(\mathbf{x}) \equiv g_p(\mathbf{x}) - \sum_{s=1}^{S} \sum_{x \in \{y,z\}} \beta^{s-p} \frac{\tau_{s,x_s}}{1-\tau_{s,x_s}} \eta^{(\mathbf{x})}_{x_s,R_p},$$

and let $\bar{b}_p \equiv \mathbb{E}[b_p(\mathbf{x})]$. Using the Slutsky equations and rearranging the previous equation, we obtain, for all p, x,

$$\sum_{s=1}^{S} \sum_{x \in \{y,z\}} \beta^{s-p} \frac{\tau_{s,x_s}}{1-\tau_{p,x_p}} \bar{\zeta}^{c}_{x_s,q_{x_p}} = 1 - \bar{b}_p - \bar{b}_p \cdot \operatorname{cov}\left(\frac{b_p}{\bar{b}_p}, \frac{x_p}{\bar{x}_p}\right).$$

This is Ramsey's formula with several consumers, first obtained by Diamond (1975).

5.2 Optimal Age-Independent Capital Income Tax Rates

In this section we study capital income taxes that are restricted to be linear and constant over many periods. Such taxes arise naturally in several cases. First, many applications impose an a priori assumption that capital taxes do not depend on the time period, e.g., Conesa, Kitao and Krueger (2009). Second, the optimal asymptotic capital tax rate in infinite horizon economies, analyzed by Chamley (1986) and Judd (1985), is equivalent to a tax that is constant across time after the economy reaches the steady-state.

For our analysis we abstract from income effects on labor supply and assume that preferences are of the form:

$$U = \sum_{s=1}^{S} \beta^{s-1} u\left(c_s - v\left(\frac{z_s}{\theta_s}\right)\right). \tag{22}$$

When labor supply has no income effects, the form of labor income taxes (linear, separable non-linear, or even history-dependent) is irrelevant for our main result.

First, note that when capital taxes can be chosen freely in each period, the optimal tax rate in period p satisfies Ramsey formula (21), where the cross-partial elasticities $\bar{\zeta}^{u}_{z_s,r_p}$ between labor income and the capital income tax rates are equal to zero. Then by exogenously restricting the tax rates to be constant across time and applying the general formulas (11) and (14), we obtain the following result, which characterizes the optimal *age-independent* capital income tax rate τ_z:

Proposition 4. *Suppose that the utility function has no income effects on labor supply as defined in (22). The optimal age-independent capital income tax rate τ_z is then given by:*

$$\frac{\tau_z}{1-\tau_z} = \left(1 - \sum_{p=2}^{S} \tilde{\beta}^{p-1} \mathbb{E}\left[\frac{z_p}{\bar{z}_p} g_p\right]\right) \frac{1}{\sum_{p=2}^{S} \sum_{s=2}^{S} \tilde{\beta}^{s-1} \bar{\zeta}^{u}_{z_s,r_p}}, \tag{23}$$

where the normalized discount factor is equal to $\tilde{\beta}^{s-1} \equiv \frac{\beta^{s-1}}{\sum_{p=2}^{S} \beta^{p-1}}$.

Proof. See Appendix. □

This proposition shows that the optimal tax, when restricted to be constant over time, is determined by the *compounded* uncompensated elasticity of savings, that is the sum of the per-period elasticities. Specifically, the capital income elasticities are summed both over periods s, because the tax rate in any given period p affects the savings choices in all periods s, and over periods p, because the tax is implemented in every period p.

We now compare this compounded elasticity with the behavioral response of capital income to a *one-period* change in the tax rate. We show that compounding the elasticities over a longer horizon can either increase or lower the elasticity that is relevant for the optimal tax rate, depending on the the magnitude of the elasticity of intertemporal substitution. First, observe that the compounded *uncompensated* elasticity can be written, using the Slutsky equation, as the sum of the compounded *compensated* elasticity and the compounded *income effect* parameter:

$$\sum_{p=2}^{S}\sum_{s=2}^{S} \tilde{\beta}^{s-1} \bar{\zeta}^{u}_{z_s,r_p} = \sum_{p=2}^{S}\sum_{s=2}^{S} \tilde{\beta}^{s-1} \left[\bar{\zeta}^{c}_{z_s,r_p} + \bar{\eta}_{z_s,R_p}\right].$$

The compenstated elasticies of capital income are always positive, while the income effect parameter $\bar{\eta}_{z_s,R_p}$ is negative for $s \leq p$ and positive otherwise. The size of the compounded uncompensated elasticity thus depends on whether the substitution effect dominates the net income effect of a tax change. The analysis is the most stark if we follow Judd (1985) and assume that capital is being held only by the agents who have no labor income, and whose utility is then $u(c) = c^{1-\sigma}/(1-\sigma)$. To simplify calculations, assume further that $S = \infty$. Computing directly the compensated elasticities $\bar{\zeta}^{c}_{z_s,r_p}$ and income effect parameters $\bar{\eta}_{z_s,R_p}$ when after-tax interest rates are equal to β^{-1}, we can compare the compounded elasticity to the elasticity of one-time tax change in period two. We obtain:

Proposition 5. *Assume that all the assumptions of this section are satisfied. Then the elasticity of capital income with respect to a change in the capital income tax rate in period two only, satisfies*

$$\begin{aligned}
\sum_{p=2}^{\infty}\sum_{s=2}^{\infty} \tilde{\beta}^{s-1} \bar{\zeta}^{u}_{z_s,r_p} &\geq \sum_{s=2}^{\infty} \tilde{\beta}^{s-1} \bar{\zeta}^{u}_{z_s,r_2}, \quad \textit{if } \sigma \leq 1, \\
\sum_{p=2}^{\infty}\sum_{s=2}^{\infty} \tilde{\beta}^{s-1} \bar{\zeta}^{u}_{z_s,r_p} &< \sum_{s=2}^{\infty} \tilde{\beta}^{s-1} \bar{\zeta}^{u}_{z_s,r_2}, \quad \textit{if } \sigma \textit{ is sufficiently large.}
\end{aligned}$$

Proof. See Appendix. □

Proposition 5 shows that compounding the elasticities may either increase or decrease the effective behavioral effect of capital income depending on the value of the intertemporal elasticity of substitution σ. Note that with our preferences, we have $\bar{\eta}_{z_s,R_p} = -\sigma\bar{\zeta}^u_{z_s,r_p}$ if $s \leq p$. The intuition thus becomes particularly transparent when we allow taxes to change only after some period P and take the limit as $P \to \infty$. In this case the positive income effects become negligible and we obtain

$$\lim_{P\to\infty} \sum_{p=P}^{\infty} \sum_{s=2}^{\infty} \tilde{\beta}^{s-p} \bar{\zeta}^u_{z_s,r_p} = \begin{cases} \infty, & \text{if } \sigma < 1, \\ -\infty, & \text{if } \sigma > 1. \end{cases} \tag{24}$$

Straub and Werning (2014) use this insight to provide an intuition for their results that the optimal capital taxes converge to zero in the long run steady state only if $\sigma < 1$ and that they remain positive and may even converge to infinity for $\sigma \geq 1$. Proposition 5 shows that the mechanisms they emphasize continue to operate for age-independent taxes even in the short run.

5.3 Optimal Non-Linear Labor Income Taxation

As a third application of our theory, consider the static model of optimal income taxation analyzed by Mirrlees (1971). Suppose that $S = 1$, there is no capital income and the individual utility is given by $u(c, y/\theta)$. We derived expressions for the matrices $\boldsymbol{D}^{-1}(\mathbf{x})\boldsymbol{E}^{c,(\mathbf{x})}_{\mathbf{x},\tau_p}$ and $\boldsymbol{D}^{-1}(\mathbf{x})\boldsymbol{I}^{(\mathbf{x})}_{\mathbf{x},R_p}$ for this case in Section 3. Applying formula (16) to the regions $\mathrm{V} = [\hat{y}, \infty)$,[21] we obtain immediately that the optimal non-linear labor income tax schedule $T(z)$ must satisfy

$$\begin{aligned} 0 =& \mathbb{E}_{y\geq\hat{y}}[1-g] - \frac{T'(\hat{y})}{1-T'(\hat{y})+\hat{y}\zeta^{c,(\hat{y})}_{y,w}T''(\hat{y})}\zeta^{c,(\hat{y})}_{y,w}\mathcal{H}_y(\hat{y}) \\ & - \mathbb{E}_{y\geq\hat{y}}\left[\frac{T'(y)}{1-T'(y)+y\zeta^{c,(y)}_{y,w}T''(y)}\eta^{(y)}_{y,R}\right]. \end{aligned} \tag{25}$$

Equation (25) is the formula obtained by Saez (2001). It formalizes his heuristic arguments that the optimal marginal tax rate on labor income $\hat{y}$ is driven by three forces: (i) the compensated elasticity of labor income $\zeta^c_{y,w}$ and the hazard rate $\mathcal{H}_y(\hat{y})$, which measure the distortions induced by the marginal tax rate at the income level $\hat{y}$; (ii) the average income effect parameter $\eta_{y,R}$ for incomes above $\hat{y}$, which measure the behavioral effects of increased average taxes on those incomes; and (iii) the value of redistributing income away from individuals above $\hat{y}$, captured by $\mathbb{E}_{y\geq\hat{y}}(1-g)$.

[21] We can apply Proposition 3 to the unbounded region $[\hat{y}, \infty)$ by considering an increasing sequence of compact sets of the form $[\hat{y}, y']$, and let $y' \to \infty$.

Using this expression, we can substitute (19) into (18) to obtain the following formula:

$$\begin{aligned} 0 = & \left(1 - \frac{1-\alpha}{\lambda}\mathcal{G}'(u)\,u_c\right) f_\theta(\theta) - \frac{T'(x_\theta)}{1-T'(x_\theta)}\frac{\dot{y}_\theta}{y_\theta}\frac{\eta_{y,R}}{1+\zeta^u_{y,w}}\theta f_\theta(\theta) \\ & + \frac{d}{d\theta}\left\{\frac{T'(x_\theta)}{1-T'(x_\theta)}\frac{\zeta^c_{y,w}}{1+\zeta^u_{y,w}}\theta f_\theta(\theta)\right\}. \end{aligned} \tag{26}$$

Integrating this differential equation yields the characterization of the optimal marginal income tax rates $T'(x_\theta)/(1-T'(x_\theta))$ derived by Mirrlees (1971). If in addition the utility function has no income effects on labor supply, so that $\eta_{x,R} = 0$, we obtain the formula derived by Diamond (1998). An advantage of writing the formula for the optimal tax schedule in the form (26) rather than in the original form of Mirrlees (1971) is that the explicit notation for the income and the substitution effects makes transparent the underlying economic effects that determine the optimal marginal taxes. Diamond (1998) and Saez (2001) further quantify these formulas and show in particular that for realistic preference parameters and hazard rates, they imply U-shaped optimal marginal tax rates.

Equation (25) finally allows us to obtain asymptotic tax rates. Suppose that $\mathbb{E}_{y\geq\hat{y}}g$, $\zeta^{c,(\hat{y})}_{y,w}$, $\eta^{(\hat{y})}_{y,R}$, and $\mathcal{H}_y(\hat{y})$ converge to the respective limits $g^{(\infty)}$, $\zeta^{c,(\infty)}_{y,w}$, $\eta^{(\infty)}_{y,R}$, and $\mathcal{H}^{(\infty)}_y$ as $\hat{y}\to\infty$, and suppose moreover that $\hat{y}T''(\hat{y})\to 0$ as $\hat{y}\to\infty$. We then obtain the top marginal tax rates as:

$$\lim_{\hat{y}\to\infty}\frac{T'(\hat{y})}{1-T'(\hat{y})} = \frac{1-g^{(\infty)}}{\zeta^{c,(\infty)}_{y,w}\mathcal{H}_y(\infty)+\eta^{(\infty)}_{y,R}}. \tag{27}$$

This expression is derived by Saez (2001).

5.4 Optimal Non-linear Labor and Capital Income Taxation

The examples of the previous Section 5.3 showed how to derive well-known results in the literature, namely those of Mirrlees (1971), Diamond (1998) and Saez (2001). We now show how to apply our theory to settings that are difficult to analyze with alternative techniques. In this section we apply our approach to illustrate several general lessons for optimal taxation that will carry over to more general settings. Suppose that $S = 2$ and the utility function has no income effects, as defined in (22). We are interested in deriving properties of the optimal taxes that are *separable*, *age-dependent*, and *non-linear*, so that the tax system consists of a non-linear labor income tax schedule $T_{1,y}(y_1)$ in period one, and of separable non-linear labor and capital tax schedules $T_{2,y}(y_2) + T_{2,z}(z_2)$ in period two.

This is a simplified version of environment considered by Conesa, Kitao and Krueger (2009). These authors impose parametric restrictions on tax functions and numerically optimize over those parameters. One limitation of this approach is that it is hard to know a priori whether a given parametric restriction is a good approximation of the fully optimal tax rates. For instance, we saw in the previous section that the optimal non-linear income tax schedule in the

static model is driven by the properties of the hazard function $\mathcal{H}_y(\hat{y})$, which for many realistic parameters implies that the optimal marginal tax rates are U-shaped. Such taxes are not allowed by conventional parametrizations of tax functions.

We start by applying our general formulas (11) and (14) to the tax schedule on labor income in period $s \in \{1,2\}$, restricting the tax system to be separable between the various incomes. We obtain that the optimal labor income tax rate in period s at the income level $\hat{y}_s$ is given by

$$\begin{aligned} 0 =& \mathbb{E}_{y_s \geq \hat{y}_s}[1-g_s] - \frac{T'_{s,z}(\hat{y}_s)}{1-T'_{s,y}(\hat{y}_s) - \hat{y}_s \zeta^{c,(\hat{y}_s)}_{y_s,w_s} T''_{s,y}(\hat{y}_s)} \bar{\zeta}^{c,(\hat{y}_s)}_{y_s,w_s} \mathcal{H}_{y_s}(\hat{y}_s) \\ & - \beta^{2-s} \mathbb{E}_{y_s \geq \hat{y}_s} \left[\frac{T'_{2,z}(z)}{1-T'_{2,z}(z) - z\zeta^c_{z_2,r_2} T''_{2,z}(z)} \hat{\eta}_{z_2,R_s} \right], \end{aligned} \tag{28}$$

where $\hat{\eta}_{z_2,R_s} = \eta_{z_2,R_2}$ if $s=2$, and $\hat{\eta}_{z_2,R_s} = (1-\tau_{2,z_2})\eta_{z_2,R_1}$ if $s=1$. Since we assume that there are no income effects on labor supply, the first line of this expression is simply a dynamic version of (25). The second line of (28) shows that in the dynamic environment, additional considerations play a role in the determination of the optimal labor income tax rates in either period, namely the effects that the labor income taxes induce on the capital income decisions of individuals with period-s labor income $y_s \geq \hat{y}_s$. Specifically, an increase in labor income taxes in period one leads to a reduction in savings, and hence capital income in period two. If the optimal marginal tax rate on capital income is positive, this reduces government revenue and creates a force to lower the labor income taxes in period one, relative to the static model. The opposite effect holds for labor income taxes in period two. Therefore, formula (28) indicates benefits of age-dependent labor income taxation.

Next, we apply formulas (11) and (14) to the tax schedule on capital income in period two. We obtain that the optimal capital income tax rate at the income level $\hat{z}$ is characterized by

$$\begin{aligned} 0 =& \mathbb{E}_{z \geq \hat{z}}[1-g_2] - \mathbb{E}_{z=\hat{z}} \left[\frac{T'_{2,z}(\hat{z})}{1-T'_{2,z}(\hat{z}) - \hat{z}\zeta^c_{z_2,r_2} T''_{2,z}(\hat{z})} \bar{\zeta}^c_{z_2,r_2} \frac{\hat{z} f_{\mathbf{x}}(y_1,y_2,\hat{z})}{1-F_{z_2}(\hat{z})} \right] \\ & - \mathbb{E}_{z \geq \hat{z}} \left[\frac{T'_{2,z}(z)}{1-T'_{2,z}(z) - z\zeta^c_{z_2,r_2} T''_{2,z}(z)} \eta_{z_2,R_2} \right] \end{aligned} \tag{29}$$

The expectation operator in the second term of equation (29) appears because the elasticity $\bar{\zeta}^c_{z_2,r_2}$ may be different for agents with a given value of capital income $z_2 = \hat{z}$, if they have different labor incomes y_1 and y_2. If the utility function is CARA, then the elasticities in the integrals do not depend on labor income and (29) is then identical to (25). Therefore, formula (29) illustrates that the same general mechanisms that determine optimal labor income taxation also determine optimal capital income taxation. As in the case of labor income taxes, the size and the shape of the capital income tax schedule are determined by the hazard rates of the capital income distribution, and by the income and substitution effects of capital income in response to changes in the capital income tax rates.

The asymptotic marginal tax rate on capital income is given by the analogue of (27),

$$\lim_{\hat{z}\to\infty}\frac{T_{2,z}'(\hat{z})}{1-T_{2,z}'(\hat{z})}=\frac{1-g_2^{(\infty)}}{\zeta_{z_2,r_2}^{c,(\infty)}\mathcal{H}_{z_2}(\infty)-\eta_{z_2,R_2}^{(\infty)}}.$$

It can be futher shown (see Appendix) that if mobility at the top of the capital income distribution converges to zero, the same formula continues to apply for the top marginal tax rates in arbitrary S period economies. If, in addition, the capital income tax schedule T_z is restricted to be age-independent, all the parameters are replaced with their compounded analogues along the lines of the analysis in Section 5.2.

5.5 Optimal Joint Taxation

We finally apply our theory to the analysis of the optimal *non-separable*, *non-linear* tax system. We illustrate this approach in a simple static framework of optimal taxation of couples. We assume that the household's goal is to maximize their total surplus, i.e., the total consumption minus the sum of disutilities of labor. Both individuals choose their labor supply on the intensive margin.[22] The couple's preferences over consumption and labor income are given by

$$\max_{c_1,c_2,y_1,y_2}\ u\left(c_1+c_2-\frac{1}{1+1/\zeta}\left(\frac{y_1}{\theta_1}\right)^{1+1/\zeta}-\frac{1}{1+1/\zeta}\left(\frac{y_2}{\theta_2}\right)^{1+1/\zeta}\right),$$

and its budget constraint is

$$c_1+c_2=y_1+y_2-T\left(y_1,y_2\right).$$

Applying the general formula (18) to this environment, we obtain that the optimal tax system is characterized by the following partial differential equation: for all $\boldsymbol{\theta}=(\theta_1,\theta_2)\in\mathbb{R}_+^2$,

$$0=\left(1-g\left(\boldsymbol{\theta}\right)\right)f_{\boldsymbol{\theta}}\left(\boldsymbol{\theta}\right)+\frac{\zeta}{1+\zeta}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial y_{-i}}{\partial\theta_{-j}}\frac{\partial}{\partial\theta_j}\left\{\frac{\frac{\tau_1}{1-\tau_i}\frac{\partial y_{\boldsymbol{\theta},1}}{\partial\theta_i}+\frac{\tau_2}{1-\tau_i}\frac{\partial y_{\boldsymbol{\theta},2}}{\partial\theta_i}}{\frac{\partial y_{\boldsymbol{\theta},1}}{\partial\theta_1}\frac{\partial y_{\boldsymbol{\theta},2}}{\partial\theta_2}-\frac{\partial y_{\boldsymbol{\theta},1}}{\partial\theta_2}\frac{\partial y_{\boldsymbol{\theta},2}}{\partial\theta_1}}\theta_i f_{\boldsymbol{\theta}}\left(\boldsymbol{\theta}\right)\right\},\tag{30}$$

where we the components of the Jacobian matrix $\frac{\partial y_{\boldsymbol{\theta},i}}{\partial\theta_j}$ evaluated at the type $\boldsymbol{\theta}$ can be easily expressed explicitly as a function of the tax rates (see Appendix for details). Formula (30) is useful because it allows us to reduce the problem of finding the optimal joint tax system in the economy as the solution to a PDE which can be computed numerically, without the need to solve for a complicated individual optimization problem. Note finally that this PDE generalizes to the two-dimensional environment the differential equation obtained in the static model of

[22] Saez, Kleven and Kreiner (2009) characterize the optimal joint tax system in the case where the secondary earner chooses labor supply on the participation margin only.

individual-based taxation of Mirrlees (1971), that is

$$0 = (1 - g(\theta)) f_\theta(\theta) + \frac{\partial}{\partial \theta} \left\{ \frac{\tau}{1-\tau} \frac{\zeta}{1+\zeta} \theta f_\theta(\theta) \right\}$$

which can be easily solved analytically to obtain the optimal tax rates of Diamond (1998).

6 Applications to Tax Reforms

In the previous Section 5, we characterized the optimal taxes in various settings, within given classes of tax functions. We showed that this approach requires solving a system of partial differential equations (19). Moreover, its solution depends on the elasticities evaluated endogenously at the *optimum* tax system. The values of these elasticities can be computed either by a priori restricting attention to a specific functional form for the utility function (as done, for example, by Mirrlees 1971), or by inferring through some alternative methods what the values of those elasticites are at the optimum (as is often implicitly assumed in the "sufficient statistics" approach, e.g., Chetty 2009).

In the present Section 6, we use the tools developed in Section 4 to explore another question: we analyze the welfare gains of small reforms of the existing tax system. The approach used to answer this question is closely related to that involved in designing the optimal tax system, since the welfare effects of local reforms are determined by the same underlying economic forces as the optimal tax system. However, evaluating the effects of local tax reforms is substantially simpler. These gains depend on the labor and capital income elasticties that can be readily estimated empirically under the *current*, rather than the optimal, tax system. Once these elasticities are known, the welfare effects of tax reforms can be computed directly using Proposition 2 without any need to solve differential equations. The sources of welfare gains can be expressed in easily interpretable and empirically measurable terms.

Throughout this section we illustrate this approach by considering a simple version of a lifecycle model. We assume that individuals live for S periods and have Greenwood, Hercowitz and Huffman (1988) preferences

$$\frac{1}{1-\sigma} \sum_{s=1}^{S} \beta^{s-1} \left(c_s - \frac{1}{1+1/\zeta} l_s^{1+1/\zeta} \right)^{1-\sigma}.$$

We assume that the baseline tax system does not depend on the age of the individual, that it is separable between labor and capital incomes within and across time periods, and that it is linear in capital income:

$$T_s(\mathbf{x}) = T_y(y_s) + \tau_z(z_s), \text{ for all } s = 1, ..., S. \tag{31}$$

We let the tax schedule on labor income, $T_y(\cdot)$, be either linear or non-linear. We further

assume that all agents face the same after-tax interest rate equal to β^{-1}. We choose this specification both because it allows us to illustrate the main effects transparently, and because these assumptions on preferences and taxes are often used in applied works on optimal taxation. It is straightforward to apply our method to compute the welfare effects of tax reforms for alternative specifications of baseline taxes. Finally, to evaluate the effects of our reforms, we can directly plug into our formulas the empirical estimates of the relevant elasticities and income distributions.

In the paragraphs that follow, we sequentially add more sophisticated elements to our stylized version of the U.S. tax system, namely age-dependence (Section 6.1), non-linear capital taxes (Section 6.2), history-dependence and joint taxation of labor and capital incomes (Section 6.3). Our goal is to use the general perturbation method described in Section 4 to show qualitatively and quantitatively the effects that each additional element of the more sophisticated tax system has on government revenue and social welfare.

6.1 Labor Income Taxation

We start by analyzing reforms of the marginal labor income tax rates. Specifically, we increase the marginal tax rates, in every period $p \in \{1, \ldots, S\}$, on a small interval around some labor income level $\hat{y}$ and take the limit as this interval goes to zero. Formally, we fix the numbers $\hat{y} > 0$, $\hat{y}' > \hat{y}$, $u > 0$, and $\Delta\tau$, and define a perturbation $h \in \mathcal{C}^2(\mathbb{R}_+)$, with $h(y) = 0$ on $[0, \hat{y}]$, $h'(y) = \Delta\tau$ on $[\hat{y} + u, \hat{y}' - u]$, and $h'(y) = 0$ on $[\hat{y}', \infty)$. We let h be monotonic on $[\hat{y}, \hat{y} + u]$ and $[\hat{y}' - u, \hat{y}']$. We can then compute the Gateaux differential of social welfare in the direction of this perturbation. Next, we consider a sequence $\{h^n\}_{n\in\mathbb{N}}$ of such perturbations, with $\hat{y}' \to \hat{y}$, $u \to 0$, $u = o(\hat{y}' - \hat{y})$, and $\Delta\tau \to 0$. At each point in the sequence, compute the Gateaux differential of social welfare in that direction and focus on the limit as $n \to \infty$ and hence $||h^n|| \to \infty$.

To report the welfare effects of this (limiting) perturbation, it is convenient to normalize the gains per dollar of statutory tax change, which we define as:

$$\Gamma_{\hat{y}} = \frac{\lim_{n\to\infty} \sum_{p=1}^{S} \delta\mathscr{W}\left(\mathscr{T}, h_p^n\right)}{\sum_{p=1}^{S} \beta^{p-1}\left(1 - F_{y,p}(\hat{y})\right)}.$$

The denominator is the present value of the statutory increase in tax revenue due to the tax reform (as every individual with income above $\hat{y}$ pays an additional unit of their income in taxes, and there are $(1 - F_{y,p}(\hat{y}))$ such individuals) and the numerator is the welfare effect of the reform, as defined in (11).

Note that the perturbation we defined restricts attention to reforms of the labor income tax schedule that do not depend on the individual's age. The total effect of this tax reform can be thought of as a sum of S age-dependent reforms, each of which increases marginal taxes at income $\hat{y}$ only for inviduals of age $p \in \{1, \ldots, S\}$. Applying Proposition 2, we obtain the

(normalized) welfare effect of this reform as:

$$\Gamma_{\hat{y}} = \sum_{p=1}^{S} \gamma_{p,\hat{y}} \left\{ \mathbb{E}_{y_p \geq \hat{y}} \left[1 - g_p \right] - \frac{T'_y(\hat{y})\,\zeta}{1 - T'_y(\hat{y}) + \hat{y}\zeta T''_y(\hat{y})} \mathcal{H}_{y,p}(\hat{y}) - \frac{\tau_z}{1-\tau_z} \mathring{\eta}_p \right\}, \tag{32}$$

where $\mathring{\eta}_p$ is the compounded income effect due to an increase in the period-p virtual income, defined as

$$\mathring{\eta}_p = \sum_{s=2}^{S} \beta^{s-p} \bar{\eta}_{z_s, R_p},$$

and $\gamma_{p,\hat{y}}$ is the weight given by

$$\gamma_{p,\hat{y}} = \frac{\beta^{p-1}\left(1 - F_{y,p}(\hat{y})\right)}{\sum_{p=1}^{S} \beta^{p-1}\left(1 - F_{y,p}(\hat{y})\right)}.$$

The expression in the curly brackets of formula (32) is the welfare effect of a tax reform that affects the marginal labor income tax rates in period p only. The welfare effect of the age-independent tax reform is thus a weighted sum of these age-dependent reforms. The weights $\gamma_{p,\hat{y}}$ depend on the time period p (through the discount factor β^p which accounts for the time period in which the tax revenue is collected) and on the fraction of individuals of age p who are affected by the reform, $(1 - F_{y,p}(\hat{y}))$. The periods p that occur later in life, and those where there are relatively few individuals with income above the threshold $\hat{y}$, receive smaller weights.

We make several observations about formula (32). First, if the distribution of labor income is independent of age, then the hazard rate $\mathcal{H}_{y,p}(\hat{y})$ is independent of the time period p, and the last term of (32), which captures the behavioral effects of the tax reform on savings, is equal to zero. Equation (32) then reduces to the formula derived in the static framework by Saez (2001). When the distribution $\mathcal{H}_{y,p}(\hat{y})$ varies with age, for example because earnings earlier in life are more compressed relative to the earnings later in life, then the behavioral effects on capital income will be non-zero. For example, we can show that for $S = 2$, this term reduces to

$$\frac{\tau_z}{1-\tau_z} \sum_{p=1}^{2} \gamma_{p,\hat{y}} \beta^{2-p} \bar{\eta}_{z_1, R_p} \propto \left\{ \bar{F}_{y_1}(\hat{y}) - \bar{F}_{y_2}(\hat{y}) \right\} \frac{\beta r}{1+\beta},$$

so that the sign of the behavioral response of capital income depends on whether there are more individuals above the $\hat{y}$ threshold in period one or in period two (see details in the Appendix).

If the baseline labor income tax schedule is linear, i.e., $T'_y(y) = \tau_y$ for all y, then the term $\hat{y}\zeta T''_y(\hat{y})$ in equation (32) is equal to zero. Suppose in particular that the baseline tax system is the *optimal* linear age-independent tax code, as we studied in Section 5.1. This restricted

optimal tax system is given by:

$$\frac{\tau_y}{1-\tau_y} = \frac{1}{\zeta}\sum_{p=1}^{S}\tilde{\beta}^{p-1}\left(\mathbb{E}\frac{y_p}{\bar{y}_p}(1-g_p) - \frac{\tau_z}{1-\tau_z}\mathring{\eta}_p\right).$$

The effect of a *linear* tax reform h_L (uniform increase in the marginal tax rate τ_y) around this baseline tax system is equal to zero, and it is given by:

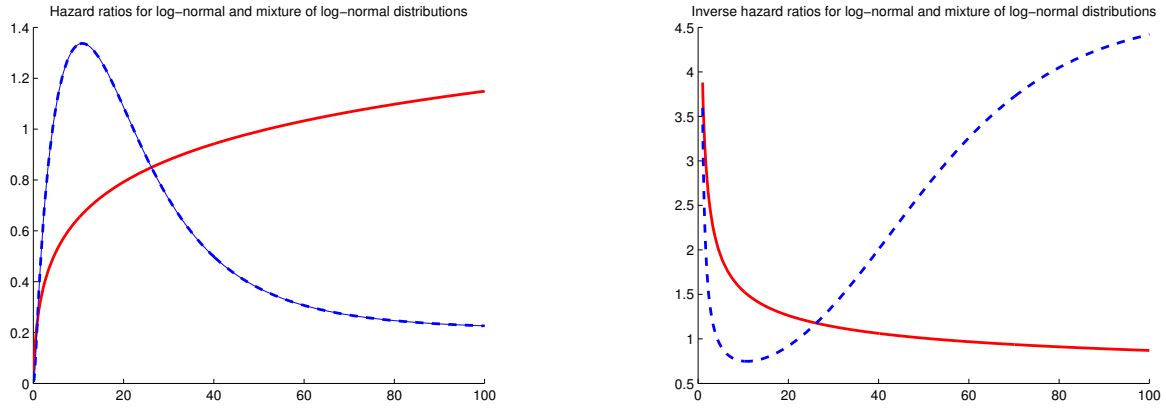
$$\delta\mathscr{W}(\mathscr{T}, h_L) = \int_0^\infty \delta\mathscr{W}(\mathscr{T}, h_{\hat{y}})\, d\hat{y} = 0, \tag{33}$$

where $h_{\hat{y}}$ denotes the non-linear perturbation at the point $\hat{y}$ defined above. This expression shows that the effect on social welfare of the linear tax reform h_L is equal to the sum of the welfare effects of the non-linear perturbations at each income level $\hat{y}$. The linear tax system is constrained to maximizing an *average* of these effects. On the other hand, a non-linear tax system can choose the tax rates so that the welfare effects $\delta\mathscr{W}(\mathscr{T}, h_{\hat{y}})$ are equal to zero pointwise, i.e., at each income level $\hat{y}$.

The gains from non-linear perturbations are the largest if the redistributive objective $\mathbb{E}_{y_p \geq \hat{y}}[1-g_p]$ and the hazard rate $\mathcal{H}_{y,p}(\hat{y})$ vary highly with income $\hat{y}$. The amount of this variation depends on the higher moments of the income distribution. To illustrate this point, we plot in Figure 2 the hazard rates of a log-normal distribution and of a distribution generated from a mixture of log-normals,[23] chosen in such a way that the first three moments of these two distributions (mean, variance and skewness) are equal. The mixture of log-normals has a higher kurtosis. The hazard rate of the high kurtosis distribution varies much more with income than that of the log-normal distribution, indicating higher welfare gains from non-linear taxation in the former case. This also provides an intuition for the small welfare gains from non-linear taxation found in the literature when shocks are log-normally distributed (Mirrlees 1971, Farhi and Werning 2013) and the much larger gains obtained when the shocks are drawn from distributions with larger higher moments or fatter tails (Saez 2001, Golosov, Troshkin and Tsyvinski 2013). Since the analysis in Section 5 shows that the shape of the optimal taxes is driven in part by the inverse hazard rates, the second panel of 2 plots the inverse hazards in these two cases. Note finally that for more general preferences, the elasticities and income effect parameters $\zeta, \mathring{\eta}_p$ would also depend on income, which constitutes an additional force in favor of imposing different taxes to different individuals.

[23] See Guvenen et al. (2014) for evidence that the mixture of lognormal distributions with high kurtosis provides a good approxiation for the empirical income distributions.

Figure 2: Hazard Rates and Inverse Hazard Rates



Solid lines: hazard and inverse hazard of log-normal distribution. Dotted line: hazard and inverse hazard of a mixture of log-normal distribution. The parameters are chosen so that the mean, variance and skewness of both distributions are the same. The excess kurtosis is equal to 18 for the mixture of log-normal distributions, and to 0 for the log-normal distribution.

Formula (32) also indicates the sources of welfare gains from allowing taxes to depend on age. The magnitude of the gains of intoducing age-independence into the tax system is determined by weighted average of the age-dependent hazard rates $\mathcal{H}_{y,p}$ and the income effects $\mathring{\eta}_p$. Since these variables generally depend on the individual's age, the welfare gains given by (32) are an average of the low and high values of $\mathcal{H}_{y,p}$ and $\mathring{\eta}_p$, corresponding to the periods where the behavioral responses to taxes are weak (so that the per-period welfare gains are large) and strong, respectively. Age-dependent taxes give the government some flexibility to fine-tune the tax rates so that the distortions are the largest for the ages in which the behavioral responses to taxes are the smallest.[24] The larger the age variation in these behavioral responses to taxes, the higher the gains from age-dependent taxation. Note finally that if the elasticities and income effect parameters $\zeta, \mathring{\eta}_p$ also depend on age, the planner has an additional reason to impose different taxes to individuals of different ages (see Kremer, 1999).

With our specific functional form for the individual preferences, we can directly compute income effects of savings. We show in the Appendix that

$$\frac{1}{1-\tau_z}\mathring{\eta}_p = \beta r\left(\frac{1}{1-\beta} - \frac{\beta^S}{1-\beta^S}S - p\right).$$

Therefore this effect is positive early in life, and negative later in life. Suppose that the capital income tax rate is positive in the baseline tax system, $\tau_z > 0$. This implies that, all things being equal, the welfare gains from increasing marginal tax rates on labor income are higher for older individuals. To give a sense of the magnitude of these numbers, take $S = 40$ and $\beta = 0.97$, we

[24] There is an additional source of benefits of age-dependence, which comes from being able to target redistribution. This effect depends on the Pareto weights that society assigns to different agents.

get:

$$-\frac{1}{1-\tau_z}\mathring{\eta}_p = \begin{cases} -0.75 & \text{if } p = 1 \text{ (age 21)}, \\ 0.17 & \text{if } p = 20 \text{ (age 40)}, \\ 1.14 & \text{if } p = 40 \text{ (age 60)}. \end{cases}$$

Therefore, the behavioral effects on savings from higher labor income taxes at age p increase revenue by $\tau_z \mathring{\eta}_p$. This indicates that, all else being equal, the gains from higher taxes are the highest later in lifetime. Note that these numbers are substantial, reaching an additional benefit of $1.14 \times \tau_z$ per dollar increase in the statutory tax liability.

The welfare effect of labor income tax reforms also depend on how the hazard rate $\mathcal{H}_{y,p}$ varies with age, which should be straighforward to compute from the same admistrative income tax data that have been used in empirical literature to address other questions (e.g., Guvenen et al 2014, Chetty **??**). We can conduct a tentative back-of-the-envelope excercise to evaluate the behavioral response to tax changes. Saez (2001) documents that in the cross-section of US taxpayers the wage income hazard rate is around 2,[25] which suggests that at least in some periods p we have $\mathcal{H}_{y,p} \approx 2$ for high income levels. At the high levels of the income distribution, the effective marginal labor income tax rates are flat and equal to about 50% (Prante and John 2012).[26] The behavioral labor income response to tax changes of top incomes is $\zeta \times \mathcal{H}_{y,p}$. There is substantial controversy in the literature about the value of the compensated labor income elasticity; the micro literature typically finds values around 0.3 and lower, while the macro literature and some structural estimates find it to be closer to 1 (see Chetty 2012, and Keane and Rogerson 2012, for an overview of the two strands). If the hazard ratio $\mathcal{H}_{y,p}$ is about 2, the behavioral response of labor income fully offsets the statutory tax reform if $\zeta > 0.5$, so that the revenue gains of the tax reform are then mostly driven by the dynamic effects on savings. The revenues from this tax reform, ignoring the redistributive effect, are given at age p by

$$1 - \zeta \mathcal{H}_{y,p} + \tau_z \left(-\frac{1}{1-\tau_z}\mathring{\eta}_p \right)$$

for a tax rate of 50%. Note that this formula provides an upper bound for the welfare effects of the tax reform when we take into account its redistributive effect.

6.2 Non-Linear Capital Income Taxation

We next apply our theory to the analysis of tax reforms of the baseline capital income tax schedule. Specifically, we characterize the welfare effects of introducing non-linearities into the baseline linear tax schedule, keeping it age-independent. We consider the effect of separable

[25] Saez (2001) estimates $a \equiv \mathbb{E}_{y \geq \hat{y}} y/\hat{y} \approx 2$ for high $\hat{y}$, which implies that the cross-sectional hazard rate $\mathcal{H}_y^{cs}(\hat{y}) = a/(a-1) \approx 2$.

[26] In particular, Prante and John (2012) document that the sum of top federal and state taxes on wages is on average 47.9% across all states. For our purposes these numbers understate effective labor income tax rates since they do not include consumption taxes.

non-linear perturbations at the capital income level $\hat{z}$, analogous to the perturbations of the labor income tax schedule that we defined in Section 6.1. Applying our general formula (11) yields the following welfare effect:

$$\Gamma_{\hat{z}} = \sum_{p=1}^{S} \gamma_{p,\hat{z}} \Gamma_{p,\hat{z}} \equiv \sum_{p=1}^{S} \gamma_{p,\hat{z}} \left\{ \mathbb{E}_{z_p \geq \hat{z}} \left[1 - g_p \right] - \frac{\tau_z}{1-\tau_z} \mathring{\zeta}_p^{c,(\hat{z})} \mathcal{H}_{z,p} \left(\hat{z} \right) - \frac{\tau_z}{1-\tau_z} \mathring{\eta}_p \right\}, \tag{34}$$

where

$$\gamma_{p,\hat{z}} = \frac{\beta^{p-1} \left(1 - F_{p,z} \left(\hat{z} \right) \right)}{\sum_{p=1}^{S} \beta^{p-1} \left(1 - F_{z,p} \left(\hat{z} \right) \right)}, \quad \text{and} \quad \mathring{\zeta}_p^{c} = \sum_{s=2}^{S} \beta^{s-p} \bar{\zeta}_{z_s, r_p}^{c,(z_p = \hat{z})}.$$

The first difference between formula (34), which gives the welfare effects of age-independent capital income tax reforms, and formula (32) which gives the welfare effects of age-independent labor income tax reforms, is the effect of compounding the relevant compensated income elasticities. Mirroring our discussion of Section 5.2, the compounding effect may a priori increase or decrease the effective behavioral response depending on the intertemporal elasticity of substitution. Second, the relevant hazard rates and the weights on the age-dependent perturbations are different in formulas (34) and (32), to reflect the fact that it is the distribution of capital income, $F_{p,z}(z)$, rather than that of labor income, $F_{p,y}(y)$, which determines the welfare effects of reforming the capital income tax schedule.

To get a sense about the magnitude of these effects, assume that the capital income distribution $F_{p,z}(z)$ does not depend on age, in which case $\sum_{p=1}^{S} \gamma_{p,\hat{z}} \mathring{\eta}_p = 0$ and $F_{p,z}(z)$ is equal to the cross-sectional distribution of capital income. In the data this distribution is more skewed than that of labor income. For example, Nirei and Souma (2007) use the cross-section of tax returns in the US to estimate the hazard rate $\mathcal{H}_{z,p}(\hat{z})$ to be equal to 1.5 at the top of the distribution. There are relatively few empirical estimates of the compounded capital income elasticity $\mathring{\zeta}_p^c$. Theoretically, if labor income is a small fraction of total consumption and the horizon S is large, then $\mathring{\zeta}_p^c \approx \sigma^{-1}$, so that $\mathring{\zeta}_p^c$ is approximately equal to the coefficient of the intertemporal substitution. Supposing that the marginal capital income tax rate is 50%,[27] the revenue effect (per dollar of statutory increase in taxes) of an age-independent tax reform of the capital income tax rate at the top of the distribution is given by $1 - 1.5 \times \sigma^{-1}$. This is an upper bound for the welfare gains, which also include the redistributive effect.

Formula (34) also emphasizes the sources of gains coming from more sophisticated reforms of the capital income tax rates. If we restrict attention to *linear* perturbations h_L of the capital tax schedule, the same reasoning as in Section 6.1 shows that the welfare effect equal to

$$\delta \mathscr{W} \left(\mathscr{T}, h_L \right) = \int_{\mathbb{R}} \delta \mathscr{W} \left(\mathscr{T}, h_{\hat{z}} \right) d\hat{z},$$

[27] In the U.S. the effective marginal tax rates on capital income vary by the source of income. The top marginal tax rate on interest income is about 50% (see Prante and John, 2012). The top tax rates on capital gains or qualified dividends are lower, but for our purposes the effective rate on capital saved in a form of corporate equity should also include the corporate taxation.

where $h_{\hat{z}}$ is the *non-linear* perturbation at the point $\hat{z}$ analyzed in this section. Unless the baseline tax rate τ_z is far from the optimum, this integral will generally sum both positive and negative terms, depending on the income level $\hat{z}$. As we discussed in Section 6.1, the non-linear reform further improves welfare by increasing the marginal taxes for those values of $\hat{z}$ where $\delta\mathscr{W}(\mathscr{T}, h_{\hat{z}}) > 0$, and decreasing them where $\delta\mathscr{W}(\mathscr{T}, h_{\hat{z}}) < 0$. The shape of the redistributive objective $\left(1 - \mathbb{E}_{z_p \geq \hat{z}} g_p\right)$, of the hazard rate $\mathcal{H}_{z,p}(\hat{z})$, and of the income and substitution effects $\mathring{\zeta}_p^{c,(\hat{z})}, \mathring{\eta}_p$ as a function of income, determine the regions in which introducing non-linear capital income tax rates brings the largest improvements in welfare.

6.3 Joint Income Taxation

The baseline tax system (31) is initially separable between incomes, both across periods (there is no history-dependence) and within periods (labor and capital incomes are not jointly taxed). In this section, we characterize the welfare gains obtained from introducing joint taxation, where taxes on income x_i depend not only on the level of x_i, but also on the levels of other incomes x_j. Such taxes arise in several different contexts. In the U.S., many social insurance programs and the Social Security system condition their payments both on current labor earnings and on the history of past earnings. Some programs are also often asset-tested, i.e., individuals are eligible to participate if their labor earnings are low and their assets are below a certain treshold. Finally, the individual tax bill depends jointly on income from labor and capital.

The non-separable tax reforms that we consider consist of increasing the marginal tax rate on income x_i at the level $\hat{x}_i$ (hence the average tax rates increase on $x_i \geq \hat{x}_i$) *conditional* on earning more than the threshold $\hat{x}_j$ of income x_j, i.e., $x_j \geq \hat{x}_j$. Since we consider perturbations that leave the tax function continuous, this reform also raises marginal tax rates on income x_j at level $\hat{x}_j$, conditional on $x_i \geq \hat{x}_i$. This joint pertubation is shown in Figure 3, where the dark (resp., light) surface represents the baseline (resp., perturbed) tax function.

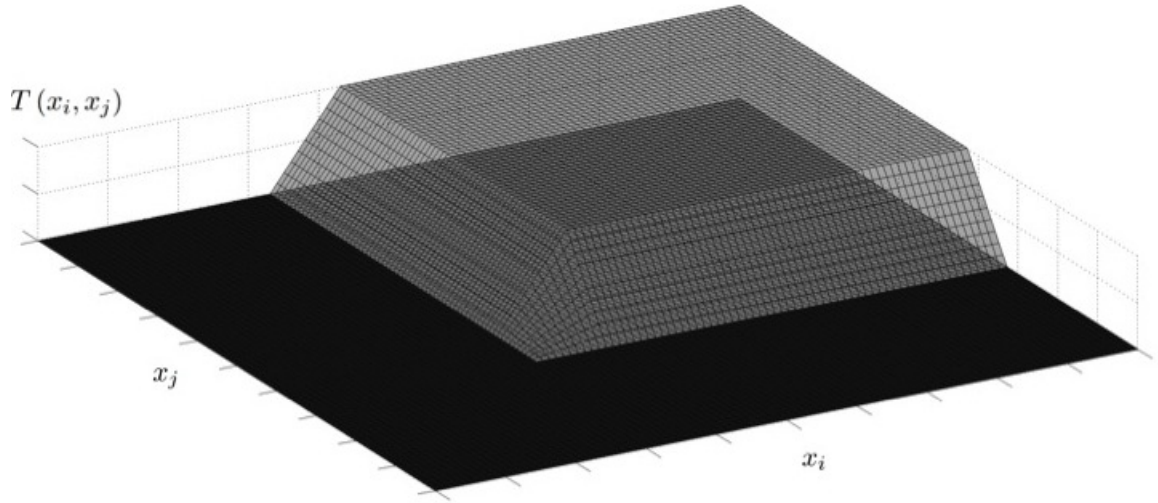
We showed in the previous sections that the welfare effects of *separable* perturbations are determined by the fraction of individuals above the base of the perturbation relative to the fraction at the base, summarized by the hazard rate $\mathcal{H}_{x_i}$ of the distribution of income x_i. The generalization of the hazard rate to two dimensions of income (x_i, x_j) is captured by

$$\mathcal{H}_{x_i,x_j}(\hat{x}_i \,|\hat{x}_j) \equiv \frac{\hat{x}_i \int_{\hat{x}_j}^{\infty} f(\hat{x}_i, x_j)\, dx_j}{\int_{\hat{x}_i}^{\infty} \int_{\hat{x}_j}^{\infty} f(x_i, x_j)\, dx_i dx_j}.$$

The denominator is the fraction of agents who face an increase in their average tax liability. The numerator is the fraction of agents who face an increase in their marginal tax rate on income x_i, scaled by the income threshold $\hat{x}_i$.

We first consider introducing joint taxation of labor income across periods, i.e., history-dependence. That is, we increase the marginal tax rate on labor income $\hat{y}_p$ in period p conditional

Figure 3: Joint Perturbations



on $y_{p-1} \geq \hat{y}_{p-1}$. The welfare gain of this tax reform is given by

$$\begin{aligned}\Gamma_{\hat{y}_p,\hat{y}_{p-1}} =& \mathbb{E}_{\substack{y_{p-1}\geq\hat{y}_{p-1}\\ y_p\geq\hat{y}_p}}\left[1-g_p\right] - \frac{\tau_z}{1-\tau_z}\mathring{\eta}_p \\ &- \frac{T'\left(\hat{y}_p\right)\zeta}{1-T'\left(\hat{y}_p\right)+\hat{y}_p\zeta T''\left(\hat{y}_p\right)}\mathcal{H}_{y_{p-1},y_p}\left(\hat{y}_p\,|\hat{y}_{p-1}\right) \\ &- \frac{T'\left(\hat{y}_{p-1}\right)\zeta}{1-T'\left(\hat{y}_{p-1}\right)+\hat{y}_{p-1}\beta\zeta T''\left(\hat{y}_{p-1}\right)}\mathcal{H}_{y_{p-1},y_p}\left(\hat{y}_{p-1}\,|\hat{y}_p\right).\end{aligned} \tag{35}$$

The first three terms of expression (35) are the analogue of the period-p age-dependent perturbation of the labor income tax rates discussed in Section 6.1, the main difference being that now the region over which the individual effects of the perturbation are summed is further restricted to the households earning more than $\hat{y}_{p-1}$ in period $(p-1)$. The last term is a novel term that appears because this perturbation distorts the labor supply decisions around the $\hat{y}_{p-1}$ treshold in period $(p-1)$.

The benefits of the joint pertubation come from two sources. First, by conditioning redistribution on past income, the government can better target its redistributive effort, as summarized in the term $\mathbb{E}_{y_{p-1}\geq\hat{y}_{p-1},y_p\geq\hat{y}_p}\left[1-g_p\right]$. Conditional on a given level of earnings y_p in period p, society generally values differently the welfare of households who have a different history of labor earnings in the previous periods. History-dependence in taxation allows the government to tailor taxes to those social preferences. Second, conditioning taxes on past earnings allows the government to raise more tax revenue with less distortions.

To illustrate the latter effect, suppose that for all p, the marginal distribution of income y_p has a Pareto tail with coefficient a_p, so that for high $\hat{y}_p$ we have

$$\mathbb{P}\left(y_p \geq \hat{y}_p\right) = c_p \cdot \left(\hat{y}_p\right)^{-a_p}.$$

Furthermore, assume that joint distribution of y_{p-1} and y_p at the tails can be summarized by the (survival) Clayton copula[28]

$$\mathbb{P}\left(y_{p-1} \geq \hat{y}_{p-1}, y_p \geq \hat{y}_p\right) = \left(\left[\mathbb{P}\left(y_{p-1} \geq \hat{y}_{p-1}\right)\right]^{-\rho} + \left[\mathbb{P}\left(y_p \geq \hat{y}_p\right)\right]^{-\rho} - 1\right)^{-\rho}$$

for $\rho > 0$. The limit as $\rho \to 0$ represents the case where y_p and y_{p-1} are comonotone (in particular, perfectly correlated), that is, all the agents with a given income in period $p-1$ also earn the same income in period p. The limit as $\rho \to \infty$ represents the case where labor earnings in the two periods are drawn independently from each other. In this case the conditional hazard rates are given by

$$\mathcal{H}_{y_{p-1},y_p}\left(\hat{y}_p|\hat{y}_{p-1}\right) = \frac{a_p \left[\mathbb{P}\left(y_p \geq \hat{y}_p\right)\right]^{-1/\rho}}{\left[\mathbb{P}\left(y_p \geq \hat{y}_p\right)\right]^{-1/\rho} + \left[\mathbb{P}\left(y_{p-1} \geq \hat{y}_{p-1}\right)\right]^{-1/\rho} - 1}.$$

Suppose for simplicity that the Pareto coefficient a_p is independent of age p, that there are no savings (or that $\tau_z = 0$), and that the baseline labor income taxes are chosen to maximize tax revenue collected from the agents with sufficiently high earnings. Under these assumptions, using the analysis of Section 5, the marginal tax rates on high incomes are constant and satisfy $T'\left(y\right)/\left(1-T'\left(y\right)\right) = \left(a\zeta\right)^{-1}$. In this case the joint perturbation for sufficiently high labor incomes in both periods yields a revenue effect equal to

$$\Gamma_{\hat{y}_p,\hat{y}_{p-1}} = 1 - \left(a\zeta\right)^{-1} \frac{\left[\mathbb{P}\left(y_{p-1} \geq \hat{y}_{p-1}\right)\right]^{-1/\rho} + \left[\mathbb{P}\left(y_p \geq \hat{y}_p\right)\right]^{-1/\rho}}{\left[\mathbb{P}\left(y_{p-1} \geq \hat{y}_{p-1}\right)\right]^{-1/\rho} + \left[\mathbb{P}\left(y_p \geq \hat{y}_p\right)\right]^{-1/\rho} - 1}.$$

This expression implies that $\Gamma_{\hat{y}_p,\hat{y}_{p-1}} < 0$ for all ρ, which implies that the separable tax system is not optimal. Specifically, a joint perturbation that decreases the marginal tax rates on incomes $\hat{y}_{p-1}$ and $\hat{y}_p$, and hence reduces the average tax rates for individuals with incomes $y_{p-1} \geq \hat{y}_{p-1}$ in period $p-1$ *and* $y_p \geq \hat{y}_p$ in period p, jointly allows to raise additional revenue, starting from the optimal separable tax schedule. Note also that $\Gamma_{\hat{y}_p,\hat{y}_{p-1}} \to 0$ and as $\rho \to 0$, so that the gains from history-dependence disappear if each agent's income is the same in both periods.

The arguments above can be generalized to other forms of joint taxation. For example, the welfare effects of jointly taxing labor and capital incomes within period p at the joint income threshold $(\hat{y}_p, \hat{z}_p)$ are given by

$$\begin{aligned}\Gamma_{\hat{y}_p,\hat{z}_p} =& \mathbb{E}_{\substack{y_p \geq \hat{y}_p \\ z_p \geq \hat{z}_p}}\left[1 - g_p\right] - \frac{\tau_z}{1-\tau_z}\mathring{\eta}_p \\ &- \frac{T'\left(\hat{y}_p\right)\zeta}{1 - T'\left(\hat{y}_p\right) + \hat{y}_p\zeta T''\left(\hat{y}_p\right)}\mathcal{H}_{y_{p-1},y_p}\left(\hat{y}_p\,|\hat{z}_p\right) \\ &- \frac{\tau_z}{1-\tau_z}\left\{\sum_{s=2}^{S}\beta^{s-p}\bar{\zeta}^{c,(z_p=\hat{z}_p,y_p\geq\hat{y}_p)}_{z_s,r_p}\right\}\mathcal{H}_{y_{p-1},y_p}\left(\hat{z}_p\,|\hat{y}_p\right).\end{aligned} \tag{36}$$

[28]This joint distribution is a generalization of the bivariate Pareto distribution, obtained for $a_p = a_{p-1} = \rho$.

Formula (36) is formally similar to equation (35), the relevant conditional hazard rates in this case being those of the joint distribution of labor and capital incomes, and the relevant elasticities being those of capital income with the usual compounding effect discussed in Sections 5.2 and 6.2. Note that these elasticities are in general different for individuals with different labor and capital incomes, and are therefore averaged over the region where the capital income tax rate is perturbed. More generally, for different preferences or a non-separable baseline income tax system, the elasticity parameters $\bar{\zeta}^c_{y_s,w_p}, \bar{\zeta}^c_{z_s,r_p}, \eta_{z_s,R_p}$ would all depend on the individuals' earnings histories, implying an additional source of benefits from using a non-separable tax system: the government can impose higher distortions in the regions where these elasticities are smaller, with an additional degree of fine-tuning relative to the separable case.

To conclude this section, note that we can easily estimate empirically the welfare gains of joint tax reforms using the formulas (35) and (36). Using an approach based on copulas to estimate the joint income distributions in the data (see, e.g., Bonhomme and Robin 2003, and Dearden, Fitzsimons, Goodman and Kaplan 2006), we can compute the conditional hazard rates and plug them into the formulas along with the elasticities and income effect parameters estimated at the current tax system. The size of the welfare gains from tax reforms will depend on the points $(\hat{y}_{p-1}, \hat{y}_p)$ or $(\hat{y}_p, \hat{z}_p)$ at which they are evaluated.

7 Conclusion

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A Proofs of Sections 3 and 4

In this section, we provide the proofs of the results of Sections 3 and 4 from the main text. We first derive analytical expressions for all the elasticities and income effect parameters (7,6) in the general model, as well as under the assumptions of Section 6. We then prove the results of Propositions 1, 2 and 3, and provide details for the derivation of various results in the text.

A.1 Elasticities and Income Effect Parameters

We start by providing analytical expressions for the elasticities and income effect parameters in the general model of Sections 4 to 2. To derive these expressions, we differentiate the system of first-order conditions (4) of the individual's problem with respect to the marginal tax rates $\left\{\hat{\tau}_{s,x_j}\right\}_{1\leq s,j\leq S}$ and the virtual incomes $\{R_s\}_{1\leq s\leq S}$. We can write these first-order conditions as

$$\begin{aligned} &U_{x_j}\left(\left\{-\sum_{i=1}^{2S}\tilde{\tau}_{t,x_i}x_{\boldsymbol{\theta},i}+R_t\right\}_{1\leq t\leq S},\{z_{\boldsymbol{\theta},t}\}_{1\leq t\leq S},\{r_{t+1}k_{\boldsymbol{\theta},t}\}_{1\leq t\leq S},\boldsymbol{\theta}\right) \\ &=\sum_{s=1}^{S}\tilde{\tau}_{s,x_j}U_{c_s}\left(\left\{-\sum_{i=1}^{2S}\tilde{\tau}_{t,x_i}x_{\boldsymbol{\theta},i}+R_t\right\}_{1\leq t\leq S},\{z_{\boldsymbol{\theta},t}\}_{1\leq t\leq S},\{r_{t+1}k_{\boldsymbol{\theta},t}\}_{1\leq t\leq S},\boldsymbol{\theta}\right), \end{aligned} \tag{37}$$

where $\tilde{\tau}_{s,z_s}=-\left(1-\tau_{s,z_s}\right)$ and $\tilde{\tau}_{s,z_t}=\tau_{s,z_t}$ if $t\neq s$, and where $\tilde{\tau}_{s,r_sk_{s-1}}=-\left(r_s^{-1}+1-\tau_{s,r_sk_{s-1}}\right)$, $\tilde{\tau}_{s,r_{s+1}k_s}=r_{s+1}^{-1}+\tau_{s,r_{s+1}k_s}$, and $\tilde{\tau}_{s,r_{t+1}k_t}=\tau_{s,r_{t+1}k_t}$ if $t\neq s-1,s$. We then use the Slutsky equations to obtain the compensated elasticities from the uncompensated elasticities and the income effect parameters. Define the $2S\times 2S$ matrix A by:

$$[A]_{s,j}\equiv -U_{x_s,x_j}-\sum_{t=1}^{S}\sum_{q=1}^{S}\tilde{\tau}_{t,x_s}\tilde{\tau}_{q,x_j}U_{c_t,c_q}+\sum_{t=1}^{S}\tilde{\tau}_{t,x_s}U_{c_t,x_j}+\sum_{q=1}^{S}\tilde{\tau}_{q,x_j}U_{x_s,c_q}.$$

Define also the $2S$-vectors $B_{\tau_{p,x_i}}$, B_{R_p}, and $B^c_{\tau_{p,x_i}}$, for any $1\leq p\leq S$ and $1\leq i\leq 2S$, by

$$\begin{aligned} \left[B^u_{\tau_{p,x_i}}\right]_s &\equiv \sum_{t=1}^{S}\tilde{\tau}_{t,x_s}U_{c_t,c_p}x_i-U_{x_s,c_p}x_i-U_{c_p}\mathbf{1}_{\{i=s\}},\quad \forall s\in\{1,\ldots,2S\},\\ \left[B_{R_p}\right]_s &= -\sum_{t=1}^{S}\tilde{\tau}_{t,x_s}U_{c_t,c_p}+U_{x_s,c_p},\quad \forall s\in\{1,\ldots,2S\},\\ \left[B^c_{\tau_{p,x_i}}\right]_s &\equiv -U_{c_p}\mathbf{1}_{\{i=s\}},\quad \forall s\in\{1,\ldots,2S\}. \end{aligned}$$

Letting $\hat{\tau}_{s,x_j}$ denote the period-s marginal or net-of-tax rate in the direction of income x_j, we can then write the the uncompensated and compensated income elasticities and the income effect

parameters as

$$\zeta^{u,c}_{x_j,\hat{\tau}_{s,x_t}} = \pm \frac{\hat{\tau}_{s,x_t}}{x_j} \left[A^{-1} B^{u,c}_{\tau_{s,x_t}} \right]_j, \quad \text{and} \quad \eta_{x_j,R_s} = \hat{\tau}_{s,x_j} \left[A^{-1} \times B_{R_s} \right]_j, \tag{38}$$

where $\pm = +$ if $x_t \in \{z_s, r_s k_{s-1}\}$, and $\pm = -$ otherwise. Note that the components of the elasticity matrices and the income effect vectors (8) are the *partial derivatives* of the (compensated or uncompensated) demands, and not directly the *elasticities* and income effect parameters we just derived.

We now show how these expressions simplify under the assumptions of Section 6. That is, we assume that the utility function is time-separable, has no income effects on labor supply, and that the baseline tax system is separable and linear in capital income. The only non-zero labor income elasticities are:

$$\zeta^{c}_{z_s,1-\tau_{s,z_s}} = \frac{v'(z_s/\theta_s)}{(z_s/\theta_s)\, v''(z_s/\theta_s)}.$$

Suppose either that the utility function is CRRA, i.e. $u(x) = x^{1-\sigma}/(1-\sigma)$, and let $\alpha \equiv \beta^{-1/\sigma} R^{1-1/\sigma}$, or that the utility function is CARA, i.e. $u(x) = -\gamma^{-1} \exp(-\gamma x)$, and let $\alpha = R$. We obtain that the only non-zero compensated capital income elasticities are given by

$$\frac{\partial \left(r_s k^c_{s-1}\right)}{\partial \left(1 - \tau_{p,r_p k_{p-1}}\right)} = \left(\frac{-u'_p}{u''_p}\right) \frac{r^2 R^{s-p-2}}{\sum_{i=0}^{S-1} \alpha^i} \begin{cases} \left(\sum_{i=0}^{S-p} \alpha^i\right) \left(\sum_{i=p-s+1}^{p-1} \alpha^i\right), & \text{if } s \leq p, \\ \left(\sum_{i=0}^{S-s} \alpha^i\right) \left(\sum_{i=1}^{p-1} \alpha^i\right), & \text{if } s \geq p+1, \end{cases}$$

and the only non-zero income effect parameters are given by

$$\frac{\partial \left(r_s k_{s-1}\right)}{\partial R_p} = \frac{r R^{s-p-1}}{\sum_{i=0}^{S-1} \alpha^i} \begin{cases} -\left(\sum_{i=S-s+1}^{S-1} \alpha^i\right), & \text{if } s \leq p, \\ \left(\sum_{i=0}^{S-s} \alpha^i\right), & \text{if } s \geq p+1. \end{cases}$$

Note that in the CARA case, $\frac{-u'_p}{u''_p}$ is simply equal to γ^{-1}.

A.2 Proofs of Propositions 1 to 3

We first prove the existence of the Gateaux differential of the income functional and show Proposition 1.

Proof of Proposition 1. We first show that the income functional $\mathbf{x}_{\boldsymbol{\theta}}(\cdot)$ is Gateaux differentiable around the initial tax system T_p. Denote by $\mathbf{x}_{\boldsymbol{\theta}} \equiv \mathbf{x}_{\boldsymbol{\theta}}(T_p)$, resp. $\tilde{\mathbf{x}}_{\boldsymbol{\theta}} \equiv \mathbf{x}_{\boldsymbol{\theta}}(T_p + \mu h)$, the income vector chosen by an individual $\boldsymbol{\theta}$ given the baseline tax system T_p, resp. the perturbed tax system in the direction h, $T_p + \mu h$. The vectors $\mathbf{x}_{\boldsymbol{\theta}}$ and $\tilde{\mathbf{x}}_{\boldsymbol{\theta}}$ are the solution to the respective systems of the first-order conditions (4), where the map $F : \mathbb{R}^{2S} \times \mathbb{R}^{2S} \times \mathbb{R} \to \mathbb{R}^{2S}$ is continuously differentiable. Assumption 2 implies that $\|\tilde{\mathbf{x}}_{\boldsymbol{\theta}} - \mathbf{x}_{\boldsymbol{\theta}}\| = O(\mu)$ as $\mu \to 0$. Moreover, we have $\|\mu h(\tilde{\mathbf{x}}_{\boldsymbol{\theta}}) - \mu h(\mathbf{x}_{\boldsymbol{\theta}})\| = o(\mu)$ as $\mu \to 0$. Let F_j denote the j^{th} component of F, and define the

matrix $A = (a_{j,s})_{1\leq j,s\leq 2S}$ as

$$a_{j,s} = \frac{\partial F_j}{\partial x_{\boldsymbol{\theta},s}} + \sum_{t=1}^{2S} \frac{\partial F_j}{\partial \tau_{p,x_t}} \frac{\partial \tau_{p,x_t}(\mathbf{x}_{\boldsymbol{\theta}})}{\partial x_{\boldsymbol{\theta},s}} + \frac{\partial F_j}{\partial T_p} \frac{\partial T_p(\mathbf{x}_{\boldsymbol{\theta}})}{\partial x_{\boldsymbol{\theta},s}},$$

the vectors $B_{x_t} = (b_{j,x_t})_{1\leq j\leq 2S}$ for all $t \in \{1,\ldots,2S\}$ as $b_{j,x_t} = \frac{\partial F_j}{\partial \tau_{p,x_t}}$, and the vector $B_T = (b_{j,T})_{1\leq j\leq 2S}$ as $b_{j,T} = \frac{\partial F_j}{\partial T_p}$. A first-order Taylor expansion of the perturbed system around the initial tax system as $\mu \to 0$ thus writes:

$$\begin{aligned}
0 = & F_j\left(\tilde{\mathbf{x}}_{\boldsymbol{\theta}}, \left\{\tau_{p,x_t}(\tilde{\mathbf{x}}_{\boldsymbol{\theta}}) + \mu \frac{\partial h}{\partial x_t}(\tilde{\mathbf{x}}_{\boldsymbol{\theta}})\right\}_{1\leq t\leq 2S}, T_p(\tilde{\mathbf{x}}_{\boldsymbol{\theta}}) + \mu h(\tilde{\mathbf{x}}_{\boldsymbol{\theta}})\right) \\
& - F_j\left(\mathbf{x}_{\boldsymbol{\theta}}, \{\tau_{p,x_t}(\mathbf{x}_{\boldsymbol{\theta}})\}_{1\leq t\leq 2S}, T_p(\mathbf{x}_{\boldsymbol{\theta}})\right), \quad \forall j = 1,\ldots,2S \\
\Rightarrow & \frac{1}{\mu}(\tilde{\mathbf{x}}_{\boldsymbol{\theta}} - \mathbf{x}_{\boldsymbol{\theta}}) = -\sum_{t=1}^{2S} \left\{A^{-1}B_{x_t}\right\} \frac{\partial h(\mathbf{x}_{\boldsymbol{\theta}})}{\partial x_t} - \left\{A^{-1}B_T\right\} h(\mathbf{x}_{\boldsymbol{\theta}}) + o_{\mu\to 0}(1),
\end{aligned}$$

which shows the existence of the Gateaux differential $\delta\mathbf{x}_{\boldsymbol{\theta}}(T_p, h) \in \mathbb{R}^{2S}$ of the income functional $\mathbf{x}_{\boldsymbol{\theta}}(\cdot)$ at T_p with increment h. We then express the Gateaux differential of the income functional as a function of the elasticity matrices and vectors of income effect parameters. To do so, we derive the change $\tilde{\mathbf{x}}_{\boldsymbol{\theta}} - \mathbf{x}_{\boldsymbol{\theta}}$ in the individual's choice vector by writing the first-order Taylor approximation of the post-perturbation system of first order conditions (37) around the solution $\mathbf{x}_{\boldsymbol{\theta}}$ to the initial system. Using the explicit expressions for the elasticities and income effect parameters derived in (38), we obtain

$$(\tilde{\mathbf{x}}_{\boldsymbol{\theta}} - \mathbf{x}_{\boldsymbol{\theta}}) = \left[\mathrm{i}_{2S} - \sum_{s=1}^{S} \boldsymbol{E}^{c,(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},\tau_s}\left(D^2 T_s(\mathbf{x}_{\boldsymbol{\theta}})\right)\right]^{-1} \left\{\boldsymbol{E}^{c,(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},\tau_p} \nabla h(\mathbf{x}_{\boldsymbol{\theta}}) + \boldsymbol{I}^{(\mathbf{x}_{\boldsymbol{\theta}})}_{\mathbf{x},R_p} h(\mathbf{x}_{\boldsymbol{\theta}})\right\}.$$

This concludes the proof of Proposition 1. □

We then show Proposition 2, which gives expressions for the Gateaux differentials of the tax revenue and social welfare functionals.

Proof of Proposition 2. Consider an admissible perturbation h of the baseline tax function T_p, so that the perturbed tax function is $T_p + \mu h$. We then have, for any $\boldsymbol{\theta}$, letting $\mathbf{x}_{\boldsymbol{\theta}} \equiv \mathbf{x}_{\boldsymbol{\theta}}(T_p)$ and $\tilde{\mathbf{x}}_{\boldsymbol{\theta}} \equiv \mathbf{x}_{\boldsymbol{\theta}}(T_p + \mu h)$,

$$\begin{aligned}
& [T_p + \mu h](\tilde{\mathbf{x}}_{\boldsymbol{\theta}}) - T_p(\mathbf{x}_{\boldsymbol{\theta}}) \\
= & \langle \nabla T_p(\mathbf{x}_{\boldsymbol{\theta}}), \delta\mathbf{x}_{\boldsymbol{\theta}}(T_p, h)\rangle \mu + h(\mathbf{x}_{\boldsymbol{\theta}}(T_p))\mu + o(\mu),
\end{aligned}$$

and similarly, using the envelope theorem and the local Lipschitz continuity of the income

function (Assumption 2),

$$\mathcal{G}\left(\mathscr{U}_{\boldsymbol{\theta}}\left(T_p+\mu h\right)\right)-\mathcal{G}\left(\mathscr{U}_{\boldsymbol{\theta}}\left(T_p\right)\right)=\left(-\lambda\delta^{p-1}g_p\left(\mathbf{x}_{\boldsymbol{\theta}}\right)h\left(\mathbf{x}_{\boldsymbol{\theta}}\right)\right)\mu+o\left(\mu\right).$$

Using Assumption **????**, we obtain that the change in social welfare is equal to

$$\begin{aligned}&\mathscr{W}\left(T_p+\mu h\right)-\mathscr{W}\left(T_p\right)\\=&\mu\delta^{p-1}\int_{\mathbb{R}_+^S\times\mathbb{R}^S}\left\{\left(1-g_p\left(\mathbf{x}\right)\right)h\left(\mathbf{x}\right)+\left\langle\nabla T_p\left(\mathbf{x}\right),\delta\mathbf{x}\left(T_p,h\right)\right\rangle\right\}f_{\mathbf{x}}\left(\mathbf{x}\right)d\mathbf{x}+o\left(\mu\right).\end{aligned}$$

This proves formula (11). Letting $\tilde{\boldsymbol{T}}'\left(\mathbf{x}\right)\equiv\boldsymbol{T}'\left(\mathbf{x}\right)\boldsymbol{D}^{-1}\left(\mathbf{x}\right)$ and using the fact that the density of incomes is equal to zero on the boundary ∂X of the set X, we can integrate by parts the integral involving $\nabla h\left(\mathbf{x}\right)$ in this expression to get

$$\int_{\mathrm{X}}\left[\tilde{\boldsymbol{T}}'\left(\mathbf{x}\right)\boldsymbol{E}_{\mathbf{x},\tau_p}^{c,(\mathbf{x})}f_{\mathbf{x}}\left(\mathbf{x}\right)\right]\nabla h\left(\mathbf{x}\right)d\mathbf{x}=-\int_{\mathrm{X}}\nabla\cdot\left[\tilde{\boldsymbol{T}}'\left(\mathbf{x}\right)\boldsymbol{E}_{\mathbf{x},\tau_p}^{c,(\mathbf{x})}f_{\mathbf{x}}\left(\mathbf{x}\right)\right]h\left(\mathbf{x}\right)d\mathbf{x}.$$

This proves formula (12). □

Next we prove Proposition 3.

Proof of Proposition 3. A necessary condition for the social welfare functional $\mathscr{W}\left(\cdot\right)$ to have an extremum at T_p is $\delta\mathscr{W}\left(T_p,h\right)=0$, for all h (see, e.g., Luenberger 1969). From equation (12), this implies that the integrand must be equal to zero pointwise, that is for all $\mathbf{x}\in\mathrm{X}$,

$$\left(1-g_p\left(\mathbf{x}\right)-\boldsymbol{T}'\left(\mathbf{x}\right)\boldsymbol{D}^{-1}\left(\mathbf{x}\right)\boldsymbol{I}_{\mathbf{x},R_p}^{(\mathbf{x})}\right)f_{\mathbf{x}}\left(\mathbf{x}\right)-\nabla\cdot\left(\boldsymbol{T}'\left(\mathbf{x}\right)\boldsymbol{D}^{-1}\left(\mathbf{x}\right)\boldsymbol{E}_{\mathbf{x},\tau_p}^{c,(\mathbf{x})}f_{\mathbf{x}}\left(\mathbf{x}\right)\right)=0.$$

Integrating this equation on the volume V with closed boundary $\mathrm{S}=\partial\mathrm{V}$ and ysing the divergence theorem, we obtain formula (16). Finally, we obtain formula (15) by using the separable linear perturbations $h_p\left(\mathbf{x}\right)=x_p$ and equation (11). □

We finally prove the formulas (18) and (19), which express the optimal tax system as a function of the distribution of types $\boldsymbol{\theta}$.

Proof of formulas (18) and (19). Differentiating the i^{th} first-order condition (4) with respect to θ_j for $j\in\{1,\ldots,2S\}$ yields

$$\sum_{s=1}^{2S}a_{i,s}\frac{\partial x_{\boldsymbol{\theta},s}}{\partial\theta_j}=-\frac{\partial F_i}{\partial\theta_j}\Rightarrow J_{\mathbf{x}}\left(\boldsymbol{\theta}\right)=-A^{-1}J_{\mathbf{F}}\left(\boldsymbol{\theta}\right),$$

where the matrix A is the same as in the proof of Proposition 1, and $J_{\mathbf{x}}\left(\boldsymbol{\theta}\right),J_{\mathbf{F}}\left(\boldsymbol{\theta}\right)$ are the matrices $\left[\partial x_{\boldsymbol{\theta},i}/\partial\theta_j\right]_{1\leq i,j\leq 2S}$ and $\left[\partial F_i/\partial\theta_j\right]_{1\leq i,j\leq 2S}$ respectively. Similarly, differentiating the

first-order conditions (4) with respect to the variables $\left\{\tau_{p,x_j}\right\}_{1\leq j\leq 2S}$ and T_p yields:

$$J_{\mathbf{x}}\left(\boldsymbol{\tau}_p\right) = -A^{-1}J_{\mathbf{F}}\left(\boldsymbol{\tau}_p\right), \text{ and } J_{\mathbf{x}}\left(T_p\right) = -A^{-1}J_{\mathbf{F}}\left(T_p\right),$$

where $J_{\mathbf{x}}\left(\boldsymbol{\tau}_p\right), J_{\mathbf{F}}\left(\boldsymbol{\tau}_p\right)$ are the matrices $\left[\partial x_{\boldsymbol{\theta},i}/\partial\tau_{p,x_j}\right]_{1\leq i,j\leq 2S}$ and $\left[\partial F_i/\partial\tau_{p,x_j}\right]_{1\leq i,j\leq 2S}$ respectively, and $J_{\mathbf{F}}\left(T_p\right)$ is the vector $\left[\partial F_i/\partial T_p\right]_{1\leq i\leq 2S}$. But from Proposition , we have $J_{\mathbf{x}}\left(\boldsymbol{\tau}_p\right) = \boldsymbol{D}^{-1}\left(\mathbf{x}\right)\boldsymbol{E}^{c,(\mathbf{x})}_{\mathbf{x},\tau_p}$ and $J_{\mathbf{x}}\left(T_p\right) = \boldsymbol{D}^{-1}\left(\mathbf{x}\right)\boldsymbol{I}^{(\mathbf{x})}_{\mathbf{x},R_p}$. We use these expressions to express the deformation matrix $\boldsymbol{D}\left(\mathbf{x}\right)$ as a function of the Jacobian matrix $J_{\mathbf{x}}\left(\boldsymbol{\theta}\right)$, and $J_{\mathbf{F}}\left(\boldsymbol{\theta}\right), J_{\mathbf{F}}\left(\boldsymbol{\tau}_p\right), J_{\mathbf{F}}\left(T_p\right)$. Using the change of variables formula $f_{\boldsymbol{\theta}}\left(\boldsymbol{\theta}\right) = \det\left(J_{\mathbf{x}}\left(\boldsymbol{\theta}\right)\right) f_{\mathbf{x}}\left(\mathbf{x}\left(\boldsymbol{\theta}\right)\right)$ in equation (17) and the chain rule, we obtain (18).

Now, in the model with idiosyncratic productivities $\{\theta_1,\ldots,\theta_S\}$ and interest rates $\{\theta_{S+1},\ldots,\theta_{2S-1}\}$, let $\boldsymbol{l}_{\boldsymbol{\theta}}$ denote the vector of labor supplies $z_{\boldsymbol{\theta},s}/\theta_s$ and capital stocks $k_{\boldsymbol{\theta},s}$. We can write the first-order conditions of the individual problem as

$$\mathbf{x}_{\boldsymbol{\theta}} = \boldsymbol{\theta}\circ\boldsymbol{l}_{\boldsymbol{\theta}}\left(\left\{\theta_j\hat{\tau}_{s,z_j}\right\}_{\substack{1\leq j\leq 2S\\1\leq s\leq S}}, \left\{\theta_{S+j-1}\hat{\tau}_{s,r_jk_{j-1}}\right\}_{\substack{2\leq j\leq S\\1\leq s\leq S}}, \left\{R_s\right\}_{1\leq s\leq S}\right),$$

where $\hat{\tau}_{s,x_j}$ is the marginal tax rate on income x_j (if $x_j\in\{z_s, r_sk_{s-1}\}$) or the next-of-tax rate otherwise, and $\circ$ is the element-wise multiplication. Differentiating this system of equations with respect to θ_j for $1\leq j\leq 2S$ yields the Jacobian matrix

$$J_{\mathbf{x}}\left(\boldsymbol{\theta}\right) = \boldsymbol{D}^{-1}\left(\mathbf{x}\right)\left[\frac{\mathbf{x}}{\boldsymbol{\theta}}\circ\left(\mathbf{i}_{2S} + \sum_{s=1}^{S}\zeta^{u,(\mathbf{x})}_{\mathbf{x},\hat{\tau}_s}\right)\right].$$

Changing variables as before yields (19). □

B Proofs of Sections 5 and 6

B.1 Proofs of Section 5

Proofs of Section **??**. Formula (21) follows from using the Slutsky equation and rearranging the terms in equation (21). Formulas (25) and (**??**) follow from (17) applied to the regions $[\bar{z},\infty)$ and $[\bar{z}_2,\infty)\times\left[r\bar{k}_2,\infty\right)$, respectively. □

We now characterize the optimum linear capital income tax schedule.

Proof of Propositions 4 and 5. Assume that the capital income distribution has mean $\overline{rk}$ in every period. Consider a revenue-neutral separable affine perturbation h of the capital income tax rate in every period $p\geq 2$. This perturbation is the sum of the linear perturbations $h_{p,1}\left(\mathbf{x}\right) = r_pk_{p-1}d\tau$, for some $d\tau>0$, and the uniform lump-sum rebate $h_{p,2}\left(\mathbf{x}\right) = dR$, where

the revenue-neutrality of h implies

$$dR = -\frac{1-\frac{\tau_{rk}}{1-\tau_{rk}}\left\{\sum_{p=2}^{S}\sum_{s=2}^{S}\frac{\delta^{s-1}}{\sum_{p=2}^{S}\delta^{p-1}}\bar{\zeta}^{u,\left(\mathbf{x}\in\mathbb{R}^{2S-1}\right)}_{r_s k_{s-1},1-\tau_{p,r_p k_{p-1}}}\right\}}{1-\frac{\tau_{rk}}{1-\tau_{rk}}\left\{\sum_{p=2}^{S}\sum_{s=2}^{S}\frac{\delta^{s-1}}{\sum_{p=2}^{S}\delta^{p-1}}\bar{\eta}^{(\mathbf{x}\in\mathbb{R}^{2S-1})}_{r_s k_{s-1},R_p}\right\}}\overline{rk}d\tau.$$

Applying Proposition 2, we obtain that the welfare effect of this perturbation is given by

$$\delta\mathscr{W}(T,h) \propto \left\{1-\frac{\sum_{p=2}^{S}\delta^{p-1}\bar{g}^{(r_p k_{p-1}\in\mathbb{R})}_{p,\mathtt{L}_p}}{\sum_{p=2}^{S}\delta^{p-1}\bar{g}^{(r_p k_{p-1}\in\mathbb{R}),\mathtt{n}}_{p,\mathtt{NL}}}\right\}-\frac{\tau_{rk}}{1-\tau_{rk}}\left\{\sum_{p=2}^{S}\sum_{s=2}^{S}\frac{\delta^{s-1}}{\sum_{p=2}^{S}\delta^{p-1}}\bar{\zeta}^{u,\left(\mathbf{x}\in\mathbb{R}^{2S-1}\right)}_{r_s k_{s-1},1-\tau_{p,r_p k_{p-1}}}\right\}.$$

Equating this expression to zero leads the optimal capital income tax rate (23).

Now consider the case where the perturbation is implemented in every period $p = p_1, \ldots, p_2$. Under the assumptions of Proposition **5**, the behavioral effect is then given by

$$\begin{aligned}\mathscr{U} \equiv \sum_{p=p_1}^{p_2}\sum_{s=2}^{\infty}\delta^{s-1}\bar{\zeta}^{u,\left(\mathbf{x}\in\mathbb{R}^{2S-1}\right)}_{r_s k_{s-1},1-\tau_{p,r_p k_{p-1}}} &= \left[\sigma^{-1}\delta+\left(\sigma^{-1}-1\right)(1-\delta)(p_1-1)\right]\frac{\delta^{p_1-1}}{1-\delta}\\ &\quad-\left[\sigma^{-1}\delta+\left(\sigma^{-1}-1\right)(1-\delta)p_2\right]\frac{\delta^{p_2}}{1-\delta}.\end{aligned}$$

For $p_2 = \infty$, we have $\delta^{-(p_1-1)}\mathscr{U} \to \infty$ (resp., $\delta^{-(p_1-1)}\mathscr{U} \to -\infty$) as $p_1 \to \infty$ if $\sigma < 1$ (resp., $\sigma > 1$). For $p_1 = 2$, we obtain that $\mathscr{U}$ is decreasing with the horizon p_2 if $\sigma > 1$ and $p_2 > 1 + \frac{\sigma}{\sigma-1}\frac{\delta}{1-\delta}$. □

We finally derive the optimal asymptotic capital income tax rate in a non-linear tax system.

Proof of Proposition **??**. Consider a sequence, indexed by $\bar{x} > 0$, of separable perturbations h of the capital income tax rate in every period $p \geq 2$, that are linear above the threshold $\bar{x}$. That is, for all p we define $h_p(\mathbf{x}) = \max\{r_p k_{p-1} - \bar{x}, 0\}\, d\tau$. Here we let the utility function have income effects on labor supply, so that the formula we derive is more general than (**??**). Assume that the labor income tax schedule is age-independent and linear, and denote by τ_z the marginal tax rate, which we assume constant for simplicity. The Gateaux differential of social welfare writes:

$$\begin{aligned}\frac{\delta\mathscr{W}\left(T,\{h_p\}_{p\geq 2}\right)}{d\tau} &= \sum_{p=2}^{S}\delta^{p-1}\left\{\int_{r\bar{k}}^{\infty}\left(1-g_p(\mathbf{x})\right)\left(r_p k_{p-1}-\bar{x}\right)f_{\mathbf{x}}(\mathbf{x})\,d\mathbf{x}\right\}+\ldots\\
&+\sum_{p=2}^{S}\left\{\int_{\bar{x}}^{\infty}\int_{\mathbb{R}_+^S\times\mathbb{R}^{S-2}}\left[\boldsymbol{T}'_{rk}(\mathbf{x})\right]\left[\mathrm{i}_{S-1}+\boldsymbol{F}_{rk}(\mathbf{x})\right]^{-1}\left[\boldsymbol{E}^{c,(\mathbf{x})}_{\mathbf{x},\tau_{p,r_p k_{p-1}}}-\left(r_p k_{p-1}-\bar{x}\right)\boldsymbol{I}^{(\mathbf{x})}_{\mathbf{x},R_p}\right]f_{\mathbf{x}}(\mathbf{x})\,d\mathbf{x}\right\}\\
&+\sum_{p=2}^{S}\left\{\int_{\bar{x}}^{\infty}\int_{\mathbb{R}_+^S\times\mathbb{R}^{S-2}}\left[\tau_z\sum_{s=2}^{S}\delta^{s-1}\left(-\frac{z_s\zeta^{c,(\mathbf{x})}_{z_s,1-\tau_{p,r_p k_{p-1}}}}{1-\tau_{p,r_p k_{p-1}}}-\frac{\eta^{(\mathbf{x})}_{z_s,R_p}}{1-\tau_{p,z_s}}\left(r_p k_{p-1}-\bar{x}\right)\right)\right]f_{\mathbf{x}}(\mathbf{x})\,d\mathbf{x}\right\},\end{aligned}$$

where $[\boldsymbol{T}'_{rk}(\mathbf{x})]$ is the $(S-1)$-row vector with components $\delta^{s-1}T'_{s,r_sk_{s-1}}(r_sk_{s-1})$ and $[\boldsymbol{F}_{rk}(\mathbf{x})]$ is the $(S-1)\times(S-1)$-matrix with components $T''_{r_jk_{j-1}}(r_jk_{j-1})\frac{\partial(r_ik_{i-1})^c}{\partial\tau_{j,r_jk_{j-1}}}$ for $i,j\geq 2$. We assume that $T''_{rk}(\cdot)$ converges to zero fast enough, i.e., for all $p\geq 2$,

$$\sup_{\{\mathbf{x}:r_pk_{p-1}\geq\bar{x}\}}\left|r_sk_{s-1}\zeta^{c,(\mathbf{x})}_{r_sk_{s-1},1-\tau_{p,r_pk_{p-1}}}T''_{rk}(r_pk_{p-1})\right|\xrightarrow[\bar{x}\to\infty]{}0.$$

The assumption of no mobility at the top ensures that as $r_pk_{p-1}\to\infty$ for any p, all the components of the matrix $\boldsymbol{F}_{rk}(\mathbf{x})$ converge to zero, and that all the marginal tax rates $T'_{s,r_sk_{s-1}}(r_sk_{s-1})$ converge to τ^{∞}_{rk}. Thus, letting $\bar{x}\to\infty$ and imposing $\lim\delta\mathscr{W}\left(T,\{h_p\}_{p\geq 2}\right)/\left(1-F(\bar{x})\right)\bar{x}d\tau=0$, we obtain the following characterization of the optimal asymptotic capital income tax rate:

$$\begin{aligned}\frac{\tau^{\infty}_{rk}}{1-\tau^{\infty}_{rk}}=&\frac{\left(\frac{a_{rk}}{a_{rk}-1}-1\right)\left(1-\sum_{p=2}^{S}\tilde{\delta}^{p-1}\bar{g}^{(\infty)}_{p,\mathtt{NL}}\right)}{\frac{a_{rk}}{a_{rk}-1}\sum_{p,s=2}^{S}\tilde{\delta}^{p-1}\bar{\zeta}^{u,(\infty)}_{r_sk_{s-1},1-\tau_{p,r_pk_{p-1}}}-\sum_{p,s=2}^{S}\tilde{\delta}^{p-1}\bar{\eta}^{(\infty)}_{r_sk_{s-1},R_p}}\\&-\frac{\tau_z}{1-\tau^{\infty}_{rk}}\frac{\frac{a_{rk}}{a_{rk}-1}\sum_{p,s=2}^{S}\tilde{\delta}^{p-1}\bar{\zeta}^{u,(\infty)}_{z_s,1-\tau_{p,r_pk_{p-1}}}-\sum_{p,s=2}^{S}\tilde{\delta}^{p-1}\bar{\eta}^{(\infty)}_{z_s,R_p}}{\frac{a_{rk}}{a_{rk}-1}\sum_{p,s=2}^{S}\tilde{\delta}^{p-1}\bar{\zeta}^{u,(\infty)}_{r_sk_{s-1},1-\tau_{p,r_pk_{p-1}}}-\sum_{p,s=2}^{S}\tilde{\delta}^{p-1}\bar{\eta}^{(\infty)}_{r_sk_{s-1},R_p}}.\end{aligned}$$

In the case where the utility function has no income effects on labor supply, the second line of this expression is equal to zero and we obtain (**??**). □

B.2 Proofs of Section 6

We now provide the proofs of the results of Sections 6.2, 6.3, and 6.1. We examine in turn the welfare effects of reforming the labor income tax schedule, the capital income tax schedule, and of introducing joint taxation into the baseline tax system. We assume that the utility function has no income effects on labor supply and the baseline tax system is separable, age-independent, and linear in capital income.

Proof of Proposition **??**. Fix a labor income level $\bar{z}_p$, and consider a sequence, indexed by the positive numbers $(d\tau,d\bar{z})$, of separable non-linear perturbations h_p of the period-p tax function in the direction z_p, defined by $h_p(\mathbf{x})=(z_p-\bar{z}_p)\,d\tau$ for $z_p\in[\bar{z}_p,\bar{z}_p+d\bar{z}]$ and $h_p(\mathbf{x})=d\bar{z}d\tau$ for $z_p\geq\bar{z}_p+d\bar{z}$. (More precisely, we consider a smooth approximation of these perturbations, so that $h_p(\mathbf{x})=(z_p-\bar{z}_p)\,d\tau$ for $z_p\in[\bar{z}_p+u,\bar{z}_p+d\bar{z}-u]$ with $u>0$.) We then let $d\tau,d\bar{z}\to 0$ with $u=o(d\bar{z})$. Applying formula (9), we obtain that the effect of these perturbations on the individual income choices is given by

$$\begin{aligned}\delta z^{(\bar{z}_p,\mathbf{x}_{-p})}_{\boldsymbol{\theta},p}(T_p,h_p)=&-\frac{z_p\zeta^{c,(z_p,\mathbf{x}_{-p})}_{z_p,1-\tau_{p,z_p}}}{1-T'_z(z_p)+z_p\zeta^{c,(z_p,\mathbf{x}_{-p})}_{z_p,1-\tau_{p,z_p}}T''_z(z_p)}d\tau,\ \text{for all } z_p\in[\bar{z}_p,\bar{z}_p+d\bar{z}],\\\delta r_sk^{(\mathbf{x})}_{\boldsymbol{\theta},s-1}(T_p,h_p)=&-\frac{\eta^{(\mathbf{x})}_{r_sk_{s-1},R_p}}{1-\tau_{p,r_sk_{s-1}}}d\bar{z}d\tau,\ \text{for all } z_p\geq\bar{z}_p+d\bar{z}.\end{aligned}$$

Applying formula (11) and taking the limit of the Gateaux differentials of social welfare as $d\bar{z}, d\tau \to 0$, we get

$$\begin{aligned}\frac{\delta \mathscr{W}\left(T_p, h_p\right)}{d\bar{z} d\tau} \xrightarrow[d\bar{z}, d\tau \to 0]{} \quad & \delta^{p-1} \int_{\bar{z}_p}^{\infty} \left(1 - g_p\left(z\right)\right) f_{z_p}\left(z\right) dz \\ & + \delta^{p-1} \int_{\mathbb{R}_+^{S-1} \times \mathbb{R}^{S-1}} T_z'\left(\bar{z}_p\right) \left[\delta z_{\boldsymbol{\theta}, p}^{(\bar{z}_p, \mathbf{x}_{-p})}\left(T_p, h_p\right)\right] f_{\mathbf{x}}\left(\bar{z}_p, \mathbf{x}_{-p}\right) d\mathbf{x}_{-p} \\ & + \sum_{s=2}^{S} \delta^{s-1} \int_{\bar{z}_p}^{\infty} \int_{\mathbb{R}_+^{S-1} \times \mathbb{R}^{S-1}} \tau_{rk} \left[\delta r_s k_{\boldsymbol{\theta}, s-1}^{(\mathbf{x})}\left(T_p, h_p\right)\right] f_{\mathbf{x}}\left(\mathbf{x}\right) d\mathbf{x}.\end{aligned}$$

Using the aggregate sufficient statistics defined in Section 5, we obtain formula (**??**). Assume for simplicity that the utility function is CRRA or CARA, and define the "savings effect" as $\mathscr{S}_p \equiv -\tau_{rk} \sum_{s=2}^{S} \delta^{s-1} \frac{\eta_{r_s k_{s-1}, R_p}}{1 - \tau_{p, r_s k_{s-1}}}$. Since $\partial\left(r_s k_{s-1}\right)/\partial R_p < 0$ for all $s \leq p$ and $\partial\left(r_s k_{s-1}\right)/\partial R_p > 0$ for all $s \geq p+1$, we obtain that the sequence $\left\{\delta^{-(p-1)} \mathscr{S}_p : p = 1, \ldots, S\right\}$ is increasing, with $\mathscr{S}_1 < 0$ and $\mathscr{S}_S > 0$. Hence there exists p^* such that the revenue gains of the period-p separable perturbation are strictly smaller (resp., larger) in the dynamic model than in the static model for $p \leq p^*$ (resp., $p > p^*$). Moreover, the revenue gains of the separable perturbation that increases lump-sum the tax liability above $\bar{z}$ by \$1 in period p are smaller than the gains from the perturbation that increases the tax liability above $\bar{z}$ by $\$\delta^{-(p'-p)}$ in period $p' > p$. □

We next characterize the welfare effects of non-linear perturbations of the capital income tax schedule in every period $p \geq 2$.

Proof of Propositions **??** *and* **??**. Fix a capital income level $\bar{x}$, and consider a sequence, indexed by the positive numbers $(d\tau, d\bar{x})$, of revenue-neutral separable non-linear perturbations $\{h_{p,1}, h_{p,2}\}$ of the capital income tax schedule in *every* period $p \geq 2$, defined by $h_{p,1}\left(\mathbf{x}\right) = \left(r_p k_{p-1} - \bar{x}\right) d\tau$ for $r_p k_{p-1} \in [\bar{x}, \bar{x} + d\bar{x}]$ and $h_{p,1}\left(\mathbf{x}\right) = d\bar{x} d\tau$ for $r_p k_{p-1} \geq \bar{x} + d\bar{x}$. (More precisely, these perturbations are smoothed appropriately, as in the proof of Proposition **??**.) The additional tax revenue generated by this tax increase is rebated uniformly in a lump-sum way through the perturbation $h_{p,2}\left(\mathbf{x}\right) = dR$ for all $r_p k_{p-1} \in \mathbb{R}$. Applying formula (11), we get that the Gateaux differential of the social welfare functional writes, letting $d\bar{x}, d\tau, dR \to 0$,

$$\begin{aligned}\delta \mathscr{W}\left(T, \left\{h_{p,1}, h_{p,2}\right\}_{p \geq 2}\right) \underset{d\bar{x}, d\tau, dR \to 0}{\sim} & - \left\{\int_{\mathbb{R}_+^S \times \mathbb{R}^S} \sum_{p=2}^{S} \delta^{p-1} g_p\left(\mathbf{x}\right) f_{\mathbf{x}}\left(\mathbf{x}\right) d\mathbf{x}\right\} \left(dR\right) \\ & - \left\{\int_{\bar{x}}^{\infty} \int_{\mathbb{R}_+^S \times \mathbb{R}^{S-2}} \sum_{p=2}^{S} \delta^{p-1} g_p\left(\mathbf{x}\right) f_{\mathbf{x}}\left(\mathbf{x}\right) d\mathbf{x}\right\} \left(d\bar{x} d\tau\right).\end{aligned}$$

The Gateaux differential of the individual income choices is given by:

$$\delta r_s k^{(\mathbf{x})}_{\boldsymbol{\theta},s-1}(T_p, h_{p,1}) = \begin{cases} -\frac{r_s k_{s-1}}{1-\tau_{rk}} \zeta^{c,(\mathbf{x})}_{r_s k_{s-1}, 1-\tau_{p,r_p k_{p-1}}} d\tau, & \text{if } r_p k_{p-1} \in [\bar{x}, \bar{x} + d\bar{x}], \\ -\frac{\eta^{(\mathbf{x})}_{r_s k_{s-1}, R_p}}{1-\tau_{p,r_s k_{s-1}}} d\tau d\bar{x}, & \text{if } r_p k_{p-1} \geq \bar{x} + d\bar{x}, \end{cases}$$

$$\delta r_s k^{(\mathbf{x})}_{\boldsymbol{\theta},s-1}(T_p, h_{p,2}) = -\frac{\eta^{(\mathbf{x})}_{r_s k_{s-1}, R_p}}{1 - \tau_{p,r_s k_{s-1}}} dR.$$

The revenue-neutrality implies, as $d\bar{x}, d\tau, dR \to 0$,

$$\frac{dR}{(1 - F_{rk}(\bar{x}))\, d\bar{x} d\tau} \to - \frac{1 - \frac{\tau_{rk}}{1-\tau_{rk}} \sum_{p,s=2}^{S} \tilde{\delta}^{s-1} \left[\frac{r\bar{k} f_{rk}(r\bar{k})}{1 - F_{rk}(r\bar{k})} \bar{\zeta}^{c,\left(r_p k_{p-1} = r\bar{k}\right)}_{r_s k_{s-1}, 1-\tau_{p,r_p k_{p-1}}} + \bar{\eta}^{\left(r_p k_{p-1} \geq r\bar{k}\right)}_{r_s k_{s-1}, R_p} \right]}{1 - \frac{\tau_{rk}}{1-\tau_{rk}} \sum_{p=2}^{S} \sum_{s=2}^{S} \tilde{\delta}^{s-1} \bar{\eta}^{(r_p k_{p-1} \in \mathbb{R})}_{r_s k_{s-1}, R_p}}.$$

Rearranging the terms and using the aggregate sufficient statistics defined in Section 5 yields formula (**??**).

Now consider the case where the perturbation is implemented in every period $p = 2, \ldots, p_2$. Under the assumptions of Proposition **??**, the compounded substitution and income effects are then respectively given by

$$\begin{aligned} \mathscr{E} &= \sum_{p=2}^{p_2} \sum_{s=2}^{\infty} \delta^{s-1} \bar{\zeta}^{c,\left(\mathbf{x} \in \mathbb{R}^{2S-1}\right)}_{r_s k_{s-1}, 1-\tau_{p,r_p k_{p-1}}} &&= \sigma^{-1} \frac{\delta}{1-\delta} - \left[\sigma^{-1}\delta + \sigma^{-1}(1-\delta) p_2\right] \frac{\delta^{p_2}}{1-\delta}, \\ \mathscr{I} &= \sum_{p=2}^{p_2} \sum_{s=2}^{\infty} \delta^{s-1} \bar{\eta}^{\left(r_p k_{p-1} \geq r\bar{k}\right)}_{r_s k_{s-1}, R_p} &&= p_2 \delta^{p_2} - \delta. \end{aligned}$$

The results of Proposition **??** follow. □

We now prove Proposition **??**, which characterize the welfare effects of joint perturbations of the tax system.

Proof of Proposition **??**. Fix $d \geq 2$ directions $(x_1, \ldots, x_d)$ of the space $\mathbb{R}^S_+ \times \mathbb{R}^S$ and the income threshold $\bar{\mathbf{x}}_d = (\bar{x}_1, \ldots, \bar{x}_d)$. We define the d-multilinear perturbation h_p of the baseline tax function T_p as $h_p(\mathbf{x}) = 0$ if $x_j \leq \bar{x}_j$ for all $j \in \{1, \ldots, d\}$, $h_p(\mathbf{x}) = (x_i - \bar{x}_i)\, d\tau$ if $x_i \in [\bar{x}_i, \bar{x}_i + d\bar{x}]$ for some $i \in \{1, \ldots, d\}$ *and* $x_j \geq \bar{x}_j + d\bar{x}$ for all $j \in \{1, \ldots, d\} \setminus \{i\}$, and $h_p(\mathbf{x}) = d\tau d\bar{x}$ if $x_j \geq \bar{x}_j + d\bar{x}$ for all $j \in \{1, \ldots, d\}$. We complete this definition on the remaining regions of the space (hypercubes of size $d\bar{x}$) by making h_p continuous and multilinear on each of these regions. (More precisely, we consider a smooth approximation of these perturbations, as in the proof of Proposition **??**.) For simplicity, we let $d = 2$, i.e., we analyze a sequence, indexed by $(d\bar{x}, d\tau) \in \mathbb{R}_+ \times \mathbb{R}$, of joint perturbation of the baseline tax system in the directions (x_1, x_2) at the point $(\bar{x}_1, \bar{x}_2)$.

Applying formula (11) and letting $d\bar{x}, d\tau \to 0$, we obtain that the Gateaux differential of the

social welfare functional for this joint perturbation h_p^J writes

$$\frac{\delta \mathscr{W}\left(T_p, h_p^J\right)}{\bar{F}_{x_1,x_2}\left(\bar{x}_1,\bar{x}_2\right) d\bar{x} d\tau} \xrightarrow[d\bar{x},d\tau\to 0]{} \delta^{p-1}\left(1-\bar{g}_{p,\mathtt{NL}}^{(x_1\geq \bar{x}_1, x_2\geq \bar{x}_2)}\right) - \frac{\tau_{rk}}{1-\tau_{rk}} \sum_{s=2}^{S} \delta^{s-1} \eta_{r_s k_{s-1}, R_p}^{(x_1\geq \bar{x}_1, x_2\geq \bar{x}_2)} - \dots$$
$$- \sum_{i=1}^{2} \left\{ \frac{\delta^{p-1} T'\left(\bar{x}_i\right) \zeta_{x_i, 1-\tau_{p,x_i}}^{c,(x_i=\bar{x}_i, x_j\geq \bar{x}_j)}}{1-T'_{p,x_i}\left(\bar{x}_i\right) + \bar{x}_i \zeta_{x_i,1-\tau_{p,x_i}}^{c,(\bar{x}_1)} T''_{p,x_i}\left(\bar{x}_i\right)} \mathbf{1}_{\{x_1=z_s\}} + \frac{\tau_{rk}}{1-\tau_{p,x_i}} \sum_{s=2}^{S} \delta^{s-1} \bar{\zeta}_{r_s k_{s-1}, 1-\tau_{p,x_i}}^{c,(x_i=\bar{x}_i, x_j\geq \bar{x}_j)} \mathbf{1}_{\{x_i=r_t k_{t-1}\}} \right\} \mathcal{H}^i_{(\bar{x}_1,\bar{x}_2)}.$$

The formulas (**??**) and (**??**) give the welfare effects of the corresponding separable perturbations $h_p^{S_1}, h_p^{S_2}$. They write:

$$\frac{\delta \mathscr{W}\left(T_p, h_p^{S_i}\right)}{\bar{F}_{x_i}\left(\bar{x}_i\right) d\bar{x} d\tau} \xrightarrow[d\bar{x},d\tau\to 0]{} \delta^{p-1}\left(1-\bar{g}_{p,\mathtt{NL}}^{(x_i\geq \bar{x}_i, x_j\geq 0)}\right) - \frac{\tau_{rk}}{1-\tau_{rk}} \sum_{s=2}^{S} \delta^{s-1} \bar{\eta}_{r_s k_{s-1}, R_p}^{(x_i\geq \bar{x}_i, x_j\geq 0)} - \dots$$
$$- \left\{ \frac{\delta^{p-1} T'\left(\bar{x}_i\right) \zeta_{x_i, 1-\tau_{p,x_i}}^{c,(x_i=\bar{x}_i, x_j\geq 0)}}{1-T'_{p,x_i}\left(\bar{x}_i\right) + \bar{x}_i \zeta_{x_i,1-\tau_{p,x_i}}^{c,(\bar{x}_1)} T''_{p,x_i}\left(\bar{x}_i\right)} \mathbf{1}_{\{x_i=z_s\}} + \frac{\tau_{rk}}{1-\tau_{p,x_i}} \sum_{s=2}^{S} \delta^{s-1} \bar{\zeta}_{r_s k_{s-1}, 1-\tau_{p,x_i}}^{c,(x_i=\bar{x}_i, x_j\geq 0)} \mathbf{1}_{\{x_i=r_t k_{t-1}\}} \right\} \mathcal{H}_{\bar{x}_i}.$$

Using these equations to substitute for the ratios of marginal tax rates in the second line of the previous expression and rearranging the terms, we obtain formula (**??**). □

We finally prove Proposition **??**, which provides a closed-form solution fot the revenue gains of joint perturbations at the tail.

Proof of Proposition **??**. Consider the generalized Clayton copula with correlation parameters (d, ρ), defined, for $d \geq 1$ and $\rho \in (0, \infty)$, as

$$C\left(u, v\right) = \left\{ \left[\left(u^{-1/\rho} - 1\right)^d + \left(v^{-1/\rho} - 1\right)^d \right]^{1/d} + 1 \right\}^{-\rho}.$$

Kendall's tau[29] and the coefficients of lower and upper tail dependence are given by:

$$\rho_\tau = 1 - \frac{2}{\left(2+\frac{1}{\rho}\right) d}, \quad \lambda_l = \lim_{q\to 0} \frac{C\left(q,q\right)}{q} = 2^{-\rho/d}, \quad \lambda_u = 2 + \lim_{q\to 0} \frac{C\left(1-q,1-q\right)-1}{q} = 2 - 2^{1/d}.$$

The marginal distributions are Pareto distributed, $\bar{F}_{x_j}\left(x_j\right) = \alpha_j \left(\frac{x_j}{c_j}\right)^{-a_j}$, hence the log-survival c.d.f., obtained from the generalized Clayton copula, writes

$$\ln \bar{F}_{x_1,x_2}\left(x_1, x_2\right) = -\rho \ln \left\{ 1 + \left[\left(\alpha_1^{-1/\rho} \left(\frac{x_1}{c_1}\right)^{a_1/\rho} - 1 \right)^d + \left(\alpha_2^{-1/\rho} \left(\frac{x_2}{c_2}\right)^{a_2/\rho} - 1 \right)^d \right]^{1/d} \right\}.$$

[29] Kendall's tau is defined as follows. Consider two random variables $\tilde{x}_1, \tilde{x}_2$, independent of x_1, x_2, but with the same joint distribution. Then $\rho_\tau\left(x_1, x_2\right) \equiv \mathbb{E}\left[\text{sign}\left(\left(x_1 - \tilde{x}_1\right) \cdot \left(x_2 - \tilde{x}_2\right)\right)\right]$.

The calculation of $\tilde{\mathbf{h}}(x_1, x_2) = 1 - \sum_i \frac{\partial \ln \bar{F}_{x_1,x_2}/\partial x_i}{\partial \ln \bar{F}_{x_i}/\partial x_i}$ then yields

$$\tilde{\mathbf{h}}(\bar{x}_1, \bar{x}_2) = \frac{1 - \left[A_1(\bar{x}_1)^d + A_2(\bar{x}_2)^d\right]^{1/d-1} \left[A_1(\bar{x}_1)^{d-1} + A_2(\bar{x}_2)^{d-1}\right]}{1 + \left[A_1(\bar{x}_1)^d + A_2(\bar{x}_2)^d\right]^{1/d}},$$

where $A_j(\bar{x}_j) \equiv \left(\bar{F}_{x_j}(\bar{x}_j)^{-1/\rho} - 1\right)$ for $j \in \{1, 2\}$. □