Asset Pricing with Horizon-Dependent Risk Aversion

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PRELIMINARY. COMMENTS WELCOME.

Abstract

We study general equilibrium asset prices in a multi-period endowment economy when agents have dynamically inconsistent preferences with respect to risk-return tradeoffs, or "horizon-dependent risk aversion" (HDRA). We find that HDRA preferences generate a decreasing term structure of risk premia if and only if volatility is stochastic. The HDRA model therefore implies that recent empirical results on the term structure of risk premia are driven by a downward-sloping term structure of the price of volatility risk. We confirm this prediction in the cross-section of stock returns, showing that the value premium is significantly larger for stocks with higher exposure to volatility risk.

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1 Introduction

It is widely believed that dynamically inconsistent preferences have no meaningful impact on general equilibrium asset prices. More specifically, what has been shown is that dynamically inconsistent time preferences have no meaningful impact on cross-sectional variation in asset returns (Harris and Laibson, 2001; Luttmer and Mariotti, 2003). Given that cross-sectional asset pricing involves intra-period risk-return tradeoffs, it is indeed quite intuitive that horizon-dependent time preferences are not suitable to address puzzles related to cross-sectional variation in high-frequency Sharpe ratios. However, a less well-known fact is that preferences can be dynamically inconsistent not only in the time dimension, but also in the risk dimension. Whether such preferences can generate interesting asset pricing questions is an open question.

Motivated by ample experimental evidence that subjects are more risk averse to immediate than to delayed risks,[1] Eisenbach and Schmalz (2014) introduce a two-period model with horizon-dependent risk aversion (HDRA) and show that HDRA is conceptually orthogonal to other non-standard preferences such as non-exponential time discounting (Phelps and Pollak, 1968; Laibson, 1997), time-varying risk aversion (Constantinides, 1990; Campbell and Cochrane, 1999), and a preference for the timing of resolution of uncertainty (Kreps and Porteus, 1978; Epstein and Zin, 1989). In the present paper, we investigate the impact of HDRA preferences on asset prices in a multi-period framework. The conceptual difficulties of solving a multi-period model with dynamically inconsistent preferences are numerous. The commonly used recursive techniques in finance and macroeconomics only apply to dynamically consistent preferences. In addition, combining time-separability with HDRA necessarily introduces inconsistent time preferences which precludes isolating the effects of HDRA. In the present paper, we therefore introduce techniques in the spirit of Strotz (1955) to solve the problem of a rational agent with HDRA preferences in a setting without time separability. Unable to commit to future behavior, the agent optimizes today, taking into account reoptimization in future periods. Solving for the subgame-perfect equilibrium of the intra-personal game yields a stochastic discount factor (SDF) that nests the standard Epstein and Zin (1989) case.

We use this SDF primarily to investigate implications of HDRA on the term structure of risk premia. We find that the term structure of equity risk premia is non-trivial only if

[1] See, e.g., Jones and Johnson (1973); Onculer (2000); Sagristano, Trope, and Liberman (2002); Noussair and Wu (2006); Coble and Lusk (2010); Baucells and Heukamp (2010); Abdellaoui, Diecidue, and Onculer (2011). See Eisenbach and Schmalz (2014) for a more thorough review.

the economy features stochastic volatility. This insight implies that the downward-sloping term structure of equity risk premia documented empirically (see the literature review below) is driven by a downward-sloping term structure of the price of volatility risk.

We confirm that this prediction is borne out in the cross-section of stock returns. Using the link between the value premium and the term structure of risk premia proposed by Lettau and Wachter (2007), we compare the value premium among stocks with low and high exposure to volatility risk. Since growth stocks have more payoff uncertainty at longer horizons relative to value stocks, they load relatively more on the lower risk premia at longer horizons. Since our HDRA model predicts the term structure of risk premia to be driven by volatility risk, the value premium should be greater for stocks with more exposure to volatility risk. Our empirical analysis confirms this prediction, with the value premium for stocks with high exposure to volatility risk being 28 percent greater than for stocks with low exposure to volatility risk.

We show in related work that S&P 500 index options are priced consistent with the model predictions as well (Andries, Eisenbach, Schmalz, and Zhang, 2014). Specifically, the GMM estimate of the price of volatility risk in a Heston (1993) model is negative for short maturities but close to zero for longer maturities, and Sharpe ratios of short-term holding returns of at-the-money straddles show the same pattern, indicating a downward-sloping term structure of the price of volatility risk. While less tightly linked to the model, the results in Dew-Becker, Giglio, Le, and Rodriguez (2014) are consistent with the predictions as well.[2]

The paper proceeds as follows. Section 2 describes the relation of the present paper to the existing literature. Section 3 presents a two-period model that illustrates the intuition of some of our results but also the inherent limitations of a static approach to investigate the implications of HDRA on asset prices. Section 4 presents the dynamic model. Section 5 analyzes the pricing of risk and the term-structure implications. Section 6 has the empirical analysis. Section 7 concludes.

[2]These authors document a downward-sloping term structure of the volatility risk premium, which is the product of price and quantity of risk at every maturity. Our model's predictions are specifically about the price of volatility risk. A direct test therefore requires a separation of the price and quantity components. That is the objective of Andries, Eisenbach, Schmalz, and Zhang (2014).

2 Related Literature

The present paper is most closely connected to a recent literature examining the term structure of the price of risk in several markets both empirically and theoretically. Binsbergen, Brandt, and Koijen (2012) show empirically that Sharpe ratios for short-term dividend strips are higher than Sharpe ratios for long-term dividend strips (see also Boguth, Carlson, Fisher, and Simutin, 2012; van Binsbergen and Koijen, 2011; Van den Steen, 2004). Giglio, Maggiori, and Stroebel (2013) show that a similar puzzle exists in housing markets. These empirical findings are inconsistent with traditional general equilibrium asset pricing models such as Campbell and Cochrane (1999); Bansal and Yaron (2004) which predict a flat or upward-sloping term structure of risk prices, and have therefore led to a vigorous debate in the literature.

Theoretical models with a connection to these questions include Lettau and Wachter (2007), who show that exogenous variation of the stochastic discount factor can be used to answer the question "Why is long-horizon equity less risky?"[3] By contrast, we derive a stochastic discount factor based on preferences that can address similar phenomena without imposing exogenous variation. Muir (2013) uses time-variation in institutional frictions to explain why the term structure of risky asset returns changes over time, whereas we keep the economy simple but allow for a non-standard preference to derive a non-trivial term structure of risk prices. Croce, Lettau, and Ludvigson (2007) use a learning friction to generate a downward-sloping equity term structure, whereas no such informational constraints exist in our model. Santos and Veronesi (2010) add individual asset cash flow risk to the model of Campbell and Cochrane (1999) to generate a value premium for short-horizon assets. Ai, Croce, Diercks, and Li (2013) develop a production-based RBC model in which capital vintages face heterogeneous shocks to aggregate productivity to derive the same result. Their result relies on long-run risk, whereas long-run risk is orthogonal to our results. Also, our model only requires an endowment economy to generate a non-trivial term structure. The latter difference is also the distinction from other production-based models with implications for the term structure of equity risk, such as Kogan and Papanikolaou (2010, 2014); Favilukis and Lin (2013). Our paper is related to Campbell, Giglio, Polk, and Turley (2012) in that volatility risk is an important driver of asset returns.

The predictions of our model are distinguishable from the above alternative expla-

[3] Dechow, Sloan, and Soliman (2004) and Da (2009) present empirical results consistent with the idea that the value premium exists because value stocks have a shorter duration than growth stocks.

nations for a downward-sloping term structure of equity risk premia. In our model, a downward-sloping term structure of the price of volatility risk is the only driver for a downward-sloping term structure of equity risk in our model. Andries, Eisenbach, Schmalz, and Zhang (2014) show how this driver can be isolated from other potential explanations.

This literature review is limited to existing contributions with a relation to the term structure of the price of risk. In an effort to focus the paper, we do not discuss the relation of auxiliary implications of our model to existing work in those domains, such as differences in expected returns of value versus growth stocks (see, e.g., Gârleanu, Kogan, and Panageas, 2012).

3 Simple Model

As discussed in Eisenbach and Schmalz (2014), introducing HDRA into a time separable model with more than two periods necessarily introduces horizon-dependent intertemporal tradeoffs similar to quasi-hyperbolic discounting. This is undesirable if we want to study the effects of HDRA in isolation. Our general model in Section 4 solves this problem by dropping time separability but this comes at the cost of gaining analytical complexity and losing intuitive clarity. To preserve the same for a few more paragraphs, here we present a simple model with time separability and uncertainty both in the immediate and proximate future to illustrate the effect of HDRA on risk pricing.

Consider a two-period model with uncertainty in both periods. The agent has time separable expected utility U_t in period $t = 0, 1$ given by

$$U_0 = E\big[v(c_0) + \delta u(c_1)\big] \quad \text{and} \quad U_1 = E\big[v(c_1)\big],$$

where v and u are von Neumann-Morgenstern utility indexes and v is more risk averse than u. At the beginning of period 0 the agent forms a portfolio of two risky assets and a risk free bond. Asset 0 is a claim on consumption in period 0 while asset 1 is a claim on consumption in period 1. Consumption in the two periods is i.i.d. Denoting the prices of the two assets by p_0 and p_1, respectively, the first-order conditions for the agent's portfolio choice yield:

$$\begin{aligned} & E\big[v'(c_0)\left(c_0 - p_0\right)\big] = 0 \\ \text{and} \quad & E\big[\delta u'(c_1)\left(c_1 - (1+r)\,p_1\right)\big] = 0. \end{aligned}$$

Eisenbach and Schmalz (2014) show that the equilibrium prices p_0, p_1 and r satisfy:

$$p_0 < (1+r)\, p_1.$$

In this two-period setting, HDRA therefore leads to an equilibrium term-structure of risk premia that is downward sloping.

This simple example illustrates how HDRA naturally affects the pricing of risk at different horizons. There are, however, important limitations to this example. The setting is subtly different from standard asset pricing models with two periods $t = 0, 1$ since there is uncertainty in both periods and a period's decisions are taken before the uncertainty of the respective period resolves. This allows for HDRA to have an effect since there is uncertainty at two different horizons without worrying about inconsistent inter-temporal tradeoffs since only one such tradeoff arises. An implication of the non-standard timing of uncertainty is that it complicates a dynamic analysis with the agent making choices both in period 0 and period 1. The period-0 portfolio choice problem above implicitly assumes that the agent has no opportunity to retrade the claim to period-1 consumption at the beginning of period 1, i.e. that the agent can commit to a portfolio in period 0.

To resolve these limitations of the two-period setting, the next section presents our main model which is no longer time-separable but in exchange is fully dynamic, has no uncertainty in the current period, allows for retrading in every period and is dynamically consistent for inter-temporal consumption tradeoffs.

4 Fully Dynamic Model

Our approach is to generalize the model of Epstein and Zin (1989) (hereafter EZ) to allow for horizon-dependent risk aversion without affecting intertemporal substitution. Let $\{\gamma_h\}_{h\geq 0}$ be a decreasing sequence representing risk aversion at horizon h. In period t, the agent evaluates a consumption stream starting in period $t+h$ by

$$V_{t,t+h} = \left((1-\beta)\, C_{t+h}^{1-\rho} + \beta E_{t+h}\left[V_{t,t+h+1}^{1-\gamma_h} \right]^{\frac{1-\rho}{1-\gamma_h}} \right)^{\frac{1}{1-\rho}}. \tag{1}$$

As in the EZ model, utility in period t, i.e. for $h = 0$, depends on (deterministic) consumption in period t and a certainty equivalent of (uncertain) continuation values in period $t+1$ and the aggregation is with constant elasticity of intertemporal substitution given by

$1/\rho$. However, in contrast to the EZ model, certainty equivalents at different horizons h are formed with different levels of risk aversion γ_h. Imminent uncertainty is treated with risk aversion γ_0, uncertainty one period removed is treated with γ_1 and so on. We assume that the agent is fully rational when making choices in period t to maximize $V_{t,t}$. This means self t realizes that its evaluation of future consumption given by $V_{t,t+1}$ differs from the objective function of $V_{t+1,t+1}$ which self $t+1$ will maximize.

For asset pricing purposes, the object of interest is the stochastic discount factor (SDF) resulting from the preferences in (1). We can arrive at the SDF using a standard derivation based on the intertemporal marginal rate of substitution:

$$\frac{\Pi_{t+1}}{\Pi_t} = \frac{dV_{t,t}/dW_{t+1}}{dV_{t,t}/dC_t}.$$

The derivative of current utility $V_{t,t}$ with respect to current consumption C_t is standard and given by:

$$\frac{dV_{t,t}}{dC_t} = V_{t,t}^{\rho}\,(1-\beta)\,C_t^{-\rho}. \tag{2}$$

The derivative of current utility $V_{t,t}$ with respect to next-period wealth W_{t+1} is almost standard:

$$\frac{dV_{t,t}}{dW_{t+1}} = \frac{dV_{t,t}}{dV_{t,t+1}} \times \frac{dV_{t,t+1}}{dW_{t+1}} = V_{t,t}^{\rho}\beta E_t\left[V_{t,t+1}^{1-\gamma_0}\right]^{\frac{\gamma_0-\rho}{1-\gamma_0}} V_{t,t+1}^{-\gamma_0} \times \frac{dV_{t,t+1}}{dW_{t+1}}. \tag{3}$$

At this point, however, we cannot appeal to the envelope condition at $t+1$ to replace the term $dV_{t,t+1}/dW_{t+1}$ by $dV_{t,t+1}/dC_{t+1}$. This is because $V_{t,t+1}$ is the value self t attaches to all future consumption while the envelope condition at $t+1$ is in terms of the objective function of self $t+1$ which is given by $V_{t+1,t+1}$:

$$\frac{dV_{t+1,t+1}}{dW_{t+1}} = \frac{dV_{t+1,t+1}}{dC_{t+1}} = V_{t+1,t+1}^{\rho}\,(1-\beta)\,C_{t+1}^{-\rho}. \tag{4}$$

The disagreement between selves t and $t+1$ requires us to take an extra step. Due to the homotheticity of our preferences, we can rely on the fact that both $V_{t,t+1}$ and $V_{t+1,t+1}$ are homogeneous of degree one which implies that

$$\frac{dV_{t,t+1}/dW_{t+1}}{dV_{t+1,t+1}/dW_{t+1}} = \frac{V_{t,t+1}}{V_{t+1,t+1}}. \tag{5}$$

This relationship captures a key element of our model: The marginal benefit of an extra

unit of wealth in period $t+1$ differs whether evaluated by self t (the numerator on the left hand side of (5)) or by self $t+1$ (the denominator on the right hand side of (5)). Due to the homotheticity of preferences, the ratio of marginal benefits equals the ratio of the values self t and self $t+1$ attach to consumption starting in period $t+1$.

To arrive at the stochastic discount factor, we first combine the relationship (5) with the envelope condition (4) to eliminate the term $dV_{t,t+1}/dW_{t+1}$ in the derivative of current utility with respect to next-period wealth (3). Then we can combine (3) with (2) to form the marginal rate of substitution and arrive at:

$$\frac{\Pi_{t+1}}{\Pi_t} = \underbrace{\beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho}}_{\text{(I)}} \times \underbrace{\left(\frac{V_{t,t+1}}{E_t\left[V_{t,t+1}^{1-\gamma_0}\right]^{\frac{1}{1-\gamma_0}}}\right)^{\rho-\gamma_0}}_{\text{(II)}} \times \underbrace{\left(\frac{V_{t,t+1}}{V_{t+1,t+1}}\right)^{1-\rho}}_{\text{(III)}}.$$

The SDF consists of three multiplicative parts:

(I) The first part is the the same as in a standard time-separable model with discount factor β and constant relative risk aversion ρ (which equals the inverse of the intertemporal elasticity of substitution).

(II) The second part originates in the fact that our model is not time separable and distinguishes between risk aversion and intertemporal substitution. It is very similar to the new part the EZ model adds to the standard part (I) but differs since it incorporates the horizon-dependent risk aversion $\{\gamma_h\}_{h\geq 0}$.

(III) The third part is fundamentally new to our model and originates in the fact that with HDRA different selves disagree about the evaluation of a given consumption stream, depending on their relative horizon. Since the SDF Π_t captures trade-offs between periods t and $t+1$, the key disagreement is how selves t and $t+1$ evaluate consumption starting in period $t+1$.

If we set $\gamma_h = \gamma$ for all horizons h, our SDF for HDRA preferences simplifies to the SDF for recursive preferences in EZ since then $V_{t,t+h} = V_{t+h,t+h}$ for all h so part (III) of the SDF equals 1 and part (II) equals the corresponding part in EZ. Our model therefore nests the model of EZ which, in turn, nests the standard time-separable model for $\gamma = \rho$.

Interestingly, note that the way our SDF captures the dynamic inconsistency for risk tradeoffs inherent in our HDRA preferences mirrors closely the way the SDF of Luttmer

and Mariotti (2003) captures the dynamic inconsistency for consumption tradeoffs inherent in their generalized discounting. In both models the disagreement between selves t and $t+1$ shows up in the SDF as the ratio of their utilities.

5 Pricing of Risk, and the Term Structure

To derive the pricing of risk under HDRA preferences, we consider a simplified version of the model where risk aversion for immediate risk is given by γ, and by $\tilde{\gamma}$ for all future risks. This framework, and our derivations for risk pricing, easily extends to a case where risk aversion is decreasing up to a given horizon, after which, for risks beyond, it remains constant ($\tilde{\gamma}$).

Our general model (1) thus becomes:

$$V_t = \left((1-\beta)\, C_t^{1-\rho} + \beta E_t \left[\tilde{V}_{t+1}^{1-\gamma} \right]^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}}$$
$$\tilde{V}_t = \left((1-\beta)\, C_t^{1-\rho} + \beta E_t \left[\tilde{V}_{t+1}^{1-\tilde{\gamma}} \right]^{\frac{1-\rho}{1-\tilde{\gamma}}} \right)^{\frac{1}{1-\rho}}$$

The second equation is simply the standard EZ framework with risk aversion $\tilde{\gamma}$. If solutions for the recursion on the continuation value $\tilde{V}$ are derived, the value function V is automatically obtained from the first equation.

5.1 Term Structure under Unit-EIS

The continuous-time counterpart to these two equations, with $\beta = 1 - \delta dt$, is derived, and yields

$$\left(\frac{V_t}{\tilde{V}_t} \right)^{1-\rho} - 1 = \delta dt \left(\left(\frac{C_t}{\tilde{V}_t} \right)^{1-\rho} - 1 \right) + (1-\rho) \frac{\Lambda_{t,\gamma}(\tilde{V})}{(1-\gamma)\, \tilde{V}_t^{1-\gamma}}\, dt,$$

as well as

$$0 = \frac{\delta}{1-\rho} \left(C_t^{1-\rho} - \tilde{V}_t^{1-\rho} \right) + \frac{\Lambda_{t,\tilde{\gamma}}(\tilde{V})}{(1-\tilde{\gamma})\, \tilde{V}_t^{1-\tilde{\gamma}}}\, \tilde{V}_t^{1-\rho},$$

where, for any parameter λ, and process X,

$$\Lambda_{t,\lambda}(X) = \lim_{\epsilon \to 0^+} E_t \frac{X_{t+\epsilon}^{1-\lambda} - X_t^{1-\lambda}}{\epsilon}.$$

Denoting logs by lowercase letters, we consider a Lucas-tree economy, with an exogenous endowment process given by

$$dc_t = \beta_{0,c} + \beta_{1,c} x_t + \alpha_c \sqrt{z_t} dW_t,$$

where the time varying drift, x_t, and the time varying volatility, z_t, have evolutions

$$\begin{aligned} dx_t &= \mu_1 x_t dt + \sigma_1 \sqrt{z_t} dW_t, \\ dz_t &= \mu_2 (z_t - 1) dt + \sigma_2 \sqrt{z_t} dW_t. \end{aligned}$$

We guarantee that the scalar state variable z_t remains always positive by imposing the restriction $\mu_2 + \frac{1}{2} |\sigma_2|^2 < 0$. To further guarantee the stationarity of the distributions, we suppose μ_1 has eigenvalues with strictly negative real parts. To simplify the model, we assume α_c, σ_1 and σ_2 are orthogonal.

Under these specifications for the endowment economy, a closed-form solution for the recursion on $\tilde{V}$ can be obtained, when the elasticity of intertemporal substitution is set to $\rho = 1$ (see Hansen, Heaton, Lee, and Roussanov, 2007). For our model, we find:

$$\tilde{v}_t - c_t = \tilde{\beta} + \tilde{\beta}_x x_t + \tilde{\beta}_z z_t$$

and

$$v_t - \tilde{v}_t = -\frac{1}{2} (\gamma - \tilde{\gamma}) z_t \left(\alpha_c^2 + \left(\tilde{\beta}_x \sigma_1 \right)^2 + \left(\tilde{\beta}_z \sigma_2 \right)^2 \right) dt, \tag{6}$$

where

$$\tilde{\beta}_x = (\delta I - \mu_1)^{-1} \beta_{1,c}$$

and

$$\tilde{\beta}_z = \frac{-(\delta - \mu_2) + \sqrt{(\delta - \mu_2)^2 - (\tilde{\gamma} - 1)^2 \sigma_2^2 \left(\alpha_c^2 + \left(\tilde{\beta}_x \sigma_1 \right)^2 \right)}}{(\tilde{\gamma} - 1) \sigma_2^2} < 0.$$

Observe from equation (6) that $v_t < \tilde{v}_t$ at all times, as should be expected. Indeed, $\tilde{v}_t$ is derived from the standard EZ model with risk-aversion $\tilde{\gamma}$, whereas v_t is derived from our horizon-dependent risk aversion model with a higher risk-aversion $\gamma > \tilde{\gamma}$ for immediate risk. For identical endowment processes, the value of future consumption under our HDRA framework is lower than under standard EZ with long-term risk aversion $\tilde{\gamma}$. Observe further, that the difference in the valuations under the two models is constant

when volatility z_t is constant. This result is central to our analysis for $\rho = 1$: it implies that, under constant volatility in the consumption process, the stochastic discount factor under our HDRA framework is the same as under the EZ framework with risk aversion γ, up to a constant. This implies, the pricing of risk and the term-structure of risk prices is unaffected by our horizon-dependent risk aversion structure, when volatility is constant, and is simply derived from the value of immediate risk aversion.

Suppose now that volatility is time varying. The evolution for the risk adjusted payoffs $\Pi_t C_t$ is:

$$\frac{\Pi_{t+1}C_{t+1}}{\Pi_t C_t} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{1-\rho} \left(\frac{\tilde{V}_{t+1}}{E_t\left[\tilde{V}_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\rho-\gamma} \left(\frac{\tilde{V}_{t+1}}{V_{t+1}}\right)^{1-\rho} \tag{7}$$

Under the restrictive assumption $\rho = 1$ for which the solutions for v_t and $\tilde{v}_t$ are derived in closed-form, this evolution becomes:

$$\frac{\Pi_{t+1}C_{t+1}}{\Pi_t C_t} = \beta \underbrace{\left(\frac{\tilde{V}_{t+1}^{1-\gamma}}{E_t\left[\tilde{V}_{t+1}^{1-\gamma}\right]}\right)}_{\text{Multiplicative martingale}}$$

Even though the $\tilde{V}_{t+1}/V_{t+1}$ term in (7) disappears when $\rho = 1$, observe that our HDRA model does affect the level of prices, since the risk-adjusted payoffs are determined by the immediate risk aversion γ, whereas the continuation value $\tilde{V}$ is determined by the EZ model with risk aversion $\tilde{\gamma}$. However, we find the knife-edge case $\rho = 1$ again yields a flat term-structure for the pricing of risk, even under our HDRA specification, and with time-varying volatility for the consumption process. Borovicka, Hansen, Hendricks, and Scheinkman (2011) develop a methodology for the derivation of the term-structure of risk-prices, and formally show the pricing of consumption risk, at time t, and at horizon h is determined by $E_t[\Pi_{t+h}C_{t+h}]$ as a function of the horizon h. Under the $\rho = 1$ case, the evolution as multiplicative martingales, for the risk adjusted payoffs, yields $E_t[\Pi_{t+h}C_{t+h}]$ independent of the horizon h and thus a flat term-structure of risk prices.

We thus find that the implications of the simple two-period HDRA model of Section 3 do not trivially extend to analytically tractable versions of the dynamic model, in which prices are set in general equilibrium at each time t. When our time-inconsistent representative agent is aware prices will be set, the following period, by her next-period self then the term-structure of prices is unaffected by risk-horizon dependent risk-aversion

as long as the consumption process does not feature time-varying volatility. Even under a stochastic volatility framework, the knife-edge case $\rho = 1$ for which closed-form solutions obtain, yields a flat term-structure of risk prices.

In the following section, we therefore relax the assumption $\rho = 1$, and analyze the term-structure impact of our horizon-dependent risk-aversion model under stochastic volatility more generally.

5.2 General Case and Role of Volatility Risk

Our model satisfies the commonly used relation between wealth and value:

$$\frac{W_t}{C_t} = \frac{1}{1-\beta}\left(\frac{V_t}{C_t}\right)^{1-\rho}$$

Using this relation, we can express the return on wealth as a function of consumption growth and the valuations V_{t+1} and $\widetilde{V}_{t+1}$:

$$R_{w,t+1} = \left(\frac{C_{t+1}}{C_t}\right)^{\rho} \frac{\tilde{V}_{t+1}^{1-\rho}}{\beta E_t\left[\tilde{V}_{t+1}^{1-\gamma}\right]^{\frac{1-\rho}{1-\gamma}}} \frac{V_{t+1}^{1-\rho}}{\tilde{V}_{t+1}^{1-\rho}},$$

so that the stochastic discount factor becomes:

$$\frac{\Pi_{t+1}}{\Pi_t} = R_{w,t+1}^{\frac{\rho-\gamma}{1-\rho}} \left(\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\left(\frac{\widetilde{V}_{t+1}}{V_{t+1}}\right)^{1-\rho}\right)^{\frac{1-\gamma}{1-\rho}}$$

As before, we consider a Lucas-tree economy – now in discrete time – with an exogenous endowment process given by

$$c_{t+1} - c_t = \beta_c + \alpha_c \sigma_t W_{t+1},$$

where the time varying volatility σ_t has evolution

$$\sigma_{t+1}^2 - \sigma^2 = \nu_\sigma\left(\sigma_t^2 - \sigma^2\right) + \alpha_\sigma \sigma_t W_{t+1}.$$

To guarantee the stationarity of the state variable, we suppose $|\nu_\sigma| < 1$. To simplify the model, we assume α_c and α_σ are orthogonal.

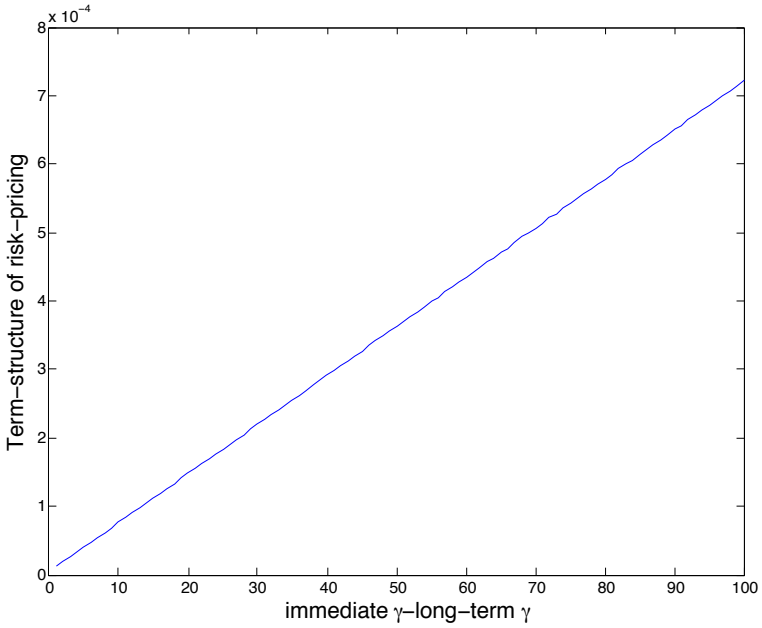


Figure 1: Simulated term-structure. We use the parameters from Bansal and Yaron (2004) for μ, ν_σ, α_c and α_σ and $\beta = 0.9$, $\tilde{\gamma} = 2$, $\rho = 0.75$.

Note that we have simplified the model by shutting down variations in the drift of consumption. Indeed, we can easily show that the results of the previous section, in which the pricing of drift shocks were unaffected by the horizon dependent variations in risk-aversion in our model, extend to the case $\rho \neq 1$. Any novel pricing effects and term-structure effects we obtain must therefore come from the pricing of the volatility shocks, and they are the only ones, beyond the immediate consumption shocks, we consider here.

As a first-pass, we simulate a 4-period model, $t = 0, 1, 2, 3$, such that $V_3 = \tilde{V}_3 = C_3$, and analyze the term-structure for the pricing of the volatility shocks, determined by:

$$\Delta = \frac{1}{3} \log P_{0,3} - \frac{1}{2} \log P_{0,2}$$

The results for the relation between the horizon-dependent drop in risk aversion $\gamma - \tilde{\gamma}$ and the term-structure of risk pricing Δ are given in Figure 1. For the assumed $\rho < 1$, i.e. an EIS greater than 1, our simulation results in a downward sloping term-structure of risk prices – the long-term claim is more expensive than the short-term claim, $\Delta > 0$. In addition, the slope becomes steeper as the difference in risk aversion increases. We see therefore that departing from the knife-edge case of a unit elasticity of intertemporal

substitution yields a relation between horizon-dependent variations in risk aversion and the term-structure of prices.

In order to formally analyze the solutions when $\rho \neq 1$, we follow the methodology of Bansal and Yaron (2004) and assume a log-linear approximation for the returns of wealth, similar to Campbell and Shiller (1988). We approximate the accounting equation for the return on wealth $R_{w,t+1} = \frac{W_{t+1}}{W_t - C_t}$ with

$$r_{w,t+1} = k_o + k_1 z_{t+1} - z_t + c_{t+1} - c_t,$$

where $z_t = w_t - c_t$ is the log wealth-to-consumption ratio, and k_o and k_1 are constant. We guess and verify the two linear equations:

$$\begin{aligned} z_t &= a + A\sigma_t^2 \\ \tilde{v}_{t+1} - \tilde{v}_t &= \beta_{v,t} + \left(\alpha_{v,c}\alpha_c + \alpha_{v,\sigma}\alpha_\sigma\right)\sigma_t W_{t+1} \end{aligned}$$

To derive the pricing equations, we further observe, that for β close to 1,

$$\frac{V_t}{\tilde{V}_t} \approx \frac{E_t\left[\tilde{V}_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}{E_t\left[\tilde{V}_{t+1}^{1-\tilde{\gamma}}\right]^{\frac{1}{1-\tilde{\gamma}}}},$$

and therefore

$$v_t - \tilde{v}_t \approx -\frac{1}{2}\left(\gamma - \tilde{\gamma}\right)\left(\alpha_{v,c}^2\alpha_c^2 + \alpha_v^2\alpha_\sigma^2\right)\sigma_t^2.$$

We assume this approximate result is exact and obtain

$$\begin{aligned} \alpha_{v,c} &= 1 \\ \alpha_{v,\sigma} &= \frac{1 - \sqrt{1 - 2\left(\gamma - \tilde{\gamma}\right)\alpha_\sigma^2\left(\frac{k_1 A}{1-\rho} + \frac{1}{2}\left(\gamma - \tilde{\gamma}\right)\alpha_c^2\right)}}{\left(\gamma - \tilde{\gamma}\right)\alpha_\sigma^2} \end{aligned}$$

under the constraint

$$\frac{k_1 A}{1-\rho} + \frac{1}{2}\left(\gamma - \tilde{\gamma}\right)\alpha_c^2 < 0,$$

which guarantees the existence of $\alpha_{v,\sigma} < 0$ (as we observe in the data). Because $\alpha_{v,c} = 1$, we obtain

$$\tilde{v}_t - c_t = \beta_v + \alpha_{v,\sigma}\sigma_t^2.$$

To arrive at the solutions for the pricing of risk, determined by the loading of the wealth-to-consumption ratio on the volatility shocks, i.e. by A, we use the Euler equation

$$E_t \left[\Pi_{t,t+1} R_{w,t+1} \right] = 1$$

which yields

$$A = (1-\rho)\,\alpha_{v,\sigma}\,\frac{\nu_\sigma\,(\gamma-\tilde{\gamma}) + (1-\gamma)}{(\gamma-\tilde{\gamma}) + k_1\,(1-\gamma)}$$

and a unique solution $A < 0$ and $\alpha_{v,\sigma} < 0$ under the two conditions $(\gamma-\tilde{\gamma})+k_1\,(1-\gamma) < 0$ and $\nu_\sigma\,(\gamma-\tilde{\gamma}) + (1-\gamma) < 0$.[4]

Under this framework, we can analyze the pricing of volatility risk in the term-structure. Denote $P_{t,h}$ the price at time t for claim to the endowment consumption in h periods. For all t, $P_{t,0} = C_t$. We guess and verify:

$$\frac{P_{t,h}}{C_t} = \exp\left(a_h + A_h \sigma_t^2\right).$$

For $h \geq 1$:

$$\frac{P_{t,h}}{C_t} = E_t\left[\Pi_{t,t+1}\frac{C_{t+1}}{C_t}\frac{P_{t+1,h-1}}{C_{t+1}}\right]$$

which yields a recursion in a_h and A_h. The term-structure for the pricing of volatility shocks is determined by the horizon dependent solutions for A_h, where A_h is determined by the initial condition $A_0 = 0$ and the recursion:

$$\begin{aligned} A_h = &\left(A_{h-1}\nu_\sigma + \frac{\rho-\gamma}{1-\rho}\left(k_1\nu_\sigma - 1\right)A + (1-\gamma)\,\frac{1}{2}\,(\gamma-\tilde{\gamma})\left(\alpha_c^2 + \alpha_v^2\alpha_\sigma^2\right)\nu_\sigma\right) \\ &+ \left(A_{h-1} + \frac{\rho-\gamma}{1-\rho}k_1 A + (1-\gamma)\,\frac{1}{2}\,(\gamma-\tilde{\gamma})\left(\alpha_c^2 + \alpha_v^2\alpha_\sigma^2\right)\right)^2 \alpha_\sigma^2 + \frac{1}{2}\,(1-\gamma)^2\,\alpha_c^2. \end{aligned}$$

In Figure 2, we show the horizon-dependent pricing of risk we obtain in our model. We obtain a downward slopping pricing of risk (in absolute value) for consumption maturities beyond one year. Periods with higher volatility yield steeper term-structures, as is observed in the data.

[4] The solution is given by $\alpha_{v,\sigma} = \frac{(1-k_1\nu_\sigma) - \sqrt{(1-k_1\nu_\sigma)^2 - ((\gamma-\tilde{\gamma})+k_1(1-\gamma))^2\alpha_c^2\alpha_\sigma^2}}{((\gamma-\tilde{\gamma})+k_1(1-\gamma))\alpha_\sigma^2}$.

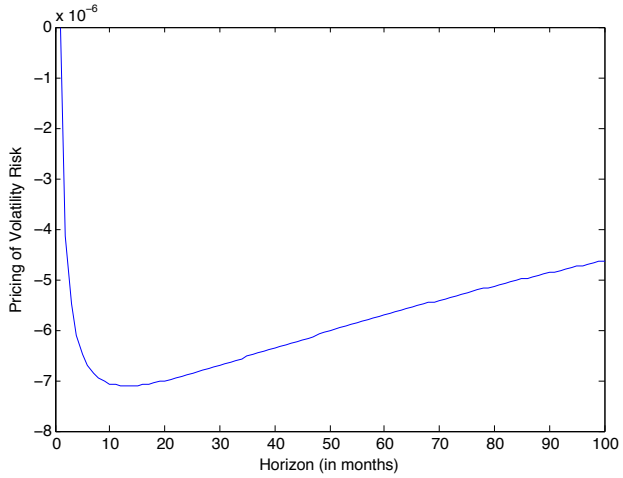


Figure 2: Calibrated term-structure. We use the parameters from Bansal and Yaron (2004) for μ, ν_σ, α_c and α_σ and $\beta = 0.9$, $\tilde{\gamma} = 2$, $\rho = 0.75$.

6 Empirical Analysis

A key implication of our model is the role of volatility risk in the term structure of risk premia. In this section we document empirically that this effect is present in the cross-section of stock returns, i.e. we show that stocks with more exposure to volatility risk show a stronger term structure for risk premia.

Our approach is based on the link between the value premium and the term structure of risk premia proposed by Lettau and Wachter (2007). The idea is based on the difference in the term structure of payoff uncertainty between growth and value stocks: relative to value stocks, growth stocks have more payoff uncertainty at longer horizons. If market risk premia are decreasing in horizon, then growth stocks have lower returns since they load relatively more on the lower risk premia at longer horizons.

Since our model predicts volatility risk to be the driver of the term structure of risk premia, we would expect the value premium to be stronger among stocks with higher exposure to volatility risk. To test this prediction, we sort stocks into portfolios along two dimensions: (i) their exposure to volatility shocks and (ii) their book-to-market ratio. Then we calculate returns for each of the four portfolios and check if the well-known premium of high book-to-market stocks is greater for the portfolios with high exposure to volatility risk.

To construct the exposure to volatility risk, we first calculate volatility shocks based on an ARMA(4,4) process fitted to the daily time series of the VIX from January 2nd,

Table 1: Average monthly returns

	High β^{vol}	Low β^{vol}
High BE/ME	0.047	0.039
	(0.0063)	(0.0056)
Low BE/ME	–0.027	–0.020
	(0.0056)	(0.0035)

1990 to June 30th, 2014.[5] We then estimate volatility betas for all stocks i by regressing daily returns from CRSP on the VIX shocks:

$$r_{it} = \alpha_i + \beta_i^{\text{vol}} \times \text{VIX shock}_t + \varepsilon_{it}$$

Finally, we sort stocks based on their volatility betas and keep only the top and bottom quartile.

To sort based on book-to-market ratios, we use monthly CRSP data to calculate market equity (ME) as well as Compustat data to construct book equity (BE). We then sort based on BE/ME and keep only the top and bottom decile. This leaves us with four portfolios: (i) high volatility beta and high book to market, (ii) high volatility beta and low book to market, (iii) low volatility beta and high book to market, and (iv) low volatility beta and low book to market.

Table 1 shows the average monthly returns of the four portfolios. Table 2 calculates the difference in returns between the high book-to-market stocks and the low book-to-market stocks (HML) separately for the high volatility beta stocks and the low volatility beta stocks. We see that the well-known value premium is positive and significant for stocks with both high and low exposure to volatility risk. Our main interest, however, is in the difference between the value premium among the two categories. As shown in Table 2, the value premium for stocks with high exposure to volatility risk is 28 percent higher than for stocks with low exposure to volatility risk and this difference is statistically significant.

[5] We use the BIC criterion to determine the number of lags and check up to 8 lags for both MA and AR.

Table 2: Differences in returns

	High β^{vol}	Low β^{vol}	Difference
HML	0.074	0.058	0.016
t-statistic	16.42	14.67	2.63

7 Conclusion

We solve for general equilibrium asset prices in an endowment economy in which assets are priced by an agent with dynamically inconsistent preferences with respect to risk-return tradeoffs. We find that horizon-dependent preferences have a meaningful impact on asset prices, and have the ability to address cross-sectional puzzles in general equilibrium asset pricing. In particular, we show that the price of risk depends on the horizon if and only if volatility is stochastic. This insight leads to numerous testable predictions. Recent empirical research by the authors and other researchers confirms the prediction that the term structure of the price of volatility risk is downward-sloping. In the present paper, we show that the value premium is significantly higher for stocks with higher exposure to aggregate volatility shocks than for shocks with lower volatility-beta. In sum, we conclude that dynamically inconsistent risk preferences can address general equilibrium asset pricing puzzles that existing models leave unexplained.

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