Time Consistency and the Duration of Government Debt: A Signalling Theory of Quantitative Easing*

Saroj Bhattarai
Penn State

Gauti B. Eggertsson
Brown University

Bulat Gafarov†
Penn State

Abstract

We present a signalling theory of quantitative easing in which open market operations that change the duration of outstanding nominal government debt affect the incentives of the central bank in determining the real interest rate. In a time consistent (Markov-perfect) equilibrium of a sticky-price model with coordinated monetary and fiscal policy, we show that shortening the duration of outstanding government debt provides an incentive to the central bank to keep short-term real interest rates low in future in order to avoid capital losses. In a liquidity trap situation then, where the current short-term nominal interest rate is up against the zero lower bound, quantitative easing can be effective to fight deflation and a negative output gap as it leads to lower real long-term interest rates by lowering future expected real short-term interest rates. We show illustrative numerical examples that suggest that the benefits of quantitative easing in a liquidity trap can be large in a way that is not fully captured by some recent empirical studies.

JEL Classification: E31, E52, E58; E61; E63

Keywords: Time-consistent equilibrium; Duration of government debt; Quantitative easing; Liquidity trap; Signalling; Zero lower bound; Sticky prices

*Preliminary. We thank seminar participants at Brown University of helpful comments and suggestions. This version: December 2013.

†Bhattarai: 615 Kern Building, University Park, PA 16802; sub31@psu.edu. Eggertsson: Robinson Hall, 64 Waterman Street, Providence, RI 02912; gauti_eggertsson@brown.edu. Gafarov: Also affiliated with National Research University Higher School of Economics; 605 Kern Building, University Park, PA 16802; bzg134@psu.edu.

1 Introduction

During the recent global financial crisis, central banks in many advanced economies such as the United States engaged in various forms of "unconventional" monetary policy actions as the short-term interest rate, the traditional policy instrument, was up against the zero lower bound. One form of such policy actions, often called "quantitative easing," involved changes in the size and/or composition of the central bank's balance sheet. In particular, the Federal Reserve in the United States carried out several large scale purchases of assets in recent years, a program often referred to collectively as Large Scale Asset Purchases (LSAPs). A considerable part of LSAPs involved buying long term government bonds, or "quantitative easing," a naming convention we use here.

What is quantitative easing? It is when the government buys long-term government debt with money. Since the nominal interest rate was zero in recent years when this policy was implemented in the United States, it makes no difference if this was done by printing money (or more precisely bank reserves) or by issuing short-term government debt: both are government issued papers that yield a zero interest rate. From the perspective of the government as a whole, at least, quantitative easing at zero nominal interest rates can then simply be thought of as shortening the maturity of outstanding government debt.[1] The government is simply exchanging long term liabilities in the hands of the public with shorter term ones.

The main goal of quantitative easing in the United States was to reduce long-term interest rates, even when the short-term nominal interest rate could not be reduced further, and thereby, stimulate the economy. Indeed, several empirical studies find evidence of reduction in long-term interest rates following these policy interventions by the Federal Reserve. For example, Gagnon et a (2011) estimate that the 2009 program that involved buying various types of debt worth $1.75 trillion reduced long-term interest rates by 58 basis points while Krishnamurthy and Vissing-Jorgensen (2011) estimate that the 2010 program that involved buying long-term government debt worth $600 billion reduced long-term interest rates by 33 basis points. In addition, Swanson and Williams (2013) and Hamilton and Wu (2012) also find similar effects on long-term interest rates.[2]

From a theoretical perspective however, the effect of such policy is not obvious since open market operations of this kind are neutral (or irrelevant) in standard macroeconomic models. This was pointed out first in a well-known contribution by Wallace (1981) and further extended by Eggertsson and Woodford (2003) to a model with sticky prices and an explicit zero lower bound on nominal interest rates. These papers showed how absent some restrictions in asset trade that prevent arbitrage, a change in the relative supplies of various assets in the hands of the private sector has no effect on equilibrium quantities and asset prices in prototypical macroeconomic models.

For this reason, some papers have recently incorporated frictions such as participation constraints due to "preferred habitat" motives in order to make assets of different maturities imperfect substitutes. This in turn negates the neutrality of open market operations as in such an environment, quantitative easing can reduce long-term interest rates because it decreases the risk-premium. For example, Chen, Curdia, and Ferrero (2012) augment a quantitative sticky price model with such segmented market frictions and show that purchases of long-term bonds by the central bank can reduce long-term interest rates by decreasing the risk-premium. They nevertheless find the effects to be fairly modest: based on their estimated model,

[1] Indeed our model will be one where we will consider a consolidated government budget constraint and joint conduct of optimal monetary and tax policy.

[2] Note however that empirical studies typically measure nominal interest rates, while theoretically, it is the ability to influence real interest rates that matter. This issue is quite important and we emphasize it in detail later.

they find that a $600 billion reduction in outstanding long-term debt, along with a commitment to keep short-term interest rates at zero for 4 quarters, increases inflation by 3 basis points (annualized) and GDP growth by 0.13% (annualized).

As Woodford (2012) argues however (building on Eggertsson and Woodford (2003)), quantitative easing need not be effective only because it reduces risk premiums. It can also reduce long-term interest rates if such policy intervention signals to the private sector that the central bank will keep the short-term interest rates low once the zero lower bound is no longer a constraint in the future. In fact, arguably, much of the findings of the empirical literature on reduction of long-term interest rates due to quantitative easing can be attributed to expectations of low future short-term interest rates. Indeed, Krishnamurthy and Vissing-Jorgensen (2011) find evidence in support of this channel in their study of the 2010 quantitative easing program.[3]

Our contribution is to provide a formal theoretical model of such a "signalling" role of quantitative easing in a standard general equilibrium sticky price model.[4] In particular, we consider coordinated optimal monetary and fiscal policy under discretion and show that in a Markov-perfect (time-consistent) equilibrium, shortening the duration of outstanding government debt provides an incentive to the central bank to keep the short-term real interest rate low in the future. This constitutes optimal policy under discretion because it avoids capital losses on the government's balance sheet, which if realized, would entail raising taxes that are costly in the model. The key intuition for this result is that if the government holds larger amount of short-term debt then current real short term rate directly affects the cost of rolling the debt over period by period, while the cost of rolling over long-term debt is not affected as strongly period by period (since the interest rate on that debt are predetermined at the time the policy is set). This implies that shortening the maturity of outstanding debt increases the incentive of the government to keep real rates low.

In a liquidity trap situation, where the current short-term nominal interest rate is up against the zero lower bound and the economy suffers from deflation and a large negative output gap, it is well known that signalling about future policy can be very effective. For example, Krugman (1998) emphasizes the importance of raising inflation expectations and Eggertsson and Woodford (2003) emphasize the commitment to lower future short-term nominal interest rate and allowing output to overshoot its steady state level. But are these type of commitments credible? In fact, it is well-understood that commitment to future expansionary policy at the zero lower bound is difficult due to time-inconsistency problems. While the public understands the benefits today of committing to lower future real interest rates, it also appreciates the government's incentive in the future to renege on these promises once the economy has recovered (this leads to the so called "deflation bias" developed in Eggertsson (2006)). Our main result, then, is that quantitative easing, or shortening the duration of government debt, makes promises of expansionary future policy "credible" because it provides an incentive to the central bank to keep short-term real interest rates low in future to avoid balance sheet losses. This mitigates the extent of deflation and negative output gap that would occur otherwise in the Markov-perfect equilibrium.[5]

[3] Woodford (2012) makes a similar argument regarding empirical evidence on most recent balance sheet policies by the Federal Reserve.

[4] Gagnon et al (2011) label this role played by the expected path of future short-term interest rates as a "signalling" role for quantitative easing. In the literature on central bank intervention in foreign exchange markets, this term has been used often and appears to have been first coined by Mussa (1981) to discuss how foreign exchange interventions might be used to signal future changes in monetary policy.

[5] Jeanne and Svensson (2007) assume net worth concerns on the part of the central bank directly and show how buying foreign exchange is useful during a liquidity trap as it commits the central bank to not depreciating the exchange rate in future (since doing so would entail capital losses on the central bank's balance sheet). Berriel, Bhattarai, and Mendes (2013) show that buying long-term bonds acts as a commitment device for the same reason. The main difference in this paper is the consideration of joint conduct of monetary and fiscal policy along with a welfare-theoretic loss function for the government.

In our calibrated model, we show numerical experiments in which these effects of quantitative easing can be very large. For example, with government debt maturity of 16 quarters, we calibrate the size of the negative demand shock (that makes the zero lower bound binding) such that output drops by 10%. In such a case, reducing the duration of government debt by 7 months decreases deflation and the negative effect on output by 42 and 281 basis points respectively.[6] This suggests that the signalling channel may explain all the effects of quantitative easing found in the empirical literature. In addition, we show that if the duration were to be reduced by 15 months, that is double the size of our baseline experiment for quantitative easing, the negative output gap at the zero lower bound would be completely eliminated.

Before presenting the model we want to clarify an important element of our analysis. A key to understanding optimal policy in the New Keynesian model at the zero bound is that it involves committing to lower future short-term *real* interest rates.[7] The short term real interest rate is the difference between future short-term nominal interest rate and expected inflation. A lower short-term real interest rate can be achieved by either a lower future short-term nominal interest rate or higher future expected inflation. A commitment of this kind will under some parameterization of the model be achieved via higher expected inflation *and higher future short-term nominal interest rates* – even if the difference between the two (the real rate) is going down. An implication of this is that a successful policy of quantitative easing aimed at lowering long-term real interest rate may involve an *increase in expected future nominal interest rates* and therefore, an increase in long-term nominal rates at the zero lower bound. This is relevant because some empirical analyses of the effect of quantitative easing focus only on long-term *nominal interest rates*, which our analysis suggests is not a sufficient statistic. Even if long-term nominal interest rate decline very little in response to quantitative easing, or even if they increase, this does not by itself suggest that the policy is ineffective, as long as the real interest rate is declining.

2 Model

The model is a standard general equilibrium sticky-price closed economy set-up with an output cost of taxation, along the lines of Eggertsson (2006). The government conducts coordinated monetary and fiscal policy under discretion. The main difference from the literature is the introduction of long-term government debt.

2.1 Private sector

We start by describing the environment faced by the private sector, its optimization problem, and the equilibrium conditions.

2.1.1 Households

A representative household maximizes expected discounted utility over the infinite horizon

$$E_t \sum_{s=0}^{\infty} \beta^s U_{t+s} = E_t \sum_{s=0}^{\infty} \beta^s \left[u\left(C_{t+s}, \xi_{t+s}\right) + g\left(G_{t+s}, \xi_{t+s}\right) - \int_0^1 v\left(h_{t+s}(i), \xi_{t+s}\right) di \right] \tag{1}$$

[6] We use these numbers for illustration based on estimates in Chadha, Turner, and Zampoli (2013), which suggest that the average maturity of treasury debt held outside the Federal Reserve was around 4 years in the last 10 years and that recent Federal Reserve balance sheet policies reduced the maturity by around 7 months.

[7] Werning (2012) makes a related point regarding the ultimate goal of optimal policy as one of generating an output boom once the trap is over.

where β is the discount factor, C_t is household consumption of the composite final good, G_t is government consumption of the composite final good, $h_t(i)$ is quantity suppled of labor of type i, and ξ_t is a vector of aggregate exogenous shocks. E_t is the mathematical expectation operator conditional on period-t information, $u(.)$ is concave and strictly increasing in C_t for any possible value of ξ_t, $g(.)$ is concave and strictly increasing in G_t for any possible value of ξ_t, and $v(.)$ is increasing and convex in $h_t(i)$ for any possible value of ξ_t.[8]

The composite final good is an aggregate of a continuum of varieties indexed by i, $C_t = \int_0^1 \left[c_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}}$, where $\varepsilon > 1$ is the elasticity of substitution among the varieties. The optimal price index for the composite final good is given by $P_t = \left[\int_0^1 p_t(i)^{1-\varepsilon} di\right]^{\frac{1}{1-\varepsilon}}$, where $p_t(i)$ is the price of the home variety i. The demand for the individual varieties is then given by $\frac{c_t(i)}{C_t} = \left(\frac{p_t(i)}{P_t}\right)^{-\varepsilon}$. Finally, G_t is defined analogously to C_t and so we omit detailed description of government spending.

The household is subject to a sequence of flow budget constraints

$$P_t C_t + B_t^S + S_t(\rho) B_t + E_t\{Q_{t,t+1} A_{t+1}\} \leq \int_0^1 n_t(i) h_t(i) di + (1 + i_{t-1}) B_{t-1}^S + (1 + \rho S_t(\rho)) B_{t-1} + A_t - P_t T_t + \int_0^1 Z_t(i) di \quad (2)$$

where $n_t(i)$ is nominal wage of labor of type i, $Z_t(i)$ is nominal profit of home firm i, B_t^S is the household's holding of one-period risk-less nominal government bond at the beginning of period $t+1$, B_t is a perpetuity government nominal bond issued by the government in period t which pays ρ^j dollars $j+1$ periods later, for each $j \geq 0$ and some decay factor $0 \leq \rho < \beta^{-1}$, and T_t is (real) government taxes. More details on the perpetuity are given in the appendix, where we follow Woodford (2001) in modelling the term-structure.[9] A_{t+1} is the value of the complete set of state-contingent securities at the beginning of period $t+1$, denominated in home currency for simplicity. Finally, i_{t-1} is the nominal interest rate on government bond holdings at the beginning of period t (which is subject to the zero lower bound $i_t \geq 0$), $S_t(\rho)$ is the price of the perpetuity nominal bond which depends on the decay factor ρ, and $Q_{t,t+1}$ is the stochastic discount factor between periods t and $t+1$ that is used to value random nominal income in period $t+1$ in monetary units at date t.[10]

The problem of the household is thus to choose $\{C_{t+s}, h_{t+s}(i), B_{t+s}^S, B_{t+s}, A_{t+s}\}$ to maximize (1) subject to a sequence of flow budget constraints given by (2), while taking as exogenously given initial wealth and $\{P_{t+s}, n_{t+s}(i), i_{t+s}, S_{t+s}(\rho), Q_{t,t+s}, \xi_{t+s}, Z_{t+s}(i), T_{t+s}\}$.

2.1.2 Firms

There is a continuum of monopolistically competitive firms indexed by i. Each firm produces a variety i according to the production function

$$y_t(i) = f(h_t(i), \xi_t) \quad (3)$$

where $f(.)$ is an increasing concave function for any ξ_t and where ξ_t is again a vector of aggregate exogenous shocks.

As in Rotemberg (1983), firms face a cost of changing prices given by $d\left(\frac{p(i)}{p_{t-1}(i)}\right)$. This adjustment cost makes the firm's pricing problem dynamic. The demand function for variety i is given by

$$\frac{y_t(i)}{Y_t} = \left(\frac{p_t(i)}{P_t}\right)^{-\varepsilon} \quad (4)$$

[8] We abstract from money in the model and are thus directly considering the "cash-less limit."

[9] Our discussion in the appendix will also make it clear why the flow budget constraint can be written as in (2).

[10] The household is subject to a standard no-Ponzi game condition.

where Y_t is total demand for goods. The firm maximizes expected discounted profits

$$E_t \sum_{s=0}^{\infty} Q_{t,t+s} Z_{t+s}(i) \tag{5}$$

where $Q_{t,t+s}$ is the stochastic discount factor between periods t and $t+s$. The period profits $Z_t(i)$ are given by

$$Z_t(i) = \left[(1+s)\, p_t(i) y_t(i) - n_t(i) h_t(i) - d\left(\frac{p_t(i)}{p_{t-1}(i)} \right) P_t \right]$$

where s is a production subsidy. We can now re-write period profits using (3) and (4) as

$$Z_t(i) = \left[(1+s)\, Y_t p_t(i)^{1-\varepsilon} P_t^{\varepsilon} - n_t(i) f^{-1}\left(Y_t p_t(i)^{-\varepsilon} P_t^{\varepsilon} \right) - d\left(\frac{p_t(i)}{p_{t-1}(i)} \right) P_t \right].$$

The problem of the firm is thus to choose $\{p_{t+s}(i)\}$ to maximize (5), while taking as exogenously given $\{P_{t+s}, Y_{t+s}, n_{t+s}(i), q_{t,t+s}, \xi_{t+s}\}$.

2.1.3 Private sector equilibrium conditions

We can now derive the necessary conditions for equilibrium that arise from the maximization problems of the private sector described above. Note that these conditions hold for any government policy.

Households optimality conditions are given by

$$\frac{v_h\left(h_t(i), \xi_t\right)}{u_C\left(C_t, \xi_t\right)} = \frac{n_t(i)}{P_t} \tag{6}$$

$$Q_{t,t+s} = \beta \frac{u_C\left(C_{t+s}, \xi_{t+s}\right)}{u_C\left(C_t, \xi_t\right)} \Pi_{t+s}^{-1} \tag{7}$$

$$\frac{1}{1+i_t} = E_t \left[\beta \frac{u_C\left(C_{t+1}, \xi_{t+1}\right)}{u_C\left(C_t, \xi_t\right)} \Pi_{t+1}^{-1} \right] \text{ with } i_t \geq 0. \tag{8}$$

$$S_t(\rho) = E_t \left[\beta \frac{u_C\left(C_{t+1}, \xi_{t+1}\right)}{u_C\left(C_t, \xi_t\right)} \Pi_{t+1}^{-1} \left(1 + \rho S_{t+1}(\rho)\right) \right] \tag{9}$$

where $\Pi_t = \frac{P_t}{P_{t-1}}$ and (7) holds for each state of nature.[11]

The firm's optimality condition from price-setting is given by

$$\begin{aligned} &-(1-\varepsilon)(1+s) Y_t p_t(i)^{-\varepsilon} P_t^{\varepsilon} + \varepsilon n_t(i) f_y^{-1}\left(y_t(i)\right) Y_t p_t(i)^{-\varepsilon-1} P_t^{\varepsilon} + d'\left(\frac{p_t(i)}{p_{t-1}(i)} \right) \frac{P_t}{p_{t-1}(i)} \\ &= E_t \left[Q_{t,t+1} d'\left(\frac{p_{t+1}(i)}{p_t(i)} \right) \frac{p_{t+1}(i)}{p_t(i)^2} P_{t+1} \right]. \end{aligned}$$

Next, we will focus on a symmetric equilibria where all firms charge the same price and produce the same amount of output

$$p_t(i) = P_t,\ y_t(i) = Y_t,\ h_t(i) = h_t, \text{ and } n_t(i) = n_t.$$

Then, the firm's optimality condition from price-setting can be written, after using the optimality condition from the household regarding labor supply and asset holdings, as

[11] A standard transversality condition is also a part of these conditions.

$$-(1-\varepsilon)(1+s)Y_t+\varepsilon\frac{v_h(h_t,\xi_t)}{u_C(C_t,\xi_t)}P_t f_y^{-1}(Y_t)Y_t P_t^{-1}+d'(\Pi_t)\Pi_t=E_t\left[\beta\frac{u_C(C_{t+1},\xi_{t+1})}{u_C(C_t,\xi_t)}\Pi_{t+1}^{-1}d'(\Pi_{t+1})\Pi_{t+1}^2\right].$$

Lets manipulate further to obtain

$$\varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}(1+s)u_C(C_t,\xi_t)-v_h(h_t,\xi_t)f_y^{-1}(Y_t)\right]+u_C(C_t,\xi_t)d'(\Pi_t)\Pi_t=E_t\left[\beta u_C(C_{t+1},\xi_{t+1})d'(\Pi_{t+1})\Pi_{t+1}\right].$$

Finally, we can replace $v_h(h_t,\xi_t)f_y^{-1}(Y_t)$ by $\tilde{v}_y(Y_t,\xi_t)$ where $\tilde{v}(y_t(i),\xi_t)=v\left(f^{-1}(y_t(i),\xi_t\right)$ (note $y_t(i)=f(h_t(i),\xi_t)$) to get

$$\varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}(1+s)u_C(C_t,\xi_t)-\tilde{v}_y(Y_t,\xi_t)\right]+u_C(C_t,\xi_t)d'(\Pi_t)\Pi_t=E_t\left[\beta u_C(C_{t+1},\xi_{t+1})d'(\Pi_{t+1})\Pi_{t+1}\right]. \tag{10}$$

2.2 Government

There is an output cost of taxation (for example, as in Barro (1979)) captured by the function $s(T_t-T)$ where T is the steady-state level of taxes. Thus, in steady-state, there is no tax cost. Total government spending is then given by

$$F_t=G_t+s(T_t-T)$$

where G_t is aggregate government consumption of the composite final good defined before.

It remains to write down the (consolidated) flow budget constraint of the government. Note that the government issues both a one-period bond B_t^S and the perpetuity B_t. We can write the flow budget constraint as

$$B_t^S+S_t(\rho)B_t=(1+i_{t-1})B_{t-1}+(1+\rho S_t(\rho))B_{t-1}+P_t(F_t-T_t)$$

which in turn, can be written in real terms as

$$b_t^s+S_t(\rho)b_t=(1+i_{t-1})b_{t-1}^S\Pi_t^{-1}+(1+\rho S_t(\rho))b_{t-1}\Pi_t^{-1}+(F_t-T_t). \tag{11}$$

where $b_t^S=\frac{B_t^S}{P_t}$ and $b_t=\frac{B_t}{P_t}$. Next, assume that the one-period bond is in net-zero supply ($B_t^S=0$), which gives as the flow budget constraint of the government

$$S_t(\rho)b_t=(1+\rho S_t(\rho))b_{t-1}\Pi_t^{-1}+(F_t-T_t). \tag{12}$$

2.3 Market clearing

The goods market clearing condition gives the overall resource constraint as

$$Y_t=C_t+F_t+d(\Pi_t). \tag{13}$$

2.4 Private sector equilibrium

We are now ready to define the private sector equilibrium, that is the set of possible equilibria that are consistent with household and firm maximization and the technological constraints of the model. A private sector equilibrium is a collection of stochastic processes $\{Y_{t+s},\ C_{t+s},\ b_{t+s},\ S_{t+s}(\rho),\ \Pi_{t+s},\ i_{t+s},\ T_{t+s},\ F_{t+s},\ G_{t+s}\}$ for $s\geq 0$ that satisfy equations (3), (6)-(9), (10), (12), and (13), for each $s\geq 0$, given b_{t-1} and an exogenous stochastic process for $\{\xi_{t+s}\}$.

3 Equilibrium

3.1 Recursive representation

It is useful to first derive a recursive representation of the equilibrium conditions in order to facilitate our discussion of the discretionary equilibrium. (12) is given as

$$S_t(\rho)b_t = (1+\rho S_t(\rho))\, b_{t-1}\Pi_t^{-1} + (F - T_t)\,.$$

where we assume $F_t = F$ at all times. Define the expectation variable $f_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\right]$ to write (8) as

$$1 + i_t = \frac{u_C\left(C_t,\xi_t\right)}{\beta f_t^e}$$

and define another expectation variable $g_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\left(1+\rho S_{t+1}(\rho)\right)\right]$ to write (9) as

$$S_t(\rho) = \frac{1}{u_C\left(C_t,\xi_t\right)}\beta g_t^e.$$

Finally, define a third expectation variable $h_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)d'\left(\Pi_{t+1}\right)\Pi_{t+1}\right]$ to write (10) as

$$\varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_C\left(C_t,\xi_t\right) - \tilde{v}_y\left(Y_t,\xi_t\right)\right] + u_C\left(C_t,\xi_t\right)d'\left(\Pi_t\right)\Pi_t = \beta h_t^e.$$

This means that the necessary and sufficient condition for a private sector equilibrium is that variables $\{Y_t,\, C_t,\, b_t,\, S_t(\rho),\, \Pi_t,\, i_t,\, T_t\}$ satisfy: (a) the following conditions

$$S_t(\rho)b_t = (1+\rho S_t(\rho))\, b_{t-1}\Pi_t^{-1} + (F - T_t) \tag{14}$$

$$1 + i_t = \frac{u_C\left(C_t,\xi_t\right)}{\beta f_t^e},\ i_t \geq 0 \tag{15}$$

$$S_t(\rho) = \frac{1}{u_C\left(C_t,\xi_t\right)}\beta g_t^e \tag{16}$$

$$\beta h_t^e = \varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}u_C\left(C_t,\xi_t\right) - \tilde{v}_y\left(Y_t,\xi_t\right)\right] + u_C\left(C_t,\xi_t\right)d'\left(\Pi_t\right)\Pi_t \tag{17}$$

$$Y_t = C_t + F + d\left(\Pi_t\right) \tag{18}$$

given b_{t-1} and the expectations $f_t^e,\ g_t^e,$ and h_t^e; (b) expectations are rational so that

$$f_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\right] \tag{19}$$

$$g_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\left(1+\rho S_{t+1}(\rho)\right)\right] \tag{20}$$

$$h_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)d'\left(\Pi_{t+1}\right)\Pi_{t+1}\right]. \tag{21}$$

Note that the possible private sector equilibrium defined above depends only on the endogenous state variable b_{t-1} and shocks ξ_t.

3.2 Markov perfect equilibrium

We characterize a Markov-perfect (time-consistent) equilibrium where the government cannot commit and acts with discretion every period.[12] In particular, we consider coordinated monetary and fiscal policy, where the central bank and the treasury conduct optimal monetary and fiscal policy under discretion. Note however, that there is limited commitment in the model as the treasury can commit to paying the nominal value of the debt that it issues. This is a standard assumption made in the literature following Lucas and Stokey (1983).

We can then write the discretionary government's optimization problem recursively as a dynamic programming problem

$$V(b_{t-1}, \xi_t) = \max_{i_t, T_t} [U(.) + \beta E_t V(b_t, \xi_{t+1})]$$

subject to the private sector equilibrium conditions (14)-(18) and the rational expectations restrictions (19)-(21). Here, $U(.)$ is the utility function of the household in (1) and $V(.)$ is the value function.[13] The detailed formulation of this maximization problem and the associated first-order necessary conditions, as well as their linear approximation, are provided in the appendix.[14]

4 Linear-quadratic approach

We take a linear-quadratic approach to studying this time-consistent equilibrium.

4.1 Linear approximation and quadratic loss function

We approximate our model around an efficient non-stochastic steady-state with zero inflation.[15] Moreover, there are no tax collection costs in steady-state. Thus, there is a non-zero steady-state level of debt.[16] In steady-state, the following relationships also hold

$$1 + i = \beta^{-1},\ S = \frac{\beta}{1 - \rho\beta}, \text{and } T = F + \frac{1 - \beta}{1 - \rho\beta} b.$$

We then log-linearize the private sector equilibrium conditions around the steady state above to get the relationships

$$\hat{Y}_t = E_t \hat{Y}_{t+1} - \sigma(\hat{\imath}_t - E_t \pi_{t+1} - r_t^e) \tag{22}$$

$$\pi_t = \kappa \hat{Y}_t + \beta E_t \pi_{t+1} \tag{23}$$

$$\hat{b}_t = \beta^{-1} \hat{b}_{t-1} - \beta^{-1} \pi_t - (1 - \rho) \hat{S}_t - \frac{T}{bS} \hat{T}_t \tag{24}$$

$$\hat{S}_t = -\hat{\imath}_t + \rho\beta E_t \hat{S}_{t+1} \tag{25}$$

where κ and σ are a function of structural model parameters and r_t^e is the efficient rate of interest that is a function of the shock ξ_t.[17] Here, (22) is the linearized household Euler equation, (23) is the linearized

[12] See Maskin and Tirole (2001) for a formal definition of the Markov-perfect equilibrium.

[13] Using compact notation, note that we can write the utility function as $u(C_t, \xi_t) + g(F - s(T_t - T)) - \tilde{v}(Y_t, \xi_t)$

[14] Note here that we assume that the government and the private-sector move simultaneously.

[15] Variables without a t subscript denote a variable in steady state. Note that output is going to be at the efficient level in steady state because of the assumption of the production subsidy (appropriately chosen) we have made before.

[16] The steady-state is efficient even with non-zero steady-state debt because of our assumption that taxes do not entail output loss in steady-state.

[17] The details of the derivation are in the appendix.

Phillips curve, (24) is the linearized government budget constraint, and (25) is the linearized forward-looking asset-pricing condition.[18]

(22) and (23) are standard relationships depicting how current output depends on expected future output and the current real interest rate gap and how current inflation depends on expected future inflation and the current output respectively.[19] (24) shows that since debt is nominal, its real value is decreased by inflation. Higher taxes also reduce the debt burden. Moreover, an increase in the price of the perpetuity bond decreases the real value of debt, with the effect depending on the duration of debt: longer the duration, lower is the effect of the bond price on debt. Finally, (25) shows that the price of the perpetuity bond is determined by (the negative of) expected present value of future short-term interest rates. Note that when $\rho = 0$, all debt is of one-period duration and (24) reduces to the standard linearized government budget constraint while (25) reduces to $\hat{S}_t = -\hat{\imath}_t$.

Moreover, a second-order approximation of household utility around the efficient non-stochastic steady state gives

$$U_t = -\left[\lambda_\pi \pi_t^2 + \hat{Y}_t^2 + \lambda_T \hat{T}_t^2\right] \tag{26}$$

where λ_π and λ_T are a function of structural model parameters.[20] Compared to the standard loss-function in models with sticky prices that contains inflation and output, (26) features losses that arise from output costs of taxation outside of steady-state.

4.2 Approximate Markov perfect equilibrium

We will next consider optimal monetary and fiscal policy under discretion as a linear-quadratic problem: minimizing the loss-function, given by the quadratic approximation to household utility in (26), subject to the private sector equilibrium conditions, given by the linear approximations in (22)-(25). This approach makes our results particularly transparent. Moreover, we prove in the proposition below that this linear-quadratic approach gives identical linear optimality conditions as the one obtained by linearizing the non-linear optimality conditions of the original non-linear government maximization problem that we described above

Proposition 1 *The linearized dynamic system of the non-linear Markov perfect equilibrium is equivalent to the linearized dynamic system of the linear-quadratic Markov perfect equilibrium.*

Proof. In Appendix. ■

4.2.1 At positive interest rates

We use a guess-and-verify approach in this approximated model to characterize the Markov perfect equilibrium at positive interest rates. We first guess that the solution is of the form

$$\begin{aligned} &\pi_t = \pi_b \hat{b}_{t-1} + \pi_r r_t^e,\ \hat{Y}_t = Y_b \hat{b}_{t-1} + Y_r r_t^e,\ \hat{S}_t = S_b \hat{b}_{t-1} + S_r r_t^e, \\ &\hat{\imath}_t = i_b \hat{b}_{t-1} + i_r r_t^e,\ \hat{T}_t = T_b \hat{b}_{t-1} + T_r r_t^e, \text{ and } \hat{b}_t = b_b \hat{b}_{t-1} + b_r r_t^e \end{aligned} \tag{27}$$

[18] Variables with hats denote log-deviations from steady state except for the nominal interest rate, which is given as $\hat{\imath}_t = \frac{i_t - i}{1+i}$. Since in the non-stochastic steady state with zero inflation, $1 + i = \frac{1}{\beta}$, this means that the zero lower bound on nominal interest rates imposes the following bound on $\hat{\imath}_t$: $\hat{\imath}_t \geq -(1-\beta)$.

[19] We write directly in terms of output rather than the output gap since we will not be considering shocks that perturb the efficient level of output in the model.

[20] The details of the derivation are in the appendix. In particular, $\lambda_\pi = \frac{\varepsilon}{k}$.

where π_b, Y_b, S_b, i_b, b_b, T_b, π_r, Y_r, S_r, i_r, b_r, and T_r are unknown coefficients to be determined. The guess in (27) implies that the expectation functions take the form

$$E_t\pi_{t+1} = \pi_b\hat{b}_t + \pi_r E_t r^e_{t+1},\ E_t\hat{Y}_{t+1} = Y_b\hat{b}_t + Y_r E_t r^e_{t+1},\ \text{and}\ E_t\hat{S}_{t+1} = S_b\hat{b}_t + S_r E_t r^e_{t+1}. \tag{28}$$

The discretionary government's optimization problem can then be written recursively as a linear-quadratic dynamic programming problem

$$V(\hat{b}_{t-1}, r^e_t) = \min[\lambda_\pi\pi_t^2 + \hat{Y}_t^2 + \lambda_T\hat{T}_t^2 + \beta E_t V(\hat{b}_t, r^e_{t+1})]$$

subject to (22)-(25), where $V(\hat{b}_{t-1}, r^e_t)$ is the value function. We can then formulate the Lagrangian where the expectation functions are substituted using (28) and the shock is suppressed

$$\begin{aligned} L_t = &\frac{1}{2}(\lambda_\pi\pi_t^2 + \hat{Y}_t^2 + \lambda_T\hat{T}_t^2) + \beta E_t V(\hat{b}_t, r^e_{t+1}) \\ &+ \phi_{1t}[\hat{Y}_t - Y_b\hat{b}_t + \sigma\hat{i}_t - \sigma\pi_b\hat{b}_t] + \phi_{2t}[\pi_t - \kappa\hat{Y}_t - \beta\pi_b\hat{b}_t] \\ &+ \phi_{3t}[\hat{b}_t - \beta^{-1}\hat{b}_{t-1} + \beta^{-1}\pi_t + (1-\rho)\hat{S}_t + \frac{T}{bS}\hat{T}_t] + \phi_{4t}[\hat{S}_t + \hat{\imath}_t - \rho\beta S_b\hat{b}_t]. \end{aligned}$$

Note also that from now on, we are also not analyzing the effects of the shock, and hence not carrying around π_r, Y_r, S_r, i_r, b_r, and T_r since they are not very relevant for our eventual experiment at the zero lower bound.

The associated first order necessary conditions and the envelope condition of the minimization problem above are provided in the appendix. Our first substantiative result is that the equilibrium conditions can be simplified, in particular by eliminating the Largrange multipliers $\phi_{1t} - \phi_{4t}$, to get

$$\lambda_\pi\pi_t + \kappa^{-1}\hat{Y}_t = [\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]\frac{bS}{T}\lambda_T\hat{T}_t \tag{29}$$

$$[1 - \beta\pi_b\kappa^{-1}(1-\rho)\sigma^{-1} - (Y_b + \sigma\pi_b)(1-\rho)\sigma^{-1} + \rho\beta S_b(1-\rho)]\hat{T}_t = -\left(\frac{T}{bS}\right)\lambda_T^{-1}\beta\pi_b\kappa^{-1}\hat{Y}_t + E_t\hat{T}_{t+1} \tag{30}$$

along with (22)-(25). The final step to computing the solution is then to plug in the conjectured solution and to match coefficients on various variables in (22)-(25), (29), and (30) to determine π_b, Y_b, S_b, i_b, b_b,and T_b. The details of this step are in the appendix.

The most important relationships emerging from our analysis are captured by (29) and (30). (29) is the so called "targeting rule" of our model, which represents the equilibrium (static) relationship among the three target variables π_t, $\hat{Y}_t$, and $\hat{T}_t$ that emerges from the optimization problem of the government. (29) thus captures how target variables are related in equilibrium as governed both by the weights they are assigned in the loss function (λ_π and λ_T) as well as the trade-offs among them as given by the private sector equilibrium conditions (κ^{-1} and $[\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]\frac{bS}{T}$). Note in particular that κ^{-1} represents the trade-off between π_t and $\hat{Y}_t$ as given by (23) while $[\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]\frac{bS}{T}$ represents the trade-off between π_t and $\hat{Y}_t$ vs. $\hat{T}_t$ as given by the combination of (22), (23), and (24). (30) is another optimality condition characterizing the Markov-perfect equilibrium and represents the "tax-smoothing objective" of the government. In contrast to similar expressions following the work of Barro (1979), which would lead to taxes being a martingale, output appears in (30) because of sticky-prices, which makes output endogenous. Finally, because of the dynamic nature of (30), as opposed to (29), the unknown coefficients that are critical

for expectations of variables, π_b, Y_b, and S_b, appear in (30).

Real interest rate incentives and duration of outstanding debt How does the duration of debt affect the real interest rate incentives of the central bank?[21] This is at the heart of the matter because what influences output is eventually the real interest rate and the aim in a zero lower bound situation is precisely to be able to decrease the short-term real interest rate even when the short-term nominal interest rate is stuck at zero. That is, we are interested in the properties of

$$\hat{r}_t = \hat{\imath}_t - E_t\pi_{t+1} = (i_b - \pi_b b_b)\,\hat{b}_{t-1} = r_b\hat{b}_{t-1}$$

where $\hat{r}_t$ is the short-term real interest rate. This is important because in a liquidity trap situation, as is well-known, decreasing future real interest rate is key to mitigating negative effects on output, if r_b depends negatively on duration, then we are able to provide a theoretical rationale for quantitative easing actions by the government.

Since most of our results below are numerical, we discuss our calibration next. We want to emphasize however, that after a series of such numerical experiments, we have found our results below to be fully robust. Moreover, numerically, we have not found multiplicity of bounded equilibria. The parameter values we pick are given in Table 1. Following Eggertsson (2006), we choose our baseline parameters as follows: $\beta = 0.99$, $\sigma = 1$, $\kappa = 0.02$, and $\varepsilon = 8$. λ_T is a free parameter and we set it at 0.8. For the steady-state level of debt-to-taxes, $\frac{bS}{T}$, we use data from the Federal Reserve Bank of Dallas and NIPA and choose 7.2.

Before doing this exercise for the general model, it is useful to consider two special cases: fully flexible prices and fully rigid prices.

a) Fully flexible prices When prices are fully flexible, then as is well-known, monetary policy cannot control the (ex-ante) real interest rate $\hat{r}_t$. Then, the only way monetary policy can affect the economy is through surprise inflation as debt is nominal. In fact there is a well-known literature that addresses the issue of how the duration of nominal debt in a flexible price environment affects allocations under time consistent optimal monetary policy. For example, Calvo and Guidotti (1990 and 1992) address optimal maturity of nominal government debt in a flexible price environment while Sims (2013) explores how the response of inflation to fiscal shocks depends on the maturity of government debt under optimal monetary policy. We conduct a complementary exercise here and want to characterize how inflation incentives depend on the duration of debt under optimal monetary and fiscal policy under discretion. Thus, we are interested in how π_b depends on duration of debt.

For this exercise, we can think of this special case of fully flexible prices as $\kappa \to \infty$. Note however, that from the two optimality conditions (29) and (30), while under flexible prices $T_b = 0$, there is indeterminacy in terms of inflation and nominal interest rate dynamics.[22] This is a well-known result in monetary economics under discretion in a flexible price environment. To show our result on the role of the duration of debt, we follow the literature such as Calvo and Guidotti and Sims (2012) and include a (very small) aggregate social cost of inflation that is independent of the level of price stickiness. Then, the objective of the government under flexible prices will be given by

$$U_t = -\left[\lambda_\pi^{'}\pi_t^2 + \hat{Y}_t^2 + \lambda_T\hat{T}_t^2\right]$$

[21] Note that in our model, the duration of debt is given by $(1-\beta\rho)^{-1}$.
[22] Note that $\lambda_\pi = \frac{\varepsilon}{k}$.

where $\lambda_\pi^{'}$ parameterizes the cost of inflation that is independent of sticky prices. Using $\lambda_\pi^{'} =$ and the rest of the parameter values from Table 1, Fig (1) shows how π_b depends on duration of debt (quarters). We see clearly that with shorter duration, there is more of an incentive to use current inflation. The intuition for this result is that from (24), we see that everything else the same, when ρ (and thereby, duration) decreases, then there will be an incentive to decrease $\hat{S}_t$, that is keep interest rates low, to manage the debt burden. This will then increase inflation in equilibrium.

b) Fully rigid prices Next we can consider the other extreme case: that of fully rigid prices. In this case, inflation is zero in equilibrium and hence $\pi_b = 0$. Then, one can directly consider the effects on the ex-ante real interest rate by analyzing the effect on the nominal interest rate since $r_b = i_b - \pi_b b_b = i_b$. Thus, we are interested in how i_b depends on the duration of debt. Using the parameters from Table 1, Fig (2) shows how i_b depends on duration of debt (quarters). We see clearly that with shorter duration, there is more of an incentive to keep the nominal interest rate higher. The intuition for this result is again that from (24), when ρ (and thereby, duration) decreases, then there will be an incentive to decrease $\hat{S}_t$, that is keep interest rates low, to manage the debt burden.

c) Partially rigid prices Having established the results in these two special cases, we now move on to the main mechanism in our paper in the intermediate and quantitatively relevant case of partially rigid prices. Fig (3) shows how r_b depends on duration of debt (quarters) at different levels of κ. As is clear, when the duration of debt in the hands of the public is shorter, it unambiguously provides an incentive to the central bank to keep the short-term real interest rate lower in future. The intuition again is that doing so allows it to avoid costly capital losses in its portfolio, which now contains more long-term bonds. That is, if the debt is short-term, then current real short rates will more directly affect the cost of rolling the debt over period by period, while the cost of rolling over long-term debt is not affected in the same way period by period. Also note that when prices are more flexible, that is when κ is bigger, r_b is affected less typically and as we saw above, in the extreme of fully flexible prices, (ex-ante) real interest rate is not controlled by policy at all.

Figs.(4) and (5) show how r_b depends on duration of debt at different levels of ε and λ_T respectively. We do this since these parameters affect the two weights in the loss-function and help us understand better the mechanics of the equilibrium. First, note that since $\lambda_\pi = \frac{\varepsilon}{k}$, changing ε will only affect the weight on inflation in the loss-function without affecting any private sector equilibrium conditions. The effect of a higher ε, while not very important quantitatively, is to increase the response of the real interest rate. This is because with a higher weight on inflation, since inflation responds less (we show this below), the real interest rate gets affected by a higher degree. Finally, the comparative statics with respect to λ_T are intuitive: as the weight on taxes in the loss-function increases, the real interest rate decreases by more in order to lower interest payments and reduce the need to raise taxes.

Inflation incentives and duration of outstanding debt While the dependence of real interest rate incentive of the government on the duration of debt is the main mechanism of our paper, given the attention inflation incentives receive in the literature, we now study how π_b varies with the duration of outstanding government debt?[23] Moreover, it helps us emphasize a point that just focusing on inflation incentives might not be sufficient to understand the nature of optimal monetary and fiscal policy in a liquidity trap situation.

[23] For some recent discussion and analysis of how inflation dynamics in sticky-price models could depend on duration of government debt, see Sims (2011) and Faraglia et al (2012).

While an analytical expression for π_b is available in the appendix and it is possible to show that $\pi_b > 0$ for all $\rho < 1$, a full analytical characterization of the comparative statics with respect to the duration of debt is not available and so we rely on numerical results. Fig (6) below shows how π_b depends on duration of debt (quarters) at different levels of κ. As expected, π_b is positive at all durations in the figure.[24] More importantly, note that that at our baseline parameterization of $\kappa = 0.02$, π_b does decrease with duration over a wide range of maturity.[25] At the same time, however, there is a hump-shaped behavior, with π_b increasing when duration is increased at very short durations.

What drives this result? First, note from (24) that everything else the same, when ρ (and thereby, duration) decreases, then there will be an incentive to decrease $\hat{S}_t$, that is keep interest rates low to manage the debt burden. This will then increase inflation in equilibrium. At the same time however, the government's incentives on inflation are not fully/only captured by this reasoning. This is because what ultimately matters for the cost of debt is the real interest rate, which because of sticky prices, is endogenous and under the control of the central bank, and which we have seen before robustly depends negatively on the duration of debt. Therefore, to understand the overall effect on the government's inflation incentives, it is critical to analyze the targeting rule, as discussed above and given by (29). Here, the term $[\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]$ plays a key role as it captures the trade-off between π_t and $\hat{Y}_t$ vs. $\hat{T}_t$. Note first that β^{-1} here simply captures the role of surprise inflation in reducing the debt burden as government debt is nominal. This term would be present even when prices are completely flexible. The term $\kappa^{-1}(1-\rho)\sigma^{-1}$ however, appears because of sticky prices. This reflects how the real interest rate is affected by manipulation of the nominal interest rate and how it in turn affects output.

Thus, in situations where either κ^{-1}, σ^{-1}, or $(1-\rho)$ is high, this term can dominate and it can be the case that decreasing the duration (or decreasing ρ) actually leads to a lower π_b. This means, for example, that when κ^{-1} is low (or when prices are more flexible), the hump-shaped behavior of π_b gets restricted to very short maturities only as the channel coming from sticky-prices is not that influential. This is exactly what is seen in Fig 1 : as κ increases, the range at which π_b increases with duration gets narrower and the hump disappears for $\kappa = 3$.[26] This is also completely consistent with what we in fact showed above in the case of fully flexible prices, where inflation response depends negatively throughout on duration of debt.

Moreover, note that not surprisingly, π_b is higher at a given duration for higher κ. This simply reflects the fact that prices are now more flexible and inflation thus responds by more to a given level of outstanding debt. We also present some additional properties of π_b with respect to the two weights in the loss-function in order to understand the mechanics of the equilibrium. In Fig. (7) we see that as ε and thereby λ_π increases, as expected, inflation responds less to outstanding debt as inflation now has a greater weight in the loss-function. Fig. (8) presents results of changing the value of λ_T. As expected, now that taxes have a greater weight in the loss-function, inflation responds by more to outstanding debt in order to devalue debt and reduce the tax burden.

Debt dynamics and duration of outstanding debt While the primary focus so far is on properties of r_b as it determines the real interest rate incentives of the central bank, it is also interesting to consider the

[24] Again, it is possible to show analytically that for all $\rho < 1$, $\pi_b > 0$. In a very similar model but one with no steady-state debt, Eggertsson (2006) proved that $\pi_b > 0$ for $\rho = 0$ (that is, for one period debt).

[25] Some limited analytical results on the properties of π_b with respect to ρ are available in the appendix. For example, it can be shown that π_b is positive for all $\rho < 1$ and that when $\rho = 1 + \beta^{-1}\sigma\kappa$ (and thus, $\rho > 1$), $\pi_b = 0$. In this sense, for a specific case, one can show that π_b is declining in ρ by comparing some extreme cases (such as $\rho = 0$ with $\rho = 1 + \beta^{-1}\sigma\kappa$). Please see the appendix for details. Note also here that the upper bound on ρ is β^{-1}. So this case of $\pi_b = 0$ is not necessarily always reached.

[26] A similar picture is obtained when increasing σ. We do not present the figure to conserve space.

properties of b_b, the parameter governing the persistence of government debt. This exercise is interesting in its own right, but more importantly, it is also worth exploring because as explained before, what is critical is the behavior of the real interest rate, and that gets affected by b_b as $E_t\pi_{t+1} = \pi_b\hat{b}_t = \pi_b b_b \hat{b}_{t-1}$. Unlike for π_b, it is not possible to show a tractable analytical solution (or any property) for b_b as it is generally a root of a fourth-order polynomial equation. We thus rely fully on numerical solutions.

Figs. (9)-(11) show how b_b depends on duration of debt at different levels of κ, ε, and λ_T respectively. It is clear that the persistence of debt increases monotonically as the duration increases.[27] In fact, for a high enough duration, debt dynamics approach that of a random walk ($b_b = 1$), as in the analysis of Barro (1979). The persistence of debt increases with duration mainly because the response of the short-term real interest rate decreases, as we will discuss below. Some of this effect is reflected in the response of inflation decreasing as discussed above. Thus, the existence of long-term nominal debt as well as sticky prices has an important impact on the dynamics of debt under optimal policy. Finally, intuitive comparative static results are in Fig. (11): as the loss-function weight on taxes increases, debt is less persistent at any level of debt duration.

Having established that at positive interest rates, decreasing the duration of debt increases the incentives of the government to lower short-term real interest rates, we now move on to analyzing the case where the nominal interest rate is at the zero lower bound.

4.2.2 At the zero lower bound

To model the case of a liquidity trap, we follow Eggertsson and Woodford (2003) and suppose that a large enough negative shock to the efficient rate of interest r_t^e, driven by an increase in the desire to save, makes the zero lower bound binding.[28] Moreover, we also assume that r_t^e follows a two-state Markov process with an absorbing state: every period, with probability μ, r_t^e takes a (large enough) negative value of $-r_L^e$ while with probability $1-\mu$, it goes back to steady-state and stays there forever after. This means that the economy will exit the liquidity trap with a constant probability of $1-\mu$ every period and that once it exits, it does not get into the trap again. The appendix contains details about the computation algorithm.

With this structure, we next consider the following policy experiment. At the liquidity trap, the level of debt is constant at b_L, while out of the trap, it is optimally determined by the government according to the Markov-perfect equilibrium described previously. Then, we analyze what would be the effect of changing the duration of debt once-and-for-all, while the zero lower bound is binding. In other words, we are interested in the comparative statics of the model as we vary the duration of debt at the liquidity trap, without changing the real value of debt issued.

Our calibrated parameter values for this experiment are given in Table 1, where we pick values in a similar strategy to Eggertsson (2006). We pick μ and r_L^e to get a drop in output of 10%, to make the experiment relevant for the recent "Great Recession" in the Untied States. We allow for debt while at the liquidity trap to be 10% above steady state. We will consider the experiment of reducing the duration of government debt by 7 months, starting from 16 quarters. For average maturity of government debt and the reduction in the duration of government debt due to quantitative easing by the Federal Reserve, we

[27] There is thus, no hump-shaped pattern, unlike for inflation. The reason is that what matters directly for persistence of debt is the real interest rate and there is no hump-shaped pattern there as we show later. Note also that for one-period debt, often b_b is negative (even here, recall that π_b is still positive). A negative b_b is not very interesting empirically as it implies oscillatory behavior of debt.

[28] For an alternate way of generating a liquidity trap in monetary models, based on an exogenous drop in the borrowing limit, see Eggertsson and Krugman (2012).

use the recent estimates of Chadha, Turner, and Zampoli (2013) which suggest that the average maturity of treasury debt held outside the Federal Reserve was around 4 years in the last 10 years and that recent Federal Reserve balance sheet policies reduced the maturity by around 7 months.

Initial duration of government debt Figs. (12) and (13) show the response of inflation and output to a negative shock to the efficient rate of interest when the duration of government debt is 16 quarters. As is clear, because of the zero lower bound constraint, the economy suffers from deflation and because of the increase in the real interest rate that it creates (that is, a gap between the real interest rate and its efficient counterpart), also from a negative effect on output. This reflects what Eggertsson (2006) labelled the "deflation bias" of a discretionary central bank at the zero lower bound. In such a case, generating expectations of future inflation would be very beneficial as it would decrease the extent of deflation and the rise of the real interest rate, as stressed by Krugman (1998) and Eggertsson (2006).

In particular, it would be beneficial for the central bank to commit to keeping the short-term real interest rates lower in future, once the zero lower bound is no longer binding. This would decrease real long-term interest rates today and help spur the economy. Thus, one main goal of monetary policy would be to decrease the extent of increase in real interest rates (after the shock is over) that is seen in Fig. (14). In other words, as Krugman (1998) and Eggertsson (2006) emphasize, the central bank needs to "commit to being irresponsible."

Shorter duration of government debt A reduction in the duration of government debt outstanding helps achieve this goal. Figs. (15) and (16) show the response of inflation and output to a negative shock to the efficient rate of interest when the duration of government debt is 13.7 quarters. As is clear by comparing with the case when the duration is 16 quarters, at the trap, the extent of deflation is reduced by 42 basis points as well as the negative effect on output by 281 basis points. Note in particular that once the shock is over and the zero lower bound is no longer a constraint, the response of inflation and output is higher compared to Figs. (12) and (13).

The main reason why this is achieved is that because the government's balance sheet has more long-term bonds (or its debt is more short-term), the central bank keeps the short-term real interest rates lower in future, especially once the zero lower bound is not binding, in order to avoid capital losses. Thus, quantitative easing indeed provides a signal about the future conduct of monetary policy and in particular, the future path of short-term interest rates. This then enables it to have effect on macroeconomic prices and quantities at the zero lower bound, as real interest rates also decline at the liquidity trap because of higher inflationary expectations once the trap is over. The response of the real interest rate when the duration of debt is shorter is given in Fig. (17), where by comparing with Fig. (14), one can see that the real interest rate increases by less at the trap and decreases by more out of the trap. This, then, is the central result of our paper: quantitative easing acts as a commitment device during a liquidity trap situation.[29]

We also conduct an experiment of how long the duration reduction has to be in order for the output gap to be closed completely at the zero lower bound. We find that approximately doubling the size of the quantitative easing, that is, reducing the maturity by 15 months, would make the response of output zero at the liquidity trap. Figs. (18)-(20) show the responses of inflation, output, and the real interest rate

[29] This result thus connects our paper with Persson, Persson, and Svensson (1987 and 2006), who show in a flexible price environment that a manipulation of the maturity structure of both nominal and indexed debt can generate an equivalence between discretion and commitment outcomes.

respectively in this case. As expected, this leads to a much bigger reduction in the real interest rate and much more of a boom in inflation and output out of the trap.

Finally, we want to clarify an important element of our analysis: the distinction between nominal and real interest rates. A key to understanding optimal policy in the sticky price model at the zero bound is that it involves committing to lower future short-term *real* interest rates. The short term real interest rate is the difference between future short-term nominal interest rate and expected inflation. A lower short-term real interest rate can be achieved by either a lower future short-term nominal interest rate or higher future expected inflation. A commitment of this kind will under some parameterization of the model be achieved via higher expected inflation and higher future short-term nominal interest rates – even if the difference between the two (the real rate) is going down. An implication of this is that a successful policy of quantitative easing aimed at lowering long-term real interest rate may involve an increase in expected future nominal interest rates and therefore, an increase in long-term nominal rates at the zero lower bound.

In fact, in our numerical experiments, this indeed does happen. Figs. (21) and (22) show the responses of the short and the long nominal interest rate when debt duration is 16 quarters while Figs. (23) and (24) show the responses when debt duration is 13.7 quarters.[30] As can be seen, the short-term nominal interest rate in fact rises by more once the trap is over when debt duration is 13.7 quarters and that the long-term nominal interest rate at the zero lower bound is higher by around 23 basis points. This aspect is relevant because some empirical analyses of the effect of quantitative easing focus only on long-term nominal interest rates, which our analysis suggests is not a sufficient statistic. Even if long-term nominal interest rate decline very little in response to quantitative easing, or even if they increase (like in our example), this does not by itself suggest that the policy is ineffective, as long as the real interest rate is declining.[31]

Capital losses from reneging on optimal policy We have emphasized and illustrated so far that the reason why lowering the duration of debt during a liquidity trap situation is beneficial is that it provides incentives for the government to keep the real interest rate low in future because otherwise it would suffer capital losses on its balance sheet. These losses then would have to be accounted for by raising costly taxes. We have shown these results by comparing the path of the real interest rate under optimal policy at a baseline and lower duration of debt.

Another way to illustrate the mechanism behind this result is to conduct the following thought experiment: suppose that once the liquidity trap is over, the government reneges on the path for inflation and output dictated by optimal policy and instead perfectly stabilizes them at zero. In such a situation, how large are capital losses, or equivalently, how high do taxes have to rise compared to if the government had continued to follow optimal policy? In particular, is this increase in taxes more when debt is of shorter duration than longer duration? We show in Figs. (25) and (26) the change in taxes if the government were to renege on optimal policy at a duration of 16 and 13.7 quarters respectively. As is clear, the increase in taxes are higher at a shorter duration of outstanding debt. Thus indeed, *lowering* the duration of government debt provides the government with *more* an incentive to keep the real interest rate low in future in order to avoid having to raise costly taxes.

Robustness We now conduct some robustness exercises. In particular, compared to the literature, we calibrated some new parameters in this paper: λ_T (at 0.8) and b_L (at 0.10). We show in Fig. (27) the effect

[30] Note that since we are plotting $\hat{\imath}_t$, the zero lower bound implies a bound of $-(1-\beta) = -0.01$.

[31] Admittedly, an increase in the long-term nominal interest rate because of quantitative easing is perhaps not empirically consistent. We just want to make the point that looking at the long-term nominal interest rate is not sufficient.

of quantitative easing on output when we vary λ_T and in Fig. (28) the effect of quantitative easing on output when we vary b_L.[32] Clearly, a higher λ_T increases the effect of quantitative easing as it leads to more of an incentive for the central bank to keep the real interest rate low (to avoid costly taxes) while a higher b_L also increases the effect of quantitative easing as it increases the debt burden and thereby, the incentive to keep real interest rates low. The Figures show that our results continue to hold qualitatively for a several of these parameter values.

5 Conclusion

We present a theoretical model where open market operations that reduce the duration of outstanding government debt, so called "quantitative easing," are not neutral because they affect the incentive structure of the central bank. In particular, in a Markov-perfect equilibrium of our model, reducing the duration of outstanding government provides an incentive for the central bank to keep short-term interest rates low in future in order to avoid balance sheet losses. When the economy is in a liquidity trap, such a policy is thus effective at generating inflationary expectations and lowering long-term interest rates, which in turn, helps mitigate the deflation and negative output gap that would ensue otherwise. In other words, quantitative easing is effective because it provides a "signal" to the private sector that the central bank will keep the short-term real interest rates low even when the zero lower bound is no longer a constraint in future.

In future work, it would be of interest to evaluate fully the quantitative importance of our model mechanism in a medium-scale sticky price model, along the lines of Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2007). Computation of Markov-perfect equilibrium under coordinated monetary and fiscal policy at the ZLB appears to not have been investigated for such models in the literature If one departs from the assumption of an efficient steady-state, which would preclude a linear-quadratic approach, this extension is likely to involve a substantial computation innovation. Moreover, as a methodological extension, it would be fruitful to allow for dynamic manipulation of the duration of government debt as a policy instrument. To do so, it will be necessary to take a higher order approximation of the equilibrium conditions and modify the guess-and-verify algorithm to compute the Markov-perfect equilibrium accordingly.

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6 Tables and Figures

6.1 Tables

Table 1: Calibration of model parameters

Parameter	Value
β	0.99
σ	1
κ	0.02
ε	8
λ_T	0.8
$\frac{bS}{T}$	7.2

Table 2: Calibration of model parameters for the ZLB experiment

Parameter	Value
r_L^e	0.0137
μ	0.86
b_L	0.10

6.2 Figures

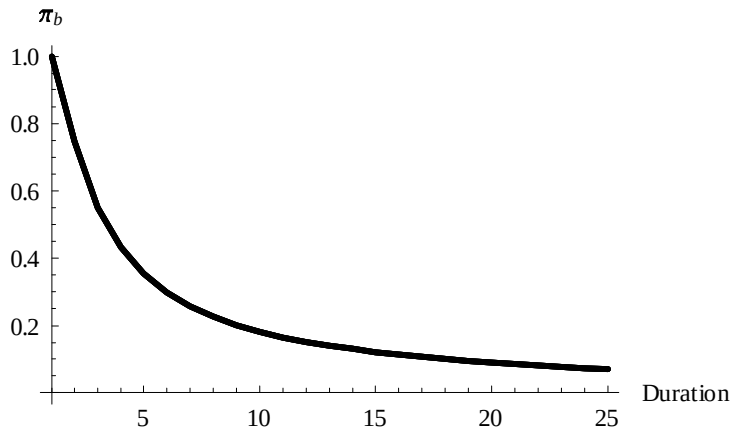


Figure 1: The effect on π_b of changing the duration (quarters) of outstanding nominal government debt under fully flexible prices.

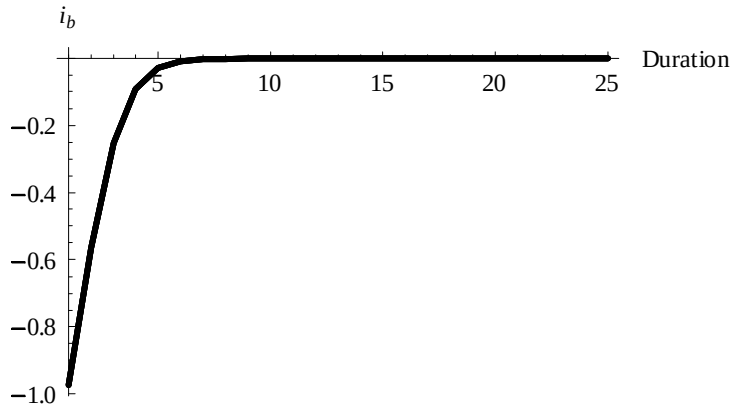


Figure 2: The effect on i_b of changing the duration (quarters) of outstanding nominal government debt under fully rigid prices.

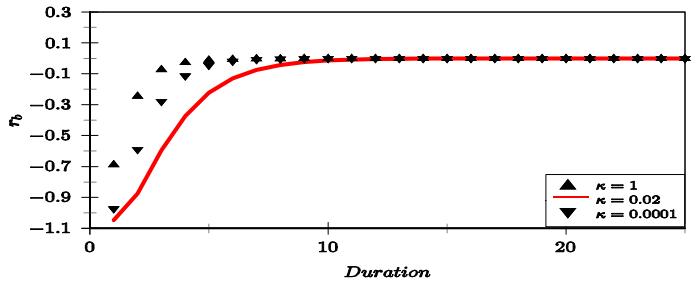


Figure 3: The effect on r_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the slope of the Phillips curve.

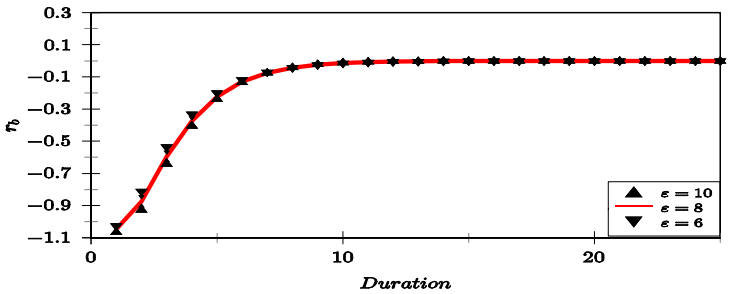


Figure 4: The effect on r_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the elasticity of substitution among varieties.

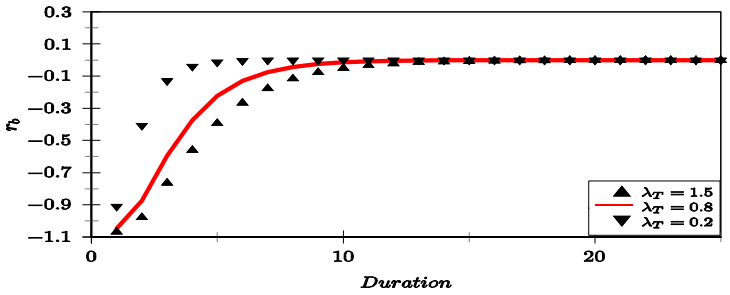


Figure 5: The effect on r_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the weight on taxes in the loss function.

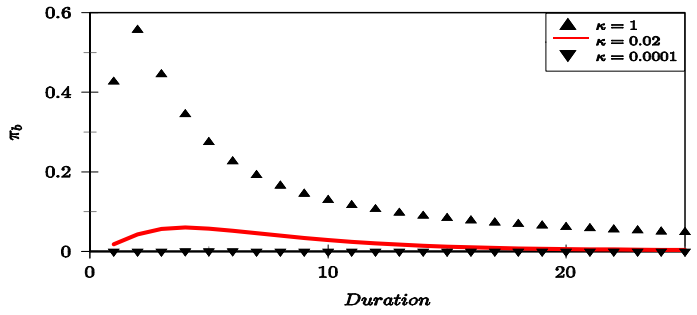


Figure 6: The effect on π_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the slope of the Phillips curve.

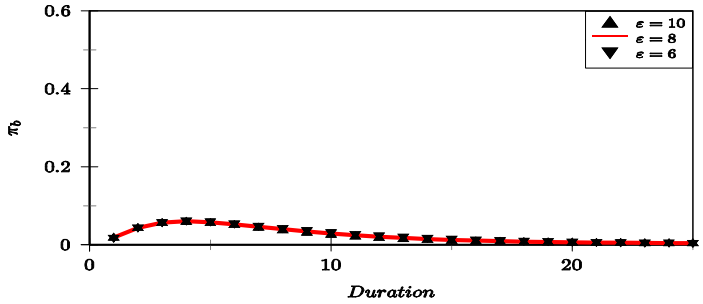


Figure 7: The effect on π_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the elasticity of substitution among varieties.

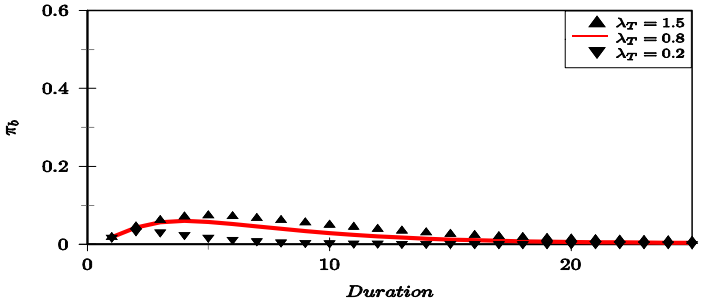


Figure 8: The effect on π_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the weight on taxes in the loss function.

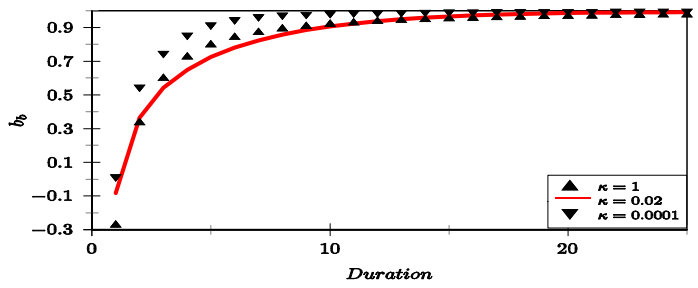


Figure 9: The effect on b_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the slope of the Phillips curve.

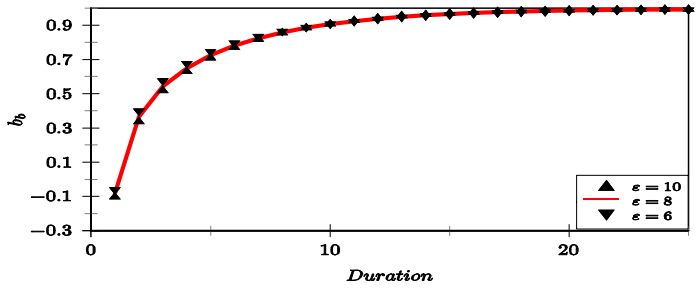


Figure 10: The effect on b_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the elasticity of substitution among varieties.

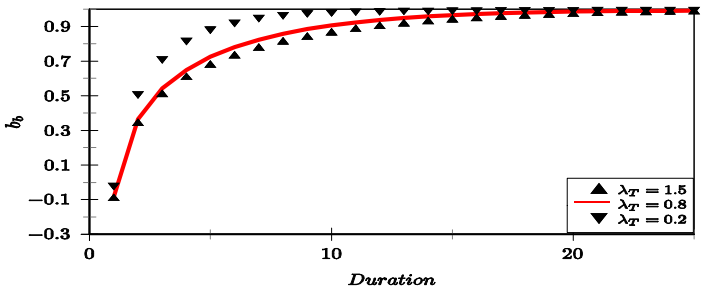


Figure 11: The effect on b_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the weight on taxes in the loss function.

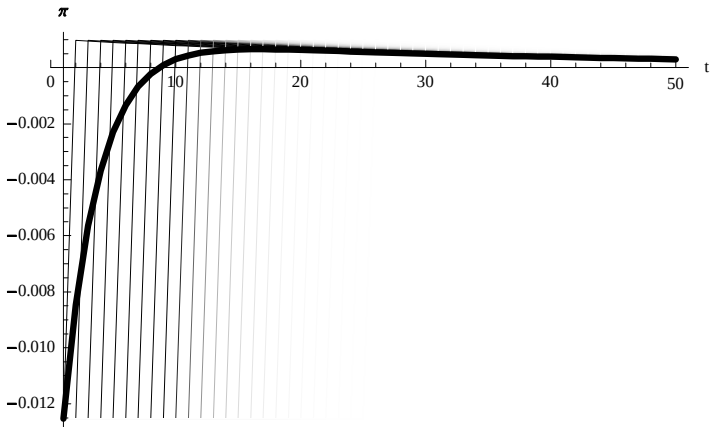


Figure 12: Reponse of inflation when the duration of government debt is 16 quarters. Each line represents the response of inflation when the efficient rate of interest returns to its steady-state in that period.

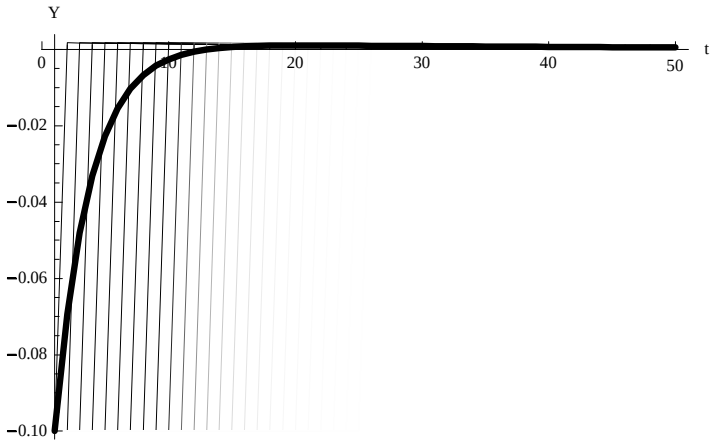


Figure 13: Reponse of output when the duration of government debt is 16 quarters. Each line represents the response of output when the efficient rate of interest returns to its steady-state in that period.

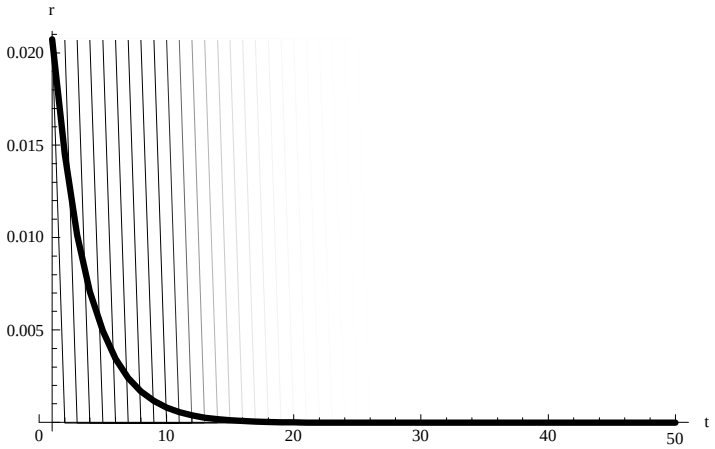


Figure 14: Reponse of the real interest rate when the duration of government debt is 16 quarters. Each line represents the response of the real interest rate when the efficient rate of interest returns to its steady-state in that period.

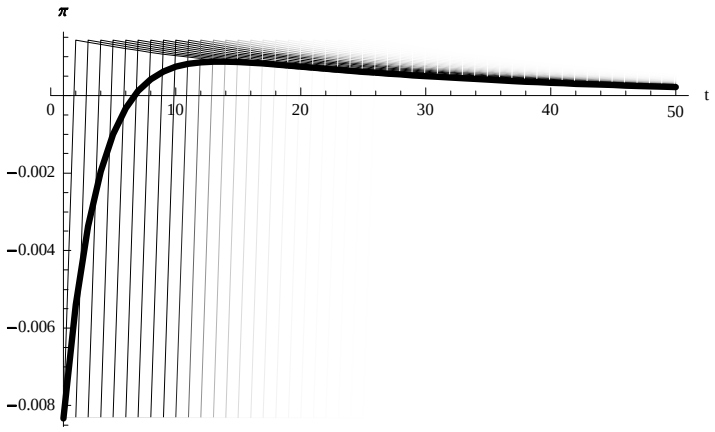


Figure 15: Reponse of inflation when the duration of government debt is 13.7 quarters. Each line represents the response of inflation when the efficient rate of interest returns to its steady-state in that period.

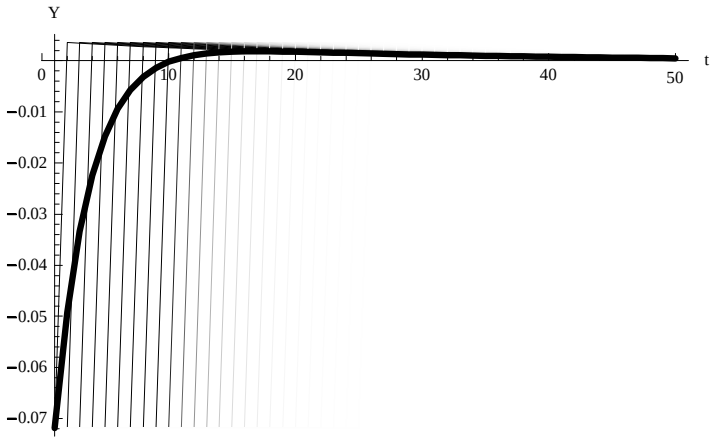


Figure 16: Reponse of output when the duration of government debt is 13.7 quarters. Each line represents the response of output when the efficient rate of interest returns to its steady-state in that period.

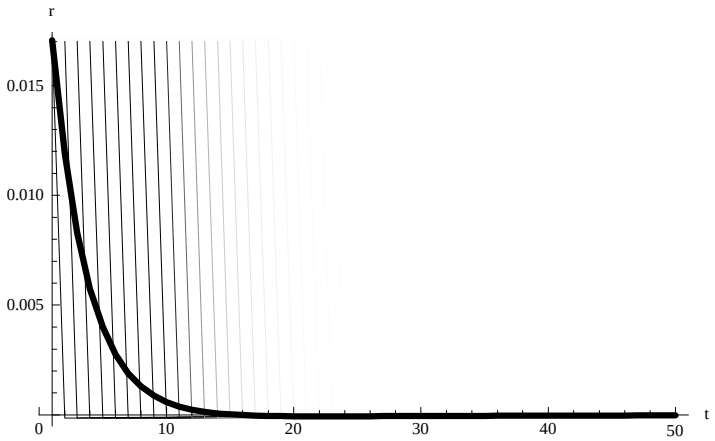


Figure 17: Reponse of the real interest rate when the duration of government debt is 13.7 quarters. Each line represents the response of the real interest rate when the efficient rate of interest returns to its steady-state in that period.

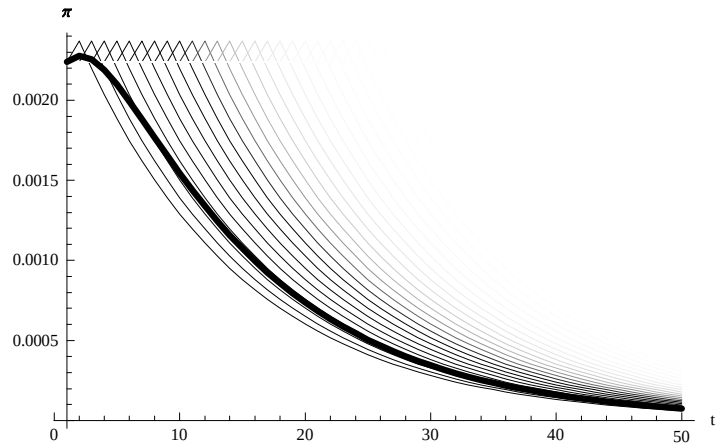


Figure 18: Reponse of inflation when the duration of government debt is 10.85 quarters. Each line represents the response of inflation when the efficient rate of interest returns to its steady-state in that period.

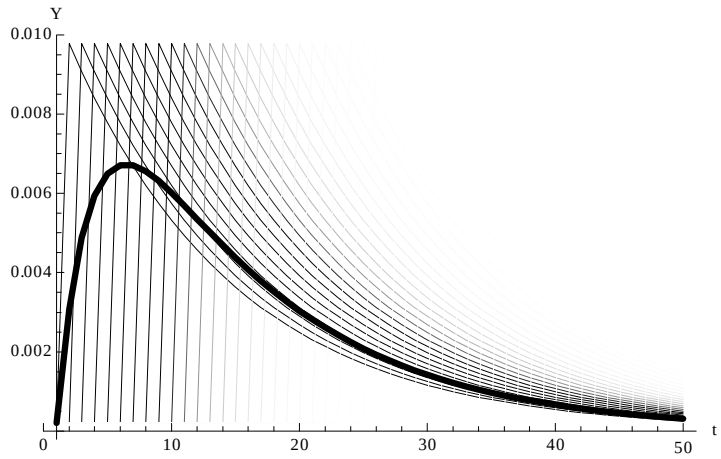


Figure 19: Reponse of output when the duration of government debt is 10.85 quarters. Each line represents the response of inflation when the efficient rate of interest returns to its steady-state in that period.

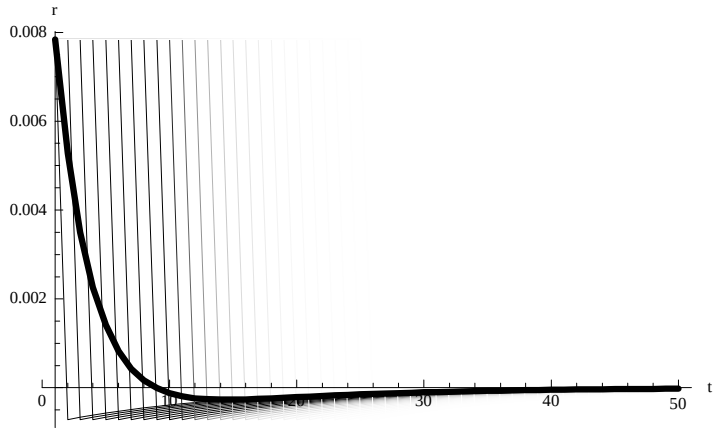


Figure 20: Reponse of the real interest rate when the duration of government debt is 10.85 quarters. Each line represents the response of the real interest rate when the efficient rate of interest returns to its steady-state in that period.

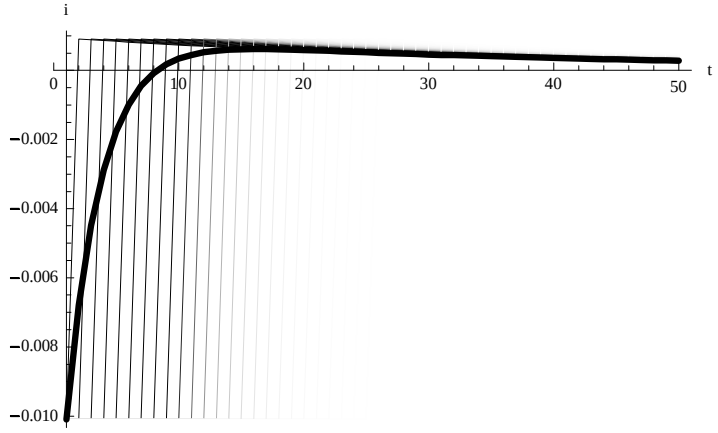


Figure 21: Reponse of the short-term nominal interest rate when the duration of government debt is 16 quarters. Each line represents the response of the short-term interest rate when the efficient rate of interest returns to its steady-state in that period.

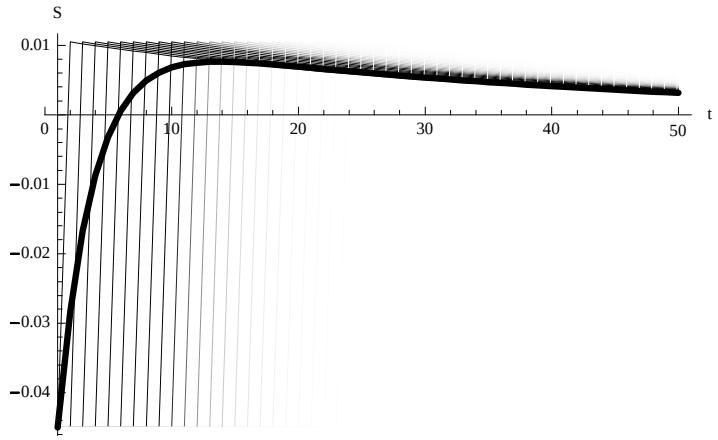


Figure 22: Reponse of the long-term nominal interest rate when the duration of government debt is 16 quarters. Each line represents the response of the long-term interest rate when the efficient rate of interest returns to its steady-state in that period.

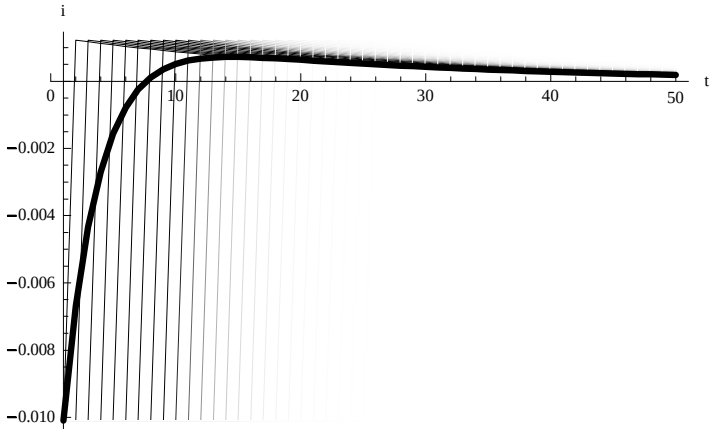


Figure 23: Reponse of the short-term nominal interest rate when the duration of government debt is 13.7 quarters. Each line represents the response of the short-term interest rate when the efficient rate of interest returns to its steady-state in that period.

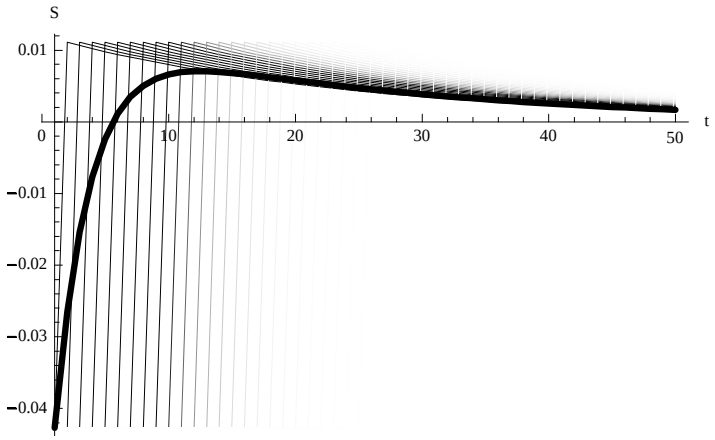


Figure 24: Reponse of the long-term nominal interest rate when the duration of government debt is 13.7 quarters. Each line represents the response of the long-term interest rate when the efficient rate of interest returns to its steady-state in that period.

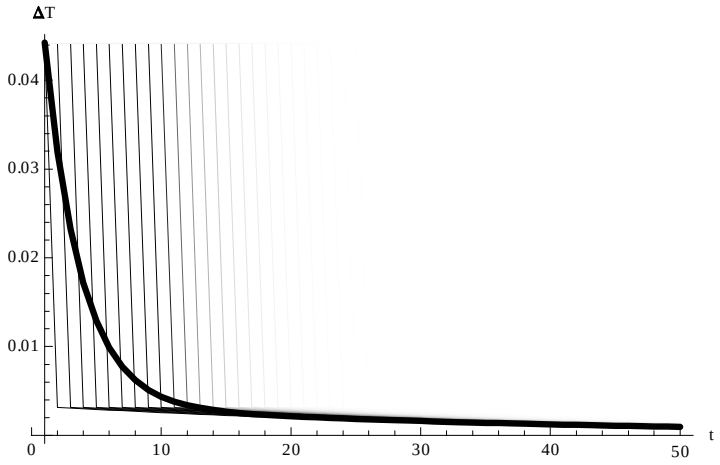


Figure 25: Increase in taxes a result of reneging on optimal policy when the duration of government debt is 16 quarters. Each line represents the response of the long-term interest rate when the efficient rate of interest returns to its steady-state in that period.

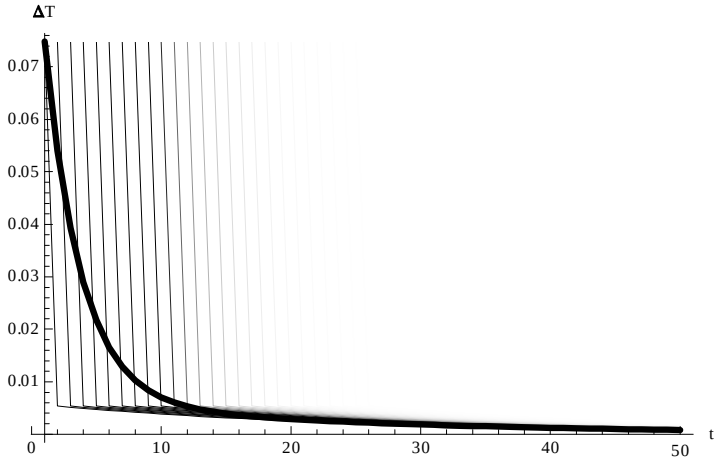


Figure 26: Increase in taxes a result of reneging on optimal policy when the duration of government debt is 13.7 quarters. Each line represents the response of the long-term interest rate when the efficient rate of interest returns to its steady-state in that period.

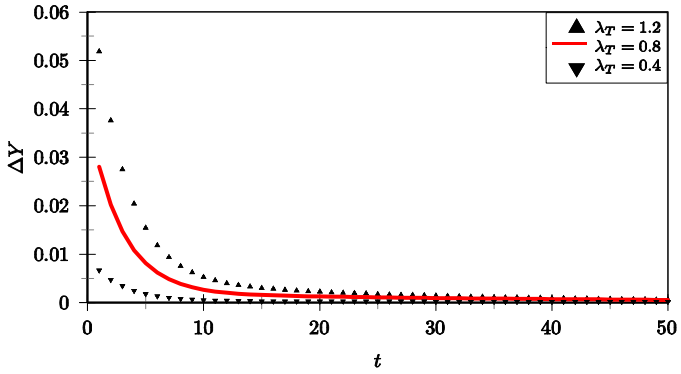


Figure 27: Effects of quantitative easing on output at different values of λ_T.

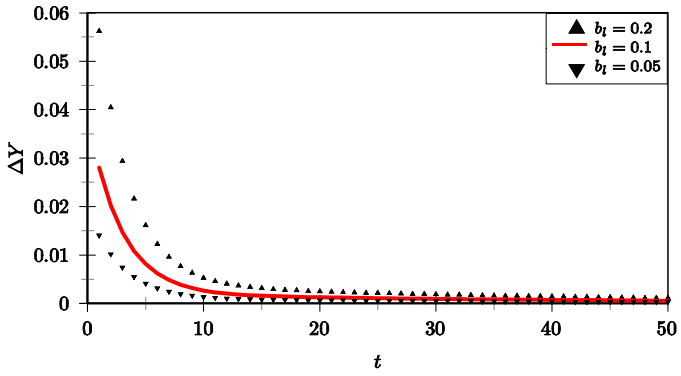


Figure 28: Effects of quantitative easing on output at different levels of initial debt.

7 Appendix

7.1 Model

7.1.1 The perpetuity nominal bond and the flow budget constraint

Following Woodford (2001), the perpetuity issued in period t pays ρ^j dollars $j+1$ periods later, for each $j \geq 0$ and some decay factor $0 \leq \rho < \beta^{-1}$. The implied steady-state duration of this bond is then $(1-\beta\rho)^{-1}$.

Let the price of a newly issued bond in period t be $S_t(\rho)$. Given the existence of the unique stochastic discount factor $Q_{t,t+j}$, we can write this price as

$$S_t(\rho) = E_t \sum_{j=1}^{\infty} Q_{t,t+j}\rho^{j-1}.$$

Now consider the period $t+1$ price of such a bond that was issued in period t. We can then write the price $S^O_{t+1}(\rho)$

$$S^O_{t+1}(\rho) = E_{t+1} \sum_{j=2}^{\infty} Q_{t+1,t+j}\rho^{j-1}.$$

Note first that

$$S^O_{t+1}(\rho) = \rho S_{t+1}(\rho) \tag{31}$$

since

$$S_{t+1}(\rho) = E_{t+1} \sum_{j=2}^{\infty} Q_{t+1,t+j}\rho^{j-2}.$$

This is highly convenient since it implies that one needs to keep track, at each point in time, of the equilibrium price of only one type of bond.

Next, we derive an arbitrage condition between this perpetuity and a one-period bond. By simple expansion of the infinite sums above and manipulation of the terms, one gets

$$S_t(\rho) = [E_t Q_{t,t+1}] + E_t\left[Q_{t,t+1}S^O_{t+1}(\rho)\right].$$

Since

$$E_t Q_{t,t+1} = \frac{1}{1+i_t}$$

we get

$$S_t(\rho) = \frac{1}{1+i_t} + E_t\left[Q_{t,t+1}S^O_{t+1}(\rho)\right].$$

Substituting further for $S^O_{t+1}(\rho) = \rho S_{t+1}(\rho)$, we then derive

$$S_t(\rho) = \frac{1}{1+i_t} + \rho E_t\left[Q_{t,t+1}S_{t+1}(\rho)\right]. \tag{32}$$

Finally, consider the flow budget constraint of the government

$$B_t^S + S_t(\rho)B_t = (1 + i_{t-1}) B_{t-1}^S + \left(\rho^0 + S_t^O(\rho)\right) B_{t-1} + P_t (F_t - T_t).$$

This can be simplified using $S_{t+1}^O(\rho) = \rho S_{t+1}(\rho)$ to

$$B_t^S + S_t(\rho)B_t = (1 + i_{t-1}) B_{t-1}^S + (1 + \rho S_t(\rho)) B_{t-1} + P_t (F_t - T_t).$$

This is the form in which we write down the flow budget constraint of the household and the government in the main text.

7.1.2 Functional forms

We make the following functional form assumptions on preferences and technology

$$\begin{aligned} u(C, \xi) &= \frac{C^{1-\tilde{\sigma}^{-1}}}{1 - \tilde{\sigma}^{-1}} \bar{C}^{\tilde{\sigma}^{-1}} \\ g(G, \xi) &= \frac{G^{1-\tilde{\sigma}^{-1}}}{1 - \tilde{\sigma}^{-1}} \bar{G}^{\tilde{\sigma}^{-1}} \bar{b}^{-1} \\ v(H, \xi) &= \frac{\lambda_1}{1 + \omega} H^{1+\omega} \bar{H}^{-\lambda_2} \\ y &= \bar{A} h^{\theta} \end{aligned}$$

where variables with bars are shocks that take a value of 1 in steady state. This implies that we can derive

$$\tilde{v}(Y, \xi_t) = \frac{\lambda_1}{1 + \lambda_2} Y^{1+\lambda_2} \bar{H}^{-\omega}$$

Moreover, we assume that the cost of price adjustment takes the form

$$d(\Pi) = d_1 \Pi^2$$

and that the cost of taxation takes the form

$$s(T_t - T) = s_1 (T_t - T)^2.$$

7.1.3 Efficient equilibrium

As benchmark, we first derive the efficient allocation.

Using $G_t = F_t - s(T_t - T) = F - s(T_t - T)$, the social planner's problem can be written as

$$\max u(C_t, \xi_t) + g(F - s(T_t - T)) - \tilde{v}(Y_t)$$

st

$$Y_t = C_t + F.$$

Formulate the Lagrangian

$$\begin{aligned} L_t = {} & u(C_t, \xi_t) + g(F - s(T_t - T)) - \tilde{v}(Y_t) \\ & + \phi_{1t} (Y_t - C_t - F) \end{aligned}$$

FOCs (where all the derivatives are to be equated to zero)

$$\begin{aligned} \frac{\partial L_t}{\partial Y_t} &= -\tilde{v}_Y + \phi_{1t} \\ \frac{\partial L_t}{\partial C_t} &= u_C + \phi_{1t} [-1] \\ \frac{\partial L_t}{\partial T_t} &= g_G \left(-s'(T_t - T)\right) \end{aligned}$$

Eliminating the Lagrange multiplier gives

$$\begin{aligned} u_C &= \tilde{v}_Y \\ g_G \left(-s'(T_t - T)\right) &= 0. \end{aligned}$$

Note that we make the following functional form assumptions on the tax collection cost

$$s(0) = 0;\ s'(0) = 0.$$

Thus, when taxes are at steady state, that is, $T_t = T$, then $s(T_t - T) = s'(T_t - T) = 0$. But note that we will allow for $s''(0) > 0$.

Efficient allocation thus requires

$$u_C = \tilde{v}_Y$$

$$T_t = T.$$

In steady state, without aggregate shocks, we have

$$Y = C + F$$

$$u_C = \tilde{v}_Y$$

$$T_t = T.$$

7.2 Non-linear Markov equilibrium

7.2.1 Equilibrium conditions

The policy problem can be written as

$$J\left(b_{t-1}, \xi_t\right) = \max\left[U\left(\Lambda_t, \xi_t\right) + \beta E_t J\left(b_t, \xi_{t+1}\right)\right]$$

st

$$S_t(\rho) b_t = \left(1 + \rho S_t(\rho)\right) b_{t-1} \Pi_t^{-1} + \left(F - T_t\right).$$

$$1 + i_t = \frac{u_C\left(C_t, \xi_t\right)}{\beta f_t^e}$$

$$i_t \geq 0$$

$$S_t(\rho) = \frac{1}{u_C\left(C_t, \xi_t\right)} \beta g_t^e$$

$$\varepsilon Y_t \left[\frac{\varepsilon - 1}{\varepsilon}\left(1 + s\right) u_C\left(C_t, \xi_t\right) - \tilde{v}_y\left(Y_t, \xi_t\right)\right] + u_C\left(C_t, \xi_t\right) d'\left(\Pi_t\right) \Pi_t = \beta h_t^e$$

$$Y_t = C_t + F + d\left(\Pi_t\right)$$

$$f_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right) \Pi_{t+1}^{-1}\right] = \bar{f}^e\left(b_t, \xi_t\right)$$

$$g_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right) \Pi_{t+1}^{-1}\left(1 + \rho S_{t+1}(\rho)\right)\right] = \bar{g}^e\left(b_t, \xi_t\right)$$

$$h_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right) d'\left(\Pi_{t+1}\right) \Pi_{t+1}\right] = \bar{h}^e\left(b_t, \xi_t\right)$$

Formulate the period Lagrangian

$$\begin{aligned}
L_t = & u\left(C_t, \xi_t\right) + g\left(F - s(T_t - T)\right) - \tilde{v}\left(Y_t\right) + \beta E_t J\left(b_t, \xi_{t+1}\right) \\
& + \phi_{1t}\left(S_t(\rho) b_t - \left(1 + \rho S_t(\rho)\right) b_{t-1} \Pi_t^{-1} - \left(F - T_t\right)\right) \\
& + \phi_{2t}\left(\beta f_t^e - \frac{u_C\left(C_t, \xi_t\right)}{1 + i_t}\right) \\
& + \phi_{3t}\left(\beta g_t^e - u_C\left(C_t, \xi_t\right) S_t(\rho)\right) \\
& + \phi_{4t}\left(\beta h_t^e - \varepsilon Y_t\left[\frac{\varepsilon - 1}{\varepsilon}\left(1 + s\right) u_C\left(C_t, \xi_t\right) - \tilde{v}_y\left(Y_t, \xi_t\right)\right] - u_C\left(C_t, \xi_t\right) d'\left(\Pi_t\right) \Pi_t\right) \\
& + \phi_{5t}\left(Y_t - C_t - F - d\left(\Pi_t\right)\right) \\
& + \psi_{1t}\left(f_t^e - \bar{f}^e\left(b_t, \xi_t\right)\right) \\
& + \psi_{2t}\left(g_t^e - \bar{g}^e\left(b_t, \xi_t\right)\right) \\
& + \psi_{3t}\left(h_t^e - \bar{h}^e\left(b_t, \xi_t\right)\right) \\
& + \gamma_{1t}\left(i_t - 0\right)
\end{aligned}$$

First-order conditions (where all the derivatives should be equated to zero)

$$\frac{\partial L_s}{\partial \Pi_t} = \phi_{1t}\left[\left(1+\rho S_t(\rho)\right) b_{t-1}\Pi_t^{-2}\right] + \phi_{4t}\left[-u_C d^{''}\Pi_t - u_C d^{'}\right] + \phi_{5t}\left[-d'\right]$$
$$\frac{\partial L_s}{\partial Y_t} = -\tilde{v}_Y + \phi_{4t}\left[-\varepsilon\left(\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_C\right) + \varepsilon Y_t \tilde{v}_{yy} + \varepsilon\tilde{v}_y\right] + \phi_{5t}$$
$$\frac{\partial L_s}{\partial i_t} = \phi_{2t}\left[u_C\left(1+i_t\right)^{-2}\right] + \gamma_{1t}$$
$$\frac{\partial L_s}{\partial S_t} = \phi_{1t}\left[b_t - \rho b_{t-1}\Pi_t^{-1}\right] + \phi_{3t}\left[-u_C\right]$$
$$\frac{\partial L_s}{\partial C_t} = u_C + \phi_{2t}\left[-u_{CC}\left(1+i_t\right)^{-1}\right] + \phi_{3t}\left[-u_{CC}S_t(\rho)\right] + \phi_{4t}\left[-\varepsilon Y_t\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_{CC} - u_{CC}d'\Pi_t\right] + \phi_{5t}\left[-1\right]$$
$$\frac{\partial L_s}{\partial T_t} = g_G\left(-s'(T_t - T)\right) + \phi_{1t}$$
$$\frac{\partial L_s}{\partial b_t} = \beta E_t J_b\left(b_t, \xi_{t+1}\right) + \phi_{1t}\left[S_t(\rho)\right] + \psi_{1t}\left[-\bar{f}_b^e\right] + \psi_{2t}\left[-\bar{g}_b^e\right] + \psi_{3t}\left[-\bar{h}_b^e\right]$$
$$\frac{\partial L_s}{\partial f_t^e} = \beta\phi_{2t} + \psi_{1t}$$
$$\frac{\partial L_s}{\partial g_t^e} = \beta\phi_{3t} + \psi_{2t}$$
$$\frac{\partial L_s}{\partial h_t^e} = \beta\phi_{4t} + \psi_{3t}$$

The complementary slackness conditions are

$$\gamma_{1t} \geq 0,\ i_t \geq 0,\ \gamma_{1t} i_t = 0$$

While the envelope condition is

$$J_b\left(b_{t-1}, \xi_t\right) = \phi_{1t}\left[-\left(1+\rho S_t(\rho)\right)\Pi_t^{-1}\right]$$

This also implies that

$$\beta E_t J_b\left(b_t, \xi_{t+1}\right) = \beta E_t \phi_{1t+1}\left[-\left(1+\rho S_{t+1}(\rho)\right)\Pi_{t+1}^{-1}\right]$$

7.2.2 Steady-state

A Markov-perfect steady-state is non-trivial to characterize because generally, we need to take derivates of an unknown function, as is clear from the FOCs. Here, we will rely on the fact that given an appropriate production subsidy, the Markov-perfect steady-state will be the same as the efficient steady-state derived above.

First, note that this requires no resource loss from price-adjustment costs, which in turn requires

$$d\left(\Pi\right) = 0$$

and thereby ensures

$$Y = C + F$$

This means that we need

$$\Pi = 1.$$

Also, this implies

$$d'\left(\Pi\right) = 0.$$

Next, note from the Phillips curve that this means, we need

$$\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_C - \tilde{v}_y = 0.$$

Now, since the efficient steady-state has $u_C = \tilde{v}_Y$, the production subsidy then has to satisfy

$$\frac{\varepsilon-1}{\varepsilon}\left(1+s\right) = 1.$$

We will be looking at a steady-state with positive interest rates

$$1 + i = \frac{1}{\beta}$$

which means that

$$\gamma_1 = 0$$

and that from the FOC wrt i_t we have

$$\phi_2 = 0.$$

Also, given that taxes are at steady-state, $g_G\left(-s'(T_t - T)\right) = 0$, from the FOC wrt T_t

$$\phi_1 = 0.$$

Given this, in turn, we have from the FOC wrt S_t

$$\phi_3 = 0.$$

Then, given that $d' = 0$ in steady-state and d'' is not, and since $\phi_1 = 0$, it gives from the FOC wrt to Π_t

$$\phi_4 = 0.$$

Note that this is highly convenient since these Largange multipliers being zero implies

$$\psi_1 = \psi_2 = \psi_3 = 0.$$

Thus, we do not need to worry about the derivatives of the unknown functions.

This proposed steady-state is consistent with other FOCs. For example, the FOC wrt Y_t is now given by

$$\phi_5 = \tilde{v}_Y$$

and that the FOC wrt C_t is given by

$$u_C = \phi_5$$

which implies

$$\tilde{v}_Y = u_C.$$

Finally, FOC wrt b_t implies

$$\beta J_b = \beta\phi_1\left[-\left(1 + \rho S(\rho)\right)\Pi^{-1}\right] = 0$$

which is also consistent with the conjectured guess.

Finally, the guess of the steady-state is also consistent with the other model equilibrium conditions, with $S(\rho)$ given by

$$S(\rho) = \beta\left[(1 + \rho S(\rho))\right]$$

that is

$$S(\rho) = \frac{\beta}{1 - \rho\beta}.$$

Then b and F are linked by

$$S(\rho)b = \left(1 + \rho S(\rho)\right)b + (F - T)$$

that is

$$T = F + \frac{1 - \beta}{1 - \rho\beta}b.$$

7.2.3 First-order approximation

We now take a log-linear approximation of the FOCs and the private sector equilibrium conditions around the steady-state above. Also, lets normalize the scale of the economy (with appropriate scaling of hours) so that $\bar{Y} = 1$. This implies $\bar{C} = 1 - F$. Also the shock ξ_t takes a value of 1 in steady-state.

Private sector equilibrium conditions We first start with the private sector equilibrium conditions. We denote variables that are in log-deviations from their respective steady-states by hats, except for $\hat{\imath}_t$.We denote variables in steady-state by bars.

First,

$$Y_t = C_t + F + d\left(\Pi_t\right)$$

gives

$$\hat{Y}_t = \bar{C}\hat{C}_t$$

Second,

$$\varepsilon Y_t \left[\frac{\varepsilon - 1}{\varepsilon}(1+s)\, u_C(C_t, \xi_t) - \tilde{v}_y(Y_t, \xi_t)\right] + u_C(C_t, \xi_t)\, d'(\Pi_t)\,\Pi_t = \beta E_t\left[u_C(C_{t+1}, \xi_{t+1})\, d'(\Pi_{t+1})\,\Pi_{t+1}\right]$$

gives

$$\bar{u}_C d''\hat{\pi}_t + \varepsilon \bar{u}_{CC}\bar{C}\hat{C}_t - \varepsilon \bar{v}_{yy}\hat{Y}_t - \varepsilon \bar{v}_{y\xi}\hat{\xi}_t + \varepsilon \bar{u}_{C\xi}\hat{\xi}_t = \beta \bar{u}_C d'' E_t\hat{\pi}_{t+1}$$

which can be simplified by making use of the log-linearized resource constraint above to yield

$$\bar{u}_C d''\hat{\pi}_t + \varepsilon(\bar{u}_{CC} - \bar{v}_{yy})\hat{Y}_t = \beta \bar{u}_C d'' E_t\hat{\pi}_{t+1}$$

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \frac{\varepsilon(\bar{v}_{yy} - \bar{u}_{CC})}{\bar{u}_C d''}\hat{Y}_t.$$

Third,

$$1 + i_t = \frac{u_C(C_t, \xi_t)}{\beta E_t\left[u_C(C_{t+1}, \xi_{t+1})\,\Pi_{t+1}^{-1}\right]}$$

gives

$$\bar{u}_{CC}\bar{C}\hat{C}_t + \bar{u}_{C\xi}\hat{\xi}_t = \bar{u}_C\hat{\imath}_t + \bar{u}_{CC}E_t\bar{C}\hat{C}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C E_t\hat{\pi}_{t+1}$$

which can be simplified by making use of the log-linearized resource constraint above to yield

$$\bar{u}_{CC}\hat{Y}_t + \bar{u}_{C\xi}\hat{\xi}_t = \bar{u}_C\hat{\imath}_t + \bar{u}_{CC}E_t\hat{Y}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C E_t\hat{\pi}_{t+1}$$

and

$$\hat{Y}_t = E_t\hat{Y}_{t+1} + \frac{\bar{u}_C}{\bar{u}_{CC}}\left[\hat{\imath}_t - E_t\hat{\pi}_{t+1}\right] + \frac{\bar{u}_{C\xi}}{\bar{u}_{CC}}\left[E_t\hat{\xi}_{t+1} - \hat{\xi}_t\right]$$

Note here that this implies that the efficient rate of interest is given by

$$r_t^e = -\frac{\bar{u}_{C\xi}}{\bar{u}_C}\left[E_t\hat{\xi}_{t+1} - \hat{\xi}_t\right].$$

Fourth,

$$S_t(\rho) = \frac{1}{u_C(C_t, \xi_t)}\beta E_t\left[u_C(C_{t+1}, \xi_{t+1})\,\Pi_{t+1}^{-1}\left(1 + \rho S_{t+1}(\rho)\right)\right]$$

gives

$$\bar{S}\bar{u}_{CC}\bar{C}\hat{C}_t + \bar{S}\bar{u}_{C\xi}\hat{\xi}_t + \bar{u}_C\bar{S}\hat{S}_t = \beta\left(1+\rho\bar{S}\right)\bar{u}_{CC}E_t\bar{C}\hat{C}_{t+1} + \beta\left(1+\rho\bar{S}\right)\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \beta\left(1+\rho\bar{S}\right)\bar{u}_C E_t\hat{\pi}_{t+1} + \beta\rho\bar{S}\bar{u}_C E_t\hat{S}_{t+1}$$

which can be simplifies by making use of the log-linearized resource constraint above to yield

$$\bar{S}\bar{u}_{CC}\hat{Y}_t + \bar{S}\bar{u}_{C\xi}\hat{\xi}_t + \bar{u}_C\bar{S}\hat{S}_t = \beta\left(1+\rho\bar{S}\right)\bar{u}_{CC}E_t\hat{Y}_{t+1} + \beta\left(1+\rho\bar{S}\right)\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \beta\left(1+\rho\bar{S}\right)\bar{u}_C E_t\hat{\pi}_{t+1} + \beta\rho\bar{S}\bar{u}_C E_t\hat{S}_{t+1}.$$

Note here that by using the log-linearized Euler equation above, one can further simplify as

$$\bar{S}\left[\bar{u}_C\hat{\imath}_t + \bar{u}_{CC}E_t\hat{Y}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C E_t\hat{\pi}_{t+1}\right] + \bar{u}_C\bar{S}\hat{S}_t = \beta\left(1+\rho\bar{S}\right)\bar{u}_{CC}E_t\hat{Y}_{t+1} + \beta\left(1+\rho\bar{S}\right)\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \beta\left(1+\rho\bar{S}\right)\bar{u}_C E_t\hat{\pi}_{t+1} + \beta\rho\bar{S}\bar{u}_C E_t\hat{S}_{t+}$$

or

$$\left[\bar{u}_C\hat{\imath}_t + \bar{u}_{CC}E_t\hat{Y}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C E_t\hat{\pi}_{t+1}\right] + \bar{u}_C\hat{S}_t = \frac{\beta\left(1+\rho\bar{S}\right)}{\bar{S}}\bar{u}_{CC}E_t\hat{Y}_{t+1} + \frac{\beta\left(1+\rho\bar{S}\right)}{\bar{S}}\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \frac{\beta\left(1+\rho\bar{S}\right)}{\bar{S}}\bar{u}_C E_t\hat{\pi}_{t+1} + \beta\rho\bar{u}_C E_t\hat{S}_{t+1}.$$

Moreover, since

$$\frac{\beta\left(1+\rho\bar{S}\right)}{\bar{S}} = 1$$

we have finally as the asset-pricing condition

$$\hat{\imath}_t + \hat{S}_t = \beta\rho E_t\hat{S}_{t+1}.$$

Fifth,

$$S_t(\rho)b_t = \left(1 + \rho S_t(\rho)\right)b_{t-1}\Pi_t^{-1} + (F - T_t)$$

gives

$$\hat{b}_t + \hat{S}_t = \rho\hat{S}_t + \frac{\left(1+\rho\bar{S}\right)}{\bar{S}}\hat{b}_t - \frac{\left(1+\rho\bar{S}\right)}{\bar{S}}\hat{\pi}_t - \frac{\bar{T}}{\bar{S}\bar{b}}\hat{T}_t$$

which is simplified further as

$$\hat{b}_t = \beta^{-1}\hat{b}_t - \beta^{-1}\hat{\pi}_t - (1-\rho)\,\hat{S}_t - \frac{\bar{T}}{\bar{S}\bar{b}}\hat{T}_t.$$

Then, finally, the expectation functions are given by

$$\hat{f}_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\right] = \bar{u}_{CC}E_t\hat{Y}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C\hat{\pi}_{t+1}$$

$$\hat{g}_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\left(1+\rho S_{t+1}(\rho)\right)\right] = \left(1+\rho\bar{S}\right)\bar{u}_{CC}E_t\hat{Y}_{t+1} + \left(1+\rho\bar{S}\right)\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \left(1+\rho\bar{S}\right)\bar{u}_CE_t\hat{\pi}_{t+1} + \rho\bar{S}\bar{u}_CE_t\hat{S}_{t+1}$$

$$\hat{h}_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)d'\left(\Pi_{t+1}\right)\Pi_{t+1}\right] = \bar{u}_Cd''E_t\hat{\pi}_{t+1}$$

Markov-perfect FOCs Here, note that since all the Lagrange multipliers except one are zero in steady-state, what we mean by hats will in fact only be deviations from steady-state for all the lagrange multipliers (as opposed to log-deviations).

First,

$$\phi_{1t}\left[\left(1+\rho S_t(\rho)\right)b_{t-1}\Pi_t^{-2}\right] + \phi_{4t}\left[-u_Cd^{''}\Pi_t - u_Cd^{'}\right] + \phi_{5t}\left[-d'\right] = 0$$

gives

$$\left(1+\rho\bar{S}\right)\bar{b}\hat{\phi}_{1t} - \bar{u}_Cd^{''}\hat{\phi}_{4t} - \bar{\phi}_5d''\hat{\pi}_t = 0$$

and since $\bar{\phi}_5 = \tilde{v}_y = \bar{u}_C$

$$\left(1+\rho\bar{S}\right)\bar{b}\hat{\phi}_{1t} - \bar{u}_Cd^{''}\hat{\phi}_{4t} - \bar{u}_Cd''\hat{\pi}_t = 0.$$

Second,

$$-\tilde{v}_Y + \phi_{4t}\left[-\varepsilon\left(\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_C\right) + \varepsilon Y_t\tilde{v}_{yy} + \varepsilon\tilde{v}_y\right] + \phi_{5t} = 0$$

gives

$$-\bar{v}_{YY}\hat{Y}_t + \left[\varepsilon\bar{Y}\bar{v}_{yy}\right]\hat{\phi}_{4t} + \hat{\phi}_{5t} = 0.$$

Third,

$$\phi_{2t}\left[u_C\left(1+i_t\right)^{-2}\right] + \gamma_{1t} = 0$$

gives

$$\bar{u}_C\beta^2\hat{\phi}_{2t} + \hat{\gamma}_{1t} = 0.$$

Fourth,

$$\phi_{1t}\left[b_t - \rho b_{t-1}\Pi_t^{-1}\right] + \phi_{3t}\left[-u_C\right] = 0$$

gives

$$\left[\left(1-\rho\right)\bar{b}\right]\hat{\phi}_{1t} - \bar{u}_C\hat{\phi}_{3t} = 0.$$

Fifth,

$$u_C + \phi_{2t}\left[-u_{CC}\left(1+i_t\right)^{-1}\right] + \phi_{3t}\left[-u_{CC}S_t(\rho)\right] + \phi_{4t}\left[-\varepsilon Y_t\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_{CC} - u_{CC}d'\Pi_t\right] + \phi_{5t}\left[-1\right] = 0$$

gives

$$\hat{Y}_t + \frac{\bar{u}_{C\xi}}{\bar{u}_{CC}}\hat{\xi}_t - \beta\hat{\phi}_{2t} - \bar{S}\hat{\phi}_{3t} - \varepsilon\hat{\phi}_{4t} - \frac{1}{\bar{u}_{CC}}\hat{\phi}_{5t} = 0.$$

Sixth,

$$g_G\left(-s'(T_t - \bar{T})\right) + \phi_{1t} = 0$$

gives

$$-\bar{g}_Gs^{''}\bar{T}\hat{T}_t + \hat{\phi}_{1t} = 0.$$

Seventh (after some replacements),

$$\beta E_t\phi_{1t+1}\left[-\left(1+\rho S_{t+1}(\rho)\right)\Pi_{t+1}^{-1}\right] + \phi_{1t}\left[S_t(\rho)\right] + \beta\phi_{2t}\left[\bar{f}_b^e\right] + \beta\phi_{3t}\left[\bar{g}_b^e\right] + \beta\phi_{4t}\left[\bar{h}_b^e\right] = 0$$

gives

$$-\bar{S}E_t\hat{\phi}_{1t+1} + \bar{S}\hat{\phi}_{1t} + \beta\bar{f}_b\hat{\phi}_{2t} + \beta\bar{g}_b\hat{\phi}_{3t} + \beta\bar{h}_b\hat{\phi}_{4t} = 0.$$

7.3 Linear-quadratic approach

7.3.1 Approximation of equilibrium conditions and household utility

We approximate around an efficient non-stochastic steady-state where $\Pi = 1$. For simplicity, from here on we will assume that the only shock that hits the economy is a discount factor shock given by ψ. Standard manipulations that are prevalent in the literature, for example in Woodford (2003), give (22) and (23) where $\sigma = \tilde{\sigma}\frac{C}{Y}$ and $k = \varepsilon\frac{\left(\sigma^{-1}+\lambda_2\right)}{d''}$. Here we detail the derivations of (24) and (25). Given

$$S_t(\rho) = \frac{1}{u_C\left(C_t,\xi_t\right)}\beta E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\left(1+\rho S_{t+1}(\rho)\right)\right],$$

and

$$1 + i_t = \frac{u_C\left(C_t,\xi_t\right)}{\beta E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\right]}$$

and the functional form assumptions above together with in steady state $1 + i = \beta^{-1}$, log-linearization gives immediately

$$\hat{S}_t = -\hat{\imath}_t + \rho\beta E_t\hat{S}_{t+1}.$$

Next, given

$$S_t(\rho)b_t = \left(1+\rho S_t(\rho)\right)b_{t-1}\Pi_t^{-1} + \left(F_t - T_t\right)$$

and that we assume $F_t = F$ and have from steady state $S = \frac{\beta}{1-\rho\beta},\ T = F + \frac{1-\beta}{1-\rho\beta}b,$ log-linearization gives immediately

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} - \beta^{-1}\pi_t - (1-\rho)\hat{S}_t - \frac{T}{bS}\hat{T}_t.$$

Note that also the following relationship holds in steady state $F = G$. We finally derive an expression for $r_t^e,$ the efficient rate of interest

$$r_t^e = \tilde{\sigma}^{-1}\left(\psi_t - E_t\psi_{t+1}\right).$$

For household utility, we need to approximate the following three components

$$u\left(Y_t - F - d(\Pi_t),\xi_t\right);\ g\left(F - S(T_t - T),\xi_t\right);\ v\left(Y_t,\xi_t\right).$$

Standard manipulations that are prevalent in the literature, for example in Woodford (2003), give as household utility

$$-\left[\lambda_\pi\pi_t^2 + \hat{Y}_t^2 + \lambda_T\hat{T}_t^2\right]$$

where $\lambda_\pi = \frac{\varepsilon}{k}$ and $\lambda_T = \frac{s''\bar{T}^2}{b(\phi+\sigma)}$.

7.3.2 Markov-perfect equilibrium at positive interest rates

Given the Lagrangian where the expectation functions are substituted and the shocks are suppressed

$$\begin{aligned}L_t = {} & \frac{1}{2}(\lambda_\pi\pi_t^2 + \hat{Y}_t^2 + \lambda_T\hat{T}_t^2) + \beta E_t V(b_t, r_{t+1}^e) + \phi_{1t}[\hat{Y}_t - Y_b b_t + \sigma\hat{\imath}_t - \sigma\pi_b b_t - \sigma r_t^e] \\ & + \phi_{2t}[\pi_t - \kappa\hat{Y}_t - \beta\pi_b b_t] + \phi_{3t}[b_t - \beta^{-1}b_{t-1} + \beta^{-1}\pi_t + (1-\rho)\hat{S}_t + \frac{T}{bS}\hat{T}_t] + \phi_{4t}[\hat{S}_t + \hat{\imath}_t - \rho\beta S_b b_t]\end{aligned}$$

the first-order necessary conditions are given by

$$\frac{\partial L}{\partial \pi_t} = \lambda_\pi\pi_t + \phi_{2t} + \beta^{-1}\phi_{3t} = 0$$

$$\frac{\partial L}{\partial Y_t} = Y_t + \phi_{1t} - \kappa\phi_{2t} = 0$$

$$\frac{\partial L}{\partial T_t} = \lambda_T T_t + \frac{T}{bS}\phi_{3t} = 0$$

$$\frac{\partial L}{\partial i_t} = \sigma\phi_{1t} + \phi_{4t} = 0$$

$$\frac{\partial L}{\partial S_t} = \phi_{3t}(1-\rho) + \phi_{4t} = 0$$

$$\frac{\partial L}{\partial b_t} = \beta E_t V_b(b_t, r_{t+1}^e) - (Y_b + \sigma\pi_b)\phi_{1t} - \beta\pi_b\phi_{2t} + \phi_{3t} - \rho\beta S_b\phi_{4t} = 0$$

while the envelope condition is given by

$$V_b(b_{t-1}, r_t^e) = -\beta^{-1}\phi_{3t}$$

which implies

$$E_t V_b(b_t, r_{t+1}^e) = -\beta^{-1} E_t \phi_{3t+1}.$$

We can then combine the envelope condition with the last FOC to yield

$$-E_t\phi_{3t+1} - (Y_b + \sigma\pi_b)\phi_{1t} - \beta\pi_b\phi_{2t} + \phi_{3t} - \rho\beta S_b\phi_{4t} = 0.$$

To summarize, we have

$$\lambda_\pi \pi_t + \phi_{2t} + \beta^{-1}\phi_{3t} = 0$$

$$Y_t + \phi_{1t} - \kappa\phi_{2t} = 0$$
$$\lambda_T T_t + \frac{T}{bS}\phi_{3t} = 0$$
$$\sigma\phi_{1t} + \phi_{4.t} = 0$$
$$\phi_{3t}(1-\rho) + \phi_{4t} = 0$$
$$-E_t\phi_{3t+1} - (Y_b + \sigma\pi_b)\phi_{1t} - \beta\pi_b\phi_{2t} + \phi_{3t} - \rho\beta S_b\phi_{4t} = 0$$
$$\hat{Y}_t = E_t\hat{Y}_{t+1} - \sigma(\hat{\imath}_t - E_t\pi_{t+1} - r_t^e)$$
$$\pi_t = \kappa\hat{Y}_t + \beta E_t\pi_{t+1}$$
$$b_t = \beta^{-1}b_{t-1} - \beta^{-1}\pi_t - (1-\rho)\hat{S}_t - \frac{T}{bS}\hat{T}_t$$
$$\hat{S}_t = -\hat{\imath}_t + \rho\beta E_t\hat{S}_{t+1}$$

which can be simplified to get

$$\lambda_\pi\pi_t + \kappa^{-1}Y_t = [\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]\frac{bS}{T}\lambda_T\hat{T}_t$$
$$[1 - \beta\pi_b\kappa^{-1}(1-\rho)\sigma^{-1} - (Y_b + \sigma\pi_b)(1-\rho)\sigma^{-1} + \rho\beta S_b(1-\rho)]\hat{T}_t = -\left(\frac{T}{bS}\right)\lambda_T^{-1}\beta\pi_b\kappa^{-1}Y_t + E_t\hat{T}_{t+1}$$
$$\hat{Y}_t = E_t\hat{Y}_{t+1} - \sigma(\hat{\imath}_t - E_t\pi_{t+1} - r_t^e)$$
$$\pi_t = \kappa\hat{Y}_t + \beta E_t\pi_{t+1}$$
$$b_t = \beta^{-1}b_{t-1} - \beta^{-1}\pi_t - (1-\rho)\hat{S}_t - \frac{T}{bS}\hat{T}_t$$
$$\hat{S}_t = -\hat{\imath}_t + \rho\beta E_t\hat{S}_{t+1}.$$

The final step is then to match coefficients after replacing the conjectured solutions

$$\lambda_\pi\pi_b b_{t-1} + \kappa^{-1}(Y_b b_{t-1}) = [\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]\frac{bS}{T}\lambda_T(T_b b_{t-1})$$
$$[1 - \beta\pi_b\kappa^{-1}(1-\rho)\sigma^{-1} - (Y_b + \sigma\pi_b)(1-\rho)\sigma^{-1} + \rho\beta S_b(1-\rho)][T_b b_{t-1}] = -\left(\frac{T}{bS}\right)\lambda_T^{-1}\beta\pi_b\kappa^{-1}[Y_b b_{t-1}] + [T_b(b_b b_{t-1})]$$
$$Y_b b_{t-1} = Y_b[b_b b_{t-1}] - \sigma((i_b b_{t-1}) - [\pi_b[b_b b_{t-1}]])$$
$$\pi_b b_{t-1} = \kappa[Y_b b_{t-1}] + \beta\pi_b[b_b b_{t-1}]$$
$$b_b b_{t-1} = \beta^{-1}b_{t-1} - \beta^{-1}[\pi_b b_{t-1}] - (1-\rho)[S_b b_{t-1}] - \frac{T}{bS}[T_b b_{t-1}]$$
$$S_b b_{t-1} = -(i_b b_{t-1}) + \rho\beta S_b[b_b b_{t-1}]$$

which in turn can be simplified to get

$$\lambda_\pi\pi_b + \kappa^{-1}Y_b = [\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]\frac{bS}{T}\lambda_T T_b \tag{33}$$

$$[(1-\rho)^{-1} - \beta\pi_b\kappa^{-1}\sigma^{-1} - (Y_b + \sigma\pi_b)\sigma^{-1} + \rho\beta S_b](1-\rho)T_b = -\left(\frac{T}{bS}\right)\lambda_T^{-1}\beta\pi_b\kappa^{-1}Y_b + T_b b_b \tag{34}$$

$$Y_b = Y_b b_b - \sigma(i_b - \pi_b b_b) \tag{35}$$

$$\pi_b = \kappa Y_b + \beta\pi_b b_b \tag{36}$$

$$b_b = \beta^{-1} - \beta^{-1}\pi_b - (1-\rho)S_b - \frac{T}{bS}T_b \tag{37}$$

$$S_b = -i_b + \rho\beta S_b b_b. \tag{38}$$

We now show some properties of π_b analytically. First, note that we will be restricting to stationary solutions, that is one where $\mid b_b \mid < 1$. Manipulations of (33)-(38) above lead to the following closed-form expression for π_b

$$\pi_b = \frac{(1-\beta b_b)}{\beta\left[\frac{T}{lV}\chi\left[\lambda_\pi + \kappa^{-1}\kappa^{-1}\left(1-\beta b_b\right)\right] + \left[(1-\rho)\sigma^{-1}\kappa^{-1}\left(1-b_b\right) + \beta^{-1}\right]\left(1-\beta b_b\right)\left(1-\rho\beta b_b\right)^{-1}\right]}$$

$$\text{where } \chi = \left[\left[\kappa^{-1}\sigma^{-1}(1-\rho) + \beta^{-1}\right]\frac{lV}{T}\lambda_T\right]^{-1}.$$

Since $\mid b_b \mid < 1$, it is clear that $\pi_b > 0$ for all $\rho < 1,$. What happens when$\rho > 1$? Numerically, we have found that often $\pi_b > 0$ still, but sometimes in fact it can hit zero (and then actually go negative if ρ is increased further). It is in fact possible to pin-down analytically when $\pi_b = 0$. Note that from (36), if $\pi_b = 0$, then $Y_b = 0$. This then implies that for this to be supported as a solution for all possible values of T_b, from (33), it must be the case that

$$[\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}] = 0$$

which in turn implies that

$$\rho = 1 + \beta^{-1}\sigma\kappa.$$

In this knife-edge case, note that one needs $\rho > 1$. Moreover, since the upper bound on ρ is β^{-1}, one needs to ensure that

$$1 + \beta^{-1}\sigma\kappa < \beta^{-1} \text{ or } \sigma\kappa < 1 - \beta.$$

This is a fairly restrictive parameterization (not fulfilled in our baseline, for example). Still, it is instructive to note that in this case of when π_b does reach 0, then it is indeed possible to show analytically that π_b declines as duration is increased while comparing $\rho = 0$ with $\rho = 1 + \beta^{-1}\sigma\kappa$ or $\rho = 1$ with $\rho = 1 + \beta^{-1}\sigma\kappa$.

7.4 Equivalence of the two approaches

We now show the equivalence of the the linearized dynamic systems for non-linear and linear-quadratic approaches.

Consider the system of equations describing private sector equilibrium and optimal government policy

$$\hat{Y}_t = E_t\hat{Y}_{t+1} + \frac{\bar{u}_C}{\bar{u}_{CC}}\left[\hat{\imath}_t - E_t\hat{\pi}_{t+1} - r_t^e\right]$$

$$\hat{\pi}_t = \frac{\varepsilon\left(\bar{v}_{yy} - \bar{u}_{CC}\right)}{\bar{u}_C d''}\hat{Y}_t + \beta E_t\hat{\pi}_{t+1}.$$

$$\hat{\imath}_t + \hat{S}_t = \beta\rho E_t\hat{S}_{t+1}.$$

$$\hat{b}_t = \beta^{-1}\hat{b}_t - \beta^{-1}\hat{\pi}_t - (1-\rho)\hat{S}_t - \frac{\bar{T}}{\bar{S}\bar{b}}\hat{T}_t.$$

$$\left(1+\rho\bar{S}\right)\bar{b}\hat{\phi}_{1t} - \bar{u}_C d^{''}\hat{\phi}_{4t} - \bar{\phi}_5 d''\hat{\pi}_t = 0$$

$$-\bar{v}_{YY}\overline{Y}\hat{Y}_t + \left[\varepsilon\bar{Y}\bar{v}_{yy}\right]\hat{\phi}_{4t} + \hat{\phi}_{5t} = 0$$

$$\bar{u}_C\beta^2\hat{\phi}_{2t} + \hat{\gamma}_{1t} = 0$$

$$\left[(1-\rho)\,\overline{b}\right]\hat{\phi}_{1t} - \bar{u}_C\hat{\phi}_{3t} = 0.$$

$$\hat{Y}_t + \frac{\bar{u}_{C\xi}}{\bar{u}_{CC}}\hat{\xi}_t - \beta\hat{\phi}_{2t} - \bar{S}\hat{\phi}_{3t} - \varepsilon\hat{\phi}_{4t} - \frac{1}{\bar{u}_{CC}}\hat{\phi}_{5t} = 0.$$

$$-\bar{g}_G s^{''}\bar{T}\hat{T}_t + \hat{\phi}_{1t} = 0.$$

$$-\bar{S}E_t\hat{\phi}_{1t+1} + \bar{S}\hat{\phi}_{1t} + \beta\bar{f}_b\hat{\phi}_{2t} + \beta\bar{g}_b\hat{\phi}_{3t} + \beta\bar{h}_b\hat{\phi}_{4t} = 0.$$

Under additional funcitonal assumptions outlined in previous sections we get

$$\bar{v}_{YY} = \phi \overline{Y}^{-1}$$

$$\bar{u}_{CC} = -\sigma \overline{Y}^{-1}$$

$$\bar{u}_{C\xi} = 1$$

$$\bar{u}_C = 1$$

$$\bar{g}_G = \bar{b}^{-1}$$

$$\bar{\phi}_5 = \tilde{\bar{y}}$$

$$\bar{Y} = 1$$

The first four equatilons are equivalent to their counterparts in the LQ-approach once one use the funcitonal assumption

$$\frac{\bar{u}_C}{\bar{u}_{CC}} = -\sigma$$

and introduce new notation

$$\frac{\varepsilon \left(\bar{v}_{yy} - \bar{u}_{CC} \right)}{\bar{u}_C d''} = \kappa$$

The latter relation implies

$$\frac{\varepsilon \left(\phi + \sigma \right)}{\kappa} = d''.$$

Let us guess solutions for all variables for the case when the ZLB is slack as a linear function of $\hat{b}_{t-1}$ and $\hat{r}_t^e$. Then the expectations will take form

$$\hat{f}_t^e = \bar{u}_{CC} E_t \hat{Y}_{t+1} + \bar{u}_{C\xi} E_t \hat{\xi}_{t+1} - \bar{u}_C E_t \hat{\pi}_{t+1} = \hat{f}_b^e \hat{b}_{t-1} + \hat{f}_r^e \ \hat{r}_t^e$$

$$\hat{g}_t^e = \left(1 + \rho \bar{S}\right) \bar{u}_{CC} E_t \hat{Y}_{t+1} + \left(1 + \rho \bar{S}\right) \bar{u}_{C\xi} E_t \hat{\xi}_{t+1} - \left(1 + \rho \bar{S}\right) \bar{u}_C E_t \hat{\pi}_{t+1} + \rho \bar{S} \bar{u}_C E_t \hat{S}_{t+1} = \hat{g}_b^e \hat{b}_{t-1} + \hat{g}_r^e \ \hat{r}_t^e$$

$$\hat{h}_t^e = \bar{u}_C d'' E_t \hat{\pi}_{t+1} = \hat{h}_b^e \hat{b}_{t-1} + \hat{h}_r^e \ \hat{r}_t^e$$

where

$$\hat{f}_b^e = -\sigma Y_b - \pi_b$$

$$\hat{g}_b^e = -\sigma \left(1 + \rho \bar{S}\right) Y_b - \left(1 + \rho \bar{S}\right) \pi_b + \rho \bar{S} S_b$$

$$\hat{h}_b^e = \frac{\varepsilon \left(\phi + \sigma \right)}{\kappa} \pi_b$$

Also under the assumption about the process $\hat{\xi}_t$ we get

$$E_t \hat{\xi}_{t+1} = \mu \hat{\xi}_t$$

and

$$\hat{r}_t^e = -\frac{\bar{u}_{C\xi}}{\bar{u}_C} \left[\mu \hat{\xi}_t - \hat{\xi}_t \right] = \hat{\xi}_t (1 - \mu).$$

When the economy is out of ZLB $\hat{r}_t^e = 0$.

Now we can find explicit representation for the lagrange multipliers

$$\hat{\phi}_{1t} = \bar{b}^{-1} s'' \bar{T} \hat{T}_t.$$

$$\hat{\phi}_{2t} = 0$$

$$\hat{\phi}_{3t} = \left[(1 - \rho) \right] s'' \bar{T} \hat{T}_t.$$

$$\hat{\phi}_{4t} = \frac{\kappa}{\varepsilon \left(\phi + \sigma \right)} \left(1 + \rho \bar{S}\right) s'' \bar{T} \hat{T}_t - \hat{\pi}_t$$

$$\hat{\phi}_{5t} = \phi \hat{Y}_t - \phi \frac{\kappa}{(\phi + \sigma)} \left(1 + \rho \bar{S}\right) s'' \bar{T} \hat{T}_t + \varepsilon \phi \hat{\pi}_t$$

So the last two equations take form

$$\varepsilon\hat{\pi}_t = -\hat{Y}_t + \frac{1}{(\phi+\sigma)}\left[\sigma\bar{S}\left(1-\rho\right) + \kappa\left(1+\rho\bar{S}\right)\right] s^{''}\bar{T}\hat{T}_t.$$

$$\left[\bar{S}\bar{b}^{-1} + \beta\left[-\sigma\left(1+\rho\bar{S}\right)Y_b - \left(1+\rho\bar{S}\right)\pi_b + \rho\bar{S}S_b\right]\left[\left(1-\rho\right)\right] + \beta\pi_b\left(1+\rho\bar{S}\right)\right] s^{''}\bar{T}\hat{T}_t - \beta\bar{b}^{-1}\frac{\varepsilon\left(\phi+\sigma\right)}{\kappa}\pi_b\hat{\pi}_t = \bar{b}^{-1}s^{''}\bar{T}\bar{S}E_t\hat{T}_{t+1}$$

which after straightforward manipulations become

$$\kappa^{-1}\varepsilon\hat{\pi}_t + \kappa^{-1}\hat{Y}_t = \left[\kappa^{-1}\left(1-\rho\right)\sigma + \beta^{-1}\right]\left(\frac{s^{''}\bar{T}^2}{\bar{b}\left(\phi+\sigma\right)}\right)\frac{\bar{b}\bar{S}}{\bar{T}}\hat{T}_t.$$

$$\left[\bar{b}^{-1} - \beta\pi_b\kappa^{-1}\left(1-\rho\right)\sigma - \left\{Y_b + \sigma^{-1}\pi_b\right\}\left(1-\rho\right)\sigma + \rho\beta S_b\left(1-\rho\right)\right]\hat{T}_t = -\frac{\bar{T}}{\bar{S}\bar{b}}\left(\frac{\bar{b}\left(\phi+\sigma\right)}{s^{''}\bar{T}^2}\right)\beta\pi_b\kappa^{-1}\hat{Y}_t + \bar{b}^{-1}E_t\hat{T}_{t+1}$$

These two equations are equivalent for the last two equations from the dynamic system obtained in LQ–approach once we introduce additional notation

$$\lambda_T = \frac{s^{''}\bar{T}^2}{\bar{b}\left(\phi+\sigma\right)}$$
$$\lambda_\pi = \kappa^{-1}\varepsilon.$$

7.5 Computation at ZLB

In our experiment the debt is kept fixed at the zero lower bound at b_L . Moreover, at the zero lower bound, $\hat{\imath}_t = 1-\beta^{-1}$. Given the specific assumptions on the two-state Markov shock process, the equilibrium is described by the system of equations

$$\begin{aligned}
\hat{Y}_t &= -\sigma\left(-\pi_b(1-\mu)\hat{b}_t - r_t^e - \mu\pi_L + \hat{\imath}_t\right) + (1-\mu)\hat{b}_t Y_b + \mu Y_L,\\
\hat{\pi}_t &= \beta\left(\pi_b(1-\mu)\hat{b}_t + \mu\pi_L\right) + \kappa Y_t,\\
\hat{\imath}_t &= 1-\beta^{-1},\\
\hat{b}_t &= b_L,\\
\hat{S}_t &= -\hat{\imath}_t + \rho\beta\left((1-\mu)S_b b_L + \mu S_L\right).
\end{aligned}$$

where variables with a b subscript denote the solution we compute at positive interest rates while variables with a L subscript denote values at the ZLB. The solution to this system is

$$\begin{aligned}
\hat{Y}_t = Y_L =& \frac{\beta\pi_b(\mu-1)b_L}{\kappa} - \frac{(1-\beta\mu)\left(\beta\pi_b(1-\mu)^2 b_L - \kappa\left(b_L\left(\pi_b(\mu-1)\sigma + (\mu-1)Y_b\right) - \sigma r_t^e + \sigma i_L\right)\right)}{\kappa(\kappa\mu\sigma - (1-\mu)(1-\beta\mu))},\\
\hat{\pi}_t = \pi_L =& -\frac{\beta\pi_b(1-\mu)^2 b_L - \kappa\left(b_L\left(\pi_b(\mu-1)\sigma + (\mu-1)Y_b\right) - \sigma r_t^e + \sigma i_L\right)}{\kappa\mu\sigma - (1-\mu)(1-\beta\mu)},\\
\hat{\imath}_t = i_L =& 1-\beta^{-1},\ \hat{b}_t = b_L,\\
\hat{S}_t = S_L =& \frac{\rho\beta(1-\mu)S_b b_L - \hat{\imath}_t}{1-\rho\beta\mu}.
\end{aligned}$$

where the solution for variables with a b subscript has already been provided above.