Information Aggregation in a DSGE Model*

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– Preliminary –

Abstract

We solve and quantitatively analyze a canonical noisy rational expectations model (Hellwig, 1980) within the framework of a conventional real business cycle model. Each household receives a private signal about future productivity. In equilibrium, the stock price serves to aggregate and transmit this information. We find that dispersed information about future productivity affects the quantitative properties of our real business cycle model in three dimensions. First, households' ability to learn about the future affects their consumption-savings decision. The equity premium falls and the risk-free interest rate rises when the stock price perfectly reveals innovations to future productivity. Second, when noise trader demand shocks limit the stock market's capacity to aggregate information, households hold heterogeneous expectations in equilibrium. However, for a reasonable size of noise trader demand shocks the model cannot generate the kind of disagreement observed in the data. Third, even moderate heterogeneity in the equilibrium expectations held by households has a sizable effect on the level of all economic aggregates and on the correlations and standard deviations produced by the model. For example, the correlation between consumption and investment growth is 0.29 when households have no information about the future, but 0.41 when information is dispersed.

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1 Introduction

An efficient allocation of goods and services depends on the ability of financial markets to aggregate dispersed information that is not available to any single economic agent in its totality (Hayek, 1945). The way in which asset prices and economic aggregates reflect this dispersed information has been the object of a large theoretical literature on "noisy rational expectations" going back to Grossman (1976) and Hellwig (1980).

This literature has produced a rich set of predictions. For example, inefficiencies may arise when prices transmit information that is costly to acquire (Grossman and Stiglitz, 1980; Verrecchia, 1982), when agents must form expectations about the expectations of others (Allen, Morris, and Shin, 2006; Bacchetta and Van Wincoop, 2008; Qiu and Wang, 2010; Angeletos, Lorenzoni, and Pavan, 2012), or when agents learn about actions that are strategic complements to their own actions (Amador and Weill, 2012; Goldstein, Ozdenoren, and Yuan, 2013).[1]

While this field of research has received a great deal of attention in recent years, it has proven difficult to quantify its results within the standard framework of dynamic stochastic general equilibrium (DSGE) models. This relative scarcity of quantitative applications is the result of a methodological barrier: the tools used to solve noisy rational expectations models are generally perceived to be incompatible with the quantitative machinery of DSGE models. Noisy rational expectations models typically combine normally distributed shocks with policy functions that are (log) linear in the expectation of the shocks that agents learn about. This combination ensures that aggregate variables (such as prices or the capital stock) inherit the normal distribution, allowing a straightforward application of the projection theorem when solving for equilibrium expectations. In contrast, policy functions in contemporary DSGE models are inherently non-linear in expectations of future variables (such as consumption, labor supply, and portfolio returns). As a result, aggregate variables are not normally distributed in these models, even if they feature normally distributed shocks. Consequently, the standard approach for solving noisy rational expectations models is not directly applicable.

In this paper we develop an approach that breaks this methodological barrier and allows us to apply the standard toolkit for solving noisy rational expectations models *within* the standard DSGE framework that is typically used for welfare and policy analysis. We use our approach to quantitatively estimate the seminal noisy rational expectations model by Hellwig (1980).[2]

In noisy rational expectations models, households receive a private signal about future pro-

[1] See Brunnermeier (2001) and Veldkamp (2011) for excellent textbooks on the subject.

[2] We choose the framework by Hellwig (1980) mainly because the results are classics and require relatively little introduction. However, we believe that our methodological approach is applicable more broadly and hope that it paves the way for a more general, quantitative evaluation of the noisy rational expectations literature. In addition to the noisy rational expectations literature cited above, our approach may also be applicable to a range of related dispersed information models in which agents learn from observing aggregate variables such as the capital stock or investment flows.

ductivity and can infer information about the signals received by others from observing the equilibrium stock price. The model has three main qualitative predictions: First, in a frictionless environment in which the trading motives of all economic agents are known, asset prices perfectly reveal all dispersed information (the Grossman (1976) equilibrium). Second, the trading activity of "noise traders" who randomly demand an unknown quantity of stocks interferes with the market's capacity to aggregate information and prevents the perfect revelation of information in equilibrium. When information is not perfectly revealed, households differ in their equilibrium expectation of future productivity. Third, noise trader demand that is solely contingent on variables that are observable by households does not affect equilibrium expectations as long as the equilibrium price function remains monotonic in the productivity shock.

In the first part of the paper we set up a real business cycle model with capital accumulation, flexible labor supply, and Epstein-Zin utility in which households receive private signals about a future productivity shock and can learn from the equilibrium stock price. In addition, outside "noise traders" engage in stochastic asset purchases in the stock market, as in Hellwig (1980).

We then introduce our solution method and formally derive the conditions under which the three main predictions of the classical noisy rational expectations model carry over to the quantitative model. The main insight from this analysis is that, under an appropriate assumption on the distribution of noise trader demand shocks, we can solve for households' equilibrium expectations as a function of only two variables: the standard deviation of noise trader demand and the standard deviation of idiosyncratic noise in households' private signal. Jointly, the two variables fully determine the equilibrium posterior distribution of households' expectations of the productivity shock. In fact, we are able to derive a monotonic transformation of noise trader demand such that households' equilibrium expectations take the same form as in the linear model.

As a result, the three main qualitative insights from the linear model seamlessly carry over to the quantitative model. In particular, the equilibrium stock price becomes perfectly revealing of the productivity shock if there are no noise traders. In addition, any policy interventions (such as open market operations) in the stock market do not affect the equilibrium informativeness of stock prices as long as they are fully contingent on observables.

In the third part of the paper we estimate and quantitatively evaluate the model. Relative to a conventional DSGE model, we have two additional parameters: the standard deviation of noise trader demand and the standard deviation of idiosyncratic noise in households' private signal. To get a benchmark value for the size of noise trader demand shocks, we calibrate their size to the asset purchase program of the Federal Reserve. While this may not be an accurate estimate, this benchmark value serves as an interpretable reference. We then choose the remaining parameter to match the cross-sectional variance of the expectation of annual GDP growth from the Survey of Professional Forecasters. Based on these estimates, we analyze the comparative statics in the

model. In a final step we estimate both the standard deviation of noise trader demand and the standard deviation of noise in the private signal to match a set of financial and business cycle moments.

We find that dispersed information about future productivity affects the quantitative results of our DSGE model in four ways. First, households' ability to learn about future productivity affects their behavior, and especially their consumption-savings decision. In our calibration, households receive private signals about the innovation to the long-run productivity shock one month in advance. In the limit in which the standard deviation of the noise trader demand shock goes to zero, the equilibrium stock price becomes perfectly revealing and all households hold the same expectation in equilibrium. We compare this limiting case of perfect information aggregation with a standard calibration in which households receive no information at all about future productivity and find that the equity premium falls by 1%, the risk-free interest rate rises by 2% when the stock price perfectly reveals the productivity shock.

Second, when noise trader demand shocks limit the stock market's capacity to aggregate information, the model cannot generate the kind of disagreement observed in the data for any reasonable size of noise trader demand shocks. For example, in our standard specification, the cross-sectional standard deviation in households' expectations of future GDP growth is 0.05%, compared to 0.27% in the survey of economic forecasters. Similarly, when we choose the precision of households' private signal to maximize the cross-sectional variance in household expectations, the cross-sectional variation in households' holdings of stocks does not exceed 0.04% in our standard specification.

Third, even moderate heterogeneity in household behavior affects the correlations and standard deviations produced by the model as well as the level of all economic aggregates. For example, the correlation between consumption growth and investment is 0.29 when households have no information about the future, 0.27 when they have perfect information about the future, but 0.41 in our standard specification.

Fourth, aside from their effect on information aggregation, noise trader demand shocks affect the model in their own right. When we choose the standard deviation of noise trader demand shocks to maximize the overall fit of the model we find that, with sufficiently (albeit unreasonably) large shocks, the model comes closer to matching the standard deviation of stock returns and stock prices, as well as the cross-sectional standard deviation of expectations of future GDP growth observed in the data.

This last finding illustrates a number of difficulties that arise when transporting the concept of noise traders into a general equilibrium setting. The first difficulty is that the non-fundamental asset demand of noise traders has no clear counterpart in the data. As a reference point, we calibrate the size of demand shocks by noise traders to trading activity by the Federal Reserve to get into a sensible range. In a different specification, we estimate noise trader shocks along

with other parameters of the model.

Second, noise traders effectively act as foreign agents, adding and subtracting resources from the economy at different points in time. When we change the standard deviation of noise trader demand shocks we also directly change the volatility of all macroeconomic aggregates, which makes comparative statics with respect to the standard deviation of noise trader demand shocks somewhat difficult to interpret.

Finally, heterogeneous classes of agents, and in particular non-maximizing agents such as noise traders, make it difficult to think about welfare. For example, it is unclear whether one should include noise traders in a welfare measure, and if so, how to calculate their utility. On the other hand, performing welfare analysis based on the utility of rational households alone may lead to counterintuitive results as wealth transfers occur between the two groups in equilibrium.

In a related paper (Hassan and Mertens, 2014) we address these issues by removing noise traders from the model altogether, and instead work only with a single class of "near-rational" households. Near-rational households are perfectly rational in all dimensions of their behavior, but make small correlated errors when forming their expectation about future productivity (they are on average slightly to optimistic in some states of the world and slightly too pessimistic in others). By the envelope theorem, such small errors are of little concern to an individual household as they have only a lower-order effect on its own utility. However, since these small errors are correlated across households, they affect the equilibrium stock price. As households (rationally) attempt to learn from the equilibrium price, the common component of households' errors is amplified and has a first-order external effect on the market's capacity to aggregate information. The model thus endogenously generates non-fundamental noise in the stock price through an information externality: a given household does not internalize the effect of its own error on the equilibrium expectation of other households in the economy. Having eliminated non-maximizing agents from the model, we are then able to address a number of welfare-related questions which we side-step in the current paper. In particular, we show that a deterioration of the market's capacity to aggregate information may cause large (first-order) aggregate welfare losses. Mertens (2009) analyzes a range of policies that may mitigate the information externality and improve the market's capacity to aggregate information.

In the interest of maintaining a maximum comparability to the most standard noisy rational expectations model we maintain the noise trader assumption in the current paper despite the difficulties with its normative interpretation and focus on the positive predictions of the model throughout.

The solution method we use in this paper generalizes a solution method we use in earlier work (Mertens, 2009; Hassan and Mertens, 2014). It builds on Judd (1998) and Judd and Guu (2001) in using an asymptotically valid higher-order expansion in all state variables around the deterministic steady state of the model in combination with a nonlinear change of variables (Judd

(2002)). Tille and van Wincoop (2013) independently develop a similar solution method.[3]

This paper is part of a large literature that strives for a richer representation of financial markets within otherwise standard DSGE models. Examples include Bernanke and Gertler (1989), Bernanke et al. (1999), Woodford and Curdia (2009), He and Krishnamurthy (2011), and Brunnermeier and Sannikov (2013). We add to this literature by providing a simple way to incorporate the notion that stock markets aggregate and transmit information into otherwise standard quantitative models.

Another closely related literature considers the effect of information about future fundamentals in the form of exogenous news shocks within DSGE models (Jaimovich and Rebelo, 2009; Barsky and Sims, 2011; Schmitt-Grohé and Uribe, 2012). Blanchard et al. (2013) estimate the relative contributions of noise shocks and information in the data. We add to this literature by providing a means to quantify the extent to which asset markets endogenously transmit news about future fundamentals.

The remainder of the paper is structured as follows. Section 2 introduces the classic noisy rational expectations model and describes the usual steps in solving the linear model. Section 3 sets up the quantitative DSGE model and shows how we can follow the same steps when solving for equilibrium expectations in the non-linear model. Section 5 quantitatively analyzes and estimates the model. Section 6 concludes.

2 The Classic Noisy Rational Expectations Model

The model economy exists at two points in time $t = 1, 2$. It is populated by a continuum of identical households indexed by $i \in [0, 1]$. At $t = 1$ each household is endowed with N units of a numéraire good which can be stored until $t = 2$ and K units of capital. At $t = 2$, each unit of capital returns η units of the numéraire

$$Y = \eta K, \ \eta \sim N\left(\bar{\eta}, \sigma_\eta^2\right). \tag{1}$$

Before trading commences, each household receives a private signal about productivity:

$$s_i = \eta + \nu_i, \tag{2}$$

where ν_i represents *i.i.d.* draws from a normal distribution with zero mean and variance σ_ν^2. Given s_i, each household chooses to purchase z_i units of capital to maximize expected utility from terminal wealth,

$$E_{1i}\left[U_i\right] = E_{1i}\left[w_{2i}\right] - \frac{\rho}{2} V_1[w_{2i}], \tag{3}$$

[3]Also see Tille and van Wincoop (2008) and Devereux and Sutherland (2011).

where $E_{1i}[\eta] = E[\eta|s_i, Q]$ and $V_1[w_{2i}]$ is the posterior variance of w_{2i} conditional on the private signal s_i and the stock price Q. The parameter $\rho > 0$ determines the degree of absolute risk aversion. Terminal wealth is given by

$$w_{2i} = z_i(\eta - Q) + N. \tag{4}$$

Households thus optimally demand

$$z_i = \frac{E_{1i}[\eta] - Q}{\rho V_1[\eta]} \tag{5}$$

units of capital.

Market clearing requires that aggregate asset demand is equal to aggregate supply:

$$\int_0^1 z_i di + T(Q) + \tau = K, \tag{6}$$

where $T(Q)$ reflects asset purchases by noise traders that follow some policy rule contingent on Q and τ are asset purchases by noise traders that are stochastic from the perspective of households. Throughout, we focus on symmetric equilibria.

Assumption L1 Contingent asset purchases by noise traders are a linear function of Q

$$T(Q) = b_0 + b_1 Q. \tag{7}$$

Assumption L2 Random asset purchases by noise traders follow the normal distribution

$$\tau \sim N\left(0, \sigma_\tau^2\right). \tag{8}$$

Assumption L3 Households can condition their expectation on prices (Q) but not on aggregate quantities ($\int_0^1 z_i di$).

2.1 Solving the Model

The standard approach to solving this "CARA-Gaussian" model proceeds in three steps. As the steps in the solution of the quantitative model are generalized versions of these same steps, we derive them in detail in order to show the analogy in the procedure.

Step 1: Guess Equilibrium Price Function

We guess that the equilibrium price function is linear in η and τ

$$Q = \pi_0 + \pi_1 \eta + \gamma \tau, \tag{9}$$

where π_0, π_1, and γ are endogenous coefficients to be solved below.

Step 2: Bayesian Expectation Formation

Plugging (5) into (6) and re-arranging terms yields

$$\hat{q} \equiv \int_0^1 E_{1i}\left[\eta\right] di + \tau \rho V_1[\eta] = \rho V_1[\eta]\left(K - T\left[Q\right]\right) + Q, \tag{10}$$

Note that all variables on the right hand side of the equation are known at $t = 1$. Observing Q is thus equivalent to observing the average expectation of η across households with some error $\tau \rho V_1[\eta]$. Without loss of generality we refer to this monotonic transformation of the stock price as $\hat{q}$.

Assuming that our guess in (9) is correct, we can show that the rational expectation of η given the private signal and $\hat{q}$ takes a linear form

$$E_{1i}[\eta] = \alpha_0 + \alpha_1 s_i + \alpha_2 \hat{q} \tag{11}$$

where the constants α_0, α_1 and α_2 are Bayesian weights that households give to the prior, the private signal, and the monotonic transformation of the market price of capital respectively.

Plugging the definition in (10) into (11), integrating on both sides, solving for $\int E_{1i}[\eta] di$, and then adding $\tau \rho V_1[\eta]$ on both sides yields

$$\hat{q} = \frac{\alpha_0}{1 - \alpha_2} + \frac{\alpha_1}{1 - \alpha_2}\eta + \frac{1}{1 - \alpha_2}\tau \rho V_1[\eta].$$

Observing the stock price is thus also equivalent to observing an unbiased signal of η

$$\left(\hat{q} - \frac{\alpha_0}{1 - \alpha_2}\right)\frac{1 - \alpha_2}{\alpha_1} = \eta + \frac{\rho V_1[\eta]}{\alpha_1}\tau. \tag{12}$$

We thus have a prior and two independent, unbiased signals of η. It follows that the posterior

variance is the inverse of the sum of the precision of these three elements[4]

$$V[\eta|s_i, Q] = V_1[\eta] = \left(\sigma_\eta^{-2} + \sigma_\nu^{-2} + \left(\frac{\alpha_1}{\rho V_1[\eta]}\right)^2 \sigma_\tau^{-2}\right)^{-1}. \tag{13}$$

Moreover, the conditional expectation of η is the precision-weighted sum of the signals and the prior mean divided by the posterior precision

$$E[\eta|s_i, Q] = \frac{\sigma_\eta^{-2}\bar{\eta} + \sigma_\nu^{-2} s_i + \left(\frac{\alpha_1}{\rho V_1[\eta]}\right)^2 \sigma_\tau^{-2}\left(\eta + \frac{\rho V_1[\eta]}{\alpha_1}\tau\right)}{V_1[\eta]^{-1}}. \tag{14}$$

Matching coefficients (14) with (11) yields

$$\alpha_1 = \frac{V_1[\eta]}{\sigma_\nu^2}. \tag{15}$$

Plugging into (13) yields

$$V_1[\eta] = \left(\frac{1}{\sigma_\eta^2} + \frac{1}{\rho^2 \sigma_\nu^4 \sigma_\tau^2} + \frac{1}{\sigma_\nu^2}\right)^{-1}. \tag{16}$$

Step 3: Verification of the price guess

Plugging (7), (14) and (15) into both sides of the second equality in (10), solving for Q and collecting terms by η and τ yields

$$Q = \frac{\left(\sigma_\eta^{-2}\bar{\eta} - \rho\left(K - b_0\right)\right) V_1[\eta]}{1 - \rho V_1[\eta] b_1} + \frac{\sigma_\nu^{-2}\left(1 + \rho^{-2}\sigma_\nu^{-2}\sigma_\tau^{-2}\right) V_1[\eta]}{1 - \rho V_1[\eta] b_1}\eta + \frac{\left(\rho^{-1}\sigma_\nu^{-2}\sigma_\tau^{-2} + \rho\right) V_1[\eta]}{1 - \rho V_1[\eta] b_1}\tau \tag{17}$$

Matching coefficients with (9), plugging in (16) and simplifying yields

$$\pi_1 = \frac{\sigma_\eta^2\left(\rho^2\sigma_\nu^2\sigma_\tau^2 + 1\right)}{\sigma_\eta^2 - \rho^2\sigma_\nu^2\sigma_\tau^2\left(\sigma_\eta^2(b_1\rho\sigma_\nu^2 - 1) - \sigma_\nu^2\right)} \text{ and } \quad \gamma = \rho\sigma_\nu^2\pi_1\ , \tag{18}$$

verifying the guess (9).

2.2 Three Classical Results

From (16) we can readily read off the three main qualitative predictions of the noisy rational expectations model.

[4]See Appendix A for a formal derivation using the projection theorem.

Result 1. *Absent stochastic noise trader demand, the stock price becomes perfectly revealing.*

$$\lim_{\sigma_\tau \to 0} V_1[\eta] = 0.$$

Result 2. *Noise trading reduces the market's capacity to aggregate information,*

$$\frac{\partial V_1[\eta]}{\partial \sigma_\tau} > 0.$$

When information aggregation is less than perfect, households put weight on their private signal and thus hold different expectations in equilibrium. (This follows directly from (15).)

Result 3. *Noise trader's contingent asset purchases have no effect on price informativeness. (Note that $T(Q)$ does not feature in (16).)*

3 A DSGE Model with Dispersed Information

We now generalize Results 1-3 to the context of a quantitative DSGE model. To this end we set up a de-centralization of the real business cycle model by Croce (2013). We choose this model because it performs well on matching quantities as well as asset prices and can readily be solved using perturbation methods. However, in principle, we believe our method is applicable to any DSGE model in which equilibrium policy functions are smooth.

3.1 Setup

Technology is characterized by a linear homogeneous production function that uses capital, K_t, and labor, N_t as inputs

$$Y_t = K_t^\alpha (e^{a_t} N_t)^{1-\alpha}, \tag{19}$$

where Y_t stands for output of the consumption good. The growth of labor productivity, $\Delta a_{t+1} = log(A_{t+1}/A_t)$, has a long-run risk component, ω, and a short-run risk component, φ,

$$\Delta a_{t+1} = \mu_a + \omega_t + \sigma_\varphi \varphi_{t+1}. \tag{20}$$

where the long-run component follows

$$\omega_t = \rho \omega_{t-1} + \sigma_\eta \eta_t. \tag{21}$$

Both shocks to productivity, φ and η, are i.i.d. standard normal.

The equation of motion of the capital stock is

$$K_{t+1} = (1 - \delta_k)K_t + I_t - G_t K_t \tag{22}$$

where I_t denotes aggregate investment and δ_k is the rate of depreciation. Furthermore, there are convex adjustment costs to capital following Jermann (1998)

$$G_t = \frac{I_t}{K_t} - \left(\frac{v_1}{1 - \frac{1}{\xi}} \left(\frac{I_t}{K_t} \right)^{1 - \frac{1}{\xi}} + v_0 \right), \tag{23}$$

where v_1, v_2, and ξ are positive constants.

A representative firm purchases capital and labor services from households. As it rents services from an existing capital stock, the firm's objective collapses to a period-by-period maximization problem

$$\max_{K_t, N_t} Y_t - D_t K_t - w_t N_t \tag{24}$$

where K_t and N_t denote factor demands for capital and labor respectively. First order conditions with respect to capital and labor pin down the market clearing wage,

$$w_t = (1 - \alpha)\frac{Y_t}{N_t} \tag{25}$$

and the rental rate of capital, $\alpha \frac{Y_t}{K_t}$. Both factors receive their marginal product. As the production function is linear homogeneous, the representative firm makes zero economic profits from producing the consumption good.

The representative firm owns an investment goods sector which converts the consumption good into units of capital, while incurring adjustment costs. It takes the price of capital as given and then performs instantaneous arbitrage:

$$\max_{I_t} Q_t \left(I_t - G_t K_t \right) - I_t \tag{26}$$

Taking the first order condition of (26), gives us the equilibrium price of capital.

$$Q_t = \frac{1}{1 - G_t'}. \tag{27}$$

Since there are decreasing returns to scale in converting consumption goods to capital, the investment goods sector makes positive profits in each period. Profits are paid to shareholders

as a part of dividends per share:[5]

$$D_t = \alpha \frac{Y_t}{K_t} + Q_t \left(G_t' \frac{I_t}{K_t} - G_t \right). \tag{28}$$

A continuum of households on the interval $i \in [0,1]$ has Epstein and Zin (1989) preferences over the consumption bundle $\tilde{C}_{it}$

$$U_{it} = \left((1-\delta)\tilde{C}_{it}^{1-\frac{1}{\psi}} + \delta E_{it} \left[U_{it+1}^{1-\gamma} \right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right)^{\frac{1}{1-\frac{1}{\psi}}}, \tag{29}$$

where the parameters ψ and γ measure the households' intertemporal elasticity of substitution and relative risk aversion, respectively. The consumption bundle $\tilde{C}_{it}$ is a CES aggregate of consumption and leisure

$$\tilde{C}_{it} = \left(o C_{it}^{1-\frac{1}{\xi_l}} + (1-o)(A_{t-1}(1-n_{it}))^{1-\frac{1}{\xi_l}} \right)^{\frac{1}{1-\frac{1}{\xi_l}}}, \tag{30}$$

where the weight of leisure scales with aggregate productivity, A, to ensure the existence of a balanced growth path. At the beginning of every period each household receives a private signal about the shock to long-run productivity:

$$s_{it} = \eta_{t+1} + \nu_{it}. \tag{31}$$

where ν_{it} represents *i.i.d.* draws from a normal distribution with zero mean and variance σ_ν^2.

Given s_{it} and their knowledge about the state of the economy, households maximize lifetime utility (29) by choosing a time path for consumption, labor, and their holdings of stocks $\{\tilde{C}_{it}, n_{it}, k_{it}\}_{t=0}^{\infty}$. Each household's optimization is subject to a budget constraint:

$$Q_t k_{it+1} = Q_{t-1} R_t k_{it} + H_{it} - C_{it} + w_t n_{it}, \tag{32}$$

where Q_t denotes the price of capital, H_{it} transfers from state-contingent claims discussed below, and w_t the wage rate. The returns to capital are defined as

$$R_{t+1} = \frac{(1-\delta_k)Q_{t+1} + \alpha \frac{Y_{t+1}}{K_{t+1}} + Q_{t+1}\left(G_{t+1}' \frac{I_{t+1}}{K_{t+1}} - G_{t+1} \right)}{Q_t}. \tag{33}$$

Foreign noise traders engage in open market operations in the stock market using an inde-

[5] Alternatively, profits may be paid to households as a lump-sum transfer; this assumption matters little for the quantitative results of the model.

pendent source of wealth. In each period they purchases $T^k(S_t) + T^\tau_{t+1}$ units of capital, where the first term follows a policy rule that is a function of the vector of observable state variables S_t, and the second term, T^τ_{t+1}, is random from the perspective of the households. The market clearing conditions for the stock, labor, and goods markets are

$$K_{t+1} = \int k_{it+1} di + T^k(S_t) + T^\tau_{t+1}, \tag{34}$$

$$N_t = \int n_{it} di, \tag{35}$$

and

$$Y_t + \varsigma_t = C_t + I_t \tag{36}$$

where ς_t stands for the net value of noise traders' purchases,

$$\varsigma_t = Q_t \left(T^k(S_t) + T^\tau_{t+1}\right) - Q_{t-1} R_t \left(T^k(S_{t-1}) + T^\tau_t\right). \tag{37}$$

Solving this model poses two main challenges. First, equilibrium quantities are non-linear functions of the underlying shocks. Maintaining the assumption of normally distributed shocks inevitably leads to a non-normal distribution of equilibrium variables. As a result, we can no longer solve for conditional distributions in closed form based on these quantities. Second, information dispersion results in a distribution of choices and state variables, which generates a non-degenerate distribution of wealth across households. In order to focus our attention on the first challenge, we side-step the second challenge by introducing contingent claims that households can use to insure against the idiosyncratic risk they face due to their private signal.[6]

At the beginning of each period (and before receiving their private signal), households can trade claims that are contingent on the state of the economy and on the realization of the noise they receive in their private signal, ν_{it}. These claims are in zero net supply and pay off at the beginning of the next period. As they are traded before any information about η_{t+1} is known, their prices cannot reveal any information about future productivity. Contingent claims trading thus completes markets between periods, without impacting households' signal extraction problem. In equilibrium all households choose to hold these securities with net payoff

$$H_{it} = \begin{cases} Q_{t-1} R_t \left(K_t - T^k[S_{t-1}] - T^\tau_t\right) - Q_{t-1} R_t k_{it} & \\ & \text{if } \left\{\tilde{C}_{it}, n_{it}, k_{it}\right\} = \arg\max \ (29)|_{H_{it}=0}, \\ 0 & \text{otherwise} \end{cases} \tag{38}$$

[6] See Mertens and Judd (2013) for a perturbation-based approach to solving incomplete markets models with substantial heterogeneity.

such that all households enter each period with the same amount of wealth. From (34) it follows immediately that these claims are in zero net supply

$$\int H_{it} di = 0. \tag{39}$$

3.2 Equilibrium

Definition 1. *Given a time path of shocks an equilibrium in this economy is a time path of quantities, signals, and prices with the following properties:*

1. *the time path of households' consumption, stock holdings, and labor supply maximizes (29) given the vector of prices and signals;*
2. *the demand for capital and labor services solve the representative firm's maximization problem (24) given the vector of prices;*
3. *the time path of capital investment solves the investment good sector's problem (26) given the vector of prices;*
4. *the time paths of stock prices, wages, and stock returns clear the markets for stocks, labor services, and capital services, respectively.*

We compute the first-order and envelope conditions for households in Appendix B.2. After taking the ratio of the first-order conditions with respect to labor and consumption, we get the marginal rate of substitution between labor supply and consumption

$$\frac{1-o}{o} \frac{A_{t-1}^{1-\frac{1}{\xi_l}} (1-n_{it})^{-\frac{1}{\xi_l}}}{C_{it}^{-\frac{1}{\xi_l}}} = w_t. \tag{40}$$

The optimal choice of stock holdings is determined by the familiar asset pricing equation

$$E_{it}[M_{it+1} R_{t+1}] = 1, \tag{41}$$

where the stochastic discount factor, M_{it+1}, is given by

$$M_{it+1} = \delta \left(\frac{C_{it+1}}{C_{it}} \right)^{-\frac{1}{\xi_l}} \left(\frac{\tilde{C}_{it+1}}{\tilde{C}_{it}} \right)^{\frac{1}{\xi_l} - \frac{1}{\Psi}} \left(\frac{U_{it+1}}{E_{it}\left[U_{it+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right)^{\frac{1}{\Psi} - \gamma}, \tag{42}$$

and returns R_{t+1} are defined in (33). Given these conditions of optimality, consumption follows from the household's budget constraint (32).

For convenience we write the average of stock holdings across households as

$$\int k_{it+1} di = \bar{K}_{t+1}, \tag{43}$$

such that we can re-write the market clearing condition for capital (34) as $K_{t+1} = \bar{K}_{t+1} + T^k(S_t) + T^{\tau}_{t+1}$.

The labor market clears when condition (35) holds. Again for convenience we write aggregate consumption and labor supply as

$$X_t = \int X_{it} di, \quad X = C, N. \tag{44}$$

Note that noise trader demand shocks affect this system of equations via the resource constraint (36) which determines the amount of aggregate investment. Investment, in turn, influences the price of capital Q via equation (27).

3.3 Solving the model

In this section we generalize the three solution steps for the linear model in section 2 to solve the quantitative model with dispersed information. The main trick is again to define a statistic $\hat{q}$ that aggregates all relevant information about η_{t+1}. We conjecture that this statistic takes analogous form to the linear model

$$\hat{q}_t \equiv \bar{E}_t + \frac{V_1[\eta]}{\sigma^2_{\eta}} \tau_t. \tag{45}$$

where $\bar{E}_t = \int E_{it}[\eta_{t+1}] di$ denotes the average expectation of η_{t+1} across households. We guess that the equilibrium stock price is a function of known state variables of the economy and $\hat{q}$, where the mapping from $\hat{q}$ into Q can now be non-linear. Using this guess, we show how households form expectations in the non-linear model. We then show that the signal extraction problem remains linear and Gaussian and can thus be solved in closed form. As an immediate consequence, the three main theoretical results from the linear setting carry over to the dynamic economy. In a third step, we confirm our initial guess and show that we find a recursive equilibrium. To complete each of these steps, we need three assumptions, which generalize each of the assumptions L1-L3 in the linear model of section 2.

Assumption Q1 Contingent stock purchases by noise traders are a function of known state variables

$$T^k_t = T^k(S_t),$$

where

$$S_t = \{K_t, T_t, \omega_{t-1}, \eta_t, \varphi_t\}.$$

Assumption Q2 The stochastic component of noise trader demand T^τ, is a function of a normally distributed shock to noise trader demand $\tau \sim N(0, \mu)$ and follows a state-dependent distribution such that the following property holds

$$\hat{I}(S_t, \bar{E}_t, \hat{q}_t) + Q_t T^\tau(S_t, \hat{q}_t, \bar{E}_t, \tau_t) = I(S_t, \hat{q}_t), \tag{46}$$

where $\hat{I}_t = Y_t - Q_{t-1} R_t T_t^\tau - C_t$ is the "fundamental" component of aggregate investment.

Just like we needed stochastic noise trader demand to be a linear function of a normally distributed shock in section 2, we now need it to be a particular (non-linear) function of a normally distributed shock. This function adds noise to the second argument of $\hat{I}$ such that aggregate investment (and thus the equilibrium stock price) is a function of only $\hat{q}$ and S_t. We show below that this assumption boils down to assuming that the T^τ is a particular multivariate normal polynomial of τ.

Assumption Q3 When forming expectations about future productivity households condition their expectations on their private information, the equilibrium stock price, and the commonly known state variables of the economy S_t but not on other quantities or prices.

$$E_{it}[\eta_{t+1}] = E[\eta_{t+1} | S_t, s_{it}, Q_t] \tag{47}$$

As L3 in the linear model, this assumption rules out that households condition their expectation on the "fundamental" demand for stocks or any quantity that reflects this information. In the context of this general equilibrium model it also implies that households cannot condition their expectation of η on the wage rate or other prices. Households in our model thus act as "Lucas households" where the member of the household charged with forecasting η_{t+1} observes S_t, s_{it} and Q, forms her expectation of based on this information, and then communicates it to another household member who decides on the household's labor supply.

We make this assumption mainly for convenience. In the more general case in which households can condition their expectation on two prices, the model simply requires an additional source of non-fundamental noise.

In addition to assumptions Q1-Q3 we require the following condition.

Condition 1. *The equilibrium stock price, Q, is a strictly monotonic function of $\hat{q}$.*

This condition typically holds in a wide range of models. It states that, other things equal, the stock price either strictly increases or strictly decreases when households expect higher productivity in the future. While we cannot solve for the mapping of $\hat{q}$ into Q in closed form, we verify this condition ex-post using the numerical solution of the model.

3.3.1 Step 1: Guess for state space

Analogously to the linear case, we start by guessing the form of equilibrium variables. The solution of the model is a set of equilibrium quantities and prices which are functions of the underlying state variables. We split these state variables into the commonly known set, S_t, and a set that depends on expectations about η.

Lemma 1. *There exists a recursive equilibrium satisfying the system of equations defined in section 3.2 with the following properties:*

1. *The equilibrium stock price, Q_t, is a function of the current (commonly known) state of the economy, S_t, and $\hat{q}_t$.*

2. *All other prices and aggregate quantities depend on S_t, the average expectation, $\bar{E}_t$, and $\hat{q}_t$*

$$X_t = X(\bar{S}_t), X = C, N, \bar{K}', w, R \tag{48}$$

where we abbreviate $\bar{S}_t = \{S_t, \hat{q}_t, \bar{E}_t\}$.

3. *A household's optimal behavior is a function of S_t, the household's conditional expectation of next period's innovation to long-run productivity $E_{it}[\eta_{t+1}]$, the average expectation, $\bar{E}_t$, and the demand statistic $\hat{q}_t$. The conditional expectation, in turn, depends on the private signal s_{it} as well as the demand statistic $\hat{q}_t$.*

$$x_{it} = x(S_{it}), x = c, n, k. \tag{49}$$

where, for convenience, we write $S_{it} = \{\bar{S}_t, E_{it}\}$. In a slight abuse of notation, $E_{it} = E_{it}[\eta_{t+1}]$.

In the remainder of the section, we verify the validity of the lemma.

Step 2: Bayesian Expectation Formation

Given Lemma 1, we invoke condition 1 (which ensures that the stock price is a strictly monotonic function of $(\hat{q})$ and assumption Q3 to arrive at the main insight of our solution method: learning from the stock price Q is just as informative as learning from its monotonic transformation, $\hat{q}$. Since there is a one-to-one mapping between the stock price and $\hat{q}$ conditional on all commonly known state variables, S_t, households can infer the only unknown quantity $\hat{q}$ from observing the stock price. In other words, conditioning on Q and $\hat{q}$ spans the same σ-algebra.

$$E_{it}[\eta_{t+1}] = E[\eta_{t+1}|S_t, s_{it}, Q_t] = E[\eta_{t+1}|S_t, s_{it}, \hat{q}_t] \tag{50}$$

The signal extraction problem is thus isomorphic to the one in section 2, despite the non-linear nature of the quantitative model. As a result, we can solve for equilibrium expectations independently of the aggregate dynamics of the model.

Lemma 2. *Households' equilibrium expectations of η_{t+1} are independent of the aggregate dynamics of the model. Conditioning on the (normally-distributed) demand statistic $\hat{q}$ results in expectations of the form*

$$E_{it}[\eta_{t+1}] = \frac{\sigma_\nu^{-2}\left(\eta_{t+1} + \nu_{it}\right) + \frac{\sigma_\eta^4}{\sigma_\nu^4 \sigma_\tau^2}\left(\eta_{t+1} + \frac{\sigma_\nu^2}{\sigma_\eta^2}\tau_t\right)}{V_t[\eta_{t+1}]^{-1}}. \tag{51}$$

Moreover, the conditional expectation of η_{t+1} is the precision-weighted sum of the signals and the prior mean divided by the posterior precision,

$$V_t[\eta_{t+1}] = \left(\sigma_\eta^{-2} + \sigma_\nu^{-2} + \frac{\sigma_\eta^4}{\sigma_\nu^4}\sigma_\tau^{-2}\right)^{-1} \tag{52}$$

The coefficient on the private signal is given by

$$\alpha_1 = \frac{V_t[\eta_{t+1}]}{\sigma_\nu^2} \tag{53}$$

and the coefficient on $\hat{q}$ is given by

$$\alpha_2 = \frac{1}{\sigma_\eta^{-4}\sigma_\nu^2\sigma_\tau^2 + 1}. \tag{54}$$

Proof: *See Appendix B.5.*

As an immediate consequence, we get the following corollary.

Corollary 1. *Given condition 1 and assumptions Q1-Q3, results 1-3 hold in the dynamic economy.*

Corollary 1 shows that the main insights from the linear setting survive in the non-linear economy. First, without stochastic shocks to noise trader demand, the economy becomes perfectly revealing. In the context of a dynamic economy, this result means that households know innovations to the long-run component of productivity one period in advance. The expectations held by households then act as they would in a "news shocks" model.

The second result states that stochastic shocks to noise trader demand increase the conditional variance of η_{t+1}. As a result, stock prices and all economic aggregates reflect both information about the future and noise. Moreover, when information aggregation is less than

perfect, households put weight on their private signal and thus hold heterogeneous expectations in equilibrium. Heterogeneity in equilibrium expectations in turn generates cross-sectional variation in consumption, investment, and labor supply decisions.

Third, noise trading will only have an impact on expectations and information aggregation insofar as it adds noise to the price process. Noise trades based on observable variables will not impact equilibrium expectations.

Since the part of noise trading conditioning only on observable variables, T^k, does not affect equilibrium expectations, we set it to zero for the remainder of the paper.

3.3.2 Step 3: Verification of guess

While the previous two steps give us a framework for thinking about the equilibrium, we need to make sure that the guess for the equilibrium structure is indeed correct. We proceed in three steps that demonstrate the consistency of each of the three statements in Lemma 2.

The second and third statements in Lemma 1 posit that a household's consumption, labor supply, and investment decisions can only depend on the variables included in the vector $S_{it} = \{\bar{S}_t, E_{it}\}$, while all aggregate variables and prices are determined by $\bar{S}_t$. Assuming that this is indeed the case, we can write any individual choice as a function $x_i(S_{it})$ and all aggregate variables and prices as functions as $X\left(\bar{S}_t\right)$. Plugging this structure into the model's equilibrium conditions and re-arranging terms results in a form

$$g_l(S_{it}) = E_{it}\left[g_r(S_{it}, S_{it+1})\right], \tag{55}$$

where $g_l(\cdot)$ and $g_r(\cdot)$ are analytic, continuously differentiable functions and we have used the fact that $\bar{S}_t \subset S_{it}$. (See Appendix B.4 for the full list of equilibrium conditions and analytical details.)

Since the functions $x_i(S_{it})$ are the solution to the system of equations (55), the key to showing the first part of Lemma 1 is to show that we can re-write $E_{it}\left[g_r(S_{it}, S_{it+1})\right]$ as a function of S_{it}. If all equilibrium conditions depend only on S_{it}, the same must be true for the the optimal behavior that is derived from these conditions. The proof follows from Taylor's theorem and the properties of the multivariate normal distribution.

We begin by re-writing g_r as a function of all elements of S_{it}.

$$\begin{aligned} g_r\left[S_{it}, K_{t+1}, T_{t+1}, \omega_t, \eta_{t+1}, \varphi_{t+1}, \hat{q}_{t+1}, \bar{E}_{t+1}, E_{it+1}\right] \\ &= \hat{g}_r\left[S_{it}, K_{t+1}, \tau_{t+1}, \omega_t, \eta_{t+1}, \varphi_{t+1}, \hat{q}_{t+1}, \bar{E}_{t+1}, E_{it+1}\right] \\ &= \sum_{\mathbf{j}} \frac{c_{\mathbf{j}}(S_{it})}{\mathbf{j!}} (K_{t+1} - K_0)^{j_1} \tau_{t+1}^{j_2} \omega_t^{j_3} \eta_{t+1}^{j_4} \varphi_{t+1}^{j_5} \hat{q}_{t+1}^{j_6} \bar{E}_{t+1}^{j_7} E_{it+1}^{j_8} \end{aligned}$$

The first equality follows from assumptions Q1 and Q3, which implies that that we can re-write T_{t+1} as a function of state variables known in period t and the noise shock τ_{t+1}. For the second equality we replace g_r with its infinite-order Taylor series expansion, where K_0 is the level of capital at the deterministic steady state, $c_{\mathbf{j}}(S_{it})$ denotes the (state-t dependent) coefficients of the Taylor series, and $\mathbf{j} = (j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8)$ is a multi-index for the expansion. (Note that to save space we have absorbed all the terms depending on S_{it} into the coefficients of the expansion.)

Now we take the expectation of g_r conditional on S_t, s_{it}, and Q_t. The terms depending on K_{t+1} and ω_t are known at time t and can thus be taken outside the expectations operator. Moreover, we get a series of terms depending on the conditional expectation of φ_{t+1}. Since φ_{t+1} is unpredictable for an investor at time t and all shocks are uncorrelated with each other, the first-order term is 0, and all the higher-order terms are known moments of the unconditional distribution of φ $E_{it}[\varphi^j_{t+1}] = E[\varphi^j_{t+1}]$ (recall that all higher order moments of the normal distribution are fully determined by its mean and variance). The same is true for the terms depending on $E_{it}[\hat{q}^j_{t+1}] = E[\hat{q}^j_{t+1}]$, $E_{it}[\bar{E}^j_{t+1}] = E[\bar{E}^j_{t+1}]$, and $E_{it}[E^j_{it+1}] = E[E^j_{it+1}]$, as all three arguments are normally distributed and have an unconditional mean and variance that follows directly from Lemma 2. The only terms remaining inside the expectations operator are then those depending on τ_t and η_{t+1}. We can thus write

$$\begin{aligned} E_{it}\left[\hat{g}_r[S_{it}, S_{it+1}]\right] &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\hat{c}_{jk}(S_{it}, K_{t+1}, \rho\omega_{t-1} + \sigma_\eta \eta_t)}{j!k!} E_{it}[\tau_t^j \eta_{t+1}^k] \\ &= g_l(K_t, T_t, \omega_{t-1}, \eta_t, \varphi_t, \hat{q}_t, \bar{E}_t, E_{it}), \end{aligned} \tag{56}$$

where the coefficients $\hat{c}_{jk}(S_{it}, K_{t+1}, \omega_t)$ collect all the terms depending on K_{t+1}, ω_t, and the higher unconditional moments of η_{t+1}, $\bar{E}_{t+1}$, and E_{it+1}.

To deal with the remaining conditional term, $E_{it}[\tau_t^j \eta_{t+1}^k]$, we use the fact that the two random variables are independent. As a result, their monomials τ_t^j and η_{t+1}^k do not co-vary and we can calculate them independently.

$$E_{it}[\tau_t^j \eta_{t+1}^k] = Cov_{it}\left[\tau_t^j, \eta_{t+1}^k\right] + E_{it}[\tau_t^j] E_{it}\left[\eta_{t+1}^k\right] = E_{it}[\tau_t^j] E_{it}\left[\eta_{t+1}^k\right]$$

Now note from Lemma 2 that η_{t+1} is conditionally normal distributed, where its mean and variance are given by (51) and (52), respectively. All higher moments, i.e. the conditional expectation of monomials η^j_{t+1} for $j > 2$ are fully determined by the conditional mean and variance. Since the conditional variance does not vary over time, we merely have to keep track of the conditional mean to recover the entire conditional distribution.

Similarly, the conditional distribution of τ_t is normal as we show in Appendix B.5. The conditional variance is again constant and the conditional mean of τ_t is simply a function of $\hat{q}$

and the conditional mean of η_{t+1}. From (27), we can write

$$E_{it}[\tau_t] = (1-\alpha_2)\hat{q}_t - \alpha_1 E_{it}[\eta_{t+1}]. \tag{57}$$

Finally, in deriving the set of individual state variables, we notice that contingent claims trading eliminates any meaningful distribution of capital across time, such that any heterogeneity in equilibrium behavior arises exclusively due to cross-sectional variation in E_{it}. It follows that we can write $E_{it}\left[g_r(S_{it}, S_{it+1})\right]$ as a function of S_{it}, concluding the proof of the first statement in Lemma 1.

The second part of Lemma 1 states that aggregate quantities and prices depend on S_{it}, $\bar{E}_t$, and $\hat{q}$. To see this, consider an aggregate variable of the form

$$X(\bar{S}_t) = \int x_i(S_{it})di \tag{58}$$

where X can represent aggregate capital (43), labor, or consumption (44). Again, without loss of generality we re-write the right hand side of the equation as an infinite order Taylor series expansion in the elements of S_{it}

$$\int x_i(S_{it})di = \int \sum_{\mathbf{j}} \frac{c_{\mathbf{j}}}{\mathbf{j}!}(K_t - K_0)^{j_1}\omega_{t-1}^{j_2}\eta_t^{j_3}\varphi_t^{j_4}\hat{q}_t^{j_5}\bar{E}_t^{j_6}E_{it}^{j_7}di. \tag{59}$$

Only the last term differs across households and thus all other variables can be taken outside the integral. The integral over E_{it} can be rewritten as

$$\int E_{it}^j = \int (E_{it} - \bar{E}_t + \bar{E}_t)^j di = \sum_{k=0}^{j} \left(\begin{array}{c} j \\ k \end{array} \right) \bar{E}_t^{j-k} \int (E_{it} - \bar{E}_t)^k di \tag{60}$$

Now we can use the fact that $E_{it} - \bar{E}_t = \alpha_1 \nu_i$, such that we can replace the last remaining term in the integral with the unconditional moments of ν_i, which are known. Therefore, all aggregate variables and prices are a function of $\hat{q}_t$ and $\bar{E}_t$, but not a function of E_{it} such that equation (58) holds.

The first part of Lemma 1 states that Q_t depends only on S_t and $\hat{q}$ but not on the average expectation, $\bar{E}_t$. We can see from (27) and (23) that Q_t is fully determined by I_t/K_t. Since K_t is itself a commonly known state variable (a part of S_t), it follows that the equilibrium stock price function will depend on the same arguments as aggregate investment. From assumption Q2 we know that the state dependent distribution of noise trader demand is such that $I_t = I(S_t, \hat{q}_t)$ and therefore $Q_t = Q(S_t, \hat{q}_t)$, concluding the proof of the Lemma.

T^τ has a state dependent distribution such that the noise trading shock τ adds linearly to

the average expectation, thus forming. The only remaining question is then what this state-dependent distribution of noise trader demand must look like and how it can be computed. The resource constraint in equation (46) links aggregate variables and noise trading with investment.

To compute the state-dependent distribution explicitly, we re-write (46) as

$$\hat{I}(S_t, \bar{E}_t, \hat{q}_t) + \hat{T}^\tau(S_t, \hat{q}_t, \bar{E}_t, \tau_t) = I(S_t, \hat{q}_t), \tag{61}$$

where $\hat{T}^\tau \equiv T^\tau / Q_t$. We can then replace the expressions on both sides of the equation with their Taylor series and iteratively match the coefficients of $\hat{T}^\tau$ such that the equation holds.

$$\hat{T}^\tau_0 + \hat{T}^\tau_S \hat{S}_t + \hat{T}^\tau_{\bar{E}} \bar{E} + \hat{T}^\tau_{\hat{q}} \hat{q} + \hat{T}^\tau_\tau \tau_t + \ldots + \hat{I}_0 + \hat{I}_S \hat{S}_t + \hat{I}_{\bar{E}} \bar{E} + \hat{I}_{\hat{q}} \hat{q} + \ldots = I_0 + I_S \hat{S}_t + I_{\hat{q}} \hat{q} + \ldots, \tag{62}$$

where we denote F_x as the derivative of the function F with respect to x. In particular, F_S is the vector of derivatives of the F with respect to each element of the vector of commonly known state variables S_t and $\hat{S}_t$ refers to deviations of the elements of S_t from their deterministic steady state values. To a first order, the implied distribution of $\hat{T}^\tau_t$ only needs to add noise to average expectation,

$$\hat{T}^\tau_0 = 0 \qquad \hat{T}^\tau_S = 0 \qquad \hat{T}^\tau_{\bar{E}} = 0 \qquad \hat{T}^\tau_{\hat{q}} = 0 \qquad \hat{T}^\tau_\tau = \hat{I}_{\bar{E}} \frac{V_t[\eta_{t+1}]}{\sigma^2_\eta} \tag{63}$$

which then determines aggregate investment as a function of known state variables and $\hat{q}$, such that

$$I_0 = \hat{I}^\tau_0 \qquad I_S = \hat{T}^\tau_S \qquad I_{\hat{q}} = \hat{I}_{\hat{q}} + \hat{T}^\tau_{\hat{q}} + \hat{I}_{\bar{E}} + \hat{T}^\tau_\tau. \tag{64}$$

The matching of higher-order coefficients is analogous.

Using these insights we solve the model using standard perturbation techniques. Perturbation methods approximate equilibrium policy functions by their Taylor series around the deterministic steady-state. To arrive at the coefficients of the Taylor series, we bring all equilibrium conditions into the appropriate form shown in equation (55). Successively differentiating the equation, evaluating at the steady-state, and solving the resulting system of equations for the coefficients in the Taylor series delivers the approximate solutions for the equilibrium policy functions and prices.

4 Qualitative Results

Upshot from the analysis above is that given Assumptions Q1-Q3, we can separate the solution of the dynamic model from its information microstructure. This allows us to characterize the the dynamic effects of dispersed information with some generality.

Contrary to a representative agent setting, the model with dispersed information leads to a

cross-sectional distribution of choices. The guess for equilibrium functions holds that individual expectations determine individual choices. Hence, a key statistic for the cross-sectional dispersion of choices is the cross-sectional variance of posterior expectations

$$V_t\left[E_{it}\left[\eta_{t+1}\right]-\int E_{it}\left[\eta_{t+1}\right]di\right]=\alpha_1^2\sigma_\nu^2. \tag{65}$$

Recall from Lemma 1 that households' policy functions can be characterized as a function of the state vector $\{S_t,\hat{q}_t,\bar{E}_t,E_{it}\}$. Without loss of generality we can use equation (45) and the fact that $E_{it}[\eta_{t+1}]=\alpha_0+\alpha_1 s_{it}+\alpha_2\hat{q}_t$ to re-write this state vector as $\{S_t,\tau,\bar{E}_t,\nu_{it}\}$. Without using any further information about the information microstructure we can then use a standard perturbation approach to solve for all endogenous variables of the model as a function of these state variables, the coefficients α_1, α_2, and the conditional variance $V_t\left[\eta_{t+1}\right]$. Equation (66) shows the first-order terms of the equilibrium aggregate consumption function,

$$\begin{aligned} C_t \;=\; & \underbrace{c_0+c_S'\hat{S}_t+\frac{1}{2}c_{\sigma^2}}_{\text{standard RBC}}+\underbrace{c_\tau\tau+\frac{1}{2}c_{\sigma^2}^{NT}\sigma_\tau^2}_{\text{direct effect of noise trading}}+\underbrace{c_{\bar{E}}\eta_{t+1}+\frac{1}{2}c_{\sigma^2}^{ns}}_{\text{News Shock}} \\ & \underbrace{+c_{\bar{E}}\left(-\left(1-(\alpha_1+\alpha_2)\right)\eta_{t+1}+\alpha_2\frac{V_t\left[\eta_{t+1}\right]}{\sigma_\eta^2}\tau\right)+\frac{1}{2}c_{\sigma^2}^{disp}\alpha_1^2\sigma_\nu^2}_{\text{Dispersed Information}}+\text{h.o.t.} \end{aligned} \tag{66}$$

where the coefficients denoted c on the right hand side are numbers that are determined by the usual "macroeconomic" parameters of the model but independent of the parameters relating to information and noise. In other words, any representative agent DSGE model implies such a series of c's, where we denote c_x as the derivative of the equilibrium aggregate consumption function with respect to x. In particular, c_S is the vector of derivatives of the aggregate production function with respect to each element of the vector of commonly known state variables S_t and $\hat{S}_t$ refers to deviations of the elements of S_t from their deterministic steady state values.

The grouping of terms on the right hand side in (66) isolates the effect of the information microstructure on aggregate consumption. (The effects on all other aggregate variables have the same form, except that the coefficients c will be different when we consider output, investment, or aggregate labor supply.) The first three terms on the right hand side give the solution for aggregate consumption that would obtain in a standard RBC model in which there is no information about the future $\sigma_\nu=\infty$ and no noise trading $\sigma_\tau=0$. $c_0+c_S'\hat{S}_t$ is the first-order Taylor series in deviations of the commonly known state variables from their deterministic steady state values and $\frac{1}{2}c_{\sigma^2}$ is the effect of risk on the level of consumption in the standard RBC model.

The second set of terms gives the direct effect of noise trading. Even when households learn nothing about the future ($\sigma_\nu=\infty$), τ has a direct effect on consumption (and all business

cycle moments) as asset purchases and sales of stocks effectively add and subtract resources from the economy. The second term in the grouping shows that when noise traders are active, they effectively increase the risk faced by households and thus add to the level effect of risk on consumption.

The third grouping shows the effect of information provision in a model in which $\sigma_\nu = 0$. If households are perfectly informed about η_{t+1} they fully react to this information in t, altering the dynamic response of economic aggregates to the long-run productivity shock. In addition, learning about the future reduces the uncertainty faced by households and thus may mitigate the effect of risk on the level of economic aggregates (the second term in the grouping).

The fourth grouping gives the dynamic and level effects of dispersed information. When information is dispersed and $\sigma_\nu > 0$, households need to rely on $\hat{q}_t$ when learning about η_{t+1}. When $\sigma_\tau > 0$ this inference is not perfect such that $(\alpha_1 + \alpha_2) < 1$. Dispersion in households' information then dampens the reaction of economic aggregates to η_{t+1} (see the first term in the round brackets). Instead, when $\alpha_2 > 0$, households also react to τ when making their equilibrium consumption, labor supply and investment decisions (the second term in the round brackets).

In addition, when $\alpha_1 > 0$, households rely on their (now noisy) private signal when making decisions such that equilibrium consumption will differ across households. Since households' policy functions are non-linear, this dispersion in equilibrium behavior has itself a level-effect on all economic aggregates. In particular, if individual households' consumption is concave in $E_{it}\left[\eta_{t+1}\right]$, the dispersion in household expectations lowers the level of aggregate consumption.

5 Quantitative Analysis

5.1 Calibration and Estimation

As is standard practice in the long-run risk literature, we calibrate our model to a monthly frequency and focus on time-aggregated statistics matching the behavior of annualized moments for the US economy 1929-2008 (Bansal and Yaron, 2004).

Our model depends on 15 parameters, 13 of which are identical to the parameters in the original real business cycle model by Croce (2013). In order to focus on the incremental effect of dispersed information within this framework we set all of these 13 parameters to equal the ones in Croce's benchmark calibration.

Table 1 lists the parameter values. On the technology side, α is set to match the capital income share and δ_k is set to match the annualized capital depreciation rate in the US economy (6%). The elasticity of the adjustment cost function, ξ, is set to 7, such that investment can be sufficiently volatile. In addition, Croce sets $\mu_a = 0.0015$ to yield an annual average growth of 1.8%. The long-run component in productivity, ω_t, is calibrated so as to be relatively small but

Table 1: Calibration

Parameter	Value	Parameter	Value
α	0.34	$12\ \delta$	0.060
β^{12}	0.95	γ	10.
$12\ \mu$	0.018	Ψ	2.0
ρ^{12}	0.80	o	0.20
ξ	7.0	ϕ_{lev}	2.0
$\sqrt{12}\sigma_a$	0.033	$\sqrt{12}\sigma_\eta$	0.0036
$\sqrt{12}\sigma_\tau$	0.0051		

Notes: Values of the 13 technology and preference parameters used in all following calibrations and estimations. α: capital share; δ_k: capital depreciation rate; ξ: elasticity of the adjustment cost function; μ_a: average growth of productivity; ρ: auto-correlation of long-run component in productivity growth; σ_η: standard deviation of shock to the long-run component in productivity growth; σ_φ: standard deviation of short-run component in productivity; ξ_l: consumption bundle elasticity; o consumption share in the consumption bundle; γ: relative risk aversion; ψ: intertemporal elasticity of substitution; δ: subjective discount factor and ϕ_{lev}: leverage of market return.

persistent, where ρ is set to yield a conservative and empirically plausible annualized persistence of 0.80. To keep the long-run component as small as in the data, he imposes $\sigma_\eta = 10\%\ \sigma_\varphi$, where σ_φ is set as 1% to obtain an annual volatility of output growth of 3.34% under the benchmark calibration.

On the preference side, the consumption bundle elasticity is $\xi_l = 1$ for simplicity. The parameter o is calibrated so that the labor share is 18% at the steady state, as in Tallarini (2000). The relative risk aversion and the intertemporal elasticity of substitution are set to values of 10 and 2, respectively. The annualized subjective discount factor δ is fixed at 0.95.

We also follow Croce (2013) in calculating excess stock returns as the excess returns on a levered claim to capital

$$R_{ex,t}^{LEV} = \left(R_t - R_{t-1}^f\right)^{\phi_{lev}}. \tag{67}$$

This again is standard practice in the finance literature as, in the data, most claims to equity are levered, where we set $\phi_{lev} = 2$, consistent with the amount of financial leverage measured by Rauh and Sufi (2012).

Finally, the parameters v_1 and v_2 in the adjustment cost function are set such that, at the deterministic steady state, $G_t = 0$ and $\partial G_t / \partial \left(I_t/K_t\right) = 0$. This implies that $v_0 = \left(\frac{1}{1-\xi}\right)(\delta + e^\mu - 1)$ and $v_1 = (\delta + e^\mu - 1)^{\frac{1}{\xi}}$.

Before the introduction of dispersed information, our calibration thus exactly coincides with the one in Croce (2013).[7] The two remaining parameters, σ_ν and σ_τ determine the size of

[7]The only way in which we deviate from his framework is that we do not introduce exogenous cash-flow shocks to levered returns (67). We omit this detail for clarity as our objective here is not to argue for or against the ability of the long-run risk model to match the volatility of stock returns in the data.

noise trader demand shocks, the dispersion in private signals, and therefore the equilibrium expectations of all households. We estimate both parameters to minimize the loss function

$$(m-\theta)\,W\,(m-\theta) \tag{68}$$

where m is a vector of moments generated by the model, θ is the vector of data targets for these moments, and W is a diagonal matrix where each entry is $\frac{1}{\theta_j^2}$, and θ_j is the j'th entry in θ. We thus minimize a weighted sum of squared deviations from the target vector.

We first present a set of results for which we calibrate noise trader demand shocks, σ_τ, to a reasonable ballpark and estimate only σ_ν. In these specifications we choose σ_τ such that the unconditional variance of T^τ equals one half of the standard deviation of the growth rate of asset purchases by the Federal Reserve between 1948 and 2008 (0.43% of GDP). In a second set of results we freely estimate both σ_τ and σ_ν to minimize the deviation from the target moments given in equation (68).

5.2 Results

Table 2 shows our basic results for a range of parameters σ_ν and σ_τ. Throughout, lowercase variables denote logs, $\sigma(\cdot)$ and $\text{corr}(\cdot)$ denote standard deviations and correlations, respectively, and d stands for the first difference in the time series. (For example, $\sigma(dy)$ stands for the standard deviation of output growth.) Column 2 lists the data targets and the remaining columns list the moments generated by the model. All data targets except the one for $\sigma_{xs}(E[dy])$ are generated using annual data ranging from 1929-2008 following the procedure in Croce (2013) (see appendix C.1 for details). $\sigma_{xs}(E[dy])$ is our measure of the cross-sectional dispersion of equilibrium expectations. We calculate it as the time-series average of the cross-sectional standard deviation in annualized forecasts of GDP growth in Survey of Economic Forecasters for the period 1969-2008.

The two remaining moments in the second panel have no direct counterpart in the data. $\sigma(k_{it}/K_t)$ is the time-series average of the cross-sectional standard deviation in equity portfolio holdings, which is a simple measure of the dispersion in equilibrium behavior induced by dispersion in equilibrium expectations. In the data, differences in behavior due to differences in expectations are of course conflated with large differences in wealth across households, such that we have no obvious moment in the data to compare our model's prediction to. The same applies to our measure of information revelation, $V_t[\eta]/\sigma_\eta^2$, calculated as the ratio of the posterior variance of η, divided by its unconditional variance. When this ratio is 1, no information is revealed in equilibrium. When it is zero, households can perfectly infer η_{t+1} one period in advance.

Column 3 shows Croce's original calibration in which households receive no information about the future productivity shock ($\sigma_\nu = \infty$). In this limit, our model coincides with a standard DSGE model in which all shocks are unknown ex-ante. Column 4 shows the opposite extreme in

Table 2: Common vs. Dispersed Information

	Data	'No Info'	'News Shock'	'Grossman Eq.'	'Dispersed Info'
$\sigma(T^{\tau})/Y^{\mathrm{SS}}(\%)$		0.	0.	0.	0.43
$\sigma_{\nu}/\sigma_{\eta}$		∞	0.	0.	3.2
$V_t[\eta]/{\sigma_{\eta}}^2$		1.	0.	0.	0.81
$\sigma(k_i/(K_{\mathrm{SS}}))(\%)$		0.	0.	0.	0.04
$\sigma_{\mathrm{xs}}(\mathrm{E}[\mathrm{dy}])(\%)$	0.28	0.	0.	0.	0.05
$\sigma(\mathrm{dy})(\%)$	3.56	3.58	3.57	3.57	3.53
$\sigma(\mathrm{dc})/\sigma(\mathrm{dy})$	0.71	0.61	0.61	0.61	0.61
$\sigma(\mathrm{di})/\sigma(\mathrm{dy})$	4.49	5.27	5.25	5.25	4.97
E[I/Y](%)	20	30.54	30.54	30.54	30.55
cor(dc, di)	0.39	0.29	0.3	0.3	0.41
cor(dc, ${r_{\mathrm{ex}}}^{\mathrm{lev}}$)	0.25	0.32	0.32	0.32	0.45
E[${r_{\mathrm{ex}}}^{\mathrm{lev}}$](%)	4.71	4.54	4.49	4.49	4.53
σ[${r_{\mathrm{ex}}}^{\mathrm{lev}}$](%)	20.89	4.97	4.94	4.94	4.55
σ(q)	0.29	0.26	0.25	0.25	0.26
E[rf](%)	0.65	0.89	0.91	0.91	0.89
σ(rf)(%)	1.86	1.07	1.07	1.07	1.07

Notes: Data are annual (1929-2008). Lowercase letters denote log-units. $E[.], \sigma(.), and \mathrm{corr}(.,.)$ denote the mean, volatility, and correlation respectively. $\sigma_{\mathrm{xs}}(\mathrm{E}[\mathrm{dy}])$ is the dispersion in GDP forecasts across forecasters, calculated as the time-series average of the cross-sectional standard deviation of one-year ahead forecasts.

which households receive perfectly accurate private signals about the future productivity shock ($\sigma_{\nu} = 0$). In both specifications all households have identical information sets (no information in the former and full information in the latter). The comparison between these two specifications thus shows the effect of common information about η on the moments generated by the model (the third set of terms in (66)). Comparing the coefficients across rows shows that the provision of common information has only a small effect on conventional business cycle moments such as $\sigma\left(dy\right)$ and $\sigma\left(dc\right)$. However, it has a larger effect on all moments that relate to the consumption/savings decision. For example, the equity premium falls from 4.54% in the case without information to 4.49% in the case of perfect information. Similarly, the risk-free interest rate rises from 0.89% to 0.91% between these two scenarios. These effects of information provision make intuitive sense. In our model, households worry about shocks to long-run productivity. When these shocks are revealed one month in advance, the unconditional variance of stock returns fall from 4.97% to 4.94%. In addition, households require a lower risk-premium for holding stocks and have less incentive to engage in precautionary savings behavior. As a result, the equity premium falls and the risk-free rate rises.

In column 5, we show results for the case in which $\sigma_{\nu} < \infty$, while $\sigma_{\tau} = 0$. In this case the stock price becomes perfectly revealing of innovations to productivity η_{t+1}, as predicted by

Corollary 1. Although households receive noisy private signals ex-ante, their posterior expectation is perfectly accurate, such that the dispersion of information has no effect on households' equilibrium behavior. The moments given in this column thus are exactly the same as those in column 3.

Although the overall effect of information provision may appear small in this calibration, it is important to keep in mind that we are evaluating the effects of learning the long-run component of the productivity shock *one month*, rather than a year or a decade, in advance. While this experiment is a natural starting point as long-run risk models are typically calibrated to monthly data, there is nothing particular about the monthly frequency. In a calibration to annual data for example, households would learn a year in advance about a larger (annual-sized) shock, such that learning presumably would have a much larger effect on all aspects of the model. While an in-depth analysis of the effect of information provision at different frequencies is beyond the scope of the present paper we believe it is an important avenue for future research.

5.3 Reasonably Sized Noise Trader Shocks

In column 6, we set the annualized standard deviation of noise trader demand shocks to 0.43% of GDP and estimate σ_ν to match the cross-sectional standard deviation of expectations of GDP growth from the Survey of Economic Forecasters. The estimation returns $\sigma_\nu = 3.2 \cdot \sigma_\eta$, in which case the model is able to generate a cross-sectional dispersion in equilibrium expectations of 0.05% (compared to 0.28% in the data). Similarly, the cross-sectional dispersion of stock holdings is small (0.04%).

Figure 1 plots the comparative statics over σ_ν for the three variables in the second panel of Table 2. The comparative static for the objective of our estimation, $\sigma_{xs}(E[dy])$, is given in the middle of the figure. It shows that there is no heterogeneity across households in equilibrium expectations when households receive a perfectly revealing private signal ($\sigma_\nu/\sigma_\eta = 0$). For moderate amounts of noise in the private signal this heterogeneity then rapidly increases, peaks at around ($\sigma_\nu/\sigma_\eta = 5$), and then monotonically decreases.

To gain intuition for this result, we can write using equations (65) and (53)

$$V_t\left[\alpha_1 \nu_{it}\right] = \left(\frac{V_t[\eta_{t+1}]/\sigma_\eta^2}{\sigma_\nu/\sigma_\eta}\right)^2 \sigma_\eta^2. \tag{69}$$

To the left of the figure, $V_t[\eta_{t+1}]/\sigma_\eta^2 = 0$ and the dispersion of equilibrium expectations is zero. As we move to the right of the figure the precision of the private signal decreases, the market's capacity to aggregate information deteriorates, and $V_t[\eta_{t+1}]/\sigma_\eta^2$ rises, such that households rely more on their private signal when forming expectations. Towards the middle of the graph, $V_t[\eta_{t+1}]/\sigma_\eta^2$ begins to converge to one and the increase in the denominator, σ_ν/σ_η, begins to

dominate. As the information content of stock prices and the precision of the private signal continue to deteriorate, households rely more and more on their priors and reduce weight on their private signal. Heterogeneity in equilibrium expectations thus peaks at an intermediate value of σ_ν. For a given size of the noise trader demand shock the information microstructure thus requires that heterogeneity in equilibrium behavior is bounded from above.

The conclusion from our estimation in column 6 of Table 2 is that, for a reasonable size of noise trader demand shocks, this upper bound is about an order of magnitude smaller than the dispersion of equilibrium expectations we observe in the data.

Another interesting result in column 6 of Table 2 is that some moments, such as the equity premium and the risk-free rate, fall into the range established by the 'no information' case in column 3 and the 'full information' case in column 4. However, most other moments, such as the standard deviation of output (3.53% versus 3.58% and 3.57%) and the correlation of investment growth with consumption growth (0.41 versus 0.29 and 0.30) fall well outside of this range.

To further investigate this finding, Figure 2 plots the comparative statics of selected moments in column 6 of Table 2 over σ_ν. In each panel of the figure, the horizontal dashed line represents the moment generated in the 'no information' case (column 3) and the horizontal dotted line represents the same moment in the 'full information' case (column 4).

The figure shows that all moments coincide with the moment from the 'full information' case when the private signal is fully precise and appear to converge to the moment from the 'no information' case as σ_ν becomes large. However, for an intermediate range around $\sigma_\nu/\sigma_\eta = 3.2$, the standard deviation of output growth, the standard deviation of stock returns and the ratios of the standard deviations of output and investment growth to output growth are well below that range. Similarly, the correlation of consumption growth and investment growth, as well as the correlation of consumption growth with stock returns are well above that range.

Comparing these results with Figure 1, suggests that the same forces that drive heterogeneity in households' equilibrium expectations also affect the quantitative results for these moments. The reason is that the same value of σ_ν that maximizes dispersion in equilibrium expectations also maximizes the effect of a given shock to noise trader demand on households' expectations. To see this, note from (51) that the coefficient on τ_t in households' equilibrium expectation is $V_t[\eta_{t+1}]\alpha_1/\sigma_\tau^2$. Substituting (53) yields $V_t[\eta_{t+1}]^2/(\sigma_\nu^2\sigma_\tau^2)$, which, for a given σ_τ, has its maximum at the same σ_ν as (69). The patterns in Figure 1 can thus be explained by the effect of noise trader demand shocks on the equilibrium expectations held by households (the fourth set of terms in (66)).

Figure 3 shows the effect of a two standard deviation (positive) shock to the long-run component of productivity in month 10, where again the dashed line corresponds to the no-information case, the dotted line to the full-information case and the solid line to the specification in column 6 of Table 2. The graph shoes that the no-information and the full information case look very

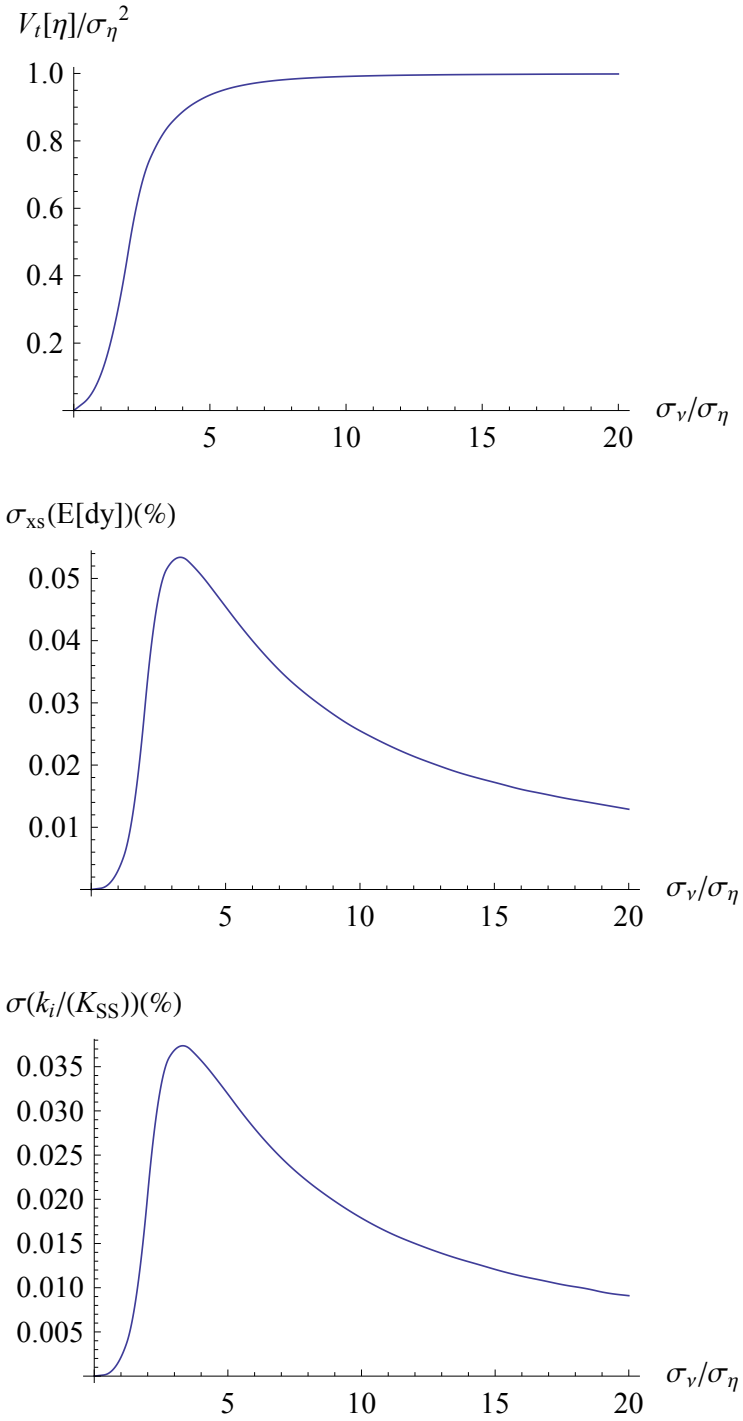


FIGURE 1: CROSS-SECTIONAL VARIATION IN EQUILIBRIUM BEHAVIOR

similar in the sense that they show large effects on output growth in the period in which the shock is revealed. In contrast, the solid line shows a partial adjustment in each period, and thus a smoother adjustment to the shock. The reason is that households who see a rise in the stock price are unsure whether this rise is a result of a noise trader demand shock or an productivity shock. They thus partially adjust to potentially good news in month 9 and adjust fully only when the shock is realized in period 10.

The two panels on the left of Figure 4 show the effect of dispersion in equilibrium behavior on the level of consumption and labor supply that corresponds to the fourth grouping of terms on the right hand side of (66). It shows that even moderate dispersion in equilibrium expectations may have a surprisingly large effect on the level of both aggregate variables. Since the consumption function is concave in $E_{it+1}[\eta_{t+1}]$, the dispersion in consumption choices lowers aggregate consumption by up to 0.015%. Similarly, dispersion in labor supply decisions may lower aggregate labor supply by up to 0.03% (note that households' labor supply decisions are convex in $E_{it+1}[\eta_{t+1}]$).

The two panels to the right of Figure 4 compare this level effect of dispersion with the level effect of risk that is present even if the model has a representative agent (the first grouping of terms in (66). It shows that in both cases the level effect of dispersion is up to an order of magnitude larger than the level effect of risk in a representative agent framework.

In column 4 of Table 3 we choose σ_ν to minimize (68) for the full vector of data moments in column 2. Since the 'no information' model in column 3 already gives a reasonably good fit to the data, it is not surprising that we obtain a relatively large estimate $\sigma_\nu/\sigma_\eta = 6.4$, such that households learn very little about future productivity in equilibrium ($V_t[\eta]/\sigma_\eta^2 = 0.97$). As a result, the macroeconomic and financial moments remain close to those in column 2. The only significant difference between the two specifications is that in column 4 the cross-sectional variance of equilibrium expectations is non-degenerate with $\sigma_{xs}(E[dy]) = 0.04\%$.

5.4 Large Noise Trader Demand Shocks

In column 5, we repeat the same exercise, but now simultaneously estimate the standard deviation of noise trader demand shocks and the precision of the private signal. Minimization the distance to target moments in equation (68) returns a standard deviation of noise trader demand shocks of 4.3% of GDP and $\sigma_\nu/\sigma_\eta = 3.2$. In equilibrium households again learn very little about the future productivity shock ($V_t[\eta]/\sigma_\eta^2 = 95$), but the very large noise trader demand shocks significantly improve the overall fit of the model. In particular, large noise trader demand shocks increase the equity premium to 4.66% (relative to 4.54% in column 2), while lowering the risk-free rate to 0.82%.

In column 6, we ask how much heterogeneity in equilibrium behavior the model can produce, by choosing both the size of the noise trader demand shocks and the precision of the private signal

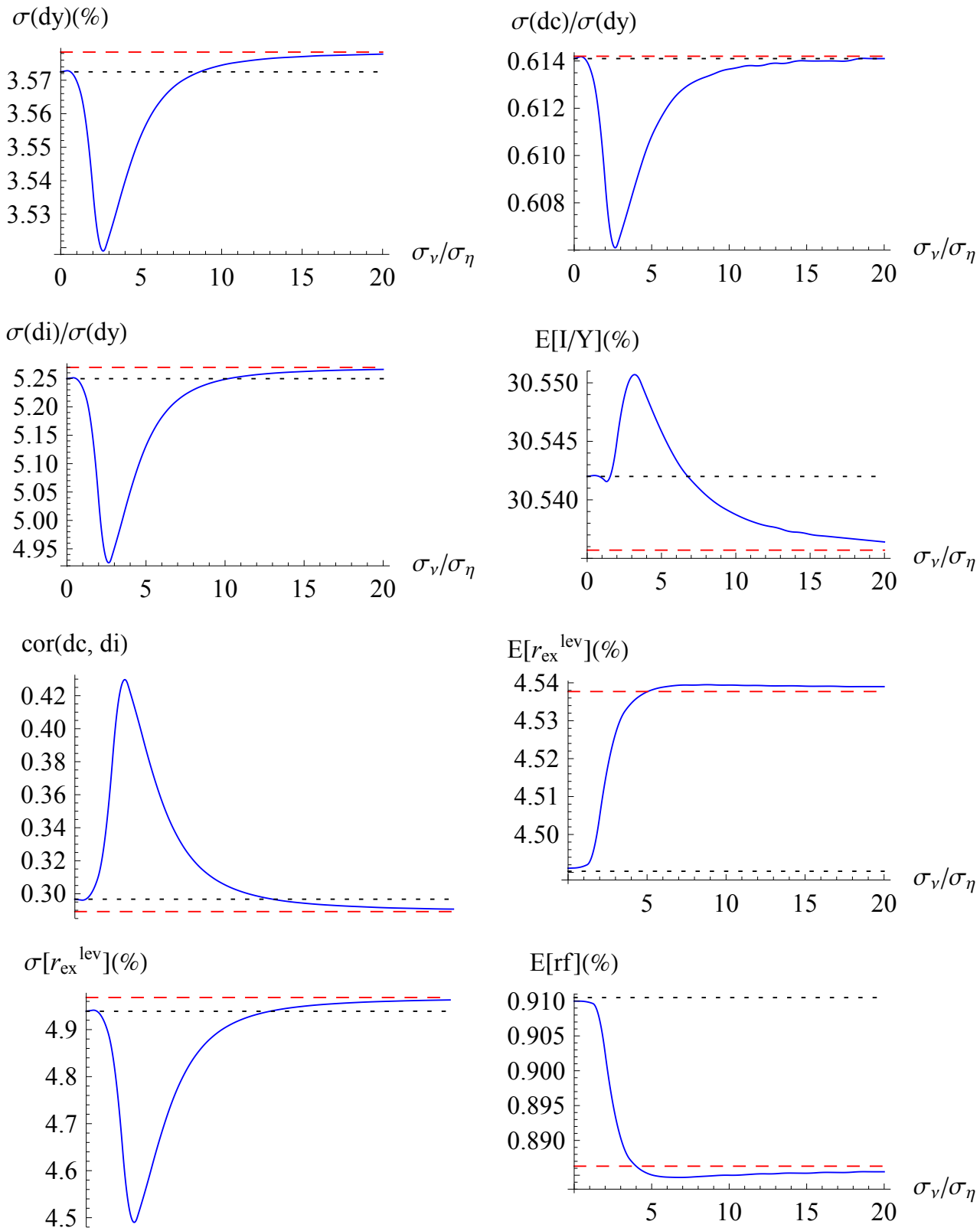


FIGURE 2: COMPARATIVE STATICS FOR SELECTED MOMENTS IN COLUMN 6 OF TABLE 2

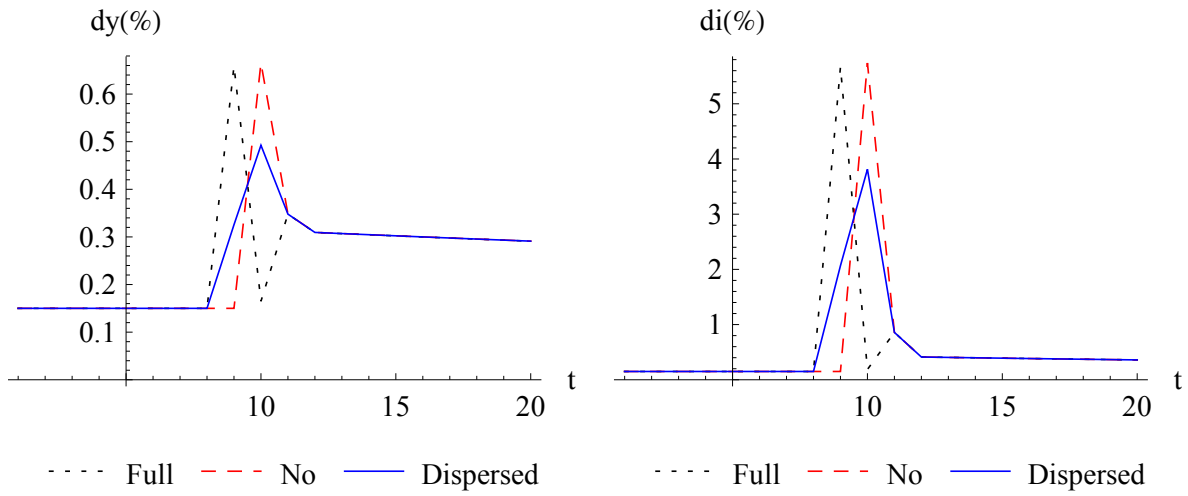


FIGURE 3: RESPONSES OF MACROECONOMIC AGGREGATES TO A 2 STANDARD DEVIATION SHOCK TO η IN MONTH 10. CALCULATIONS BASED ON SPECIFICATION IN COLUMN 6 OF TABLE 2.

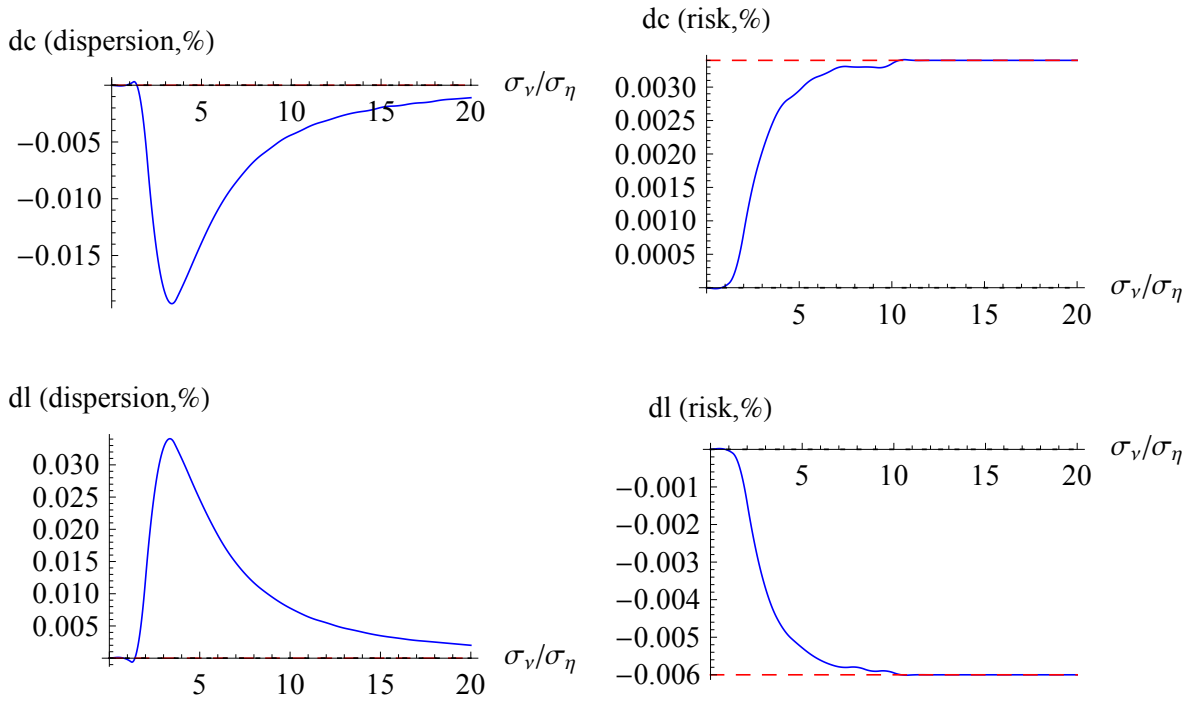


FIGURE 4: THE EFFECT OF DISPERSION ON THE LEVEL OF AGGREGATE CONSUMPTION AND LABOR SUPPLY. CALCULATIONS BASED ON SPECIFICATION IN COLUMN 6 OF TABLE 2.

TABLE 3: ESTIMATES OF σ_ν AND σ_τ

	Data	'No Info'	'Dispersed Info'	'Dispersed Info'	'Dispersed Info'
Pick σ_ν to match			all	all	σ_{xs}(E[dy])
$\sigma(T^\tau)/Y^{\text{SS}}(\%)$		0.	0.43	4.3	4.3
σ_ν/σ_η		∞	6.4	3.2	1.6
$V_t[\eta]/{\sigma_\eta}^2$		1.	0.97	0.95	0.83
$\sigma(k_i/(K_{\text{SS}}))(\%)$		0.	0.03	0.05	0.08
σ_{xs}(E[dy])(%)	0.28	0.	0.04	0.07	0.11
σ(dy)(%)	3.56	3.58	3.57	3.56	3.53
σ(dc)/σ(dy)	0.71	0.61	0.61	0.61	0.61
σ(di)/σ(dy)	4.49	5.27	5.2	5.16	4.99
E[I/Y](%)	20	30.54	30.54	30.56	30.59
cor(dc, di)	0.39	0.29	0.32	0.33	0.41
cor(dc, ${r_{\text{ex}}}^{\text{lev}}$)	0.25	0.32	0.35	0.36	0.45
E[${r_{\text{ex}}}^{\text{lev}}$](%)	4.71	4.54	4.54	4.66	4.68
σ[${r_{\text{ex}}}^{\text{lev}}$](%)	20.89	4.97	4.86	4.82	4.57
σ(q)	0.29	0.26	0.26	0.26	0.26
E[rf](%)	0.65	0.89	0.88	0.82	0.81
σ(rf)(%)	1.86	1.07	1.07	1.07	1.07

Notes: Data are annual (1929-2008). Lowercase letters denote log-units. $E[.], \sigma(.), and \text{corr}(.,.)$ denote the mean, volatility, and correlation respectively. σ_{xs}(E[dy]) is the dispersion in GDP forecasts across forecasters, calculated as the time-series average of the cross-sectional standard deviation of one-year ahead forecasts (1969-2008).

to match the cross-sectional standard deviation of equilibrium expectations. This estimation again returns $\sigma_\tau/\sigma_\eta = 4.3$ (the largest noise trader demand shocks on our grid) and $\sigma_\nu/\sigma_\eta = 1.6$. For this extreme set of parameters the model generates a cross-sectional standard deviation of expected GDP growth of 0.11% which is about factor four larger than the dispersion we were able to generate in Table 2. Nevertheless the estimate is still less then half the size of the dispersion observed in the data (0.28%).

6 Conclusion

In this paper we introduced a solution method that allows us to solve the standard noisy rational expectations model within the framework of a contemporary non-linear DSGE model. We showed that under mild assumptions on the distribution of noise trader demand shocks, the main qualitative results of the canonical noisy rational expectations model carry over to the non-linear framework. One of these findings is that any component of noise trader demand that is contingent on variables that are observable by households do not affect equilibrium expectations as long as the equilibrium price function remains monotonic in the productivity shock. An immediate

corollary to this result is that any monetary or fiscal policy that conditions only on publicly observable variables does not affect the equilibrium informativeness of stock prices.

Our quantitative exercise yields four main insights. The first main insight is that households' ability to learn about the future affects consumption-savings decision. A major shortcoming of the present framework is that it does not lend itself to a welfare analysis of these quantitative effects, mainly because noise trading implies large wealth transfers between maximizing households and non-maximizing noise traders, which are hard to judge from a normative perspective. In a related paper (Hassan and Mertens, 2014) we propose to solve this conceptual problem by introducing a single class of near-rational households.

The second main insight is that the differences in opinion about future GDP growth reflected in the Survey of Economic Forecasters are hard to rationalize within a noisy rational expectations model. For a reasonable size of noise trader demand shocks, the dispersion of equilibrium expectations in the model remains an order of magnitude smaller than that in the data.

Third, even when households have moderate differences of opinion in equilibrium, their expectations about the future, and thus their behavior, may load on non-fundamental noise trader demand shocks. When this is the case, the effect of the demand shocks on household expectations may dramatically change the moments generated by the model. While these changes did not improve the overall fit of the model to the data in the example we study in this paper, this may nevertheless be the case in a range of other applications. The main point is that the noisy rational expectations framework predicts that households rationally respond to non-fundamental noise in equilibrium. Our quantitative exercise shows that this response is large enough to affect the quantitative predictions of the DSGE model.

Fourth, when noise trader demand shocks are (unreasonably) large, these demand shocks no longer have a large effect on household expectations but nevertheless directly affect the quantitative predictions of the model.

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Appendix

A Appendix to Section 2

A.1 Deriving (13) and (14)

Using (2) and (9), the vector $(\eta, s_i, \hat{q})$ has unconditional expectation $(\bar{\eta}, \bar{\eta}, \frac{\alpha_1}{1-\alpha_2}\bar{\eta})$ and the variance covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_\eta^2 & \sigma_\eta^2 & \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 \\ \sigma_\eta^2 & \sigma_\eta^2 + \sigma_\nu^2 & \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 \\ \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 & \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 & (\frac{\alpha_1}{1-\alpha_2})^2\sigma_\eta^2 + (\frac{\rho V_1[\eta]}{1-\alpha_2})^2\sigma_\tau^2 \end{pmatrix}. \tag{70}$$

Thus by the property of the conditional variance of the multi-normal distribution,

$$V[\eta|s_i, \hat{q}] = \sigma_\eta^2 - \begin{pmatrix} \sigma_\eta^2 & \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 \end{pmatrix} \begin{pmatrix} \sigma_\eta^2 + \sigma_\nu^2 & \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 \\ \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 & (\frac{\alpha_1}{1-\alpha_2})^2\sigma_\eta^2 + (\frac{\rho V_1[\eta]}{1-\alpha_2})^2\sigma_\tau^2 \end{pmatrix}^{-1} \begin{pmatrix} \sigma_\eta^2 \\ \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 \end{pmatrix}. \tag{71}$$

Simplifying (71) and noting $V[\eta|s_i, \hat{q}] = V[\eta|s_i, Q]$ gives (13).

Similarly, by the properties of conditional mean of the multi-normal distribution,

$$E[\eta|s_i, \hat{q}] = \bar{\eta} + \begin{pmatrix} \sigma_\eta^2 & \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 \end{pmatrix} \begin{pmatrix} \sigma_\eta^2 + \sigma_\nu^2 & \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 \\ \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 & (\frac{\alpha_1}{1-\alpha_2})^2\sigma_\eta^2 + (\frac{\rho V_1[\eta]}{1-\alpha_2})^2\sigma_\tau^2 \end{pmatrix}^{-1} \begin{pmatrix} s_i - \bar{\eta} \\ \hat{q} - \frac{\alpha_1}{1-\alpha_2}\bar{\eta} \end{pmatrix}. \tag{72}$$

Substituting in $\hat{q}$ with (12) and $V[\eta|s_i, \hat{q}]$ with (71) and noting $E[\eta|s_i, \hat{q}] = E[\eta|s_i, Q]$ gives (14).

B Appendix to Section 3

B.1 Equation of motion for capital

Plugging (38) into (32) and integrating over individuals on both sides with (34) and (35) gives

$$Q_t \left(K_{t+1} - T^k\left[S_t\right] - T^\tau_{t+1}\right) = Q_{t-1}R_t \left(K_t - T^k\left[S_{t-1}\right] - T^\tau_t\right) - C_t + w_t N_t \tag{73}$$

Plugging in (25), (33), (36) and (37) yields (22).

B.2 Deriving the Equilibrium Conditions

Agents maximize utility (29) subject to budget constraint (32). State variables in individual optimization are the holdings of capital and bonds, i.e. $U_{it} = U_{it}(k_{it}, b_{it-1})$. Thus the first order

conditions and envelope conditions are:
First-order condition with respect to consumption

$$(1-\delta)\tilde{C}_{it}^{-\frac{1}{\psi}}\tilde{C}_{it}^{\frac{1}{\xi_l}}oC_{it}^{-\frac{1}{\xi_l}} = \delta E_{it}\left[U_{it+1}^{1-\gamma}\right]^{\frac{\gamma-\frac{1}{\psi}}{1-\gamma}} E_{it}[U_{it+1}^{-\gamma}U_{ikt+1}\frac{1}{Q_t}] \tag{74}$$

First-order condition with respect to labor

$$(1-\delta)\tilde{C}_{it}^{-\frac{1}{\psi}}\tilde{C}_{it}^{\frac{1}{\xi_l}}(1-o)A_{t-1}^{1-\frac{1}{\xi_l}}(1-n_{it})^{-\frac{1}{\xi_l}} = \delta E_{it}\left[U_{it+1}^{1-\gamma}\right]^{\frac{\gamma-\frac{1}{\psi}}{1-\gamma}} E_{it}[U_{it+1}^{-\gamma}U_{ikt+1}\frac{w_t}{Q_t}] \tag{75}$$

Envelope condition for capital

$$U_{ikt} = U_{it}^{\frac{1}{\psi}}\delta E_{it}\left[U_{it+1}^{1-\gamma}\right]^{\frac{\gamma-\frac{1}{\psi}}{1-\gamma}} E_{it}[U_{it+1}^{-\gamma}U_{ikt+1}\frac{Q_{t-1}}{Q_t}R_t] \tag{76}$$

Taking the ratio of first-order conditions with respect to labor (75) and to consumption (74) gives (40) where w_t is given by (25).

Plugging the first-order condition with respect to consumption (74) into the right hand side of envelope condition for capital (76) gives

$$U_{ikt} = U_{it}^{\frac{1}{\psi}}(1-\delta)\tilde{C}_{it}^{\frac{1}{\xi_l}-\frac{1}{\psi}}oC_{it}^{-\frac{1}{\xi_l}}Q_{t-1}R_t \tag{77}$$

Iterating (77) to $t+1$, plugging $\frac{U_{ikt+1}}{Q_t}$ into the first-order condition with respect to consumption (74) and re-arranging yield

$$\tilde{C}_{it}^{-\frac{1}{\psi}}\tilde{C}_{it}^{\frac{1}{\xi_l}}oC_{it}^{-\frac{1}{\xi_l}} = \delta E_t\left[U_{it+1}^{1-\gamma}\right]^{\frac{\gamma-\frac{1}{\psi}}{1-\gamma}} E_t[U_{it+1}^{-\gamma}U_{it+1}^{\frac{1}{\psi}}\tilde{C}_{it+1}^{\frac{1}{\xi_l}-\frac{1}{\psi}}oC_{it+1}^{-\frac{1}{\xi_l}}R_{t+1}] \tag{78}$$

Defining $M_{it+1} = \delta\left(\frac{C_{it+1}}{C_{it}}\right)^{-\frac{1}{\xi_l}}\left(\frac{\tilde{C}_{it+1}}{\tilde{C}_{it}}\right)^{\frac{1}{\xi_l}-\frac{1}{\Psi}}\left(\frac{U_{it+1}}{E_t\left[U_{it+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\frac{1}{\Psi}-\gamma}$ in (78) gives (41) and (42).

B.3 Programming Guide

B.3.1 Normalizing Transformation

Define normalized variables

$$\begin{aligned}&\hat{C}_t = \frac{C_t}{A_{t-1}},\ \hat{K}_t = \frac{K_t}{A_{t-1}},\ \hat{\bar{K}}_t = \frac{\bar{K}_t}{A_{t-1}},\ \hat{V}_t = \frac{V_t}{\tilde{C}_t},\ \hat{I}_t = \frac{I_t}{A_{t-1}},\\ &\hat{C}_{it} = \frac{C_{it}}{A_{t-1}},\ \hat{k}_{it} = \frac{k_{it}}{A_{t-1}},\ \hat{V}_{it} = \frac{V_{it}}{\tilde{C}_t}.\end{aligned} \tag{79}$$

B.3.2 Functional Forms

IAagg ($\hat{I}_t$) Normalizing (36) with (79) and plugging in (19) yields

$$\hat{I}_t = \hat{K}_t^{\alpha}(e^{\Delta a_t} N_t)^{1-\alpha} - \hat{C}_t + \hat{\varsigma}_t, \tag{80}$$

where applying (79) to (37) and substituting (33) in gives the last term $\hat{\varsigma}_t = \frac{\varsigma_t}{A_{t-1}}$ as

$$\hat{\varsigma}_t = \ Q_t \hat{T}_{t+1} e^{\Delta a_t} - \left(\alpha \left(\frac{e^{\Delta a_t} N_t}{\hat{K}_t} \right)^{1-\alpha} + Q_t \left[1 - \delta_k + (\frac{v_1}{1-\frac{1}{\xi}} - v_1)(\frac{\hat{I}_t}{\hat{K}_t})^{1-\frac{1}{\xi}} + v_0 \right] \right) \hat{T}_t. \tag{81}$$

G (G_t) Normalizing (23) with (79) yields

$$G_t = \frac{\hat{I}_t}{\hat{K}_t} - \left[\frac{v_1}{1-\frac{1}{\xi}} \left(\frac{\hat{I}_t}{\hat{K}_t} \right)^{1-\frac{1}{\xi}} + v_0 \right]. \tag{82}$$

Q (Q_t): Plugging (84) into (27) and applying (79) yields

$$Q_t = v_1^{-1} \left(\frac{\hat{I}_t}{\hat{K}_t} \right)^{\frac{1}{\xi}}. \tag{83}$$

Rn (R_{t+1}): Taking derivative of (23) with respect to $\frac{I_t}{K_t}$ and then applying (79) yields

$$G_t' = 1 - v_1 \left(\frac{I_t}{K_t} \right)^{-\frac{1}{\xi}} = 1 - v_1 \left(\frac{\hat{I}_t}{\hat{K}_t} \right)^{-\frac{1}{\xi}}. \tag{84}$$

Substituting Y_{t+1}, G_{t+1} and G_{t+1}' with (19), (23) and (84) separately into (33) and applying (79) yields

$$R_{t+1} = \frac{\alpha \left(\frac{e^{\Delta a_{t+1}} N_{t+1}}{\hat{K}_{t+1}} \right)^{1-\alpha} + Q_{t+1} \left[1 - \delta_k + (\frac{v_1}{1-\frac{1}{\xi}} - v_1)(\frac{\hat{I}_{t+1}}{\hat{K}_{t+1}})^{1-\frac{1}{\xi}} + v_0 \right]}{Q_t}. \tag{85}$$

SDFin (M_{it+1}): With $\xi_l = 1$, applying (79) to Cobb-Douglas form $\tilde{C}_{i,t}$

$$\tilde{C}_{i,t} = C_{i,t}^{o}(A_{t-1}(1-n_{i,t}))^{1-o} = \hat{C}_{it}^{o} A_{t-1}(1-n_{i,t})^{1-o}. \tag{86}$$

Plugging (86) into $V_{i,t+1}$ yields

$$V_{i,t+1} = \frac{V_{i,t+1}}{\tilde{C}_{i,t+1}} \frac{\tilde{C}_{i,t+1}}{\tilde{C}_{i,t}} \tilde{C}_{i,t} = \hat{V}_{it+1} \left(\frac{\hat{C}_{it+1}}{\hat{C}_{it}} \right)^{o} \left(\frac{1 - n_{i,t+1}}{1 - n_{i,t}} \right)^{1-o} e^{\Delta a_t} \tilde{C}_{i,t}. \tag{87}$$

Applying (79) to (42) with $\xi_l = 1$ and substituting in (86) and (87) yields

$$\begin{aligned} M_{it+1} =& \delta \left(\frac{\hat{C}_{it+1}}{\hat{C}_{it}} e^{\Delta a_t} \right)^{-1} \left(\left(\frac{\hat{C}_{it+1}}{\hat{C}_{it}} \right)^{o} \left(\frac{1 - n_{i,t+1}}{1 - n_{i,t}} \right)^{1-o} e^{\Delta a_t} \right)^{1-\frac{1}{\Psi}} \\ & \left(\frac{\left(\hat{V}_{it+1} \left(\frac{\hat{C}_{it+1}}{\hat{C}_{it}} \right)^{o} \left(\frac{1-n_{i,t+1}}{1-n_{i,t}} \right)^{1-o} e^{\Delta a_t} \right)^{\frac{1}{\Psi}-\gamma}}{E_{it}[\Phi_1]^{\frac{\frac{1}{\Psi}-\gamma}{1-\gamma}}} \right), \end{aligned} \tag{88}$$

where $\Phi_1 = \left(\frac{V_{i,t+1}}{\tilde{C}_{i,t}} \right)^{1-\gamma} = \left(\hat{V}_{it+1} \left(\frac{\hat{C}_{it+1}}{\hat{C}_{it}} \right)^{o} \left(\frac{1-n_{i,t+1}}{1-n_{i,t}} \right)^{1-o} e^{\Delta a_t} \right)^{1-\gamma}$ by (87).

B.3.3 Setup

Eqm1: Plugging (25) into (40) with $\xi_l = 1$ yields

$$\frac{1-o}{o} \left(\frac{\hat{C}_{it}}{1 - n_{i,t}} \right) = (1-\alpha) \hat{K}_t^{\alpha} N_t^{-\alpha} e^{(1-\alpha)\Delta a_t}. \tag{89}$$

Eqm2: This is simply (41).

Eqm3: Dividing both sides of (29) by $\tilde{C}_{i,t}$ and applying (79) yields

$$\hat{V}_{it}^{1-\frac{1}{\Psi}} = (1-\delta) + \delta \frac{E_{it}[\Phi_2]^{\frac{1-\frac{1}{\Psi}}{1-\gamma}} \left(e^{\Delta a_t} \right)^{1-\frac{1}{\Psi}}}{\left(\hat{C}_{it}^{o} (1 - n_{i,t})^{1-o} \right)^{1-\frac{1}{\Psi}}}, \tag{90}$$

where we write $U = V$ and $\Phi_2 = \left(\hat{V}_{it+1} \hat{C}_{it+1}^{o} \left(1 - n_{i,t+1}\right)^{1-o} \right)^{1-\gamma}$.

EqmKAaggn (Market clearing of capital): Applying (79) to (43) yields

$$\hat{\bar{K}}_{t+1} = \int \hat{k}_{it+1} di. \tag{91}$$

EqnCAagg (Aggregation of consumption): Applying (79) to the definition of C_t yields

$$\int \hat{C}_{it} di = \hat{C}_t. \tag{92}$$

EqnLagg (Market clearing of labor): This is simply (35).
EqmKAin ($(KA_i)_{t+1}$): Plugging (19), (25), (33) and (38) into (32) and applying (79) yields

$$Q_t\hat{k}_{it+1}e^{\Delta a_t} = \left(\alpha\left(\frac{e^{\Delta a_t}N_t}{\hat{K}_t}\right)^{1-\alpha} + Q_t\left[1-\delta_K + (\frac{v_1}{1-\frac{1}{\xi}} - v_1)(\frac{\hat{I}_t}{\hat{K}_t})^{1-\frac{1}{\xi}} + v_0\right]\right)(\hat{K}_t - \hat{T}_t)$$
$$- \hat{C}_{it} + (1-\alpha)\hat{K}_t^{\alpha}N_t^{-\alpha}e^{(1-\alpha)\Delta a_t}n_{it}. \tag{93}$$

B.4 Structure of equilibrium conditions

Rearranging (40) yields

$$\frac{1-o}{o}\frac{A_{t-1}^{1-\frac{1}{\xi_l}}(1-n_{it})^{-\frac{1}{\xi_l}}}{C_{it}^{-\frac{1}{\xi_l}}} - w_t = 0. \tag{94}$$

Rearranging (41) yields

$$Q_t C_{it}^{-\frac{1}{\xi_l}}\tilde{C}_{it}^{\frac{1}{\xi_l}-\frac{1}{\Psi}}E_{it}\left[U_{it+1}^{1-\gamma}\right]^{\frac{\frac{1}{\Psi}-\gamma}{1-\gamma}} = E_{it}\left[\delta C_{it+1}^{-\frac{1}{\xi_l}}\tilde{C}_{it+1}^{\frac{1}{\xi_l}-\frac{1}{\Psi}}U_{it+1}^{\frac{1}{\Psi}-\gamma}(\alpha\frac{Y_{t+1}}{K_{t+1}} + Q_{t+1}(1-\delta_k + G'_{t+1}\frac{I_{t+1}}{K_{t+1}} - G_{t+1}))\right] \tag{95}$$

Rearranging (32) yields

$$Q_t k_{it+1} - Q_{t-1}R_t k_{it} - T_{it} + c_{it} - w_t n_{it} = 0 \tag{96}$$

B.5 Proof of lemma 2

Given Lemma 1, it follows immediately that

$$E_{it}\left[\eta_{t+1}\right] = E\left[\eta_{t+1}|s_{it}, S_t\right] = E\left[\eta_{t+1}|s_{it}, \hat{q}_t\right], \tag{97}$$

We can thus guess that the rational expectation of η_{t+1} to be the linear function

$$E_{it}[\eta_{t+1}] = \alpha_0 + \alpha_1 s_{it} + \alpha_2\hat{q}_t, \tag{98}$$

where α_0, α_1, and α_2 are the optimal weights on the prior, the private signal and the average expectation, respectively. Substituting in (45), taking the integral across individuals, and solving for $\int E_{it}\left[\eta_{t+1}\right]$ gives

$$\int E_{it}[\eta_{t+1}]di = \frac{\alpha_0}{1-\alpha_2} + \frac{\alpha_1}{1-\alpha_2}\eta_{t+1} + \frac{\alpha_2}{1-\alpha_2}\frac{V_t\left[\eta\right]}{\sigma_\eta^2}\tau_t. \tag{99}$$

Adding $\frac{V_t[\eta]}{\sigma_\eta^2}\tau_t$ on both sides of the equation, substituting (45) and simplifying yields

$$\frac{1-\alpha_2}{\alpha_1}\hat{q}_t - \frac{\alpha_0}{\alpha_1} = \eta_{t+1} + \frac{1}{\alpha_1}\frac{V_t\left[\eta\right]}{\sigma_\eta^2}\tau_t. \tag{100}$$

Thus with the normality of the fundamental shock τ_t and the demand statistic $\hat{q}_t$, the vector $(\eta_{t+1}, s_{it}, \hat{q}_t)$ has an unconditional expectation of

$$\mu_t = \begin{pmatrix} \bar{\eta} & \bar{\eta} & \frac{\alpha_0}{1-\alpha_2} + \frac{\alpha_1}{1-\alpha_2}\bar{\eta} \end{pmatrix} \tag{101}$$

and an unconditional variance covariance matrix of

$$\Sigma_t = \begin{pmatrix} \sigma_\eta^2 & \sigma_\eta^2 & \frac{\alpha_1\sigma_\eta^2}{1-\alpha_2} \\ \sigma_\eta^2 & \sigma_\eta^2 + \sigma_\nu^2 & \frac{\alpha_1\sigma_\eta^2}{1-\alpha_2} \\ \frac{\alpha_1\sigma_\eta^2}{1-\alpha_2} & \frac{\alpha_1\sigma_\eta^2}{1-\alpha_2} & \frac{\alpha_1^2\sigma_\eta^2}{(1-\alpha_2)^2} + \frac{V_1^2\sigma_\tau^2}{(1-\alpha_2)^2\sigma_\eta^4} \end{pmatrix} \tag{102}$$

Thus by the property of the conditional variance of the multi-normal distribution, the conditional variance is

$$V[\eta_{t+1}|s_{it}, \hat{q}_t] = \sigma_\eta^2 - \begin{pmatrix} \sigma_\eta^2 & \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 \end{pmatrix} \begin{pmatrix} \sigma_\eta^2 + \sigma_\nu^2 & \frac{V_1\sigma_\eta^2}{(1-\alpha_2)\sigma_\nu^2} \\ \frac{V_1\sigma_\eta^2}{(1-\alpha_2)\sigma_\nu^2} & \frac{V_1^2\sigma_\eta^2}{(1-\alpha_2)^2\sigma_\nu^4} + \frac{V_1^2\sigma_\tau^2}{(1-\alpha_2)^2\sigma_\eta^4} \end{pmatrix}^{-1} \begin{pmatrix} \sigma_\eta^2 \\ \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 \end{pmatrix} \tag{103}$$

which simplifies to (52). Similarly, the conditional mean is

$$E[\eta_{t+1}|s_{it}, \hat{q}_t] = \bar{\eta} + \begin{pmatrix} \sigma_\eta^2 & \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 \end{pmatrix} \begin{pmatrix} \sigma_\eta^2 + \sigma_\nu^2 & \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 \\ \frac{\alpha_1}{1-\alpha_2}\sigma_\eta^2 & (\frac{\alpha_1}{1-\alpha_2})^2\sigma_\eta^2 + (\frac{\rho V_1[\eta]}{1-\alpha_2})^2\sigma_\tau^2 \end{pmatrix}^{-1} \begin{pmatrix} s_{it} - \bar{\eta} \\ \hat{q} - (\frac{\alpha_0}{1-\alpha_2} + \frac{\alpha_1}{1-\alpha_2}\bar{\eta}) \end{pmatrix} \tag{104}$$

which simplifies to (51).

Matching coefficients with (98) gives (53) and (54).

Taking conditional expectation on both sides of (100) and solving for $E_{it}[\tau_t]$ yields

$$E_{it}\left[\tau_t\right] = \left(\left(1-\alpha_2\right)\hat{q}_t - \alpha_0 - \alpha_1 E_{it}\left[\eta_{t+1}\right]\right)\frac{\sigma_\eta^2}{V_t\left[\eta\right]}.$$

Similarly, taking conditional variance on both sides of (100) and plugging (53) gives

$$V_{it}[\tau_t] = \frac{\sigma_\eta^4}{\sigma_\nu^4} V_{it}[\eta_{t+1}]$$

C Appendix to Section 5

C.1 Data Sources

Consumption (C_t). Per capita consumption data are from the National Income and Product Accounts (NIPA) annual data reported by the Bureau of Economic Analysis (BEA). The data are constructed as the sum of consumption expenditures on nondurable goods and services (Table 1.1.5, Lines 5 and 6) deflated by corresponding price deflators (Table 1.1.9, Lines 5 and 6).

Physical Investment (I_t). Per-capita Physical investment data are also from the NIPA tables. We measure physical investment by fixed investment (Table 1.1.5, Line 8) minus information processing equipment (Table 5.5.5, Line 3) deflated by its price deflator (Table 1.1.9, Line 8). Information processing equipment is interpreted as investment in intangible capital and is therefore subtracted from fixed investment.

Output (Y_t). It is the sum of total consumption and investment, that is, $C_t + I_t$. We exclude government expenditure and net export because not explicitly modeled in our economy.

Labor (N_t). It is measured as the total number of full-time and part-time employees as reported in the NIPA Table 6.4. Data are annual.

Stock market returns (R_t) and **Risk-free rate** (R_t^f) The stock market return are from the Fama-French dataset available online on K. French′s webpage at

http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/ftp/F-F_Research_Data_Factors.zip.

The nominal risk-free rate is measured by the annual 3-month T-bill return. The real stock market returns and risk-free rate are computed by subtracting realized inflation (annual CPI through FRED) from the nominal risk-free rate.

Price-dividends (pd_t) ratio and **Tobin′s Q** (Q_t). Data on annual price-dividend ratio, and dividend are obtained from CRSP. Annual dividends are obtained by time-aggregating monthly dividends. Nominal dividends are turned into real dividends using the CPI index. Data on Tobin′s Q are from the Flow of Funds (FoF) and are obtained directly from the St. Louis Fed by dividing the variable MVEONWMVBSNNCB (Line 35 of Table B.102 in the FoF report) by TNWMVBSNNCB (Line 32 of table B.102 in the FoF report).

GDP Forecast ($\hat{y}_{it}$ and $\hat{y}_t$) GDP forecast data during 1969-2010 are from the Survey of Professional Forecasters provided by the Philadelphia Federal Reserve at

http://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/