HOW TO CONTROL CONTROLLED SCHOOL CHOICE

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Abstract. We characterize choice rules for schools that regard students as substitutes while at the same time expressing preferences for a diverse student body. The stable (or fair) assignment of students to schools requires the latter to regard the former as substitutes. Such a requirement is in conflict with the reality of schools' preferences for diversity. We show that the conflict can be useful, in the sense that certain unique rules emerge from imposing both considerations.

1. Introduction

Recent school choice programs seek to install a stable (or fair) assignment of students to schools (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005). This objective is severely compromised by school districts' concern for diversity. Under diversity considerations, a stable assignment may not exist, and the mechanisms used in reformed school districts may not work.

There is a very basic tension between diversity considerations and the requirements of stable matching: diversity considerations introduce complementarities in schools' preferences; but the theory of stable matchings requires substitutability. If a school is concerned with gender balance, for example, then it may admit a mediocre male applicant only to maintain gender balance because it has admitted an excellent female applicant. The two students are thus complements, not substitutes, for the school. Complementarities in the school's choices of

Date: February 15, 2013.

We thank Estelle Cantillon, Lars Ehlers, Juan Pereyra Barreiro, Ariel Rubinstein, and Yves Sprumont for their helpful comments, as well as seminar audiences at MIT/Harvard, ECARES, Sciences-Po, University of Arizona, University of California-Davis, University of California-Irvine and University of Montreal.

students are a problem because both the theory and the mechanism proposed in school choice programs require that students be substitutes in schools' choices. We are far from the first to recognize this problem: Section 1.1 below discusses the relevant literature. The idea that diversity clashes with stability is very easy to recognize; in Section 1.2 we present a particularly simple example of the incompatibility between stability and diversity concerns.

Our paper seeks to reconcile diversity with the objective of seeking a stable matching of students to schools. We characterize the schools' choices that are compatible with *both* diversity considerations and the theory of stable matchings. There is so much tension between substitutability and diversity that one might think no choice rule can satisfy both. We prove that this need not be the case: we study the choices that satisfy certain normative axioms, one of them being substitutability, and show how combinations of axioms give rise to unique choice procedures, some of which are already implemented in practice. Our procedures allow schools to express concerns for diversity while allowing the standard mechanism (the one used in the school choice programs guided by stable matching theory) to install a stable assignment of students to schools.[1]

We assume that students belong to one of multiple types. Types could be categories of gender, socioeconomic status, race, or ethnicity. In all our results, an "ideal" or "target" distribution plays a crucial role. For example, we axiomatize a rule that tries to minimize the (Euclidean) distance between the distribution over types in the student body and some ideal distribution over types (see Section 4). A common consideration is that each school should have a share of white, black, Hispanic, etc. children that matches, as closely as possible, the distribution of races and ethnicities in the relevant population (Alves and Willie, 1987). Our rule operationalizes this consideration, where

[1]We do not propose any new mechanisms: we want a theory that will work with the mechanisms that have already been accepted and adopted by multiple school districts. Indeed, these mechanisms have been accepted across many different market design problems (Roth, 2008), such as markets for entry-level professional jobs.

matching the population distribution is the targeted outcome. The axiomatization tells us what such a rule means in terms of normative qualitative axioms.

In two other rules, the school sets aside a number of seats for each type of students (quotas and reserves; see Section 5). The number of seats set aside for each type is related to the target distribution over types. Yet another rule (Section 4) seeks to maximize a measure of diversity (e.g., the Theil measure of diversity: see Theil (1967); Foster and Sen (1997)); the target distribution enters as a parameter in the measure of diversity.

In Section 3 we give a brief overview of these rules and the corresponding axioms. The point of these results is that the rules are uniquely determined by the normative considerations underlying our axioms.

We imagine that a school district can discuss a menu of axioms, and settle on the axioms that it deems most desirable. Basically, schools have two sources of preferences. They have given "priorities," which are preferences over *individual* students. These priorities can result from test scores, from the distance of the student's residence to the school, or some other objective criteria. The school also has preferences over the *composition* of the student body: these preferences come from concerns about diversity. The school or the district may combine these two preferences in different ways. Our results give recommendations on how the combination should be carried out so that the standard mechanism in matching theory will work. If a school or a district agrees on a set of axioms, then there will be a unique way of combining priorities and diversity preferences into a choice procedure for the school. As we shall see, substitutability, imposed as an axiom, has very strong implications for how to combine priorities and preferences for the composition of the student body.

As we explain in Sections 7 and 8, our rules are very similar to policies being implemented around the world. We describe specific examples taken from the US and other countries. We argue that the rules of reserves and quotas are similar in spirit and implementation to many

actual school admissions policies. Our results provide a guide to the normative content of these policies, and show how they can be tweaked to achieve a rule satisfying the axiom of gross substitutes. Ultimately we hope our paper can help guide the design of school admissions for districts that wish to use stable matching mechanisms.

1.1. **Related literature.** Abdulkadiroğlu and Sönmez (2003) introduced matching theory as a tool in school choice and noted the problem with diversity concerns; they offer a solution based on quotas, one of the models we axiomatize below.

The last two years have seen multiple explorations into controlled school choice and diversity concerns. Kojima (2010) shows that affirmative action policies based on majority quotas may hurt minority students. To overcome this difficulty, Hafalir, Yenmez, and Yildirim (2011) propose affirmative action based on minority reserves. They show that the outcome of the deferred acceptance algorithm (DA) with minority reserves Pareto dominates DA with majority quotas. More generally, Ehlers, Hafalir, Yenmez, and Yildirim (2011) study affirmative action policies when there are both upper and lower type-specific bounds. They propose solutions based on whether the bounds are hard or soft. In contrast, our paper seeks to endogenize the rules and consider (possibly) all of them. Part of our research deals with the results uncovered by Hafalir, Yenmez, and Yildirim (2011). Other papers consider specific choice rules (Westkamp, 2010; Kominers and Sönmez, 2012; Erdil and Kumano, 2012).

In contrast with other papers in the literature, our focus is not on the market as a whole, but rather on the preferences or choices of individual schools. We imagine that the mechanism is fixed (the deferred acceptance algorithm) and that we can design schools' choices to satisfy certain normative axioms.

We focus on school preferences, but student preferences may also induce problems: for example, students may have preferences over their colleagues. These problems are treated by Echenique and Yenmez

(2007) and Pycia (2012); they are outside the scope of the present analysis. We focus here on diversity and on its effects on standard stable matching theory. Our exercise pins down reasonable circumstances in which schools may be concerned about diversity but where the theory still remains useful because schools satisfy the axiom of gross substitutes.

Our paper is also related to the literature on choice and social choice since we study choice rules based on axiomatic properties. In particular, Rubinstein and Zhou (1999) characterize choice rules on Euclidean space given a reference point that chooses the closest point in the set to the reference point. Our ideal-point model is similar to this in the sense that the distribution of the chosen set is closest to the ideal point. However, the choice rule based on ideal points also needs to specify the set of chosen students rather than just their distribution. Some of the rules considered by Masatlioglu, Nakajima, and Ozbay (2012) are related to ours. For example we also consider the "top N" rule for students of a given type. They do not, however, discuss type composition as a consideration in choice.

1.2. **Motivating example.** In this example, we demonstrate the basic conflict between diversity concerns and the existence of stable matching. Suppose there are two schools, c_1 and c_2, and two students, s_1 and s_2. The students are of different "types." For example, s_1 and s_2 could be of different gender, race, or ethnicity.

School c_1 can admit two students, but it is constrained to mimic the population representation of each type. So it must admit either both students or none. School c_2 has a single empty seat. It prefers to admit student s_1 over student s_2.

The students have preferences over schools as well: s_1 prefers c_1 over c_2, while s_2's favorite school is c_2. The table below summarizes the agents' preferences.

c_1	c_2	s_1	s_2
$\{s_1, s_2\}$	s_1	c_1	c_2
	s_2	c_2	c_1

It is easy to see that no matching of students to schools is stable (or "fair," to use the terminology in school choice). For example, if both students are assigned to school c_1, then s_2 might request the empty slot in school c_2. School c_2 finds s_2 acceptable, so the pair (c_2, s_2) can "block" this assignment (equivalently, s_2's claim to the empty seat is "justified"). Similarly, if s_2 is assigned to c_2 then s_1 would have no place, as school c_1 cannot admit an unbalanced student body. Then s_1 would claim s_2's spot in school c_2. Since s_1 has a higher priority than s_2 at that school, (c_2, s_1) can "block" this assignment. Thus the assignment of s_2 to c_2 is unstable. Finally, if s_1 is assigned to school c_2 and s_2 is unassigned, then both students would prefer school c_1, and school c_1 prefers to get both of them. Therefore, $(c_1, \{s_1, s_2\})$ can block the assignment.

Thus, there exists no stable or fair assignment of students to schools in this example. The reason is that c_1's preferences for diversity cause the students s_1 and s_2 to be complements. Complementarities in the school's preference make it impossible to have a stable assignment.

2. Model

2.1. **Notational conventions.** For any vector $x \in \mathbf{Z}_+^d$, let $||x|| \equiv \sum_{i=1}^d x_i$ be the sum of its coordinates. For any $x, y \in \mathbf{Z}_+^d$, let $x \wedge y \equiv (\min\{x_1, y_1\}, \ldots, \min\{x_d, y_d\})$ and $x \vee y \equiv (\max\{x_1, y_1\}, \ldots, \max\{x_d, y_d\})$ be the infimum and supremum of x and y, respectively.

For a finite set A, $|A|$ denotes the cardinality of A, and $\mathcal{P}(A)$ denotes the power set of A.

2.2. **Admissions choices.** We consider the admissions choices of an individual school or college. A school's admissions policy is described by a choice rule that determines which students to admit from a pool of applicants. Our model is therefore the model of abstract choice, one

of the most basic models in microeconomics: see, for example, Moulin (1991), or Chapter 2 in Mas-Colell, Whinston, and Green (1995). Later in the paper we study the market-wide implications of our results.

Let $\mathcal{S}$ be a nonempty finite set of all ***students***. A ***choice rule*** is a function $C : \mathcal{P}(\mathcal{S}) \setminus \{\emptyset\} \to \mathcal{P}(\mathcal{S})$ such that $C(S)$ is a subset of S, for all $S \subseteq \mathcal{S}$. The interpretation of C is that, if a school had the ability to admit its students out of the set S of students, then it would choose $C(S)$ to be its student body.

We shall assume that there is a positive number q such that $|C(S)| \leq q$ for all $S \subseteq \mathcal{S}$. The number q is the ***capacity*** of the school: the number of available seats that it has.

A ***priority*** or a ***preference*** on $\mathcal{S}$ is a binary relation on $\mathcal{S}$ that is complete, transitive, and antisymmetric (often called a linear order or a strict preference in the literature).

The set of students $\mathcal{S}$ is partitioned into students of different "types," which can be based on gender, socioeconomic factors, race, or ethnicity. Formally, there exists a set $T \equiv \{t_1, \ldots, t_d\}$ of ***types***, and a type function $\tau : \mathcal{S} \to T$; $\tau(s)$ is the type of student s. Let $\mathcal{S}^t$ be the set of type-t students, i.e., $\mathcal{S}^t \equiv \{s \in \mathcal{S} : \tau(s) = t\}$. Similarly, for any set of students $S \subseteq \mathcal{S}$, let $S^t \equiv S \cap \mathcal{S}^t$.

We use a function $\xi : \mathcal{P}(\mathcal{S}) \to \mathbf{Z}_+^d$ to describe how many students of each type a given set of students has. So we let

$$\xi(S) \equiv (|S^{t_1}|, \ldots, |S^{t_d}|) \in \mathbf{Z}_+^d,$$

which consists of the number of students of each type in S. We term $\xi(S)$ the ***distribution*** of students in S.

We assume that the school is not large enough to admit all students of a given type: $q \leq |\mathcal{S}^t|$ for all $t \in T$.[2]

[2]This assumption is reasonable, but it is not important for our results. We only use it because it makes it easier to write some of our proofs. As far as we know, none of our results depend on it.

3. Characterizations of choice rules: Overview

Our paper deals with how to combine two considerations in forming a school's choice function. One consideration stems from the school's priorities over individual students; the second is the composition of the student body, motivated by diversity concerns. There is obviously a need to balance these two considerations because they are not a priori compatible. As we have seen in Example 1.2, the conflict between these two considerations can be resolved in a way that makes stable matching impossible. Our guiding principle in performing this balance is that stable matchings exist and the mechanism used in school choice programs be made to work as intended.

We achieve this principle by imposing the axiom of gross substitutes:

Axiom 1. *Choice rule C satisfies* ***gross substitutes (GS)*** *if $s \in S \subseteq S'$ and $s \in C(S')$ imply that $s \in C(S)$.*

Equivalently, the axiom of gross substitutes says that if student s is rejected from S, and $S \subseteq S'$, then s must also be rejected from S'. It says that no student should be chosen because he or she complements another student. If all schools' choices satisfy GS, then a stable matching exists and the mechanism proposed by the recent school choice literature works well.[3]

Now, it so happens that GS plays more than one role in our theory. First, it guarantees that a stable matching exists and that the mechanisms used in school choice behave as they should. Second, it also helps us give very sharp rules for how schools should behave. All our rules allow a school to choose a high-priority student over a low-priority student when they are of the same type, but they differ on when to use

[3]GS was first studied by Kelso and Crawford (1982) and Roth (1984). GS is sufficient for the existence of stable matchings and for the Gale-Shapley deferred acceptance algorithm to find a stable matching. It is also in some sense necessary for these properties to hold (Hatfield and Milgrom, 2005; Hatfield and Kojima, 2008). Note that GS is formally identical to Sen's α. The interpretation is different, though, because here $C(S)$ is the chosen subset of S, not a set of alternatives that are "equally good."

priorities in the choice between students of different types. GS plays a role in such choices.

Model		Diversity				Rationality		
	GS	Mon	Dep	Eff	RM	t-WARP	A-SARP	E-SARP
Ideal point	✓	✓				✓		
Schur	✓		✓	✓		✓		
Reserves	✓			✓		✓	✓	
Quotas	✓				✓	✓		✓

We discuss four different rules in the paper: see the table for a summary. One rule is "reserves," in which a school reserves a number of seats for each type. The school then tries to fill these reserved seats; some of them may be unfilled if there are not enough applicants of a given type. For the remaining seats, students compete "openly." The rule uses reserves to limit when a high-priority student of one type is chosen over a low-priority student of another type. The reserves emerge partially through the use of a revealed-preference axiom (A-SARP), but it is GS that ensures that they are used consistently by the rule. The reserves rule illustrates well how GS helps pin down a specific rule because the other axioms, A-SARP and efficiency (Eff), are relatively weak.

In the model of quotas, instead of seats being reserved for a type, there is an upper bound, or quota, on how many students of a given type may be accepted. Students compete "openly" for seats until they hit the quota on their types. After that happens, a student may be turned down in favor of a lower-priority candidate whose quota has not yet been obtained. As we explain in Sections 7 and 8, quotas are used in the Chicago school district in the US and in college admissions in India. We argue that our results imply that these school districts would be better off switching to a system of reserves.

Aside from GS, we can sort the axioms into two categories: diversity and rationality axioms. The diversity axioms constrain distributions over types. We have looked at four such axioms: The distribution-monotonicity axiom (Mon) says that an increase in the distribution over the set of applicants should result in an increase in the distribution over

the admitted students. The distribution-dependence axiom (Dep) is a weaker form of monotonicity, and requires that if two sets of applicants have the same distribution, then the sets of admitted students should also have the same distribution. The efficiency axiom (Eff) states that a student should never be rejected if there is an empty seat. Rejection maximality (RM) requires that if a student is rejected from a school that has space for him, then a maximal number of students of his type must have been achieved.

Rationality axioms deal with individual priorities. They are versions of standard axioms from revealed preference theory. We use them here to elicit a priority order over individual students that is compatible with the choice rule. The type-weak axiom of revealed preference (t-WARP) deals with comparisons between students of the same type. We look at two versions of the strong axiom of revealed preference (A-SARP and E-SARP), which become relevant once we compare students of different types.[4]

We have described our two most flexible models, quotas and reserves. We also characterize two rigid models. The "ideal point" model tries first to achieve a distribution over types that is as close as possible (in Euclidean distance) to some ideal distribution over types. Given such a choice, it selects the best available, or highest-priority, students of each type. In the Schur model, the distribution does not try to approximate some ideal point but instead seeks to maximize some measure of the degree of diversity of the school. These models do not allow the school to use priorities in choosing between students of different types.

We start with the two rigid rules and then turn to the two flexible rules. A common approach in how we characterize these rules involves mapping a choice C into a function that depends on distributions. This approach is developed in Section 9. For our flexible rules, we need a correspondence instead of a function (see the definition in Section 11).

[4]One final axiom, irrelevance of rejected students, was omitted from the table. It is a basic rationality criterion, and it is satisfied by all our models. Indeed, it is hard to imagine a normative model of choice that would violate irrelevance of rejected students.

4. Ideal points and Schur concavity

We begin by analyzing the two least flexible rules discussed in Section 3. These rules never allow a school to choose a high-priority student of one type over a low-priority student of another.

A priority order over individual students still plays a role in the school's choices. It matters for choosing students of the same type. We need to make sure that the choice satisfies basic revealed preference axioms. The two axioms we use are versions of standard properties in the decision theoretic literature, used in the study of abstract choice rules and social choice (see, e.g., Moulin (1991)).[5]

Axiom 2. *Choice rule C satisfies the* ***type-weak axiom of revealed preference (t-WARP)*** *if, for any s, s', S, and S' such that $\tau(s) = \tau(s')$ and $s, s' \in S \cap S'$,*

$$s \in C(S) \text{ and } s' \in C(S') \setminus C(S) \text{ imply } s \in C(S').$$

The type-weak axiom of revealed preference is necessary for the existence of some underlying priority ordering over students. We need it to ensure that a school admits the best students of each type, given the underlying priority order.

Our second rationality axiom simply says that a rejected student may be made unavailable without affecting the set of chosen students. It has been used before in the matching context by Blair (1988), Alkan (2002), and Alkan and Gale (2003); and by Aygün and Sönmez (2012) for markets with contracts.

Axiom 3. *Choice rule C satisfies* ***irrelevance of rejected students (IRS)*** *if $C(S') \subseteq S \subseteq S'$ implies that $C(S) = C(S')$.*

To be clear, GS and the two rationality axioms capture standard properties of choice. GS is important given that we want to apply our results to matching markets.

[5]Rationality could instead require a preference relation over sets of students. We are focusing on priorities over individual students because it seems to be what most schools use in forming their preferences.

Next, we impose axioms that reflect a concern for diversity. Different axioms deliver different choice rules: we focus on those that are generated by ideal points and Schur concave monotone functions.

4.1. **Ideal points.** A school may have an ideal distribution that it tries to achieve. For example, it may strive for perfect gender balance, or for a distribution over races and ethnicities that match those in the population. Here we characterize those rules that try to minimize the Euclidean distance from the distribution of admitted students to the ideal distribution. In Sections 7 and 8 we give examples of how actual school districts' policies reflect this concern.

Choice rule C is ***generated by an ideal point*** if there is a vector $z^* \in \mathbf{Z}^d_+$ with $||z^*|| \leq q$ and a strict priority $\succeq$ over $\mathcal{S}$ such that,[6] for any $S \subseteq \mathcal{S}$, (1) $\xi(C(S))$ is the closest vector to z^* (in Euclidean distance), among those in $B(\xi(S))$ where

$$B(x) \equiv \{z \in \mathbf{Z}^d_+ : z \leq x \text{ and } ||z|| \leq q\};$$

and (2) the students of type t in $C(S)$ have higher priority than any student of type t in $S \setminus C(S)$, for any t.

Our next axiom states that an increase in the number of applicants of every type should give rise to an increase in the admissions of every type.

Axiom 4. *Choice rule C satisfies* ***distribution-monotonicity (Mon)*** *if $\xi(S) \leq \xi(S')$ implies that $\xi(C(S)) \leq \xi(C(S'))$.*

Distribution-monotonicity is responsible for the ideal-point rule's lack of flexibility. It first fixes a target distribution, and then chooses the best student body to fit that distribution. The models of reserves and quotas of Section 5 are much more flexible: they do allow priorities to affect $\xi(C(S))$. We believe that the ideal-point model may still be a good approximation to how schools operate in actuality because many

[6] In all of our models, we assume that all students are acceptable to the school, so $\succeq$ is over $\mathcal{S}$ instead of $\mathcal{S} \cup \emptyset$. The alternative can easily be incorporated such that whenever $\emptyset \succeq s$, there exists no S such that $s \in \mathcal{C}(S)$.

schools have diversity targets independently of the quality-composition of the body of applicants.[7]

Occasionally, it may simply be impossible to compare students of different types. For example, in admissions to graduate school it is often very difficult to compare students that apply for different countries.

Importantly, Mon is compatible with most forms of pure diversity concerns, including choices that fail GS. The example in Section 1.2 satisfies the Mon axiom.

Theorem 1. *A choice rule satisfies GS, t-WARP, IRS, and Mon, if and only if it is generated by an ideal point.*

The proof is provided at the end of the paper (in Section 10), as are all our proofs.

The result in Theorem 1 is surprising because GS has nothing to do with diversity. Rather, Mon says that the school has type distribution of the student body as a primary objective, but there are many ways in which diversity can be implemented. The tension between substitutability and diversity is important enough, however, that when we put the four axioms together, only ideal-point rules survive.

Remark 1. t-WARP alone does not suffice to give a rationalizing priority relation because it only rules out revealed preference cycles of length two. Normally, one needs to rule out cycles of any length, and thus we need a version of the strong axiom of revealed preference (or Richter's notion of consistency: see Richter (1966)).[8] It turns out that GS aids t-WARP in establishing a rationalizing priority relation, so we can do without a stronger axiom.

[7]In fact it is likely that schools do not care about all possible S, but only a few S in the vicinity of their current student body. In that case, they do not face very different priority distributions when they need to make actual choices.

[8]Alternatively, one may have to observe choice from all sets with two or three elements, but such a condition is not useful in the present model. Ehlers and Sprumont (2008) is one study of behavior described by WARP, allowing for cyclic choices. In our case it turns out that cycles are ruled out by the interaction of t-WARP with GS and IRS.

4.2. **Schur concave.** The distance to an ideal distribution is a reasonable criterion, but it may lead to inefficiencies. A school that is deeply committed to diversity may leave some seats empty (or may underreport its capacity) when additional students would upset the distribution over gender, or race/ethnicity, of the student body. Some inefficiency may, then, be an unavoidable consequence of a strong commitment to diversity.

That said, it may be reasonable to require schools to be efficient in the sense of never leaving a seat empty if they can fill it. To this end, we now substitute the monotonicity axiom that we used above for an efficiency axiom: the school is required to fill all seats that it can fill.

Axiom 5. *Choice rule C satisfies* ***efficiency (Eff)*** *if $C(S) = S$ when $|S| \leq q$, and $|C(S)| = q$ when $|S| > q$.*

We still need to ensure that the school cares primarily about diversity: that it sets a diversity objective independently of its body of applicants. This is the inflexibility we have talked about, and which we relax in Section 5. Thus, we impose the following weakening of the distribution-monotonicity axiom.

Axiom 6. *Choice rule C satisfies* ***distribution-dependence (Dep)*** *if $\xi(S) = \xi(S')$ implies that $\xi(C(S)) = \xi(C(S'))$.*

When a choice rule is distribution-dependent, then for any two sets with the same distribution, the sets of admitted students also have the same distribution. However, in contrast with Mon, it does not say anything about two sets that have different distributions.

As a result of these axioms, the choice of a distribution is guided by a measure of diversity. Researchers studying diversity often consider numerical measures of diversity (such as entropy, or Theil's index). For example, the ecological diversity studied by Weitzman (1992) is a special case of entropy. These numerical measures are often Schur concave, a property that we shall not define here. Instead, we shall use a canonical construction of a Schur concave function (see Marshall, Olkin, and Arnold (2010)). A school satisfying our axioms will seek

to maximize the sum of values of a monotone increasing and concave function.[9]

Say that C is ***Schur-generated*** if there is a point $z^* \in \mathbf{Z}^d_+$ with $||z^*|| \leq q$, a strictly increasing and concave function $g : \mathbf{R} \to \mathbf{R}$, and a strict priority $\succeq$ over $\mathcal{S}$ such that

(1) $\sum_{t=1}^{d} g(x_t - z_t^*)$ achieves a maximum in $B(\xi(S))$ at $\xi(C(S))$;
(2) $\xi(C(S)) = \xi(C(S'))$ for any S and S' with $\xi(S) = \xi(S')$; and
(3) students of type t in $C(S)$ have higher priority than any student of type t in $S \setminus C(S)$.

Theorem 2. *A choice rule satisfies GS, t-WARP, IRS, Eff, and Dep, if and only if it is Schur-generated.*

The interpretation of this model is as follows. Suppose that $z^* = 0$. Then, since g is concave, the maximization of $\sum_{t=1}^{d} g(x_t)$ involves values of x_t that are as close to each other as possible. That is, it seeks to obtain equal representation of all types in the school. Otherwise, when $z^* > 0$, then the maximum is going to be achieved at a point when $x \geq z^*$ and $x_t - z_t^*$ are as close to each other as possible. Equivalently, the school tries to achieve a distribution of students z^* and tries to get the same number of students in excess.

5. Reserves and quotas

We now turn to two models that allow priorities to guide the choice between students of different types, as long as it does not violate the school's (flexible) diversity objectives.

We use revealed preference axioms that infer a revealed preference in some circumstances but not others. In standard choice theory, one alternative s is revealed preferred to s' when s is chosen, and s' is not chosen, from a set that contains both. Here we shall not infer a revealed preference from all such situations, only in situations that are tied to our ultimate goal of finding either reserves or quotas.

[9]Note that monotonicity of the measure is an important difference with the objective of minimizing distance in the ideal-point rule. The latter will not be a special case of Schur-generated choice rules.

It is important to emphasize, though, that the revealed preference axioms (E-SARP and A-SARP below) alone do not suffice to establish our rules. The GS axiom makes sure that the reserves/bounds that we infer from revealed preferences are used consistently by the school's choices. Much as in the results of Section 4, GS has strong implications for the form of the choice rule.

5.1. **Quotas.** A school may want to limit the number of admitted students who have the same type, so it may have a type-specific limit or quota for each type. However, as long as these quotas are not exceeded, the school does not differentiate between students with different types. This is the model studied in Abdulkadiroğlu and Sönmez (2003). As we explain in Sections 7 and 8, quotas are used in the US in Chicago and in Indian college admissions.

Choice rule $\mathcal{C}$ is ***generated by quotas*** if there exist a strict priority $\succeq$ over $\mathcal{S}$ and a vector $(r_t)_{t\in T} \in \mathbf{Z}_+^d$ such that for any $S \subseteq \mathcal{S}$,

(1) $|C(S)^t| \leq r_t$;
(2) if $s \in C(S)$, $s' \in S \setminus C(S)$ and $s' \succ s$, then it must be the case that $\tau(s) \neq \tau(s')$ and $\left|C(S)^{\tau(s')}\right| = r_{\tau(s')}$; and
(3) if $s \in S \setminus C(S)$, then either $|C(S)| = q$ or $\left|C(S)^{\tau(s)}\right| = r_{\tau(s)}$.

In this case, r_t is an upper bound on the number of students of type t that the school can accept. The school considers all students and chooses the highest-ranked ones conditional on not exceeding any of the quotas. In particular, if $||r|| \leq q$ then this model is equivalent to the ideal-point model.

Say that S is ***ineffective*** for t if there is S' such that $|S^t| = |S'^t|$ with $|C(S)^t| < |C(S')^t|$. In words, a set is ineffective for type t when the school does not accept the maximum number of type t students among the sets with the same number of type t applicants. This notion of ineffectiveness is crucial in our axiom below.

Axiom 7. *Choice rule C satisfies the* ***effective strong axiom of revealed preference (E-SARP)*** *if there are no sequences $\{s_k\}_{k=1}^K$ and $\{S_k\}_{k=1}^K$, of students and sets of students, respectively, such that, for all k*

(1) $s_{k+1} \in C(S_{k+1})$ *and* $s_k \in S_{k+1} \setminus C(S_{k+1})$*;*
(2) $\tau(s_{k+1}) = \tau(s_k)$ *or* S_{k+1} *is ineffective for* $\tau(s_k)$*.*
(using addition mod K*).*

E-SARP rules out cycles in the revealed preference of the choice rule, but it is careful as to where it infers a revealed preference from choice. The subtlety in the definition is the second part that requires either $\tau(s_{k+1}) = \tau(s_k)$ or that S_{k+1} is ineffective for $\tau(s_k)$. In the first case, when s_{k+1} and s_k have the same type, it is revealed that s_{k+1} has a higher priority than s_k. However, when they have different types, s_{k+1} is revealed preferred to s_k only when S_{k+1} is ineffective for $\tau(s_k)$ implying that the school could admit more students of type $\tau(s_k)$. It is easy to see that E-SARP implies t-WARP.

Our next axiom is a diversity axiom. It states that whenever a type t student is rejected when there is an empty seat in the school, then it must be the case that the school has admitted the most number of type t students.

Axiom 8. *Choice rule* C *satisfies* ***rejection maximality (RM)*** *if* $s \in S \setminus C(S)$ *and* $|C(S)| < q$ *imply that* $\left|C(S)^{\tau(s)}\right| \geq \left|C(S')^{\tau(s)}\right|$ *for every* S'*.*

Rejection maximality is the main axiom we use to construct the quotas.

Theorem 3. *Choice rule* C *satisfies GS, E-SARP, and RM if and only if* C *is generated by quotas.*

Remark 2. We characterize choice rules that are generated by quotas in terms of GS, E-SARP and RM. However, it is easy to see that if a choice rule C is generated by quotas then it also satisfies IRS. We reconcile this observation with the result above by showing that if a choice rule satisfies GS and RM then it also satisfies IRS.

Suppose that choice rule C satisfies GS and RM. Let S' and S be such that $C(S') \subseteq S \subseteq S'$. By GS, $C(S') \subseteq C(S)$. Suppose for contradiction that there exists $s \in C(S) \setminus C(S')$. This implies that $s \in S' \setminus C(S')$

and $|C(S')| < q$. By RM, $\left|C(S')^{\tau(s)}\right| \geq \left|C(\hat{S})^{\tau(s)}\right|$ for any $\hat{S}$. Letting $\hat{S} = S$ yields $\left|C(S')^{\tau(s)}\right| \geq \left|C(S)^{\tau(s)}\right|$, which is a contradiction since $C(S') \subseteq C(S)$ and $s \in C(S) \setminus C(S')$. Therefore, there does not exist $s \in C(S) \setminus C(S')$, so $C(S) = C(S')$ and IRS is satisfied.

Even when $||r|| > q$ a choice rule that is generated by quotas r can be inefficient. For example, suppose that all applicants have the same type and their quota is less than the capacity of the school. In this case, the school is not going to fill its capacity. Next, we impose efficiency and IRS instead of rejection maximality and get a different model that can be characterized by *reserves*.

5.2. **Reserves.** Our second flexible rule is based on reserving seats for each type, instead of bounding the number of seats that each type can get. The rule offers flexibility to choose the highest-priority students, regardless of type, as long as a minimum number of students for each type has been reached. Reserves is the model studied by Ehlers, Hafalir, Yenmez, and Yildirim (2011).[10] We argue in Sections 7 and 8 that the districts now using quotas, such as Chicago and Indian college admissions, may be better off using reserves.

A choice rule C is ***generated by reserves*** if there exist a strict priority $\succeq$ over $\mathcal{S}$ and a vector $(r_t)_{t \in T} \in \mathbf{Z}_+^d$ with $||r|| \leq q$ such that for any $S \subseteq \mathcal{S}$,

(1) $|C(S)^t| \geq r_t \wedge |S^t|$;
(2) if $s \in C(S)$, $s' \in S \setminus C(S)$ and $s' \succ s$, then it must be the case that $\tau(s) \neq \tau(s')$ and $\left|C(S)^{\tau(s)}\right| \leq r_{\tau(s)}$; and
(3) if $\emptyset \neq S \setminus C(S)$, then $|C(S)| = q$.

In words, the school reserves r_t seats for students of type t. Given a pool of students S, it admits the best $r_t \wedge |S^t|$ students according to priority $\succ$. In a second stage, it admits the best (according to priority $\succ$) students regardless of type among the remaining students.

[10] To be more precise, Ehlers, Hafalir, Yenmez, and Yildirim (2011) study a more general model with both lower and upper bounds that could be either hard or soft.

Say that $t \in T$ is ***saturated*** at S if there is S' such that $|S^t| = |S'^t|$ with $S'^t \setminus C(S')^t \neq \emptyset$. The interpretation is that when t is saturated at S then the school is not obliged to accept all S^t out of diversity considerations.

Axiom 9. *Choice rule C satisfies the **adapted strong axiom of revealed preference (A-SARP)** if there are no sequences $\{s_k\}_{k=1}^K$ and $\{S_k\}_{k=1}^K$, of students and sets of students, respectively, such that, for all k*

(1) $s_{k+1} \in C(S_{k+1})$ and $s_k \in S_{k+1} \setminus C(S_{k+1})$;
(2) $\tau(s_{k+1}) = \tau(s_k)$ or $\tau(s_{k+1})$ is saturated at S_{k+1}

(using addition mod K).

The adapted SARP rules out the existence of certain cycles in revealed preference, where again we are careful as to when we infer the existence of a revealed preference. It is stronger than t-WARP. The difference between E-SARP and A-SARP is the second component of the definition, when we require that $\tau(s_{k+1})$ is saturated at S_{k+1}. When this happens, even though the school could admit fewer type $\tau(s_{k+1})$ students, it accepts more. Thus, in the revealed preference, s_{k+1} is preferred to s_k even if they have different types. This axiom allows us to construct a priority order over students.

Note that the notion of A-SARP, and our definition of saturation, already incorporates something similar to reserves. GS is crucial, however, to ensure that the school uses reserves consistently, in the matter described in the definition of reserves.[11]

Theorem 4. *Choice rule C satisfies GS, A-SARP, IRS, and Eff if and only if C is generated by reserves.*

6. Implications for school choice and matching markets

A ***matching market*** is a tuple $\langle \mathcal{C}, \mathcal{S}, (\succ_s)_{s\in\mathcal{S}}, (C_c)_{c\in\mathcal{C}} \rangle$, in which $\mathcal{C}$ is a finite set of ***schools***, $\mathcal{S}$ is a finite set of ***students***, for each $s \in \mathcal{S}$;

[11]This is achieved by Lemma 6 in the proof of Theorem 4.

$\succeq_s$ is a strict preference order over $\mathcal{C} \cup \{s\}$ where $\{s\}$ is the outside option for student s,[12] and for each $c \in \mathcal{C}$; C_c is a choice rule over $\mathcal{S}$.

A ***matching*** (or an ***assignment***) μ is a function on the set of agents such that

(1) $\mu(c) \subseteq \mathcal{S}$ for all $c \in \mathcal{C}$ and $\mu(s) \in \mathcal{C} \cup \{s\}$ for all $s \in \mathcal{S}$;
(2) $s \in \mu(c)$ if and only if $\mu(s) = c$ for all $c \in \mathcal{C}$ and $s \in \mathcal{S}$.

In a matching market, we would like to find *stable* matchings that satisfy individual rationality and fairness properties that we formalize below.

Definition 1. *A matching μ is **stable** if*

*(1) (**individual rationality**) $C_c(\mu(c)) = \mu(c)$ for all $c \in \mathcal{C}$, $\mu(s) \succeq_s \{s\}$ for all $s \in \mathcal{S}$; and*
*(2) (**no blocking**) there exists no (c, S') such that $S' \not\subseteq \mu(c)$ such that $S' \subseteq C_c(\mu(c) \cup S')$ and $c \succeq_s \mu(s)$ for all $s \in S'$.*

Stability requires both individual rationality and no blocking. First, individual rationality for schools means that no school can be better off by rejecting some of the admitted students, whereas for students it means that each student prefers her assigned school to her outside option. Second, no blocking implies that there exists no coalition of agents who can beneficially rematch among themselves. This is the standard definition of stability used in many-to-one matching problems (Roth and Sotomayor, 1990).

For matching markets, stability has proved to be a useful solution concept because mechanisms that find stable matchings are successful in practice (Roth, 2008). Moreover, finding stable matchings is relatively easy. In particular, the deferred acceptance algorithm (DA) of Gale and Shapley (1962) finds a stable matching, and DA has other attractive properties.[13] Therefore, it also serves as a recipe for market design. For example, it has been adapted by the New York and Boston school districts (see Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005)

[12]The outside option for student s can be going to a private school or being homeschooled.

[13]For a history of the deferred acceptance algorithm, see Roth (2008).

and Abdulkadiroğlu, Pathak, and Roth (2005)). For completeness, we provide a description of the student-proposing deferred acceptance algorithm.

Deferred Acceptance Algorithm (DA)

Step 1: Each student applies to her most preferred school. Suppose that S_c^1 is the set of students who applied to school c. School c tentatively admits students in $C_c(S_c^1)$ and permanently rejects the rest. If there are no rejections, stop.

Step k: Each student who was rejected at Step $k-1$ applies to her next preferred school. Suppose that S_c^k is the set of new applicants and students tentatively admitted at the end of Step $k-1$ for school c. School c tentatively admits students in $C_c(S_c^k)$ and permanently rejects the rest. If there are no rejections, stop.

The algorithm ends in finite time since at least one student is rejected at each step.

Usually the only strategic component of a matching market is the student preference profile $(\succ_s)_{s\in\mathcal{S}}$; schools' choice rules are fixed by laws and regulations. Therefore, our only concern is that students and their families state their preferences truthfully. To this end, we consider a group strategy-proofness concept for students.

Let $P_{\mathcal{S}}$ be the set of student preference profiles $(\succ_s)_{s\in\mathcal{S}}$ and $\mathcal{M}$ be the set of matchings between $\mathcal{C}$ and $\mathcal{S}$. A ***mechanism*** is a function $\Phi : P_{\mathcal{S}} \to \mathcal{M}$.

Definition 2. *Mechanism Φ is **group strategy-proof for students** if there exists no group of students S', preference profiles $(\succ_s)_{s\in\mathcal{S}}$, and $(\succ'_s)_{s\in S'}$ such that*

$$\Phi((\succ'_s)_{s\in S'}, (\succ_s)_{s\in\mathcal{S}\setminus S'}) \succ_s \Phi((\succ_s)_{s\in\mathcal{S}})$$

for all $s \in S'$.

Informally, a mechanism is group strategy-proof for students if there exists no group of students who can jointly manipulate their preferences to be matched with better schools.

The following axiom for choice rules plays a critical role in establishing the desirable properties of the deferred acceptance algorithm.

Axiom 10. *Choice rule C satisfies* ***the law of aggregate demand (LAD)*** *if $S \subseteq S'$ implies $|C(S)| \leq |C(S')|$.*

The law of aggregate demand requires that excluding some students should not increase the number of chosen students. This property was first introduced by Alkan (2002) and Alkan and Gale (2003) for matching markets without transfers (or contracts), and by Hatfield and Milgrom (2005) for markets with contracts. All of the choice rules that we have studied in Sections 4 and 5 satisfy LAD. In particular, it is easy to see that distribution-monotonicity or efficiency implies LAD.

The following result is well known (see Roth and Sotomayor (1990), Hatfield and Milgrom (2005), Hatfield and Kojima (2009), and Aygün and Sönmez (2012)). We state it here to highlight the role of the properties that we have studied in Sections 4 and 5.

Theorem 5. *Suppose that schools' choice rules satisfy IRS and GS, then DA produces the stable matching that is simultaneously the best stable matching for all students. Suppose, furthermore, that choice rules satisfy LAD; then DA is group strategy-proof for students and each school is matched with the same number of students in any stable matching.*

We refer to the outcome of DA as the ***student-optimal stable matching***. Below we study how the student-optimal stable matching changes with the choice rules.

Theorem 6. *Consider two choice rule profiles $(C_c)_{c\in\mathcal{C}}$ and $(C'_c)_{c\in\mathcal{C}}$ such that, for each school c, C_c and C'_c satisfy IRS and GS and, furthermore, $C_c(S) \subseteq C'_c(S)$ for every set of students S. Let μ and μ' be the student-optimal stable matchings with $(C_c)_{c\in\mathcal{C}}$ and $(C'_c)_{c\in\mathcal{C}}$, respectively. Then $\mu'(s) \succeq_s \mu(s)$ for all s.*

Informally, if choice rule C' selects more students than choice rule C, then all students weakly prefer DA with C' to DA with C. In fact, a slightly more general statement is true: Suppose that, for each c, C'_c satisfies IRS and GS and μ' is the student-optimal stable matching with $(C'_c)_{c\in\mathcal{C}}$. Then if $(C)_{c\in\mathcal{C}}$ is a choice rule profile such that $C_c(S) \subseteq C'_c(S)$ for every $S \subseteq \mathcal{S}$, $c \in C$ and μ is a stable matching with $(C)_{c\in\mathcal{C}}$, then $\mu'(s) \succeq_s \mu(s)$ for all s. In other words, we do not need to make any assumptions about $(C)_{c\in\mathcal{C}}$ as long as there is a stable matching. However, we make such assumptions in Theorem 6 to make the statement easier. Otherwise, we need to assume that there exists a stable matching with respect to $(C)_{c\in\mathcal{C}}$.

In particular, we can compare the outcome of DA with choice rules that are generated by reserves and quotas:

Corollary 1. *Suppose that C_c is generated by quotas $(r^c_t)_{t\in T}$ and C'_c is generated by reserves $(r^c_t)_{t\in T}$ where $||r^c|| \leq q_c$ for every school c with the same priority $\succeq$. Let μ and μ' be the student-optimal stable matchings with $(C_c)_{c\in\mathcal{C}}$ and $(C'_c)_{c\in\mathcal{C}}$, respectively. Then $\mu'(s) \succeq_s \mu(s)$ for all s.*

This follows directly from Theorem 6 above. Indeed, if student s is rejected by C'_c from S, i.e., if $s \in S \setminus C'_c(S)$, then $\xi(C'_c(S))_{\tau(s)} \geq r_{\tau(s)}$ and for any $s' \in C'_c(S)$ with $\tau(s') = \tau(s)$ we have $s' \succeq_c s$. Therefore, s cannot be chosen by choice rule C from S since there are at least $r_{\tau(s)}$ students of type $\tau(s)$ who have a higher ranking than s in S.

Corollary 1 also implies that students weakly prefer Schur-generated choice rules to ideal-point-generated choice rules with the same reference point whenever DA is used. In fact, we can say more than that, for particular reference points:

Theorem 7. *Suppose that C_c satisfies GS and distribution-monotonicity for each c and that there are two types ($d = 2$). Let $z^*_c = \xi(C_c(\mathcal{S}))$ for each c. Furthmore, let μ be the DA outcome using choice profile $(C_c)_{c\in C}$, μ^i the DA outcome using choice profile with rules that are generated by ideal points z^*_c for each c, and μ^s be the DA outcome using*

choice rules that are Schur-generated z_c^* for each c. Then

$$\mu^s(s) \succeq_s \mu(s) \succeq_s \mu^i(s),$$

for all s.

Therefore, with this particular choice of z^*, we get the result that students prefer the outcome of DA using the Schur-generated choice rules, to DA using the original choice profile, to DA using the ideal-point-generated choice rules.

Finally, in the next result, we consider matching markets in which schools' choice rules are generated by quotas. For this market, we establish a type-specific "rural hospitals theorem":

Theorem 8. *Suppose that choice rule C_c is generated by quotas r_c for each school c. If there exists a stable matching μ such that $|\mu^t(c)| < r_c^t$ and $|\mu(c)| < q_c$ then for any stable matching μ', $\mu'^t(c) = \mu^t(c)$.*

This result follows immediately from Kojima (2012). He shows that the same conclusion holds when school preferences satisfy a more general condition called *separable with affirmative action constraints*.

In particular, if schools' choice rules are generated by ideal points, then each school's student distribution is the same in all stable matchings, i.e., the diversity of schools is the same in all stable matchings. Therefore, as long as the school district implements a stable matching, each school's incoming student distribution is fixed and does not depend on the stable matching.

7. Controlled school choice in the US

The legal background on diversity in school admissions is complicated. Since the landmark 1954 *Brown v. Board of Education* supreme court ruling, which ended school segregation, many school districts have attempted to achieve more integrated schools. The current legal environment is summarized in the 2011 guidelines issued by the US departments of justice and education: "Guidance on the voluntary use of race to achieve diversity and avoid racial isolation in elementary and secondary schools." (There is a separate set of guidelines for college

admissions.) We shall not summarize these guidelines here, but suffice it to say that they are perfectly compatible with the theory developed in this paper.

In particular, the "race neutral" approaches described in the guidelines can be carried out through our methods (race neutrality goes into the definition of types). We proceed to briefly describe some of the best-known programs in the US.

7.1. **Jefferson County.** The Jefferson County (KY) School District is prominent in promoting diversity among its schools, and the litigation surrounding its admissions policies serves partly as a basis for the 2011 US government guidelines mentioned above. The rules proposed by the county violate the GS axiom, and therefore would be incompatible with an assignment mechanism based on the deferred acceptance algorithm, like that in use in Boston, New York City, or Chicago. We believe that Jefferson County's objectives could be satisfied by using one of the rules we propose in this paper–for example, reserves.

Starting from the early 1970s, the student assignment plan used in Jefferson County went through major changes. First, in order to avoid segregation, a racial assignment plan was used and students were bussed to their schools. In the early 1990s a school choice system was implemented, allowing parents to state their preferences over schools. In 1996, schools were required to have between 15 and 50 percent of African American students. In 2002 a lawsuit was filed against the Jefferson County School District because it had a racial admissions policy. After a litigation process, the case came before the US Supreme Court. The Supreme Court in 2007 ruled in favor of the plaintiffs, and decided that race cannot be the only factor to use for admissions.

Following this ruling, Jefferson County switched to an assignment plan that considered the socioeconomic status of parents: Using census data, the school district divided the county into two regions and required all schools to have between 15 and 50 percent of their students to be from the first region. This rule violates GS.

Jefferson County is undergoing yet another change at the moment. The new assignment plan, which was accepted by the school district to be implemented in 2013/14 admissions cycle, divides students into three types: Type 1, Type 2, and Type 3. These types are determined by educational attainment, household income, and percentage of white residents in the census block group that the student lives in. Then each school is assigned a diversity index, defined as the average of student types. The new admissions policy requires each school to have a diversity index between 1.4 and 2.5.

These two assignment policies are in conflict with the GS axiom, so they are incompatible with a school choice plan that would seek to install a stable (or fair) matching. It should be clear, however, that the rules proposed in our paper can achieve similar objectives to the ones in the current policies, while satisfying GS.

7.2. **Chicago.** Chicago public schools also strive for diversity (Pathak and Sönmez, 2012; Kominers and Sönmez, 2012). To this end, they have an affirmative action policy that uses socioeconomic status to divide students into four types: Tier 1, Tier 2, Tier 3, and Tier 4. Students who would like to attend selective high schools take a centralized exam that is used to determine a score for each student. Each school allocates 40 percent of their seats to the students with the highest rank and then 15 percent of the seats are allocated to each tier separately. In particular, if a school is divided into two fictitious schools, one representing open seats and the other representing the rest, this affirmative action policy can be viewed as the quotas model as follows: For the first fictitious school, the quota for each type can be the capacity of the school, whereas for the second fictitious school the quota for each type can be 15 percent of the total school capacity. In particular, the actual implementation of the affirmative action policy is very similar to this. By Corollary 1, if the Chicago school district switched to a

reserves rule, which uses choice rules generated by reserves for the second fictitious school with the same quotas, all students would weakly benefit.[14]

Alternatively, the Chicago system could be modified to satisfy a simple model of reserves. This would involve first filling the spaces that are reserved for each type, and then having types compete openly for the remaining spaces. Depending on the distribution of scores for each type, the quotas could be calibrated to achieve the effect desired by the school district. The rule has the advantage of fitting directly into the existing stable matching school choice mechanism.

8. Controlled School Choice in Other Countries

Policies to enhance diversity can be found in many countries around the world (Sowell, 2004). Some of these policies implement preferential policies, whereas some of them implement policies based on quotas. The former resemble the reserves rule that we have studied, while the latter are similar to the quotas model (with regional variations in actual implementation). Many countries have similar policies, including, but not limited to, Brazil, China, Germany, Finland, Macedonia, Malaysia, Norway, Romania, Sri Lanka, and the United States. Below we discuss two particular examples: college admissions in India and high school admissions in French-speaking Belgium.

8.1. **Indian College Admissions.**

In India the caste system divides society into hereditary groups or castes ("types" in our model). Historically, it has enforced a particular division of labor and power in society, and placed severe limits on socioeconomic mobility. To overcome this, the Indian Constitution has since 1950 implemented affirmative action. It prescribes that the "scheduled castes" (SCs) and "scheduled tribes" (STs) be represented in government jobs and public universities proportional to their population percentage in the state that they belong to. These percentages change from state to state. For example, in

[14]Kominers and Sönmez (2012) have a different counterfactual in which students either rank the fictitious school representing the open slots first or last.

Andhra Pradesh, each college allocates 15 percent of its seats for SCs, 6 percent for STs, 35 percent for other "backward classes," and the remaining 44 percent is left open for all students.

The college admissions to these public schools is administered by the state, and it works as follows. Students take a centralized exam that determines their ranking. Then students are called one by one to make their choices from the available colleges. In each college, first the open seats are filled. Afterward the reserved seats are filled only by students for whom the seats are reserved. This model corresponds to the situation described above for Chicago. Therefore, this affirmative action policy fits into our quotas-ideal-point model in which we replace each school with two copies, the first representing the open seats and the second representing the rest. For the first copy of the college, each student is treated the same and the choice rule picks the best available students regardless of their caste. For the second copy, a choice model based on quotas-ideal-point model is used. Similarly to the Chicago school district, if a soft quota policy were used, all students would be weakly better off.

The Indian choice rule can also be generated by reserves in which each bound is greater than the school's capacity. But the second copy of the college implements a choice rule that is generated by the quotas rule described in the previous paragraph.[15]

8.2. **High Schools in French-Speaking Belgium.** In French-speaking Belgium, high school admissions are set up to promote diversity. However, in contrast with many examples we have seen thus far, the target of affirmative action policy is the set of students who have attended "disadvantaged primary schools." The administration announces these primary schools, which may change each year depending on supply and demand. Each school is required to reserve at least 15 percent of its seats for students from disadvantaged primary schools, and also some seats for students living in the neighborhood of the school. If a reserved

[15]For an empirical study of affirmative action policies in Andhra Pradesh see Bagde, Epple, and Taylor (2011).

seat for either group cannot be filled then it can also be allocated to other students so long as there is no student from the privileged group willing to take that seat. This choice corresponds to the reserves model described above.

9. Auxiliary lemmas

Our general approach is the following: We translate considerations related to distributions into results on functions on $\mathbf{Z}_+^d$. Most of our results follow from mapping C into a function or a correspondence defined on $\mathbf{Z}_+^d$. In particular, it turns out that some of our axioms have interesting counterparts as properties of such functions and correspondences.

9.1. **Notational Conventions.** For $A \subseteq \mathbf{Z}_+^d$, $\partial A \equiv \{z \in A : z' \gg z \Rightarrow z' \notin A\}$, where $z' \gg z$ if and only if $z'_t > z_t$ for all t and, similarly, let $\partial^M A \equiv \{z \in A : z' > z \Rightarrow z' \notin A\}$.

Let e_t denote the unit vector in $\mathbf{Z}_+^d$. It is the vector with 0 in all its entries except that corresponding to t, in which it has 1.

9.2. **Ideal point model.** We shall first introduce some simple lemmas related to functions on $\mathbf{Z}_+^d$. Hopefully the discussion is suggestive of how we use these lemmas in proving our main results: to this end we use suggestive names for the relevant properties of these functions.

Let $f : \{x \in \mathbf{Z}_+^d : 0 \leq x \leq \xi(\mathcal{S})\} \to \mathbf{Z}_+^d$. We say that f is ***monotone increasing*** if $y \leq x$ implies that $f(y) \leq f(x)$; ***monotone strictly increasing*** if $y < x$ implies that $f(y) < f(x)$; f is ***within budget*** if $f(x) \in B(x) = \{y : 0 \leq y \leq x, ||y|| \leq q\}$; that f satisfies ***gross substitutes*** if

$$y \leq x \Rightarrow f(x) \wedge y \leq f(y).$$

A function f is ***generated by an ideal point*** if there is $z^* \in \mathbf{Z}_+^d$ such that $||z^*|| \leq q$, and $f(x)$ minimizes the Euclidean distance to z^* among the vectors in $B(x)$.

Lemma 1. *Function f is monotone increasing, within budget and satisfies gross substitutes if and only if it is generated by an ideal point.*

We need the following lemma:

Lemma 2. *Let $z^* \in \mathbf{Z}_+^d$ satisfy $||z^*|| \leq q$. Then $x \wedge z^*$ is the unique minimizer of the Euclidean distance to z^* among the vectors in $B(x)$.*

Proof of Lemma 2. First note that $x \wedge z^* \in B(x)$. The distance from z to z^* is minimized if $\sum_t (z_t - z_t^*)^2$ is minimized. The lemma follows from the observation that one can minimize, for each t, $(z_t - z_t^*)^2$ by setting $z_t = \min\{x_t, z_t^*\}$: when $\min\{x_t, z_t^*\} = z_t^*$ this is trivial, and when $\min\{x_t, z_t^*\} = x_t$ then there are no $z \in B(x)$ with $z_t > x_t$. Since $z_t = \min\{x_t, z_t^*\}$ for every t, we get $z = x \wedge z^*$. □

Proof of Lemma 1. We first show that if f is generated by an ideal point z^* with $||z^*|| \leq q$, then it is monotone increasing, within budget and it satisfies gross substitutes. Suppose that the ideal point is z^*. By Lemma 2, $f(x) = x \wedge z^*$. Then $f(x) \leq x$ and $||f(x)|| \leq ||z^*|| \leq q$, so f is within budget. Next we show monotonicity:

$$y \leq x \Rightarrow y \wedge z^* \leq x \wedge z^* \Rightarrow f(y) \leq f(x).$$

Last we show gross substitutes. Let $y \leq x$. Then,

$$f(x) \wedge y = (x \wedge z^*) \wedge y = (x \wedge y) \wedge z^* = y \wedge z^* = f(y).$$

It will be useful to consider an additional property. We say that f satisfies the ***boundary condition*** if $f(x) \in \partial B(x)$.

We now turn to proving that the axioms are sufficient for generation by an ideal point. We suppose that $f : \mathbf{Z}_+^d \to \mathbf{Z}_+^d$ is a function satisfying monotonicity, gross substitutes, and it is within budget. We show that it must be generated by some ideal point. We consider two different cases, the case when f satisfies the boundary condition and when it does not.

First, suppose that f satisfies the boundary condition. Let $\hat{x}$ such that $\hat{x}_t \geq q$ for all t and $z^* \equiv f(\hat{x})$. Note that $\sum_t z_t^* = q$ because

$$f(\hat{x}) \in \partial B(\hat{x}) = \{z \in \mathbf{Z}_+^d : \sum_t z_t = q\},$$

by the choice of $\hat{x}$ and the boundary condition. We show that, for all y, $f(y)$ minimizes the distance to z^* in $B(y)$.

Note if $y \leq x$ then the monotonicity of f, and that $f(y) \leq y$ implies that $f(y) \leq y \wedge f(x)$. Thus the gross substitutes axiom becomes:

$$y \leq x \Rightarrow f(x) \wedge y = f(y). \tag{1}$$

Let y be arbitrary. We shall prove that $f(y) = y \wedge z^*$. Now, $\hat{x} \leq \hat{x} \vee y$, so $z^* = f(\hat{x}) \leq f(\hat{x} \vee y)$, as f is monotone increasing. Then $\sum_t z_t^* = q$ and $f(\hat{x} \vee y) \in \partial B(\hat{x} \vee y)$ implies that $z^* = f(\hat{x} \vee y)$. Now, $y \leq \hat{x} \vee y$ and the substitutes condition (1) gives us that

$$f(y) = y \wedge f(\hat{x} \vee y) = y \wedge z^*.$$

By Lemma 2, $f(y)$ minimizes the distance to z^* in $B(y)$.

We finish the proof by considering the case when f does not satisfy the boundary condition. In this case there is z^* such that $f(z^*) \notin \partial B(z^*)$. We shall prove that f is generated by ideal point $f(z^*)$.

Let $x \in \mathbf{Z}_+^d$. Note that $z^* \wedge x \leq z^*$, monotonicity, and gross substitutes, imply (using equation (1)) that

$$f(z^* \wedge x) = (z^* \wedge x) \wedge f(z^*) = (z^* \wedge f(z^*)) \wedge x = f(z^*) \wedge x.$$

Similarly, $z^* \wedge x \leq x$ gives us that

$$f(z^* \wedge x) = (z^* \wedge x) \wedge f(x) = (x \wedge f(x)) \wedge z^* = f(x) \wedge z^*.$$

Thus, $f(z^*) \wedge x = f(x) \wedge z^*$. Now observe that $f(z^*) \notin \partial B(z^*)$ means that $f(z^*) \ll z^*$; so $f(z^*) \wedge x \ll z^*$. Then $f(z^*) \wedge x = f(x) \wedge z^*$ is only possible if $f(z^*) \wedge x = f(x)$. By Lemma 2 $f(x)$ minimizes the distance to $f(z^*)$ in $B(x)$. □

Remark 3. There are ideal point rules that violate the boundary condition, but when the ideal point z^* is such that $\sum_t z_t^* = q$, then the ideal point rule satisfies the boundary condition. To see this, suppose first that $z^* \leq x$. Then $\sum_t z_t^* = q$ implies that $f(x) = x \wedge z^* = z^* \in \partial B(x)$. If, on the other hand, $z^* \not\leq x$ then there is t such that $x_t = (x \wedge z^*)_t = f(x)_t$; so if $z \gg f(x)$ then $z \notin B(x)$, as $z_t > f(x)_t = x_t$. Therefore, $f(x) \in \partial B(x)$.

9.3. **Schur concavity.** We say that $f : \{x \in \mathbf{Z}_+^d : 0 \leq x \leq \xi(\mathcal{S})\} \to \mathbf{Z}_+^d$ is ***Schur-generated*** if there is $z^* \in \mathbf{Z}_+^d$ such that $||z^*|| \leq q$ and a monotone strictly increasing and concave function $g : \mathbf{R} \to \mathbf{R}$, such that $f(x)$ is a maximizer of $\sum_{t=1}^d g(x_t - z_t^*)$ in the set $B(x)$ for all x. Similarly, f is ***efficient*** if $f(x) \in \partial^M(B(x))$ for all x.

Lemma 3. *A function f is efficient and satisfies gross substitutes if and only if it is Schur-generated.*

Proof. Let f satisfy the two axioms. Let $\hat{x}$ be such that $\hat{x}_t > q$ for all t, and let $z^* \equiv f(\hat{x})$. For any $\alpha \in \mathbf{R}$, let

$$g(\alpha) \equiv \alpha \wedge q + (\alpha - \alpha \wedge q)/2.$$

Notice that g is strictly monotone increasing and concave. Let $\nu(x) \equiv \sum_t g(x_t + q - z_t^*)$. Since g is monotone increasing and concave, so is ν.

We shall prove that $f(y)$ maximizes ν in $B(y)$. To prove this, we show that $f(y) \geq z^* \wedge y$. Note that this suffices because it says that $f(y)_t \geq z_t^*$ when $y_t > z_t^*$, and $f(y)_t = y_t$ (as $f(y) \leq y$) when $y_t \leq z_t^*$; by definition of ν and the axiom of efficiency, ν is maximized by such an f.

Now, that $f(y) \geq z^* \wedge y$ follows from gross substitutes in the case that $y \leq \hat{x}$. Suppose then that $y \not\leq \hat{x}$.

First, $y \wedge \hat{x} \leq \hat{x}$ and gross substitutes imply that $f(y \wedge \hat{x}) \geq z^* \wedge (y \wedge \hat{x}) = z^* \wedge y$, as $z^* = z^* \wedge \hat{x}$.

Second, $y \wedge \hat{x} \leq y$ and gross substitutes imply that

$$f(y \wedge \hat{x}) \geq f(y) \wedge (y \wedge \hat{x}) = (f(y) \wedge y) \wedge \hat{x} = f(y) \wedge \hat{x} = f(y).$$

Then $y \not\leq \hat{x}$ implies that $||y|| > q$, so efficiency of f implies that $||f(y)|| = q$. Then $f(y \wedge \hat{x}) \geq f(y)$ and $||f(y \wedge \hat{x})|| \leq q$ give us that $f(y \wedge \hat{x}) = f(y)$. We showed above that $f(y \wedge \hat{x}) \geq z^* \wedge y$, so we obtain that $f(y) \geq z^* \wedge y$ as desired. □

10. Proofs of results from Section 4

10.1. **Proof of Theorem 1.** Suppose that C satisfies the axioms. We shall prove that it is generated by an ideal point. To this end, we show

that there exist an ideal point z^* and a strict priority $\succ$ such that the choice function created by these coincides with C. The result follows essentially from Lemma 1 above.

Define f as follows. For any $x \leq \xi(\mathcal{S})$, let S be such that $x = \xi(S)$ and let $f(x) = \xi(C(S))$. By distribution-monotonicity we know that $\xi(S) = \xi(S') \Rightarrow \xi(C(S)) = \xi(C(S'))$, so the particular choice of S does not matter; thus f is well defined. Moreover, when $y \leq x$ we have $f(y) \leq f(x)$, again by distribution-monotonicity. So f is a monotone increasing function. In addition, $f(x) \leq x$ and $||f(x)|| \leq q$, so f is within budget. Let z^* be as defined in the proof of Lemma 1. Since $f(z^*) = z^*$, we have that $||z^*|| \leq q$.

Define a binary relation R by saying that $s \, R \, s'$ if $\tau(s) = \tau(s')$ and there is some $S \ni s, s'$ such that $s \in C(S)$ and $s' \notin C(S)$. We shall prove that R is transitive.

Lemma 4. *If C satisfies GS, t-WARP and IRS, then R is transitive.*

Proof. Let $s \, R \, s'$ and $s' \, R \, s''$; we shall prove that $s \, R \, s''$. Let S' be such that $s', s'' \in S'$, $s' \in C(S')$, and $s'' \notin C(S')$. Consider the set $S' \cup \{s\}$. First, note that $s \in C(S' \cup \{s\})$. The reason is that if $s \notin C(S' \cup \{s\})$ then $C(S' \cup \{s\}) = C(S') \ni s'$, by irrelevance of rejected students. Thus $s' \, R \, s$, in violation of t-WARP. Second, note that $s'' \notin C(S' \cup \{s\})$, as $s'' \notin C(S')$ and C satisfies gross substitutes. □

The relation R is transitive. Thus it has an extension to a linear order $\succ$ over S. For any S, and any $s, s' \in S$ with $\tau(s) = \tau(s')$ we have that $s \succ s'$ when $s \in C(S)$ while $s' \notin C(S)$.

Lemma 5. *If C satisfies GS then $\xi(\mathcal{S}) \geq x \geq y$ implies $f(x) \wedge y \leq f(y)$.*

Proof. Suppose that C satisfies GS. Let $\xi(\mathcal{S}) \geq x \geq y$ and $S' \subseteq \mathcal{S}$ be such that $\xi(S') = x$. Construct S with $\xi(S) = y$ as follows. If $y_t \geq \xi(C(S'))_t$, then $S^t \supseteq C(S')^t$. However, if $y_t < \xi(C(S'))_t$, then $S^t \subseteq C(S')^t$. In the former case, $C(S)^t \supseteq C(S')^t$ by gross substitutes. In the later case, $C(S)^t = S^t$ by gross substitutes. In both cases, $\xi(C(S))_t \geq \min\{\xi(S)_t, \xi(C(S'))_t\}$, which implies $f(y) \geq f(x) \wedge y$. □

By Lemma 5, that C satisfies gross substitutes implies that f satisfies gross substitutes. In addition, f is also monotone increasing and within budget, as was shown above. Therefore, f is generated by an ideal point rule with z^* by Lemma 1. Then C is generated by the ideal point z^* and priority order $\succ$.

Conversely, let C be generated by an ideal point z^* and $\succeq$. It is immediate that C satisfies t-WARP. Define f as above. Here, f is well defined because for any S and S' such that $\xi(S) = \xi(S') = x$, $\xi(C(S))$ is the closest vector to z^* among those in $B(x)$ and $\xi(C(S'))$ is the closest vector to z^* among those in $B(x)$. Therefore, $\xi(C(S)) = \xi(C(S'))$ and so f is well defined.

To show that C satisfies distribution-monotonicity, let $y = \xi(S)$ and $x = \xi(S')$ such that $y \leq x$. By Lemma 2, $f(x) = x \wedge z^*$ and $f(y) = y \wedge z^*$. Then, $f(x) = x \wedge z^* \leq y \wedge z^* = f(y)$, and, therefore, $\xi(C(S)) \leq \xi(C(S'))$. Hence, C satisfies distribution-monotonicity.

To show that C satisfies irrelevance of rejected students, let $C(S') \subseteq S \subseteq S'$, $\xi(S') = x$ and $\xi(S) = y$. By construction, $\xi(C(S')) = x \wedge z^*$ and $\xi(C(S)) = y \wedge z^*$. Therefore,

$$\xi(C(S')) \geq \xi(C(S)).$$

On the other hand, by assumption $\xi(C(S')) \leq \xi(S) = y$ and also $\xi(C(S')) = x \wedge z^* \leq z^*$. The last two inequalities imply that

$$\xi(C(S')) \leq y \wedge z^* = \xi(C(S)).$$

Hence, $\xi(C(S')) = \xi(C(S))$, which means that the same number of type t students are chosen from S and S' for each t. Since students are chosen according to their priorities in $\succeq$ and $C(S') \subseteq S \subseteq S'$, we conclude that $C(S') = C(S)$.

To see that C satisfies gross substitutes, let $s \in S \subseteq S'$, $\tau(s) = t$, $\xi(S) = y$ and $\xi(S') = x$. As we have shown above, $f(x) = x \wedge z^*$ and $f(y) = y \wedge z^*$. If $f(y)_t \geq f(x)_t$, then more type t students are chosen in S compared to S'. Since $s \in C(S')$, and C is generated by an ideal point, we derive that $s \in C(S)$. On the other hand, if $f(y)_t < f(x)_t$, then $f(y)_t < z^*_t$ since $f(x)_t = (x \wedge z^*)_t \leq z^*_t$. Since $f(y)_t = (y \wedge z^*)_t$,

we derive that $f(y)_t = y_t$. That means all type t students are chosen from S, so $s \in C(S)$. Hence, C satisfies gross substitutes.

10.2. **Proof of Theorem 2.** Let C satisfy the axioms. By the distribution-dependence axiom we can define f as in the proof of Theorem 1. We can also define a strict preference $\succ$ to act as priority order, by the same argument as in the proof of Theorem 1.

Now, by Lemma 5, f satisfies the axiom of gross substitutes for functions. The axiom of efficiency for C implies that f satisfies the boundary condition, efficiency for functions. By Lemma 3, f is Schur-generated. This implies that C is Schur-generated.

Conversely, suppose that C is Schur-generated. It is easy to see that C satisfies t-WARP and distribution-dependence. We show that C also satisfies GS, IRS and efficiency. For any $x \leq \xi(\mathcal{S})$, let S be such that $\xi(S) = x$ and define $f(x) \equiv \xi(C(S))$. Since C is Schur-generated, f is well defined and also Schur-generated. By Lemma 3, f is efficient and satisfies gross substitutes. That f is efficient implies C is efficient.

To see that C satisfies gross substitutes, let $s \in S \subseteq S'$, $\tau(s) = t$, $\xi(S) = y$ and $\xi(S') = x$. Since f satisfies GS, we have

$$\min\{f(x)_t, y_t\} \leq f(y)_t.$$

If $y_t \leq f(x)_t$, then GS implies $y_t \leq f(y)_t$, which is equivalent to $y_t = f(y)_t$. Hence, $s \in C(S)$. On the other hand, if $y_t \geq f(x)_t$, then $f(x)_t \leq f(y)_t$, so more type t students are chosen from S compared to S'. Since $s \in C(S')$ and C satisfies t-WARP, this implies $s \in C(S)$.

To show that C satisfies irrelevance of rejected students, let $C(S') \subseteq S \subseteq S'$, $\xi(S') = x$ and $\xi(S) = y$. Since C satisfies GS, $C(S') \subseteq C(S)$. Suppose that there exists $s \in C(S) \setminus C(S')$. Then $\xi(C(S)) > \xi(C(S'))$ but this is a contradiction since $\xi(C(S))$ maximizes $\sum_{t=1}^{d} g(x_t - z_t^*)$ in $B(\xi(S))$ and $\xi(C(S'))$ maximizes $\sum_{t=1}^{d} g(x_t - z_t^*)$ in $B(\xi(S'))$ with $B(\xi(S')) \supseteq B(\xi(S))$. Therefore, $C(S) \setminus C(S') = \emptyset$, which implies $C(S) = C(S')$.

11. Proofs from Section 5

11.1. **Proof of Theorem 3.** Suppose that C satisfies the axioms. We start by showing that C is generated by quotas.

Let $r_t \equiv |C(\mathcal{S}^t)|$ and $S \subseteq \mathcal{S}$. First we prove that $|C(S)^t| \leq r_t$. If $r_t = q$, then $|C(S)^t| \leq r_t$ holds trivially. Suppose that $r_t < q$. Since $|\mathcal{S}^t| > q$, there exists $s \in \mathcal{S}^t \setminus C(\mathcal{S}^t)$. By rejection maximality, for every S, $r_t = |C(\mathcal{S}^t)| \geq |C(S)^t|$. Therefore, $|C(S)^t| \leq r_t$.

Let $\succ^*$ be defined as follows: $s \succ^* s'$ if there exists $S \supseteq \{s, s'\}$ such that $s \in C(S)$, $s' \notin C(S)$ and either $\tau(s) = \tau(s')$ or S is ineffective for $\tau(s')$. By the effective strong axiom of revealed preference, $\succ^*$ has a linear extension $\succ$ to $\mathcal{S}$.

We now show that if $s \in C(S)$, $s' \in S \setminus C(S)$ and $s' \succ s$, then it must be the case that $\tau(s) \neq \tau(s')$ and $\left|C(S)^{\tau(s')}\right| = r_{\tau(s')}$. If $\tau(s) = \tau(s')$, then $s \succ^* s'$ and $s \succ s'$, which is a contradiction with the fact that $\succ$ is an extension of $\succ^*$. So $\tau(s) \neq \tau(s')$. To prove that $\left|C(S)^{\tau(s')}\right| = r_{\tau(s')}$ suppose, towards a contradiction, that $\left|C(S)^{\tau(s')}\right| \neq r_{\tau(s')}$, so $\left|C(S)^{\tau(s')}\right| < r_{\tau(s')}$.

We shall prove that S is ineffective for $\tau(s')$, which will yield the desired contradiction, as $\succ$ is an extension of $\succ^*$. Let $S' \equiv S^{\tau(s')}$. We consider three cases. First, $|C(S')| = q$ then $|C(S)| < |C(S')|$ (as there is $s \in C(S)$ with $\tau(s) \neq \tau(s')$), so S is ineffective for $\tau(s')$. Second, consider the case when $|C(S')| < q$ and $|C(S')| < |S'|$. Then, by rejection maximality,

$$|C(S')| = r_{\tau(s')} > \left|C(S)^{\tau(s')}\right|.$$

Hence S is ineffective for $\tau(s')$. Third, consider the case when $|C(S')| < q$, and $|C(S')| = |S'|$. Then $|C(S')| > \left|C(S)^{\tau(s')}\right|$, as $s' \in S^{\tau(s')} \setminus C(S)^{\tau(s')}$. Thus S is ineffective for $\tau(s')$. In all three cases we conclude that $s \succ^* s'$. Since $\succ$ is a linear extension of $\succ^*$, we get $s \succ s'$, a contradiction.

Finally, we need to show that if $s \in S \setminus C(S)$, then either $|C(S)| = q$ or $\left|C(S)^{\tau(s)}\right| = r_{\tau(s)}$. By rejection maximality, $s \in S \setminus C(S)$ and $|C(S)| < q$ implies that $\left|C(S)^{\tau(s')}\right| \geq \left|C(S')^{\tau(s')}\right|$ for any S'. We already

proved that r_t is the supremum of $\{\left|C(\hat{S})^t\right| : \hat{S} \subseteq \mathcal{S}\}$. So we get that $\left|C(S)^{\tau(s')}\right| = r_{\tau(s')}$. Hence, $s \in S \setminus C(S)$ implies either $|C(S)| = q$ or $\left|C(S)^{\tau(s')}\right| = r_{\tau(s')}$.

To finish the proof, suppose that C is generated by quotas. Then it is easy to see that C satisfies the effective strong axiom of revealed preference and rejection maximality. We show that it also satisfies gross substitutes. Suppose that $s \in S \subseteq S'$ and $s \in C(S')$. For each type t, let $S(t; r_t) \subseteq S^t$ be the r_t highest ranked type t students in S (if $|S^t| \leq r_t$ then $S(t; r_t) = S^t$). Define $S'(t; r_t)$ analogously. Since $s \in C(S')$, we have $s \in S'(\tau(s), r_{\tau(s)})$ and the ranking of s in $\cup_t S'(t; r_t)$ is no more than q. Since $S \subseteq S'$, the preceding statements also hold for S instead of S', which implies that $s \in C(S)$.

11.2. **Proof of Theorem 4.** For any $x \leq \xi(\mathcal{S})$, let $F(x) \equiv \{\xi(C(S)) : \xi(S) = x\}$ and

$$\hat{f}(x) = \bigwedge_{f(x) \in F(x)} f(x).$$

The proof requires the following lemma.

Lemma 6. *Let C satisfy GS. If $y \in \mathbf{Z}_+^d$ is such that $\hat{f}(y)_t < y_t$ then $\hat{f}(y + e_{t'})_t < y_t + 1_{t=t'}$*

Proof. Let y and t be as in the statement of the lemma. Let S be such that $\xi(S) = y$ and $\xi(C(S))_t < \xi(S)_t = y_t$. Such a set S exists because $\hat{f}(y)_t < y_t$. Let $s' \notin S$ be an arbitrary student with $\tau(s') = t'$. Note that

$$\emptyset \neq S^t \setminus C(S)^t \subseteq (S \cup \{s'\})^t \setminus C(S \cup \{s'\})^t,$$

as C satisfies GS. Then we cannot have $\xi(C(S \cup \{s'\}))_t = y_t + 1_{t=t'}$ because that would imply $(S \cup \{s'\})^t \setminus C(S \cup \{s'\})^t = \emptyset$. Then

$$y_t + 1_{t=t'} > \xi(C(S \cup \{s'\}))_t \geq \hat{f}(y + e_{t'})_t.$$

□

Suppose that C satisfies the axioms. Using Lemma 6, we can construct the vector r of minimum quotas as follows. Let $\bar{x} = \xi(\mathcal{S})$. The lemma implies that if $\hat{f}(y_t, \bar{x}_{-t})_t < y_t$ then $\hat{f}(y'_t, \bar{x}_{-t})_t < y'_t$ for

all $y'_t > y_t$. Then there is $r_t \in \mathbf{N}$ such that $y_t > r_t$ if and only if $\hat{f}(y_t, \bar{x}_{-t}) < y_t$. This uses the assumption we made on the cardinality of $\mathcal{S}^t$, which ensures that $\hat{f}(y)_t < y_t$ if y_t is large enough. Note that we may have $r_t = 0$.

First we prove that $S \subseteq \mathcal{S}$ with $|S^t| \leq r_t$ then $S^t = C(S)^t$. Observe that, for any x and t, $\hat{f}(r_t, x_{-t}) = r_t$. To see this note that if there is x and t such that $\hat{f}(r_t, x_{-t}) < r_t$ then Lemma 6 would imply that $\hat{f}(r_t, \bar{x}_{-t}) < r_t$, in contradiction with the definition of r. In fact, we can say more: For any x, t, and y_t, if $y_t \leq r_t$ then $\hat{f}(r_t, x_{-t}) = r_t$ and Lemma 6 imply that $\hat{f}(y_t, x_{-t}) = y_t$. Therefore, letting $S \subseteq \mathcal{S}$ with $|S^t| \leq r_t$ we have that

$$\left|C(S)^t\right| \geq \hat{f}(y)_t = y_t,$$

where $y = \xi(S)$. Since $y_t = |S^t| \geq |C(S)^t|$ we have that $S^t = C(S)^t$.

Second we prove that, if $|S^t| > r_t$, then $|C(S)^t| \geq r_t$. Let $\tilde{S} = C(S)$. Assume, towards a contradiction, that $\left|\tilde{S}^t\right| < r_t$. Let $S' = \tilde{S} \cup S''$, where $S'' \subseteq S^t \setminus \tilde{S}^t$ is such that $|S'^t| = r_t$. By irrelevance of rejected students, $C(S') = C(S)$. Thus,

$$\hat{f}(\xi(S'))_t \leq \left|C(S')^t\right| = \left|C(S)^t\right| < r_t.$$

Since $\xi(S')_t = |S'^t| = r_t$, we obtain a contradiction with the definition of r_t above.

Consider the following binary relation. Let $s \succ^* s'$ if there is S, at which $\{s\} = \{s, s'\} \cap C(S)$ and $\{s, s'\} \subseteq S$, and either $\tau(s) = \tau(s')$ or $\tau(s)$ is saturated at S. By the adapted strong axiom of revealed preference, $\succ^*$ has a linear extension $\succeq$ to $\mathcal{S}$.

Third we prove that C is consistent with $\succeq$, as stated in the definition. Let $s \in C(S)$ and $s' \in S \setminus C(S)$. If $\tau(s) = \tau(s')$ then $s \succ^* s'$ by definition of $\succ^*$; hence $s \succeq s'$. If $\tau(s) \neq \tau(s')$ then we need to consider the case when $|S^t| > r_t$ where $t = \tau(s)$. The construction of r_t implies that $r_t = \hat{f}(|S^t|, \bar{x}_{-t}) < |S^t|$. Therefore, there exists $S' \subseteq \mathcal{S}$ such that if

$$S' = S^t \cup \left(\cup_{\tilde{t} \neq t} \mathcal{S}^{\tilde{t}}\right)$$

then $S'^t \setminus C(S')^t \neq \emptyset$. Thus t is saturated at S. Since $s \in C(S)$ and $s' \in S \setminus C(S)$, we get $s \succeq s'$, as $\succeq$ extends $\succ^*$.

It remains to show that if C is generated by reserves, then it satisfies the axioms. It is immediate that it satisfies Eff, IRS and A-SARP.

To see that it satisfies gross substitutes, let $S \subseteq S'$ and $s \in S \setminus C(S)$. Then $\left|S^{\tau(s)}\right| > r_{\tau(s)}$, so $\left|S'^{\tau(s)}\right| > r_{\tau(s)}$. Moreover, $s \in S \setminus C(S)$ implies that there are $r_{\tau(s)}$ students in $S^{\tau(s)}$ ranked above s. So s could only be admitted at the second step in the construction of C. Let $C^{(1)}(S)$ be the set of students that are accepted in the first step, S^* be the set of students that are considered in the second step and q^* be the number of remaining seats to be allocated in the second step. Again, $s \in S \setminus C(S)$ implies that there are q^* students ranked above s in S^*. Consider the following procedure for S'. In the first step for each t we accept $\xi(C^{(1)}(S))_t$ highest ranked students of type t. And in the second step we consider all remaining students. It is clear that s cannot be admitted in the first step since $S'^{\tau(s)} \supseteq S^{\tau(s)}$ and that there are at least $r_{\tau(s)}$ students ranked above s in $S^{\tau(s)}$. Moreover, in the second step of the new procedure, there are more higher ranked students of each type compared to S^*, so s can also not be admitted in the second step since there are only q^* seats left. If s cannot be admitted with this procedure, then it cannot be in $C(S')$ because for each $t \neq \tau(s)$, $\xi(C^{(1)}(S))_t \leq r_t$. Therefore, $s \in S' \setminus C(S')$.

12. Proofs from Section 6

12.1. **Proof of Theorem 6.** We start with the following lemma.

Lemma 7. *If C satisfies GS and (c, S) blocks a matching μ, then for every $s \in S \setminus \mu(c)$ $(c, \{s\})$ blocks μ.*

Proof. Since (c, S) blocks μ, we have $S \subseteq C_c(\mu(c) \cup S)$. Let $s \in S \setminus \mu(c)$, by substitutability $s \in C(\mu(c) \cup S)$ implies $s \in C(\mu(c) \cup \{s\})$. Therefore, $(c, \{s\})$ blocks μ. □

We proceed with the proof of Theorem 6.

Since we use two different choice rule profiles and stability depends on the choice rules, we prefix the choice rule profile to stability, individual

rationality and no blocking to avoid confusion. For example, we use C-stability, C-individual rationality and C-no blocking.

By Theorem 5, DA produces the student-optimal stable matching. Denote the student-optimal stable matching with C and C' by μ and μ', respectively. Since $C_c(\mu(c)) = \mu(c)$ by C-individual rationality of μ by every school c, $C'_c(\mu(c)) \supseteq C_c(\mu(c))$ by the assumption, and $C'_c(\mu(c)) \subseteq \mu(c)$ by definition of the choice rule we get $C'(\mu(c)) = \mu(c)$. Therefore, μ is also C'-individually rational for schools. Since student preference profile is fixed, μ is also C'-individually rational for students. If μ is a C'-stable matching, then μ' Pareto dominates μ since μ' is the student-optimal C'-stable matching. Otherwise, if μ is not a C'-stable matching, then there exists a C'-blocking pair. Whenever there exists such a blocking pair, there also exists a blocking pair consisting a school and a student by Lemma 7. In such a situation, we apply the following improvement algorithm. Let $\mu^0 \equiv \mu$.

> **Step k:** Consider blocking pairs involving school c_k and students who would like to switch to c_k, say $S^k_{c_k} \equiv \{s : c_k \succ_s \mu^{k-1}(s)\}$. School c_k accepts $C'_{c_k}(\mu^{k-1}(c_k) \cup S^k_{c_k})$ and rejects the rest of the students. Let $\mu^k(c_k) \equiv C'_{c_k}(\mu^{k-1}(c_k) \cup S^k_{c_k})$ and $\mu^k(c) \equiv \mu^{k-1}(c)$ for $c \neq c_k$. If there are no more blocking pairs, then stop and return μ^k, otherwise go to Step $k+1$.

We first prove by induction that no previously admitted student is ever rejected in the improvement algorithm. For the base case when $k = 1$ note that $C'_{c_1}(\mu(c_1) \cup S^1_{c_1}) \supseteq C_{c_1}(\mu(c_1) \cup S^1_{c_1})$ by assumption and $C_{c_1}(\mu(c_1) \cup S^1_{c_1}) = \mu(c_1)$ since μ is C-stable. Therefore, $C'(\mu(c_1) \cup S^1_{c_1}) \supseteq \mu(c_1)$, which implies that no students are rejected at the first stage of the algorithm. Assume, by mathematical induction hypothesis, that no students are rejected during Steps 1 through $k-1$ of the improvement algorithm. We prove that no student is rejected at Step k of the algorithm. There are two cases to consider.

First, consider the case when $c_n \neq c_k$ for all $n \leq k-1$. Since μ is C-stable, we have $C_{c_k}(\mu(c_k) \cup S^k_{c_k}) = \mu(c_k)$ (as students in $S^k_{c_k}$ prefer c_k to their schools in μ). By assumption, $C'_{c_k}(\mu(c_k) \cup S^k_{c_k}) \supseteq C_{c_k}(\mu(c_k) \cup S^k_{c_k})$

which implies $C'_{c_k}(\mu(c_k) \cup S^k_{c_k}) \supseteq \mu(c_k)$. Since $\mu(c_k) \supseteq \mu^{k-1}(c_k)$ we have $C'_{c_k}(\mu^{k-1}(c_k) \cup S^k_{c_k}) \supseteq \mu^{k-1}(c_k)$ by substitutability. In this case no student is rejected at Step k.

Second, consider the case when $c_k = c_n$ for some $n \leq k-1$. Let n^* be the last step smaller than k in which school c_k was considered. Since each student's match is either the same or improved at Steps 1 through $k-1$, we have $\mu^{n^*-1}(c_k) \cup S^{n^*}_{c_k} \supseteq \mu^{k-1}(c_k) \cup S^k_{c_k}$. By construction, $\mu^{n^*}(c_k) = C'_{c_k}(\mu^{n^*-1}(c_k) \cup S^{n^*}_{c_k})$ which implies $\mu^{k-1}(c_k) \subseteq C'_{c_k}(\mu^{k-1}(c_k) \cup S^k_{c_k})$ by substitutability and the fact that $\mu^{n^*}(c_k) \supseteq \mu^{k-1}(c_k)$ (since n^* is the last step before k in which school c_k is considered). Therefore, no student is rejected at Step k.

Since no student is ever rejected by the improvement algorithm, it ends in a finite number of steps. Moreover, the resulting matching does not have any C'-blocking pair. By construction, it is also C'-individually rational. This shows that there exists a C'-stable matching that Pareto dominates μ. Since μ' is the student-optimal C'-stable matching, we have that μ' Pareto dominates μ for students.

12.2. **Proof of Theorem 7.** Fix a school c. Let f be defined as in the proof of Theorem 1 for choice rule profile C_c: such f is well defined because C_c satisfies distribution-dependence. Similarly, let f^i be the corresponding function in the ideal point model, given ideal point z^*_c; and let f^s be the f corresponding function in the Schur model, given parameter z^*_c.

Let S be a set of students and $y \equiv \xi(S) \leq \xi(\mathcal{S})$ be the type distribution of S. Then, by gross substitutes of C_c and Lemma 2 we have that

$$f^i(y) = y \wedge z^* \leq f(y).$$

Moreover, we show that $f(y) \leq f^s(y)$. When $||y|| \leq q$ we have $f(y) \leq f^s(y)$ because $f(y) \in B(y)$ and $f^s(y) \in \partial^M B(y) = \{y\}$. Consider the other case when $||y|| > q$. First, if $z^* \leq y$ then $f^s(y) \geq z^*$ by definition of f^s. By distribution-monotonicity $z^* = f(\xi(C(\mathcal{S}))) \geq f(y)$, so we obtain that $f(y) \leq z^* \leq f^s(y)$. In second place, assume that $z^* \not\leq y$. Suppose, without loss of generality that $z^*_1 > y_1$. Then by

definition of f^s we have that $f^s(y) = (y_1, q - y_1)$; the reason is that for any other $(x_1, x_2) \in \partial^M B(y)$ we have $x_1 < y_1 < z_1^*$ and $x_2 > z_2^*$ so that

$$g(x_1 + 1 - z_1^*) + g(x_2 - 1 - z_2^*) > g(x_1 - z_1^*) + g(x_2 - z_2^*)$$

for any increasing and concave g under our assumptions. Now note that $f^s(y)_1 = y_1 = (y \wedge z^*)_1 = f^i(y)_1$. So $f(y)_1 = y_1$, as $f(y) \geq f^i(y)$. Now, $f(y)_2 \leq f^s(y)_2 = q - y_1$, as $||f(y)|| \leq q$.

Since $f^i(y) \leq f(y) \leq f^s(y)$, the conclusion follows from Theorem 6.

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