Strategic Trading in Informationally Complex Environments*

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Abstract

We study trading behavior and the properties of prices in informationally complex markets. Our model is based on the single-period version of the linear-normal framework of Kyle (1985). We allow for essentially arbitrary correlations among the random variables involved in the model: the true value of the traded asset, the signals of strategic traders, the signals of competitive market makers, and the demand coming from liquidity traders. We first show that there always exists a unique linear equilibrium, which can be characterized analytically, and illustrate its properties in a series of examples. We then use this equilibrium characterization to study the informational efficiency of prices as the number of strategic traders becomes large. If the demand from liquidity traders is uncorrelated with the true value of the asset or is positively correlated with it (conditional on other signals), then prices in large markets aggregate all available information. If, however, the demand from liquidity traders is negatively correlated with the true value of the asset, then prices in large markets aggregate all available information *except* that contained in liquidity demand.

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1 Introduction

Whether and how dispersed information enters into market prices is one of the central questions of information economics. A key obstacle to full information revelation and aggregation in markets is the strategic behavior of informed traders. A trader who has private information about the value of an asset has an incentive to trade in the direction of that information. However, the more he trades, the more he reveals his information, and the more he moves the prices closer to the true value of an asset. Thus, to maximize his profits, an informed trader may stop short of fully revealing his information, and thus the informational efficiency of market prices may fail.

Thus, an important and natural question is when we should expect market prices to in fact reflect all information available to market participants. One stream of literature considers trading in dynamic environments, with informed traders having multiple opportunities for trading.[1] In these settings, in each period, traders may have an incentive to withhold some of their information in order not to eliminate their profits. However, over time, traders will gradually reveal all of their information, and in many (although not all) cases, by the end of trading, market prices will in fact aggregate all available information.

One issue with the case of dynamic trading is that while at the end, market prices accurately reflect all available information, that is generally not the case during most of the time the market is in operation—and thus much of the trading may happen at prices that are far away from the ones that would prevail if all private information was publicly available to all market participants. Therefore, another important stream of research abstracts away from the time dimension and repeated trading in markets, and considers instead an alternative intuition for when market prices may accurately reflect information: when the number of informed traders is large, and each one of them is informationally small. In that case, each of the informed traders has limited impact on market prices, but their aggregate behavior does in fact reflect the aggregate information available in the market. As a result market prices are close to those that would prevail if all private information were publicly available, and all trades happen at those prices.

Non-strategic explorations of this intuition go back to Hayek (1945), Grossman (1976), and Radner (1979). Subsequently, a line of research (which we discuss in detail in Section 2) has considered strategic foundations for this intuition, studying strategic behavior of informed agents in finite markets, and then considering the properties of prices as the number of these agents becomes large. This stream of work, however, imposes very strict assumptions on how information is distributed among the agents, typically assuming that the signals of informed agents are symmetrically distributed, or satisfy other related restrictions so that in equilibrium, the strategies of all informed traders are identical (see Section 2). In practice, however, the distribution of information in the economy can be much more complex. Some agents may be strictly more informed than others. Groups of agents may have access to different sources of information, so that the correlations of signals within a group are very different from correlations across groups (and the sizes of the groups

[1] See, e.g., Hellwig (1982), Dubey et al. (1987), Kyle (1985), Wolinsky (1990), Foster and Viswanathan (1996), Back et al. (2000), Ostrovsky (2012), and Golosov et al. (2013), among others.

may be different, and the correlations of signals between different groups may be different as well). Some agents may be informed about the fundamental value of the security, while others may be uninformed about the fundamentals but possess some "technical" information about the market or other traders. And of course all such possibilities may be present in a market at the same time.

Our paper makes two main contributions.

First, we present an analytically tractable framework that makes it possible to study trading in such informationally complex environments. Our model is based on the single-period version of the model of Kyle (1985). As in that paper, an important assumption that makes our model analytically tractable is the assumption of joint normality of random variables involved in the setting: the true value of the traded asset, the signals of strategic traders, the signals of competitive market makers, and the demand coming from liquidity traders. Beyond that assumption, however, we impose essentially no restrictions on the joint distribution of these variables, making it possible to model informationally rich situations such as those described above. In this framework, we show that there always exists a unique linear equilibrium, which can be computed analytically.

Second, we explore the informational properties of equilibrium prices as the number of informed agents becomes large. We assume that there are several types of agents, with each agent of a given type receiving the same information, and fix the matrix of correlations of signals across the types (and other random variables in the model). We then allow the numbers of agents of every type to grow (without restricting the rates of growth in any way; e.g., the number of agents of one type may grow much faster than the number of agents of another type). We find that the informational properties of prices in these large markets depend on the informativeness of the demand from liquidity traders. If the demand from liquidity traders is uncorrelated with the true value of the asset or is positively correlated with it (conditional on other signals), then prices in large markets aggregate all available information. If, however, the demand from liquidity traders is negatively correlated with the true value of the asset, then prices in large markets aggregate all available information *except* that contained in liquidity demand.

We also illustrate our model with several applications. One example shows that under fairly simple (but, crucially, asymmetric) information structures, an informed trader may choose to trade "against" his information, i.e., sell the asset when his signal implies that the expected value of the asset is positive, and vice versa. Two examples explore the profitability of "technical" trading, and show that a trader may be able to make substantial positive expected profit even if he has no information about the value of the asset, provided there is at least one other ("fundamental") trader who does, and provided that the technical trader has information about the demand from liquidity traders or about the mistakes of the fundamental trader. Finally, our last set of examples shows how equilibrium trading and outcomes depend on the amount of private information available to the market maker (beyond the aggregate market demand).

The remainder of this paper is organized as follows. In Section 2 we discuss the related literature. In Section 3 we present the model. In Section 4 we state and prove our first main result, on the existence and uniqueness of linear equilibrium, and we show how to compute this equilibrium

analytically. In Section 5, we illustrate our result with several applications in informationally complex settings. In Section 6, we present our second main result, on information aggregation in large markets.

2 Related Literature

The literature on strategic foundations of information aggregation and revelation in markets goes back to Wilson (1977), who considers an auction-based model in which multiple partially informed agents bid on a single object. Other work in this tradition includes Milgrom (1981), Pesendorfer and Swinkels (1997), Kremer (2002), and Reny and Perry (2006). These papers find that under various suitable conditions, information does get aggregated (and revealed in winning bids) when the number of bidders becomes large. However, these results depend critically on strong symmetry assumptions on the bidders' signals and strategies.

Another related stream of literature, going back to Kyle (1989), considers equilibria in demand and supply functions, where bidders specify how many units of an asset they demand or supply for each possible price level, and then the market maker picks the price that clears the market.[2] The papers in this tradition also assume a very high degree of symmetry among the trading agents, typically assuming that these agents are ex ante identical, receive symmetrically distributed information, and employ identical strategies in equilibrium. A recent paper by Rostek and Weretka (2012) partially relaxes this symmetry assumption, and replaces it with a somewhat weaker "equicommonality" assumption on the matrix of correlations among the agents' values. This assumption states that the sum of correlations in each column (or, equivalently, each row) of the correlation matrix is the same, and that the variances of all traders' values are also the same. While this assumption is more general than full symmetry among the agents, it is still quite restrictive: for example, the equilibrium in this model is still symmetric, with all traders using identical strategies.

Finally, a closely related stream of literature is the work building on Kyle's (1985) model. In that literature, as in our paper, one or more strategic traders, fully or partially informed about the value of the traded asset, are present in the market. These strategic traders submit market orders to centralized market makers. There are also liquidity traders who submit exogenously determined market orders. The market makers set the price of the asset equal to their Bayesian estimate of its value, given their prior information, the knowledge of strategic traders' strategies, and the observed order flow. Our paper borrows much of its analytical framework from this literature. The key difference is that while many of the papers in this area consider both static and dynamic models of trading but place restrictive assumptions on the information structure, our paper places virtually no restrictions on the information structure (beyond joint normality), and focuses on the one-period model of trading and on the informational properties of prices as the number of strategic traders becomes large.

In the original model of Kyle (1985), there is only one informed trader, who knows the value of

[2] See Vives (2008) for a textbook treatment of that literature.

the asset. Holden and Subrahmanyam (1992) study a generalization with multiple fully informed traders. Foster and Viswanathan (1996) further extend the model by allowing these traders to observe imperfect signals about the value of the asset. Different traders may observe different signals, but the distribution of these signals across the traders has to be symmetric, as are the traders' strategies. Back et al. (2000) consider a continuous-time analog of the model of Foster and Viswanathan (1996). Caballé and Krishnan (1994) and Pasquariello (2007) consider multi-asset versions of the one-period model with multiple traders, but still maintain the assumption of symmetry of information among the traders.

A small number of papers goes beyond the fully symmetric case. Foster and Viswanathan (1994) consider a model with two strategic traders in which one trader is strictly more informed than the other. Colla and Mele (2010) consider a model in which informed traders are located on a circle, with the correlations of signals being stronger for traders who are closer to each other (in this model, as in the Rostek and Weretka (2012) model discussed above, all traders use identical strategies in equilibrium). Bernhardt and Miao (2004) consider a model with a very general information structure, allowing, as our paper does, for an asymmetric variance-covariance matrix of traders' signals.[3] However, while Bernhardt and Miao (2004) characterize necessary and sufficient conditions for linear equilibria, and use these conditions to study the properties of such equilibria analytically and numerically in some specific examples, they do not provide any general results on equilibrium existence or uniqueness and do not provide general closed-form equilibrium characterizations.

3 Model

There is a security traded in the market, whose value v is not initially known to market participants. There are n strategic traders, $i = 1, \ldots, n$. Prior to trading, each strategic trader i (he) privately observes a multidimensional signal $\theta_i \in \mathbb{R}^{k_i}$, where $k_i \geq 1$ is the dimensionality of the signal. For convenience, we will denote by $\theta = (\theta_1; \theta_2; \cdots; \theta_n)$ the vector[4] summarizing the signals of all strategic traders. The dimensionality of vector θ is $K = \sum_{i=1}^{n} k_n$. There is also a market maker (she), who privately observes signal $\theta_M \in \mathbb{R}^{k_M}$, $k_M \geq 0$ (when $k_M = 0$, the market maker does not receive any signals, as in the standard Kyle (1985) model).[5,6] Finally, there are liquidity traders,

[3] There are several differences between the models. Unlike Bernhardt and Miao (2004), we allow liquidity demand to be correlated with the value of the asset and/or the signals of informed traders. We also allow the market maker to observe signals correlated with the value of the security, the demand from noise traders, and/or the signals of informed traders. Finally, we do not impose any special structure on how the informed traders' signals are related to the value of the asset (and other random variables in the model), beyond joint normality. On the other hand, Bernhardt and Miao (2004) consider a model with multiple trading periods, while we restrict attention to one period.

[4] We denote the row vector with elements $x_1, \ldots, x_k$ by $(x_1, \ldots, x_k)$, and the column vector with the same elements by $(x_1; \ldots; x_k)$. All vectors are column vectors unless specified otherwise.

[5] Strictly speaking, θ_i and θ_M are random variables whose realizations are in $\mathbb{R}^{k_i}$ and $\mathbb{R}^{k_M}$.

[6] The multidimensionality of the traders' and the market maker's signals allows our model to incorporate complex relationships among their information sets: for example, one trader can observe strictly more information than another trader; one trader can observe the union of two other traders' signals; some information may be common to several players while some other information is not; and so forth. We illustrate the richness of the model with several applications in Section 5.

whose exogenously given random demand, denoted by u, is in general not directly observed by either the strategic traders or the market maker.

The key assumption that makes the model analytically tractable is that all of the random variables mentioned above—v, θ, θ_M, and u—are jointly normally distributed. Specifically, we assume that the vector $\mu = (v; \theta; \theta_M; u)$ is drawn randomly from the multivariate normal distribution with expected value 0 and variance-covariance matrix Ω. The assumption that the expected value of vector μ is equal to zero is simply a normalization that allows us to simplify the notation. We will also assume that every variance-covariance matrix for signal θ_i of strategic trader i and the variance-covariance matrix of the marker maker's signal θ_M are full rank. This assumption is without loss of generality; it simply eliminates redundancies in each trader's signals. Note that we do not place a full rank restriction on the matrix Ω itself: For instance, two different strategic traders are allowed to have perfectly correlated signals. The only substantive restrictions that we place on matrix Ω are as follows.

Assumption 1 At least one strategic trader receives at least some information about the value of the security, beyond that contained in the market maker's signal. Formally:

$$Cov(v, \theta | \theta_M) \neq 0. \tag{1}$$

Assumption 2 The market maker does not perfectly observe the demand from liquidity traders. Formally:

$$Var(u | \theta_M) > 0. \tag{2}$$

3.1 Trading and Payoffs

After observing his signal θ_i, each strategic trader i submits his demand $d_i(\theta_i)$ to the market. In addition, the realized demand from liquidity traders, u, is also submitted to the market. The market maker observes her signal θ_M and the total demand $D = \sum_{i=1}^{n} d_i(\theta_i) + u$, and subsequently sets the price of the security, $P(\theta_M, D)$, based on these observations. Securities are traded at this price $P(\theta_M, D)$ (with each strategic trader getting his demand $d_i(\theta_i)$, liquidity traders getting u, and the market maker taking the position of size $-D$ to clear the market). At a later time, the true value of the security is realized, and each strategic trader i obtains profit $\pi_i = d_i(\theta_i) \cdot (v - P(\theta_M, D))$.

3.2 Linear Equilibrium

Our solution concept is essentially the same as that in Kyle (1985). We say that a profile of demand functions $d_i(\cdot)$ and pricing rule $P(\cdot, \cdot)$ form an *equilibrium* if

(i) on the equilibrium path, the price P set by the market maker is equal to the expected value of the security conditional on θ_M and D, given the primitives and the demand functions $d_i(\cdot)$; and

(ii) for every player i, for every realization of signal θ_i, the expected payoff from submitting demand $d_i(\theta_i)$ is at least as high as the expected payoff from submitting any alternative demand d_i', given the realization of signal θ_i, the pricing pricing rule $P(\cdot,\cdot)$ and the profile of strategies of other players $(d_j(\cdot))_{j\neq i}$.

The equilibrium is *linear* if functions d_i (for all i) and pricing rule P are linear functions of their arguments, i.e., $d_i(\theta_i) = \alpha_i^T\theta_i$ for some $\alpha_i \in \mathbb{R}^{k_i}$ and $P(\theta_M, D) = \beta_M^T\theta_M + \beta_D D$ for some $\beta_M \in \mathbb{R}^{k_M}$ and $\beta_D \in \mathbb{R}$.[7,8]

4 Equilibrium Existence and Uniqueness

We can now state and prove our first main result.

Theorem 1 *There exists a unique linear equilibrium.*

The proof of Theorem 1 is in Appendix B. In Appendix A, we summarize the notation used in the proof, as well as in some of the sections below.

The proof consists of five steps. The first two steps are fairly standard, and are essentially the same as in the earlier literature on linear-normal equilibria: They show that if all strategic traders follow linear strategies, then the pricing rule resulting from Bayesian updating is also linear; and that if all strategic traders other than trader i follow linear strategies, and the market maker is also using a linear pricing rule (with a positive coefficient β_D on aggregate demand D), then the best response of trader i is also linear and is uniquely determined by the other traders' strategies and the pricing rule. The substantively novel parts of the proof are the next three steps, which show that the conditions derived in the first two steps allow us to express all parameters of the pricing rule and traders' strategies as functions of "market depth" $\gamma = 1/\beta_D$, and the entire system of equations from the first two steps collapses into one quadratic equation in γ—which has exactly one positive root.

The proof is constructive, producing the following formula for the parameters of interest, as a function of various components of matrix Ω. The notation is described in Appendix A.

[7]Note that condition (i) is required to hold only on the equilibrium path. In the standard Kyle (1985) model and many of its generalizations, every observation of the market maker can be rationalized as being on the equilibrium path, and thus this qualifier is not needed. In our case, it is in general possible that for some strategy profiles $d_i(\cdot)$, only some realizations of aggregate demand D can be observed by the market maker if the strategic traders follow those strategies. E.g., if one strategic trader observes liquidity demand u and submits demand $-u$, and all other strategic traders submit zero demands, then on the equilibrium path, the market maker can only observe zero demand. The full description of equilibrium needs to specify the behavior of the market maker off the equilibrium path (to make it possible for the strategic traders to analyze the profitability of potential deviations). By analogy with perfect Bayesian equilibrium, our definition restricts the beliefs of the market maker on the equilibrium path, where they are pinned down by Bayes rule, and does not restrict them off the equilibrium path.

[8]In principle, we could consider a more general definition of linear equilibrium and allow the strategies and the pricing rule to potentially have nonzero intercepts. However, one can show that in our setting, linear equilibria with nonzero intercepts do not exist. The proof of this statement is available upon request.

Depth $\gamma = -\left(b + \sqrt{b^2 - 4ac}\right)/2a$, where

$$\begin{aligned} a &= -A_v^T \Sigma_{diag} A_v, \\ b &= A_v^T \left(2\Sigma_{diag} + \Lambda\right) A_u + \Sigma_{uM} \Sigma_{MM}^{-1} \Sigma_{Mv} - \sigma_{uv}, \\ c &= Var(A_u^T \theta - u | \theta_M). \end{aligned}$$

(The proof shows that $a < 0$, $c > 0$, and thus $\gamma > 0$.) Equilibrium pricing rule and strategies are then as follows:

$$\begin{aligned} \beta_D &= \frac{1}{\gamma}, \\ \beta_M &= \Sigma_{MM}^{-1} \left(\Sigma_{Mv} - \Sigma_{\theta M}^T A_v\right) - \beta_D \Sigma_{MM}^{-1} \left(\Sigma_{Mu} - \Sigma_{\theta M}^T A_u\right); \\ \alpha &= \frac{1}{\beta_D} A_v - A_u. \end{aligned}$$

These expressions are simplified in the case $k_M = 0$, when the market maker does not observe any private signals (other than the aggregate demand D).[9] In that case,

$$\begin{aligned} a &= -A_v^T \Sigma_{diag} A_v, \\ b &= A_v^T \left(2\Sigma_{diag} + \Lambda\right) A_u - \sigma_{uv}, \\ c &= Var(A_u^T \theta - u), \end{aligned}$$

where

$$\begin{aligned} \Lambda &= \Sigma_{\theta\theta} + \Sigma_{diag}, \\ A_u &= \Lambda^{-1} \Sigma_{\theta u}, \\ A_v &= \Lambda^{-1} \Sigma_{\theta v}. \end{aligned}$$

The expressions are simplified even further if, in addition, the demand from liquidity traders, u, is uncorrelated with other random variables. Then $b = 0$ and $\gamma = \sqrt{\frac{\sigma_{uu}}{A_v^T \Sigma_{diag} A_v}}$, and so

$$\beta_D = \sqrt{\frac{A_v^T \Sigma_{diag} A_v}{\sigma_{uu}}} \quad \text{and} \quad \alpha = \sqrt{\frac{\sigma_{uu}}{A_v^T \Sigma_{diag} A_v}} A_v.$$

[9] Strictly speaking, our proof does not apply directly to the case $k_M = 0$ since, for example, it uses the inverse of the variance-covariance matrix of θ_M. However, one can drop all terms related to θ_M from the proof and immediately obtain the proof for that case. Alternatively, one can consider a model in which the market maker observes a signal that is independent of all other random variables. The equilibrium in that model will be economically equivalent to one in which $k_M = 0$.

5 Applications

In this section, we illustrate the general framework presented above with several specific applications. We first present a simple yet seemingly counterintuitive example in which a trader informed about the value of the security trades in the direction opposite to that value. Next, we study what happens when one of the strategic traders is informed about the demand of liquidity traders. We conclude by analyzing two examples in which the market maker possesses private information about the value of the security (beyond the aggregate market demand), and study how this information gets incorporated into the price of the security and how it affects equilibrium trading strategies and the sensitivity of equilibrium prices to market demand.

5.1 Trading "against" own signal

In this section, we present an example of information structure under which a trader who receives a signal about the value of the security trades in the opposite direction: i.e., when based on his information the value of the security is positive, he shorts the security, and when it is negative, he buys it. Note that since our model is a one-shot game, there cannot be any incentives to do that of the form "I will try to mislead others first, and then take advantage of the mispricing."

Example 1 *The value of the security is distributed as* $v \sim N(0,1)$*. There are two strategic traders. Trader 1 observes a noisy estimate of* v*:* $\theta_1 = v + \rho_1 \xi$*, where* ξ *is a normally distributed random variable (*$\xi \sim N(0,1)$*), independent of* v*, and* ρ_1 *is a parameter that determines how accurate trader 1's signal is (e.g., if* $\rho_1 = 0$*, then trader 1 observes* v *exactly, and if* ρ_1 *is very large, then trader 1's signal is not very accurate). Trader 2 also observes a noisy estimate of* v*:* $\theta_2 = v + \rho_2 \xi$*. Finally, there is demand from liquidity traders,* $u \sim N(0,1)$*, which is independent of all other random variables. Formally, the resulting correlation matrix is*

$$\Omega = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1+\rho_1^2 & 1+\rho_1\rho_2 & 0 \\ 1 & 1+\rho_1\rho_2 & 1+\rho_2^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

From the analysis in the preceding section, we know that in the unique linear equilibrium, the pricing rule is characterized by some $\beta_D > 0$. We also know that the strategies of traders 1 and 2 are characterized by:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{\beta_D} \Lambda^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{3}$$

where $\Lambda = \begin{pmatrix} 2+2\rho_1^2 & 1+\rho_1\rho_2 \\ 1+\rho_1\rho_2 & 2+2\rho_2^2 \end{pmatrix}$.

Using the matrix inversion formula and setting $\delta = \frac{1}{\beta_D \cdot \det(\Lambda)}$ (which is positive, since Λ is positive definite), we get

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \delta \begin{pmatrix} 2+2\rho_2^2 & -1-\rho_1\rho_2 \\ -1-\rho_1\rho_2 & 2+2\rho_1^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \delta \begin{pmatrix} 1+2\rho_2^2-\rho_1\rho_2 \\ 1+2\rho_1^2-\rho_1\rho_2 \end{pmatrix}. \tag{4}$$

Thus, if $\rho_1 = 2\rho_2 + \frac{1}{\rho_2}$, trader 1 *never trades*, despite θ_1 being informative about the value of the security, and for $\rho_1 > 2\rho_2 + \frac{1}{\rho_2} > 0$, trader 1 *always trades in the direction opposite to his signal* θ_1, despite θ_1 being positively correlated with the value of the security, v. Similarly, if ρ_2 is equal to or greater than $2\rho_1 + \frac{1}{\rho_2}$, then trader 2 does not trade or trades in the direction opposite to his signal.

To get the intuition behind this equilibrium, consider a slight variation of Example 1.

Example 2 *The value of the security is $v \sim N(0,1)$. There are two strategic traders. Trader 1 observes a noisy estimate of v: $\theta_1 = v + \xi$, where $\xi \sim N(0,1)$, independent of v. Trader 2 observes ξ: $\theta_2 = \xi$. The demand from liquidity traders, $u \sim N(0,1)$, is independent of all other random variables. The resulting correlation matrix is*

$$\Omega = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case, $\Lambda = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$ and

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{\beta_D}\Lambda^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \delta \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \delta \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{5}$$

for some $\delta > 0$, and thus trader 2 trades in the direction opposite to his signal. Note that in this example, trader 2 is not informed about the value of the security: his signal ξ is independent of v. However, he is informed about the bias in trader 1's signal, and thus knows in which direction trader 1 is likely to "err" when submitting his demand. Thus, trader 2, by partly "undoing" this error (i.e., trading against it), can in expectation make positive profits, despite not having any direct information about the value of the security. In a sense, while trader 1 trades on "fundamental" information, trader 2 trades on "technical" information: trader 1 would be able to make money even without having trader 2 around, but trader 2's ability to make a profit depends critically on having trader 1 around and on exploiting that trader's mistakes.

In Example 1, the intuition is similar. If, say, ρ_1 is large relative to $2\rho_2 + \frac{1}{\rho_2}$, then the main "chunk" of trader 1's information is about the mistake that trader 2 makes, and not about the fundamental value of the security. This causes trader 1 to want to "undo" that mistake and trade "against" his signal, while trader 2 continues to trade in a natural direction. When $\rho_1 = 2\rho_2 + \frac{1}{\rho_2}$,

the incentives of trader 1 to trade on "fundamental" information (the positive correlation of his signal with the value of the security) and on the "technical" information (the positive correlation of his signal with the mistake of trader 2) cancel out, and trader 1 ends up completely staying away from the market.

5.2 Trading in the presence of information about liquidity demand

In this section, we study what happens when one of the strategic traders does not know anything about the value of the security, but is informed about the amount of liquidity trading, and compare the equilibrium to that of the standard model without such a trader.

Example 3 *The value of the security is distributed as* $v \sim N(0, \sigma_{vv})$*, and the demand from liquidity traders is distributed as* $u \sim N(0, \sigma_{uu})$*, independently of* v*. There are two strategic traders. Trader 1's signal is equal to* v*:* $\theta_1 = v$*. He is fully informed about the value of the security, just like in the standard Kyle model. Trader 2 is uninformed about the value of the security, but has insider information about the demand from liquidity traders:* $\theta_2 = u$*. Formally, the correlation matrix is*

$$\Omega = \begin{pmatrix} \sigma_{vv} & \sigma_{vv} & 0 & 0 \\ \sigma_{vv} & \sigma_{vv} & 0 & 0 \\ 0 & 0 & \sigma_{uu} & \sigma_{uu} \\ 0 & 0 & \sigma_{uu} & \sigma_{uu} \end{pmatrix}.$$

The auxiliary matrices in this example are:

$$\Lambda = \begin{pmatrix} 2\sigma_{vv} & 0 \\ 0 & 2\sigma_{uu} \end{pmatrix}, \; A_u = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \text{ and } A_v = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}.$$

Coefficient b in the quadratic equation is equal to zero, and therefore

$$\begin{aligned} \gamma &= \sqrt{-\frac{c}{a}} = \sqrt{\frac{\sigma_{uu}}{\sigma_{vv}}}; \\ \alpha_1 &= \frac{1}{2}\gamma = \frac{1}{2}\sqrt{\frac{\sigma_{uu}}{\sigma_{vv}}}; \\ \alpha_2 &= -\frac{1}{2}. \end{aligned}$$

For comparison, if the second strategic trader was not present, the model would reduce to the standard model of Kyle (1985), and the equilibrium would be characterized by

$$\begin{aligned} \gamma &= 2\sqrt{\frac{\sigma_{uu}}{\sigma_{vv}}}; \\ \alpha_1 &= \sqrt{\frac{\sigma_{uu}}{\sigma_{vv}}}. \end{aligned}$$

In other words, when the second strategic trader (who is informed about the demand from liquidity traders) is present in the market, that trader "takes away" one half of that "liquidity" demand. As a result, the first strategic trader, who knows the value of the security, trades half as much as he would in the absence of that second trader, and the market maker's pricing rule is twice as sensitive. As a result, for any realization of v and u, the price in the market with the second strategic trader will be exactly the same as that in the market without that trader—and thus the informativeness of prices in not affected in either direction by whether there is a trader in that market who observes the trading flow from liquidity traders. Likewise, the expected loss of liquidity traders is also unaffected by the presence of a trader who observes their demand. Since, by construction, the market maker in expectation breaks even, it has to be the case that the profit of the second strategic trader comes out of the first trader's pocket. In fact, the second trader takes away exactly one half of the first trader's profit.[10] Also, as in Example 2, the second trader is trading on "technical" information, and is only able to make a profit because a "fundamental" trader is also present in the market.

5.3 Trading in the presence of an informed market maker

In the preceding examples, as in much of the literature, the market maker does not receive any information other than the aggregate demand coming from strategic and liquidity traders. In this subsection, we turn to examples in which the market maker does possess some additional information. We show how this information affects the strategies of other traders and illustrate the interplay between the weight the market maker places on this additional information and the weight he places on market demand.

Example 4 *Let ϵ_1, ϵ_2, ϵ_3, ϵ_4, and ϵ_5 be independent random variables, each distributed normally with mean 0 and variance 1. The value of the security is $v = \epsilon_1 + \epsilon_2$. The demand from liquidity traders is $u = \epsilon_5$. There are two partially informed strategic traders and a partially informed market maker. Trader 1's signal is $\theta_1 = \epsilon_1 + \epsilon_3$. Trader 2's signal is $\theta_2 = \epsilon_2 + \epsilon_4$. Market maker's signal is $\theta_M = \epsilon_2$. Note that while Trader 1 possesses some "unique" information about the value of the security, Trader 2 does not (because ϵ_2 is observed by the market maker, and ϵ_4 is pure noise). Formally, the correlation matrix is*

$$\Omega = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

[10] To see this, note that prices in the two market are always the same, realization by realization, while the demand of the first strategic trader, in the presence of the second one, is exactly one half of what it would be in the absence of that trader.

The auxiliary matrices in this example are:

$$\Lambda = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix},\ A_u = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \text{and } A_v = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}.$$

Therefore, in this case, we have

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{\beta_D} \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix},$$

and so $\alpha_2 = 0$. Thus, Trader 2 does not trade in equilibrium. This illustrates a more general phenomenon: In equilibrium, a strategic trader cannot make a positive profit (and does not trade) if his information is the same as or worse than (in information-theoretic sense) that of the market maker.[11]

Our final example considers a sequence of markets, indexed by the number of strategic traders, n. All traders receive the same information, which is imperfectly correlated with both the value of the asset and the market maker's information.

Example 5 *The value of the security (v), the demand from liquidity traders (u), and two information shocks (ϵ_1 and ϵ_2) are all distributed normally with mean 0 and variance 1, independently of each other. There are n identically informed strategic traders and a partially informed market maker. Each strategic trader observes a signal $\theta_1 = v + \epsilon_1$. The market maker observes a signal $\theta_M = v + \epsilon_2$. Formally (indexing all matrices by the number of strategic traders in the market, n), the correlation matrix is*

$$\Omega^n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 2 & 2 & \cdots & 2 & 1 & 0 \\ 1 & 2 & 2 & \cdots & 2 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & 2 & \cdots & 2 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

The auxiliary matrices are:

$$\Lambda^n = \begin{pmatrix} 3\frac{1}{2} & 1\frac{1}{2} & \cdots & 1\frac{1}{2} & 1\frac{1}{2} \\ 1\frac{1}{2} & 3\frac{1}{2} & \cdots & 1\frac{1}{2} & 1\frac{1}{2} \\ 1\frac{1}{2} & 1\frac{1}{2} & \cdots & 1\frac{1}{2} & 1\frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1\frac{1}{2} & 1\frac{1}{2} & \cdots & 3\frac{1}{2} & 1\frac{1}{2} \\ 1\frac{1}{2} & 1\frac{1}{2} & \cdots & 1\frac{1}{2} & 3\frac{1}{2} \end{pmatrix},\ \text{so that } (\Lambda^n)^{-1} = \begin{pmatrix} \frac{3n+1}{6n+8} & \frac{-3}{6n+8} & \cdots & \frac{-3}{6n+8} & \frac{-3}{6n+8} \\ \frac{-3}{6n+8} & \frac{3n+1}{6n+8} & \cdots & \frac{-3}{6n+8} & \frac{-3}{6n+8} \\ \frac{-3}{6n+8} & \frac{-3}{6n+8} & \cdots & \frac{-3}{6n+8} & \frac{-3}{6n+8} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-3}{6n+8} & \frac{-3}{6n+8} & \cdots & \frac{3n+1}{6n+8} & \frac{-3}{6n+8} \\ \frac{-3}{6n+8} & \frac{-3}{6n+8} & \cdots & \frac{-3}{6n+8} & \frac{3n+1}{6n+8} \end{pmatrix};$$

[11] We omit the proof of this statement; it is available upon request.

$$A_u^n = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}; \text{ and } A_v^n = \begin{pmatrix} \frac{1}{3n+4} \\ \vdots \\ \frac{1}{3n+4} \end{pmatrix}.$$

Coefficient b in the quadratic equation is equal to zero, and so

$$\begin{aligned} \gamma^n &= \sqrt{-\frac{c}{a}} = \frac{3n+4}{\sqrt{2n}}; \\ \alpha_i^n &= \gamma A_{vi} = \frac{1}{\sqrt{2n}}; \\ \beta_M^n &= \frac{1}{2}\big(1 - \frac{n}{3n+4}\big) = \frac{2n+4}{6n+8} = \frac{n+2}{3n+4}. \end{aligned}$$

Note that the weight β_M that the market maker places on his own signal is not constant in n. If there were no strategic traders at all, and only noise traders, it would be equal to $\frac{1}{2} = \frac{Cov(v,\theta_M)}{Var(\theta_M)}$. As n grows, this weight is monotonically decreasing (converging to $\frac{1}{3}$ in the limit). Thus, it is not the case that the market maker simply combines the information contained in his own signal and the additional information contained in the aggregate demand D "additively"—the interplay between the two sources of information is more intricate, and the weight that the market maker places on his own signal depends on the information structure of the strategic traders.

The second observation concerns the informativeness of prices. Take any n, and consider a realization of θ_1, θ_M, and u. In this realization, demand D is equal to $n\alpha_i^n\theta_1 + u = \frac{n}{\sqrt{2n}}\theta_1 + u$, and the market price P set by the market maker is equal to $\beta_D^n D + \beta_M^n\theta_M = \frac{n}{3n+4}\theta_1 + \frac{n+2}{3n+4}\theta_M + \frac{\sqrt{2n}}{3n+4}u$. Now, fix the realization of random variables, and let the number of strategic traders, n, grow to infinity. Then price P converges to $\frac{1}{3}\theta_1 + \frac{1}{3}\theta_M$. But notice that this expression is precisely the expected value of the asset, v, conditional on the information available in the market: u is uninformative, because it is independent of all other random variables, and $E[v|\theta_1,\theta_M] = Cov\left(v, \begin{pmatrix} \theta_1 \\ \theta_M \end{pmatrix}\right)^T Var\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}^{-1}\begin{pmatrix} \theta_1 \\ \theta_M \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} \theta_1 \\ \theta_M \end{pmatrix} = \frac{1}{3}\theta_1 + \frac{1}{3}\theta_M$. Hence, as the number of strategic traders becomes large, their information and the information of the market maker get incorporated into the market price with precisely the weights that a Bayesian observer with access to *all* information available in the market would assign. In other words, as the number of strategic traders becomes large, all information available in the market is aggregated and revealed by the market price. In the next section, we show that this is not a coincidence: the information aggregation result holds very generally.

6 Information aggregation

We now turn to the second main result of our paper: the aggregation of dispersed information when the number of traders becomes large.

Specifically, consider a sequence of markets, indexed by $m = 1, 2, \ldots$. Every market is in the

general framework of Section 3. In every market, there are n groups of strategic traders, with at least one trader in each group. Index i, $1 \leq i \leq n$, now denotes a group of traders. The size of group i in market m is denoted by $\ell_i^{(m)}$. All traders in the same group i receive the same signal $\theta_i \in \mathbb{R}^{k_i}$. The notation from the preceding sections carries over, except that θ_i now denotes the signal common to all the traders in group i.

The variance-covariance matrix Ω of vector $\mu = (v; \theta; \theta_M; u)$ is the same in all markets m. The number of traders in each group, however, changes with m: specifically, we assume that for every i, $\lim_{m\to\infty} \ell_i^{(m)} = \infty$, i.e., all groups become large as m becomes large. Note, however, that we do not impose any restrictions on the rates of growth of those groups: e.g., the sizes of some groups may grow much faster than those of other groups.

We impose the same two conditions on matrix Ω as in Section 3, and, in addition, we place the following restriction:

Assumption 3 The variance-covariance matrix of random vector $(\theta; \theta_M; u)$ is full rank.

It follows from Theorem 1 that for each m, there exists a unique linear equilibrium in the corresponding market. Let $p^{(m)}$ denote the random variable that is equal to the resulting price in the unique linear equilibrium of market m.

We can now state and prove our main result on information aggregation in large markets: If the demand from liquidity traders is positively correlated with the true value of the asset (conditional on other signals), then prices in large markets aggregate all available information: $p^{(m)}$ converges to $E[v|\theta, \theta_M, u]$. If the demand from liquidity traders is negatively correlated with the true value of the asset, then prices in large markets aggregate all available information *except* that contained in liquidity demand: $p^{(m)}$ converges to $E[v|\theta, \theta_M]$. If the demand from liquidity traders is uncorrelated with the true value of the asset, then both statements are true: $p^{(m)}$ converges to $E[v|\theta, \theta_M, u] = E[v|\theta, \theta_M]$.

Theorem 2

- *If* $Cov(u, v|\theta, \theta_M) \geq 0$, *then* $\lim_{m\to\infty} E\left[\left(p^{(m)} - E[v|\theta, \theta_M, u]\right)^2\right] = 0$.
- *If* $Cov(u, v|\theta, \theta_M) \leq 0$, *then* $\lim_{m\to\infty} E\left[\left(p^{(m)} - E[v|\theta, \theta_M]\right)^2\right] = 0$.

The proof of Theorem 2 is in Appendix C. Intuitively, when the number of informed traders of each type is large, the information of each strategic "type" has to be (almost) fully incorporated into the market price, since otherwise each trader of that type would be able to make a non-negligible profit, which cannot happen in equilibrium. The signal of the market maker gets incorporated into the market price by construction. With liquidity demand, however, the situation is more subtle. When liquidity demand is positively correlated with the asset value ($Cov(u, v|\theta, \theta_M) > 0$), equilibrium strategies and market depth adjust precisely in a way that makes liquidity demand get incorporated into the market price "correctly," i.e., with the same weight as it would be incorporated into the market price by a Bayesian observer who was fully informed about all the random

variables in the model (except value v). As a result, price $p^{(m)}$ converges to $E[v|\theta, \theta_M, u]$, and so all information available in the market is incorporated into the market price. However, when liquidity demand is negatively correlated with the value of the asset ($Cov(u, v|\theta, \theta_M) < 0$), this cannot happen. In equilibrium, aggregate demand always enters the market price with a positive sign (sensitivity β_D is positive). Thus, liquidity demand also enters the market price with a positive sign. However, a fully informed Bayesian observer would put a *negative* weight on liquidity demand—which cannot happen in any linear equilibrium, for any parameter values. So what happens instead as m becomes large is that the variance of the aggregate demand from informed traders grows to infinity (in contrast to the case $Cov(u, v|\theta, \theta_M) > 0$, in which it converges to a finite value). And thus as m grows, liquidity demand u has less and less impact on the market price, and in the limit it has no impact at all: price $p^{(m)}$ converges to $E[v|\theta, \theta_M]$. The same happens in the case $Cov(u, v|\theta, \theta_M) = 0$, for the same reason, but in that case $E[v|\theta, \theta_M]$ is equal to $E[v|\theta, \theta_M, u]$, and so price $p^{(m)}$ does converge to the expected value of the asset given all the information available in the market.

Appendix A: Notation

We decompose the variance-covariance matrix Ω of the vector $(v; \theta_1; \ldots; \theta_n; \theta_M; u)$ as follows:

$$\begin{pmatrix} \sigma_{vv} & \Sigma_{v1} & \cdots & \Sigma_{vn} & \Sigma_{vM} & \sigma_{vu} \\ \Sigma_{1v} & \Sigma_{11} & \cdots & \Sigma_{1n} & \Sigma_{1M} & \Sigma_{1u} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \Sigma_{nv} & \Sigma_{n1} & \cdots & \Sigma_{nn} & \Sigma_{nM} & \Sigma_{nu} \\ \Sigma_{Mv} & \Sigma_{M1} & \cdots & \Sigma_{Mn} & \Sigma_{MM} & \Sigma_{Mu} \\ \sigma_{uv} & \Sigma_{u1} & \cdots & \Sigma_{un} & \Sigma_{uM} & \sigma_{uu} \end{pmatrix}.$$

In this matrix, every σ represents a (scalar) variance or covariance of the asset value and/or the demand of liquidity traders, and every Σ represents a (generally non-scalar) variance-covariance matrix of an element of the vector $(v; \theta_1; \ldots; \theta_n; \theta_M; u)$ with another element. We also introduce notation for variance-covariance matrices of the entire vector of strategic traders' signals, $\theta = (\theta_1; \cdots; \theta_n)$, with itself and with other elements of vector μ. Specifically:

$$\Sigma_{\theta\theta} = Var\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \cdots & \Sigma_{1n} \\ \vdots & \ddots & \vdots \\ \Sigma_{n1} & \cdots & \Sigma_{nn} \end{pmatrix}, \qquad \Sigma_{\theta M} = Cov\left(\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \theta_M\right) = \begin{pmatrix} \Sigma_{1M} \\ \vdots \\ \Sigma_{nM} \end{pmatrix},$$

$$\Sigma_{\theta v} = Cov\left(\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, v\right) = \begin{pmatrix} \Sigma_{1v} \\ \vdots \\ \Sigma_{nv} \end{pmatrix}, \qquad \Sigma_{\theta u} = Cov\left(\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, u\right) = \begin{pmatrix} \Sigma_{1u} \\ \vdots \\ \Sigma_{nu} \end{pmatrix}.$$

In addition, we will use the following matrices:

$$\Sigma_{diag} = \begin{pmatrix} \Sigma_{11} & 0 & 0 & 0 \\ 0 & \Sigma_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \Sigma_{nn} \end{pmatrix},$$

$$\Lambda = \Sigma_{diag} + \Sigma_{\theta\theta} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{\theta M}^{T},$$

$$A_u = \Lambda^{-1}(\Sigma_{\theta u} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{Mu}),$$

$$A_v = \Lambda^{-1}(\Sigma_{\theta v} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{Mv}).$$

(We will show in the proof of our main result that Λ is invertible).

Appendix B: Proof of Theorem 1

The proof of Theorem 1 is constructive. For convenience, it is broken into several steps. Step 1 expresses the linear relationship implied by condition (i) of the definition of equilibrium, that price must be equal to the expected value of the security conditional on the information available to the market maker. Step 2 derives the best response of a strategic trader to a linear pricing rule and linear strategies of other strategic traders, and shows that this best response is linear. It also establishes that in equilibrium, coefficient β_D has to be positive. Step 3 summarizes the equations in Steps 1 and 2 and reorganizes them in a system of three "almost" linear equations (they are all linear if one scalar variable, $\gamma = 1/\beta_D$, is fixed). Step 4 reduces this system of equations to one quadratic equation in γ. Step 5 shows that this quadratic equation has exactly one positive root, thus completing the proof.

Step 1. Let $\alpha = (\alpha_1; \ldots; \alpha_n)$ be a profile of linear strategies for the strategic traders. Each α_i in this profile is a vector $(\alpha_i^1; \ldots; \alpha_i^{k_i}) \in \mathbb{R}^{k_i}$, corresponding to linear strategy

$$d_i(\theta_i) = \alpha_i^1\theta_i^1 + \cdots + \alpha_i^{k_i}\theta_i^{k_i},$$

where $\theta_i^1, \ldots, \theta_i^{k_i}$ are the elements of vector $\theta_i \in \mathbb{R}^{k_i}$.

Take any linear pricing rule $(\beta_M; \beta_D)$, $\beta_M \in \mathbb{R}^{k_M}$, $\beta_D \in \mathbb{R}$. For convenience, let vector $\beta = (\beta_M; \beta_D)$ summarize the pricing rule and let random vector $\eta = (\theta_M; D = \alpha^T\theta + u)$ denote the information available to the market maker when she sets the price. Then for this pricing rule to be consistent with profile α, condition (i) of the definition of equilibrium requires that

$$\beta^T\eta = E[v|\eta],$$

which is equivalent to the following condition:[12]

$$Cov(v, \eta) = \beta^T Var(\eta).$$

Expressing $Cov(v, \eta)$ and $Var(\eta)$ using the notation introduced in Appendix A, we thus get the following equivalent characterization of condition (i) of the definition of equilibrium:

$$\left(\beta_M^T, \beta_D\right) \begin{pmatrix} \Sigma_{MM} & \Sigma_{\theta M}^T \alpha + \Sigma_{Mu} \\ \alpha^T \Sigma_{\theta M} + \Sigma_{Mu}^T & \alpha^T \Sigma_{\theta\theta} \alpha + 2\Sigma_{\theta u}^T \alpha + \sigma_{uu} \end{pmatrix} = \left(\Sigma_{vM}, \Sigma_{\theta v}^T \alpha + \sigma_{vu}\right). \tag{6}$$

Step 2. We now consider the optimization problem of a strategic trader i. Suppose he observes signal realization $\tilde{\theta}_i$ of signal θ_i, and subsequently submits demand d. Assuming that other traders $j \neq i$ follow linear strategies α_j, and that the market maker follows a linear pricing rule $(\beta_M; \beta_D)$, the expected profit of trader i from submitting demand d when observing realization $\tilde{\theta}_i$ is equal to

$$E\left[d\left(v - \beta_M^T \theta_M - \beta_D\left(d + \sum_{j \neq i} \alpha_j^T \theta_j + u\right)\right) \middle| \theta_i = \tilde{\theta}_i\right]. \tag{7}$$

Using the fact that d is a choice variable, and thus d and d^2 are constants from the point of view of taking expectations, we can rewrite equation (7) as

$$d \cdot E\left[v - \beta_M^T \theta_M - \beta_D\left(\sum_{j \neq i} \hat{\alpha}_j^T \theta_j + u\right) \middle| \theta_i = \tilde{\theta}_i\right] - d^2 \cdot \beta_D. \tag{8}$$

Now, if $\beta_D < 0$, trader i can make an arbitrarily large expected profit, and no single d maximizes it—hence, β_D cannot be negative in equilibrium.

If $\beta_D = 0$, and $E\left[v - \beta_M^T \theta_M | \theta_i = \tilde{\theta}_i\right] \neq 0$, then again trader i can make an arbitrarily large expected profit, and no single d maximizes it. But it follows from Assumption 1 in the model[13] that for at least one trader i, for at least some (in fact, for almost all) realizations $\tilde{\theta}_i$, we have $E\left[v - \beta_M^T \theta_M | \theta_i = \tilde{\theta}_i\right] \neq 0$—hence, β_D cannot be equal to zero in equilibrium.

[12] To see the equivalence, note first that $\beta^T \eta = E[v|\eta] \implies Cov(v, \eta) = Cov(E[v|\eta], \eta) = Cov(\beta^T \eta, \eta) = \beta^T Var(\eta)$. To go in the opposite direction, note that $Cov(v, \eta) = \beta^T Var(\eta) = Cov(\beta^T \eta, \eta) \implies Cov(v - \beta^T \eta, \eta) = 0$. Since variables $v - \beta^T \eta$ and η are jointly normal, $Cov(v - \beta^T \eta, \eta) = 0$ implies that they are independent, and thus for every realization $\tilde{\eta}$ of random variable η, $E[v - \beta^T \eta | \eta = \tilde{\eta}] = E[v - \beta^T \eta] = 0$, which implies that for every realization $\tilde{\eta}$, $E[v|\eta = \tilde{\eta}] = E[\beta^T \eta | \eta = \tilde{\eta}] = \beta^T \tilde{\eta}$.

[13] Assumption 1 says that at least one strategic trader i has some useful information beyond that contained in the market maker's signal: $Cov(v, \theta | \theta_M) \neq 0$.

Finally, if $\beta_D > 0$, then there is a unique d maximizing the expected profit:

$$d^* = \frac{1}{2\beta_D} E\left[v - \beta_M^T \theta_M - \beta_D \left(\sum_{j \neq i} \alpha_j^T \theta_j + u \right) \middle| \theta_i = \tilde{\theta}_i \right] \tag{9}$$

$$= \frac{1}{2\beta_D} \left(\Sigma_{iv}^T - \beta_M^T \Sigma_{iM}^T - \beta_D \left(\sum_{j \neq i} \alpha_j^T \Sigma_{ij}^T + \Sigma_{iu}^T \right) \right) \Sigma_{ii}^{-1} \tilde{\theta}_i, \tag{10}$$

where equation (10) is the standard projection/signal extraction formula, which can be used because of the joint normality of the relevant variables. Note that d^* is a linear function of $\tilde{\theta}_i$, and vector α_i is uniquely determined by pricing rule $(\beta_M; \beta_D)$ and strategies α_j for $j \neq i$.

Step 3. It therefore follows from the arguments in Steps 1 and 2 that profile of strategies α and pricing rule $(\beta_M; \beta_D)$ form a linear equilibrium if and only if $\beta_D > 0$ and the following two conditions hold:

(i) $\left(\beta_M^T, \beta_D\right) \begin{pmatrix} \Sigma_{MM} & \Sigma_{\theta M}^T \alpha + \Sigma_{Mu} \\ \alpha^T \Sigma_{\theta M} + \Sigma_{Mu}^T & \alpha^T \Sigma_{\theta\theta} \alpha + 2\Sigma_{\theta u}^T \alpha + \sigma_{uu} \end{pmatrix} = \left(\Sigma_{vM}, \Sigma_{\theta v}^T \alpha + \sigma_{vu}\right);$

(ii) for all i, $\alpha_i = \frac{1}{2\beta_D} \left(\Sigma_{iv}^T - \beta_M^T \Sigma_{iM}^T - \beta_D \left(\sum_{j \neq i} \alpha_j^T \Sigma_{ij}^T + \Sigma_{iu}^T \right) \right) \Sigma_{ii}^{-1}$.

We will now show that there is a unique profile (α, β) satisfying these conditions, thus proving the existence and uniqueness of linear equilibrium.

First, we re-write condition (ii), for all i, as:

$$2\Sigma_{ii}\alpha_i = \frac{1}{\beta_D}(\Sigma_{iv} - \Sigma_{iM}\beta_M) - \sum_{j \neq i} \Sigma_{ij}\alpha_j - \Sigma_{iu}$$

or equivalently

$$\Sigma_{ii}\alpha_i + \sum_j \Sigma_{ij}\alpha_j = \frac{1}{\beta_D}(\Sigma_{iv} - \Sigma_{iM}\beta_M) - \Sigma_{iu}. \tag{11}$$

"Stacking" equations (11) for all i one under another, and rewriting the resulting system of equations in matrix form using the notation defined in Appendix A, we obtain the following condition (equivalent to condition (ii)):

$$(\Sigma_{diag} + \Sigma_{\theta\theta})\alpha = \gamma\Sigma_{\theta v} - \Sigma_{\theta M}\beta_M' - \Sigma_{\theta u}, \tag{12}$$

where for convenience we define $\gamma = 1/\beta_D$, $\beta_M' = \beta_M/\beta_D$.

Next, using this notation, and transposing the matrix equation in condition (i), that condition can be written as a system of two equations:

$$\Sigma_{MM}\beta_M' + \Sigma_{\theta M}^T \alpha + \Sigma_{Mu} = \gamma\Sigma_{Mv}, \tag{13}$$

$$\alpha^T \Sigma_{\theta M}\beta_M' + \Sigma_{uM}\beta_M' + \alpha^T \Sigma_{\theta\theta}\alpha + 2\Sigma_{\theta u}^T \alpha + \sigma_{uu} = \gamma(\Sigma_{\theta v}^T \alpha + \sigma_{vu}). \tag{14}$$

Step 4. We will now solve the system of equations (12), (13), and (14). Equation (13) allows us to express β'_M as a function of α and γ:

$$\beta'_M = \Sigma_{MM}^{-1}\left(\gamma\Sigma_{Mv} - \Sigma_{\theta M}^T\alpha - \Sigma_{Mu}\right). \tag{15}$$

We then plug this expression of β'_M into equation (12):

$$(\Sigma_{diag} + \Sigma_{\theta\theta})\alpha = \gamma\Sigma_{\theta v} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\left(\gamma\Sigma_{Mv} - \Sigma_{\theta M}^T\alpha - \Sigma_{Mu}\right) - \Sigma_{\theta u},$$

or, isolating α on the left-hand side and collecting the terms with γ,

$$\left(\Sigma_{diag} + \Sigma_{\theta\theta} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{\theta M}^T\right)\alpha = \left(\Sigma_{\theta v} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{Mv}\right)\gamma - \left(\Sigma_{\theta u} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{Mu}\right).$$

Note that

$$\begin{aligned}\Sigma_{\theta\theta} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{\theta M}^T &= Var(\theta) - Cov(\theta,\theta_M)Var(\theta_M)^{-1}Cov(\theta_M,\theta)\\ &= Var(\theta|\theta_M),\end{aligned}$$

where the last equation follows from the standard projection formula for multivariate normal distributions. Thus, matrix $\Sigma_{\theta\theta} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{\theta M}^T$ is positive semidefinite, and matrix $\Sigma_{diag} + \Sigma_{\theta\theta} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{\theta M}^T$ is positive definite (and thus invertible). Letting

$$\begin{aligned}\Lambda &= \Sigma_{diag} + \Sigma_{\theta\theta} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{\theta M}^T,\\ A_u &= \Lambda^{-1}\left(\Sigma_{\theta u} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{Mu}\right),\\ A_v &= \Lambda^{-1}\left(\Sigma_{\theta v} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{Mv}\right),\end{aligned}$$

we can express α as a linear function of γ:

$$\alpha = \gamma A_v - A_u.$$

Plugging this expression into equation (15), we can also express β'_M as a linear function of γ:

$$\begin{aligned}\beta'_M &= \Sigma_{MM}^{-1}\left(\gamma\Sigma_{Mv} - \Sigma_{\theta M}^T\left(\gamma A_v - A_u\right) - \Sigma_{Mu}\right)\\ &= \gamma\Sigma_{MM}^{-1}\left(\Sigma_{Mv} - \Sigma_{\theta M}^T A_v\right) - \Sigma_{MM}^{-1}\left(\Sigma_{Mu} - \Sigma_{\theta M}^T A_u\right).\end{aligned}$$

Using these expressions, we can now rewrite equation (14) as a quadratic equation of just one scalar variable, γ:

$$a\gamma^2 + b\gamma + c = 0, \tag{16}$$

where

$$\begin{aligned}
a &= A_v^T \Sigma_{\theta M} \Sigma_{MM}^{-1} \left(\Sigma_{Mv} - \Sigma_{\theta M}^T A_v \right) + A_v^T \Sigma_{\theta\theta} A_v - \Sigma_{\theta v}^T A_v, \\
b &= -A_v^T \Sigma_{\theta M} \Sigma_{MM}^{-1} \left(\Sigma_{Mu} - \Sigma_{\theta M}^T A_u \right) - A_u^T \Sigma_{\theta M} \Sigma_{MM}^{-1} \left(\Sigma_{Mv} - \Sigma_{\theta M}^T A_v \right) \\
&\quad + \Sigma_{uM} \Sigma_{MM}^{-1} \left(\Sigma_{Mv} - \Sigma_{\theta M}^T A_v \right) - 2 A_v^T \Sigma_{\theta\theta} A_u + 2 \Sigma_{\theta u}^T A_v + \Sigma_{\theta v}^T A_u - \sigma_{vu}, \\
c &= A_u^T \Sigma_{\theta M} \Sigma_{MM}^{-1} \left(\Sigma_{Mu} - \Sigma_{\theta M}^T A_u \right) - \Sigma_{uM} \Sigma_{MM}^{-1} \left(\Sigma_{Mu} - \Sigma_{\theta M}^T A_u \right) \\
&\quad + A_u^T \Sigma_{\theta\theta} A_u - 2 \Sigma_{\theta u}^T A_u + \sigma_{uu}.
\end{aligned}$$

Therefore, finding a linear equilibrium is equivalent to finding a positive root of equation (16). To prove that this equation has a unique such root, we first simplify the expressions for a, b, and c. (For the proof, it is sufficient to simplify a and c, but getting a simplified expression for b is useful for deriving an explicit analytic characterization of the equilibrium.) Starting with a:

$$\begin{aligned}
a &= A_v^T \Sigma_{\theta M} \Sigma_{MM}^{-1} \left(\Sigma_{Mv} - \Sigma_{\theta M}^T A_v \right) + A_v^T \Sigma_{\theta\theta} A_v - \Sigma_{\theta v}^T A_v, \\
&= A_v^T \left[\left(\Sigma_{\theta M} \Sigma_{MM}^{-1} \Sigma_{Mv} - \Sigma_{\theta v} \right) + \left(\Sigma_{\theta\theta} - \Sigma_{\theta M} \Sigma_{MM}^{-1} \Sigma_{\theta M}^T \right) A_v \right] \\
&= A_v^T \left[(-\Lambda A_v) + (\Lambda - \Sigma_{diag}) A_v \right] \\
&= -A_v^T \Sigma_{diag} A_v.
\end{aligned}$$

Next,

$$\begin{aligned}
b &= -A_v^T \Sigma_{\theta M} \Sigma_{MM}^{-1} \left(\Sigma_{Mu} - \Sigma_{\theta M}^T A_u \right) - A_u^T \Sigma_{\theta M} \Sigma_{MM}^{-1} \left(\Sigma_{Mv} - \Sigma_{\theta M}^T A_v \right) \\
&\quad + \Sigma_{uM} \Sigma_{MM}^{-1} \left(\Sigma_{Mv} - \Sigma_{\theta M}^T A_v \right) - 2 A_v^T \Sigma_{\theta\theta} A_u + 2 \Sigma_{\theta u}^T A_v + \Sigma_{\theta v}^T A_u - \sigma_{vu}, \\
&= 2 A_v^T \left(\Sigma_{\theta u} - \Sigma_{\theta M} \Sigma_{MM}^{-1} \Sigma_{Mu} \right) + A_u^T \left(\Sigma_{\theta v} - \Sigma_{\theta M} \Sigma_{MM}^{-1} \Sigma_{Mv} \right) \\
&\quad + 2 A_v^T \left(\Sigma_{\theta M} \Sigma_{MM}^{-1} \Sigma_{\theta M}^T - \Sigma_{\theta\theta} \right) A_u + \Sigma_{uM} \Sigma_{MM}^{-1} \Sigma_{Mv} - \sigma_{uv} \\
&= 2 A_v^T \Lambda A_u + A_u^T \Lambda A_v \\
&\quad + 2 A_v^T \left(\Sigma_{\theta M} \Sigma_{MM}^{-1} \Sigma_{\theta M}^T - \Sigma_{\theta\theta} \right) A_u + \Sigma_{uM} \Sigma_{MM}^{-1} \Sigma_{Mv} - \sigma_{uv} \\
&= A_v^T \left(2\Sigma_{diag} + \Lambda \right) A_u + \Sigma_{uM} \Sigma_{MM}^{-1} \Sigma_{Mv} - \sigma_{uv}.
\end{aligned}$$

Finally,

$$\begin{aligned}
c &= A_u^T \Sigma_{\theta M} \Sigma_{MM}^{-1} \left(\Sigma_{Mu} - \Sigma_{\theta M}^T A_u \right) - \Sigma_{uM} \Sigma_{MM}^{-1} \left(\Sigma_{Mu} - \Sigma_{\theta M}^T A_u \right) \\
&\quad + A_u^T \Sigma_{\theta\theta} A_u - 2 \Sigma_{\theta u}^T A_u + \sigma_{uu} \\
&= - \left(\Sigma_{uM} - A_u^T \Sigma_{\theta M} \right)^T \Sigma_{MM}^{-1} \left(\Sigma_{Mu} - \Sigma_{\theta M}^T A_u \right) \\
&\quad + A_u^T \Sigma_{\theta\theta} A_u - 2 \Sigma_{\theta u}^T A_u + \sigma_{uu} \\
&= \begin{pmatrix} A_u \\ -1 \end{pmatrix}^T C \begin{pmatrix} A_u \\ -1 \end{pmatrix},
\end{aligned}$$

where, letting $\hat{\theta}$ denote vector $(\theta; u)$,

$$\begin{aligned} C &= \begin{pmatrix} \Sigma_{\theta\theta} & \Sigma_{\theta u} \\ \Sigma_{\theta u}^T & \sigma_{uu} \end{pmatrix} - \begin{pmatrix} \Sigma_{\theta M} \\ \Sigma_{uM} \end{pmatrix} \Sigma_{MM}^{-1} \begin{pmatrix} \Sigma_{\theta M} \\ \Sigma_{uM} \end{pmatrix}^T \\ &= Var\left(\hat{\theta}\right) - Cov\left(\hat{\theta}, \theta_M\right) Var\left(\theta_M\right)^{-1} Cov\left(\theta_M, \hat{\theta}\right) \\ &= Var\left(\hat{\theta}|\theta_M\right). \end{aligned}$$

Step 5. We will now determine the signs of coefficients a and c.

Matrix Σ_{diag} is positive definite, by construction. Vector A_v is not equal to zero: Matrix Λ^{-1} is positive definite, and vector $\Sigma_{\theta v} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{Mv} = Cov(\theta, v|\theta_M)$ is not equal to zero (by Assumption 1 of the model). Thus, $a = -A_v^T \Sigma_{diag} A_v < 0$.

To determine the sign of coefficient c, note first that $c = Var(A_u^T\theta - u|\theta_M)$. So if we show that $c \neq 0$, it will immediately follow that $c > 0$.

If $A_u = 0$, then $c \neq 0$ follows from Assumption 2 of the model.

Suppose $A_u \neq 0$. It is convenient to introduce an auxiliary random variable, ϕ, drawn randomly from the normal distribution with mean zero and variance-covariance matrix Σ_{diag}, independent of all other random variables in the model. Note that matrix A_u now has a simple interpretation:

$$A_u = Var(\theta + \phi|\theta_M)^{-1} Cov(\theta, u|\theta_M) = Var(\theta + \phi|\theta_M)^{-1} Cov(\theta + \phi, u|\theta_M).$$

Let $\epsilon = u - A_u^T(\theta + \phi)$. Then $c = Var(\epsilon + A_u^T\phi|\theta_M)$. To show that $c > 0$, it is thus sufficient to show that $\epsilon + A_u^T\phi$ is not constant, conditional on θ_M. Consider $Cov(\epsilon + A_u^T\phi, A_u^T(\theta + \phi)|\theta_M)$.

On one hand, $Cov(\epsilon, A_u^T(\theta+\phi)|\theta_M) = Cov(u - A_u^T(\theta+\phi), A_u^T(\theta+\phi)|\theta_M) = Cov(u, \theta+\phi|\theta_M)A_u - A_u^T Var(\theta + \phi|\theta_M)A_u = 0$.

On the other hand, $Cov(A_u^T\phi, A_u^T(\theta+\phi)|\theta_M) = Var(A_u^T\phi|\theta_M) = A_u^T\Sigma_{diag}A_u$, which is not equal to zero, because $A_u \neq 0$ and Σ_{diag} is positive definite. Therefore, $Cov(\epsilon + A_u^T\phi, A_u^T(\theta + \phi)|\theta_M) \neq 0$, and thus $\epsilon + A_u^T\phi$ is not constant conditional on θ_M, and so $c > 0$.

Thus, $a < 0$, $c > 0$, and hence equation (16) has exactly one positive root. Therefore, there exists a unique linear equilibrium.

Appendix C: Proof of Theorem 2

Step 1. Consider first a specific market m, and, for convenience, drop superscript (m). We know there exists a unique linear equilibrium. It then has to be the case that any two strategic traders in the same group have the same linear strategy (otherwise, by swapping the strategies of these two traders, we would be able to obtain a different linear equilibrium). Let us denote by α_i the aggregate demand multiplier, in equilibrium, of group i: That is, given signal θ_i, each trader in the group submits demand $\frac{1}{\ell_i}\alpha_i^T\theta_i$.

With this notation, note that the expression for condition (i)—the market maker's inference

given his information—remains unchanged: Equations (13) and (14) remain valid. Condition (ii)—the best response of a strategic trader—is now slightly different. In this notation, it becomes: For all i,

$$\frac{1}{\ell_i}\alpha_i = \frac{1}{2\beta_D}\left(\Sigma_{iv}^T - \beta_M^T\Sigma_{iM}^T - \beta_D\left(\sum_{j\neq i}\alpha_j^T\Sigma_{ij}^T + \frac{\ell_i - 1}{\ell_i}\alpha_i\Sigma_i^T i + \Sigma_{iu}^T\right)\right)\Sigma_{ii}^{-1}.$$

As in Step 3 of the proof of Theorem 1, this condition can be rewritten as

$$(\widehat{\Sigma}_{diag} + \Sigma_{\theta\theta})\alpha = \gamma\Sigma_{\theta v} - \Sigma_{\theta M}\beta_M' - \Sigma_{\theta u}, \tag{17}$$

where γ and β_M' are defined as before, and instead of Σ_{diag} as in equation (12), we now have

$$\widehat{\Sigma}_{diag} = \begin{pmatrix} \frac{1}{\ell_1}\Sigma_{11} & 0 & \dots & \dots \\ 0 & \frac{1}{\ell_2}\Sigma_{22} & 0 & \dots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{\ell_n}\Sigma_{nn} \end{pmatrix}.$$

Then, again by analogy with the proof of Theorem 1, we define

$$\begin{aligned} \widehat{\Lambda} &= \widehat{\Sigma}_{diag} + \Sigma_{\theta\theta} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{\theta M}^T, \\ \widehat{A}_u &= \widehat{\Lambda}^{-1}\left(\Sigma_{\theta u} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{Mu}\right), \\ \widehat{A}_v &= \widehat{\Lambda}^{-1}\left(\Sigma_{\theta v} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{Mv}\right), \end{aligned}$$

and then finding a linear equilibrium is equivalent to solving the quadratic equation

$$a\gamma^2 + b\gamma + c = 0,$$

where

$$\begin{aligned} a &= -\widehat{A}_v^T\widehat{\Sigma}_{diag}\widehat{A}_v, \\ b &= \widehat{A}_v^T\left(2\widehat{\Sigma}_{diag} + \widehat{\Lambda}\right)\widehat{A}_u + \Sigma_{uM}\Sigma_{MM}^{-1}\Sigma_{Mv} - \sigma_{uv}, \\ c &= Var(\widehat{A}_u^T\theta - u|\theta_M). \end{aligned}$$

Since by definition $\gamma = 1/\beta_D$, solving the above quadratic equation is equivalent to solving the quadratic equation

$$c\beta_D^2 + b\beta_D + a = 0,$$

which turns out to be a more convenient characterization that we will proceed with. Similarly to the proof of Theorem 1, we also have a simple expression for α, $\alpha = \widehat{\Lambda}_v/\beta_D - \widehat{\Lambda}_u$.

Step 2. Let us now consider the entire sequence of markets, and restore superscript (m) for the variables. We will show the following:

$$\begin{aligned} a^{(m)} &\to 0, \\ b^{(m)} &\to -Cov(u, v|\theta, \theta_M), \\ c^{(m)} &\to Var(u|\theta, \theta_M). \end{aligned}$$

We note that, by our Assumption 3, both $Var(\theta|\theta_M)$ and $Var(\theta_M|\theta)$ are positive definite and thus invertible. Also, as $m \to \infty$, $\widehat{\Sigma}^{(m)}_{diag} \to 0$. Thus,

$$\begin{aligned} \widehat{\Lambda}^{(m)} &\to \Sigma_{\theta\theta} - \Sigma_{\theta M}\Sigma^{-1}_{MM}\Sigma^T_{\theta M} = Var(\theta|\theta_M), \\ \widehat{A}^{(m)}_u &\to Var(\theta|\theta_M)^{-1}\left(\Sigma_{\theta u} - \Sigma_{\theta M}\Sigma^{-1}_{MM}\Sigma_{Mu}\right) = Var(\theta|\theta_M)^{-1}Cov(\theta, u|\theta_M), \\ \widehat{A}^{(m)}_v &\to Var(\theta|\theta_M)^{-1}\left(\Sigma_{\theta v} - \Sigma_{\theta M}\Sigma^{-1}_{MM}\Sigma_{Mv}\right) = Var(\theta|\theta_M)^{-1}Cov(\theta, v|\theta_M). \end{aligned}$$

We immediately get that

$$\begin{aligned} a^{(m)} &\to 0, \\ b^{(m)} &\to Cov(v, \theta|\theta_M)Var(\theta|\theta_M)^{-1}Var(\theta|\theta_M)Var(\theta|\theta_M)^{-1}Cov(\theta, u|\theta_M) + \Sigma_{uM}\Sigma^{-1}_{MM}\Sigma_{Mv} - \sigma_{uv}, \\ c^{(m)} &\to Var\left(Cov(u, \theta|\theta_M)Var(\theta|\theta_M)^{-1}\theta - u|\theta_M\right). \end{aligned}$$

Then, by the projection formula,

$$\Sigma_{uM}\Sigma^{-1}_{MM}\Sigma_{Mv} - \sigma_{uv} = -Cov(u, v|\theta_M),$$

and

$$Cov(v, \theta|\theta_M)Var(\theta|\theta_M)^{-1}Cov(\theta, u|\theta_M) - Cov(u, v|\theta_M) = -Cov(u, v|\theta, \theta_M),$$

thus $b^{(m)} \to -Cov(u, v|\theta, \theta_M)$.

Finally, projecting u on θ and θ_M yields

$$u = Cov(u, \theta|\theta_M)Var(\theta|\theta_M)^{-1}\theta + Cov(u, \theta_M|\theta)Var(\theta_M|\theta)^{-1}\theta_M + \epsilon_{u,\theta,\theta_M},$$

and

$$Var(Cov(u, \theta|\theta_M)Var(\theta|\theta_M)^{-1}\theta - u|\theta_M) = Var(\epsilon_{u,\theta,\theta_M}) = Var(u|\theta, \theta_M),$$

thus $c^{(m)} \to Var(u|\theta, \theta_M)$.

Note that these limits imply that $\beta^{(m)}_D$ converges, since by Assumption 3, $Var(u|\theta, \theta_M) > 0$. Thus $\beta^{(m)}_D\alpha^{(m)}$ converges, and, in particular, the sequence is bounded.

Step 3. We now introduce, for each market m, a random vector $\widehat{\theta}^{(m)}$, which is independent of the other random variables of the model, and is distributed normally with mean 0 and variance-

covariance matrix $\widehat{\Sigma}^{(m)}_{diag}$.

By condition (i), or equivalently by equations (13) and (14), $\beta_D^{(m)}$ and $\beta_M^{(m)}$ are the coefficients of the projection of v on the total demand $(\alpha^{(m)})^T\theta + u$ and the market maker's signal θ_M:

$$v = \beta_D^{(m)}((\alpha^{(m)})^T\theta + u) + (\beta_M^{(m)})^T\theta_M + \epsilon_{v,(\alpha^{(m)})^T\theta+u,\theta_M}. \tag{18}$$

Also, $\beta_D^{(m)}\alpha^{(m)}$ is equal to the coefficient of the projection of $v - \beta_D^{(m)}u - (\beta_M^{(m)})^T\theta_M$ on $\theta + \widehat{\theta}^{(m)}$:

$$v - \beta_D^{(m)}u - (\beta_M^{(m)})^T\theta_M = \beta_D^{(m)}(\alpha^{(m)})^T(\theta + \widehat{\theta}^{(m)}) + \epsilon_{v,\theta+\widehat{\theta}^{(m)}}. \tag{19}$$

We show that θ_M is independent of $\epsilon_{v,\theta+\widehat{\theta}^{(m)}}$. Because all the random variables involved in these equations are jointly normal, it suffices to show that $Cov(\epsilon_{v,\theta+\widehat{\theta}^{(m)}}, \theta_M) = 0$. From equation (19), and using the independence of $\widehat{\theta}^{(m)}$ and θ_M,

$$Cov(\epsilon_{v,\theta+\widehat{\theta}^{(m)}}, \theta_M) = Cov(v, \theta_M) - Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta, \theta_M) - Cov(\beta_D^{(m)}u, \theta_M) - Cov((\beta_M^{(m)})^T\theta_M, \theta_M).$$

But from equation (18), we get that

$$Cov(v, \theta_M) = Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta, \theta_M) + Cov(\beta_D^{(m)}u, \theta_M) + Cov((\beta_M^{(m)})^T\theta_M, \theta_M),$$

hence establishing the equality $Cov(\epsilon_{v,\theta+\widehat{\theta}^{(m)}}, \theta_M) = 0$.

In the remainder of the proof, $p^{(m)}$ is the random variable that corresponds to the equilibrium price of the asset in market m, i.e.,

$$p^{(m)} = \beta_D^{(m)}(\alpha^{(m)})^T\theta + \beta_D^{(m)}u + (\beta_M^{(m)})^T\theta_M. \tag{20}$$

Step 4. As $m \to \infty$, the behavior of the market depends on the sign of $Cov(u, v|\theta, \theta_M)$. Let us first consider the case in which $Cov(u, v|\theta, \theta_M) > 0$.

In this case, $\beta_D^{(m)} \to Cov(u, v|\theta, \theta_M)/Var(u|\theta, \theta_M)$, which is immediately seen from the limits of the coefficients of the quadratic equation that $\beta_D^{(m)}$ solves, established in Step 2. We will show that

$$E\left[(p^{(m)} - E[v|\theta_M, \theta, u])^2\right] \to 0,$$

i.e., $p^{(m)}$ converges to $E[v|\theta_M, \theta, u]$ in L^2.

We observe that

$$E[v|\theta_M, \theta, u] - p^{(m)} = E[v - p^{(m)}|\theta_M, \theta, u]$$

and, from equations (19) and (20),

$$v - p^{(m)} = \beta_D^{(m)}(\alpha^{(m)})^T\widehat{\theta}^{(m)}\epsilon_{v,\theta+\widehat{\theta}^{(m)}},$$

and since $\widehat{\theta}^{(m)}$ is independent of θ, θ_M, u, we get that

$$E[v - p^{(m)}|\theta_M, \theta, u] = E[\epsilon_{v,\theta+\widehat{\theta}^{(m)}}|\theta_M, \theta, u] = E[\epsilon_{v,\theta+\widehat{\theta}^{(m)}}|\theta, u],$$

where the second inequality comes from the fact that $\epsilon_{v,\theta+\widehat{\theta}^{(m)}}$ and θ_M are independent, as shown in Step 3.

We observe that, to get

$$E\left[E[\epsilon_{v,\theta+\widehat{\theta}^{(m)}}|\theta, u]^2\right] \to 0$$

it is enough to show that, for every realization $\tilde{\theta}, \tilde{u}$,

$$E[\epsilon_{v,\theta+\widehat{\theta}^{(m)}}|\theta = \tilde{\theta}, u = \tilde{u}] \to 0$$

which is implied by $Cov(\epsilon_{v,\theta+\widehat{\theta}^{(m)}}, \theta) \to 0$ and $Cov(\epsilon_{v,\theta+\widehat{\theta}^{(m)}}, u) \to 0$.

We first show that $Cov(\epsilon_{v,\theta+\widehat{\theta}^{(m)}}, \theta) \to 0$. From equation (19), we have that, using that $\widehat{\theta}^{(m)}$ and θ are independent,

$$Cov(\epsilon_{v,\theta+\widehat{\theta}^{(m)}}, \theta) = Cov(v, \theta) - Cov(\beta_D^{(m)} u, \theta) - Cov((\beta_M^{(m)})^T \theta_M, \theta) - Cov((\beta_M^{(m)})^T, \theta_M).$$

We also have that, noting that $\epsilon_{v,\theta+\widehat{\theta}^{(m)}}$ and $\theta + \widehat{\theta}^{(m)}$ are independent, and $\widehat{\theta}^{(m)}$ is independent of v, u, θ and θ_M,

$$Cov(v, \theta) = Cov(\beta_D^{(m)}(\alpha^{(m)})^T \theta, \theta) + Cov(\beta_D^{(m)} u, \theta) + Cov(\beta_M^{(m)})^T \theta_M, \theta) + \beta_D^{(m)}(\alpha^{(m)})^T Var(\widehat{\theta}^{(m)})$$

Thus,

$$Cov(\epsilon_{v,\theta+\widehat{\theta}^{(m)}}, \theta) = \beta_D^{(m)}(\alpha^{(m)})^T Var(\widehat{\theta}^{(m)}) = \beta_D^{(m)}(\alpha^{(m)})^T \widehat{\Sigma}_{diag}^{(m)}.$$

Now, since $\beta_D^{(m)}(\alpha^{(m)})$ converges to a finite limit, we get that

$$Cov(\epsilon_{v,\theta+\widehat{\theta}^{(m)}}, \theta) \to 0.$$

We observe that to establish this convergence, we have not yet used the fact that $\beta_D^{(m)}$ converges to a positive value.

Next, we show that $Cov(\epsilon_{v,\theta+\widehat{\theta}^{(m)}}, u) \to 0$. From equation (18), we get that

$$\begin{aligned} Cov(v, (\alpha^{(m)})^T \theta) + Cov(v, u) &= Cov(\beta_D^{(m)}(\alpha^{(m)})^T \theta, (\alpha^{(m)})^T \theta) + Cov(\beta_D^{(m)}(\alpha^{(m)})^T \theta, u) \\ &+ Cov(\beta_D^{(m)} u, (\alpha^{(m)})^T \theta) + Cov(\beta_D^{(m)} u, u) + Cov((\beta_M^{(m)})^T \theta_M, (\alpha^{(m)})^T \theta) + Cov((\beta_M^{(m)})^T \theta_M, u). \end{aligned}$$

As $\beta_D^{(m)}$ converges to a positive limit, $\alpha^{(m)}$ converges. As we have just shown that

$$Cov(v, \theta) - Cov(\beta_D^{(m)} \alpha^{(m)})^T \theta, \theta) - Cov(\beta_D^{(m)} u, \theta) - Cov(\beta_M^{(m)})^T \theta_M, \theta) = \beta_D^{(m)}(\alpha^{(m)})^T Var(\widehat{\theta}^{(m)}) \to 0,$$

we get

$$Cov(v,(\alpha^{(m)})^T\theta) - Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta,(\alpha^{(m)})^T\theta) - Cov(\beta_D^{(m)}u,(\alpha^{(m)})^T\theta) \\ - Cov((\beta_M^{(m)})^T\theta_M,(\alpha^{(m)})^T\theta) \to 0,$$

and thus

$$Cov(v,u) - Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta,u) - Cov(\beta_D^{(m)}u,u) - Cov((\beta_M^{(m)})^T\theta_M,u) \to 0,$$

hence $Cov(\epsilon_{v,\theta+\widehat{\theta}^{(m)}},u) \to 0$.

Step 5. In this final step, we examine the remaining case in which $Cov(u,v|\theta,\theta_M) \leq 0$. In this case $\beta_D^{(m)} \to 0$, which again is seen from the limits of the coefficients $a^{(m)}, b^{(m)}, c^{(m)}$ in Step 2. We will show that

$$E\left[(p^{(m)} - E[v|\theta,\theta_M])^2\right] \to 0,$$

i.e., $p^{(m)}$ converges to $E[v|\theta,\theta_M]$ in L^2.

We observe that, from equation (19),

$$p^{(m)} - E[v|\theta,\theta_M] = \beta_D^{(m)}(u - E[u|\theta,\theta_M]) - E[\epsilon_{v,\theta+\widehat{\theta}^{(m)}}|\theta,\theta_M].$$

As $\beta_D^{(m)} \to 0$, the first term converges to 0 in L^2. In Step 3 we showed that $\epsilon_{v,\theta+\widehat{\theta}^{(m)}}$ is independent of θ_M, so $E[\epsilon_{v,\theta+\widehat{\theta}^{(m)}}|\theta,\theta_M] = E[\epsilon_{v,\theta+\widehat{\theta}^{(m)}}|\theta]$. Finally, in Step 4, we showed that $Cov(\epsilon_{v,\theta+\widehat{\theta}^{(m)}},\theta) \to 0$, implying that $E[\epsilon_{v,\theta+\widehat{\theta}^{(m)}}|\theta,\theta_M]$ converges to 0 in L^2. Thus $p^{(m)}$ converges to $E[v|\theta,\theta_M]$ in L^2.

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