Learning whether other Traders are Informed

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Abstract

Standard rational expectations (RE) models of financial markets assume that market participants know whether other traders are informed, and therefore *know* how to update their beliefs using prices. We relax this assumption to study return dynamics when investors *learn* how to use the information in prices. In this setting, the price is a non-linear function of the underlying signal, and, as a result, expected returns and volatility are stochastic and persistent, even when fundamental shocks are i.i.d. Moreover, the price reaction to information about dividends is asymmetric: the price reacts stronger to bad news than it does to good news. In fact, the price can even decrease following positive signals about dividends. The model also generates volatility clustering in which surprises in return realizations (or dividend realizations) are followed by higher volatility and higher expected returns.

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1 Introduction

Standard rational expectations (RE) models assume that all participants know whether other agents in the market are informed. This implies that all agents, including those who are uninformed about fundamentals, understand how informative prices are about fundamentals.[1] Arguably, this requires an unrealistic degree of sophistication on the part of uninformed investors. That is, it seems unlikely that investors who are *uncertain* about fundamentals, know with *certainty* whether other investors are privately informed. As such, the standard RE approach is unable to study return dynamics when investors *learn* how to use the information in prices to update their valuations.

We develop a dynamic framework to address this question. There are two groups of investors: the uninformed and the potentially informed. The potentially informed investors receive a signal in each period which they believe is informative about next period's dividends. The uninformed investors use prices and realized dividends to learn about whether the potentially informed investors are actually informed and, consequently, to update their beliefs about future dividends. As such, the uninformed investors in our model *learn* about whether the price is informative about fundamentals.[2]

Unlike traditional RE models, the price is non-linear in the aggregate information available to investors. This implies that expected returns and the volatility in returns are stochastic and persistent, even though all fundamental shocks are assumed to be homoskedastic and independent over time. The model predicts that the price reacts more strongly to negative news than it does to positive news. Moreover, the model can generate volatility clustering in which large (positive or negative), unexpected return realizations in the current period are followed by higher return volatility, and higher expected returns, in the next period.

The intuition for these results follows from how the uninformed investors update their beliefs. Conditional on the signal being informative, the uninformed investor's inference problem is linear — a positive surprise (i.e., good news) leads to a higher conditional expectation about fundamentals, and a negative surprise (i.e., bad news) leads to a lower conditional expectation. However, since the uninformed investor is unsure about whether

[1] For instance, in Kyle (1985), the uninformed market maker knows that there exists an informed strategic trader as well as the precision of that trader's private information. Similarly, in Grossman and Stiglitz (1980), uninformed investors know the number of informed investors in the market and the precision of their signals.

[2] In our setting, the presence of of noise traders is unnecessary for generating non-trivial outcomes. Although the price fully reveals the signal of the potentially informed, uninformed investors are unsure whether the signal conveys payoff relevant information and trade occurs due to the lack of a common prior. As we discuss in Remark 1, one can re-interpret the model as one in which investors share a common prior, but uninformed investors are uncertain about potentially informed investors trading motives. In this case, the uninformed investors must learn about whether the potentially informed investor is an informed investor or a liquidity trader.

the signal is informative, a surprise in the signal (in either direction) increases the uninformed investor's uncertainty, or posterior variance, about fundamentals. As a result, the posterior variance of uninformed investors is stochastic, which, in turn, implies that expected returns and volatility in returns are stochastic.

Due to the multi-dimensionality of the uninformed investor's learning process, prices react asymmetrically to good news versus bad news. When there is a negative surprise, the uninformed investor's conditional expectation is lower and her conditional variance is higher, both of which lead to a decrease in the price. However, when there is a positive surprise, the conditional expectation is higher but so is the conditional variance, and these have off-setting effects on the price. As a result, prices respond more to negative surprises, or bad news, than they do to good news. In the extreme, when the magnitude of positive surprise in the signal is small relative to the overall risk concerns, the effect on the conditional variance can dominate the effect on the conditional expectation, and the price reaction to additional good news is negative.

Volatility clustering follows from how the uninformed investor updates her beliefs in response to realized dividends. Since the uninformed investor forms her conditional expectation of next period's dividends based on the potentially informed investor's signal, a dividend realization that is far from her conditional expectation (i.e., a large surprise) leads her to revise her belief about the informativeness of the signal downwards. In other words, big surprises in dividend realizations, which are accompanied by large surprises in returns, reduce the likelihood that the potentially informed investors are actually informed. This, in turn, can increase the uninformed investor's uncertainty about fundamentals, which leads to higher volatility and higher expected returns in future periods.

While the majority of the RE literature has focused on linear-normal equilibria, a number of papers, including Ausubel (1990), Foster and Viswanathan (1993), Rochet and Vila (1994), DeMarzo and Skiadas (1998), Barlevy and Veronesi (2000), Spiegel and Subrahmanyam (2000), Breon-Drish (2010), and Albagli, Hellwig, and Tsyvinski (2011), have explored the effects of relaxing the assumption that fundamental shocks and signals are normally distributed. Our paper contributes to this literature by developing a model in which non-normality in signals is endogenously generated. Specifically, even though fundamental shocks and signal errors are normally distributed in our model, since the uninformed investor is uncertain about whether other investors are informed, her beliefs about the price signal are given by a mixture of normals distribution.

Within the microstructure literature, our work is related to Li (2011), who considers an similar generalization of Back (1992), which itself is a continuous time version of the Kyle (1985) model. Li (2011) develops a finite-horizon model in which a risk-neutral, competitive

market maker must learn about whether the strategic trader is informed or not, based on the total market order flow. This implies that the market maker's filtering problem is multi-dimensional, and, as in our model, the resulting pricing rule is nonlinear. Our analysis differs both in its motivation and execution. Whereas Li (2011) focuses on primarily the market microstructure implications (e.g., market depth, insider's profit), we focus primarily on the asset pricing implications. Importantly, since our model considers risk-averse investors, we are able to analyze the effect of the uninformed investor's non-linear learning problem on risk-premia and expected returns.

Finally, our model contributes to the differences of opinion (DO) literature, which has been important in generating empirically observed features of price and volume dynamics (e.g., Harrison and Kreps, 1978; Harris and Raviv, 1993; Kandel and Pearson, 1995; Scheinkman and Xiong, 2003; Banerjee and Kremer, 2010). With the exception of Banerjee, Kaniel, and Kremer (2009), the DO models in the literature have largely ignored the role of learning from prices, since investors agree to disagree about fundamentals, and therefore find the information in the price irrelevant.[3] In our model, investors may exhibit differences of opinion (since all potentially informed investors believe their signals are payoff relevant), but uninformed investors still condition on prices to update their beliefs about fundamentals. In this sense, our model bridges the gap between the RE and DO approaches.

2 Model Setup and Equilibrium

2.1 Payoffs and Preferences

There are two assets: a risk-free asset and a risky asset. The gross risk-free rate is normalized to $R \equiv 1 + r > 1$. The risky asset pays a stream of dividends $d_t \sim \mathcal{N}(\mu, \sigma^2)$, which are i.i.d., and investors observe the realization of d_t at date t. The aggregate supply of the risky asset is constant and equal to Z. Denote the price of the risky asset at date t by P_t, and denote the dollar return on the risky asset by Q_{t+1}, where

$$Q_{t+1} = P_{t+1} + d_{t+1} - RP_t.$$

Investor i has mean-variance preferences over next period's wealth. In particular, she submits a limit order, $x_{i,t}$, such that

[3] In the DO model of Banerjee et al. (2009), investors agree to disagree but use the price to update their beliefs about higher order expectations, which is useful for them to speculate against each other.

$$x_{i,t} = \arg\max_{x} \mathbb{E}_{i,t}[W_{i,t}R + xQ_{t+1}] - \tfrac{\alpha}{2}\text{var}_{i,t}[W_{i,t}R + xQ_{t+1}], \tag{1}$$

and where $\mathbb{E}_{i,t}[\cdot]$ and $\text{var}_{i,t}[\cdot]$ denote her conditional expectation and variance, respectively, given date t information, $W_{i,t}$ denotes her wealth, α denotes her risk-aversion, and $x_{i,t}$ denotes her trading position in the risky asset at date t. This implies that her optimal demand for the risky asset is given by

$$x_{i,t} = \frac{\mathbb{E}_{i,t}[Q_{t+1}]}{\alpha \text{var}_{i,t}[Q_{t+1}]} = \frac{\mathbb{E}_{i,t}\left[P_{t+1} + d_{t+1} - RP_t\right]}{\text{var}_{i,t}\left[P_{t+1} + d_{t+1}\right]} \tag{2}$$

The optimal demand is analogous to an investor's demand in an overlapping generations (OLG) model (e.g., Spiegel, 1998; Banerjee, 2011), but as discussed below, some investors have persistent beliefs across periods. The assumption is made primarily for tractability, but retains the key feature of a dynamic framework — namely, an investor's optimal portfolio depends not only on her beliefs about the fundamental dividend, but also on her beliefs about future prices.

2.2 Information and Beliefs

There are two (groups of) investors: the uninformed (denoted by U) and the potentially informed (denoted by θ). Investor θ is either informed (i.e., $\theta = I$) or not informed (i.e., $\theta = NI$), where the prior probability of being informed is $\pi_0 \equiv \Pr(\theta = I)$. Specifically, at date t, investor θ receives a signal $S_{\theta,t}$ of the form:

$$S_{\theta,t} = \begin{cases} d_{t+1} + \varepsilon_t & \text{if} \quad \theta = I \\ u_{t+1} + \varepsilon_t & \text{if} \quad \theta = NI, \end{cases}$$

where $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ and $u_{t+1} \sim \mathcal{N}(0, \sigma^2)$ are independent of d_{t+1}.

It is convenient to parameterize the information quality of the I investor's signal (i.e., $S_{I,t}$) by the Kalman gain, λ, where

$$\lambda \equiv \frac{\text{cov}[S_{I,t}, d_{t+1}]}{\text{var}[S_{I,t}]} = \frac{\sigma^2}{\sigma^2 + \sigma_\varepsilon^2}.$$

Note that λ is decreasing in the noise of the signal (i.e., σ_ε^2) and takes values between zero and one. When $\lambda = 0$, the signal $S_{I,t}$ is completely uninformative; investors learn nothing about future dividends by observing it. Conversely, when $\lambda = 1$, $S_{I,t}$ perfectly reveals the

realization of next period's dividend.

Investor U does not know θ and so is unsure whether the other investors have payoff relevant information about the asset. However, she has rational expectations about the joint distribution of signals and fundamentals. Since there are no additional sources of noise, in equilibrium, U will be able to infer $S_{\theta,t}$ from P_t and use this to update her beliefs about fundamentals. Let π_t denote the probability that investor U attributes to investor θ being informed at date t: $\pi_t \equiv \Pr_{U,t}(\theta = I)$. Then, investor U's conditional beliefs about the value d_{t+1} next period are given by

$$\mathbb{E}_{U,t}[d_{t+1}] = \pi_t \lambda S_{\theta,t}, \text{ and} \tag{3}$$

$$\text{var}_{U,t}[d_{t+1}] = \pi_t \sigma^2(1-\lambda) + (1-\pi_t)\sigma^2 + \pi_t(1-\pi_t)(\lambda S_{\theta,t})^2. \tag{4}$$

Investor U will also update her beliefs about whether θ is informed conditional on the realization of d_{t+1} using Bayes rule. In particular, the posterior probability of $\theta = I$, conditional on a realization of d_{t+1}, is given by

$$\pi_{t+1} = \frac{\pi_t \Pr(S_{\theta,t}, d_{t+1} | \theta = I)}{\pi_t \Pr(S_{\theta,t}, d_{t+1} | \theta = I) + (1-\pi_t)\Pr(S_{\theta,t}, d_{t+1} | \theta = NI)} = \frac{\frac{\pi_t}{\sigma_\varepsilon}\phi\left(\frac{S_{\theta,t} - d_{t+1}}{\sigma_\varepsilon}\right)}{\frac{\pi_t}{\sigma_\varepsilon}\phi\left(\frac{S_{\theta,t} - d_{t+1}}{\sigma_\varepsilon}\right) + \frac{1-\pi_t}{\sqrt{\sigma^2+\sigma_\varepsilon^2}}\phi\left(\frac{S_{\theta,t} - 0}{\sqrt{\sigma^2+\sigma_\varepsilon^2}}\right)}, \tag{5}$$

where $\phi(\cdot)$ is the probability distribution function for a standard normal random variable.

For simplicity, we assume that investor θ believes that her signal is informative, and in particular, this implies that investor $\theta = NI$ has *incorrect* beliefs. This assumption is made for tractability since it implies that the conditional beliefs of the potentially informed investor are symmetric across type. For $\theta \in \{I, NI\}$, investor θ's conditional beliefs about the value d_{t+1} next period are given by

$$\mathbb{E}_{\theta,t}[d_{t+1}] = \lambda S_{\theta,t}, \quad \text{and} \quad \text{var}_{\theta,t}[d_{t+1}] = \sigma^2(1-\lambda). \tag{6}$$

As Remark 1 suggests, there are equivalent specifications with alternative trading motives that avoid the assumption of heterogeneous prior beliefs.

Remark 1. *In our model, investors exhibit differences of opinion. This lack of a common prior generates trade (e.g., Milgrom and Stokey, 1982). One could instead interpret the model as one in which investors share a common prior, but θ investors may trade for liquidity reasons. In particular, suppose that investor θ could either be informed (i.e., $\theta = I$), or a liquidity trader (i.e., $\theta = L$), where the informed trader is identical to the type described above. Instead of receiving a signal about the asset, the liquidity trader expects to receive an*

endowment shock χ_{t+1} *at date* $t+1$ *if trading ends, where* $\text{cov}_t(\chi_{t+1}, d_{t+1}) = \tilde{S}_{NI,t}$ *is known at date* t*. Moreover, suppose investor* L*'s risk aversion coefficient is normalized to* $\alpha(1-\lambda)$*. Then, if we set the distribution of* $\tilde{S}_{NI,t}$ *to* $\tilde{S}_{NI,t} = -\frac{\lambda}{\alpha\sigma^2(1-\lambda)}(u_{t+1} + \varepsilon_t)$*, then the optimal demand of investor* $\theta = L$ *is identical to the optimal demand of* $\theta = NI$ *described above.*

2.3 Characterizing the Equilibrium

We define the equilibrium as follows.

Definition 1. *The equilibrium consists of the price* P_t *of the risky asset, investor demands* $x_{i,t}$*, and investor beliefs such that:*

1. *Investor* i*'s demand,* $x_{i,t}$ *is optimal given her beliefs and preferences; i.e.,* $x_{i,t}$ *solves equation* (1)*,*
2. *Investor* i*'s posterior beliefs are given by Bayes rule, conditional on her prior beliefs; i.e., beliefs are characterized by equations* (3)–(6)*,*
3. *The price* P_t *clears the market for the risky asset; i.e.,*

$$x_{\theta,t} + x_{U,t} = Z. \tag{7}$$

Our first result establishes the existence of an equilibrium and provides a characterization.

Proposition 1. *There exists an equilibrium. Investor* i*'s optimal demand is given by expression* (2)*, investor beliefs are given by*

$$\begin{aligned} \mathbb{E}_{U,t}[d_{t+1}] &= \pi_t \lambda S_{\theta,t}, \quad \mathbb{E}_{\theta,t}[d_{t+1}] = \lambda S_{\theta,t}, \\ \text{var}_{U,t}[d_{t+1}] &= \sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)(\lambda S_{\theta,t})^2, \text{ and } \text{var}_{\theta,t}[d_{t+1}] = \sigma^2(1-\lambda) \end{aligned}$$

and the price of the risky asset is given by

$$P_t = \tfrac{1}{R}\left(\bar{\mathbb{E}}_t[P_{t+1} + d_{t+1}] - \alpha\kappa_t \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]Z\right), \tag{8}$$

where $\bar{\mathbb{E}}_t[\cdot] \equiv \kappa_t \mathbb{E}_{\theta,t}[\cdot] + (1-\kappa_t)\mathbb{E}_{U,t}[\cdot]$*, and* κ_t *and* λ *are given by*

$$\kappa_t = \frac{\text{var}_{U,t}[P_{t+1} + d_{t+1}]}{\text{var}_{U,t}[P_{t+1} + d_{t+1}] + \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]} \quad \text{and} \quad \lambda = \frac{\sigma^2}{\sigma^2 + \sigma_\varepsilon^2}. \tag{9}$$

The equilibrium price has an intuitive form; it is given by a weighted average of investors' conditional expectations about future payoffs, adjusted for a risk-premium. The weight

of each investor's expectation in $\bar{\mathbb{E}}_t[\cdot]$ depends on the conditional variance of her beliefs relative to those of the others.[4] The price reflects only the market expectation (i.e., $P_t = \frac{1}{R}\bar{\mathbb{E}}_t[P_{t+1} + d_{t+1}]$) if either investors are risk-neutral (i.e., $\alpha \to 0$), or, if the aggregate supply of the risky asset is zero (i.e., $Z = 0$).

The characterization of the price in Proposition 1 implies that (dollar) returns can be expressed as

$$Q_{t+1} = P_{t+1} + d_{t+1} - \bar{\mathbb{E}}_t[P_{t+1} + d_{t+1}] + \alpha\kappa_t \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]Z.$$

Since the U investor has rational expectations, we can condition down from her information set to characterize expected returns and volatility in returns from the perspective of the econometrician. Conditional on the econometrician's information set $\mathcal{I}$, expected returns can be decomposed into an expectations component and a risk-premium component as follows:

$$\mathbb{E}[Q_{t+1}|\mathcal{I}] = \mathbb{E}[P_{t+1} + d_{t+1} - \bar{\mathbb{E}}_t[P_{t+1} - d_{t+1}] + \alpha\kappa_t \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]Z|\mathcal{I}], \tag{10}$$

$$= \underbrace{\mathbb{E}\left[\kappa_t(\mathbb{E}_{U,t}[P_{t+1} + d_{t+1}] - \mathbb{E}_{\theta,t}[P_{t+1} - d_{t+1}])|\mathcal{I}\right]}_{\text{expectations}} + \underbrace{E[\alpha\kappa_t \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]Z|\mathcal{I}]}_{\text{risk premium}}, \tag{11}$$

Equation (11) highlights two key difference between our results and the standard linear models. First, in most linear models with normally distributed payoffs, the conditional variance of investors' beliefs is deterministic, and thus that the risk-premium component of expected returns is deterministic. In contrast, since investor U's conditional variance of d_{t+1} depends on the realization of $S_{\theta,t}$, the conditional risk-premium in our model is stochastic. Second, unlike linear rational expectations models, the expectations component of expected returns need not always be zero in our model. In rational expectations models, since *every* investor's beliefs satisfies the Law of Iterated Expectations, and investors share a common prior, investors cannot disagree conditional on the same information set, and so the expectations component is zero. In our model, since NI investors have incorrect beliefs, investors do not share a common prior, and as a result, can agree to disagree. Hence, the expectations component of expected returns need not be zero provided $\pi_t < 1$. The expectations component essentially reflects the component of expected returns that is driven by disagreement across investors.[5]

Similarly, the Law of Total Variance can be used to condition down to the econometri-

[4]Specifically, the weight on investor i is given by the precision of her conditional beliefs divided by the sum of the precisions of all investors, i.e., $\kappa_{i,t} = \frac{1/\text{var}_{i,t}[P_{t+1}+d_{t+1}]}{\sum_j 1/\text{var}_{j,t}[P_{t+1}+d_{t+1}]}$.

[5] Under the liquidity trader interpretation of the model described in Remark 1, $S_{\theta,t}$ represents liquidity shocks to the demand of $\theta = L$ traders, which leads to predictability in returns. See Banerjee et al. (2009) for a more general discussion of this result and of the role of differences of opinion and noise traders in generating predictability in returns.

cian's information set. In particular, the variance in returns can be expressed as the sum of the expected conditional variance and the variance of the conditional expected return i.e.,

$$\text{var}\left[Q_{t+1}|\mathcal{I}\right] = \mathbb{E}[\text{var}_{U,t}\left[P_{t+1}+d_{t+1}\right]|\mathcal{I}] + \text{var}\left[\mathbb{E}_{U,t}[P_{t+1}+d_{t+1}-RP_t]|\mathcal{I}\right]. \tag{12}$$

The decompositions in equations (11) and (12) will be useful in characterizing the properties of conditional and unconditional expected returns and variance in returns for the benchmark models in the next subsection and for the general model in Sections 3 and 4.

2.4 Benchmark Cases

In this subsection, we consider two natural benchmarks which arise as special cases of the model. In both benchmarks, U is *not* uncertain about θ. This exercise helps us to isolate this uncertainty as the key feature driving the main results in later sections. First, we characterize the equilibrium for the case in which $\pi_t = 0$. In this case, U investors do not condition on the price when updating their beliefs about the fundamental value of the asset and thus it is analogous to a Walrasian setting (or a standard difference of opinions setting). Next, we consider the other extreme, when $\pi_t = 1$. This setting is analogous to a standard rational expectations environment. In both of these special cases, and in contrast to the general model, the equilibrium price is linear in the signal $S_{\theta,t}$ of the θ investor.

Corollary 1. *Suppose $\pi_0 = 0$, and all θ investors are not informed. Then, in the unique equilibrium, investor i's optimal demand is given by expression (2), the price of the risky asset is given by*

$$P_t = A_0 S_{\theta,t} + B_0, \tag{13}$$

where $B_0 = -\frac{1}{r}\kappa_t var_{\theta,t}[P_{t+1}+d_{t+1}]\alpha Z$ and A_0 is the unique real root to the following cubic equation:

$$RA_0\left(2A_0^2 + (2-\lambda)\lambda\right) - \lambda\left(A_0^2+\lambda\right) = 0. \tag{14}$$

Investor beliefs are given by

$$\begin{aligned}
\mathbb{E}_{U,t}[P_{t+1}+d_{t+1}] &= B_0,\\
\mathbb{E}_{\theta,t}[P_{t+1}+d_{t+1}] &= B_0 + \lambda S_{\theta t},\\
\text{var}_{U,t}[P_{t+1}+d_{t+1}] &= A_0^2\left(\sigma^2+\sigma_\varepsilon^2\right)+\sigma^2, \text{ and}\\
\text{var}_{\theta,t}[P_{t+1}+d_{t+1}] &= A_0^2\left(\sigma^2+\sigma_\varepsilon^2\right)+\sigma^2(1-\lambda),
\end{aligned}$$

where λ and κ_t are given by equation (9). Since $\mathbb{E}_t[d_{t+1}|S_{\theta,t}] = 0$, conditional expected returns and variance in returns are given by

$$\mathbb{E}_t[Q_{t+1}|S_{\theta,t}] = \kappa_t \alpha \text{var}_t\left[Q_{t+1}|S_{\theta,t}\right] Z - RA_0 S_{\theta,t} \tag{15}$$

$$\text{var}_t[Q_{t+1}|S_{\theta,t}] = A_0^2\left(\sigma^2 + \sigma_\varepsilon^2\right) + \sigma^2 = \text{var}_{U,t}[P_{t+1} + d_{t+1}] \tag{16}$$

Though θ investors are not informed, they believe they have payoff relevant information. As a result, the price responds to realizations of $S_{\theta,t}$. However, since the signals are spurious, prices are expected to mean-revert in the next period, and this induces mean reversion in expected returns through the $-RA_0 S_{\theta,t}$ term in expression (15).[6] Finally, since the equilibrium is linear, the risk-premium component of expected returns and the conditional volatility of returns are constant.

Corollary 2. *Suppose $\pi_0 = 1$ and all θ are informed. Then, in the unique equilibrium, investor i's optimal demand is given by expression (2), the price of the risky asset is given by*

$$P_t = A_1 S_{\theta,t} + B_1,$$

where $A_1 = \frac{\lambda}{R}$, and $B_1 = -\frac{1}{2r}\alpha \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]Z$. Investor beliefs are given by

$$\mathbb{E}_{U,t}[P_{t+1} + d_{t+1}] = \mathbb{E}_{\theta,t}[P_{t+1} + d_{t+1}] = B_1 + \lambda S_{\theta,t}, \text{ and}$$
$$\text{var}_{U,t}[P_{t+1} + d_{t+1}] = \text{var}_{U,t}[P_{t+1} + d_{t+1}] = A_1^2(\sigma^2 + \sigma_\varepsilon^2) + \sigma^2(1 - \lambda),$$

where λ and κ_t are given by equation (9). Since $\mathbb{E}_t[d_{t+1}|S_{\theta,t}] = \lambda S_{\theta,t}$, conditional expected returns and variance in returns are given by

$$\mathbb{E}_t[Q_{t+1}|S_{\theta,t}] = \tfrac{1}{2}\alpha \text{var}_t[Q_{t+1}|S_{\theta,t}]Z \tag{17}$$

$$\text{var}_t[Q_{t+1}|S_{\theta,t}] = A_1^2(\sigma^2 + \sigma_\varepsilon^2) + \sigma^2(1 - \lambda) = \text{var}_{U,t}[P_{t+1} + d_{t+1}] \tag{18}$$

When all θ investors are informed, prices respond efficiently to the information in $S_{\theta,t}$. As a result, expected returns are constant, and reflect only the risk-premium that investors require for holding the risky asset. The conditional volatility of returns is also constant, since the equilibrium is linear in $S_{\theta,t}$.

[6] As mentioned earlier, under the liquidity trader interpretation of the model described in Remark 1, $S_{\theta,t}$ represents i.i.d. liquidity shocks to the demand of $\theta = L$ traders, which lead to temporary price pressure and mean-reversion in expected returns.

3 The Static Setting

In this section, we focus on the static version of the model. This allows us to solve for equilibrium prices in closed form and develop the underlying intuition for the model more transparently. In Subsection 3.1, we characterize how the equilibrium price changes as a function of the underlying parameters. In Subsection 3.2, we describe the model's predictions for observables like expected returns and volatility.

3.1 Properties of the price function

We begin with an explicit characterization of the price function. Note that since there is only one trading period, U investors do not update their beliefs about π_t, and so $\pi_t = \pi_0$. For notational consistency, we retain the t subscript, which can be interpreted as time zero.

Proposition 2. *In the static model, the equilibrium price is given by*

$$P_t = \tfrac{1}{R}\left(\left(\pi_t + (1-\pi_t)\kappa_t\right)\lambda S_{\theta,t} - \alpha\sigma^2(1-\lambda)\kappa_t Z\right), \tag{19}$$

where the weight κ_t is given by

$$\kappa_t = \frac{\sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)(\lambda S_{\theta,t})^2}{\sigma^2(1-\lambda) + \sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)(\lambda S_{\theta,t})^2} \in [0,1]. \tag{20}$$

Returns are given by

$$Q_{t+1} \equiv d_{t+1} - RP_t = d_{t+1} - \left(\pi_t + (1-\pi_t)\kappa_t\right)\lambda S_{\theta,t} + \alpha\sigma^2(1-\lambda)\kappa_t Z. \tag{21}$$

The result follows from Proposition 1. As in equation (10), the price can be decomposed into a market expectations component and a risk-premium component, i.e.,

$$P_t = \tfrac{1}{R}\left(\underbrace{\left(\pi_t + (1-\pi_t)\kappa_t\right)\lambda S_{\theta,t}}_{\text{expectations}} - \underbrace{\alpha Z\sigma^2(1-\lambda)\kappa_t}_{\text{risk premium}}\right) \tag{22}$$

Since the risk-aversion coefficient, α, and the aggregate supply of the asset, Z, affect the risk-premium component, but not the expectations component, the product of these parameters determine the relative role of each component in the price. On the one hand, when risk aversion is low or the aggregate supply of the asset is small, the price is primarily driven by the expectations component. On the other hand, when risk aversion is high, or the aggregate supply of the asset is large, the risk-premium component drives the price. As such, it is useful to characterize separately how each component of the price depends on the

underlying parameters. The following corollary presents these results.

Corollary 3. *In the static model:*

(i) The expectations component of the price is increasing in $S_{\theta,t}$, and increasing in both λ and π_t for $S_{\theta,t} > 0$, but decreasing in λ and π_t for $S_{\theta,t} < 0$.

(ii) The risk-premium component of the price is hump-shaped in $S_{\theta,t}$ around $S_{\theta,t} = 0$, U-shaped in π_t around $\frac{1}{2}\left(1 - \frac{\sigma^2}{\lambda S_{\theta,t}^2}\right)$, increasing in λ for small $|S_{\theta,t}|$, but decreasing in λ for large $|S_{\theta,t}|$ and $\lambda > 0$.

The intuition for the expectations component is straightforward. The average conditional expectation about d_{t+1} is increasing in the signal $S_{\theta,t}$. Moreover, the average expectation is more sensitive to the signal $S_{\theta,t}$ when either π_t or λ are larger, and so the derivatives of this component with respect to π_t and λ, respectively, depend on the sign of $S_{\theta,t}$.

The risk-premium component of prices depends on the uncertainty that investors face. In particular, note that the risk-premium component is given by

$$-\alpha\sigma^2(1-\lambda)\kappa_t Z = -\alpha\left(\frac{1}{\text{var}_{\theta,t}[d_{t+1}]} + \frac{1}{\text{var}_{U,t}[d_{t+1}]}\right)^{-1} Z, \tag{23}$$

which is increasing in the conditional variance of the U and θ investors. Unlike standard asymmetric information models, since the conditional variance of the uninformed investor is stochastic, the risk-premium component of the price is also stochastic in our model. Recall that the conditional variance $\text{var}_{U,t}[d_{t+1}]$ given in expression (4) is given by

$$\text{var}_{U,t}[d_{t+1}] = \pi_t\sigma^2(1-\lambda) + (1-\pi_t)\sigma^2 + \pi_t(1-\pi_t)(\lambda S_{\theta,t})^2. \tag{24}$$

This implies that $\text{var}_{U,t}[d_{t+1}]$ increases in $|S_{\theta,t}|$, is hump-shaped in π_t around $\frac{1}{2}\left(1 - \frac{\sigma^2}{\lambda S_{\theta,t}^2}\right)$, and decreases in λ for small $|S_{\theta,t}|$, but can increase in λ for large $|S_{\theta,t}|$. Intuitively, large realizations of $|S_{\theta,t}|$ increase the uninformed investor's uncertainty, since she is unsure about whether the signal is informative (since $\pi_t \in (0,1)$). Moreover, in the limit, as $|S_{\theta,t}|$ becomes very large, this uncertainty is highest when π_t is close to $\frac{1}{2}$. Finally, for small realizations of $|S_{\theta,t}|$, a more informative signal (i.e., high λ) reduces the posterior variance (for $\pi_t > 0$), but when $|S_{\theta,t}|$ is large enough, a more informative signal can increase the uninformed investor's uncertainty.

The overall effect of $S_{\theta,t}$ on the price in our model distinguishes it from linear rational expectations and difference of opinions models. Note that the expectations component of price is monotonic in $S_{\theta,t}$, but the risk-premium component is hump-shaped in $S_{\theta,t}$ around zero. This implies that the two components reinforce each other when $S_{\theta,t} < 0$, but offset

each other when $S_{\theta,t} > 0$. In other words, the market reacts *asymmetrically* to news about fundamentals: the price responds more strongly to bad news (i.e., $S_{\theta,t} < 0$) than to good news (i.e., $S_{\theta,t} > 0$).

Since the risk-premium component is bounded by $-\alpha\sigma^2(1-\lambda)Z$, the expectations component dominates when $|S_{\theta,t}|$ is large enough and the price. However, for $S_{\theta,t}$ small enough, the risk-premium component dominates. This means that for small, good news (i.e., small $S_{\theta,t} > 0$), if the overall risk concerns in the market are large enough (i.e., αZ is large relative to $S_{\theta,t}$), the price actually *decreases* as the news gets better. Intuitively, when investors are very risk-averse or the aggregate supply of the asset is large, a small, positive surprise about fundamentals has a bigger impact on prices through the uncertainty it generates for uninformed investors than through its effect on the market's expectations.

To summarize, the uncertainty that uninformed investors face about whether θ investors are informed leads to prices that react more strongly to bad news than to good news. Moreover, when the surprise is positive, but small, the price can decrease even when the news is good. These results are consistent with empirical evidence documented in the literature. For instance, using a sample of voluntary disclosures, Skinner (1994) documents that the price reaction to bad news is, on average, twice as large as that for good news.

3.2 Expected returns and volatility

Given the results from Proposition 2, we now turn to investigating the moments of returns.

Proposition 3. *In the static model, the expected return and volatility, conditional on date t information, is given by*

$$\mathbb{E}_t[Q_{t+1}] = -(1-\pi_t)\lambda\kappa_t S_{\theta,t} + \alpha\sigma^2(1-\lambda)\kappa_t Z, \text{ and} \tag{25}$$

$$\text{var}_t[Q_{t+1}] = \sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)(\lambda S_{\theta,t})^2, \tag{26}$$

respectively. The unconditional expected return and volatility are given by

$$\mathbb{E}[Q_{t+1}] = \sigma^2(1-\lambda)\alpha Z\mathbb{E}[\kappa_t], \text{ and} \tag{27}$$

$$\text{var}[Q_{t+1}] = \sigma^2(1-\pi_t^2\lambda) + (1-\pi_t)^2\lambda^2\text{var}[\kappa_t S_{\theta,t}] + (\sigma^2(1-\lambda)\alpha Z)^2\text{var}[\kappa_t], \tag{28}$$

respectively.

The expression for the conditional expected return is a special case of expression (10). In particular, the conditional expected return consists of an expectations component, $-(1-\pi_t)\lambda\kappa_t S_{\theta,t}$, and a risk-premium component, $\alpha\sigma^2(1-\lambda)\kappa_t Z$. Recall that the expectations component is non-zero in our model since investors exhibit differences of opinion — the

uninformed θ investors (who believe they are informed) incorrectly condition on their signal, and this introduces a mean-reverting shock to expected returns.[7]. However, since the $S_{\theta,t}$ signals are mean-zero and i.i.d., the expectations component does not have an effect on the unconditional expected return.

In a static model, the only source of volatility in returns, conditional on date t information, is the dividend shock d_{t+1}. Hence, the conditional volatility is driven by the conditional variance of d_{t+1}, given the information at date t. The expression for the unconditional volatility of returns given in equation (28) is a special case of equation (12), each term captures a different source of risk,

$$\text{var}[Q_{t+1}] = \underbrace{\sigma^2(1-\pi_t^2\lambda)}_{\text{fundamental}} + \underbrace{(1-\pi_t)^2\lambda^2\text{var}[\kappa_t S_{\theta,t}]}_{\text{expectations}} + \underbrace{(\sigma^2(1-\lambda)\alpha Z)^2\text{var}[\kappa_t]}_{\text{risk premium}}. \tag{29}$$

The first term is the expectation of the conditional variance in returns and so captures the volatility in returns due to uncertainty about next period's fundamental dividend shock d_{t+1}. Note that this uncertainty disappears when both information is perfect (i.e., $\lambda = 1$) and it is commonly known that θ investors are informed (i.e., $\pi_t = 1$). The sum of the second and third terms are the variance of the conditional expectation of returns. The second term of (29) represents the variance in the expectations component of conditional expected returns. As discussed in Section 2.3, the expectations component of conditional expected returns need not be zero in our model since investors exhibit differences of opinion when $\pi_t < 1$ — the second term in (28) reflects the volatility in returns due to variation in this component of expected returns. Finally, the third term is volatility due to variation in the risk-premium component of conditional expected returns, and is zero when information quality is perfect (i.e., $\lambda = 1$). Much like prices, each of these components behaves differently with changes π_t and key parameters of interest. To further investigate these comparative statics, we start by presenting the following results.

Proposition 4. *In the static model,*

(i) The unconditional expected return is homogeneous of degree 1 (HD1) in σ^2 and αZ.

(ii) The unconditional volatility component due to fundamental shocks is HD1 in σ^2 and HD0 in αZ.

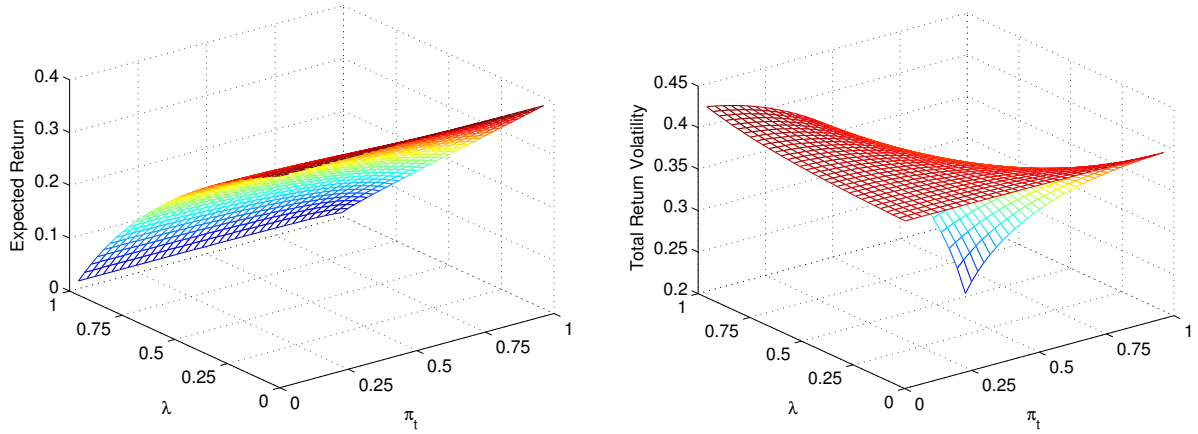
(iii) The unconditional volatility component due to the expectations component of returns is HD1 in σ^2 and HD0 in αZ.

[7] As mentioned earlier, in a noisy rational expectations interpretation of the model, liquidity shocks or noise traders induce a similar mean-reversion in expected returns

(iv) The unconditional volatility component due to the risk premium component of returns is HD2 in σ^2 and αZ.

As expected, (i) implies that unconditional expected returns are increasing in the fundamental volatility and the overall risk concerns in the market (as captured by αZ). Results (ii) through (iv) are also fairly intuitive, but they have important implications for which component drives overall volatility. In particular, when overall concerns about risk in the market are relatively high, the risk premium component is the key driver of overall return volatility. When αZ and σ^2 are relatively small, the first and second components drive overall volatility.

Proposition 4 is also useful for exploring comparative static results with respect to λ and π_t. For example, (i) implies that when exploring how expected returns change with λ and π_t, it is without loss to normalize σ^2 and αZ. By doing so, we are left with a two-dimensional parameter space (i.e., $(\pi_t, \lambda) \in [0,1]^2$), over which the expected return can be plotted to obtain comparative-static results that obtain *for any parameter specification* of the model. Figure 1(a) illustrates the result. It implies that both higher quality information (as measured by λ) and greater likelihood of an informed trader (as measured by π_t) decrease the expected return. This is because both higher quality information and a higher likelihood of an informed trader imply that the price is more informative about the fundamentals, and the uncertainty faced by the uninformed investor is lower.

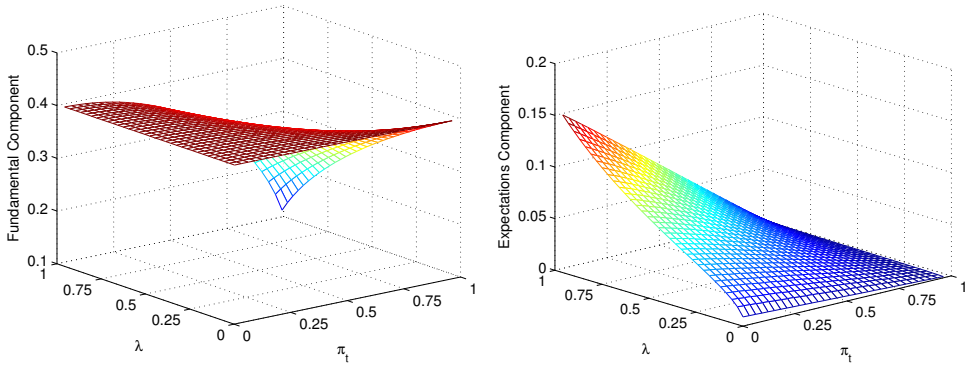


(a) Expected Return (b) Volatility

FIGURE 1: Illustration of expected returns (a) and total volatility (b) as they depend on the quality of information λ and the probability of a θ being informed, i.e., π_t. The other parameters are $\sigma = 40\%$, and $\alpha Z = 3$.

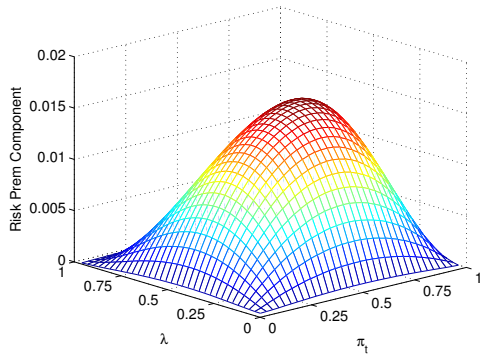
Using (ii) through (iv), we can conduct a similar exercise to to characterize the comparative static effects of each of the individual components of volatility. Figure 2(a) shows the volatility in returns due to fundamental dividend shocks is decreasing in π_t and λ, since an

increase in either parameter reduces the uncertainty that investors face about next period's dividend. Figure 2(b) shows that the variance in the expectations component of conditional expected returns is decreasing in π_t but increasing in λ. Recall that the expectations component of the conditional expected returns is non-zero because investors exhibit differences of opinion, and in particular, because uninformed θ investors believe they are informed. This effect is larger when π_t is smaller (since θ investors are less likely to actually be informed) and when λ is larger (since uninformed θ investors put more weight on their signals), which leads to the effect on volatility. Figure 2(c) shows the risk-premium component of volatility is non-monotonic in both π_0 and λ. This is because the risk-premium component of returns is stochastic only when both λ and π_t are strictly between zero and one. As discussed in Section 2.4, for $\pi_t = 0$ and $\pi_t = 1$, the risk-premium component of expected returns is constant. Similarly, when $\lambda = 1$, the risk-premium is zero, while when $\lambda = 0$, all θ investors are (effectively) uninformed, and so the risk-premium is constant.



(a) Fundamental Component

(b) Expectations Component



(c) Risk-Premium Component

FIGURE 2: The three components of volatility as they depend on the quality of information λ and the probability of a θ being informed, i.e., π_t. Panel (a) plots the fundamental component of volatility (i.e., $\sigma^2(1-\pi_t^2\lambda)$), panel (b) plots the expectations component (i.e., $(1-\pi_t)^2\lambda^2\text{var}[\kappa_t S_{\theta,t}]$), and panel (c) plots the risk-premium component (i.e., $(\sigma^2(1-\lambda)\alpha Z)^2\text{var}[\kappa_t]$). The other parameters are set as in Figure 1.

As implied by Proposition 4, the first two components of volatility do not depend on αZ,

but the third does. The sum of the first two components is decreasing in λ when π_t is large but increasing in λ when π_t is small. The effect of λ on the sum of the first two components of volatility is dominated by the fundamental component when $\pi_t = 1$ and by the expectations component when $\pi_t = 0$. Of course, comparative statics on the total return volatility depend on the relative magnitudes of σ^2 and αZ, which determine the relative weight on each component. However, we can summarize the comparative statics on each component as follows: the expected return is decreasing in both π_t and λ; the fundamental component of volatility is decreasing in both π_t and λ; the expectations component of volatility is decreasing in π_t but increasing in λ; the risk-premium component of volatility is hump-shaped in both π_t and λ.

4 The Dynamic Setting

In this section, we extend our analysis to the dynamic setting. Since the model is not tractable analytically, we solve the model numerically using an iterative procedure to compute the equilibrium price function (i.e., the fixed-point of (8)). In addition to demonstrating that key features of the static setting are robust, the analysis offers several new comparative static results. It also allows us to generate testable predictions for the time series of returns, which are discussed in Subsection 4.2.

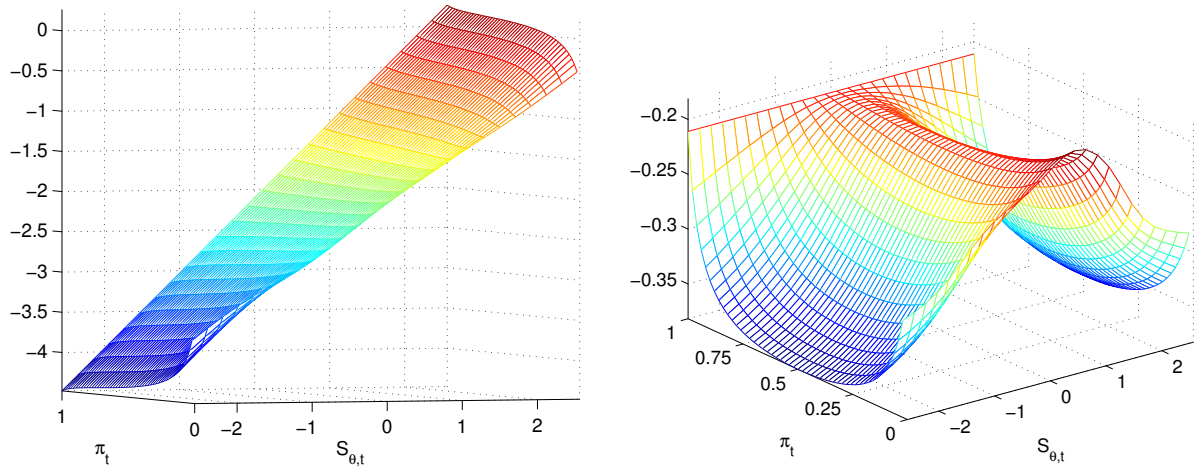
The key insight to understanding the difference between the static and dynamic settings is that in the dynamic setting, the price is affected not only by investors' beliefs about fundamentals, but also their beliefs about future prices. This dynamic consideration reinforces some of the results from the static setting, but overturns others, and is crucial to understanding the intuition behind the model's predictions.[8] It is important to recognize that the degree to which these dynamic considerations and beliefs about future prices affect current prices depend on the extent to which investor discount future payoffs. In particular, as one would expect, return characteristics (i.e., expected returns, volatility) tend toward this in the static model as R grows. However, for our discussion and analysis, we assume a small enough value of R that allows us to highlight the differences in the implications of the dynamic and static models.

4.1 Prices, Expected Returns and Volatility

We begin by discussing the two components of prices as plotted in Figure 3. As in the static setting, the price P_t—the sum of the two components—is a non-linear function of π_t and $S_{\theta,t}$ and is more sensitive to bad news (i.e., negative $S_{\theta,t}$) than it is to good news (i.e., positive

[8] As Banerjee (2011) argues, these dynamic considerations are also crucial in distinguishing standard rational expectations and difference of opinions models based on observable characteristics.

$S_{\theta,t}$). The intuition is the same as before; all else equal, investors' expectation of future dividends, and hence the expectations component of prices as in Figure 3(a), increases in $S_{\theta,t}$. However, a surprise in $S_{\theta,t}$ in either direction also leads to an increase in uncertainty for the U investor, and hence the risk-premium component is hump-shaped in $S_{\theta,t}$ as in Figure 3(b). For negative $S_{\theta,t}$ these two effects reinforce each other, while for positive $S_{\theta,t}$, the effects offset each other, and this leads to the asymmetric reaction of prices to $S_{\theta,t}$. The comparative statics with respect to π_t are familiar from the static case — the expectations component increases in π_t and the risk-premium component is U shaped in π_t.



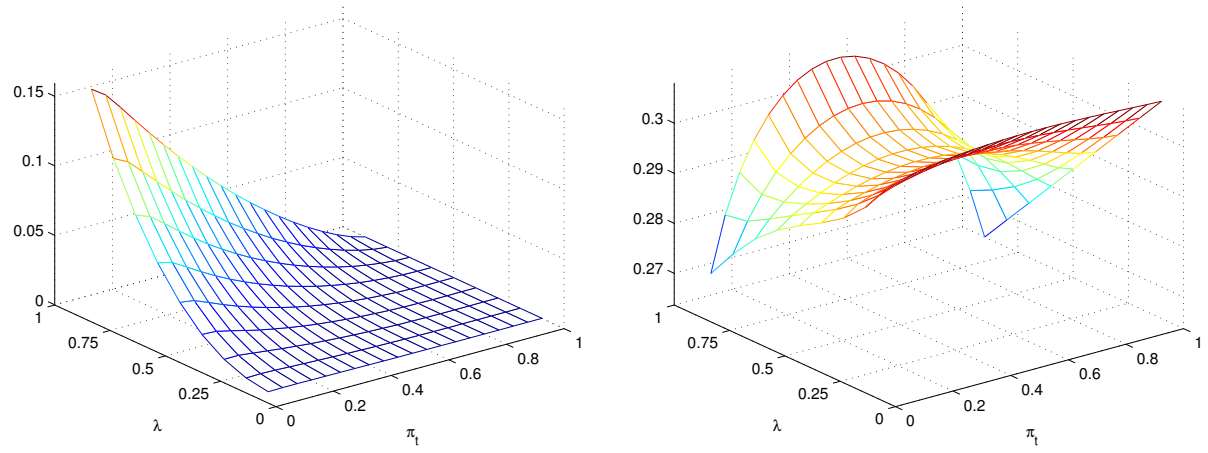
(a) Expectations Component of P_{t+1}

(b) Risk Premium Component of P_{t+1}

FIGURE 3: The two components of the equilibrium price function as they depends on the underlying state variable and the realization of information.

Unconditional expected returns can also be decomposed into an expectations component and a risk-premium component (see (11)). Unlike the result in Proposition 3, in a dynamic setting the expectations component of expected returns is not zero. As Figure 4(a) suggests, this is because the expectations component of expected returns depends not only on the difference in investors' unconditional expectations of d_{t+1} (which is zero, as in the static setting), but also the difference in their beliefs about P_{t+1}, which is not zero. The expectations component of expected returns given in (11) can be interpreted as a measure of disagreement between U and θ investors about future prices, it increases in λ and decreases in π_t. Intuitively, investors disagree more when λ is larger since each puts more weight on their own interpretation of $S_{\theta,t}$, but less when π_t is larger.

The risk-premium component of expected returns, shown in Figure 4(b), also behaves differently in the dynamic model. Recall that in the static setup, the risk-premium component decreases in λ and π_t since better information (i.e., higher λ) and a higher likelihood



(a) Expectations Component of $\mathbb{E}(Q_{t+1})$ (b) Risk Premium Component of $\mathbb{E}(Q_{t+1})$

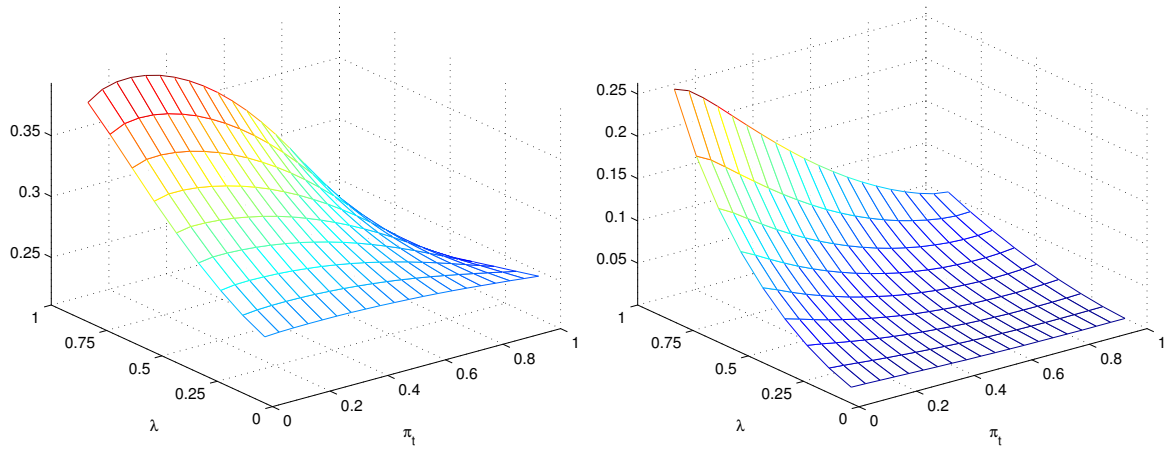
FIGURE 4: The two components of expected returns as they depends on the underlying state variable and the quality of the information.

of θ being informed both lead to less uncertainty about next period's dividend. However, in the dynamic model, the risk-premium component is non-monotonic in both π_t and λ.

The difference in these patterns are a consequence of the fact that in a dynamic model, uncertainty about future prices affects the risk-premium component of expected returns. In particular, while an increase in λ decreases investor uncertainty about next period's dividends, it also makes prices more sensitive to $S_{\theta,t}$ thereby increasing future volatility. Similarly, for a fixed λ, the risk-premium component of the price is most sensitive to $S_{\theta,t}$ for intermediate values of π_t (see Figure 3(b)), and so uncertainty about future prices is higher for intermediate values of π_t. The interaction between dividend uncertainty and price uncertainty lead to the patterns in the risk-premium component. For high and low values of π_t, the effect of λ on dividend uncertainty dominates, and so the risk-premium decreases in λ. For intermediate values of π_t, the effect of λ on dividend uncertainty dominates for low λ but the effect on price uncertainty dominates for high λ, and hence the risk-premium exhibits a U-shape in λ. Finally, all else equal, uncertainty about next period's dividend (for the U investor) and about the risk-premium component of next period's price are highest for intermediate values of π_t, and hence the risk-premium exhibits a hump-shape in π_t.

As in the static case, the relative impact of the expectations component and the risk-premium component depend on fundamental volatility and overall risk concerns (i.e., αZ). Unlike the static setting, expected returns decrease in π_t when the expectations component dominates, but they are hump-shaped in π_t if the risk-premium component dominates.

Recall from equation (12) that using the law of total variance, we can decompose the total volatility (i.e., unconditional variance) of returns into into the expectation of the conditional



(a) Expected Conditional Variance (b) Variance of the Conditional Expectation

FIGURE 5: The two components of volatility as they depends on the underlying state variable and the quality of the information.

variance, as in Figure 5(a), and the variance of the conditional expectation, plotted in Figure 5(b). Much like the analysis of expected returns, beliefs about future prices play an important role. First, note that the conditional variance in returns now depends not only on uncertainty about next period's dividend (as in the static setup) but also on uncertainty about next period's price. While uncertainty about next period's dividend is decreasing in both π_t and λ, uncertainty about next period's dividends can increase in λ (as P_{t+1} becomes more sensitive to $S_{\theta,t+1}$) and is hump-shaped in π_t, as discussed above. As a result, when dynamic considerations are important enough, the expected conditional variance component of the volatility in returns is hump-shaped in π_t and may increase or decrease in λ.

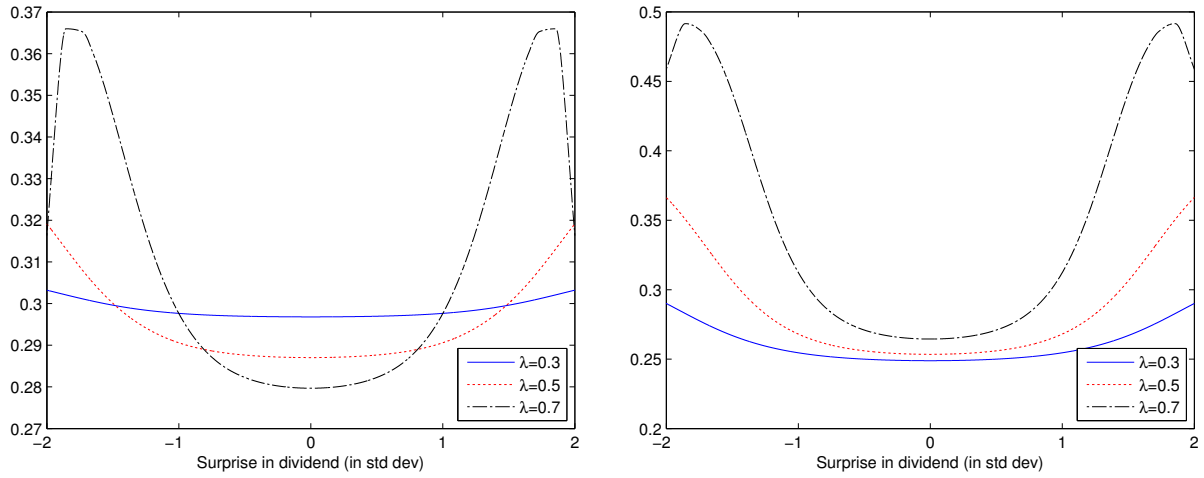
As in the static case, the variance of the conditional expectation of returns is increasing in λ — all else equal, a larger λ increases the sensitivity of investors' expectations to $S_{\theta,t}$ and, therefore, makes them more volatile. The effect of π_t on the variance of the conditional expectation of returns depends on which component dominates. In particular, as equation (11) suggests, the expectations component of conditional expected returns is driven by the disagreement between U and θ investors about expected future dividends and prices, and so the variance of this component decreases in π_t, since investors disagree less. However, the variance of the risk-premium component of expected returns is hump-shaped in π_t since the conditional risk-premium is most sensitive to $S_{\theta,t}$ when π_t is near $\frac{1}{2}$. As a result, if the risk-premium component of conditional expected returns dominates, then the variance of the conditional expectation of returns is hump-shaped in π_t. If, instead, the expectations component of returns dominates, then the variance of the conditional expectation is decreasing in π_t. The overall effect of π_t and λ on the variance of returns depend on the interaction of these components, which in turn, depend on the relative impact of the expectations and risk

premium components of returns.

4.2 Time-Series Implications

The way in which λ and π_t affect the distribution of prices and returns allow us to generate predictions of the model that distinguish it from standard (linear) rational expectations models. In the dynamic model, the probability π_t that U investors attribute to θ investors being informed, is a stochastic (endogenous) state variable of the model. Moreover, π_t is persistent over time.[9]. As a result, in addition to generating *stochastic* expected returns and volatility as in the static model, the dynamic model predicts that these conditional moments are *persistent*.

The dynamic model also implies return realizations should predict future expected returns and volatility in returns. In particular, if $\pi_t < 1$ is large enough, the model predicts *volatility clustering* — return surprises in either direction, driven by dividend surprises, are followed by an increase in volatility of returns and higher expected returns. The intuition for these results follows from the discussion in the earlier section. An unanticipated realization of d_{t+1} leads the U investor to revise her beliefs about θ being informed downwards (i.e., $\pi_{t+1} < \pi_t$)[10] This revision in beliefs generates additional uncertainty for the U investors, and as a result, leads to higher future volatility and higher expected returns going forward (see Figures 4 and 5).



(a) Future Expected Returns

(b) Future Volatility

FIGURE 6: Expected returns and volatility in period $t+1$ as they depend on dividend surprises in period t and the quality of the information. In both figures π_t is set equal to 0.9.

Figure 6 illustrates this clustering effect. Specifically, the figure plots expected returns

[9]From the perspective of the U investor it is a martingale; $\mathbb{E}_{U,t}[\pi_{t+1}] = \pi_t$

[10]This follows from the evolution of π_t in (5) and because π_t is initially large.

and volatility in period $t+1$ as a function of the current realization of d_{t+1} (scaled by its standard error). Note that most realizations of $|d_{t+1} - S_{\theta,t}|$ lead to an increase in both volatility and expected returns. However, sufficiently extreme realizations of $|d_{t+1} - S_{\theta,t}|$ can actually decrease expected returns, since the posterior π_t is almost zero (e.g., for $\lambda = 0.8$).

5 Conclusions

In trying to understand how investors learn to use the information in prices, we relax the standard assumption in RE models that all investors *know* whether other investors are informed. We show that this can have important consequences on the models' predictions on return dynamics. In our model, uninformed investors must use prices and realized dividends to *learn* about whether other investors are informed, and if so, what their information is. The uninformed investor solves a non-linear filtering problem, and this implies that the equilibrium price is a non-linear function of the underlying signal. This non-linearity naturally leads to expected returns and volatility that are stochastic and persistent, even though we assume that shocks to fundamentals and signals are i.i.d. In our model, prices react more strongly to bad news than to good news, and returns exhibit clustering — big return realizations (of either sign) are followed by higher volatility and higher expected returns, a phenomenon that has been well documented in the literature (**?**).

Our analysis provides an early step in understanding how investors may learn to use the information in prices. Although beyond the scope of the current paper, interesting theoretical extensions include exploring the effects of generalizing the dividend process to allow for persistence, generalizing the information structure, and studying the optimal information acquisition strategy of uninformed investors. Another important area for further research is to explore what we can infer about investor preferences, beliefs and learning when we compare the model's predictions about return dynamics (e.g., stochastic volatility persistence and clustering) to empirically observed patterns in asset prices.

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Appendix

Proof of Proposition 1. Optimality of $x_{i,t}$ follows from (2), the expressions for beliefs are given by (3)–(6), and the expression for the price follows from the market clearing condition, i.e., (7). Thus, it is sufficient to show that there exists a price function, $P(x)$, that satisfies the recursive equation given by (8) and where x denotes the current state, i.e., (π, D, S).

Consider the space of all functions $f : \mathbb{R}^3 \to \mathbb{R}$ such that $|f|^2$ is Lesbesgue integrable, that is all $f \in L_2$. Equipped with the typical norm and the standard equivalence relation norm, $(L_2, \|\cdot\|_2)$ forms a Banach Space (Zeidler, 1986, p. 771). Let $T(f)$ denote the operator defined by

$$T(f) = \rho\left(\kappa(f)\mathbb{E}_\theta\left[f(x') + D'|x\right] + (1-\kappa(f))\,\mathbb{E}_U\left[f(x') + D'|x\right] - \kappa(f)\text{var}_{\theta,t}\left[f(x') + D'|x\right]\alpha Z,\right.$$

where

$$\kappa(f(x)) = \frac{\text{var}_{U,t}\left[f(x') + D'|x\right]}{\text{var}_{U,t}\left[f(x') + D'|x\right] + \text{var}_{\theta,t}\left[f(x') + D'|x\right]} \in (0,1)$$

It is straightforward to verify that T is a continuous operator from $M \to M$, where $M \subset L^2$ is nonempty, compact and convex. Hence, we can apply the Schauder Fixed Point theorem (Zeidler, 1986, Theorem 2.A), which proves that their exists a fixed point and hence a stationary equilibrium.

□

Proof of Corollary 1. Given a conjecture for the price of the form $P_t = A_0 S_{\theta,t} + B_0$, the expressions for beliefs are immediate and they imply that the optimal demand for the U investor is given by

$$x_{U,t} = \frac{B_0 - RP_t}{\alpha\left(A_0^2(\sigma^2 + \sigma_\varepsilon^2) + \sigma^2\right)}.$$

and for the θ investor is given by

$$x_{\theta,t} = \frac{B_0 + \lambda S_{\theta,t} - RP_t}{\alpha\left(A_0^2(\sigma^2 + \sigma_\varepsilon^2) + \sigma^2(1-\lambda)\right)}.$$

The market clearing condition implies that

$$RP_t = B_0 + \kappa_t \lambda S_{\theta,t} - \alpha \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]\kappa_t Z$$

and matching terms with the conjectured price function, we get that A_0 solves

$$RA_0 = \frac{\lambda(A_0^2 + \lambda)}{2A_0^2 + (2-\lambda)\lambda}$$

and $B_0 = -\frac{1}{r}\alpha\kappa_t \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]Z$. This implies B solves the following cubic equation:

$$RA_0(2A_0^2 + (2-\lambda)\lambda) - \lambda(A_0^2 + \lambda) = 0$$

Moreover, since the discriminant of the above equation is less than zero, there is one real root of A_0. The above also implies that $0 \leq A_0 \leq \frac{\lambda}{R}$. The expressions for expected returns and volatility in returns follow from plugging in the expression for price and computing the moments. □

Proof of Corollary 2. Given a conjecture for the price of the form $P_t = A_1 S_{\theta,t} + B_1$, the expressions for

beliefs are immediate and they imply that thee optimal demand for both types of investors is given by

$$x_{U,t} = x_{\theta,t} = \frac{B_1 + \lambda S_{\theta,t} - RP_t}{\alpha\left(\sigma^2(1-\lambda) + A_1^2(\sigma^2+\sigma_\varepsilon^2)\right)}.$$

Since beliefs are identical, $\kappa_t = \frac{1}{2}$, and so the market clearing condition implies that

$$RP_t = B_1 + \lambda S_{\theta,t} - \tfrac{1}{2}\alpha \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]Z,$$

and matching terms with the conjectured price function, we get that $A_1 = \frac{\lambda}{R}$, $B_1 = -\frac{1}{r}\alpha\kappa_t \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]Z$. The expressions for expected returns and volatility in returns follow from plugging in the expression for price and computing the moments. □

Proof of Corollary 3. Taking derivatives of κ_t we get

$$\frac{\partial}{\partial\lambda}\kappa_t = \frac{\sigma^2(1-\pi_t)(\pi_t S_{\theta,t}^2(2-\lambda)\lambda+\sigma^2)}{(\sigma^2(1-\lambda)+\sigma^2(1-\pi_t\lambda)+\pi_t(1-\pi_t)(\lambda S_{\theta,t})^2)^2} = \frac{(1-\pi_t)(\pi_t S_{\theta,t}^2(2-\lambda)\lambda+\sigma^2)}{\sigma^2(1-\lambda)^2}(1-\kappa_t)^2 \geq 0 \tag{30}$$

$$\frac{\partial}{\partial\pi_t}\kappa_t = -\frac{\left(\sigma^2-(1-2\pi_t)\lambda S_{\theta,t}^2\right)\lambda(1-\lambda)\sigma^2}{(\sigma^2(1-\lambda)+\sigma^2(1-\pi_t\lambda)+\pi_t(1-\pi_t)(\lambda S_{\theta,t})^2)^2} = -\frac{\left(\sigma^2-(1-2\pi_t)\lambda S_{\theta,t}^2\right)\lambda}{\sigma^2(1-\lambda)}(1-\kappa_t)^2 \tag{31}$$

$$\frac{\partial}{\partial S_{\theta,t}}\kappa_t = \frac{2\pi_t(1-\pi_t)(1-\lambda)\lambda^2\sigma^2 S_{\theta,t}}{(\sigma^2(1-\lambda)+\sigma^2(1-\pi_t\lambda)+\pi_t(1-\pi_t)(\lambda S_{\theta,t})^2)^2} = \frac{2\pi_t(1-\pi_t)\lambda^2 S_{\theta,t}}{\sigma^2(1-\lambda)}(1-\kappa_t)^2 \tag{32}$$

which implies κ_t is U-shaped around $S_{\theta,t} = 0$, and is hump shaped in π_t around $\frac{1}{2}\left(1 - \frac{\sigma^2}{\lambda S_{\theta,t}^2}\right)$.

- Effect of λ: The derivative of the expectations component of P_t with respect to λ is given by

$$\frac{\partial}{\partial\lambda}\left((\kappa_t + (1-\kappa_t)\pi_t)\right)\lambda S_{\theta,t}) = \left(\pi_t + (1-\pi_t)\left(\kappa_t + \lambda\frac{\partial}{\partial\lambda}\kappa_t\right)\right)S_{\theta,t}$$

  and so this component increases with λ for $S_{\theta,t} > 0$ and decreases in λ otherwise. The derivative of risk-premium component is given by

$$\frac{\partial}{\partial\lambda}(-\alpha\sigma^2(1-\lambda)\kappa_t Z) = -\alpha\sigma^2 Z\left((1-\lambda)\frac{\partial}{\partial\lambda}\kappa_t - \kappa_t\right)$$

  The derivative can be positive or negative depending on $|S_{\theta,t}|$. For $|S_{\theta,t}|$ small enough, the derivative is strictly positive, and so the risk-premium component of price increases in λ. But for $|S_{\theta,t}|$ large enough the derivative can be negative for intermediate λ, and so the risk-premium component of price is U-shaped in λ. Therefore, the risk premium is not monotone in λ. However, the risk-premium component is always lower for $\lambda = 0$ than it is for $\lambda = 1$.

- Effect of π_t: The derivative of the expectations component of P_t with respect to π_t is given by

$$\frac{\partial}{\partial\pi_t}\left((\kappa_t + (1-\kappa_t)\pi_t)\right)\lambda S_{\theta,t}) = \left((1-\kappa_t) + (1-\pi_t)\frac{\partial}{\partial\pi_t}\kappa_t\right)\lambda S_{\theta,t}$$

  Inserting the expression from (31) for $\frac{\partial}{\partial\pi_t}\kappa_t$ gives:

$$\left((1-\kappa_t) + (1-\pi_t)\frac{\partial}{\partial\pi_t}\kappa_t\right) = \frac{\sigma^2(1-\lambda)\left(((1-\pi_t)S_{\theta,t}\lambda)^2 + 2(1-\lambda)\sigma^2\right)}{(1-\lambda)\sigma^2 + (1-\pi_t\lambda)\sigma^2 + (\lambda S_{\theta,t})^2(1-\pi_t)} > 0$$

  Therefore, the derivative of the expectations component of prices with respect to π_t has the same sign as $S_{\theta,t}$.

The risk premium component of the price is U-shaped in π_t around $\frac{1}{2}\left(1-\frac{\sigma^2}{\lambda S_{\theta,t}^2}\right)$, since

$$\frac{\partial}{\partial \pi_t}(-\alpha\sigma^2(1-\lambda)\kappa_t Z) = -\alpha\sigma^2(1-\lambda)Z\frac{\partial}{\partial \pi_t}\kappa_t$$

- Effect of $S_{\theta,t}$: The expectations component of P_t is always increasing in $S_{\theta,t}$, since

$$\frac{\partial}{\partial S_{\theta,t}}\left((\kappa_t+(1-\kappa_t)\pi_t)\right)\lambda S_{\theta,t}) = (\kappa_t+(1-\kappa_t)\pi_t))\lambda + (1-\pi_t)\lambda S_{\theta,t}\frac{\partial}{\partial S_{\theta,t}}\kappa_t > 0$$

The risk-premium component of the price is hump-shaped in $S_{\theta,t}$ around zero, since

$$\frac{\partial}{\partial S_{\theta,t}}(-\alpha\sigma^2(1-\lambda)\kappa_t Z) = -\alpha\sigma^2(1-\lambda)Z\frac{\partial}{\partial S_{\theta,t}}\kappa_t$$

This establishes the comparative static results. □

Proof of Proposition 3. Taking the expectation of the right-hand side (RHS) of (21) and using that $\mathbb{E}_t[d_{t+1}] = \pi_0\lambda S_{\theta,t}$, we have

$$\mathbb{E}[Q_{t+1}] = \mathbb{E}[\mathbb{E}_t[Q_{t+1}]] = \alpha\sigma^2(1-\lambda)Z\mathbb{E}[\kappa_t] - (1-\pi_0)\lambda\mathbb{E}[\kappa_t S_{\theta,t}]$$

Thus, it suffices to show that $\mathbb{E}[\kappa_t S_{\theta,t}] = 0$. For this, note that $\kappa_t \cdot S_{\theta,t}$ is an odd-function (of $S_{\theta,t}$) and the distribution of $S_{\theta,t}$ is symmetric around zero. Thus, $\mathbb{E}[\kappa_t S_{\theta,t}|S_{\theta,t}>0] = -\mathbb{E}[\kappa_t S_{\theta,t}|S_{\theta,t}<0]$, which implies $\mathbb{E}[\kappa_t S_{\theta,t}] = 0$.

Volatility of returns is given by

$$\begin{aligned}
\text{var}[Q_{t+1}] &= \mathbb{E}[\text{var}_t[Q_{t+1}]] + \text{var}[\mathbb{E}_t[Q_{t+1}]] \\
&= \mathbb{E}[\sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)(\lambda S_{\theta,t})^2] + \text{var}\left[\alpha\sigma^2(1-\lambda)\kappa_t Z - (1-\pi_t)\kappa_t\lambda S_{\theta,t}\right] \\
&= \sigma^2(1-\pi_t^2\lambda) + (\alpha\sigma^2(1-\lambda)Z)^2\text{var}[\kappa_t] + (1-\pi_t)^2\lambda^2\text{var}[\kappa_t S_{\theta,t}] \\
&\quad - 2\alpha\sigma^2(1-\lambda)\lambda Z(1-\pi_t)\text{cov}(\kappa_t, \kappa_t S_{\theta,t})
\end{aligned}$$

Recall that Stein's Lemma implies that for $Y \sim \mathcal{N}(0,\sigma_Y^2)$, and $g(Y)$ such that $\mathbb{E}[g(Y)Y] < \infty$ and $\sigma_Y^2\mathbb{E}[g'(Y)] < \infty$, we have $\text{cov}(g(Y),X) = \mathbb{E}[g'(Y)]\text{cov}(Y,X)$. This implies

$$\begin{aligned}
\text{cov}(\kappa_t, S_{\theta,t}) &= \mathbb{E}\left[\frac{\partial}{\partial S_{\theta,t}}\kappa_t\right]\text{var}(S_{\theta,t}) \\
\text{var}[\kappa_t S_{\theta,t}] &= \mathbb{E}[\kappa_t^2 S_{\theta,t}^2] - (\mathbb{E}[\kappa_t S_{\theta,t}])^2 \\
&= \text{cov}(\kappa_t^2 S_{\theta,t}, S_{\theta,t}) - \text{cov}(\kappa_t, S_{\theta,t}) \\
&= \left(\mathbb{E}\left[\kappa_t^2 + 2\kappa_t S_{\theta,t}\frac{\partial}{\partial S_{\theta,t}}\kappa_t\right] - \mathbb{E}\left[\frac{\partial}{\partial S_{\theta,t}}\kappa_t\right]\right)\text{var}(S_{\theta,t}) \\
\text{cov}(\kappa_t, \kappa_t S_{\theta,t}) &= \mathbb{E}[\kappa_t^2 S_{\theta,t}] - \mathbb{E}[\kappa_t]\mathbb{E}[\kappa_t S_{\theta,t}] \\
&= \text{cov}(\kappa_t^2, S_{\theta,t}) - \mathbb{E}[\kappa_t]\text{cov}(\kappa_t, S_{\theta,t}) \\
&= \left(\mathbb{E}\left[2\kappa_t\frac{\partial}{\partial S_{\theta,t}}\kappa_t\right] - \mathbb{E}[\kappa_t]\mathbb{E}\left[\frac{\partial}{\partial S_{\theta,t}}\kappa_t\right]\right)\text{var}(S_{\theta,t})
\end{aligned}$$

Since $\frac{\partial}{\partial S_{\theta,t}}\kappa_t(S_{\theta,t}) = -\frac{\partial}{\partial S_{\theta,t}}\kappa_t(-S_{\theta,t})$, we have that $\mathbb{E}\left[\frac{\partial}{\partial S_{\theta,t}}\kappa_t\right] = 0$, and $\mathbb{E}\left[\kappa_t\frac{\partial}{\partial S_{\theta,t}}\kappa_t\right] = 0$. This implies

that volatility can be expressed as:

$$\begin{aligned}\text{var}[Q_{t+1}] &= \sigma^2(1-\pi_t^2\lambda) + (\alpha\sigma^2(1-\lambda)Z)^2\text{var}[\kappa_t] + (1-\pi_t)^2\lambda^2\text{var}[\kappa_t S_{\theta,t}] \\ &= \sigma^2(1-\pi_t^2\lambda) + (\alpha\sigma^2(1-\lambda)Z)^2\text{var}[\kappa_t] + \sigma^2(1-\pi_t)^2\lambda\left(\mathbb{E}\left[\kappa_t^2 + 2\kappa_t S_{\theta,t}\tfrac{\partial}{\partial S_{\theta,t}}\kappa_t\right]\right)\end{aligned}$$

since $\lambda = \sigma^2/\text{var}(S_{\theta,t})$. □

Proof of Proposition 4. It suffices to show that $\mathbb{E}[\kappa_t]$ and $\text{var}[\kappa_t]$ are HD0 in σ^2, while $\text{var}[\kappa_t S_{\theta,t}]$ is HD1 in σ^2. Recall that

$$\kappa_t = \frac{\sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)\lambda^2 S_{\theta,t}^2}{\sigma^2(1-\lambda) + \sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)\lambda^2 S_{\theta,t}^2}$$

and by definition, $\lambda = \frac{\sigma^2}{\sigma^2+\sigma_\epsilon^2}$ and $S_{\theta,t} \sim N(0, \sigma^2+\sigma_\epsilon^2) = N(0, \sigma^2/\lambda)$, we have

$$\mathbb{E}[\kappa_t] = \frac{1}{\sqrt{2\pi\sigma^2/\lambda}}\int_{-\infty}^{\infty} \frac{\sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)\lambda^2 s^2}{\sigma^2(1-\lambda) + \sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)\lambda^2 s^2} \exp\left(\frac{-s^2}{2\sigma^2/\lambda}\right) ds$$

Using a change of variables, by letting $x \equiv \frac{\sqrt{\lambda}}{\sigma}s$, we get that

$$\begin{aligned}E[\kappa_t] &= \frac{1}{\sqrt{2\pi\sigma^2/\lambda}}\frac{\sigma}{\sqrt{\lambda}}\int \frac{\sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_0)\lambda^2\frac{\sigma^2}{\lambda}x}{\sigma^2(1-\lambda) + \sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)\lambda^2\frac{\sigma^2}{\lambda}x} \exp\left(\frac{-x^2}{2}\right) dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{(1-\pi_t\lambda) + \pi_t(1-\pi_t)\lambda x}{(1-\lambda) + (1-\pi_t\lambda) + \pi_t(1-\pi_t)\lambda x} \exp\left(\frac{-x^2}{2}\right) dx \qquad (33)\end{aligned}$$

And clearly (33) is independent of σ. To see that $\text{var}[\kappa_t]$ is also independent of σ, note that same proof as above applies to $E[\kappa_t^2]$.

For $\text{var}\left[\kappa_t S_{\theta,t}\right]$, again using the same change of variables, we have that

$$\begin{aligned}\mathbb{E}[\kappa_t S_{\theta,t}] &= \frac{1}{\sqrt{2\pi\sigma^2/\lambda}}\int_{-\infty}^{\infty} \frac{\sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)\lambda^2 s^2}{\sigma^2(1-\lambda) + \sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)\lambda^2 s^2} s\exp\left(\frac{-s^2}{2\sigma^2/\lambda}\right) ds \\ &= \frac{\sigma}{\sqrt{2\pi}}\int \frac{(1-\pi_t\lambda) + \pi_t(1-\pi_0)\lambda x}{(1-\lambda) + (1-\pi_t\lambda) + \pi_t(1-\pi_t)\lambda x}\frac{x}{\sqrt{\lambda}} \exp\left(\frac{-x^2}{2}\right) dx\end{aligned}$$

which clearly scales with σ and hence $\left(E[\kappa_t S_{\theta,t}]\right)^2$ scales with σ^2. The same change of variables can be used to show that the same is true of $E[(\kappa_t S_{\theta,t})^2]$, which completes the proof. □