The Combinatorial Clock Auction, Revealed Preference and Iterative Pricing[*]

Lawrence M. Ausubel[†] and Oleg V. Baranov[‡]

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Abstract

The combinatorial clock auction (CCA) has recently become a popular design choice for allocating valuable items such as spectrum licenses—and it has been used in many of the digital dividend auctions worldwide. However, the current implementation of the CCA has limitations stemming from two sources. First, the most common activity rules, while motivated by revealed preference, impose only some of the revealed-preference constraints on bidders. This may reduce the informativeness of the clock rounds and may leave excessive room for strategic bidding. Second, the CCA is an iterative second-price auction rather than an iterative first-price auction, unlike most other dynamic auction formats, including the English auction and the SMRA. In this paper we propose a new activity rule that can be characterized as the most restrictive activity rule that still permits straightforward bidding by bidders with quasilinear utilities. With this new activity rule, the price feedback of the CCA is greatly improved, enabling a transformation of the CCA to an iterative first-price format. The centerpiece of the innovation is a simple "bidder exposure" calculation, which provides an upper bound on the bidder's payment under second pricing. The notion of bidder exposure is broadly linked to the notion of "clinching" introduced in Ausubel (2004, 2006) for auctions without package bidding.

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[†] Department of Economics, University of Maryland, College Park, MD 20742, USA. Email: ausubel@econ.umd.edu

[‡] Department of Economics, University of Colorado, Boulder, CO 80309, USA. Email: oleg.baranov@colorado.edu

1 Introduction (INCOMPLETE)

The combinatorial clock auction (CCA) has recently become a popular auction design choice for allocating highly valuable items. This design has already been used in countries like Germany, Austria, UK, Australia, Ireland, Canada and Switzerland to allocate the spectrum licenses for a buildup of the next generation wireless networks. Unlike the simultaneous multiple round auction (SMRA) that has been used in most of the previous spectrum auctions, the CCA design allow bidders to express virtually any preferences over the auctioned items including complementarities and substitutes without running an exposure risk typical for SMRA.

As with any auction format in this environment, the current implementation of the CCA has a number of known problems. First of all, a typical CCA with a core-selecting payment rule is not actually a dynamic auction in the usual sense. Well-known dynamic auctions such as English auction or SMRA are all "pay-as-bid" formats in a sense that the winner always pays the full amount of its winning bid. In contrast, bidders generally would not pay the full amount of their winning bids in the CCA, making it more similar to sealed-bid formats with the second-price payment rules rather than the classical dynamic auctions. The lack of price feedback during the clock rounds significantly complicates the bidding process for bidders, especially if they are budget-constrained.

Second of all, the purpose of the supplementary round in the CCA from the auctioneer perspective is twofold. The auctioneer not only wants to collect bids that are focused on lots that were not allocated during the clock stage but also bids that can be used in price calculations. The lack of bidding incentives for the latter class of bids is highly problematic. Given the low winning probability of such bids, bidders frequently fail to submit them at all or submit bids that artificially inflated for the purposes of increasing prices for their competitors. This duality often causes internal tensions for the activity rules. A very good illustrative example of this tension is the Final Cap Activity Rule that has been used in almost all recent CCA auctions.

Finally, a set of activity rules that is most frequently used for CCA can produce anomalies such as non-monotonic bid limits. This happens when a bid amount for a package is constrained by a smaller limit than a limit for a strictly smaller package (a subset).

The first major contribution of this paper is a new set of activity rules for the CCA that is based on the well-known concept of the revealed preferences. Unlike the current activity rules, the new activity rules make use of the quasilinear version of the generalized axiom of revealed preference (QGARP). To put it differently, bidder's clock bids have to be consistent with some valuation profile. The QGARP-based activity rule eliminates many undesirable features of the current rules such as non-monotonic bid limits, endogenous constraining rounds, parking strategies and etc. All these features disappear under QGARP since it does not rely on the artificial system of eligibility points inherited by the CCA from the SMRA. More importantly, the

internal consistency of the QGARP rules allows bidders to track their exposure, the highest price they can possibly pay in case the clock stage of the CCA ends in the current round with no excess supply.

The second major contribution of this paper is a new pricing calculation approach for the clock rounds that makes use of the bidder exposure calculations that become feasible under QGARP. In any given clock round, auctioneer can calculate the bidder exposure for any package a bidder might demand. Further, the bidder is only required to bid the exposure amount rather than the full clock price for the chosen package. Due to QGARP constraints, the amount of the bidder discount does not depend on the selected package and is solely determined by the clock bidding history of other bidders. Dynamic discounting of clock bids allows to eliminate "overbidding" from the CCA when bidders are consistently asked to bid more than they need to guarantee their winnings. Since the opportunity costs are already reflected trough the bid discounts, auctioneer requires the final payments to be at least the exposure amount in case the bidder ends up winning the clock package after the supplementary round.

The paper proceeds as follows. Section 2 provides a short description of the main components of the current CCA design including activity rules. In Section 3, we introduce a new set of activity rules for the clock stage and the supplementary stage that are based on QGARP and establish its key properties. Section 4 contains welfare results of various value profiles that can be recovered from the clock round bidding history that was restricted by QGARP. Section 5 develops exposure calculation that can be used to calculate bidder discounts dynamically in each clock round. Section 6 introduces enhanced combinatorial clock auction and discuss its properties. Section 7 provides a discussion of various practical aspects of the new design. Section 8 links the bidder exposure calculation to the "crediting/debiting" formula developed in Ausubel (2006). Section 9 concludes. Most proofs are relegated to Appendix A.

2 Description of the Current CCA Rules

A typical spectrum auction that uses CCA format starts with a clock phase which is followed by a supplementary round and concludes with an assignment stage. Below we outline the main features of this three-phase design.

Clock Phase: The clock phase consists of several bidding rounds. In each round, the auctioneer announces prices and bidders reply with their quantity demands. Further, the auctioneer increments clock prices on items with positive excess demand. The clock phase ends when there are no items with positive excess demand. During the clock phase, bidders' demands are constrained by eligibility point monotonicity and/or QWARP-based activity rules.[1]

[1] See Section 2.1 for definitions of QWARP-based activity rule.

Supplementary Phase: In the supplementary phase, auctioneer conducts another round of bidding. Unlike the clock phase, bidders are allowed to name not only packages they want but also the desired prices for these packages. Supplementary round can be thought as a sealed-bid auction where bids made in the clock rounds cannot be withdrawn. Bidders can supplement their bids from the clock rounds with additional package bids that are constrained by a QWARP-based activity rule.

Winner Determination and Pricing: All bids made throughout the clock phase and supplementary phase are considered by the winner determination process. This process finds an allocation that maximizes total bid value subject to constraints that each item cannot be awarded more than once and only one bid from each bidder can be selected. Prices are typically determined by the core-selecting pricing procedure.

Assignment phase: The assignment phase is required only if some of the auctioned items were offered as generic lots. In spectrum auctions, generic blocks of spectrum are assigned to specific frequencies during the assignment phase. It is routinely done through another auction were bidders bid for the desired assignment of their winnings in the spectrum band. Bidders might be required to make additional payments as a result of the assignment phase.

In this paper we are not analyzing the assignment phase and focus solely on the clock phase, supplementary phase and the pricing determination process.

2.1 Clock Phase

Suppose that in Round 1, a bidder is allowed to place a bid on any package from a feasible set of packages X which is defined by the auctioneer, potentially by the amount of the financial deposit made by the bidder prior to the auction. For example, the set X might contain all packages that are affordable at the opening prices given the amount of the financial deposit.

The definition of the eligibility-point monotonicity rule requires some additional details on the auction process. Each product offered in the auction is assigned a number of eligibility points. Each bidder's initial eligibility is set using some exogenous factors like its financial deposit. Given the current bidder's eligibility, the bidder's eligibility for the next clock round is set to equal to the lower of its current eligibility and the eligibility of the clock package demanded by the bidder in the current clock round computed as a sum of eligibility points for items included in the demanded package.

Eligibility-Point Monotonicity Rule: In a Round t, a bidder can bid on any package $x \in X$ with equal or lower eligibility than the bidder's eligibility in Round t.

Definition: A bidder with a valuation profile $v(\cdot)$ is said to *bid straightforwardly* during the clock stage if, in every round t, her bid $x_t \in Q(p_t) = \arg\max_{x \in X} \{v(x) - p_t x\}$.

Unattractive feature of the eligibility-point monotonicity rule is given by the following theorem:

Theorem(???): For any choice of eligibility points, there exist bidder valuations and price histories such that the bidder is prevented from bidding its true valuations by an activity rule requiring monotonicity in eligibility points; and if bidders attempt to bid straightforwardly, the outcome will necessarily be inefficient.

The clock bidding history for any bidder is given by the collection of price-quantity pairs (p_t, x_t) for each Round t.

Definition: (WARP) A collection of price-quantity pairs $(p_1, x_1), ..., (p_{t-1}, x_{t-1}), (p_t, x_t)$ satisfies the *weak axiom of revealed preferences* (WARP) if and only if:

$$p_s x_s \geq p_s x_z \quad \text{implies} \quad p_z x_z \leq p_z x_s \quad \forall s, z \in \{1, ..., t\}$$

Intuitively, WARP requires that if bundle x_s revealed preferred to bundle x_z in Round s, then bundle x_z cannot be revealed preferred to bundle x_s in Round z. In other words, WARP tests limited rationality of the bidder's clock history.

A clock activity rule based on WARP alone is too weak for any practical purposes. For example, a WARP-based rule would not prevent a bidder from bidding on some package in one round and bidding on the superset of the same package in the next round in which all clock prices have increased. A natural refinement for the WARP for the auction context is to require quasilinearity on top of limited rationality.

Definition: (QWARP) A collection of price-quantity pairs $(p_1, x_1), ..., (p_{t-1}, x_{t-1}), (p_t, x_t)$ satisfies the *quasilinear weak axiom of revealed preferences* (QWARP) if and only if:

$$(p_s - p_z)(x_s - x_z) \leq 0 \quad \forall s, z \in \{1, ..., t\}$$

or, equivalently,

$$p_s(x_s - x_z) + p_z(x_z - x_s) \leq 0 \quad \forall s, z \in \{1, ..., t\}$$

Unlike WARP, QWARP is well-suited for auction applications in terms of its strengths to limit bid sniping as it only allows switches to packages that have become relatively less expensive.

In order to motivate the QWARP definition, suppose that there exists two pairs (p_s, x_s) and (p_z, x_z) such that $p_s(x_s - x_z) + p_z(x_z - x_s) > 0$. Then this bidder would agree to participate in the following transaction. It

would be acceptable for such bidder to exchange bundle x_s for bundle x_z for a net payment of $p_z(x_z - x_s)$ and then exchange bundle x_z for bundle x_s for a net payment of $p_s(x_s - x_z)$.As a result, the bidder would agree to pay a strictly positive amount $p_s(x_s - x_z) + p_z(x_z - x_s) > 0$ to stay with the same bundle. This kind of behavior is inconsistent with rationality. Next we define a QWARP-based clock activity rule.

Definition: (QWARP Clock Rule) A bidder is permitted to place a bid on a package $x \in X$ in Round t with clock prices p_t if and only if the collection of observations $(p_1, x_1), ..., (p_{t-1}, x_{t-1}), (p_t, x)$ satisfies QWARP.

This clock rule requires that bidders place clock bids that are consistent with limited rationality under quasilinearity. Unlike Eligibility-Point Monotonicity Rule, the QWARP Clock Rule always permits straightforward bidding. However, it does not provide enough protection against bidding histories with "dead-ends", i.e., a series of nonzero demand choices that deems any nonzero packages infeasible in subsequent rounds. This shortcoming stems from the fact that QWARP only tests limited rationality rather than the full rationality. The "dead-end" problem can be easily demonstrated with a simple unit-demand example provided in Table 1. It can be verified that clock bids made in first three rounds are feasible under QWARP, yet the only feasible bundle in Round 4 is the zero package (0, 0, 0).

Table 1: "Dead-End" under WARP Clock Rule

Round	Price vector	Bidder's demand
1	$p_1 = (1, 1, 1)$	$x_1 = (1, 0, 0)$
2	$p_2 = (1, 1, 4)$	$x_2 = (0, 1, 0)$
3	$p_3 = (4, 1, 4)$	$x_3 = (0, 0, 1)$
4	$p_4 = (4, 3, 5)$	…

In order to avoid "dead-ends", Ausubel and Cramton suggested combining strength of the QWARP Rule with the robustness of the eligibility-point monotonicity rule. The resulting hybrid rule allows both straightforward bidding and feasibility. However, in order to avoid "dead-ends", only a subset of the QWARP constraints is imposed.

Hybrid Revealed Preference / Eligibility-Point Monotonicity Rule

In a Round t, a bidder can bid on any package $x \in X$ with equal or lower eligibility than the bidder's eligibility in Round t or if the package x satisfies QWARP constraints with respect to all previous eligibility-reducing rounds starting from the last round the bidder had enough eligibility to bid on x.

Unfortunately, the hybrid rule still has some serious shortcomings. For example, this clock rule does not allow to impose many important revealed preference constraints in the supplementary round. This might lead to anomalies such as "non-monotonic caps" when a cap[2] for a larger package is smaller than the cap for a smaller package. Further, limiting attention to the revealed preference constraints from eligibility-reducing rounds might create incentives for bidders to create artificial eligibility-reducing rounds to lock in the favorable relative prices.

2.2 Supplementary Phase

In the supplementary phase, bidders can revise their clock bids (upwards only) and possibly add bids for packages they have not bid on during the clock phase. The maximum bid amount a bidder can bid for any package is typically constrained by QWARP-type constraints from different clock rounds. Typically, a bid for a package $x \in X$ is constrained by the bid made in the last clock round in which bidder's eligibility was higher than the eligibility associated with the package x (known as the Relative Cap) and, possibly, subject to the bid made in the final clock round (known as the Final Cap Rule).

2.3 Winner and Price Determination

All bids made throughout the clock phase and supplementary phase are considered during the winner determination. Most commonly, the auctioneer would add bids for all possible combinations of lots at reserve prices to make sure that bidders pay at least reserve for each incremental lot they win. The solution to the winner determination problem is the allocation that maximizes the total value of all bids subject to feasibility constraints. Bids are considered on the mutually exclusive basis meaning that only one bid from a bidder can be selected as part of the solution.

Once the solution is identified, the prices are calculated using one of the core-selecting pricing rules. The Nearest-Vickrey Pricing Rule, developed by Day and Cramton, is the standard choice of the auctioneers in almost all CCAs.

3 QGARP Activity Rule for the Clock Stage

This section specifies a set of suggested activity rules and corresponding results. The biggest difference from the current CCA activity rules is that the newly proposed activity rule does not utilize eligibility points and

[2] Bid cap is the maximum bid amount that the bidder can bid for a given package in the supplementary round (usually, determined with respect to the bid for the package that was demanded in the last clock round)

instead require bidders to bid consistent with some rational quasilinear valuation profile during the clock rounds.

We start our description of the activity rules with a reminder of the generalized version of the Revealed Preference Axiom.

Definition: (GARP) A collection of price-quantity pairs $(p_1, x_1),...,(p_{t-1}, x_{t-1}),(p_t, x_t)$ satisfies the *generalized axiom of revealed preferences* (GARP) if and only if:

$$p_{k_1} x_{k_1} \geq p_{k_1} x_{k_2},..., p_{k_{m-1}} x_{k_{m-1}} \geq p_{k_{m-1}} x_{k_m} \quad \text{implies} \quad p_{k_m} x_{k_m} \leq p_{k_m} x_{k_1} \quad \forall k_1,...,k_m \in \{1,...,t\}$$

Intuitively, GARP requires that if a bundle x_{k_1} is implicitly reveled preferred to bundle x_{k_m} through a series of transitive decision choices, then the bundle x_{k_m} cannot be revealed to be strictly preferred to bundle x_{k_1}.

The advantage of the GARP over WARP is that the former guarantees existence of the rational valuation profile (this result is known as Afriat's Theorem).

Afriat's Theorem (1967): A finite set of data is consistent with utility maximization if and only if it satisfies GARP.

While GARP is a lot more restrictive than WARP, it is still too weak to preclude bid sniping in the auction context. At the same time, restricting attention to quasilinear valuation profiles and requiring GARP can serve as a powerful base for the activity rule.

Definition: (QGARP) A collection of price-quantity pairs $(p_1, x_1),...,(p_{t-1}, x_{t-1}),(p_t, x_t)$ satisfies *quasilinear generalized axiom of revealed preferences* (QGARP) if and only if

$$p_{k_1}(x_{k_1} - x_{k_m}) + p_{k_2}(x_{k_2} - x_{k_1}) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) \leq 0 \qquad \forall k_1,...,k_m \in \{1,...,t\}.$$

The QGARP definition says that the bidder choices should not demonstrate a lack of rationality in the following sense. Suppose that a bidder owns x_{k_m}. Bidder choices made in rounds $k_1, k_2,..., k_m$ are consistent with a series of transactions that would eventually leave the bidder with the same bundle x_{k_m}. QGARP definition requires that the corresponding total net payment from these transactions must not be strictly positive.

Theorem 0: The following statements are equivalent:

a) $(p_1, x_1),...,(p_{t-1}, x_{t-1}),(p_t, x_t)$ satisfies QGARP.

b) There exist numbers $v_i \quad i = 1,..,t$ such that

$$v_i \leq v_j + p_j(x_i - x_j) \quad \text{for} \quad i, j = 1,...,t.$$

Proof: See Appendix.

Definition 1: (Q*GARP Clock Rule*) A bidder is permitted to place a bid on a package $x \in X$ in Round t with clock prices p_t if and only if the collection of observations $(p_1, x_1), ..., (p_{t-1}, x_{t-1}), (p_t, x)$ satisfies QGARP.

Results of the Theorem 0 can be used to motivate the QGARP Clock Rule. One of the long-standing motivations for the auction activity rules is that the activity rule should never get in a way of a bidder who bids in a straightforward manner. Taking this perspective, the QGARP activity rule is the most restrictive activity rule that never gets in a way of a straightforward bidder with a fixed quasilinear valuation profile. In other words, for any activity rule for which the set of feasible packages in a given round is smaller than that one under the QGARP Clock Rule, then there exists a valuation profile consistent with prior straightforward bidding for which truthful bidding in this round is infeasible. The QGARP Clock Rule is stronger than either the QWARP Clock Rule or the clock activity rules currently used in combinatorial clock auctions. For example, under QGARP Clock Rule a bidder might be unable to bid on the package demanded in the previous clock round.

The following theorem establishes three important properties of the QGARP Clock Rule. First, a straightforward bidder can always bid on its most-profitable package. Second, a bidder is guaranteed that at least one of the previously demanded packages is feasible in any given round. Together, these results ensure that bidders always have a viable alternative to continue their clock bidding. Finally, a bidder is always permitted to bid on any subset of any package that is feasible in the current round.

Denote X_t a set of packages that are feasible to the bidder in Round t. By construction, $X_1 = X$.

Theorem 1: The following statements are true for the set of feasible packages X_t in Round t:

a) (profit-maximization) If the bidder was bidding straightforwardly in all previous rounds, the set of feasible packages X_t contains all profit-maximizing packages for Round t;

b) (no "dead-ends") X_t contains at least one package from the set of previously demanded packages $\{x_1, ..., x_{t-1}\}$;

c) (smaller bids) X_t contains all subsets of any package x that is feasible in Round t ($x \in X_t$).

Proof: See Appendix.

Theorem 0 motivates our procedure for verifying whether an incoming bid $x_t \in X$ in Round t is consistent with QGARP. Given the collection of observations $(p_1, x_1), ..., (p_{t-1}, x_{t-1})$ that is consistent with QGARP, we

need to verify whether the extended collection $(p_1, x_1), ..., (p_{t-1}, x_{t-1}), (p_t, x_t)$ is also consistent with QGARP. This problem can be formulated as a simple linear programming problem of the following form:

$$GV_t(x_t) = \min_{\lambda, v} \lambda$$

subject to:

(1) $\lambda \geq 0$

(2) $v_i \leq v_j + p_j(x_i - x_j) \quad \forall i \in \{1, ..., t\} \quad \forall j \in \{1, ..., t-1\}$

(3) $v_i \leq v_t + p_t(x_i - x_t) + \lambda \quad \forall i \in \{1, ..., t-1\}$

The problem minimizes the QGARP violation subject to all revealed preference constraints. The choice variable $\lambda \geq 0$ plays a role of the relaxation factor that ensures that the system of revealed preference constraints (1) - (3) always has a solution. In case the objective function is positive, a positive relaxation is required in order to satisfy all revealed preference constraints which is inconsistent with QGARP. In such contingency, bidder is not allowed to bid for package x_t in Round t. If the value of the objective is zero, the clock bid for package x_t is consistent with QGARP and the bid is deemed feasible.

This formulation not only validates whether a set of bids is consistent, but also provides a size of violation in case the set of bids does violate QGARP. The size of violation has very important implications for the new pricing approach introduced below. This formulation is a very simple linear programming problem that can be easily and quickly solved by majority of solvers even for large number of price-quantity observations.

Before turning to the next section, we derive a second formulation for the QGARP validation that will be used in some proofs. The second formulation is less practical for implementation purposes as it requires searching for a sequence of rounds using a binary formulation rather than solving a simple linear programming problem.

Lemma 1: The QGARP violation for the collection of observations $(p_1, x_1), ..., (p_{t-1}, x_{t-1}), (p_t, x_t)$ is given by:

$$GV_t(x_t) = \max_{k_1, ..., k_m \in \{1, ..., t\}} \{p_{k_1}(x_{k_1} - x_t) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) + p_t(x_t - x_{k_m})\}$$

Proof: See Appendix.

4 Implied Value Profiles and Welfare Properties

For any given bidder, its clock bidding history is given by T observations of price-demand pairs $(p_1, x_1), ..., (p_T, x_T)$ that correspond to the demand choices the bidder made during all clock rounds. Denote

$V = \{(v_1,...,v_T):\quad v_i \le v_j + p_j(x_i - x_j) \quad \forall i,j \in \{1,...,T\}\}$ a set of all possible solutions to the full system of revealed preference constraints. This set is determined only up to a constant. It is convenient to assume that the implied value of the final clock package is its clock price in the final clock round plus some constant c, i.e., $v_T = p_T x_T + c$ for any $v = (v_1,...,v_T) \in V$. According to Theorem 0, set V is nonempty as long as the collection $(p_1,x_1),...,(p_T,x_T)$ satisfies QGARP. The following lemma proves that the set V also forms a lattice.

Lemma 2: Set V is a lattice. Therefore, there exists a unique minimal and a unique maximal element of V denoted $\underline{v}$ and $\overline{v}$ correspondingly.

Proof: See Appendix.

Definition: (Consistent Valuation Profile) Using any element $v = (v_1,...,v_T)$ of the lattice V where $v_T = p_T x_T + c$, define a *consistent valuation profile* $v(.)$ that is consistent with the clock bidding history $(p_1,x_1),...,(p_T,x_T)$ as follows:

$$v(x) = \begin{cases} v_t & \text{if } \exists t : x = x_t \\ \left[-\infty, \min\limits_{t\in\{1...T\}} \{v_t + p_t(x - x_t)\}\right] & \text{otherwise} \end{cases} \qquad \forall x \in X$$

Definition: (Maximal Consistent Valuation Profile) Using the maximal element $\overline{v} = (\overline{v}_1,...,\overline{v}_T)$ of the lattice V where $\overline{v}_T = p_T x_T + c$, define the *maximal consistent valuation profile* $\overline{v}(.)$ as follows:

$$\overline{v}(x) = \min_{t\in\{1...T\}} \{\overline{v}_t + p_t(x - x_t)\} \qquad \forall x \in X$$

The next theorem establishes three important properties of the maximal valuation profile $\overline{v}(.)$. First, we show that there is an alternative formula that can be used to calculate $\overline{v}(x)$ for any package $x \in X$. Intuitively, the highest possible value for package x is the highest price of this package in the last clock round which would have been consistent with QGARP, i.e., $p_T x - GV_T(x)$, adjusted by the same constant c that has been applied to the implied value of package x_T. Further, we show that the maximal valuation profile is monotonic and concave.

Theorem 2: Given the QGARP-consistent bidding history $(p_1,x_1),...,(p_T,x_T)$, the maximal consistent valuation profile $\overline{v}(.)$ has the following properties:

a) (formula) For any package $x \in X$, $\overline{v}(x)$ can be computed using the following formula:

$$\overline{v}(x) = p_T x + c - GV_T(x)$$

where $GV_T(x)$ is the QGARP violation for the counterfactual bidding history $(p_1, x_1),...,(p_{T-1}, x_{T-1}),(p_T, x)$ (as if the bidder had selected package x in Round T instead of x_T).

b) (monotonicity) For any two packages $x, x' \in X$ such that $x \geq x'$, $\overline{v}(x) \geq \overline{v}(x') + r(x - x')$.

c) (concavity) A valuation profile $\overline{v}(.)$ is concave.

Proof: See Appendix.

In order to proceed, we need to expand our notation to allow for bidder indexes. x_t^i denotes demand of bidder *i* in Round *t*, $GV_t^i(x)$ denotes the QGARP violation of bidder *i* in Round *t* for the package x, $X_t = \sum_j x_t^j$ is the aggregate demand in Round *t* , and $X_t^{-i} = \sum_{j \neq i} x_t^j$ is the aggregate demand in Round *t* that excludes demand of bidder *i*. Further, all value profiles are normalized as follows $v_T^i = \overline{v}^i(x_T^i) = p_T x_T^i + c^i$. Additionally, denote $TGV_t^C(X)$ the minimum total QGARP violation for bidders in coalition C when jointly bidding for a package X in Round t.

$$TGV_t^C(X) = \min_{\sum_{j \in C} x^j = X} \left[\sum_{j \in C} GV_t^j(x^j) \right]$$

Finally, we define a family of valuation profiles that would be typically allowed to represent bidder preferences during the supplementary sealed-bid stage of the CCA. Such valuation profiles are not restricted to be consistent or even rational.

Definition: (Permissible Valuation Profile) Given the maximal consistent valuation profile $\overline{v}^i(.)$ with normalization $\overline{v}^i(x_T) = p_T x_T^i + c^i$ such that $c^i \geq -(p_T - r)x_T^i$ if $x_T^i \neq 0$ and $\overline{v}^i(x_T^i) = 0$ if $x_T^i = 0$, a permissible valuation profile for bidder i is defined as follows:

$$v^i(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \overline{v}^i(x_T^i) & \text{if} \quad x = x_T^i \\ [0, \max\{0, \overline{v}^i(x)\}] & \text{if} \quad \text{otherwise} \end{cases} \qquad \forall x \in X^i$$

The following lemma provides an intuitive condition for Pareto dominance of the final clock allocation. (Add intuition)

Lemma 3: For any collection of permissible valuation profiles with a set of normalization constants $\left\{c^i\right\}_{i=1}^N$, the final clock allocation $x_T = (x_T^1, ..., x_T^N)$ Pareto dominates an alternative allocation $y = (y^1, ..., y^N)$ such that $Y = \sum_{j \in N} y^j \leq S$ if

$$\sum_{j:y^j \neq 0} GV_T^j(y^j) + \sum_{j:y^j=0} c^j > (p_T - r)(Y - X_T).$$

Further, the value of any alternative allocation cannot exceed the value associated with the final clock allocation x_T by more than

$$(p_T - r)U - \min_{\sum_j y^j = S} \left[\sum_{j:y^j \neq 0} GV_T^j(y^j) + \sum_{j:y^j=0} c^j \right],$$

where $U = S - X_T$ is the final clock round undersell.

Proof: See Appendix.

The next theorem establishes the welfare properties of the final clock allocation relative to permissible valuation profiles derived from the clock stage bidding history in case the clock stage ends with no excess supply.

Theorem 3: (*First Fundamental Welfare Theorem*) If all markets clear in the final clock round ($U = 0$), then any alternative allocation is Pareto dominated by the final clock allocation for any collection of permissible valuation profiles with a set of normalization constants $\left\{c^i\right\}_{i=1}^N$ such that

$$\sum_{j \in C} c^j \geq -TGV_T^{N \setminus C}(S) \quad \forall C \subset N.$$

Proof:

By Lemma 2, the welfare improvement of any alternative allocation $y = (y^1, ..., y^N)$ relative to the final clock allocation does not exceed

$$(p_T - r)U - \min_{\sum_j y^j = S} \left[\sum_{j:y^j \neq 0} GV_T^j(y^j) + \sum_{j:y^j=0} c^j \right] = 0 - \min_{\sum_j y^j = S} \left[\sum_{j:y^j \neq 0} GV_T^j(y^j) + \sum_{j:y^j=0} c^j \right] \leq 0,$$

since $\sum_{j \in C} c^j + TGV_T^{N \setminus C}(S) \geq 0$.

A weaker version of Theorem 3 (when $c^i \geq 0 \ \ \forall i$) is a well-known result in the CCA literature about the stability of the tentative final clock allocation (the allocation cannot be altered as a result of the supplementary stage) in case all markets have been cleared and the Final Cap rule applies to bids made in the supplementary round. See Ausubel and Cramton (2011) for details. Theorem 3 extends this Pareto optimality result to a richer class of permissible valuation profiles. Namely, normalization constants allowed to be negative as long as they are not too negative (i.e., if $\sum_{j \in C} c^j \geq -TGV_T^{N \setminus C}(S) \quad \forall C \subset N$). The next section describes a way in which the result of Theorem 3 can be used to enhance the CCA design.

5 Bidder Exposure, Bid Discounts and Iterative Pricing

Suppose that the clock stage ended in Round T with no undersell. According to Theorem 3, the final clock allocation will preserve its optimality property if bidder values for their respective packages are lower than the associated prices for these packages in Round T. In other words, bidders who were forced to bid the final clock price for their packages end up bidding more than the amount sufficient to guarantee their respective winnings. This observation raises an important question. What is the highest appropriate bid amount that a given bidder should have been required to submit for its final clock package? More generally, what is the highest total bid amount that a group of bidders C should have been required to bid for their final clock packages? Intuitively, the sufficient bid amount should be the maximum possible incremental value the complement group of bidders $N \setminus C$ can have for the lots contained in the final clock packages of bidders in C. In case the final clock round resulted in unallocated lots, the incremental value for lots contained in bids of bidders in C can be combined with value for unallocated lots due to package bidding. Therefore, this maximum incremental amount will include the value for unallocated lots. This incremental value amount will be referred to as the exposure of coalition C and denoted by $p_T^C(x_T^C, U)$ where $x_T^C = \sum_{j \in C} x_T^j$ is the aggregate final clock demand of bidders in C.

Lemma 4: For any collection of permissible valuation profiles with a set of normalization constants $\left\{c^i\right\}_{i=1}^N$, the exposure of coalition C for package x_T^C is given by:

$$p_T^C(x_T^C, U) = p_T x_T^C - \min_{\sum_{j \in N \setminus C} y^j = S} \left[\sum_{y^j \neq 0} GV_T^j(y^j) + \sum_{y^j = 0} c^j \right] + (p_T - r)U .$$

Proof: See Appendix.

(Add intuition for the exposure calculation)

Corollary 1:

a) For any collection of permissible valuation profiles with a set of normalization constants $\{c^i\}_{i=1}^N$, the "zero undersell" exposure of coalition C for package x_T^C is higher than the reserve price of this package (i.e. $p_T^C(x_T^C, 0) \geq r x_T^C$);

b) For any collection of permissible valuation profiles with a set of normalization constants $\{c^i\}_{i=1}^N$ such that $c^j \geq 0 \quad \forall j \in N$, the exposure of coalition C is given by

$$p_T^C(x_T^C, U) = p_T x_T^C - TGV_T^{N/C}(S) + (p_T - r)U;$$

c) For any collection of permissible valuation profiles with a set of normalization constants $\{c^i\}_{i=1}^N$ such that $\sum_{j \in C} c^j \geq -TGV_T^{N/C}(S) \quad \forall C \subset N$, the exposure of coalition C is bounded from above by:

$$p_T^C(x_T^C, U) \leq p_T x_T^C + (p_T - r)U.$$

Suppose that the final clock round concluded with zero undersell. According to Corollary 1(a), the individual bidder exposure for the final clock package exceeds the reserve price for this package. Further, suppose that a discount $0 \leq d^i \leq (p_T - r)x_T^i$ has been applied to the final clock round bid for each bidder i. Then the bid submitted by bidder i for package x_T^i would be $b_T^i(x_T^i) = p_T x_T^i - d^i$. Denote $b_T^C(x_T^C) = p_T x_T^C - d^C$ the total bid of coalition C for package x_T^C where $d^C = \sum_{i \in C} d^i$ is the total discount of coalition C.

Theorem 4: Under an assumption of zero undersell in Round T, the total bid amount $b_T^C(x_T^C)$ of any coalition C for the package x_T^C is not lower than the exposure of the coalition C for the package x_T^C (i.e. $b_T^C(x_T^C) \geq p_T^C(x_T^C, 0)$) if and only if

$$d^C \leq TGV_T^{N \setminus C}(S) \quad \forall C \subset N.$$

Proof: See Appendix.

Theorem 4 provides a rationale to limit bidder discounts such that a total discount given to any coalition of bidders does not exceed the total minimum QGARP violation of other bidders when bidding for the full supply vector in the final clock round. One can always assume that the clock stage would end in the current round, and therefore calculate a set of discounts that can be applied to bids. A reasonable set of discounts might be a set that maximizes the total discount given to bidders subject to constraint that all coalitions bid at least their exposure in round t:

$$TD = \max_{d} \sum_{i} d^i$$

subject to:

$$\sum_{i \in C} d^i \le TGV_t^{N/C}(S) \qquad \forall C \subset N$$

In case the solution $\{d_t^i\}_{i=1}^N$ to the above optimization problem is not unique, the most straightforward tie-breaking procedure would utilize the same rule as the pricing procedure that will be eventually used to calculate prices.

For example, in case the pricing rule is the Nearest-Vickrey Pricing Rule, the corresponding unique discount calculation would be the following:

$$\min_{d} \sum_{i} \left[d^i - TGV_t^{N \setminus i}(S) \right]^2$$

subject to:

$$\sum_{i \in C} d^i \le TGV_t^{N/C}(S) \qquad \forall C \subset N$$

$$\sum_{i} d^i = TD$$

Once discounts are calculated, the highest bid amount that is actually needed in round t from bidder i who wants to make a bid for a package x is only

$$b_t^i(x) = \max\{p_t x - d_t^i, rx\},$$

where the bid for package x is capped from below by the reserve price of package x.

The final logical step is to require that bidders not only bid according to their exposure levels but also pay it in case they are allocated packages that were demanded. This step allows a transition from the iterative second-

price auction, an auction where clock bids track values, to the iterative first-price auction where clock bids track required payments. We refer to the ability of the dynamic stage of the auction to communicate current prices as *iterative pricing* property.

With the new clock stage implementation that includes QGARP Activity Rule, bid discounts based on exposure and iterative pricing, there is no need for the supplementary stage in case the clock stage ends with no excess supply. Bidders are awarded their final clock packages and pay the full amount of their exposure in the last clock round. The auction outcome is fully efficient relative to the reported value profiles. Further, bidders are charged the minimum amount required to make sure that their lots cannot be redistributed among other bidders at a higher incremental value. However, clock stages of CCAs typically end with positive excess supply. In the next section, we develop a set of auction rules governing the supplementary round that takes place after the clock stage ended with unallocated lots.

6 QGARP Activity Rule for the Supplementary Round

Under typical CCA rules, in the supplementary round bidders are allowed to revise bid amounts for packages that were demanded during the clock rounds and submit bids for any other desirable packages. Without loss of generality, we consider a sealed bid stage in which bidders start with their bid for the final clock package only (i.e., no other bids made during the clock stage are added to the list of supplementary bids). A more standard approach of adding all bids made throughout the clock stage to the list of supplementary bids is considered later in Section 7.2.

In order to prevent bid sniping, all supplementary bids are limited by the activity rule that specifies the highest amount the bidder can bid for any particular package given final bid amounts for clock packages.

Our proposed activity rule for the supplementary stage is based on the bidding histories generated by bidders in clock rounds. For bidder i, its clock bidding history is given by T observations of price-demand pairs $(p_1, x_1^i), ..., (p_T, x_T^i)$ that correspond to the demand choices the bidder made in all clock rounds.

Suppose that the bidder's supplementary bid for the final clock package is $b(x_T^i) = p_T x_T^i + c^i$ where $c^i \geq -d_T^i$. Given the maximal valuation profile $\overline{v}^i(.)$ such that $\overline{v}_T^i = p_T x_T^i + c^i$, we define a simple QGARP-based activity rule for the supplementary stage. The maximum bid amount a bidder i can bid for a package $x \in X$,

$CAP^i(x)$, never exceeds the corresponding maximal valuation $\overline{v}^i(x)$ that a straightforward bidder can potentially have for the package x.

Definition: (*Simple Cap Activity Rule*) In the Supplementary Round, the bid amount for any package $x \in X$ cannot exceed the value of the associated maximal valuation $\overline{v}^i(x)$ for this package, i.e.

$$b^i(x) \le CAP^i(x) = \overline{v}^i(x).$$

In order to implement the iterative pricing principle, we need to make sure that all coalitions of bidders pay at least their exposure amounts on the final clock packages in case they win them. We can integrate this requirement into the supplementary stage by utilizing *pseudo* bidders. A pseudo bidder, one for each actual bidder, is added to the winner determination problem with the bid that duplicate the final clock bid submitted by the corresponding actual bidder[3]. Bids placed on behalf of pseudo bidders act as anchors for the final payments of the actual bidders.

We refer to the combination of the new QGARP-based activity rule and the iterative pricing procedure as an *enhanced combinatorial clock auction* (ECCA). The following theorem establishes several important properties of the ECCA design under a technical assumption that all bids placed in the final clock round were above the reserve prices. This assumption is just for convenience and clarity of exposition.

Theorem 5: The following statements are true for the ECCA when $b_T^i(x_T^i) = p_T x_T^i - d_T^i \ge r x_T^i$ for all bidders i:

a) If the clock stage ends with no excess supply, the auction concludes after the clock stage with bidder *i* winning its final clock allocation x_T^i and paying $p_T x_T^i - d_T^i$;

b) If the clock stage ends with a positive excess supply U, bidder i is guaranteed to win its final clock package if the bid for the final clock package x_T^i is raised by $(p_T - r)U$ and its final payment in this case would be not lower than $p_T x_T^i - d_T^i$;

c) If the clock stage ends with a positive excess supply U, it is possible for bidder i to win package y^i only if

$$b^i(y^i) \ge p_T y^i + c^i - (p_T - r)U + \min_{\sum_{j \in N \setminus i} y^j = S - y^i} \left[\sum_{y^j \ne 0} GV_T^j(y^j) - \sum_{y^j = 0} d_T^j \right],$$

[3] Technicalities: Any ties between real bidders and pseudo bidders should be broken in favor of actual bidders. The formulation of the WD problem that is used for pricing should allow nonzero winnings for the pseudo bidder only if the actual bidder is omitted from the WD problem.

given the supplementary bid of $b^i(x_T^i) = p_T x_T^i + c^i$ for its final clock package. If successful, the final payment for package y^i would be at least

$$p_T y^i - d_T^i - (p_T - r)U + \min_{\sum_{j \in N \setminus i} y^j = S - y^i} \left[\sum_{y^j \neq 0} GV_T^j(y^j) - \sum_{y^j = 0} d_T^j \right];$$

d) The minimum auction revenue is given by $\sum_i b_T^i(x_T^i) = p_T X_T - TD_T$.

Proof: See Appendix.

Part (a) of Theorem 5 proves that allocations and associated payments have been already established during the clock stage thus deeming the supplementary stage irrelevant. Part (b) of the Theorem 5 is a standard "knockout bid" result. Any bidder with a nonzero final clock package can always lock in its winnings by sufficiently raising the amount of the bid on the final clock package relative to zero. Part (c) provides a lower bound on the bidder payment in case the bidder manages to win a package that differs from the final clock package. This lower bound is due to anchoring effect of the pseudo bid. Finally, part (d) derives a lower limit on the auction revenue that the auctioneer will eventually collect for any possible set of supplementary bids.

7 Some Practical Aspects of the ECCA

7.1 QGARP Activity Rule with Dynamic Discounting

(incomplete)

The Section 3 of the paper develops a QGARP Activity Rule for the clock stage when the bidders face linear prices on all lots. Under ECCA, bidders place bids that are derived from linear prices by applying discounts. Here we discuss how a slight modification of the activity rule can be used when bidders report their demands based on discounted bids.

The central result behind the bidder exposure calculation is that it produces a bidder discount that does not depend on the package. With this in mind, demand data generated under ECCA can still be used in the same way as if bidders were facing regular linear prices. Technically, the discount term appears on both sides of the revealed preference constraints when two nonnull packages are compared. The only meaningful difference arises when a bidder places a bid for the null package during the clock round in which its exposure discount is positive.

Suppose that a bidder bid for the null package in round t with a positive discount d_t:

A modified formulation for the QGARP validation problem is provided below:

$$GV_t(x_t) = d_t + \min_{\lambda, v} \lambda$$

subject to:

(1) $\lambda \geq 0$

(2) $v_i \leq v_j + p_j(x_i - x_j) \quad \forall i \in \{1,...,t\} \quad \forall j \in \{1,...,t-1\}$

(3) $v_i \leq v_t + p_t(x_i - x_t) + \lambda \quad \forall i \in \{1,...,t\}$

7.2 Prior Clock Bids

For majority of practical auction designs, the auctioneer frequently takes additional steps to incentivize bidders to submit meaningful bids throughout the auction. For example, in a typical CCA, any bid placed during the clock stage can be selected as a part of the winning allocation. This is achieved by adding the highest bids made by the bidder for all packages that were demanded during the clock stage.

If the exposure discounts are not dynamically applied to clock bid amounts, the clock bids never violate the maximum valuation profile with the normalization $\overline{v}(x_T) = p_T x_T$ for nondecreasing price trajectories.

If the exposure discounts are applied in each round, this property might be violated. This can happen if at some point the exposure discount sharply increased due to favorable relative clock prices in the current round. Theoretically, it might lead to clock bids placed at higher amounts than the ones allowed by the maximum valuation profile with the normalization $\overline{v}(x_T) = p_T x_T - d_T$.

In practice, this problem can be easily solved by appropriately reducing clock bids such that they do not violate the maximum valuation profile. For example, one can simply truncate all bids that are added to the list of supplementary bids such that they do not violate Simple Cap. Alternatively, any bid received in rounds with $d_t < d_T$ can be recalculated as if a larger discount of d_T should have been applied in Round t.

7.3 Other Activity Rules for the Supplementary Round

Auctioneer can use a different activity rule for the supplementary stage to further mitigate the missing bid problem. The Simple Cap Activity Rule used in Section 6 is theoretically sound but it allows bidders to place maximum bids without appropriately adjusting their bids on packages demanded in the clock stage. We provide two versions of the activity rule that can be used instead of the Simple Cap that require bidders to revise some or all of their clock bids in order to submit high bids for different packages.

Definition: (*Full Cap Activity Rule*) In the Supplementary Round, the bid amount for any package $x \in X$ should satisfy the full set of revealed preference constraints, i.e.,

$$b(x) \leq CAP(x) = \min_{t \in \{1,\ldots,T\}} \left\{ b(x_t) + p_t(x - x_t) \right\}.$$

This rule is the most restrictive version of the activity rule. In order to bid high on any package $x \in X$, the bidder would most likely need to bid sufficiently high on all clock packages. It might be highly undesirable for budget-constrained bidders who would need to bid beyond their budgets for some packages in order to be able to submit within budget bids for other packages. The Relative Cap Activity Rule alleviates this concern while still making sure that bidder's bids does not allow gaps in bid amounts that are possible under Simple Cap Activity Rule. This rule is the closest in spirit to the activity rules employed in practical CCAs.

Definition: (*Relative Cap Activity Rule*) In the Supplementary Round, the bid amount for any package $x \in X$ should satisfy revealed preference constraints with respect to all clock rounds where package x was more expensive than the chosen package and also should be less than the associated maximal valuation $\overline{v}(x)$, i.e.,

$$b(x) \leq CAP(x) = \min\left\{ \overline{v}(x), \min_{t \in \{1,\ldots,T\}: p_t x \geq p_t x_t} \left\{ b(x_t) + p_t(x - x_t) \right\} \right\}.$$

8 Exposure Calculation and Clinching

It is important to establish the connection between the exposure calculation developed here and the similar "clinching" and "crediting/debiting" formulas developed in Ausubel (2004) and Ausubel (2006) for environments with substitutes. In order to make such comparison, we need to consider both calculations under assumptions that were made in Ausubel (2006).

Theorem 6: Consider a model with divisible commodities and continuous piecewise smooth price trajectory $p(t) : t \in [0, T]$. If the starting price vector $p(0)$ is chosen such that $X^{-i}(p(0)) \leq S$ for bidder i, both the "crediting/debiting" and exposure calculations yield the same payment amount for bidder i. Further, if each bidder $j \neq i$ bids straightforwardly, then bidder i maximizes her payoff by bidding straightforwardly receiving her payoff from the modified VCG mechanism with price of $p(0)$.

Proof: (sketch of the proof to be added)

In general, the exposure calculation and "crediting/debiting" calculation are different. First of all, "crediting/debiting" formula can be used only when it is applied to a continuous price trajectory (see Ausubel (2006) for a model with limited discretization). Even under the continuous treatment, two calculations can still produce different payment amounts depending on the selection of the starting price vector $p(0)$.

The difference between two calculations is best described as a trade-off between incentive compatibility and individual rationality. The crediting/debiting formula preserves the incentive compatibility (path independence property) at a cost of overcharging bidders for items that were not available in the auction. To remove the effect of "crediting/debiting" overcharging from bidders' payments, Ausubel (2006) developed a procedure that requires running the clock stage up to $n+1$ times (n – number of bidders). Unlike "crediting/debiting" formula, the exposure calculation maintains individual rationality and therefore does not require multiple runs of the clock stage. However, the exposure calculation still includes some amount of overcharging that does not violate individual rationality but can sometimes depend on a particular price trajectory. As a result, the incentive compatibility property (in a sense of path independence) is no longer satisfied under exposure calculation. Given the straightforward bidding behavior by opponents, a bidder might find it beneficial to deviate from straightforward bidding during the clock stage in case such deviation will result in a more favorable price trajectory that will reduce the amount of overcharging. It should be noted that such price trajectory manipulations can be dangerous in practice if the auction uses a strict activity rule like QGARP.

(Maybe write down an explanation in terms of subsets of revealed preference constraints that are used. "Crediting/debiting" – round order dictates which revealed preference constraints are used, additionally uses RP from rounds beyond S: "Exposure" – uses all revealed preference constraints, but only looks at the relevant RP constraints that limit the distance between S and the final bundle)

Finally, we note that both the "clinching" calculation and the exposure calculation are identical in environments with homogeneous products as in Ausubel (2004). The equivalence holds for both discrete and continuous versions of the model and any starting price.

9 Conclusion (TO BE ADDED)

10 References (TO BE ADDED)

11 Appendix

Proof for Theorem 0:

$(2) \Rightarrow (1)$

For any sequence $k_1, ..., k_m \in \{1, ..., t\}$

$$p_{k_1}(x_{k_1} - x_{k_m}) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) \le v_{k_1} - v_{k_m} + v_{k_2} - v_{k_1} + ... + v_{k_m} - v_{k_{m-1}} = 0$$

$(1) \Rightarrow (2)$

Normalize $v_t = C$. Define $v_j = v_t - \max_{k_1,...,k_m} \{p_{k_1}(x_{k_1} - x_j) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) + p_t(x_t - x_{k_m})\}$. We need to show that $v_i \le v_j + p_j(x_i - x_j)$ for all $i, j \in \{1, ..., t\}$.

- By construction, $v_i \le v_t + p_t(x_i - x_t)$ for all $i \in \{1, ..., t\}$.

$$\begin{aligned} v_t &= v_j + p_j(x_t - x_j) + p_j(x_j - x_t) + \max_{k_1,...,k_m} \{p_{k_1}(x_{k_1} - x_j) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) + p_t(x_t - x_{k_m})\} \\ &= v_j + p_j(x_t - x_j) + \max_{k_1,...,k_m} \{p_j(x_j - x_t) + p_{k_1}(x_{k_1} - x_j) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) + p_t(x_t - x_{k_m})\} \\ &\le v_j + p_j(x_t - x_j) + 0 \end{aligned}$$

- For any $i, j \in \{1, ..., t-1\}$: Suppose $v_j = v_t - \left[p_{q_1}(x_{q_1} - x_j) + ... + p_{q_m}(x_{q_m} - x_{q_{m-1}}) + p_t(x_t - x_{q_m}) \right]$
  - Case 1: $i \notin \{q_1, ..., q_m\}$

$$\begin{aligned} v_i &= v_t - \max_{k_1,...,k_m} \{p_{k_1}(x_{k_1} - x_i) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) + p_t(x_t - x_{k_m})\} \\ &\le v_t - \left[p_j(x_j - x_i) + p_{q_1}(x_{q_1} - x_j) + ... + p_{q_m}(x_{q_m} - x_{q_{m-1}}) + p_t(x_t - x_{q_m}) \right] \\ &= v_j + p_j(x_i - x_j) \end{aligned}$$

  - Case 2: $i \in \{q_1, ..., q_m\} : q_k = i$

$$\begin{aligned} v_i &= v_t - \left[p_{q_{k+1}}(x_{q_{k+1}} - x_i) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) + p_t(x_t - x_{k_m}) \right] \\ &= v_j + \left[p_{q_1}(x_{q_1} - x_j) + ... + p_i(x_i - x_{q_{k-1}}) \right] \\ &= v_j + p_j(x_i - x_j) + \left[p_j(x_j - x_i) + p_{q_1}(x_{q_1} - x_j) + ... + p_i(x_i - x_{q_{k-1}}) \right] \\ &\le v_j + p_j(x_i - x_j) \end{aligned}$$

Proof for Lemma 2:

Iteratively applying inequalities from (2) and (3), we show that λ should satisfy the following set of constraints:

$$\lambda \geq p_{k_1}(x_{k_1} - x_t) + p_{k_2}(x_{k_2} - x_{k_1}) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) + p_t(x_t - x_{k_m}) \qquad \forall k_1, ..., k_m \in \{1, ..., t\},$$

or, equivalently,

$$\lambda \geq \max_{k_1, ..., k_m \in \{1, ..., t\}} \{p_{k_1}(x_{k_1} - x_t) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) + p_t(x_t - x_{k_m})\}$$

Further, the minimization of the objective in the first formulation is equivalent to:

$$GV_t(x_t) = \max_{k_1, ..., k_m \in \{1, ..., t\}} \{p_{k_1}(x_{k_1} - x_t) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) + p_t(x_t - x_{k_m})\}$$

Proof for Theorem 1:

(a)

Suppose that a profit-maximizing package $x \in X$ does not belong to set X_t. Then there exists a sequence $k_1, ..., k_m \in \{1, ..., t-1\}$ such that:

$$p_{k_1}(x_{k_1} - x) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) + p_t(x - x_{k_m}) > 0$$
$$p_t(x - x_{k_m}) > p_{k_1}(x - x_{k_1}) + ... + p_{k_m}(x_{k_{m-1}} - x_{k_m})$$

Since package x is a profit-maximizing package in Round t,

$$v(x) - v(x_{k_m}) \geq p_t(x - x_{k_m}) > p_{k_1}(x - x_{k_1}) + p_{k_2}(x_{k_1} - x_{k_2}) + ... + p_{k_m}(x_{k_{m-1}} - x_{k_m})$$

Given that the bidder was bidding straightforwardly in all previous rounds:

$$v(x_{k_1}) - v(x_{k_2}) \leq p_{k_2}(x_{k_1} - x_{k_2})$$
...
$$v(x_{k_{m-1}}) - v(x_{k_m}) \leq p_{k_m}(x_{k_{m-1}} - x_{k_m})$$

Therefore:

$$v(x) - v(x_{k_m}) > p_{k_1}(x - x_{k_1}) + v(x_{k_1}) - v(x_{k_2}) + ... + v(x_{k_{m-1}}) - v(x_{k_m})$$
$$v(x) - p_{k_1}x > v(x_{k_1}) - p_{k_1}x_{k_1}$$

The last inequality proves that the bidder was not bidding optimally in Round k_1.

(b) According to Theorem 0, there exists a valuation profile v that rationalizes the choices made in all previous rounds. Note that this profile only specifies values (up to an additive constant which can be made

arbitrarily large) for the observed bundles $\{x_1,...,x_{t-1}\}$. This valuation profile can be extended to include all other bundles at the zero value. For the specified profile, the most profitable package in Round t with prices p_t is one of the bundles from the set of previously demanded packages $\{x_1,...,x_{t-1}\}$. According to part (a), a profit-maximizing package is always feasible for the straightforward bidder.

(c) If a package x is feasible in Round t, then for any possible sequence $k_1,...,k_m$, the QGARP condition is satisfied:

$$p_{k_1}(x_{k_1}-x)+...+p_{k_m}(x_{k_m}-x_{k_{m-1}})+p_t(x-x_{k_m})\le 0$$

But then for any sequence of rounds $k_1,...,k_m$ and any package $x'\le x$,

$$p_{k_1}(x_{k_1}-x')+...+p_{k_m}(x_{k_m}-x_{k_{m-1}})+p_t(x'-x_{k_m})=p_{k_1}(x_{k_1}-x)+...+p_{k_m}(x_{k_m}-x_{k_{m-1}})+p_t(x-x_{k_m})-\Delta(p_t-p_{k_1})$$

Therefore, the QGARP conditions for package x' will be satisfied since it is satisfied for package x and $\Delta(p_t-p_{k_1})\ge 0$ since $\Delta=x-x'\ge 0$ and $p_t\ge p_{k_1}$.

Proof for Lemma 2:

Suppose that $v'=(v'_1,...,v'_T)\in V$ and $v''=(v''_1,...,v''_T)\in V$.

$$\begin{array}{l} v'_i\le v'_j+p_j(x_i-x_j) \\ v''_i\le v''_j+p_j(x_i-x_j) \end{array} \quad \forall i,j\in\{1,...,T\}$$

Trivially,

$$\begin{array}{l} \max\{v'_i,v''_i\}\le \max\{v'_j,v''_j\}+p_j(x_i-x_j) \\ \min\{v'_i,v''_i\}\le \min\{v'_j,v''_j\}+p_j(x_i-x_j) \end{array} \quad \forall i,j\in\{1,...,T\}$$

Proof for Theorem 2:

(a)

Given the formula $\overline{v}(x)=\min\limits_{t\in\{1...T\}}\{\overline{v}_t+p_t(x-x_t)\}$, or equivalently,

$$\begin{array}{ll} \overline{v}(x)\le \overline{v}_t+p_t(x-x_t) & \forall t\in\{1,...,T\} \\ \overline{v}(x)\le \overline{v}_T-\left[p_{k_1}(x_{k_1}-x)+...+p_T(x_T-x_{k_m})\right] & \forall k_1,...,k_m\in\{1,...,T-1\} \end{array}$$

Given the normalization $\overline{v}_T=p_Tx_T+c$,

$$\begin{aligned}\overline{v}(x) &= p_T x_T + c \quad - \max_{k_1,\ldots,k_m \in \{1,\ldots,T-1\}} \{p_{k_1}(x_{k_1} - x) + \ldots + p_T(x_T - x_{k_m})\} \\ &= p_T x + c \quad - \max_{k_1,\ldots,k_m \in \{1,\ldots,T-1\}} \{p_{k_1}(x_{k_1} - x) + \ldots + p_T(x - x_{k_m})\} \\ &= p_T x + c - QGV_T(x)\end{aligned}$$

(b)

For any two package $x, x' \in X$ such that $x' \le x$, $\overline{v}(x') \le \overline{v}(x) - r(x - x')$.

Since $x' \le x$, we can express x' as $x' = x - \Delta$ where $\Delta \ge 0$.

$$\begin{aligned}\overline{v}(x') &= \min_t\{\overline{v}_t + p_t(x' - x_t)\} = \min_t\{\overline{v}_t + p_t(x - \Delta - x_t)\} = \min_t\{\overline{v}_t + p_t(x - x_t) - p_t\Delta\} \\ &\le \min_t\{\overline{v}_t + p_t(x - x_t)\} - r\Delta = \overline{v}(x) - r(x - x')\end{aligned}$$

(c)

$\overline{v}(.)$ is concave.

For any two bundles x, x' and $\forall \alpha \in (0,1)$, denote $\tilde{x} = \alpha x + (1-\alpha)x'$.

$$\begin{aligned}\overline{v}(\tilde{x}) &= \min_t\{\overline{v}_t + p_t(\tilde{x} - x_t)\} = \min_t\{\overline{v}_t + p_t(\alpha x + (1-\alpha)x' - x_t)\} = \\ &= \min_t\{\alpha\left[\overline{v}_t + p_t(x - x_t)\right] + (1-\alpha)\left[\overline{v}_t + p_t(x' - x_t)\right]\} \\ &\ge \alpha \min_t\{\overline{v}_t + p_t(x - x_t)\} + (1-\alpha)\min_t\{\overline{v}_t + p_t(x' - x_t)\} = \alpha\overline{v}(x) + (1-\alpha)\overline{v}(x')\end{aligned}$$

Proof for Lemma 3:

Since $c^i \ge -(p_T - r)x_T^i \quad \forall i$, the total value associated with the final clock allocation $x_T = (x_T^1, \ldots, x_T^N)$ is given by:

$$TV(x_T) = \sum_j \overline{v}^j(x_T^j) + r(S - X_T) = \sum_j\left[p_T x_T^j + c^j\right] + r(S - X_T) = (p_T - r)X_T + \sum_j c^j + rS$$

For any alternative allocation $y = (y^1, \ldots, y^N)$, the maximum possible total value is

$$\begin{aligned}TV^*(y) &= \sum_j \overline{v}^j(y^j) + r(S - Y) = \sum_{j:y^j \ne 0}\left[p_T y^j + c^j - GV_T^j(y^j)\right] + r(S - Y) = \\ &= p_T Y + \sum_{j:y^j \ne 0} c^j - \sum_{j:y^j \ne 0} GV_T^j(y^j) + r(S - Y) = (p_T - r)Y + \sum_{j:y^j \ne 0} c^j - \sum_{j:y^j \ne 0} GV_T^j(y^j) + rS\end{aligned}$$

The final clock allocation x_T dominates allocation y for any collection of permissible valuation profiles if

$$TV(x_T) > TV^*(y)$$

$$\sum_{j:y^j \neq 0} GV_T^j(y^j) + \sum_{j:y^j=0} c^j > (p_T - r)(Y - X_T)$$

Further, by monotonicity property of the maximum consistent valuation profiles, the highest total value for an alternative allocation would be achieved if all lots are allocated to bidders $\sum_j y^j = S$.

$$TV^* = \max_{\sum_j y^j = S} \sum_j \overline{v}^j(y^j) = p_T S + \max_{\sum_j y^j = S} \left[\sum_{j:y^j \neq 0} c^j - \sum_{j:y^j \neq 0} GV_T^j(y^j) \right]$$

$$TV^* - TV(x_T) = p_T S + \max_{\sum_j y^j = S} \left[\sum_{j:y^j \neq 0} c^j - \sum_{j:y^j \neq 0} GV_T^j(y^j) \right] - (p_T - r)X_T - \sum_j c^j - rS$$

$$= (p_T - r)U - \min_{\sum_j y^j = S} \left[\sum_{j:y^j=0} c^j + \sum_{j:y^j \neq 0} GV_T^j(y^j) \right]$$

Proof for Lemma 4:

$$p_T^C(x_T^C, U) = \max_{\sum_{j\in\bar{C}} x^j = S} \sum_{j\in\bar{C}} \overline{v}^j(x^j) - \left[\sum_{j\in C} \overline{v}^j(x_T^j) + rU \right]$$

$$= \max_{\sum_{j\in\bar{C}} x^j = S} \sum_{j\in\bar{C}:x^j\neq 0} \left[p_T x^j + c^j - GV_T^j(x^j) \right] - \sum_{j\in C} \left[p_T x_T^j + c^j \right] - rU$$

$$= p_T(S - X_T^{\bar{C}}) - \min_{\sum_{j\in\bar{C}} x^j = S} \left[\sum_{j\in\bar{C}:x^j\neq 0} GV_T^j(x^j) + \sum_{j\in\bar{C}:x^j=0} c^j \right] - rU$$

$$= p_T x_T^C - \min_{\sum_{j\in\bar{C}} x^j = S} \left[\sum_{j\in\bar{C}:x^j\neq 0} GV_T^j(x^j) + \sum_{j\in\bar{C}:x^j=0} c^j \right] + (p_T - r)U$$

Proof of Theorem 4:

$b_T^C(x_T^C) \geq p_T^C(x_T^C, 0)$ is equivalent to $d^C \leq \min_{\sum_{j\in\bar{C}} x^j = S} \left[\sum_{j\in\bar{C}:x^j\neq 0} GV_T^j(x^j) - \sum_{j\in\bar{C}:x^j=0} d^j \right]$.

Since $\min_{\sum_{j\in C} x^j = S} \left[\sum_{j\in C: x^j \neq 0} GV_T^j(x^j) - \sum_{j\in C: x^j = 0} d^j \right] \leq TGV_T^C(S)$, the forward direction follows. Backward direction follows immediately if $\min_{\sum_{j\in C} x^j = S} \left[\sum_{j\in C: x^j \neq 0} GV_T^j(x^j) - \sum_{j\in C: x^j = 0} d^j \right] = TGV_T^C(S)$. Otherwise, suppose that $d^C > \min_{\sum_{j\in C} x^j = S} \left[\sum_{j\in C: x^j \neq 0} GV_T^j(x^j) - \sum_{j\in C: x^j = 0} d^j \right] = TGV_T^{\bar{C}\setminus K}(S) - d^K$. But then $d^{C\cup K} > TGV_T^{\overline{C\cup K}}(S)$ leads to a contradiction.

Proof for Theorem 5:

(a)

Immediately follows from Theorem 3 and the discount calculation process.

(b)

Immediately follows from Lemma 3 and the fact that bidder i only bids for the final clock package x_T^i.

(c)

The total value associated with the final clock allocation $x_T = (x_T^1, ..., x_T^N)$ is given by:

$$TV(x_T) = \sum_j \overline{v}^j(x_T^j) + r(S - X_T) = \sum_j \left[p_T x_T^j + c^j \right] + r(S - X_T) = p_T X_T + \sum_j c^j + r(S - X_T).$$

For an alternative allocation $y = (y^1, ..., y^N)$, the maximum possible total value given a bid of $b^i(y^i)$

$$TV^* = b^i(y^i) + \max_{\sum_{j\neq i} y^j = S - y^i} \sum_{j\neq i} \overline{v}^j(y^j) = b^i(y^i) + p_T(S - y^i) + \max_{\sum_{j\neq i} y^j = S - y^i} \left[\sum_{y^j \neq 0} c^j - \sum_{y^j \neq 0} GV_T^j(y^j) \right]$$

The alternative allocation will beat the final clock allocation if:

$$TV^* \geq TV(x_T)$$

$$b^i(y^i) \geq p_T y^i + c^i - (p_T - r)U + \min_{\sum_{j\in N\setminus i} y^j = S - y^i} \left[\sum_{y^j \neq 0} GV_T^j(y^j) + \sum_{y^j = 0} c^j \right]$$

Given that $c^j \geq -d_T^j \quad \forall j$,

$$b^i(y^i) \geq p_T y^i + c^i - (p_T - r)U + \min_{\sum_{j \in N \setminus i} y^j = S - y^i} \left[\sum_{y^j \neq 0} GV_T^j(y^j) - \sum_{y^j = 0} d_T^j \right]$$

The minimum payment can be found similarly.

(d)

The coalition of pseudo bidders does not win any lots. In a core-selecting auction the coalition of winners (actual bidders) should jointly pay the value of bids placed by pseudo bidders.

Some text:

The exposure calculation includes a term that accounts for possible undersell after the clock stage. Intuition behind this formula is straightforward. Suppose that the clock stage allocated all lots such that $U = 0$. When it was possible bidders in $\bar{C}$ to redistribute lots from x_T^C among themselves without violating QGARP, then they are able to claim the full value of $p_T x_T^C$ for lots in x_T^C. To put it differently, when the sum of QGARP violations can be minimized to zero, it was feasible for bidders in $\bar{C}$ to cumulatively bid for the supply vector S in the final clock round. Therefore, the full value of $p_T x_T^C$ is required for optimality property to hold under all possible valuation profiles. In case the minimized sum of QGARP violations is positive, bidders in $\bar{C}$ are not able to bid for the supply vector S in the final clock round. Then the total value of bidders in C can be as low as $p_T x_T^C - TGV_T^{\bar{C}}(S)$, and still be sufficiently high to guarantee the optimality of the allocation for any set of permissible valuation profiles.

The Combinatorial Clock Auction, Revealed Preference and Iterative Pricing

Lawrence Ausubel

University of Maryland

Oleg Baranov

University of Colorado

Version 1.0 September 2, 2013

Combinatorial Clock Auction (current generation):

- Has been designed to allocate multiple heterogeneous items
- Has been used for digital dividend auction in Denmark, Ireland, the Netherlands, Switzerland, the UK and Australia (regional licenses)
- Has been adopted for upcoming spectrum auctions in Canada (regional licenses) and Slovakia
- In the past couple of years, CCAs have been used for more major spectrum auctions worldwide than any other auction format

Combinatorial Clock Auction - high level description:

- A three-phase auction with package bidding, consisting of a clock auction phase, followed by a supplementary round, and concluding with an assignment round

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- **Assignment Stage:** Sealed-bid auction to map generic spectrum to specific frequencies
- Allocation and pricing are given by a *core-selecting mechanism*

Diagram of the CCA

Auction Start
Auctioneer announces starting price

Diagram of the CCA

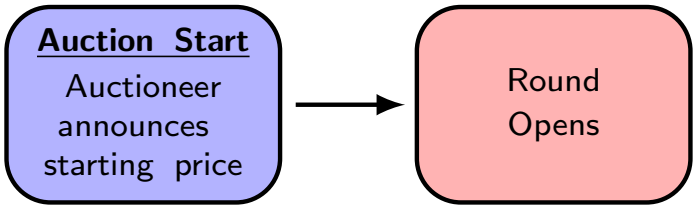


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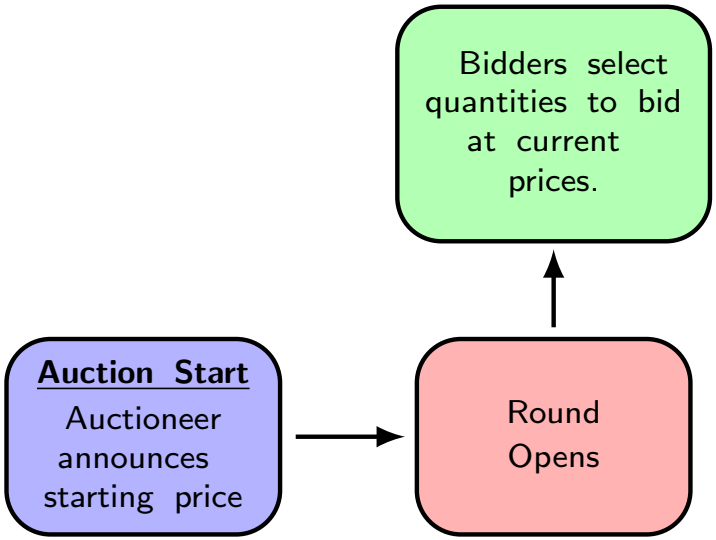


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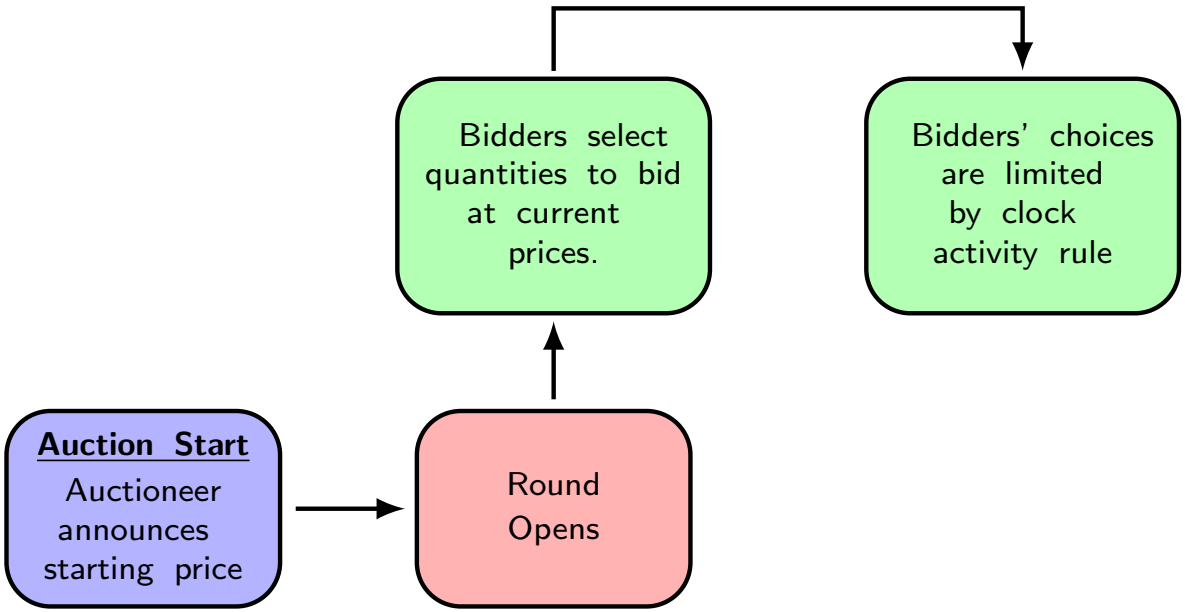


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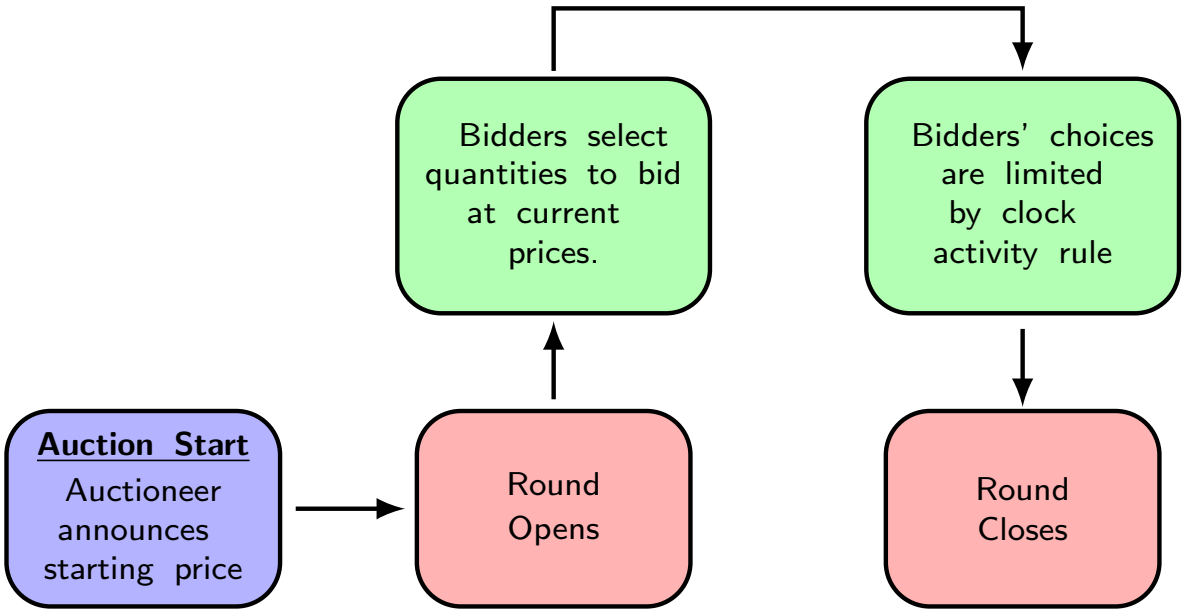


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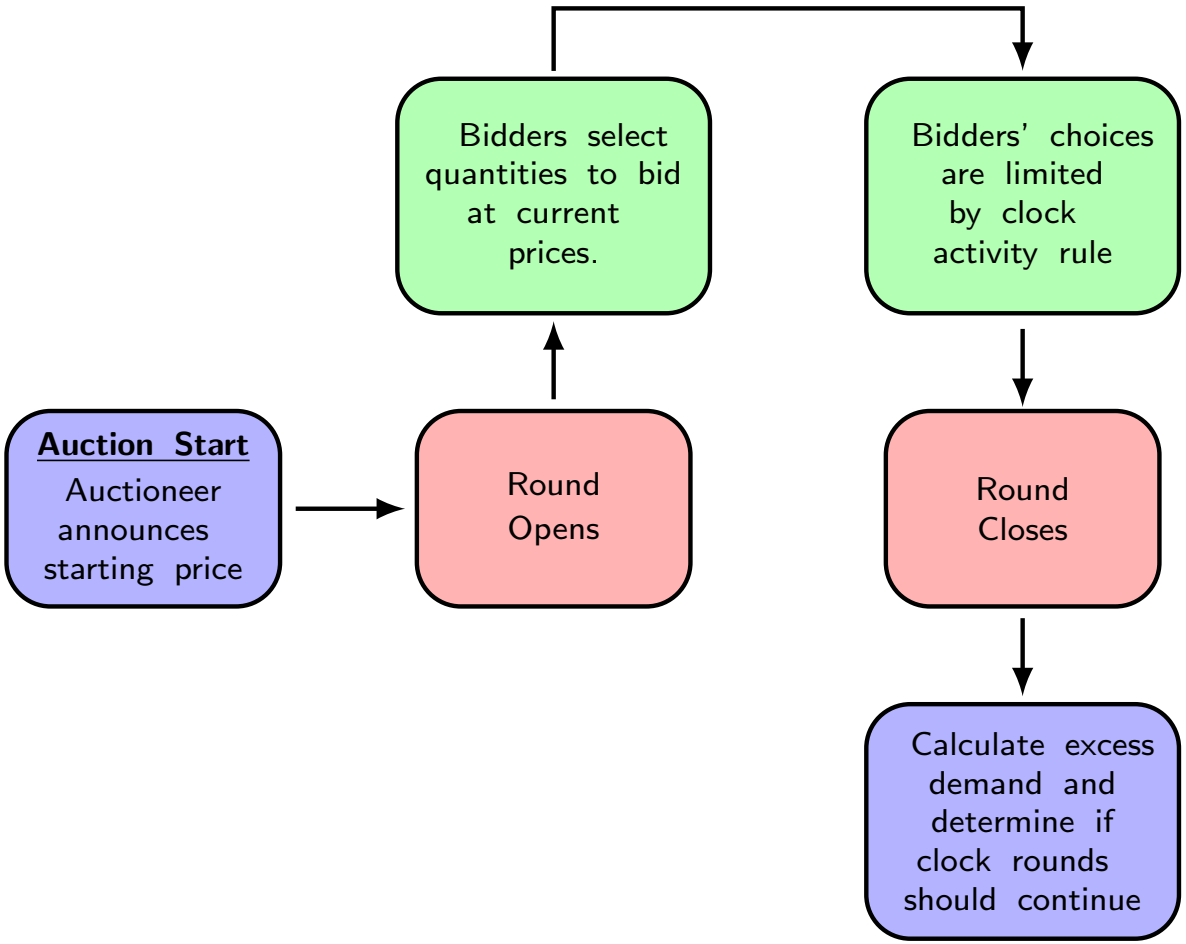


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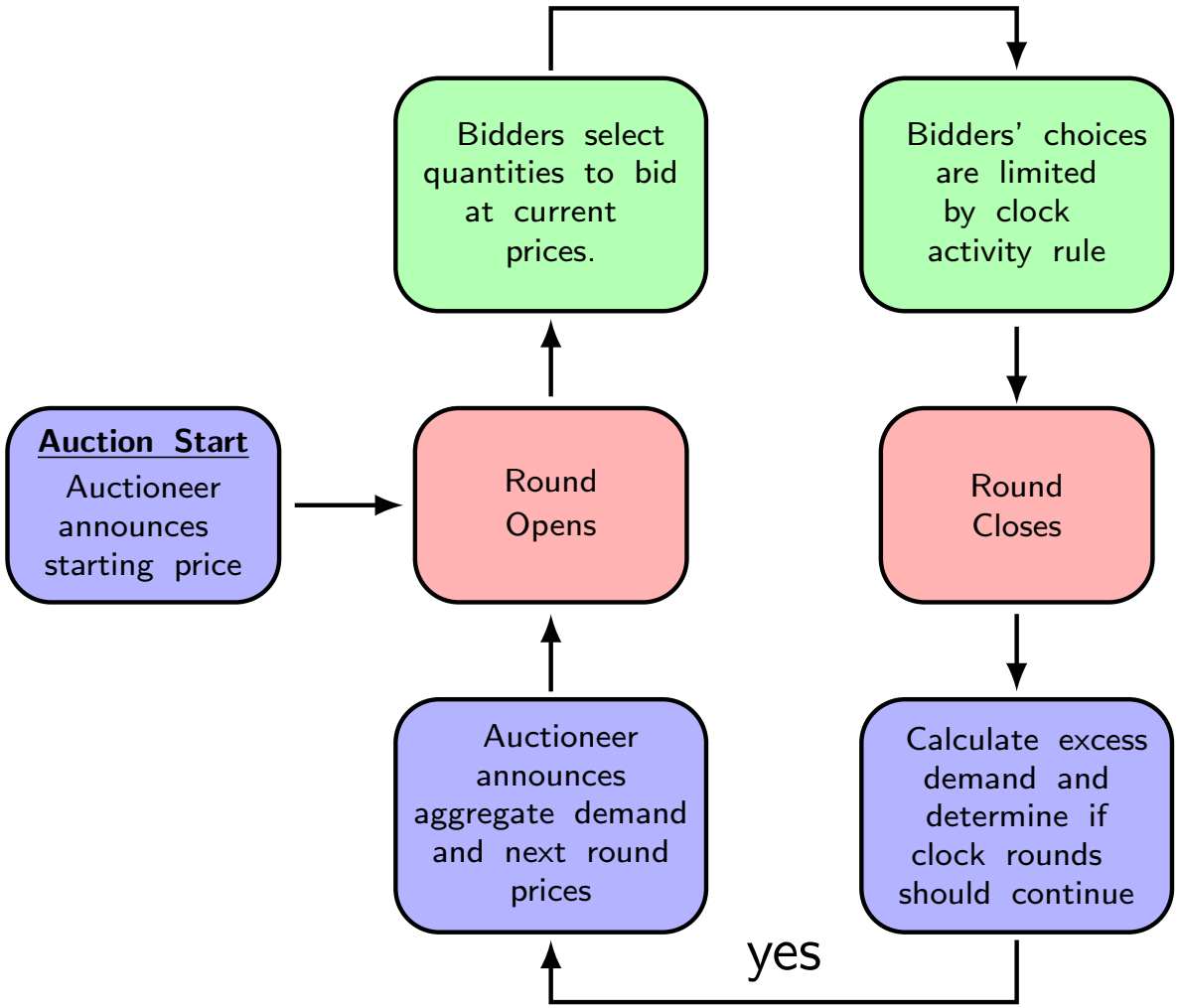


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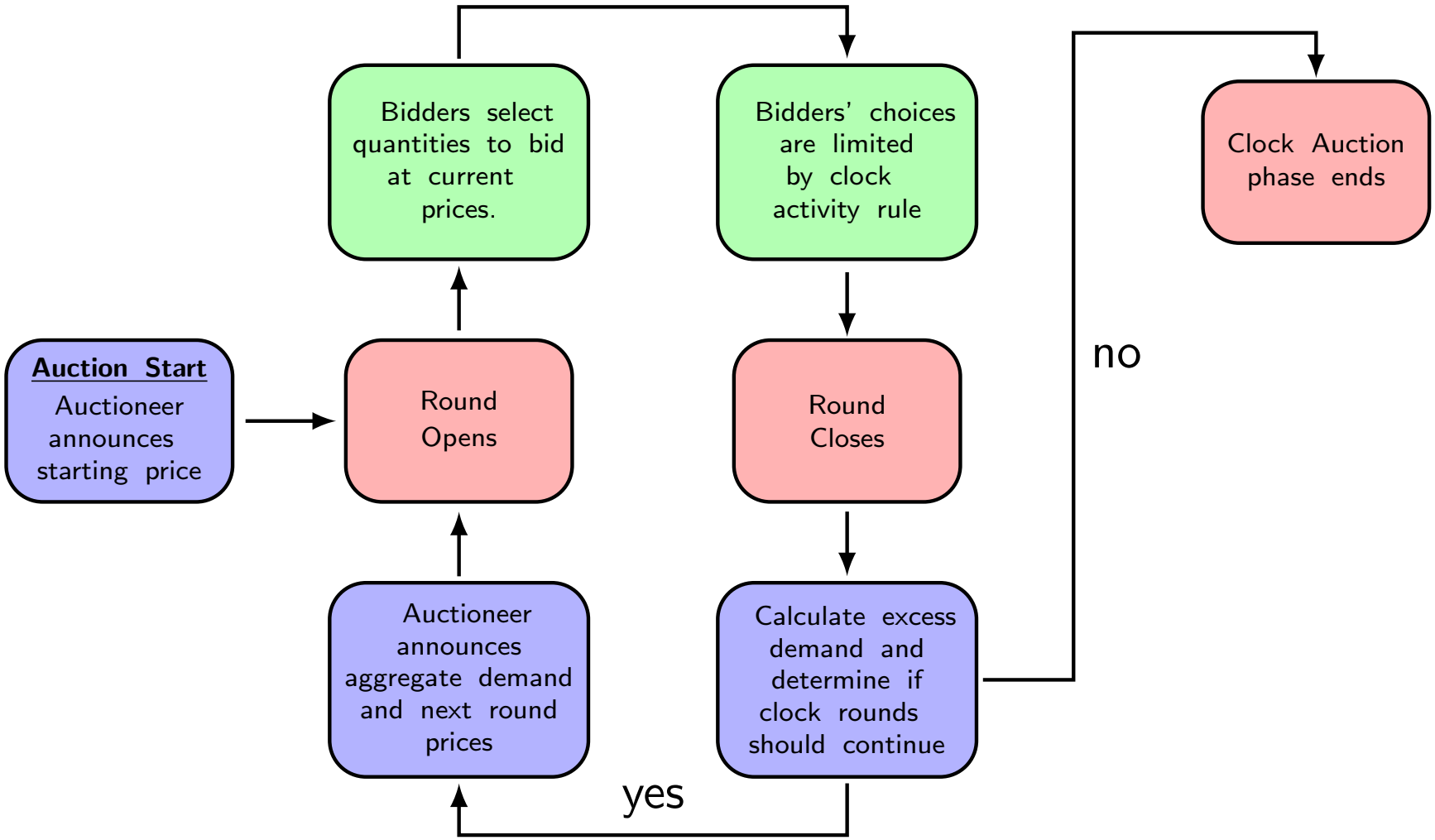


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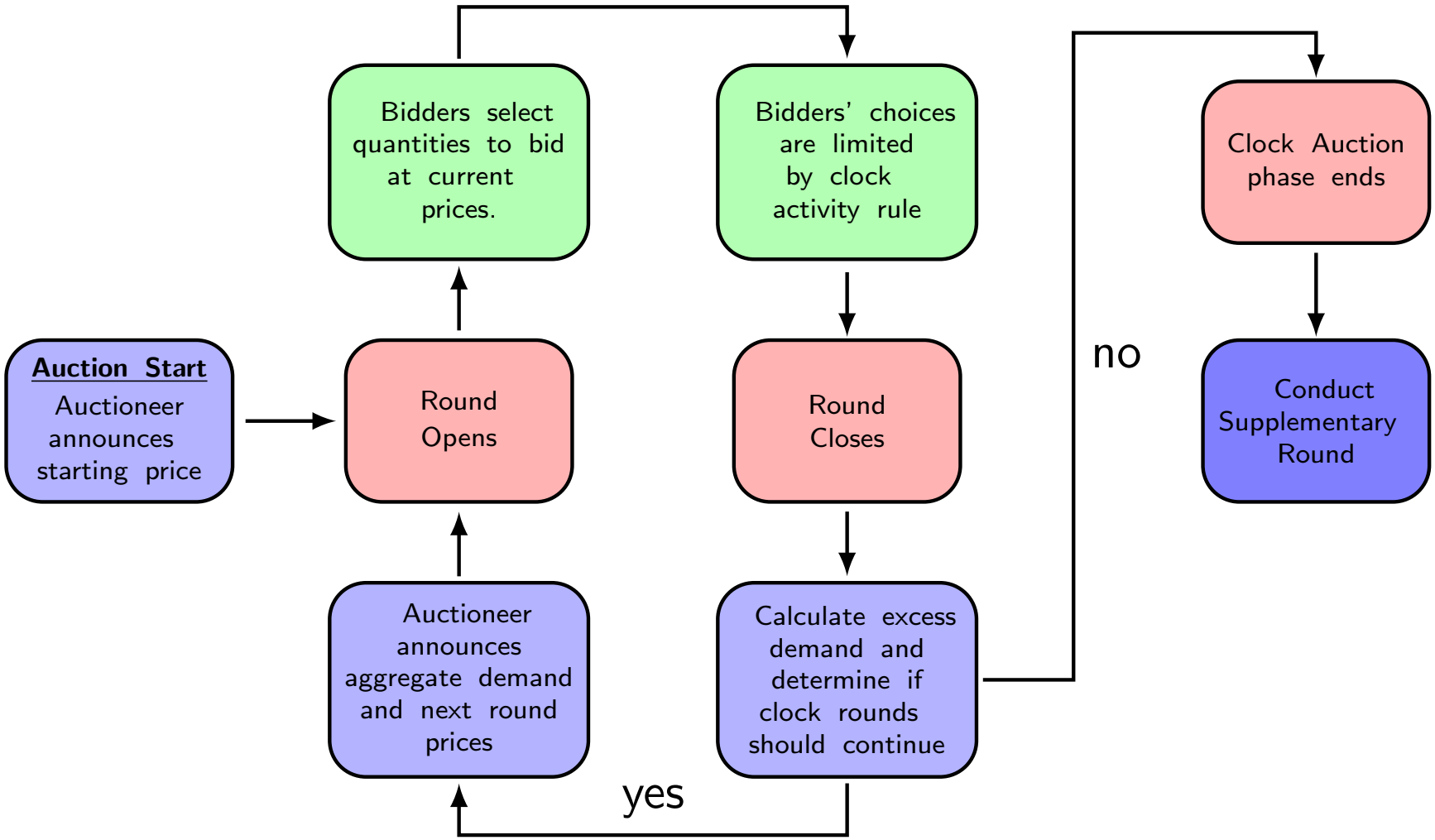
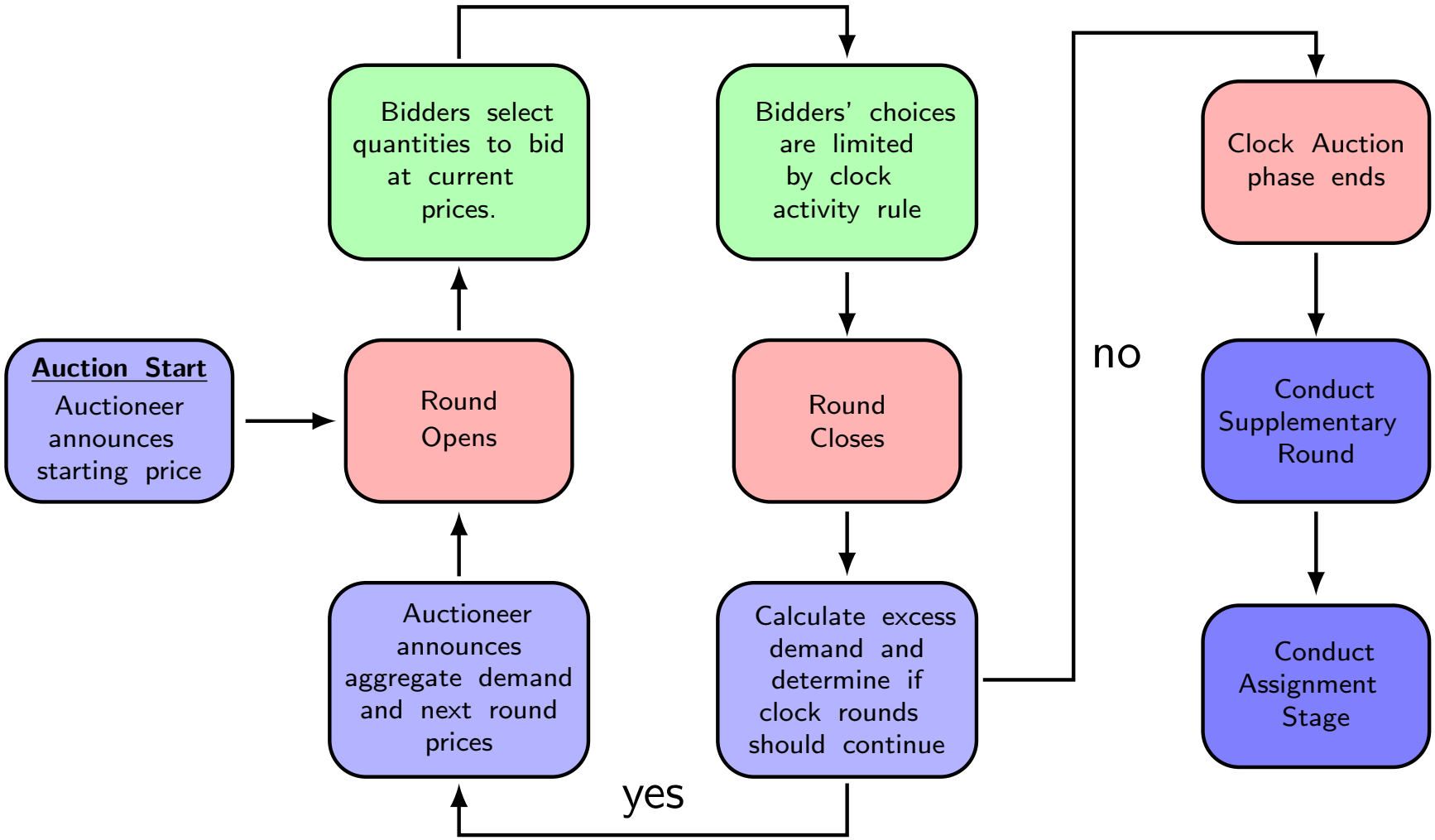


Diagram of the CCA



Combinatorial Clock Auction - Background Theory:

- **Clock Stage:** Like Walrasian Tâtonnement with ascending price trajectory
- **Activity Rule:** Eligibility Point Monotonicity, Revealed Preference
- **Closing Rule:** No Excess Demand for all goods (Excess Supply is permitted)

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Important theoretical property:

Truthful bidding (*or straightforward bidding*) is not the only weakly dominant strategy in CCA with the Vickrey pricing rule.

Activity rules in the SMRA and CCA:

Purpose:

Activity rules are intended to prevent "bid sniping": a bidder must submit serious bids in early rounds of the clock stage in order to be allowed to submit serious bids in later rounds (clock and supplementary)

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SMRAs have required monotonicity in eligibility points. Some CCAs have taken a similar approach.

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History:

SMRAs have required monotonicity in eligibility points. Some CCAs have taken a similar approach.

Problem:

Theorem: For any choice of eligibility points, there exist bidder valuations and price histories such that the bidder is prevented from bidding its true valuations by an activity rule requiring monotonicity in eligibility points; and if bidders attempt to bid straightforwardly, the outcome will necessarily be inefficient.

Activity rules in the CCA:

Adopted solution:

Some recent (and upcoming) CCAs (Ireland, Australia and Canada) have incorporated revealed preference considerations into the clock stage activity rules.

- QWARP: $(p_s - p_t)(x_s - x_t) \leq 0$, for $s < t$
- Verbally, clock bids can only include switches to packages that become relatively less expensive

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- Verbally, clock bids can only include switches to packages that become relatively less expensive

Ausubel and Cramton (2011)

A "hybrid" revealed-preference / eligibility-point approach:

A bidder can bid on package x if:

- Package x is consistent with QWARP, with respect to all prior eligibility-reducing rounds, or
- Package x is within bidder's current eligibility

Activity rules in the CCA: Supplementary Stage

Definition: Relative Cap

A bid for package x should satisfy the revealed preference constraint with respect to the last clock round in which bidder's eligibility was higher or equal to the total eligibility associated with the package x.

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Activity rules in the CCA: Supplementary Stage

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Definition: Final Cap

A bid for package x should satisfy the revealed preference constraint with respect to the final clock round.

Intuition:

- "Bid Sniping": Limit bids for more valuable packages by the bids for less valuable packages.
- Relative Cap implements this idea by substituting eligibility (exogenous measure of size) for value, e.g., limit bids for larger packages with respect to the bids for smaller packages

Critique of the current CCA:

Critique of the CCA Activity Rules:

The incorporation of revealed-preference considerations is an improvement on pure eligibility-point-based activity rules, but current versions are still imperfect.

Unresolved Issues:

- Only some of the revealed-preference constraints can be imposed, in order to avoid creating "dead-ends" and infeasibility
- It may be possible for a bidder to influence which revealed-preference constraints are imposed
- Bid anomalies (e.g., non-monotonicities may be present in the caps)
- "Parking" strategy (even easier execution than is SMRA)

Example 1: Bid Cap Non-Monotonicity

Definition: Relative Cap

A bid for package x should satisfy the revealed preference constraint with respect to the last clock round in which bidder's eligibility was higher or equal to the total eligibility associated with the package x.

- a sample clock bidding history

Round	Price	Demand	Activity	Eligibility
1	(10, 10)	(1, 1)	2	2
2	(20, 10)	(1, 1)	2	2
3	(30, 10)	(0, 1)	1	2
4	(40, 10)	(0, 1)	1	1
5	(50, 10)	(0, 1)	1	1

- Suppose that the bidder does not raise its final clock bid for (0, 1)
- Bid Limit for (1,0) = 10 + 40 = 50 (revealed preference w.r.t. Round 5)
- Bid Limit for (1,1) = 10 + 30 = 40 (revealed preference w.r.t. Round 3)
- Bidder can bid more for a smaller package!

Example 2: Selection of Revealed Preference Constraints

Definition: Relative Cap

A bid for package x should satisfy the revealed preference constraint with respect to the last clock round in which bidder's eligibility was higher or equal to the total eligibility associated with the package x.

- a sample clock bidding history

Round	Price	Demand	Activity	Eligibility
1	(100, 100, 1)	(1, 0, 1)	101	101
2	(100, 200, 1)	(1, 0, 1)	101	101
3	(100, 300, 1)	(1, 0, 0)	100	101
4	(200, 300, 1)	(1, 0, 0)	100	100
5	(300, 300, 1)	(1, 0, 0)	100	100

- Suppose that the bidder does not raise its final clock bid for (1, 0, 0)
- Bid Limit for (0, 1, 0) = 300 + 0 = 300 (revealed preference w.r.t. Round 5)
- Bid Limit for (0, 1, 1) = 300 + 201 = 501 (revealed preference w.r.t. Round 3)
- Bidder can bid substantially more for an almost equivalent package!

Example 3: Swiss Auction - February 2012

Band	Supply (Lots)	Reserve Price	Number of Lots won			Spectrum Caps
			Orange	Sunrise	Swisscom	
800 MHz	6	21.3	2	2	2	5 lots 800+900
900 MHz	7	21.3	1	3	3	4 lots 900
1800 MHz	15	7.1	5	4	6	7 lots
2.1 GHz	12	5.4	4	2	6	6 lots
2.6 GHz (p)	14	8.3	4	5	4	
2.6 GHz (u)	3	12.45	0	0	3	
Price at reserve			154.2	187.2	252.05	
Price at auction			154.7	481.7	359.8	

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1800 MHz	15	7.1	5	4	6	7 lots
2.1 GHz	12	5.4	4	2	6	6 lots
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Price at reserve			154.2	187.2	252.05	
Price at auction			154.7	481.7	359.8	

- Opportunity costs pricing is non-linear and non-anonymous
- Nevertheless, price paid by Sunrise seems to be unfairly high
- Price difference can also be explained by a insufficiently restrictive activity rule

Critique of the current CCA:

CCA Allocation and Pricing Rules:

All bids submitted throughout the clock rounds and the supplementary round are entered into the winner determination problem. The value maximizing allocation is selected. The payments are determined using the core pricing rule (most commonly Nearest-Vickrey pricing rule).

Interesting Fact:

Most dynamic auction formats are "iterative first-price" - they are *literally* first-price (while *effectively* second-price)

For example, in the English auction (as well as the SMRA, the clock auction and the ascending proxy auction), bids are submitted which, if they turn out to win, specify the amounts that will actually be paid.

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For example, in the English auction (as well as the SMRA, the clock auction and the ascending proxy auction), bids are submitted which, if they turn out to win, specify the amounts that will actually be paid.

Problem:

The CCA as currently used for spectrum auctions is closer to being an "iterative second-price" auction.

Critique of the current CCA:

Critique of the CCA Pricing Rule:

- This creates some basic tensions in the CCA, as far as stability between the clock stage and the supplementary round, on the one hand, and price determination, on the other
  - With a Final Price Cap and without undersell, each bidder is guaranteed to win its final clock package;
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  Example: With all bidders being winners, the final payments for all winners can be as low as reserve for any clock bidding history.
- It also might exacerbate the effects of budget constraints

Example 4: UK Auction - February 2013

Pre Auction Press:

The Office for Budget Responsibility (OBR) had forecast that the auction would raise £3.5bn for the Treasury...Chancellor George Osborne included £3.5bn worth of 4G auction receipts in his Autumn Statement in December...BBC political producer Andrew Fagg says this allowed Mr Osborne to play the "trump card" of predicting that the UK's budget deficit would fall in 2012-13. **(BBC News)**

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Clock Stage:

Lots	A1	A2	C	D1	D2	E	Total Price (in £bl)
Quantity	4	1	14	part of C		9	
Round 1 (in £ml)	225	250	15	3	6	0.1	1.361
Round 52 (in £ml)	423	846	92	21.9	43.8	24.4	4.046

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Mid Auction Press (after the clock stage):

UK 4G Spectrum Auction Could Raise Government As Much As £4bn ... Report suggests **fierce competition** could yield the treasury more than expected **(The Times)**

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After Auction Press :

"4G auction to be investigated by audit office after poor return" Auction generated £2.3bn in June - £1.2bn short of Treasury expectations **(The Guardian)**

"4G auction: Treasury could have raised £3bn more" Telecoms watchdog Ofcom has published data showing the theoretical maximum £5bn the chancellor could have received, rather than the £2.3bn actual total **(The Guardian)**

Example 4: UK Auction - February 2013

Example of the "Missing Bid" problem:

- *HKT (UK) Company Ltd* dropped out of the auction in Round 46 (price of E lot - 6.4 £ml)
- Final Bids of *HKT (UK) Company Ltd* (after the supplementary round):

Number of E lots	2	3	4	5	6	7	8	9
Clock Bid	10.22	8.88	9.88	10.3	10.32	10.01	9.552	8.955
Final Bid	10.25	10.7	10	10.35	10.35	10.10	10	10.75
Max Cap	12.4	15.95	18.91	21.38	23.44	25.16	26.59	27.784

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- Nonmonotonic bids
- Bids for 4,5,6,7,8,9 lots are 100% losing bids given bids for 2 and 3 and reserve of 0.1 £ml for one E lot

Main Idea - Illustration

Setup: 20 homogeneous lots with a reserve price of \$10

Bidder values: $v_1(12) = 230, v_1(10) = 200$ and $v_2(12) = 232, v_2(10) = 200$

Solution: Each bidder should win 10 lots and pay $p_1 = 112$ and $p_2 = 110$.

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Clock stage of CCA:

R	Price	Demand - Price	
		Bidder 1	Bidder 2
1	10	12 - 120	12 - 120
2	11	12 - 132	12 - 132
...	...	...	...
5	14	12 - 168	12 - 168
6	15	10 - 150	12 - 180
7	16	10 - 160	10 - 160

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Modified clock stage:

R	Price	Demand - Price	
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Both clock stages resulted in the same allocation. The CCA bid amounts overestimate the minimum value bidder has to have in order to secure its winnings.

Main Idea - Another Illustration

Setup: 20 homogeneous lots with a reserve price of \$10

Bidder values: $v_1(12) = 170, v_1(10) = 140$ and $v_2(12) = 232, v_2(10) = 200$

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Clock stages resulted in different allocations. The modified clock stage terminated at the optimal allocation.

Literature Review:

- **CCA:** *Ausubel, Cramton and Milgrom (2006)*
- **Activity Rule:** *Samuelson (1938), Afriat(1967), Varian(1982,1983), Rochet(1987), Ausubel, Cramton and Milgrom (2006), Ausubel and Cramton (2011)*
- **Exposure Formula:** *Ausubel (2004, 2006)*
- **Core-Selecting Auctions:** *Day and Raghavan (2007), Day and Milgrom (2008), Day and Cramton (2008), Parkes (2001), Parkes and Ungar (2000), Ausubel and Milgrom (2002, 2006)*
- **Practical CCA:** *Cramton (2013)*

Roadmap and Contributions:

- A new class of activity rules based on GARP
  - addresses all problems mentioned above
  - eliminates use of eligibility points
  - benefits of the "parking" strategy are severely limited
  - can be adapted for other auction formats like SMRA

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- Bidder Exposure Calculation
  - utilizes new activity rule to calculate minimum guaranteed winning bid
  - link between "clinching" and bidder exposure
- A new pricing approach
  - bid discount approach based on bidder exposure calculation
  - bidders are no longer required to make artificially high bids
  - allows us to convert CCA into an *iterative first-price auction*

Standard Model:

Environment:

Auctioneer want to auction multiple heterogeneous items (S - Supply Vector)

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N bidders with general valuation profiles that can include both substitutes and complements (private value model).

Reserve Price:

Auctioneer imposes reserve price r using the *lot reserve* approach.

Some "old" definitions:

Definition: WARP

A collection of price-quantity pairs $(p_1, x_1), ..., (p_t, x_t)$ satisfies the *weak axiom of revealed preferences* (WARP) if and only if:

$$p_s x_s \geq p_s x_z \quad \text{implies} \quad p_z x_z \leq p_z x_s \quad \forall s, z \in \{1, ..., t\}$$

WARP is too weak to be used as a base for the activity rule (demand can go up at higher prices).

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WARP is too weak to be used as a base for the activity rule (demand can go up at higher prices).

Definition: QWARP

A collection of price-quantity pairs $(p_1, x_1), ..., (p_t, x_t)$ satisfies the *quasilinear weak axiom of revealed preferences* (QWARP) if and only if:

$$(p_s - p_z)(x_s - x_z) \leq 0 \quad \forall s, z \in \{1, ..., t\}$$

Potentially, QWARP can be used as a base for the activity rule.

QWARP Clock Rule:

Definition: QWARP Clock Rule

A bidder is permitted to place a bid on a package $x \in X$ in Round t with clock prices p_t if and only if the collection of observations $(p_1, x_1), ..., (p_{t-1}, x_{t-1}), (p_t, x)$ satisfies QWARP.

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QWARP Clock Rule has an important flaw:

Round	Price	Demand
1	p_1=(1, 1, 1)	x_1=(1, 0, 0)
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3	p_3=(4, 1, 4)	x_3=(0, 0, 1)
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QWARP Clock Rule has allowed to place bids which produced a "dead-end": the only allowable bid in Round 4 is (0,0,0)

Some "old" definitions:

Definition: GARP

A collection of price-quantity pairs $(p_1, x_1), ..., (p_t, x_t)$ satisfies the *generalized axiom of revealed preferences* (GARP) if and only if:

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- **Intuition:** "cyclical consistency"
- **Afriats Theorem (1967):** A finite set of data is consistent with utility maximization if and only if it satisfies GARP.
- GARP is too weak to be used as a base for the activity rule (same as WARP).

QGARP:

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A collection of price-quantity pairs $(p_1, x_1), ..., (p_t, x_t)$ satisfies the *quasilinear generalized axiom of revealed preferences* (QGARP) if and only if:

$$p_{k_1}(x_{k_1} - x_{k_m}) + p_{k_2}(x_{k_2} - x_{k_1}) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) \leq 0$$
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Potentially, QGARP can be used as a base for the activity rule (same as QWARP).

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Potentially, QGARP can be used as a base for the activity rule (same as QWARP).

Theorem 0: Existence of the quasilinear valuation profile

A collection of price-quantity pairs $(p_1, x_1), ..., (p_t, x_t)$ satisfies QGARP if and only if there exist numbers $v_i \quad i = 1, ..., t$ such that

$$v_i \leq v_j + p_j(x_i - x_j) \quad \text{for} \quad i, j = 1, ..., t$$

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Denote X_t a set of packages that are feasible to the bidder in Round t.

Theorem 1: Properties of the QGARP Clock Rule

The following statements are true for the set of feasible packages X_t for Round t:

a) (profit-maximization) If the bidder was bidding straightforwardly in all previous rounds, the set of feasible packages X_t contains all profit-maximizing packages for Round t;

b) (no "dead-ends") X_t contains at least one package from the set of previously demanded packages $\{x_1, ..., x_{t-1}\}$;

c) (smaller bids) X_t contains all subsets of any package x which belongs to the set X_t.

QGARP Clock Rule:

"Dead-end" example for QWARP Clock Rule:

Round	Price	Demand
1	$p_1=(1, 1, 1)$	$x_1=(1, 0, 0)$
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What happens under QGARP?

- The reason for the dead-end under QWARP is that the bid placed in Round 3 violates QGARP.
  Violation: $p_1(x_1 - x_3) + p_2(x_2 - x_1) + p_3(x_3 - x_2) = 3 > 0$!!!
- Under QGARP, the bid in Round 3 would not be allowed. The dead-end would be avoided.

QGARP Clock Rule:

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- QGARP Clock Rule is more restrictive than QWARP Clock Rule which in turn is more restrictive than the rules used in today's CCAs.
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  Example: *last bid can be infeasible*
- **Pending Issue:** Budget-constrained bidders and QGARP Clock Rule *(more on this later)*

QGARP Validation:

Suppose that the collection $(p_1, x_1), ..., (p_{t-1}, x_{t-1})$ satisfies QGARP. How one can verify an extended collection $(p_1, x_1), ..., (p_t, x_t)$?

QGARP Validation: Formulation 1

A linear programming problem:

$$QGV_t(x_t) = \min_{\lambda, v} \lambda$$

subject to

$$\lambda \geq 0 \qquad (1)$$

$$v_i \leq v_j + p_j(x_i - x_j) \qquad \forall i \in \{1, ..., t\} \quad \forall j \in \{1, ..., t-1\} \qquad (2)$$

$$v_i \leq v_t + p_t(x_i - x_t) + \lambda \qquad \forall i \in \{1, ..., t-1\} \qquad (3)$$

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- If $QGV_t(x_t) = 0$, a bid for package x_t is feasible in Round t
- If $QGV_t(x_t) > 0$, a bid for package x_t violates QGARP. The value of the objective $QGV_t(x_t)$ provides the size of QGARP violation.

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QGARP Validation: Formulation 2

Putting (2) and (3) together:

$$\lambda \geq p_{k_1}(x_{k_1} - x_t) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) + p_t(x_t - x_{k_m}) \qquad \forall k_1, ..., k_m \in \{1, ..., t\}$$

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$$\lambda \geq 0 \tag{1}$$
$$v_i \leq v_j + p_j(x_i - x_j) \qquad \forall i \in \{1, ..., t\} \quad \forall j \in \{1, ..., t-1\} \tag{2}$$
$$v_i \leq v_t + p_t(x_i - x_t) + \lambda \qquad \forall i \in \{1, ..., t-1\} \tag{3}$$

QGARP Validation: Formulation 2

Putting (2) and (3) together:

$$\lambda \geq p_{k_1}(x_{k_1} - x_t) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) + p_t(x_t - x_{k_m}) \qquad \forall k_1, ..., k_m \in \{1, ..., t\}$$

$$\lambda \geq \max_{k_1, ..., k_m \in \{1, ..., t\}} \left[p_{k_1}(x_{k_1} - x_t) + ... + p_{k_m}(x_{k_m} - x_{k_{m-1}}) + p_t(x_t - x_{k_m}) \right]$$

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QGARP Validation: Formulation 1

$$QGV_t(x_t) = \min_{\lambda, v} \lambda$$

subject to

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Implied Value Profiles:

Some definitions:

- Clock bidding history for a given bidder: $(p_1, x_1), ..., (p_T, x_T)$
- Denote $V = \{(v_1, ..., v_T) : \ v_i \leq v_j + p_j(x_i - x_j) \quad \forall i, j \in \{1, ..., T\}\}$
- **Normalization:** $v_T = p_T x_T + c, \ c > 0$ for any $v = (v_1, ..., v_T) \in V$.
- **Theorem 0:** Set V is nonempty as long as $(p_1, x_1), ..., (p_T, x_T)$ satisfies QGARP.

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Lemma 1:

Set V is a lattice. Therefore, there exists a unique minimal and a unique maximal element of V denoted $\underline{v}$ and $\overline{v}$ correspondingly.

Implied Value Profiles:

A consistent valuation profile: *(consistent with clock bidding history)*

- Any element of V: $v = (v_1, ..., v_T)$ where $v_T = p_T x_T + c, \ \forall c$.
- Consistent valuation profile $v(.)$:

$$v(x) = \begin{cases} v_t & \text{if } \exists t: \ x_t = x \\ [-\infty, \min_{t \in \{1,...,T\}} \{v_t + p_t(x - x_t)\}] & \text{otherwise} \end{cases}$$

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Maximal valuation profile:

- Maximal element of V: $\overline{v} = (\overline{v}_1, ..., \overline{v}_T)$ where $\overline{v}_T = p_T x_T + c, \ \forall c$.
- Maximal Valuation profile $\overline{v}(.)$:

$$\overline{v}(x) = \min_{t \in \{1,...,T\}} \{\overline{v}_t + p_t(x - x_t)\} \qquad \forall x \in X$$

Properties of the Maximal Value Profile

Theorem 2: Properties of the Maximal Valuation Profile

Given the QGARP-consistent bidding history $(p_1, x_1), ..., (p_T, x_T)$, the normalized maximal valuation profile $\overline{v}(.)$ has the following properties:

a) (formula) For any package $x \in X$, $\overline{v}(x)$ can be computed using the following formula:

$$\overline{v}(x) = p_T x + c - QGV_T(x)$$

where $QGV_T(x)$ is the QGARP violation for the counterfactual bidding history $(p_1, x_1), ..., (p_T, x)$;

b) (monotonicity) For any two packages $x, x' \in X$ such that $x' \leq x$, $\overline{v}(x') \leq \overline{v}(x)$;

c) (concavity) A maximal valuation profile $\overline{v}(.)$ is concave.

Permissible Valuation profiles

Permissible valuation profile:

- Normalization: $c = 0$ if $x_T = 0$ and any c if $x_T \neq 0$.
- Maximum valuation profile $\overline{v}(x)$ with the normalization $\overline{v}(x_T) = p_T x_T + c$.
- Permissible valuation profile $v(.)$:

$$v(x) = \begin{cases} 0 & \text{if } x = 0 \\ \overline{v}(x_T) & \text{if } x = x_T \\ [0, \max\{0, \overline{v}(x)\}] & \text{otherwise} \end{cases}$$

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Additional Notation:

- superscript i on all bidder-specific objects like x_t^i, $QGV_t^i(x)$, $\overline{v}^i(.)$, c^i;
- aggregated demands: $X_t^C = \sum_{j \in C} x_t^j$, $\quad X_t^{-i} = \sum_{j \neq i} x_t^j$, $\quad X_t = \sum_j x_t^j$;
- The minimum total QGARP violation for bidders in coalition C when bidding for package X in Round t:

$$TV_t^C(X) = \min_{\sum_{j \in C} x^j = X} \left[\sum_{j \in C} QGV_t^j(x^j) \right]$$

Welfare properties:

Theorem 3: Welfare Properties

Suppose that $c^i \geq 0 \ \forall i$.

a) (*First Fundamental Welfare Theorem*) If all markets clear in the final clock round (no excess supply), then any alternative allocation is Pareto dominated by the final clock allocation x_T when compared under any collection of permissible valuation profiles;

b) If clock rounds end with excess supply $U = S - X_T$, then an alternative feasible allocation $y = (y^1, ..., y^N)$ is Pareto dominated by the final clock allocation $x_T = (x_T^1, ..., x_T^N)$ when compared under any collection of permissible valuation profiles if

$$\sum_{j \in N} QGV_T^j(y^j) + \sum_{j:\, y^j = 0} c^j \ > (p_T - r)(Y - X_T) \qquad \text{where} \quad Y = \sum_{j \in N} y^j$$

Further, the value of any undominated alternative allocation evaluated under any collection of permissible valuation profiles cannot exceed the value of the final clock allocation x_T by more than $(p_T - r)U - TV_T^N(S)$.

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Losing Bid Analysis for the Supplementary Round

Bidder i will never be awarded a package x if $\boxed{QGV_T^i(x) + TV_T^{-i}(S - x) > (p_T - r)U}$.

Bidder Exposure: The Case of No "Undersell" $U = 0$

What if some of the normalizing constants are negative?

- Theorem 3 (a): $\boxed{c^i \geq 0 \quad \forall i}$ is a sufficient condition.
- This means that expressed value of $v(x_T) = p_T x_T + c$ for x_T can be substantially higher than the minimum value necessary to guarantee the optimality property established by Theorem 3(a).
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Bidder Exposure Calculation (when $U = 0$)

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\begin{aligned}
p_T^i(x_T^i) &= \max_{\sum_{j\neq i} y^j = S} \left[\sum_{j\neq i} \overline{v}^j(y^j)\right] - \sum_{j\neq i} \overline{v}^j(x_T^j) \\
&= \max_{\sum_{j\neq i} y^j = S} \left[\sum_{j\neq i} \left(p_T y^j + c^j - QGV_T^j(y^j)\right)\right] - \sum_{j\neq i} \left[p_t x_T^j + c^j\right] \\
&= p_T(S - X_T^{-i}) - \min_{\sum_{j\neq i} y^j = S} \left[\sum_{j\neq i} QGV_T^j(y^j)\right] \\
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Welfare and Bidder Exposure:

Theorem 4: Welfare Properties

Suppose that $c^i \geq -TV_T^{-i}(S) \quad \forall i$ where $TV_T^{-i}(S) \quad \forall i$ have been computed under the assumption of full divisibility. Then Theorem 3(a) continues to hold, i.e., the final clock allocation is the Pareto optimal allocation under any collection of permissible valuation profiles.

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Proof:

1. By Theorem 2, maximum valuation profile $\overline{v}^i(.)$ is concave for all bidders i.
2. Under the assumption of full divisibility, the concavity of the utility functions is a sufficient condition for the core property of the Vickrey payoff. divisibility.

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Notes:

1. Note that the calculation of $TV_T^{-i}(S)$ is easier under full divisibility.
2. However, for some environments it might include overestimation of the true exposure.

Implications of Theorem 4: Bid Discount

Bid Discount Approach:

- In CCA, bidders are required to bid $p_t x$ for a package x in Round t. If the clock stage ends with no excess supply, some bidders were required to submit excessive bids.

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- Suppose that bidder i is required to bid $p_t x - TV_t^{-i}(S)$ for a package x in Round t. By Theorem 4, it is fully sufficient in case Round t ends up being the final clock round with no excess supply.
- If Round t is not the last round, the process continues.
- If Round t is the last clock round with excess supply, any issues arising from the unallocated lots can be addressed in the supplementary round.

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- If Round t is the last clock round with excess supply, any issues arising from the unallocated lots can be addressed in the supplementary round.
- Practical bid formula:

$$b_t^i(x) = \max\{rx, p_t x - TV_t^{-i}(S)\} \qquad \forall x \in X$$

Bidder Exposure: Positive "Undersell" $U > 0$

Version 1: Inclusive Bidder Exposure

- Calculate incremental value for lots in x_T^i accounting for the undersell

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Bidder Exposure Calculation (when $U > 0$)

$$\begin{aligned} p_T^i(x_T^i, U) &= \max_{\sum_{j\neq i} y^j = S} \left[\sum_{j\neq i} \overline{v}^j(y^j)\right] - \left[\sum_{j\neq i} \overline{v}^j(x_T^j) + rU\right] \\ &= \max_{\sum_{j\neq i} y^j = S} \left[\sum_{j\neq i} \left(p_T y^j + c^j - QGV_T^j(y^j)\right)\right] - \sum_{j\neq i}\left[p_t x_T^j + c^j\right] - rU \\ &= p_T(S - X_T^{-i}) - \min_{\sum_{j\neq i} y^j = S} \left[\sum_{j\neq i} QGV_T^j(y^j)\right] - rU \\ &= p_T x_T^i - TV_T^{-i}(S) + (p_T - r)U \end{aligned}$$

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Assuming bidder i made a bid of $p_T x_T^i - TV_T^{-i}(S)$ for package x_T^i in Round t, bidder i might need to raise its bid by $(p_T - r)U$ in the supplementary round in order to match its full exposure (a very familiar **knockout bid** result)

Feasibility of Bid Discount Formula

Bid Discount Formula:

$$b_t^i(x) = \max\{rx, p_t x - TV_t^{-i}(S)\} \qquad \forall x \in X$$

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- QGARP implications: Can continue to use almost the same formulas since bidders continue to face the same relative prices (minor tweak to handle absolute values)
- Issue: Bid discount does not depend on the size of the package
  - large discount can wipe out the price of the small package
  - but such situation will result in a large undersell (need to bid more in the supplementary round)
  - Intuitively, the "clinching" intuition tells us that the discount should fall with the size of the package (to eliminate incentives to engage in demand reduction)

Bidder Exposure: Positive "Undersell" $U > 0$

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- **Summary: Seems like a highly impractical way of introducing bid discount.**
- **Intuition**: *We are not actually doing "clinching" during the clock stage. The whole purpose of the clock stage is to elicit values and put bid constraints on the sealed bid auction carried out during the supplementary round.*

QGARP Activity Rule for the Supplementary Round

Supplementary Round Setup:

- The bid for the final clock package is added $b_T^i(x_T^i) = \max\{rx, p_T x_T^i - TV_T^{-i}(S)\}$ to the set of bids considered by the WD problem.

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In the Supplementary Round, the bid amount for any nonnull package $x \in X$ cannot exceed the value of the associated maximal valuation $\overline{v}^i(x)$ for this package, i.e.,

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Under Simple Cap Activity Rule, bidders can bid according to any permissible valuation profile (it does not have to be consistent or monotonic)

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How to implement:

For each actual bidder, add a *pseudo* bidder with a bid that duplicates the final clock bid (anchor bid) of the bidder to the WD problem (with some technical constraints).

Pricing

We refer to the combination of the new QGARP-based activity rule and the iterative pricing procedure as an enhanced combinatorial clock auction (ECCA).

Theorem 5: ECCA Properties

a) *(no supplementary round)* If the clock stage ends with no excess supply, the auction concludes with bidder i winning its final clock package x_T^i and paying $b_T^i(x_T^i) = p_T x_T^i - TV_T^{-i}(S)$;

b) *(knockout bid)* If the clock stage ends with a positive excess supply U, bidder i is guaranteed to win its final clock package by bidding at least $p_T^i(x_T^i, U)$ for the final clock package x_T^i and its final payment in this case would be not lower than $b_T^i(x_T^i)$;

c) *(minimum payment - anchor effect)* If the clock stage ends with a positive excess supply U and bidder i manages to win a package x^i as a result of the supplementary round, its final payment would be not lower than

$$p_T^i(x^i) - \left[(p_T - r)U - TV_T^{-i}(S - x^i)\right]$$

d) *(minimum revenue)* The minimum auction revenue after the supplementary round is given by $\sum_i b_T^i(x_T^i)$.

Exposure Calcualtion and Crediting/Debiting("Clinching")

Ausubel(2004) and Ausubel(2006)

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- Ausubel (2006) introduced a "crediting/debiting" formula for heterogeneous items

$$a_i(T) = p(0)[S - X^{-i}(0)] - \int_0^T p(t)dX^{-i}(t)$$

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Highlights of the crediting/debiting formula and model:

- "Dense" price trajectory - continuous or discrete with enough data points
- Concavity of utility functions (substitutes for discrete model)
- Incentive-compatibility
- Several price trajectories needed to recover Vickrey payments in general

Exposure Calcualtion and Crediting/Debiting("Clinching")

Crediting/Debiting Formula vs Revealed Preference Constraint:

$$a_i(T) = p(0)[S - X^{-i}(0)] - \int_0^T p(t)dX^{-i}(t)$$

$$v_i - p_j x_i \leq v_j - p_j x_j$$

$$v_i - v_j \leq p_j(x_i - x_j)$$

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Densness of the price trajectory

- "Crediting/Debiting" formula only works for continuous or sufficiently dense discrete price trajectory
- "Bidder Exposure" works for any price trajectory (extension).
- Intuitively, the more points price trajectory contains, the more of the revealed preference constraints become irrelevant. In the limit, the only revealed preference constrains that matter are indifference equalities from switch points like:

$$v(x_i) - p(t)x_i = v(x_j) - p(t)x_j$$

Exposure Calcualtion and Crediting/Debiting("Clinching")

Starting Point (actually Finisihing Point)

- Crediting/Debiting finishes necessarily at the initial price vector $p(0)$ - exogenous finishing point.

  "As if" at $p(0)$ the aggregate opponent switches from $X^{-i}(0)$ to S.

- Exposure Calculation finds the maximum value for S such that it is consistent with bidding history $X^{-i}(t)$ - endogenous finishing point $\hat{t}$

  (a point where aggregated opponent was indifferent between $X^{-i}(\hat{t})$ and S)

$$a_i(T) = p(\hat{t})[S - X^{-i}(\hat{t})] - \int_{\hat{t}}^{T} p(t)dX^{-i}(t)$$

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Implications:

- Exogenous finishing point was chosen in order to keep incentive-compatibility. However, it resulted in violation of individual rationality. Individual rationality was obtained by using a mechanism with many price trajectories ("parallel clocks")
- With endogenous finishing point, we lose incentive compatibility but keep individual rationality.
- Incentive compatibility: favorable and unfavorable price trajectories - bidder can try to induce favorable price trajectory through its bidding

Exposure Calcualtion and Crediting/Debiting("Clinching")

For the environment of Ausubel (2006):

Theorem 6: Equivalence of Crediting/Debiting and Exposure Calculations

Consider a model with divisible commodities and continuous piecewise smooth price trajectory $p(t) : t \in [0, T]$. If the starting price vector $p(0)$ is chosen such that $X^{-i}(p(0)) \leq S$ for bidder i, both the "crediting/debiting" and exposure calculations yield the same payment amount for bidder i. Further, if each bidder $j \neq i$ bids straightforwardly, then bidder i maximizes her payoff by bidding straightforwardly and receiving her payoff from the VCG mechanism with the reserve price of $p(0)$.

Practical Aspects of the ECCA

- QGARP Validation with Bid Discount
- Adding prior clock bids to the Supplementary Round
- Bidder Exposure based on Aggregate Demand
- Various Activity Rules for the Supplementary Round

QGARP Validation with Bid Discount:

Suppose that the collection $(p_1, x_1), ..., (p_{t-1}, x_{t-1})$ satisfies QGARP. How one can verify an extended collection $(p_1, x_1), ..., (p_t, x_t)$? Denote d_t a bidder discount in Round t and $T_0 = \{i \in \{1, ..., t\} : x_i = 0\}$.

QGARP Validation: Formulation 3

A linear programming problem:

$$QGV_t(x_t) = \min_{\lambda, v} \lambda$$

subject to

$$\lambda \geq 0$$
$$v_i \leq v_j + p_j(x_i - x_j) + d_j \mathbb{I}\{j \notin T_0\} - d_i \mathbb{I}\{i \notin T_0\} \qquad \forall i \in \{1, ..., t\} \quad \forall j \in \{1, ..., t-1\}$$
$$v_i \leq v_t + p_t(x_i - x_t) + d_t \mathbb{I}\{t \notin T_0\} - d_i \mathbb{I}\{i \notin T_0\} + \lambda \qquad \forall i \in \{1, ..., t-1\}$$

Prior Clock Bids

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As a matter of practical auction design, it is desirable to add clock bids placed in all clock rounds (not only the bid placed in the last clock round) to the winner determination problem. For example, all clock bids are added to the WD problem in a typical CCA.

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Since the bid discount tend to increase from one round to another round (a decrease is also possible), it is possible that adding bids at submitted levels might violate the maximum valuation profile.

Solution

Recalculate clock bid amounts from rounds with a smaller discount using the discount from the final clock round. Then bids can be safely added to the WD problem. Auctioneer also might consider to add *pseudo* bids for the corresponding *pseudo* bidder.

Bidder Exposure based on Aggregate Demand

Lemma 2:

An aggregated clock bidding history $(p_1, X_1), ..., (p_T, X_T)$ where $X_t = \sum_j x_t^j$ satisfies QGARP if all individual clock bidding histories $(p_1, x_1^j), ..., (p_T, x_T^j)$ that form the aggregated bidding history satisfy QGARP.

Denote $\overline{v}^{-i}(.)$ the maximum valuation profile for an aggregated bidder $-i$ based on aggregated bidding history $(p_1, X_1^{-i}), ..., (p_T, X_T^{-i})$.

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Bidder Exposure Calculation (when $U = 0$)

$$\begin{aligned} p_T^i(x_T^i) &= \overline{v}^{-i}(S) - \overline{v}^{-i}(X_T^{-i}) \\ &= p_T S + c^{-i} - QGV_T^{-i}(S) - p_t X_T^{-i} + c^{-i} \\ &= p_T x_T^i - QGV_T^{-i}(S) \end{aligned}$$

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Comments:

- Overcharging since many revealed preference constraints are ignored. It goes away as the density of price trajectory increases.
- Does not require full divisibility assumption made in Theorem 4.

Alternative Activity Rules for Supplementary Round

Definition: Simple Cap Activity Rule

In the Supplementary Round, the bid amount for any nonnull package $x \in X$ cannot exceed the value of the associated maximal valuation $\overline{v}^i(x)$ for this package, i.e.,

$$b^i(x) \leq CAP^i(x) = \overline{v}^i(x)$$

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Definition: Relative Cap Activity Rule

In the Supplementary Round, the bid amount for any nonnull package $x \in X$ should satisfy revealed preference constraints with respect to all clock rounds where package x was more expensive than the chosen package and also should be less than the associated maximal valuation $\overline{v}^i(x)$, i.e.,

$$b^i(x) \leq CAP^i(x) = \min\{\overline{v}^i(x), \min_{t \in \{1,\ldots,T\}: p_t x \geq p_t x_t^i} \{b^i(x_t^i) + p_t(x - x_t^i)\}\}$$

Conclusion:

- A new class of activity rules based on GARP/QGARP
  - the strongest rule in a class of activity rules that permit straightforward bidding
  - fixes many problems of currently used activity rules
  - allows various relaxations to account for "bounded" optimization faced by bidders
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- Enhanced Combinatorial Clock Auction (ECCA)
  - utilizes new activity rule to calculate bidder exposure
  - implementable bid discount approach based on bidder exposure calculation
  - converts CCA into an *iterative first-price auction*
  - restoration of "price discovery"