Stable Matching in Large Economies*

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Abstract

Complementarities of preferences have been known to jeopardize the stability of two-sided matching markets, and have thus limited the scope for applying a well-functioning centralized mechanism such as Gale and Shapley's deferred acceptance algorithm. Yet they are a pervasive feature in many markets. We seek to overcome this difficulty by considering a large market with a finite number of firms and a continuum of workers. Workers have preferences over firms while firms have preferences over distributions of workers and may exhibit complementarity. We demonstrate that if firm preferences are continuous, then there exists a stable matching even with complementarity. Building on this result, we show that there exists an approximately stable matching in any large finite economy. We apply our analysis to show the existence of stable matchings in probabilistic and time-share matching models.

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1 Introduction

Since the celebrated work by Gale and Shapley (1962), matching theory has advanced to take a center stage in market design and economic theory more generally. In particular, its successful application in medical matching and school choice has changed how these markets are organized in a fundamental way. The matching theory research by now offers a much clearer vision than before about what can be accomplished by centralizing a matching market. For instance, if Gale and Shapley's (student-proposing) deferred acceptance algorithm is employed to match students with schools or workers with firms, we now expect there to be good incentives for participants not to block assignments (stability) and to report their preferences truthfully (strategyproofness).[1]

Stability is particularly important. First, as has been argued by Roth and Sotomayor (1990), stability ensures the long-term sustainability of a mechanism by eliminating the incentives for parties to circumvent the mechanism by side contracting either during or after the market. Second, stability has a desirable fairness implication, by eliminating so called justified envy — envy that cannot be explained away by the preference of the partner of the envied agent. In the school choice application, if schools' preferences rest on the test score or other merit measures of a student, fairness appears to demand no justified envy as a desideratum.

Unfortunately, stable matching is guaranteed only in limited market conditions. It is well known (see Sönmez and Ünver (2010), Hatfield and Milgrom (2005), Hatfield and Kojima (2008), and Hatfield and Kominers (2010) for example) that the existence of a stable matching is not generally satisfied unless the preferences of participants, say firms, are substitutable. Here substitutability means that, roughly speaking, a firm's demand for a worker never grows with more workers being available.[2] In other words, complementarity can lead to the nonexistence of a stable matching.

[1] See Roth (1984) and Roth and Peranson (1999) for application to medical residency matching, and Balinski and Sönmez (1999) and Abdulkadiroglu and Sonmez (2003) for student placement.

[2] More precisely, if a firm does not wish to hire a worker from a set of workers, then it never prefers to hire the worker from a bigger (in the sense of a set inclusion) set of workers.

This is a serious limitation on the applicability of centralized matching mechanisms, since complementarities of preferences are a pervasive feature of many matching markets. Firms often seek to hire workers with complementary skills. For instance, in professional athletic leagues, teams demand athletes that complement one another in skills as well as in the positions they play. Some public schools in New York City seek diversity of their student bodies in their skill levels. Complementarities appear to be an important feature of some colleges' preferences or evaluations toward students. In particular, US colleges tend to exhibit a desire to build a class that is complementary and diverse in terms of their aptitudes, life backgrounds, and demographics.

Unless one gets a handle on complementarities, we would not know how to organize such markets, and the applicability of centralized matching will remain severely limited. The limitation is particularly pertinent for many decentralized markets that may potentially benefit from centralization. College admissions and graduate admissions are an obvious example. Decentralized matching leaves much to be desired in terms of efficiencies and fairness, and of the yield management burden put on the institutions. Even though such markets may benefit from centralization, we are still far from being able to articulate how the centralized mechanism should be designed and what benefit can be achieved from centralization, in the presence of rich preferences the participants exhibit in these markets.

This paper takes a step toward accommodating complementarities and other forms of general preferences. The general impossibility means, however, that the notion of stability needs to be weakened in some way. Our approach is to consider a large market. Specifically, we consider a market which consists of a large number of workers/students on one side and a finite number of firms/colleges with large capacities on the other, and ask whether stability can be achieved in an "asymptotic" sense — i.e., whether participants' incentives for blocking disappears as the size of workers and firms' capacities grow large. Large markets we envision approximate college admissions and other labor markets. Our notion also preserves the motivation behind the original notion of stability: as long as the incentive for blocking is suppressed sufficiently, the instability and fairness concerns will not be so serious as to jeopardize the mechanism.

We first consider a continuum model in which there is a finite set of firms and a continuum of workers. Each worker desires to be matched to at most one firm. Firms have preferences over groups of workers and, importantly, may exhibit complementarities. A matching is a feasible distribution of worker types to each firm. The model generalizes Azevedo and Leshno (2011) who assume responsive preferences (a special case of substitutable preferences) for the firms.

Our first result is that the set of stable matchings is characterized via a fixed point of a mapping defined on a functional space. While a fixed point characterization has appeared in the matching literature (see Adachi (2000), Hatfield and Milgrom (2005), Echenique and Oviedo (2006), among others), our characterization differs in that the space over which the mapping is defined is a functional space, and to invoke a relevant fixed point theorem in such a space, we construct our mapping in a novel and different manner from those used in the matching literature.

Building on the above result, we show that a stable matching exists if firms' choice functions are continuous in the appropriate sense. The continuity of choice functions allows for a rich and general class of preferences including complementarities.[3] Technically, the existence of a stable matching is established by the Fan-Glicksberg fixed point theorem — a generalization of Kakutani's fixed point theorem to functional spaces —, which appears to be novel in the matching literature. Application of the theorem, particularly establishing compactness of the relevant functional space, however, requires a particular novel construction of the relevant mapping in the functional space.

When firm preferences satisfy substitutability, we show that the set of stable matchings forms a lattice. In particular, there exist a stable matching that is most preferred (resp. least preferred) among all stable matchings by every worker (resp. firm) and a stable matching that is most preferred (resp. least preferred) among all stable matchings by every firm (resp. worker). A version of the rural hospital theorem is also established with the addition of an appropriate version of the law of aggregate demand.

[3]For instance, it allows for Leontief-type preferences with respect to alternative types of workers, desiring to hire all types in equal size (or density).

Although our basic model assumes that there are a continuum of workers, we show that there is a sense in which it serves as a legitimate approximation of large finite economies. More specifically we demonstrate that, for any large finite economy that is sufficiently close to our continuum model (in terms of the distribution of worker types and firms' preferences), there exists an approximately stable matching in the sense that the incentives for blocking is arbitrarily small.

Although the basic model assumes that firm preferences are strict, our framework can be extended to allow for indifferences in the firms' preferences. Accommodating indifferences is particularly important in the school choice context in which preferences are given by coarse priorities that put many students in the same priority class. To accommodate this extension, we represent a firm's preference as a choice correspondence (as opposed to a function). Then we extend both the fixed point characterization (via a correspondence defined on a functional space) and the proof of the existence.

Equipped with this generalization, our framework enables us to generalize the probabilistic matching and time-share matching models to accommodate general preferences. These latter models study how schools/firms can match stably with alternative students/workers for their probability/time shares (see Alkan and Gale (2003), Sotomayor (1999), Kesten and Unver (2009), among others). Our continuum model can be used to generalize these results to allow for more general preferences including complementarities as well as indifferences. Accommodating indifferences is important in school choice design, and complementarities are also relevant since some schools (as those in NYC) seek diversity in their student bodies. Our continuum model lends itself to studying such a probabilistic/time share environment; we can simply interpret types in different subsets within the type space as probabilistic/time units belonging to alternative (finite) workers. We can also establish existence of a matching satisfying a stronger notion of fairness, called no school-wise discrimination, suggested by Kesten and Unver (2009) in this more general environment.

2 Model of A Continuum Economy

A finite set $F = \{f_1, ..., f_n\}$ of firms matches with a unit mass of workers. Let $\emptyset$ be the null firm, representing the option of not being matched with any firm, and define $\tilde{F} := F \cup \{\emptyset\}$. The workers are identified with types $\theta \in \Theta = [0, 1]$, endowed with a Borel σ-algebra $\mathcal{B}([0, 1])$ and a measure that admits an absolutely continuous cdf G and positive density g in $(0, 1)$. Throughout a cumulative distribution function is denoted by a capital letter and its density is denoted by the corresponding lower case letter.

Any subset of the population, or **subpopulation**, is defined by a nondecreasing function $X : \Theta \to [0, 1]$ such that $X(0) = 0$ and that

$$X(\theta') - X(\theta) \leq G(\theta') - G(\theta), \forall \theta' \geq \theta.$$

Let $\mathcal{X}$ denote the set of all subpopulations. For any $X, X' \in \mathcal{X}$, let $(X + X')(\theta) := X(\theta) + X'(\theta)$, and $(X - X')(\theta) := X(\theta) - X'(\theta)$. (Note that $(X + X')(\theta)$ and $(X - X')(\theta)$ need not be elements of $\mathcal{X}$.) We further say $\tilde{X} \in \mathcal{X}$ is a **subpopulation of** $X \in \mathcal{X}$, denoted $\tilde{X} \sqsubset X$, if

$$\tilde{X}(\theta') - \tilde{X}(\theta) \leq X(\theta') - X(\theta), \forall \theta' \geq \theta.$$

Let x be the density function associated with X, whenever well defined. For $X_F, X'_F \in \mathcal{X}^n$, we write $X_F \sqsubset_F X'_F$ if $X_f \sqsubset X'_f$ for all $f \in F$.

The following observations are immediate:

Lemma 1. *1. $X \in \mathcal{X}$ if and only if X is continuous and admits a density function $x(\theta)$ for almost every $\theta \in \Theta$ such that $x(\theta) \in [0, g(\theta)]$ whenever $x(\theta)$ is well defined.*

2. $\tilde{X} \sqsubset X \in \mathcal{X}$, if and only if $\tilde{X} \in \mathcal{X}$ and admits a density function $\tilde{x}(\theta) \in [0, x(\theta)]$ for almost all θ.

3. If $X \in \mathcal{X}$, then $G - X \in \mathcal{X}$; and, for $X, X' \in \mathcal{X}$, $X \sqsubset X'$ if and only if $G - X' \sqsubset G - X$.

We now describe preferences on both sides of the market. A type θ agent is assumed to have a strict preference over $\tilde{F}$, given by a binary relation $\succ_\theta$. We assume that $\Theta = [0, 1]$

is partitioned into a finite number of intervals within which $\succ_\theta$ is constant; let $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{\bar{k}} = 1$ provide the partition, where $\bar{k}$ is a finite integer and, for each $k \in \{1, 2, \ldots, \bar{k}\}$, preferences of all workers of types in interval $[\theta_{k-1}, \theta_k)$ are identical to one another. Now, define for any $X, X' \in \mathcal{X}$, $(X \vee X')(\theta) := \int_0^\theta \max\{x(s), x'(s)\}ds$, $(X \wedge X')(\theta) := \int_0^\theta \min\{x(s), x'(s)\}ds$. Suppose firm f has offered to match with a subpopulation $X_f \in \mathcal{X}$, and let $X_F = (X_f)_{f \in F}$ and $X_{\tilde{F}} = (X_f)_{f \in \tilde{F}}$. We assume throughout that the offers made for workers with a given type θ are allocated to those workers by a serial dictatorship according to some arbitrary serial order.

A firm's preference is described by its choice function. A firm f's **choice function** is a mapping $C_f : \mathcal{X} \to \mathcal{X}$ such that for any $X \in \mathcal{X}$, $C_f(X)$ is a subpopulation of X and satisfies the following **revealed preference** property: for any $X, X' \in \mathcal{X}$ with $X' \sqsubset X$, if $C_f(X) \sqsubset X'$, then $C_f(X') = C_f(X)$. Note that we are assuming that the firm's demand is unique given any set of available workers. In Section 5, we consider a generalization of the model in which the firm's choice is not unique. Let $R_f : \mathcal{X} \to \mathcal{X}$ be a **rejection function** defined by $R_f(X)(\theta) := X(\theta) - C_f(X)(\theta)$. By convention, we let $C_\emptyset(X) = X, \forall X \in \mathcal{X}$, meaning that $R_\emptyset(X) = 0, \forall X \in \mathcal{X}$.

Let $C^f : \mathcal{X}^{n+1} \to \mathcal{X}$ for each $f \in F$ be given by

$$C^f(X)(\theta) := \int_0^\theta \max\{0, g(\theta) - \sum_{f' \in \tilde{F} : f' \succ_\theta f} x_{f'}(s)\}ds, \forall X \in \mathcal{X}^{n+1}.$$

In words, $C^f(X)$ denotes the subpopulation who can be attracted to firm f given X. Note that the function $C^f(\cdot)$ does not depend on x_f.

A **matching** is $M := (M_f)_{f \in \tilde{F}}$ such that $M_f \in \mathcal{X}, \forall f \in \tilde{F}$ and $\sum_{f \in \tilde{F}} M_f = G$.

Definition 1. *A matching M is **stable** if*

1. *For all $f \in F$ and θ such that $m_f(\theta) > 0$, $f \succ_\theta \emptyset$; and for each $f \in F$, $M_f = C_f(M_f)$.*

2. *There does not exist $f \in F$ and $M'_f \in \mathcal{X}, M'_f \neq M_f$ such that*

$$M'_f = C_f(M'_f \vee M_f) \sqsubset C^f(M'_f \vee M_f, M_{-f}) \tag{1}$$

Condition (1) of this definition means that a worker of each type prefers the matching over being unmatched, and that each firm never wishes to drop a subset of its matched workers. Condition (2) requires that there be no firm and a distribution of workers who are currently unmatched and want to be matched to one another.

3 Existence in The Continuum Economy

Let $\mathcal{M}$ denote the set of stable matchings. We relate each matching in $\mathcal{M}$ to a fixed point of a certain operator defined over $\mathcal{X}^{n+1}$. To this end, define a map $\Phi_f : \mathcal{X}^{n+1} \to \mathcal{X}$, where for each $X = (X_f)_{f\in\tilde{F}} \in \mathcal{X}^{n+1}$,

$$\Phi_f(X)(\theta) = G(\theta) - R_f(X_f)(\theta). \tag{2}$$

Let $\Phi := (\Phi_f)_{f\in\tilde{F}}$. To define another map, let $f_-^\theta \in \tilde{F}$ denote an immediate predecessor of f according to θ's preference order: that is, $f_-^\theta \succ_\theta f$ and there is no $f' \in \tilde{F}$ such that $f_-^\theta \succ_\theta f' \succ_\theta f$. Then, let $\Psi_f : \mathcal{X}^{n+1} \to \mathcal{X}$ be given by

$$\Psi_f(X)(\theta) = \int_0^\theta (g(s) - x_{f_-^s}(s))ds, \forall X \in \mathcal{X}^{n+1}, \tag{3}$$

with the convention that $X_{f_-^\theta}(\theta) = 0$ if f is preferred most by θ. Let $\Psi := (\Psi_f)_{f\in\tilde{F}}$. Now consider a composite map $T := \Psi \circ \Phi$. Note that the construction of the mapping is similar but different from the previous literature such as Adachi (2000), Hatfield and Milgrom (2005), Echenique and Oviedo (2006).

Theorem 1. *A matching M is stable if and only if there is a fixed point $X = T(X)$ such that $M_f = C_f(X_f), \forall f \in \tilde{F}$.*

Proof. See Appendix A.1. ∎

The fixed-point characterization in this theorem is similar in spirit to previous studies such as Adachi (2000), Hatfield and Milgrom (2005), Echenique and Oviedo (2006). However, the proof is quite different and necessitates a new method of proof. Part of the reason

is that the space of continuous type space requires the construction of the mapping T to be distinct from the ones in the previous literature.

We now introduce a condition on the firms' preferences that would ensure existence of stable matchings. Let $||\cdot||$ denote a sup norm, i.e., for any $X, X' \in \mathcal{X}$, $||X - X'|| = \sup_{\theta\in[0,1]} |X(\theta) - X'(\theta)|$.

Definition 2. *The firm f's preference is **continuous** if, for each $X \in \mathcal{X}$ and for each $\epsilon > 0$, there exists $\delta > 0$ such that*

$$||X' - X|| < \delta \Rightarrow ||C_f(X') - C_f(X)|| < \epsilon.$$

Theorem 2. *If each firm's preference is continuous, then there exists a stable matching.*

Proof. See Appendix A.2. ∎

4 Approximate Stability in Finite Economies

In this section, we build on the existence of a stable matching in the continuum economy to demonstrate when and how approximate stability can be achieved in a large but finite economy.

In this section, we need to deal with economies of finite sizes, so we allow for discrete distributions of worker types. In order to formalize the approximate stability concept, we first endow firms with cardinal utility functions over distributions of workers that may be discrete. Let u_f denote the utility function of firm f, with $u_f(M_f)$ being the firm's utility from matching with a subpopulation of workers M_f. We assume that firms' utility functions are continuous: For any M_f and $\epsilon > 0$, there exists $\delta > 0$ such that $|u_f(M_f) - u_f(M'_f)| < \epsilon$ for any M'_f with $||M_f - M'_f|| < \delta$.

Let $\epsilon > 0$ be a positive real number. A matching is **ϵ-stable** if, for any block, the utility gain for the firm involved in the block is less than ϵ. Formally, a matching M is ϵ-stable if, for any f and M'_f that blocks M, $u_f(M'_f) < u_f(M_f) + \epsilon$.

We consider a sequence of markets $(\Gamma^q)^q$ indexed by a positive integer q. In economy Γ^q, there are q workers, each with a type in Θ. There is a set of n firms $f_1, \ldots, f_n$, which is fixed

across all q. Firm f's preference is such that it evaluates the set of workers it is matched with, using f's preferences as in the continuum economy, but with the distribution of workers normalized by the economy's size. Formally, let the (normalized) population G^q of workers in Γ^q be defined so that $G^q(\theta)$ represents the number of workers with type less than or equal to θ divided by q. Similarly, any subpopulation $X^q \sqsubset G^q$ is a discrete distribution over types similarly normalized. A firm f's utility from matching with a subpopulation X^q is defined simply by using the firm f's utility function u_f. Each Firm's choice function in Γ^q is modified to choose the most preferred subpopulation among the discrete distributions– the chosen subpopulation should be a step function with each step being a multiple of $\frac{1}{q}$ that maximizes utility u_f of f. Note here that we are relaxing absolute continuity of G^q unlike the continuum economy.[4]

We say that a sequence of economies $(\Gamma^q)_q$ converges to a continuum economy Γ if the cumulative distribution function G^q of worker types converges uniformly to the distribution G of the continuum economy, that is, $||G^q - G|| \to 0$ as $q \to \infty$.

Theorem 3. *Fix any $\epsilon > 0$ and a sequence of economies $(\Gamma^q)_q$ that converges to a continuum economy. For any sufficiently large q, there exists an ϵ-stable matching in Γ^q.*

Proof. See Appendix A.3. ∎

On the other hand, no matter how large the economy is, as long as it is finite, there does not necessarily exist a stable matching. In this sense, Theorem 3 is tight and cannot be strengthened to the existence of a stable matching even in large finite economies. This point is illustrated by the following example.

Example 1. There exist two firms f and f' and two types of students θ and θ', each with q identical copies.[5] Preferences are as follows: f prefers to be matched to exactly identical

[4]The absolute continuity is used to guarantee compactness of $\mathcal{X}^{n+1}$ for the fixed point theorem. The assumption is not needed as we do not directly apply a fixed point theorem for the finite economy.

[5]Note that in the model of this section, q is the total number of workers in the model. For simplicity we abuse notation in this example, but this example fits into the model of this section by normalization. Also note that there are only two types of workers in this example, but these workers can be represented

numbers of type θ and θ' workers, and prefers to match as much as possible up to capacity q each. Formally, given the numbers of available workers x_θ and $x_{\theta'}$ for θ and θ', firm f chooses $\min\{x_\theta, x'_\theta, q\}$ of each worker types. Firm f' has the maximum capacity of q positions and prefers θ to θ' to $\emptyset$. Type θ workers prefer f to f' to $\emptyset$ while type θ' workers prefer f' to f. We shall show that there exists no stable matching if q is odd, no matter how large it is. To see this point, consider the following cases. Let m be the number of workers of each of the types θ and θ' who are matched to f.

1. Suppose $m > q/2$. For any such matching, at least one position is vacant at firm f' because f' has q positions, but strictly more than q workers are matched to f out of the total of $2q$ workers. Thus such a matching is blocked by f' and a type θ' worker who is currently matched to f.

2. Suppose $m < q/2$. Consider the following cases.

   (a) Suppose that there exists a type θ worker who is unmatched. Then such a matching is unstable because that worker and firm f' block it (note that f' prefers θ most).

   (b) Suppose that there exists no type θ worker who is unmatched. This implies that there exists a type θ' worker who is unmatched (because there are $2q$ workers in total, but firm f is matched to strictly fewer than q workers by assumption, and f' can be matched to at most q workers in any individually rational matching). Then, since f is the most preferred by all θ workers, a θ' worker prefers f to $\emptyset$, and there is some vacancy at f because $m < q/2$, the matching is blocked by a coalition of a type θ worker, a type θ' worker, and f.

by a continuum type space, in which $[0, 1/2)$ corresponds to θ and $[1/2, 1]$ corresponds to θ', so the model is within the scope of this section.

5 Stable Matching with Choice Correspondence

In this Section, we extend our analysis to the case in which a firm's choice from a distribution of available workers may not be unique. To do so, we let $C_f : \mathcal{X} \rightrightarrows \mathcal{X}$ be a **choice correspondence**: i.e. for any $X \in \mathcal{X}$, $C_f(X)$ is the set of subpopulations of X that are most preferred by f among all subpopulations of X. By convention, we let $C_\emptyset(X) = \{X\}, \forall X \in \mathcal{X}$. Then, let $R_f(X) := \{Y \in \mathcal{X} \,|\, Y = X - X' \text{ for some } X' \in C_f(X)\}$.

We assume that for any $X \in \mathcal{X}$, $C_f(X)$ is nonempty. Assume further that $C_f(\cdot)$ satisfies the reveal preference property: consider any $X, Y \in \mathcal{X}$ with $Y \sqsubset X$ and also any $X' \in C_f(X)$. Then, if $X' \sqsubset Y$, we must have $X' \in C_f(Y)$.

We define a function $C^f : \mathcal{X}^{n+1} \to \mathcal{X}$ for each $f \in F$ as: for $X \in \mathcal{X}^{n+1}$,

$$C^f(X)(\theta) := \int_0^\theta \max\{0, g(\theta) - \sum_{f' \in \tilde{F}: f' \succ_\theta f} x_{f'}(s)\} ds$$

to denote the subpopulation who can be attracted to the firm f. Note that the function $C^f(\cdot)$ does not depend on x_f.

A **matching** is $M := (M_f)_{f \in \tilde{F}}$ such that $M_f \in \mathcal{X}, \forall f \in \tilde{F}$ and $\sum_{f \in \tilde{F}} M_f = G$.

Definition 3. *A matching M is **stable** if*

1. *For all $f \in F$ and θ such that $m_f(\theta) > 0$, $f \succ_\theta \emptyset$; and for each $f \in F$, $M_f \in C_f(M_f)$.*
2. *There does not exist $f \in F$ and $M'_f \in \mathcal{X}$ such that $M'_f \sqsubset C^f(M'_f \vee M_f, M_{-f})$, $M'_f \in C_f(M'_f \vee M_f, M_{-f})$, and $M_f \notin C_f(M'_f \vee M_f)$.*

To relate each stable matching to a fixed point of a certain operator defined over $\mathcal{X}^{n+1}$, define a mapping $\Phi_f : \mathcal{X}^{n+1} \rightrightarrows \mathcal{X}$, where for each $X = (X_f)_{f \in \tilde{F}} \in \mathcal{X}^{n+1}$,

$$\Phi_f(X) := G - (X_f - C_f(X_f)), \tag{4}$$

or $\Phi_f(X) = \{X' \in \mathcal{X} \,|\, X' = G - (X_f - Y) \text{ for some } Y \in C_f(X_f)\}$. Let $\Phi := (\Phi_f)_{f \in \tilde{F}}$. To define another map, let $f^\theta_- \in \tilde{F}$ denote an immediate predecessor of f according to θ's

preference order: that is, $f_-^\theta \succ_\theta f$ and there is no $f' \in \tilde{F}$ such that $f_-^\theta \succ_\theta f' \succ_\theta f$. Then, define a map $\Psi_f : \mathcal{X}^{n+1} \to \mathcal{X}$, where, for any $X \in \mathcal{X}^{n+1}$,

$$\Psi_f(X)(\theta) = \int_0^\theta (g(s) - x_{f_-^s}(s))ds, \tag{5}$$

with the convention that $x_{f_-^\theta}(\theta) = 0$ if f is preferred most by θ. Let $\Psi := (\Psi_f)_{f \in \tilde{F}}$. Now consider a composite map $T := \Psi \circ \Phi$. Note that Ψ is a function but Φ is a correspondence, so T is also a correspondence.

Now we obtain the desired characterization of stable matching as follows:

Theorem 4. *A matching M is stable if and only if there is a fixed point $X \in T(X)$ such that $M_f = C_f(X_f), \forall f \in \tilde{F}$.*

Proof. See Appendix A.1. ∎

Theorem 5. *Suppose that for each $f \in \tilde{F}$, C_f is convex-, closed-valued, and upper hemi-continuous. Then, there exists a stable matching.*

Proof. See Appendix A.2 ∎

5.1 Probabilistic and Time Share Matching Models

In the time-share (or probabilistic) matching model, there are a finite set $W = \{1, ..., m\}$ of workers endowed with unit time, and as before a finite set $F = \{f_1, ..., f_n\}$ of firms who are looking to hire these workers for positive shares of their time. As before, $\tilde{F} := F \cup \{\emptyset\}$. Formally, a **matching** in time-share model is a $(n+1) \times m$ matrix $Z = [z_{fw}]_{w \in W, f \in \tilde{F}}$ such that $z_{fw} \geq 0$ and $\sum_{f \in \tilde{F}} z_{fw} = 1/m$ for each w. (The individual time endowment $1/m$ is a normalization, with no real consequence.) Throughout, vectors $z_f = (z_{fw})_{w \in W}$ and $z_w = (z_{fw})_{f \in \tilde{F}}$ denote respectively firm f's and worker w's time share.

For later purpose, we define

$$\tilde{C}_w^f(Z) := \max\{\frac{1}{m} - \sum_{f' \in \tilde{F}: f' \succ_w f} z_{f'w}, 0\},$$

that is the share of time of worker w available for firm f. Then, we let $\tilde{C}^f(Z) = (C^f_w(Z))_{w\in W}$.

Next each firm f's preference is described as follows. We now allow a firm to be indifferent among a set of workers. To allow for the indifference, we let a partition $P_f = \{S^1_f, ..., S^{K_f}_f\}$ (with $S^i \cap S^j = \emptyset$ and $\cup_{i=1}^{K_f} S^k_f = W$) denote the set of indifference classes for firm f, to describe a preference such that firm f is indifferent to redistributing the total time contracted with workers within each group S^k_f. Let $I_f = \{1, ..., K_f\}$ be the associated index set. We let $\mathbb{P}_f(z_f) := (\sum_{w\in S^j_f} z_{fw})_{j\in I_f}$ denote the vector of total times that firm f contracts with alternative indifference classes. The firm f's preference is then represented by a choice correspondence $\tilde{C}_f$ such that for any $y \in \tilde{C}_f(z)$, $y_w \in [0, z_w]$ and that, given $y \in \tilde{C}_f(z)$, $y' \in \tilde{C}_f(z)$ if and only if $\mathbb{P}_f(y) = \mathbb{P}_f(y')$. In other words, each firm f has uniquely optimal choice up to the time share allocation within each indifference class. It then follows that $\tilde{C}_f$ induces a function, $\Gamma_f : [0,1]^{|W|} \to \mathbb{R}^{K_f}_+$, given by

$$\Gamma_f(z) = \{u \in \mathbb{R}^{K_f}_+ \mid u = \mathbb{P}_f(y) \text{ for some } y \in \tilde{C}_f(z)\}.$$

In words, the k-th component of $\Gamma_f(z)$, denoted $\Gamma^k_f(z)$, $k \in I_f$, indicates firm f's demand for total time share from (indifference) class k of workers, given z. Given our earlier assumption, $\Gamma_f(z)$ is well (i.e., uniquely) defined. For our purpose, we also assume that Γ_f is continuous. A time share model can be in fact summarized by $(W, F, \succ, P, \Gamma)$, where $\succ := (\succ_w)_{w\in W}$ and $(P, \Gamma) := (P_f, \Gamma_f)_{f\in F}$.

We now define a notion of stability.

Definition 4. *A matching Z in a time share model is **stable** if (i) $z_{fw} > 0$ means that $f \succ_w \emptyset$; (ii) $z_f \in \tilde{C}_f(z_f)$ for every f, and if (iii) there does not exist $f \in F$ and z'_f such that $z'_f \leq \tilde{C}^f(z'_f \vee z_f, z_{-f})$, $z'_f \in \tilde{C}_f(z'_f \vee z_f)$, and $z_f \notin \tilde{C}_f(z'_f \vee z_f)$.*

Remark 1. *We say that a matching Z in a time share model is **group stable** if (i) and (ii) of Definition 4 hold and if (iii) there does not exist $F' \subseteq F$ and $(z_f)_{f\in F'}$ such that $z'_f \leq \tilde{C}^f(z' \vee z)$, $z'_f \in \tilde{C}_f(z'_f \vee z_f)$ for all $f \in F'$, and $z_f \notin \tilde{C}_f(z'_f \vee z_f)$ for some $f \notin F'$. This definition is a strengthening of our stability concept, as it requires that the matching be immune to blocks by coalitions potentially involving multiple firms. Such stability concepts*

with coalitional deviations are used by Sotomayor (1999), Echenique and Oviedo (2006), and Hatfield and Kominers (2010), among others. Since each worker has a single unit demand, group stability is clearly equivalent to our stability concept defined in Definition 4.

We now map a time share model $(W, F, \succ, P, \Gamma)$ to a continuum matching model $(G, F, \succ_\Theta, C_F)$. To begin, we set the type distribution G to be uniform with $G(\theta) = \theta$, and partition Θ into equal-sized subintervals $[\theta_0, \theta_1], ..., (\theta_{m-1}, \theta_m]$, each corresponding to a worker; specifically, subinterval $\Theta_w := (\theta_{w-1}, \theta_w]$ corresponds to worker w, where $\theta_0 = 0$ and $\theta_m = 1$. We then endow preference of each type $\theta \in \Theta$ such that $\succ_\theta := \succ_w$ if $\theta \in \Theta_w$.

We complete the description of the continuum matching model by defining a choice correspondence C_f for each $f \in \tilde{F}$. To do so, for each $X \sqsubset G$, let

$$\zeta_f(X) := (X(\theta_1), X(\theta_2) - X(\theta_1), ..., X(\theta_m) - X(\theta_{m-1}))$$

be the profile of time shares associated with alternative workers. Then, for any $f \in F$ and any $X \in \mathcal{X}$,

$$C_f(X) = \{X' \sqsubset X \,|\, \mathbb{P}_f(\zeta_f(X')) = \Gamma_f(\zeta_f(X))\}. \tag{6}$$

Let $C_\emptyset(X) = \{X\}$. It is straightforward to see that as $\tilde{C}_f$ satisfies the revealed preference, so does C_f. Define $C^f : \mathcal{X}^{n+1} \to \mathcal{X}$ for each $f \in \tilde{F}$ as before.

Now we can establish the equivalence between the set of stable matchings in time share and continuum matching models:

Theorem 6. *A matching Z in time share model $(W, F, \succ, P, \Gamma)$ is stable if and only if $Z = \zeta(M)$ for a matching M that is stable in the corresponding continuum model.*

Proof. See Appendix A.4.∎

Lemma 2. *The choice correspondence C_f defined in (6) is convex-valued. Also, if Γ_f is continuous, then C_f is closed-valued and upper-hemicontinuous.*

Proof. For any given $X \in \mathcal{X}$, consider any $X', X'' \in C_f(X)$. Then, for any $\lambda \in [0, 1]$,

$$\begin{aligned}\mathbb{P}_f(\zeta_f(\lambda X' + (1-\lambda)X'')) &= \mathbb{P}_f(\lambda\zeta_f(X') + (1-\lambda)\zeta_f(X'')) \\ &= \lambda\mathbb{P}_f(\zeta_f(X')) + (1-\lambda)\mathbb{P}_f(\zeta_f(X'')) = \Gamma_f(\zeta_f(X)),\end{aligned}$$

where the first two equalities hold due to the linearity of ζ_f and $\mathbb{P}_f$, and the third equality due to the assumption that $X', X'' \in C_f(X)$ so $\mathbb{P}_f(\zeta_f(X')) = \mathbb{P}_f(\zeta_f(X'')) = \Gamma_f(\zeta_f(X))$. This string of equalities implies that $\lambda X' + (1-\lambda)X'' \in C_f(X)$, so C_f is convex-valued.

To show the upper-hemicontinuity, consider two sequences $(X^k)_{k=1}^{\infty}$ and $(\tilde{X}^k)_{k=1}^{\infty}$ in $\mathcal{X}$ converging to some X and $\tilde{X}$ in $\mathcal{X}$, respectively, such that for each k, $\tilde{X}^k \in C_f(X^k)$, i.e. $\mathbb{P}_f(\zeta_f(\tilde{X}^k)) = \Gamma_f(\zeta_f(X^k))$. Since $\mathbb{P}_f$, ζ_f, and Γ_f are all continuous (so their compositions are also continuous), we have $\mathbb{P}_f(\zeta_f(\tilde{X})) = \lim_{k\to\infty} \mathbb{P}_f(\zeta_f(\tilde{X}^k)) = \lim_{k\to\infty} \Gamma_f(\zeta_f(X^k)) = \Gamma_f(\zeta_f(X))$, which means that $\tilde{X} \in C_f(X)$, establishing the upper-hemicontinuity of C_f. The proof of closed-valuedness is similar and hence omitted. ∎

Combining this result with Theorem 5 immediately yields the existence of stable matching in the time share model:

Theorem 7. *Suppose that Γ_f is continuous. Then, there exists a stable matching in the time share model.*

Definition 5. *A matching Z in time share model is **strongly fair** if (i) it is stable and if (ii) for any $f \in F$ and any $w, w' \in S_f^k$, if $z_{fw} < z_{fw'}$, then $\sum_{f':f' \prec_w f} z_{f'w} = 0$.*

We now show the existence of strongly fair matching by introducing another choice correspondence. To do so, let us first define two correspondences

$$A_f(X) := \{X' \sqsubset X \,|\, \mathbb{P}_f(\zeta_f(X')) = \Gamma_f(\zeta_f(X))\}$$

$$B_f(X) := \{X' \sqsubset X \,|\, \text{ for each } k, \text{ there is some } \alpha^k \in [0, \tfrac{1}{m}] \text{ such that } \zeta_{fw}(X') = \min\{\zeta_{fw}(X), \alpha^k\}, \forall w \in S_f^k\}.$$

The choice correspondence is then defined as

$$C_f(X) = A_f(X) \cap B_f(X). \tag{7}$$

Note that the correspondence $A_f(X)$ is the same as in (6). The correspondence $B_f(X)$ requires each firm to hire all workers in each indifference class k for the same amount of time α^k, except for workers whose available time $\zeta_{fw}(X)$ falls short of α^k.

Lemma 3. *For any $X \in \mathcal{X}$, $C_f(X)$ is non-empty. Also, $C_f(\cdot)$ satisfies the revealed preference property.*

Proof. For the nonemptiness, define $\hat{\alpha}^k \in [0, \frac{1}{m}]$ to satisfy $\sum_{w \in S_f^k} \min\{\zeta_{fw}(X), \hat{\alpha}^k\} = \Gamma_f^k(\zeta_f(X))$. Such $\hat{\alpha}^k$ always exists since the LHS is greater than the RHS with $\hat{\alpha}^k = 0$ while the opposite is true with $\hat{\alpha}^k = \frac{1}{m}$, so the intermediate value theorem applies. Now let for each $\theta \in \Theta_w$, $X'(\theta) = X'(\theta_{w-1}) + \min\{X(\theta) - X(\theta_{w-1}), \hat{\alpha}^k\}$ with $X'(\theta_0) = 0$. It is straightforward to verify that X' belongs to both $A_f(X)$ and $B_f(X)$, given the definition of $\hat{\alpha}^k$.

For the revealed preference property, consider any X, $X' \in C_f(X)$, and $\tilde{X} \sqsubset X$ such that $X' \sqsubset \tilde{X}$. Since we already know that $A_f(\cdot)$ satisfies the revealed preference, we have $X' \in A_f(\tilde{X})$. It thus suffices to show $X' \in B_f(\tilde{X})$. The fact that $X' \in B_f(X)$ means that for each k and $w \in S_f^k$, we have either $\zeta_{fw}(X') = \alpha^k$ or $\zeta_{fw}(X') = \zeta_{fw}(X) \leq \alpha^k$. In the latter case, $\zeta_{fw}(X') = \zeta_{fw}(\tilde{X}) = \zeta_{fw}(X) \leq \alpha^k$ since $X' \sqsubset \tilde{X} \sqsubset X$ and $\zeta_{fw}(X') = \zeta_{fw}(X)$. Thus the desired result follows. ∎

Using this choice correspondence, we can establish the equivalence between the sets of strongly fair matchings in time share and stable matchings in the continuum matching model:

Theorem 8. *A matching Z in time share model $(W, F, \succ, P, \Gamma)$ is strongly fair if and only if $Z = \zeta(M)$ for a matching M that is stable in the corresponding continuum model with the choice correspondence C_f defined in (7).*

Proof. (only if part) Consider a strongly fair matching Z in the time share model. Define a matching M for the corresponding continuum model as in (20). We show that M is stable. To show that $M_f \in C_f(M_f)$ for each $f \in F$ (i.e. Part 1 of Definition 5), note first that since $A_f(\cdot)$ is the same choice function as used in Theorem 6 and since M_f is stable with that choice function, we have $M_f \in A_f(M_f)$. To prove $M_f \in B_f(M_f)$, it suffices to set $\alpha^k = \frac{1}{m}$, since then $\zeta_{fw}(M_f) \leq \alpha^k, \forall w \in S_f^k$ so $\min\{\zeta_{fw}(M_f), \alpha^k\} = \zeta_{fw}(M_f), \forall w \in S_f^k$.

Part 2 of Definition 5 can be established following the same argument as in the proof of Theorem 6 since the argument there only relies on the revealed preference property of C_f, which is shown in Lemma 3.

(if part) For any stable matching M in the continuum model, consider an associated matching $Z = (z_f)_{f\in\tilde{F}}$ with $z_f = \zeta_f(M_f), \forall f$ in the time share model. Let us show Z is stable. First, it is straightforward, thus omitted, to check Part 1 of Definition 4. To check Part 2, suppose to the contrary that there is a blocking pair f and z'_f, and define a matching M' as in (22). Then, the argument following (22) shows that f and M'_f are a blocking pair for M under the choice function $A_f(\cdot)$, which implies: (i) $M_f \notin A_f(M'_f \vee M_f)$ or $\mathbb{P}_f(\zeta_f(M_f)) \neq \Gamma_f(\zeta_f(M'_f \vee M_f))$; (ii) $M'_f \sqsubset C^f(M'_f \vee M_f, M_{-f})$. By the nonemptiness of $C_f(M'_f \vee M_f)$, there is some $\tilde{M}_f \in C_f(M'_f \vee M_f)$. Then, since $\tilde{M}_f \sqsubset (\tilde{M}_f \vee M_f) \sqsubset (M'_f \vee M_f)$, the revealed preference implies $\tilde{M}_f \in C_f(\tilde{M}_f \vee M_f)$. Next, since $\tilde{M}_f \sqsubset C_f(M'_f \vee M_f)$ and also since both M'_f and M_f are a subpopulation of $C^f(M'_f \vee M_f, M_{-f})$, we have $\tilde{M}_f \sqsubset (M'_f \vee M_f) \sqsubset C^f(M'_f \vee M_f, M_{-f}) = C^f(\tilde{M}_f \vee M_f, M_{-f})$. Also, by (i) and the fact that $\tilde{M}_f \in \big(A_f(M'_f \vee M_f) \cap A_f(\tilde{M}_f \vee M_f)\big)$, we have $\mathbb{P}_f(\zeta_f(M_f)) \neq \Gamma_f(\zeta_f(M'_f \vee M_f)) = \mathbb{P}_f(\zeta_f(\tilde{M}_f)) = \Gamma_f(\zeta_f(\tilde{M}_f \vee M_f))$, which means that $M_f \notin A_f(\tilde{M}_f \vee M_f)$ or $M_f \notin C_f(\tilde{M}_f \vee M_f)$. It follows from our argument so far that f and $\tilde{M}_f$ block M.

To show that Z is strongly fair, suppose not. So there are some f, k, and $w, w' \in S_f^k$ for whom $\zeta_{fw}(M_f) = z_{fw} < z_{fw'} = \zeta_{fw'}(M_f)$ and $\sum_{f':f'\prec_w f} z_{f'w} > 0$. Fixing any such f, k, and w, let f' be any firm such that $f' \prec_w f$ and $z_{f'w} > 0$. Letting $W' = \arg\max_{\tilde{w}\in S_f^k} z_{f\tilde{w}}$, note that $w \notin W'$. We can define a matching for firm f as follows: For each $\tilde{w}$ and $\theta \in \Theta_{\tilde{w}}$,

$$M'_f(\theta) - M'_f(\theta_{\tilde{w}-1}) = \begin{cases} M_f(\theta) - M_f(\theta_{w-1}) + \epsilon(M_{f'}(\theta) - M_{f'}(\theta_{w-1})) & \text{if } \tilde{w} = w \\ \epsilon'(M_f(\theta) - M_f(\theta_{\tilde{w}-1})) & \text{if } \tilde{w} \in W' \\ M_f(\theta) - M_f(\theta_{\tilde{w}-1}) & \text{otherwise} \end{cases},$$

where $\epsilon, \epsilon' \in (0,1)$ are chosen to satisfy $\zeta_{f\tilde{w}}(M'_f) < \zeta_{fw'}(M'_f), \forall w' \in W', \tilde{w} \notin W'$ and

$$\zeta_{fw}(M'_f) + \sum_{\omega'\in W'} \zeta_{fw'}(M'_f) = \zeta_{fw}(M_f) + \sum_{\omega'\in W'} \zeta_{fw'}(M_f). \tag{8}$$

Note first that that $M'_f \sqsubset C^f(M) = C^f(M'_f \vee M_f, M_{-f})$, since $f' \prec_w f$ and thus $M'_f \sqsubset (M_f + M_{f'} 1_{\{\theta \in \Theta_w\}}) \sqsubset C^f(M)$. Let us show next that $M'_f \in C_f(M'_f \vee M_f, M_{-f})$. The fact that $M_f \in C_f(C^f(M))$ due to the stability of M, implies $M_f \in C_f(M_f \vee M'_f)$ by the revealed preference, since $M_f \sqsubset (M_f \vee M'_f) \sqsubset C^f(M)$. This means $M_f \in A_f(M_f \vee M'_f)$, which can be combined with (8) to yield $\mathbb{P}_f(\zeta_f(M'_f)) = \mathbb{P}_f(\zeta_f(M_f))$ and thus $M'_f \in A_f(M_f \vee M'_f)$. To show $M'_f \in B_f(M_f \vee M'_f)$, we set $\alpha^k = \max_{w \in S^k_f} \zeta_{fw}(M'_f)$ and observe that for all $w' \in W'$, $\min\{\zeta_{fw'}(M_f \vee M'_f), \alpha^k\} = \min\{\zeta_{fw'}(M_f), \alpha^k\} = \alpha^k = \zeta_{fw'}(M'_f)$ while for all $\tilde{w} \notin W'$, $\min\{\zeta_{f\tilde{w}}(M_f \vee M'_f), \alpha^k\} = \min\{\zeta_{f\tilde{w}}(M'_f), \alpha^k\} = \zeta_{f\tilde{w}}(M'_f)$, which implies $M'_f \in B_f(M_f \vee M'_f)$. That is, $M'_f \in C_f(M_f \vee M'_f)$.

We now draw a contradiction by showing that $M_f \notin B_f(M_f \vee M'_f)$ or $M_f \notin C_f(M_f \vee M'_f)$. This is shown by arguing that there is no α^k such that $\zeta_{f\tilde{w}}(M_f) = \min\{\zeta_{f\tilde{w}}(M_f \vee M'_f), \alpha^k\}, \forall \tilde{w} \in S^k_f$. If $\alpha^k < \max_{\tilde{w} \in S^k_f} \zeta_{f\tilde{w}}(M_f)$, then, for $w' \in W'$, $\min\{\zeta_{fw'}(M_f \vee M'_f), \alpha^k\} = \min\{\zeta_{fw'}(M_f), \alpha^k\} = \alpha_k < \zeta_{fw'}(M_f)$. If $\alpha^k \geq \max_{\tilde{w} \in S^k_f} \zeta_{f\tilde{w}}(M_f)$, then $\min\{\zeta_{fw}(M_f \vee M'_f), \alpha^k\} = \min\{\zeta_{fw}(M'_f), \alpha^k\} = \zeta_{fw}(M'_f) > \zeta_{fw}(M_f)$. ∎

Theorem 9. *Suppose that Γ_f is continuous. Then, there exists a strongly fair matching in the time share model.*

Proof. It suffices to show that the choice correspondence defined in (7) is closed-, convex-valued, and upper hemicontinuous. The proof of convex-valuedness is straightforward and thus omitted. We here only prove the upper hemicontinuity since the proof of closed-valuedness is similar and simpler. According to 16.25 Theorem of Aliprantis and Border, the intersection of a family of closed-valued upper hemicontinuous correspondences, one of which is also compact-valued, is upper hemicontinuous. Since we already know that $A_f(\cdot)$ is closed-, compact-valued, and upper hemicontinuous, we only need to prove that $B_f(\cdot)$ is upper hemicontinuous (given that its closed-valuedness has been proved). To do so, consider sequences $(X^m)_{m=1}^\infty$ and $(\tilde{X}^m)_{m=1}^\infty$ with $\tilde{X}^m \in B_f(X^m), \forall m$, converging to X and $\tilde{X}$, respectively. So, for each $k \in K_f$, there is a sequence $(\alpha^k_m)_{m=1}^\infty$ such that $\zeta_{fw}(\tilde{X}^m) = \min\{\zeta_{fw}(X^m), \alpha^k_m\}, \forall w \in S^k_f$. For each k, let α^k be a limit to which the sequence $(\alpha^k_m)_{m=1}^\infty$ (or its subsequence) converges. We claim that $\zeta_{fw}(\tilde{X}) = \min\{\zeta_{fw}(X), \alpha^k\}, \forall w \in$

S_f^k. If, for instance, $\zeta_{fw}(\tilde{X}) > \min\{\zeta_{fw}(X), \alpha^k\}$, then one can find sufficiently large m to make $\zeta_{fw}(\tilde{X}^m)$, $\zeta_{fw}(X^m)$, and α_m^k close to $\zeta_{fw}(\tilde{X})$, $\zeta_{fw}(X)$, and α^k, respectively, so that $\zeta_{fw}(\tilde{X}^m) > \min\{\zeta_{fw}(X^m), \alpha_m^k\}$ (since the operators ζ_f and min are continuous), which is a contradiction. The same argument applies to the case $\zeta_{fw}(\tilde{X}) < \min\{\zeta_{fw}(X), \alpha^k\}$. ∎

6 Substitutable Preferences

Definition 6. *Firm f's preference is **substitutable** if $R_f(X) \sqsubset R_f(X')$ whenever $X \sqsubset X'$.*

In words, substitutability means that a firm rejects more of any given worker types when facing a bigger set of workers. Consider a partial order $(\mathcal{X}, \sqsubset)$. For any $X, X' \in \mathcal{X}$, $X \vee X' \in \mathcal{X}$ and $X \wedge X' \in \mathcal{X}$, so it is a lattice. A partial order $(\mathcal{X}, \sqsubset)$ is a complete lattice. The next characterization is convenient for this purpose. Fix an arbitrary $\mathcal{Y} \subset \mathcal{X}$ and $\theta \in \Theta$, we let $\{x(\theta)|X \in \mathcal{Y}\}$ denote the set of all (well-defined) densities associated with X's in $\mathcal{Y}$ at θ.

Lemma 4. *For any $\mathcal{Y} \subset \mathcal{X}$, let $\overline{Y}(\theta) := \int_0^\theta \sup\{x(s)|X \in \mathcal{Y}\}ds$ and $\underline{Y}(\theta) := \int_0^\theta \inf\{x(s)|X \in \mathcal{Y}\}ds$. Then, $\overline{Y}, \underline{Y} \in \mathcal{X}$. Moreover, $\sup \mathcal{Y} = \overline{Y}$ and $\inf \mathcal{Y} = \underline{Y}$. In other words, $(\mathcal{X}, \sqsubset)$ is a complete lattice.*

Proof. Since $x(\theta)$ is well defined for almost every θ, $\sup\{x(\theta)|X \in \mathcal{Y}\}$ and $\inf\{x(\theta)|X \in \mathcal{Y}\}$ are well defined and nonnegative for almost every θ. Clearly, $\overline{Y}$ and $\underline{Y}$ are nondecreasing. Further, $\overline{Y}(0) = \underline{Y}(0) = 0$. Next observe that $\sup\{x(\theta)|X \in \mathcal{Y}\} \leq g(\theta)$ for almost every θ. If not, there exists $\hat{X} \in \mathcal{Y}$ such that $\hat{x}(\theta) > g(\theta)$, which contradicts Lemma 1. Hence, for any $\theta' > \theta$,

$$\overline{Y}(\theta') - \overline{Y}(\theta) = \int_\theta^{\theta'} \sup\{x(s)|X \in \mathcal{Y}\}ds \leq \int_\theta^{\theta'} g(s)ds = G(\theta') - G(\theta).$$

We thus conclude that $\overline{Y} \in \mathcal{X}$. Analogously, $\underline{Y} \in \mathcal{X}$.

Next, we prove $\sup \mathcal{Y} = \overline{Y}$. Fix any $\tilde{X} \in \mathcal{Y}$. Then, for any $\theta < \theta'$,

$$\tilde{X}(\theta') - \tilde{X}(\theta) = \int_\theta^{\theta'} \tilde{x}(s)ds \leq \int_\theta^{\theta'} \sup\{x(s)|X \in \mathcal{Y}\}ds = \overline{Y}(\theta') - \overline{Y}(\theta).$$

Hence, $\tilde{X} \sqsubset \overline{Y}$. Suppose $Z \sqsupset X$ for all $X \in \mathcal{Y}$. We show that $Z \sqsupset \overline{Y}$. Suppose not. Then, there exists $\theta' > \theta$ such that $Z(\theta') - Z(\theta) < \overline{Y}(\theta') - \overline{Y}(\theta)$. We must have $\hat{\theta} \in [\theta, \theta']$ such that $z(\hat{\theta}) < \overline{y}(\hat{\theta})$. There must exist $\hat{X} \in \mathcal{Y}$ such that $\hat{x}(\hat{\theta}) > z(\hat{\theta})$. Hence, for $\epsilon \in (0, \frac{\hat{x}(\hat{\theta}) - z(\hat{\theta})}{2})$, there exists $\hat{\theta}' > \hat{\theta}$ sufficiently close to $\hat{\theta}$ such that

$$\frac{\hat{X}(\hat{\theta}') - \hat{X}(\hat{\theta})}{\hat{\theta}' - \hat{\theta}} > \hat{x}(\hat{\theta}) - \epsilon > z(\hat{\theta}) + \epsilon > \frac{Z(\hat{\theta}') - Z(\hat{\theta})}{\hat{\theta}' - \hat{\theta}},$$

so

$$\hat{X}(\hat{\theta}') - \hat{X}(\hat{\theta}) > Z(\hat{\theta}') - Z(\hat{\theta}),$$

contradicting to the fact that $Z \sqsupset X$ for all $X \in \mathcal{Y}$. The proof for $\inf \mathcal{Y} = \underline{Y}$ is symmetric.

∎

It follows that a partial order $(\mathcal{X}^{n+1}, \sqsubset_{\tilde{F}})$ is also a complete lattice. We now observe that the map $T = \Psi \circ \Phi$ is monotonic in the partial order $(\mathcal{X}^{n+1}, \sqsubset_{\tilde{F}})$, given substitutability of firms' preferences. To see this, let $X_{\tilde{F}} \sqsubset_{\tilde{F}} X'_{\tilde{F}}$ both in $\mathcal{X}^n$. First, substitutability of firms' preferences (and the definition of $R_\emptyset \equiv 0$ implies that $R_f(X_f) \sqsubset R_f(X'_f)$ for $f \in \tilde{F}$, which via Lemma 1 gives $Y_{\tilde{F}} := \Phi(X_{\tilde{F}}) \sqsupset_{\tilde{F}} \Phi(X'_{\tilde{F}}) =: Y'_{\tilde{F}}$. Next, for each $f \in \tilde{F}$, $\theta' > \theta$

$$\Psi_f(Y_{\tilde{F}})(\theta') - \Psi_f(Y_{\tilde{F}})(\theta) = -\int_\theta^{\theta'} y_{f_-^\theta}(s)ds \leq -\int_\theta^{\theta'} y'_{f_-^\theta}(s)ds = \Psi_f(Y'_{\tilde{F}})(\theta') - \Psi_f(Y'_{\tilde{F}})(\theta),$$

which implies that $\Psi(Y_{\tilde{F}}) \sqsubset \Psi(Y'_{\tilde{F}})$, or $T(X_{\tilde{F}}) \sqsubset_{\tilde{F}} T(X'_{\tilde{F}})$. By Tarski's fixed point theorem, we have the following result:

Theorem 10. *When the firms' preferences are substitutable, the set $\mathcal{X}^*$ of fixed points of T is nonempty, and $(\mathcal{X}^*, \sqsubset_{\tilde{F}})$ is a complete lattice.*

Let M and M' be two matchings, and say $M' \succeq_f M$ if there exist $M' \sqsupset M$ such that $M = C_f(X)$ and $M' = C_f(X')$, and say $X' \succeq_F X$ if $X' \succeq_f X$ for all f. Likewise, we say that $M' \succeq_\theta M$ if for each $f \in \tilde{F}$, $\sum_{f'' \in \tilde{F}: f'' \succeq_\theta f} m'_{f''}(\theta) \geq \sum_{f'' \in \tilde{F}: f'' \succeq_\theta f} m_{f''}(\theta)$. We say $M' \succeq_\Theta M$ if $M' \succeq_\theta M$ for all $\theta \in \Theta$. We say that a matching $\overline{M}$ is firm optimal (resp., firm pessimal) if $\overline{M} \succeq_F M$ (resp., $\overline{M} \preceq_F M$) for all stable matching M. A matching $\underline{M}$ is worker optimal (resp., worker pessimal) if $\underline{M} \succeq_\Theta M$ (resp., $\underline{M} \preceq_\Theta M$) for all stable matching M.

Theorem 10 implies the following:

Corollary 1. *There exists a firm-optimal (worker pessimal) stable matching* $\overline{M} = (C_f(\overline{X}_f))_{f\in F}$, *where* $\overline{X} = \sup_{\sqsubset_F} \mathcal{X}^*$, *and a firm-pessimal (and worker optimal) stable matching* $\underline{M} = (C_f(\underline{X}))_{f\in F}$, *where* $\underline{X} = \inf_{\sqsubset_F} \mathcal{X}^*$.

Proof. The stable matching $\overline{M}$ is firm-optimal since $\overline{X} \sqsupset X$ for all $X \in \mathcal{X}^*$. To see it is worker-pessimal, fix any stable matching M. Then, by Theorem 1, $M = (C_f(X))_{f\in\tilde{F}}$ for some $X \in \mathcal{X}^*$. Since $\overline{X} \sqsupset X$, and firms' preferences are substitutable, $\Phi(\overline{X}) \sqsubset \Phi(X)$, which implies by Lemma 1 that $\phi_f(\overline{X})(\theta) \leq \phi_f(X)(\theta)$ for all f and a.e. θ. The worker pessimality of $\overline{M}$ then follows since

$$\sum_{f''\in\tilde{F}:f''\succeq_\theta f} \overline{m}_{f''}(\theta) = \sum_{f''\in\tilde{F}:f''\succeq_\theta f} \phi_f(\overline{X})(\theta) \leq \sum_{f''\in\tilde{F}:f''\succeq_\theta f} \phi_f(X)(\theta) = \sum_{f''\in\tilde{F}:f''\succeq_\theta f} m_{f''}(\theta)$$

for all f and for a.e. θ. ∎

Definition 7. *Firm* f*'s preference exhibits the* ***law of aggregate demand*** *if for any* $X, X' \in \mathcal{X}$, *with* $X \sqsubset X'$, $C_f(X)(1) \leq C_f(X')(1)$.

Lemma 5. *(Rural hospital) If firms' preferences exhibit substitutability and the law of aggregate demand, then for any stable matching* M, *we have* $M_f(1) = \overline{M}_f(1)$ *and* $M_\varnothing = \overline{M}_\varnothing$.

Proof. Let M be a stable matching. Then, there exists $X \in \mathcal{X}^*$ such that $M_f = C_f(X_f)$ for each $f \in F$. Since $X \sqsubset \overline{X}$, by the law of aggregate demand, we have

$$\overline{M}_f(1) = C_f(\overline{X}_f)(1) \geq C_f(X_f)(1) = M_f(1). \tag{9}$$

Next since $\overline{M}$ is worker pessimal, for all f, $\Phi_f(\overline{X}) \sqsubset \Phi_f(X)$, which means that $\phi_f(\overline{X})(\theta) \leq \phi_f(X)(\theta)$ for almost every θ for all $f \in \tilde{F}$. Let $w(\theta) := \varnothing^\theta_-$ be the immediate predecessor of $\varnothing$ (i.e., the worst individually rational firm) for type θ. It thus follows that $\phi_{w(\theta)}(\overline{X})(\theta) \leq \phi_{w(\theta)}(X)(\theta)$ for almost every θ. (We let $\phi_\varnothing(X)(\theta) = 0$.) Then,

$$\sum_{f\in F} \overline{M} = \int_0^{[\cdot]} \phi_{w(s)}(\overline{X})(s)ds \sqsubset \int_0^{[\cdot]} \phi_{w(s)}(X)(s)ds = \sum_{f\in F} M. \tag{10}$$

Combining (9) and (10), we conclude that $M_f(1) = \overline{M}_f(1)$ for each $f \in F$.

Further, we must have $\sum_{f\in F} \overline{M}_f(\theta) = \sum_{f\in F} M_f(\theta)$ for all θ. To prove this, suppose otherwise. Then, there must exist $\theta \in (0,1)$ such that $\sum_{f\in F} \overline{M}_f(\theta) \neq \sum_{f\in F} M_f(\theta)$. If $\sum_{f\in F} \overline{M}_f(\theta) < \sum_{f\in F} M_f(\theta)$, then there must be a positive measure of $\theta' \in [\theta, 1]$ such that $\sum_{f\in F} \overline{m}_f(\theta') > \sum_{f\in F} m_f(\theta')$, which contradicts (10). If $\sum_{f\in F} \overline{M}_f(\theta) > \sum_{f\in F} M_f(\theta)$, then there must exist a positive measure of $\theta' \in [0, \theta]$ such that $\sum_{f\in F} \overline{m}_f(\theta') > \sum_{f\in F} m_f(\theta')$, which again contradicts (10). ∎

We next introduce a condition that would ensure uniqueness of a stable matching. The condition refers to a new notation. For any matching M, let $M^f_\emptyset$ be a subpopulation of workers, defined by $M^f_\emptyset(\theta) := \int_0^\theta m_\emptyset \cdot \mathbf{1}_{\{f \succ_s \emptyset\}} ds$ for each θ, that are unmatched but available to firm f under M. For $X, X' \in \mathcal{X}$, we let $X + X' := \int_0^{[\cdot]} \min\{x(s) + x'(s), g(s)\} ds$. (May be we should introduce this notation before.)

Definition 8. *(Rich preferences) A firm's preference is* ***rich*** *if the following holds. Let M and $\hat{M}$ be two individually rational matchings such that $\sum_{f\in F} M_f \sqsupset \sum_{f\in F} \hat{M}_f$, and $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$ for each $f \in F$ and $M_{f'} \neq C_{f'}(\hat{M}_{f'} \vee M_{f'})$ for at least one firm $f' \in F$. Then, there exists $f^* \in F$ such that $M_{f^*} \neq C_{f^*}(M_{f^*} + \hat{M}^{f^*}_\emptyset)$.*

In words, the condition is explained as follows. Suppose there are any two matchings M and $\hat{M}$ such that all firms prefer $\hat{M}$ to M, strictly so for at least one firm. Then, there must exist a firm that would be happy to match with some workers that are unmatched but available to that firm under M or under $\hat{M}$.

Theorem 11. *If firms' preferences are rich and substitutable and exhibit the law of aggregate demand, the set of stable matchings is unique.*

Proof. Suppose otherwise. Then there exists a stable matching M that differs from the firm optimal stable matching $\overline{M}$. Let X and $\overline{X}$ be respectively such that $M_f = C_f(X_f)$, $\overline{M}_f = C_f(\overline{X}_f)$ and $X_f \sqsubset \overline{X}_f$, for each $f \in F$.

First of all, by Lemma 5, $\sum_{f\in F} M_f \sqsupset \sum_{f\in F} \overline{M}_f$. Next, since $X_f \sqsubset \overline{X}_f$ for each $f \in F$, we have $\overline{M}_f \vee M_f \sqsubset \overline{X}_f$. Revealed preference then means that, for each $f \in F$,

$$\overline{M}_f = C_f(\overline{M}_f \vee M_f).$$

Moreover, since $M \neq \overline{M}$, there is at least one firm f' such that $M_{f'} \neq C_{f'}(\hat{M}_{f'} \vee M_{f'})$. But then by the rich preferences, there exists $f^* \in F$ such that

$$M_{f^*} \neq C_{f^*}(M_{f^*} + \overline{M}_{\emptyset}^{f^*}).$$

Meanwhile, by Lemma 5, we have $M_\emptyset = \overline{M}_\emptyset$, which in turn implies that $M_\emptyset^{f^*} = \overline{M}_\emptyset^{f^*}$. It then follows that

$$M_{f^*} \neq C_{f^*}(M_{f^*} + M_{\emptyset}^{f^*}).$$

This result contradicts the stability of M, since for $\hat{M}_{f^*} := C_{f^*}(M_{f^*} + M_\emptyset^{f^*})$, we have

$$\hat{M}_{f^*} = C_{f^*}(M_{f^*} \vee \hat{M}_{f^*}) \sqsubset C^{f^*}(M_{f^*} \vee \hat{M}_{f^*}, M_{-f^*}).$$

∎

Remark 2. *Formally, what we want: In a suitably specified restriction to the AL environment, full support implies rich preferences. One way to make such a restriction: Assume that firms' preferences are responsive. Say $s_f(\theta)$ is the score of type θ worker for firm f. For a matching M, let $p_f = \inf\{s_f(\theta) | M_f(\theta) > 0\}$. Define $\hat{p}_f$ for $\hat{M}$ similarly.*

Now fix any M and $\hat{M}$ such that $\sum_{f\in F} M_f \sqsupset \sum_{f\in F} \hat{M}_f$, and $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$ for each $f \in F$ and $M_{f'} \neq C_{f'}(\hat{M}_{f'} \vee M_{f'})$ for at least one firm $f' \in F$. That $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$ for each firm f means that each firm hires weakly more workers under $\hat{M}$ than under M (or else, he will select some workers from M since all workers hired under M are acceptable by individual rationality). Since $\sum_{f\in F} M_f \sqsupset \sum_{f\in F} \hat{M}_f$, it follows that $M_f(1) = \hat{\mathcal{M}}_f(1)$ for each $f \in F$. Then, the argument used in the proof of Lemma 5 establishes $\sum_{f\in F} M_f = \sum_{f\in F} \hat{M}_f$. Since $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$ and $M_f(1) = \hat{\mathcal{M}}_f(1)$ for each $f \in F$, $p_f \leq \hat{p}_f$. Finally, we have $M'_f \neq C_{f'}(\hat{M}_{f'} \vee M_{f'})$ for at least one firm f', which implies that $p_{f'} < \hat{p}_{f'}$.

Now consider a set

$$\tilde{\Theta} := \{\theta \in \Theta | f' \succ_\theta \emptyset \succ_\theta f'', \forall f'' \neq f \text{ and } p_{f'} \leq s_{f'}(\theta) \leq \hat{p}_{f'}\}$$

of workers, and let $\tilde{M}$ *denote its distribution. The full support assumption means that* $\tilde{M}(1) > 0$*. We show that these workers are unemployed under both* $\hat{M}$ *and* M*. To see this, first individual rationality of* $\hat{M}$ *means that these workers are unemployed under* $\hat{M}$*. Since* $\sum_{f \in F} M_f = \sum_{f \in F} \hat{M}_f$*, almost all of these workers must be unemployed under* M *as well.*

Next, note that $\tilde{M} \sqsubset \hat{M}_\emptyset^{f'}$*. Since* $\hat{p}_{f'} > p_{f'}$*, firm* f' *will wish to replace some of its workers with these workers under* M*. Hence,* $M_{f'} \neq C_{f'}(M_{f'} + \hat{M}_\emptyset^{f'})$*, so the rich preferences follow.*

A Appendix: Proofs

A.1 Proof of Theorems 1 and 4

Since Theorem 1 is a special case of Theorem 4, we prove the latter.

("only if" part): Suppose M is a stable matching. Let X be such that for each $f \in \tilde{F}$, $X_f(\theta) = \int_0^\theta \sum_{f \succeq_s f'} m_{f'}(s)ds$. Note that the stability of M implies $x_f(\theta) = 0$ for $f \prec_\theta \emptyset$. Also, let $\bar{X} \in \mathcal{X}^{n+1}$ be given by $\bar{X}_f(\theta) = \int_0^\theta \sum_{f' \succeq_s f} m_{f'}(s)ds$. Since M is a matching, we have, by construction, $G - X_f + M_f = \bar{X}_f$ for all $f \in \tilde{F}$.

We now prove that X is a fixed point of T. First, we have

$$C^f(X_f, M_{-f})(\theta) = \int_0^\theta \max\{0, g(s) - \sum_{f' \succ_s f} m_{f'}(s)\}ds = \int_0^\theta \sum_{f \succeq_s f'} m_{f'}(s)ds = X_f(\theta),$$

where the first equality follows from the definition of C^f along with the fact that C^f does not depend on its first argument, and the last equality follows from the definition of X. Given this and the fact that $M_f \sqsubset X_f$, the stability of M implies that $M_f \in C_f(X_f) = C_f(X_f \vee M_f)$, or else $X'_f \in C_f(X_f)$ could serve as M'_f in Definition 3 to block M. It thus follows that $X_f - M_f \in R_f(X_f)$ and hence $\bar{X}_f = G - (X_f - M_f) \in \Phi_f(X)$ for each $f \in \tilde{F}$ or $\bar{X} \in \Phi(X)$. Note now that for each $f \in \tilde{F}$,

$$\Psi_f(\bar{X}_f)(\theta) = \int_0^\theta \{g(s) - \bar{x}_{f^s_-}(s)\}ds = \int_0^\theta (g(s) - \sum_{f' \succ_s f} m_{f'}(s))ds = X_f(\theta).$$

Thus, $X = \Psi(\bar{X}) \in \Psi(\Phi(X)) = T(X)$ so X is a fixed point of T.

("if" part): Let $X \in \mathcal{X}^{n+1}$ be a fixed point of T. Then, we must have some $X' \in \mathcal{X}^{n+1}$ such that for each $f \in \tilde{F}$, $X_f = \Psi_f(X')$ and $X'_f \in \Phi_f(X)$, i.e. $X'_f = G - (X_f - M_f)$ for some $M_f \in C_f(X_f)$. We first show that for each $f \in \tilde{F}$,

$$x_f(\theta) = g(\theta) - \sum_{f' \succ_\theta f} m_{f'}(\theta), \tag{11}$$

where x_f is the density of X_f, and m_f is the density of M_f. It is straightforward to see from (5) that if f is ranked first by θ, then $x_f(\theta) = \psi_f(X')(\theta) = g(\theta)$ so (11) holds for such f.

Consider now any $f \in \tilde{F}$ and assume that (11) holds true for f_-^θ, so $x_{f_-^\theta}(\theta) = g(\theta) - \sum_{f' \succ_\theta f_-^\theta} m_{f'}(\theta)$. We show that (11) also holds for f. To do so, observe

$$x'_{f_-^\theta}(\theta) = g(\theta) - (x_{f_-^\theta}(\theta) - m_{f_-^\theta}(\theta)) = \sum_{f' \succ_\theta f_-^\theta} m_{f'}(\theta) + m_{f_-^\theta}(\theta) = \sum_{f' \succ_\theta f} m_{f'}(\theta),$$

where the second equality follows from the induction hypothesis. Thus,

$$X_f(\theta) = \Psi_f(X')(\theta) = \int_0^\theta (g(s) - x'_{f_-^s}(s))ds = \int_0^\theta (g(s) - \sum_{f' \succ_s f} m_{f'}(s))ds,$$

so $x_f(\theta) = g(\theta) - \sum_{f' \succ_\theta f} m_{f'}(\theta)$, as was to be shown.

Now recall $M_\emptyset \in C_\emptyset(X) = \{X\}, \forall X$, and set $f = \emptyset$ in (11) to obtain

$$g(\theta) - \sum_{f' \succ_\theta \emptyset} m_{f'}(\theta) = x_\emptyset(\theta) = m_\emptyset(\theta), \tag{12}$$

or $g(\theta) = \sum_{f' \succeq_\theta \emptyset} m_{f'}(\theta)$. Then, for $f \in F$ such that $\emptyset = f_-^\theta$, (11) implies

$$x_f(\theta) = g(\theta) - \sum_{f' \succ_\theta f} m_{f'}(\theta) = g(\theta) - \sum_{f' \succeq_\theta \emptyset} m_{f'}(\theta) = 0$$

so $m_f(\theta) = 0$ for such f.

A similar argument can be used to show that $m_f(\theta) = 0$ for all other $f \prec_\theta \emptyset$. Combine this with (12) to observe that M is a matching, i.e. $\sum_{f \in \tilde{F}} m_f(\theta) = g(\theta)$ for all θ, or $\sum_{f \in \tilde{F}} M_f = G$. Also, the first part of part 1 in Definition 3 is easily satisfied. The second part (i.e. $M_f \in C_f(M_f)$) follows from the revealed preference property since $M_f \sqsubset X_f$ and $M_f \in C_f(X_f)$.

It only remains to check part 2 of Definition 3. Suppose for contradiction it fails. Then, there exist f and M'_f such that

$$M'_f \sqsubset C^f(M'_f \vee M_f, M_{-f}),\ M'_f \in C_f(M'_f \vee M_f) \text{ and } M_f \notin C_f(M'_f \vee M_f). \tag{13}$$

Observe now that for each $f \in F$,

$$C^f(M'_f \vee M_f, M_{-f})(\theta) = \int_0^\theta \max\{0, g(s) - \sum_{f' \succ_s f} m_{f'}(s)\}ds = X_f(\theta),$$

where the first equality follows from the definition of C^f and the fact that it does not depend on the first argument, and the second equality follows from (11). Thus, we have $M'_f \sqsubset C^f(M'_f \vee M_f, M_{-f}) = X_f$. Since then $M_f \sqsubset (M'_f \vee M_f) \sqsubset X_f$ and $M_f \in C_f(X_f)$, the revealed preference property implies $M_f \in C_f(M'_f \vee M_f)$, contradicting (13). We have thus proven that M is stable.

A.2 Proofs of Theorems 2 and 5

Since Theorem 2 is a special case of Theorem 5, we prove the latter.

We first prove that $\mathcal{X}$ is closed. To this end, consider any sequence $\{X^n\}$, $X^n \in \mathcal{X}$ for each n, such that $||X^n - X|| \to 0$ as $n \to \infty$. We prove that $X \in \mathcal{X}$. To show this we first show that X is nondecreasing. To see this note that $X(\theta') - X(\theta) < 0$ for some $\theta' > \theta$ otherwise. But then

$$\begin{aligned} ||X^n - X|| &\geq \max\{|X^n(\theta') - X(\theta')|, |X(\theta) - X^n(\theta)|\} \\ &\geq \frac{|X^n(\theta') - X(\theta')| + |X(\theta) - X^n(\theta)|}{2} \\ &\geq \tfrac{1}{2}|X^n(\theta') - X^n(\theta) + X(\theta) - X(\theta')| \\ &\geq \tfrac{1}{2}|X(\theta) - X(\theta')|, \end{aligned}$$

which never vanishes as $n \to \infty$, yielding a contradiction. Likewise, for each $\theta' > \theta$, we must have $X(\theta') - X(\theta) \leq G(\theta') - G(\theta)$. To see this suppose to the contrary that $X(\theta') - X(\theta) > G(\theta') - G(\theta)$. Then,

$$\begin{aligned} ||X^n - X|| &\geq \max\{|X(\theta') - X^n(\theta')|, |X^n(\theta) - X(\theta)|\} \\ &\geq \tfrac{1}{2}(|X(\theta') - X^n(\theta')| + |X^n(\theta) - X(\theta)|) \\ &\geq \tfrac{1}{2}|X(\theta') - X(\theta) - X^n(\theta') + X^n(\theta)| \\ &\geq \tfrac{1}{2}|X(\theta') - X(\theta) - (G(\theta') - G(\theta))|, \end{aligned}$$

which never vanishes as $n \to \infty$, yielding a contradiction. Combining these, we conclude that $X \in \mathcal{X}$, proving that $\mathcal{X}$ is closed. Therefore its Cartesian produce $\mathcal{X}^{n+1}$ is also closed (with respect to the product topology).

Next, we prove that $\mathcal{X}$ is compact. To show this, note that by definition of subpopulations, for any any $X \in \mathcal{X}$ and $\theta, \theta' \in [0,1]$,

$$|X(\theta') - X(\theta)| \leq |G(\theta') - G(\theta)| \leq K|\theta' - \theta|,$$

for $K := \sup_{\tilde{\theta}} g(\tilde{\theta}) < \infty$. Hence, $\mathcal{X}$ is Lipschitz continuous with Lipschitz constant K. By the Arzèla-Ascoli theorem, the set of Lipschitz continuous functions with Lipschitz constant K is compact (see Ok (2011) for instance). And since $\mathcal{X}$ is its closed subset, and any closed subset of a compact set is compact (again, see Ok (2011) for example), $\mathcal{X}$ is compact. Therefore its Cartesian product $\mathcal{X}^{n+1}$ is also compact (with respect to the product topology).

Next, we verify that Φ_f inherits all the properties of C_f. To show the convex-valuedness, consider any $X \in \mathcal{X}$ and $X', X'' \in \Phi_f(X)$. It means that there are some $Y', Y'' \in C_f(X)$ satisfying $X' = G - (X - Y')$ and $X'' = G - (X - Y'')$. Then, the convexity of $C_f(X)$ implies that for any $\lambda \in [0,1]$, $\lambda Y' + (1-\lambda)Y'' \in C_f(X)$ so $\lambda X' + (1-\lambda)X'' = G - (X - (\lambda Y' + (1-\lambda)Y'')) \in \Phi(X)$. To show the upper hemi-continuity, consider any sequence $(X^k)_{k=1}^{\infty}$ in $\mathcal{X}^{n+1}$ and $(\tilde{X}^k)_{k=1}^{\infty}$ in $\mathcal{X}$ converging to some X and $\tilde{X}$, respectively, such that $\tilde{X}^k \in \Phi_f(X^k)$ for each k. Then, for each k, there is some $Y^k \in C_f(X^k)$ satisfying $\tilde{X}^k = G - (X_f^k - Y^k)$ or $\tilde{X}^k + X_f^k - G$. Hence, Y^k converges to $Y = \tilde{X} + X_f - G$. By the upper hemicontinuity of C_f, $Y \in C_f(X)$ so $\tilde{X} = G - (X_f - Y) \in \Phi(X)$, as desired. The proof of closedness is similar and thus omitted.

Next, we prove the following claim:

Claim 1. *The mapping Ψ is continuous and hence upper hemicontinuous.*

Proof. To show that Ψ is continuous, fix any $X \in \mathcal{X}^{n+1}$ and $\epsilon > 0$. Take $\delta = \epsilon/(2\bar{k} + 3)$, where $\bar{k}$ is the number of intervals that partition the workers' type space such that the ordinal preferences of workers are constant in each of the intervals (recall that $\bar{k}$ is finite

by assumption). Note that for any $X, X' \in \mathcal{X}^{n+1}$, $f \in \tilde{F}$, and $\theta \in [0,1]$, we have

$$\begin{aligned}
\Psi_f(X)(\theta) - \Psi_f(X')(\theta) &= \int_0^\theta (g(s) - x_{f^s_-}(s))ds - \int_0^\theta (g(s) - x'_{f^s_-}(s))ds \\
&= \int_0^\theta (x'_{f^s_-}(s) - x_{f^s_-}(s))ds \\
&= \sum_{k=1}^{\hat{k}} [X'_{f^{\theta_k}_-}(\theta_k) - X_{f^{\theta_k}_-}(\theta_k) - (X'_{f^{\theta_k}_-}(\theta_{k-1}) - X_{f^{\theta_k}_-}(\theta_{k-1}))] \\
&\quad + [X'_{f^\theta_-}(\theta) - X_{f^\theta_-}(\theta) - (X'_{f^\theta_-}(\hat{\theta}_k) - X_{f^\theta_-}(\hat{\theta}_k))],
\end{aligned}$$

where $\hat{k} = \max\{k | \theta_k \leq \theta\}$. By assumption $\hat{k}$ is at most a constant $\bar{k}$. Therefore, for any $X' \in \mathcal{X}^{n+1}$ such that $||X' - X|| < \delta$, we obtain

$$|\Psi_f(X)(\theta) - \Psi_f(X')(\theta)| \leq (2\bar{k} + 2)||X' - X|| < \epsilon,$$

showing that Ψ is continuous. Noting that continuity and upper hemicontinuity are equivalent when the mapping is a function, we complete the proof. ∎

Observe now that as both Ψ and Φ are convex-valued[6] and upper hemicontinuous, so is their composition $T = \Psi \circ \Phi$.[7] Also, it is straightforward to show that T is closed-valued. We can thus invoke the Kakutani-Fan-Glicksberg's fixed point theorem to show that the set of fixed points of T is nonempty (and compact).[8]

A.3 Proof of Theorem 3

Let Γ be the limit continuum economy which the sequence $(\Gamma^q)_q$ converges to. Let $\theta_1 < \theta_2 < \cdots < \theta_{\bar{q}}$ be the types of workers in Γ^q (and let $\theta_0 = -1$ as notational convention).[9] We prove the result constructively.

[6] The mapping Ψ is a function and, as such, convex-valued in a trivial sense.

[7] For the upper hemicontinuity, refer to 16.23 Theorem in Aliprantis and Border (1997), for instance, which says that a composition of two upper hemicontinuous correspondences is upper hemicontinuous.

[8] For the Kakutani-Fan-Glicksberg's fixed point theorem, refer to 16.12 Closed Graph Theorem and 16.51 Corollary in Aliprantis and Border (1997).

[9] Note that we allow for possibility that there are more than one worker of the same type even in finite economies, so $\bar{q}$ may be strictly smaller than q.

We begin by showing a lemma that proves useful. For any population G, fix a sequence (G^q) of finite-economy populations such that $||G^q - G|| \to 0$ as $q \to \infty$. For any q and subpopulation $X \sqsubset G$, define X^q as follows.

1. $X^q(\theta_0) = 0$.

2. For any $i = 1, \ldots, \bar{q}$, inductively define

$$X^q(\theta_i) = \max \left\{ \frac{m}{q} | m \in \mathbb{N}, \frac{m}{q} \leq X(\theta_i), \frac{m}{q} - X^q(\theta_{i-1}) \leq G^q(\theta_i) - G^q(\theta_{i-1}) \right\}. \quad (14)$$

3. $X^q(\theta) = X^q(\theta_i)$ for all $\theta \in [\theta_i, \theta_{i+1})$.[10]

Lemma 6. *Suppose $||G^q - G|| \to 0$ as $q \to \infty$ and $X \sqsubset G$. For any $\epsilon > 0$, we have $||X^q - X|| < \epsilon$ for any sufficiently large q, where X^q is constructed by the above procedure.*

Proof. Let q be sufficiently large so that $q > \frac{5}{\epsilon}$ and $||G - G^q|| < \frac{\epsilon}{5}$. First note that, when constructing $X^q(\theta_i)$ from $X^q(\theta_{i-1})$, if it is the first constraint $\frac{m}{q} \leq X(\theta_i)$ that is binding in the maximization problem (14), then $|X^q(\theta_i) - X(\theta_i)| < \frac{1}{q} < \epsilon$. Consider any i such that it is the second constraint, $\frac{m}{q} - X^q(\theta_{i-1}) \leq G^q(\theta_i) - G^q(\theta_{i-1})$, that is binding in the maximization problem (14). Let j be the largest integer no greater than i such that the first constraint was binding for j. Since the second constraint is binding in the maximization problem (14) for $k \in \{j+1, j+2, \ldots, i\}$ by definitions of i and j, it follows that $X^q(\theta_k) - X^q(\theta_{k-1}) = G^q(\theta_k) - G^q(\theta_{k-1})$. Adding up these equalities for $k = j+1, \ldots, i$, we obtain $X^q(\theta_i) - X^q(\theta_j) = G^q(\theta_i) - G^q(\theta_j)$. Because $X(\theta_i) - X(\theta_j) \leq G(\theta_i) - G(\theta_j)$ by the assumption that $X \sqsubset G$, this implies $[X(\theta_i) - X^q(\theta_i)] - [X(\theta_j) - X^q(\theta_j)] \leq [G(\theta_i) - G^q(\theta_i)] - [G(\theta_j) - G^q(\theta_j)]$. Therefore we have

$$[X(\theta_i) - X^q(\theta_i)] \leq [X(\theta_j) - X^q(\theta_j)] + [G(\theta_i) - G^q(\theta_i)] - [G(\theta_j) - G^q(\theta_j)] < \frac{3\epsilon}{5}, \quad (15)$$

where the last inequality follows because $[X(\theta_j) - X^q(\theta_j)] < \frac{1}{q} < \frac{\epsilon}{5}$ by definition of j, and $|G(\theta_i) - G^q(\theta_i)|, |G(\theta_j) - G^q(\theta_j)| < \frac{\epsilon}{5}$ by $||G - G^q|| < \frac{\epsilon}{5}$.

[10] For $i = 0$, we need to set $X^q(\theta) = X^q(\theta_i)$ only for $\theta \in [\theta_i, \theta_{i+1}) \cap \Theta$.

Consider any $\theta \in [\theta_i, \theta_{i+1})$. It follows that

$$X(\theta) - X(\theta_i) \leq G(\theta) - G(\theta_i) = [G(\theta) - G^q(\theta)] - [G(\theta_i) - G^q(\theta_i)] < \frac{2\epsilon}{5}, \tag{16}$$

where the first inequality follows from $X \sqsubset G$, the second equality follows from the assumption $G^q(\theta) = G^q(\theta_i)$, and the last inequality follows from the assumption $||G - G^q|| < \frac{\epsilon}{5}$.

To finish the proof, for any $\theta \in [\theta_i, \theta_{i+1})$, we observe that

$$[X(\theta) - X^q(\theta)] = [X(\theta) - X(\theta_i)] + [X(\theta_i) - X^q(\theta_i)] < \epsilon,$$

where the equality follows because $X^q(\theta) = X^q(\theta_i)$ for any $\theta \in [\theta_i, \theta_{i+1})$ by construction, and the inequality follows from relations (15) and (16). Since $X(\theta) - X^q(\theta)$ is always nonnegative by construction, this completes the proof. ∎

By Theorem 2, there exists a stable matching M in the continuum economy. Now, construct M^q by the following procedure.

1. Define $M^q_{f_1}$ as the X^q in the above procedure with respect to $X := M_{f_1}$.

2. Define $M^q_{f_2}$ as the X^q in the above procedure with respect to $X := M_{f_2}$, while replacing G with $G - M_{f_1}$, and G^q with $G^q - M^q_{f_1}$.

3. Generally, for any $k \in \{1, 2, \ldots, n\}$, inductively define $M^q_{f_k}$ as the X^q in the above procedure with respect to $X := M_{f_k}$, while replacing G with $G - \sum_{l<k} M_{f_l}$, and G^q with $G^q - \sum_{l<k} M^q_{f_l}$.

Noting that the number of firms is finite, Lemma 6 implies that $||M^q - M|| \to 0$ as $q \to 0$. Thus, by continuity of firms' utility functions, for any $f \in F$ and $\epsilon > 0$,

$$u_f(M^q_f) > u_f(M_f) - \frac{\epsilon}{2}, \tag{17}$$

for any sufficiently large q. Let $C^f(M^q)$ be the subpopulation of workers in economy Γ^q who weakly prefer f to their match in M^q. Since $||M^q - M|| \to 0$, we have $||C^f(M^q) - C^f(M)|| \to 0$ as $q \to \infty$. Let $M'_f = C_f(C^f(M^q))$. In words, M'_f is the most profitable block of M^q for f in the continuum economy, that is, the optimal deviation in a situation where the

current matching is M^q, but the firm can deviate to any subpopulation, not just a discrete distribution. Then the above-mentioned property that $||C^f(M^q) - C^f(M)|| \to 0$ as $q \to \infty$ and continuity of C_f imply that, for any $\delta > 0$, $||M'_f - M_f|| < \delta$ for any sufficiently large q. Thus, by continuity of firms' utility functions,

$$u_f(M'_f) < u_f(M_f) + \frac{\epsilon}{2}, \tag{18}$$

for any sufficiently large q. Let M''_f be the most profitable block of M^q for f in economy Γ^q. Then M''_f is the optimal deviatin facing the same population G^q and matching M^q as when computing M'_f but with an additional constraint that the deviation is discrete, so $u_f(M''_f) \leq u_f(M'_f)$. This and inequality (18) imply

$$u_f(M''_f) < u_f(M_f) + \frac{\epsilon}{2}. \tag{19}$$

Combining inequalities (17) and (19), we obtain $u_f(M''_f) < u_f(M^q_f) + \epsilon$, completing the proof.

A.4 Proof of Theorem 6

(only if part) Consider a stable matching Z in the time share model. Then, define recursively: for each f and for each $\theta \in \Theta_w$,

$$M_f(\theta) = M_f(\theta_{w-1}) + \left(\frac{\theta - \theta_{w-1}}{\theta_w - \theta_{w-1}}\right) z_{fw}, \tag{20}$$

where $M_f(\theta_0) = 0$. Clearly, $\zeta_f(M_f) = z_f, \forall f \in \tilde{F}$. We show that M is stable in the continuum model. First, $m_f(\theta) = 0$ for any $\theta \in \Theta_w$ such that $\varnothing \succ_\theta f$, since $z_{fw} = 0$ for any such f and w. Next, since $z_f \in \tilde{C}_f(z_f)$, we have $\Gamma_f(z_f) = \mathbb{P}_f(z_f)$, which implies that $\Gamma_f(\zeta_f(M_f)) = \mathbb{P}_f(\zeta_f(M_f))$ so $M_f \in C_f(M_f)$. To show that part 2 of Definition 3 also holds, suppose for a contradiction that there exist a pair f and M'_f blocking M. Denote $z'_f = \zeta_f(M'_f)$. The fact that $M'_f \in C_f(M'_f \vee M_f)$ and $M_f \notin C_f(M'_f \vee M_f)$ implies: (i) $\mathbb{P}_f(z_f) = \mathbb{P}_f(\zeta(M_f)) \neq \mathbb{P}_f(\zeta(M'_f)) = \mathbb{P}_f(z'_f)$; and (ii) $\mathbb{P}(z'_f) = \mathbb{P}(\zeta(M'_f)) = \Gamma_f(\zeta_f(M'_f \vee M_f))$ or $z'_f \in \tilde{C}_f(\zeta_f(M'_f \vee M_f))$. Also, the fact that $M'_f \sqsubset C^f(M'_f \vee M_f, M_{-f})$ leads to (iii)

$z'_f \leq \tilde{C}^f(z'_f \vee z_f, z_{-f})$.[11] Observe now that for any w,

$$(M'_f \vee M_f)(\theta_w) - (M'_f \vee M_f)(\theta_{w-1}) = \int_{\theta_{w-1}}^{\theta_w} \max\{m'_f(\theta), m_f(\theta)\}d\theta$$
$$\geq \max\{\int_{\theta_{w-1}}^{\theta_w} m'_f(\theta)d\theta, \int_{\theta_{w-1}}^{\theta_w} m_f(\theta)d\theta\} = z'_{fw} \vee z_{fw}, \tag{21}$$

so $(z'_f \vee z_f) \leq \zeta_f(M'_f \vee M_f)$. Thus, by (ii) and the revealed preference, we have $z'_f \in \tilde{C}_f(z'_f \vee z_f)$, which, combined with (i) and (iii), contradicts that Z is stable in the time share model.

(if part) Consider a stable matching M in the continuum model. We show that an associated matching $Z = (z_f)_{f\in\tilde{F}}$ where $z_f = \zeta_f(M_f)$ for all $f \in \tilde{F}$ is stable in the time share model. It is straightforward, thus omitted, to verify part 1 of Definition 4.

To verify part 2, suppose that there exist a pair f and z'_f blocking Z. To establish a contradiction, let us consider a matching

$$M'_f(\theta) = M'_f(\theta_{w-1}) + \min\{\alpha_{fw}(M_f(\theta) - M_f(\theta_{w-1})), C^f(M)(\theta) - C^f(M)(\theta_{w-1})\} \tag{22}$$

where $\alpha_{fw} \geq 0$ is a nonnegative number satisfying

$$z'_{fw} = \min\{\alpha_{fw}(M_f(\theta_w) - M_f(\theta_{w-1})), C^f(M)(\theta_w) - C^f(M)(\theta_{w-1})\}.$$

[11] To see this, note that for all w,

$$z'_{fw} = M'_f(\theta_w) - M'_f(\theta_{w-1}) \leq \int_{\theta_{w-1}}^{\theta_w} \max\{1 - \sum_{f'\in\tilde{F}:f'\succ_\theta f} m_{f'}(\theta), 0\}d\theta$$
$$= \int_{\theta_{w-1}}^{\theta_w} (1 - \sum_{f'\in\tilde{F}:f'\succ_\theta f} m_{f'}(\theta))d\theta$$
$$= \frac{1}{m} - \sum_{f'\in\tilde{F}:f'\succ_w f} z_{f'w} = \tilde{C}^f_w(z'_f \vee z_f, z_{-f}),$$

where the second equality holds since $\sum_{f'\in\tilde{F}:f'\succ_\theta f} m_{f'}(\theta) = \sum_{f'\in\tilde{F}:f'\succ_\theta f} \frac{z_{f'w}}{\theta_w - \theta_{w-1}} \leq 1$.

Such α_{fw} always exists since $z'_{fw} \leq C^f(M)(\theta_w) - C^f(M)(\theta_{w-1})$.[12] Moreover, since $z_{fw} = M_f(\theta_w) - M_f(\theta_{w-1}) \leq C^f(M)(\theta_w) - C^f(M)(\theta_{w-1})$, we have $\alpha_{fw} \geq 1$ if and only if $z'_{fw} \geq z_{fw}$. Now note that, by the definition of M'_f above, $m'_f(\theta) \geq (\leq) m_f(\theta), \forall \theta \in \Theta_w$ if and only if $\alpha_{fw} \geq (\leq) 1$. Hence, we have $m'_f(\theta) \geq (\leq) m_f(\theta), \forall \theta \in \Theta_w$ if and only if $z'_{fw} \geq (\leq) z_{fw}$. Given this, the inequality in (21) holds as equality for any w, so $\zeta_f(M'_f \vee M_f) = z'_f \vee z_f$. Then, Z being blocked by z'_f implies $\zeta_f(M'_f) = z'_f \in \tilde{C}_f(z'_f \vee z_f) = \tilde{C}_f(\zeta_f(M'_f \vee M_f))$, which in turn implies $\mathbb{P}_f(\zeta_f(M'_f)) = \Gamma_f(\zeta_f(M'_f \vee M_f))$ or $M'_f \in C_f(M'_f \vee M_f)$. Z being blocked by z'_f also implies $\mathbb{P}_f(z'_f) \neq \mathbb{P}_f(z_f)$ or $M_f \notin C_f(M'_f \vee M_f)$. Lastly, by definition of M'_f in (22), we have $M'_f \sqsubset C^f(M) = C^f(M'_f \vee M_f, M_{-f})$. In sum, f and M'_f can serve as a pair blocking M in the continuum model, yielding the desired contradiction.

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[12] To see it, note

$$z'_{fw} \leq \tilde{C}^f_w(z'_{fw} \vee z_{fw}, z_{-f}) = \frac{1}{m} - \sum_{f' \in \tilde{F}: f' \succ_w f} z_{f'w}$$
$$= \int_{\theta_{w-1}}^{\theta_w} (1 - \sum_{f' \in \tilde{F}: f' \succ_\theta f} m_{f'}(\theta)) d\theta = C^f(M)(\theta_w) - C^f(M)(\theta_{w-1}),$$

where the first equality follows from the fact that Z is a feasible matching in the time share model, so $\sum_{f' \in \tilde{F}: f' \succ_w f} z_{f'w} \leq \frac{1}{m}$.

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