Endogenous Liquidity and Defaultable Bonds*

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Abstract

This paper studies the liquidity of defaultable corporate bonds that are traded in an over-the-counter secondary market with search frictions. Bargaining with dealers determines a bond's endogenous liquidity, which depends on both the firm fundamental and the time-to-maturity of the bond. Corporate default and investment decisions interact with the endogenous secondary market liquidity via the rollover channel. A default/investment-liquidity loop arises: Earlier endogenous default worsens a bond's secondary market liquidity, which amplifies equity holders' rollover losses, which in turn leads to earlier endogenous default. Thus, our model characterizes the full inter-dependence between liquidity premium and default premium in understanding credit spreads for corporate bonds. We also study the optimal maturity implied by the model, and an extension where worsening secondary market liquidity feeds back to endogenous under-investment.

Keywords: Positive Feedback, Fundamental and Liquidity, Over-The-Counter Market, Secondary Bond Market, Structural Models, Bid-Ask Spread

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1 Introduction

Although corporate bond markets make up a large part of the U.S. financial system,[1] it has been well documented that secondary corporate bond markets – which are mainly over-the-counter (OTC) markets – are much less liquid than equity markets. For instance, Edwards, Harris, and Piwowar (2007) study the U.S. OTC secondary trades in corporate bonds and estimate the transaction cost to be around 150 bps, and Bao, Pan, and Wang (forthcoming) find an even larger number.[2] Moreover, both papers document a strong empirical pattern that the liquidity for corporate bonds deteriorates dramatically for bonds with lower fundamental, i.e., bonds that are issued by firms with higher credit derivative swaps (CDS), and liquidity improves for bonds with little time-to-maturity left. The left hand side of Figure 1 illustrates this increasing pattern w.r.t. CDS rates by plotting the average implied transaction cost based on Bao, Pan, and Wang (forthcoming) for bonds sorted by the firm CDS rates, suggesting firm fundamental as one of the key determinants of the secondary market liquidity for corporate bonds. Similarly, the right hand side of Figure 1 illustrates the increasing pattern w.r.t. time-to-maturity based on the same illiquidity measure for bonds sorted by time-to-maturity.

Additionally, the recent financial crisis of 2007-2008 has demonstrated that the deterioration of secondary market liquidity can adversely affect the refinancing operations of firms, which in turn exacerbates the firm fundamental and drives up the credit derivative swaps. Taken together, these two observations suggest a positive feedback loop between the liquidity of secondary market and the asset fundamental for corporate bonds, and our model aims to deliver such an effect.

We model the endogenous liquidity in the secondary corporate bond market as a search-based over-the-counter (OTC) market á la Duffie, Garlenau, and Pedersen (2005). Bond investors who are hit by liquidity shocks prefer early payments, and with a certain matching technology they meet and trade with an intermediary dealer at an endogenous bid-ask spread. The resulting bid-

[1] According to flow of funds, the values of corporate bonds reaches about 4.7 trillion in the first quarter of 2011, which consists of about one third of total liabilities of U.S. corporate businesses.

[2] Empirical papers that investigate secondary bond market liquidity are Hong and Warga (2000), Schultz (2001), Harris and Piwowar (2006); Green, Hollifield, and Schurhoff (2007b,a).

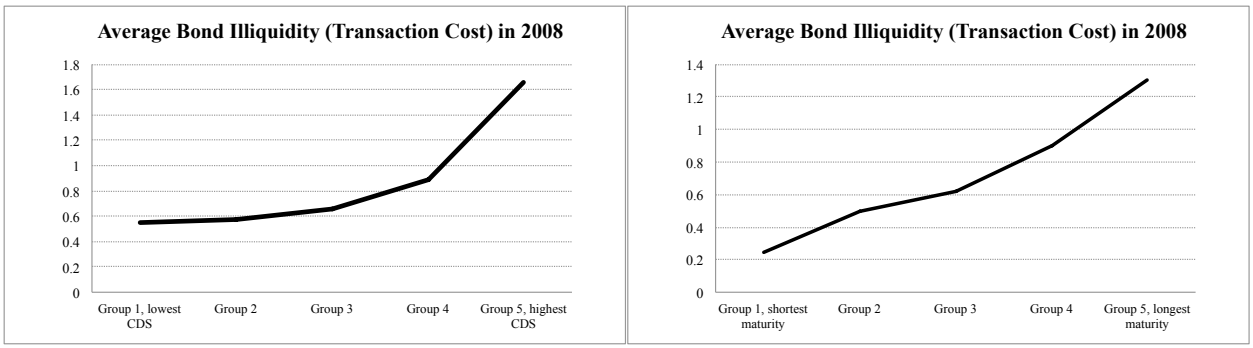


Figure 1: The median corporate bond secondary market illiquidity for quintile groups with increasing CDS (left hand side) and increasing time-to-maturity (right hand side). For each month in 2008, we first estimate the bond illiquidity measure γ_0 (which captures the mean-reversion of transaction prices) using trade-by-trade data following Bao, Pan, and Wang (forthcoming). We then form five quintiles for firms based on their 5-year CDS rates (LHS) and time-to-maturity (RHS), and calculate the median of bond illiquidity γ_0 for each quintile. The number shown in the figure is the mean γ_0 over 12 months for each quintile. Data source: TRACE.

ask spread captures the endogenous secondary market liquidity, which in equilibrium depends on both the firm's distance-to-default and the bond's time-to-maturity.

The second important ingredient for the feedback between fundamental and liquidity is the endogenous default decision by equity holders. This mechanism is borrowed from the standard Leland-type corporate finance structural models, i.e., Leland (1994) and Leland and Toft (1996) (hereafter LT96). More specifically, a firm that maintains a stationary debt maturity structure has to roll over (refinance) the maturing bonds by issuing new bonds with identical terms. When the firm fundamental deteriorates, equity holders will face heavy rollover losses due to falling prices of newly issued bonds. Equity holders choose an endogenous default threshold, at which point bond investors with defaulted bonds step in to recover part of the firm value due to dead-weight bankruptcy cost.

The liquidity of defaulted bonds is important in deriving the endogenous bond liquidity before the firm defaults. Motivated by the fact that bankruptcy leads to not only the freezing of assets within the company but also a delay in the payout of any cash depending on court proceedings,[3] we endogenize the (il)liquidity of defaulted bonds by modeling the firm default as a delayed payout.

[3]The Lehman Brothers bankruptcy in September 2008 is a good case in point. After much legal uncertainty, payouts to the debt holders only started trickling out after more than 3 years.

This assumption results in a boundary condition when we derive the bond valuations before default by solving a system of (linked) Partial Differential Equations (PDEs).[4] We solve this system of PDEs, as well as the equity valuation and the endogenous default boundary, in closed form in Section 3.

Consistent with empirical findings, we show in Section 4.1 that the endogenous bid-ask spread is increasing with the firm's distance-to-default, holding the time-to-maturity constant, and decreasing in the bond's time-to-maturity, holding the distance-to-default constant. Moreover, our model produces a novel testable empirical prediction that the slope w.r.t. time-to-maturity of the bid-ask spread will be greater for bonds with higher distance-to-default. Intuitively, as the stated maturity of corporate bonds plays no role in bankruptcy procedures, the difference between bonds with different time-to-maturities vanishes if firms are close to default.

Based on the endogenous liquidity derived in this model, Section 4.2 illustrates the positive feedback loop between fundamental and liquidity for corporate bonds that are traded in the secondary market. Essentially, the endogenous default by equity holders – via the rollover channel – feeds the worsening secondary market liquidity back to the fundamental value of corporate bonds. Imagine a negative shock to the firm cash flows. Because it pushes the firm closer to default, this represents a negative shock to the bond's fundamental value. More importantly, because bonds of defaulted firms suffer greater illiquidity, the outside option of bondholders when bargaining with the dealer declines. This worsens the secondary market liquidity and lowers the bond prices even further. The wider refinancing gap (i.e., heavier rollover losses) between the newly issued bond prices and promised principals causes equity holders to default earlier, pushing the firm even closer to default. As a result, lower distance-to-default reduces the fundamental value of the corporate bonds even further, and so forth.

The above mechanism emphasizes that the secondary market liquidity can feed back to the

[4]This arises because bond valuations depend on firm fundamental, the bond's time-to-maturity, and the liquidity state of bond holders. Another possibility for the bankruptcy boundary condition is to assume some adverse selection with regard to the bankruptcy recovery value, a path we have do not pursue in this model due to the difficulties inherent in tracking persistent private information.

bond's fundamental value via the firm's endogenous distance-to-default. Since one can broadly interpret default as a form of disinvestment, this suggests a further interaction between firm fundamental and the firm's financing liquidity. Following this idea, we consider a simple extension of our base model where equity holders make an endogenous investment decision at the initial date. Relative to the model with exogenous financing liquidity, given negative fundamental shocks, the endogenous financing liquidity gives a further adverse kick in equity holders' investment incentives, which further lowers the firm's fundamental.

This feedback loop between fundamental and liquidity implies a full inter-dependence between liquidity and default components in the credit spread for corporate bonds. The model contrasts with the widely-used reduced-form approach in the empirical research, where it is common to decompose firms' credit spreads into independent liquidity-premium and default-premium components and then assess their quantitative contributions, e.g., Longstaff, Mithal, and Neis (2005), Beber, Brandt, and Kavajecz (2009), and Schwarz (2010). Our fully solved structural model calls for more structural approaches in the future empirical study about the impact of liquidity factors upon the credit spread of corporate bonds. Our model also offers a potential resolution to the hitherto difficulty of structural models to match the AAA credit spread as pointed out by Huang and Huang (2003). As our model features a non-vanishing liquidity premium due to an illiquid secondary market, even AAA bonds feature a yield in excess of treasuries.

Additionally, our model features a trade-off between better liquidity provision by short-term bonds, and a more severe debt-equity conflict of interest caused by a higher rollover frequency. On the liquidity provision side, bond investors hit by liquidity shocks can either sell to dealers or sit out shocks by waiting to receive the face value when the bond matures. Shorter maturity improves upon the waiting option, leading to a lower bid-ask spread and hence greater secondary market liquidity. On the other hand, equity holders are absorbing rollover losses ex post. The shorter the maturity, the higher the rollover frequency, and the heavier the rollover losses. Consequently, equity holders default earlier, leading to greater inefficiencies due to dead-weight bankruptcy costs. Based on this tradeoff, we can endogenize the firm's initial choice of debt maturity in our model.

Our paper belongs to the recent literature on the role of secondary market trading frictions in the corporate finance structural models as best exemplified by Black and Cox (1976), Leland (1994) and LT96. Ericsson and Renault (2006) is an early paper which incorporates secondary bond market liquidity into LT96. Based on numerical solutions, they emphasize the interaction between secondary liquidity and the bankruptcy-renegotiation. He and Xiong (forthcoming) (hereafter HX11) take the simplified secondary market friction introduced in the classic article of Amihud and Mendelson (1986) — bond investors hit by liquidity shocks are forced sell their holdings immediately at an exogenous and constant transaction cost. Via the endogenous default channel, HX11 emphasize that liquidity shocks may lead to a significant rise of the default component in corporate bonds' credit spreads. Our paper endogenizes the secondary market liquidity by micro-founding the bond trading in a search-based secondary market, and derives the equilibrium liquidity jointly with equilibrium asset prices.[5] It is the endogenous liquidity that distinguishes our paper from HX11, which plays a crucial role for the positive feedback mechanism between fundamental and liquidity for corporate bonds.

Our paper also makes a contribution to the search based asset-pricing literature, as represented by Duffie, Garlenau, and Pedersen (2005, 2007); Weill (2007); Lagos and Rocheteau (2007, 2009). To our knowledge, this literature with concentration on OTC markets has thus far focused on the determinants of contact intensities and behavior of intermediaries, while eschewing time-varying asset fundamentals and asset maturities. Undoubtedly, asset-specific dynamics are important for the secondary corporate bond market, and we fill this gap by incorporating the firm's distance-to-default and the bond's time-to-maturity in deriving the asset (bond) valuations.[6] Moreover, our paper demonstrates that, via the rollover channel, the endogenous search-based secondary market liquidity can have significant impact on the firms' behavior on the real side, which in turn affects

[5]Two other well-known endogenous market illiquidity models based on private information are Kyle (1985) and Glosten and Milgrom (1985). We deem that the search based framework is suitable for the secondary market for corporate bonds, and also has the advantage of being integrated seamlessly into the dynamic firm setting in LT96.

[6]The existing literature often assumes constant asset payoffs and infinite maturity. As far as we know, the only paper with deterministic time dynamics in a search framework is the contemporaneous Afonso and Lagos (2011), which introduces deterministic time dynamics via an end-of-day trading close in the federal funds market. Also, because corporate bond payoffs are highly nonlinear in firm fundamentals, our closed-form solution with stochastic fundamentals is nontrivial.

secondary market liquidity.

Positive feedback is an active research topic for different research areas. For instance, the strategic complementarity naturally gives rise to positive feedback effect in the global games literature (e.g., Morris and Shin 2009), and a similar effect emerges in He and Xiong (Forthcoming) who study dynamic coordinations among creditors whose debt contracts mature at different times. Through the information channel, Goldstein, Ozdenoren, and Yuan (2011) show that market prices can feedback to firm's investment decisions. Brunnermeier and Pedersen (2009) illustrate the positive feedback loop (spiral) between funding liquidity and market liquidity. Cheng and Milbradt (forthcoming) show how managerial risk-shifting feeds back on bondholders decision to run, which in turn feeds back on managerial incentives. Manso (2011) shows how credit ratings can affect a firm's default decision, which feeds back into the rating decision. Relative to these models that utilize a static framework (with the exception of Manso (2011), He and Xiong (Forthcoming) and Cheng and Milbradt (forthcoming)), our research is cast in a standard dynamic corporate finance structural models, which has the advantage of potentially realistic calibration to data.

The paper is organized as follows. Section 2 lays out the model with a search-based secondary bond market. Section 3 solves the model in closed-form. Section 4 illustrate the positive feedback loop between fundamental and liquidity, and Section 5 provides discussions and extensions. Section 6 concludes. All proofs can be found in the Appendix.

2 The Model

2.1 Firm Cash Flows and Debt Maturity Structure

We consider a continuous-time model where the firm generates (after-tax) cash flows at a rate of $\delta_t > 0$, where $\{\delta_t : 0 \leq t < \infty\}$ follows a geometric Brownian motion in the risk-neutral probability measure:

$$\frac{d\delta_t}{\delta_t} = \mu dt + \sigma dZ_t, \tag{1}$$

where μ is the constant growth rate of cash flow rate, σ is the constant asset volatility, and $\{Z_t : 0 \leq t < \infty\}$ is a standard Brownian motion, representing random shocks to the firm fundamental.

We assume the risk-free rate r to be constant in this economy. As our focus is on the interaction between secondary market liquidity and equity holders' endogenous default decision, this treatment is innocuous.

We follow LT96 in assuming that the firm maintains a stationary debt structure. At each moment in time, the firm has a continuum of bonds outstanding with an aggregate principal of p and an aggregate coupon payment of c, where p and c are constants that we take as exogenously given. We normalize the measure of bonds to 1, so that each bond has a principal value of p and coupon of c. Each bond has a maturity T, and expirations of the bonds are uniformly spread out across time. Here, $\frac{1}{T}$ is the firm's rollover frequency on its debt; that is, during a time interval $(t, t+dt)$, a fraction $\frac{1}{T}dt$ of the bonds matures and needs to be rolled over. In the main analysis we take the firm's debt maturity T as given; Section 5.3 discusses the optimal debt maturity T^* that maximizes the firm value.

These bonds differ only in the time-to-maturity $\tau \in [0, T]$. Denote by $D(\delta, \tau)$ the value of one unit of bond as a function of the firm fundamental δ and its time-to-maturity τ. Following the LT96 framework, we assume that the firm commits to a stationary debt structure denoted (c, p, T). In other words, when a bond matures, the firm will replace it by issuing a new bond with identical (initial) maturity T, principal value p, and coupon rate c, in the primary market (to be modeled shortly).

2.2 Secondary Bond Market and Search-Based Liquidity

As in Duffie, Garlenau, and Pedersen (2005), individual bond investors are subject to idiosyncratic liquidity shocks, and once hit by shocks they need to search for market-makers/dealers for trade. More specifically, at any time with probability ξdt individual bond holders are hit by liquidity shocks and they need to turn their holding into cash. We model this sudden need for liquidity as an upward jump in the discount rate from the common interest rate r to a higher level $\overline{r} > r$. For

simplicity, this higher discount rate persists until the agent either manages to sell his debt-holdings or the debt matures and the face value p is paid out, after which the investor exits the market forever. We call an agent who has been hit by a liquidity shock as being in the liquidity state, and use L (i.e. low valuation or liquidity shocked agent) to indicate this state.

In reality, secondary corporate debt markets are less liquid than equity or primary debt markets. Thus, we assume that the secondary debt markets are subject to the following trading frictions. The L bond investor who wants to sell his debt-holdings has to wait an exponential time with intensity λ to meet a dealer who can implement the transaction. When they meet, bargaining occurs over the economic surplus generated. We follow Duffie, Garlenau, and Pedersen (2007) who show it is sufficient to define Nash-bargaining weights $1-\beta$ of the dealer and β of the investor to model this bargaining. We also make the additional assumption that each creditor only holds one unit of bond— this is for tractability of the search market in terms of constant parameters.

The illiquidity of secondary bond markets give rise to wedges in bond valuations for different investor types. Define $D_H(\delta,\tau)$ and $D_L(\delta,\tau)$ to be the valuations of debt from the high (or normal) type and the low (or liquidity) type. Suppose that a contact between a type L investor and the dealer occurs. We make the simplifying assumption that the dealer faces a competitive interdealer market with a continuum of dealers, and at any time they (collectively) can contact H type investors who are competitive as well. Therefore, the particular dealer in question can turn around and instantaneously sell directly (or through another dealer) to H type investors at a price of $D_H(\delta,\tau)$, which implies that the surplus from trade is

$$S(\delta,\tau) \equiv D_H(\delta,\tau) - D_L(\delta,\tau).$$

The transaction price at which L types sell to the dealer, $X(\delta,\tau)$, thus implements the following

splits of the surplus,

$$\begin{aligned} D_H(\delta,\tau) - X(\delta,\tau) &= (1-\beta)\, S(\delta,\tau) \\ X(\delta,\tau) - D_L(\delta,\tau) &= \beta \cdot S(\delta,\tau), \end{aligned}$$

so that

$$X(\delta,\tau) = \beta D_H(\delta,\tau) + (1-\beta)\, D_L(\delta,\tau). \tag{2}$$

The reader should observe that this formulation treats $D_L(\delta,\tau)$ as the L types outside option that he can always assure himself. It is easy to show that when $\lambda \to \infty$, then $S \to 0$, and debt values converge to the LT96 case with perfectly liquid secondary markets.

Therefore, in our model, the ask price at which dealers sell to H type investors is simply their valuation D_H, while the bid price at which L type investors sell their bond holdings to dealers is X. This implies that $D_H - X = (1-\beta)(D_H - D_L)$ is also the (dollar) bid-ask spread, as X is the dealer's purchase price while D_H is his selling price to H type investors.[7]

A schematic representation of the different bond values and the transaction price is given in Figure 2. As we will see later, the outside option D_L changes with regard to firm fundamental δ and time to maturity τ. The three panels of the figure give a preview of our results. The height of the box indicates the value D_H, the height of the shaded area indicates the outside option D_L, and the dashed line depicts the price at which the transaction of an L type with the dealer takes place. The (dollar) bid-ask spread is thus the distance between the top solid line of the rectangle, D_H, and the dashed line, X. Panel (A) shows a bond that has just been issued with maturity T and a AAA rating, panel (B) shows a bond close to maturity but away from default, and panel (C) shows a bond close to default. In a preview of results, comparing (B) to (A), we see the maturity effect: the outside option has increased as maturity has drawn closer, and the bid-ask spread has decreased. Comparing (C) to (A), we see the default effect: the outside option sustained a larger

[7] Thus, H type investors are indifferent between buying and not buying the bond, whereas for any $\beta > 0$, L type investors strictly prefer selling the bond.

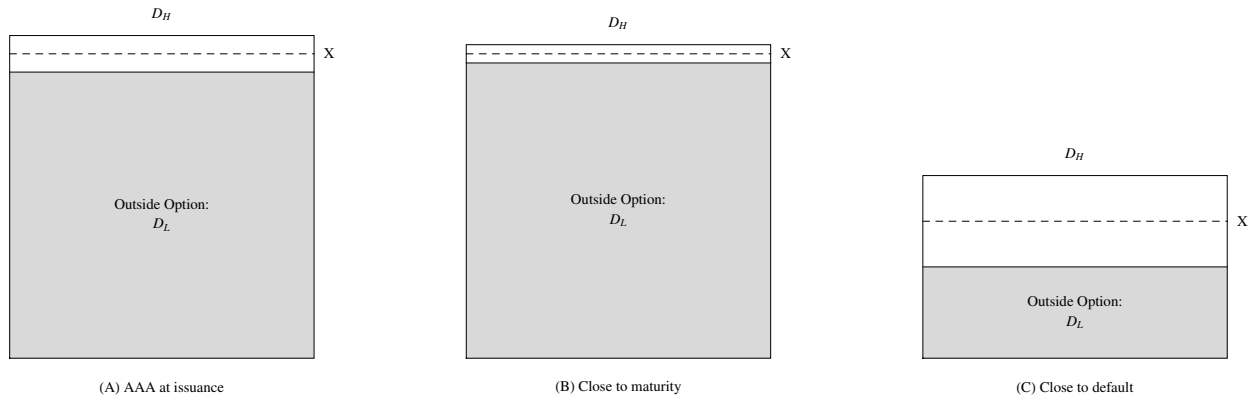


Figure 2: Schematic representation of D_H, D_L, and the transaction price X for different states (δ, τ). (A) shows a benchmark bond that is far from the bankruptcy boundary and with the maximum time-to-maturity T. (B) shows a bond that is away from the bankruptcy boundary, but close to maturity. (C) shows a bond that is with maximum maturity, but close to the bankruptcy boundary.

drop than D_H as default becomes imminent, and the bid-ask spread has increased.

2.3 Debt Rollover

As mentioned, at any time the firm replaces the maturing bonds with newly issued ones in the so-called primary market. We assume that the firm will hire a dealer who can place the new debt to H type investors, and dealers are competitive in the primary market. Thus, the firm receives the full bond value of the high type D_H.[8] Of course, the H type incorporates in his bond valuation D_H the possibility that he will be hit by a liquidity shock in the future and thus has to use the illiquid secondary market to sell the bond.

However, the newly issued bond price in the primary market can be higher or lower than the required principal payments of the maturing bonds, either due to changing firm fundamental or secondary market illiquidity. Equity holders are the residual claimants of the rollover gains/losses. Again, following LT96, we assume that any gain will be immediately paid out to equity holders and any loss will be funded by issuing more equity at the market price. Thus, over a short time

[8]Segura and Suarez (2011) present a banking model without secondary markets which concentrates on the effect of periodic shut-downs of the primary market for debt funding.

interval $(t, t+dt)$, the net cash flow to equity holders (omitting dt) is

$$NC_t = \underbrace{\delta_t}_{CF} - \underbrace{(1-\pi)\,c}_{Coupon} + \underbrace{\frac{1}{T}\left[D_H\left(\delta_t, T\right) - p\right]}_{Rollover}. \tag{3}$$

The first term is the firm's cash flow. The second term is the after-tax coupon payment to bond investors, where π denotes the marginal tax benefit rate of debt: for each dollar received by bond investors, the government is subsidizing π dollars so that equity holders only have to pay $1-\pi$ dollars.[9] The third term captures the firm's rollover gains/losses by issuing new bonds to replace maturing bonds. This term can be seen as a *repricing* of a measure $\frac{1}{T}$ bonds each t. In this transaction, there is a $\frac{1}{T}dt$ fraction of bonds maturing, which require a principal payment of $\frac{1}{T}pdt$; while the primary market value of the newly issued bonds is $\frac{1}{T}D_H(\delta_t, T)dt$. When the newly issued bond price $D_H(\delta_t, T)$ drops (say because the cash flow rate δ_t goes down) so that $D_H\left(\delta_t, T\right) < p$, equity holders have to absorb the negative cash-flow stemming from rollover, $\frac{1}{T}[D_H(\delta_t, T) - p]dt$, to prevent bankruptcy. Thus, the rollover frequency $\frac{1}{T}$ (or the inverse of debt maturity) affects the extent of rollover losses/gains.

2.4 Bankruptcy

When the firm issues additional equity to fund these rollover losses, the equity issuance dilutes the value of existing shares.[10] Equity holders are willing to buy more shares and bail out the maturing debt holders as long as the equity value is still positive (i.e. the option value of keeping the firm alive justifies absorbing the rollover losses). When the firm defaults its equity value drops to zero. The default threshold δ_B is endogenously determined by equity holders, which is an important

[9] He and Matvos (2011) present a model where corporate debt has positive externalities which offers one potential reason for the existence of this debt tax subsidy.

[10] A simple example works as follows. Suppose a firm has 1 billion shares of equity outstanding, and each share is initially valued at \$10. The firm has \$10 billion of debt maturing now, but the firm's new bonds with the same face value can only be sold for \$9 billion. To cover the shortfall, the firm needs to issue more equity. As the proceeds from the share offering accrue to the maturing debt holders, the new shares dilute the existing shares and thus reduce the market value of each share. If the firm only needs to roll over its debt once, then the firm needs to issue 1/9 billion shares and each share is valued at \$9. The \$1 price drop reflects the rollover loss borne by each share.

ingredient for the feedback loop between firm fundamentals and secondary market liquidity.[11]

When the firm goes bankrupt, we simply assume that creditors can only recover a fraction α of the firm's unlevered value from liquidation, which is $\alpha\frac{\delta_B}{r-\mu}$.[12] As usual, the bankruptcy cost is ex post borne by debt holders but represents a deadweight loss to equity holders ex ante, who cannot commit to not default. Since maturity per se does not matter in bankruptcy, for simplicity we assume equal seniority of all creditors.

Because one driving force of our model is that agents value receiving cash early, our bankruptcy treatment has to be careful in this regard. If bankruptcy leads investors to receive the proceeds immediately, L type investors who are trying to sell their bonds could view default as a beneficial outcome. In other words, bankruptcy confers a benefit similar to maturity that may outweigh the deadweight loss stemming from the bankruptcy cost $1-\alpha$. This "liquidity by default" is against the fact that in practice bankruptcy leads to a much more illiquid secondary market, the freezing of assets within the company, and a delay in the payout of any cash depending on court proceeding.[13] More importantly, this is also inconsistent with the empirical pattern shown in Figure 1 that the liquidity is lower when the firm is closer to default, which suggests a sufficient gap in the private recovery values of α_H and α_L of agents of type H and L respectively.

Motivated by above empirical facts, we make the following assumption for defaulted bonds. Suppose that after bankruptcy the cash flow stays constant at δ_B forever. To capture the uncertain timing of the court decision, the payout of cash $\alpha\frac{\delta_B}{r-\mu}$ occurs at a Poisson arrival time with intensity θ. We focus on situations where $\alpha\frac{\delta_B}{r-\mu} < p$ (which holds for all our examples) so that the recovery rate to bond holders is below 100%. Also, the secondary market for defaulted bonds is illiquid with

[11] To focus on the liquidity effect originating from the debt market, we ignore any additional frictions in the equity market such as transaction costs and asymmetric information. It is important to note that while we allow the firm to freely issue more equity, the equity value can be severely affected by the firm's debt rollover losses. This feedback effect allows the model to capture difficulties faced by many firms in raising equity during a financial-market meltdown even in the absence of any friction in the equity market.

[12] The bankruptcy cost $1-\alpha$ can be interpreted in different ways, such as loss from selling the firm's real asset to second-best users, loss of customers because of anticipation of the bankruptcy, asset fire-sale losses, legal fees, etc.

[13] The Lehman Brothers bankruptcy in September 2008 is a good case in point. After much legal uncertainty, payouts to the debt holders only started trickling out after more than 3 years.

contact intensity λ_B. Then, the defaulted bond values D_H^B and D_L^B satisfy

$$\begin{aligned} rD_H^B &= \theta\left(\alpha\frac{\delta_B}{r-\mu} - D_H^B\right) + \xi\left(D_L^B - D_H^B\right), \\ \overline{r}D_L^B &= \theta\left(\alpha\frac{\delta_B}{r-\mu} - D_L^B\right) + \lambda_B\left(X^B - D_L^B\right), \end{aligned}$$

where as before $X^B = \beta D_L^B + (1-\beta) D_H^B$ is the transaction price received by L type investors. Plugging X^B into the above equations, we can solve for $D_i^B = \alpha_i \frac{\delta_B}{r-\mu}$ for $i \in \{H, L\}$ where

$$\begin{aligned} \alpha_H &= \tfrac{\theta\alpha(\overline{r}+\theta+\lambda_B\beta+\xi)}{\overline{r}(\xi+\theta)+r(\overline{r}+\theta+\lambda_B\beta)+\theta(\xi+\theta+\lambda_B\beta)}, \\ \alpha_L &= \tfrac{\theta\alpha(r+\theta+\lambda_B\beta+\xi)}{\overline{r}(\xi+\theta)+r(\overline{r}+\theta+\lambda_B\beta)+\theta(\xi+\theta+\lambda_B\beta)}. \end{aligned}$$

One can easily see that $\alpha_H > \alpha_L$ as $\overline{r} > r$. We denote the (bold face) vector $\boldsymbol{\alpha} \equiv [\alpha_H, \alpha_L]^\top$ as the effective bankruptcy cost factors from the perspective of different bond holders. Clearly, the wedge $\alpha_H - \alpha_L$ characterizes the illiquidity of the defaulted bonds when the firm (i.e. equity holders) declares bankruptcy. Throughout the paper we will assume that this illiquidity in the default state is sufficiently high.

3 Model Solutions

3.1 Debt Valuations and Credit Spreads

We first derive bond valuations by taking the firm's default boundary δ_B as given. Recall that $D_H(\delta, \tau)$ and $D_L(\delta, \tau)$ are the value of one unit of bond with time-to-maturity $\tau \leq T$, an annual coupon payment of c, and a principal value of p to a type H and L investor, respectively. We have the following system of partial differential equation (PDE) for the values of D_H and D_L:

$$\begin{aligned} rD_H &= c - \frac{\partial D_H}{\partial \tau} + \mu\delta \cdot \frac{\partial D_H}{\partial \delta} + \frac{\sigma^2\delta^2}{2}\frac{\partial^2 D_H}{\partial \delta^2} + \underbrace{\xi\left[D_L - D_H\right]}_{Liqidity\,shock}, \\ \overline{r}D_L &= c - \frac{\partial D_L}{\partial \tau} + \mu\delta \cdot \frac{\partial D_L}{\partial \delta} + \frac{\sigma^2\delta^2}{2}\frac{\partial^2 D_L}{\partial \delta^2} + \underbrace{\lambda\left[X - D_L\right]}_{Secondary\,market}. \end{aligned} \tag{4}$$

where we omit the two-dimensional argument (δ, τ) for both debt value functions. More importantly, we omit the equity holders' default boundary δ_B in the above PDE system, which bond investors take as given.

The first equation in (4) is the type H bond valuation. The left-hand side rD_H is the required (dollar) return from holding the bond for type H investors. There are four terms on the right-hand side, capturing expected returns from holding the bond. The first term is the coupon payment. The next three terms capture the expected value change due to change in time-to-maturity τ (the second term) and fluctuation in the firm's fundamental δ_t (the third and fourth terms). The last term is a loss $D_L - D_H$ caused by the liquidity shock that transforms H investors into L investors, multiplied by the probability of the liquidity shock over dt.

The second equation in (4), the type L bond valuation, follows a similar explanation to the one above. The two differences are that the left hand side now has a higher required return $\overline{r}$, and there is the value impact of the secondary market reflected in the last term of the right hand side. A type L investor meets a dealer with probability of λdt and is then able to sell his bond (with a private value D_L) at a price of

$$X\left(\delta, \tau; \delta_B\right) = (1-\beta) D_L\left(\delta, \tau; \delta_B\right) + \beta D_H\left(\delta, \tau; \delta_B\right). \tag{5}$$

As explained in Section 2.2, this price is derived under the assumption that the dealer can turn around and immediately sell the bond to H type investors through a frictionless interdealer market.

Define the matrix $\mathbf{A}$ so that the following decomposition holds:

$$\mathbf{A} \equiv \begin{bmatrix} r+\xi & -\xi \\ -\lambda\beta & \overline{r}+\lambda\beta \end{bmatrix} = \mathbf{P}\hat{\mathbf{D}}\mathbf{P}^{-1}.$$

We let $\hat{\mathbf{D}} \equiv Diag\begin{bmatrix} \hat{r}_1 & \hat{r}_2 \end{bmatrix}$, where $\hat{r}_i = \frac{r+\xi+\overline{r}+\lambda\beta \pm \sqrt{[(r+\xi)-(\overline{r}+\lambda\beta)]^2+4\xi\lambda\beta}}{2}$ satisfying $\hat{r}_1 > \overline{r} > \hat{r}_2 > r$, be the matrix of eigenvectors of $\mathbf{A}$, and denote by $\mathbf{P}$ be the matrix of stacked eigenvalues. For a

given δ_B, we derive the closed-form solution for the bond values in the next proposition.[14]

Proposition 1 *The debt values are given by*

$$\begin{bmatrix} D_H(\delta,\tau) \\ D_L(\delta,\tau) \end{bmatrix} = \mathbf{P} \begin{bmatrix} A_1 + B_1 e^{-\hat{r}_1\tau}\left[1 - F(\delta,\tau)\right] + C_1 G_1(\delta,\tau) \\ A_2 + B_2 e^{-\hat{r}_2\tau}\left[1 - F(\delta,\tau)\right] + C_2 G_2(\delta,\tau) \end{bmatrix}. \tag{6}$$

Here, by defining $a \equiv \frac{\mu - \frac{\sigma^2}{2}}{\sigma^2}$, $\gamma_1 \equiv 0$, $\gamma_2 \equiv -2a$, $\eta_{j1,2} \equiv -a \pm \sqrt{a^2 + \frac{2}{\sigma^2}\hat{r}_j}$, *and* $q(\delta,\chi,t) \equiv \frac{\log(\delta_B) - \log(\delta) - (\chi + a)\cdot\sigma^2 t}{\sigma\sqrt{t}}$, *the constants in (6) are given by:*

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \equiv c\hat{\mathbf{D}}^{-1}\mathbf{P}^{-1}\mathbf{1}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \equiv p\mathbf{P}^{-1}\mathbf{1} - c\hat{\mathbf{D}}^{-1}\mathbf{P}^{-1}\mathbf{1}, \quad \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \equiv \frac{\delta_B}{r-\mu}\mathbf{P}^{-1}\boldsymbol{\alpha} - c\hat{\mathbf{D}}^{-1}\mathbf{P}^{-1}\mathbf{1},$$

$$F(\delta,\tau) \equiv \sum_{i=1}^{2}\left(\frac{\delta}{\delta_B}\right)^{\gamma_i} N\left[q(\delta,\gamma_i,\tau)\right], \; G_j(\delta,\tau) \equiv \sum_{i=1}^{2}\left(\frac{\delta}{\delta_B}\right)^{\eta_{ji}} N\left[q(\delta,\eta_{ji},\tau)\right],$$

where $N(x)$ *is the cumulative distribution function for a standard normal distribution.*

A closer inspection of the solution reveals a linear combination (via the matrix $\mathbf{P}$) of two sub-solutions each closely related to the original LT96 solution. The main difference to the LT96 solution is that each of these independent sub-solutions $i = \{1,2\}$ has a distorted discount rate $\hat{r}_i > r$ (for $\xi > 0$), a distorted coupon rate $\hat{c}_i \equiv \left(c\mathbf{P}^{-1}\mathbf{1}\right)_i$ and a distorted recovery rate $\hat{\alpha}_i \equiv \left(\mathbf{P}^{-1}\boldsymbol{\alpha}\right)_i$.[15]

3.2 Equity Valuation

The next step in the analysis of this problem is the equity holders decision to default. Equity holders receive the net cash flow in (3) every instant. Because equity is naturally an infinite maturity security and we are investigating a stationary setting, its value $E(\delta;\delta_B)$ satisfies the

[14] All proofs, because of the linear decomposition, would go through even if creditors would be subject to possibly different shock states, $\overline{r}_1, \overline{r}_2, ...$

[15] Here, for any matrix or vector $\mathbf{M}$, $(\mathbf{M})_i$ selects the i-th row and $(\mathbf{M})_{ij}$ selects the i-th row and j-th column of the matrix.

following ordinary differential equation:

$$rE = \delta - (1-\pi)\, c + \underbrace{\frac{1}{T}\left[D_H\left(\delta, T\right) - p\right]}_{Rollover} + \mu\delta E' + \frac{\sigma^2\delta^2}{2}E'', \tag{7}$$

where the left hand side is the required rate of return of equity holders. On the right hand side, the first three terms are the equity holders net cash flows, and the next two terms are capturing the instantaneous change of the firm fundamental. As mentioned earlier, the term involving square brackets is the cash-flow term that arises from rolling over debt (while keeping coupon, principal, and maturity stationary), with $\frac{1}{T}$ being the rollover frequency.

Similar to HX11, we cannot solve equity value as the difference between the levered firm value and debt value as in LT96, because part of the firm value goes to the dealers in the secondary bond market, and part vanishes because of inefficient holdings of debt by L types. Instead, we need to solve for $E\left(\delta\right)$ based on (7), which is non-trivial due to the highly-nonlinear form of $D_H\left(\delta, T\right)$ given in (6). The next proposition gives the equity value.[16]

Proposition 2 *Given a default boundary δ_B, the equity value is given by*

$$E\left(\delta; \delta_B\right) = K\left(\frac{\delta}{\delta_B}\right)^{\kappa_2} + \frac{\delta}{r-\mu} + K_0 - \frac{g_F\left(\delta\right)}{T}\sum_{j=1}^{2} P_{0j}B_j e^{-\hat{r}_j T} + \frac{1}{T}\sum_{j=1}^{2} P_{0j}C_j g_{G_j}\left(\delta\right), \tag{8}$$

where $P_{01} = P_{11}$ and $P_{02} = P_{12}$ and P_{ij} gives the element of $\mathbf{P}$ in row i and column j, $\kappa_{1,2} \equiv -a \pm \frac{\sqrt{a^2\sigma^4 + 2\sigma^2 r}}{\sigma^2}$, $\Delta\kappa \equiv \kappa_1 - \kappa_2$, and

$$\begin{aligned}
K_0 &\equiv \tfrac{1}{r}\left\{-(1-\pi)\, c + \tfrac{1}{T}\left[\textstyle\sum_{j=1}^{2} P_{0j}A_j + \sum_{j=1}^{2} P_{0j}B_j e^{-\hat{r}_j T} - p\right]\right\},\\
K &\equiv -\left[\delta_B + K_0 - \tfrac{1}{T}g_F\left(\delta_B\right)\textstyle\sum_{j=1}^{2} P_{0j}B_j e^{-\hat{r}_j T} + \tfrac{1}{T}\sum_{j=1}^{2} P_{0j}C_j g_{G_j}\left(\delta_B\right)\right],\\
g_F\left(x\right) &\equiv \tfrac{1}{-\Delta\kappa}\tfrac{2}{\sigma^2}\textstyle\sum_{i=1}^{2}\left\{\frac{x^{\kappa_2}}{\delta_B^{\gamma_i}}H\left(x, \gamma_i, \kappa_2, T\right) - \frac{x^{\kappa_1}}{\delta_B^{\gamma_i}}H\left(x, \gamma_i, \kappa_1, T\right)\right\},\\
g_{G_j}\left(x\right) &\equiv \tfrac{1}{-\Delta\kappa}\tfrac{2}{\sigma^2}\textstyle\sum_{i=1}^{2}\left\{\frac{x^{\kappa_2}}{\delta_B^{\eta_{ij}}}H\left(x, \eta_{ij}, \kappa_2, T\right) - \frac{x^{\kappa_1}}{\delta_B^{\eta_{ij}}}H\left(x, \eta_{ij}, \kappa_1, T\right)\right\},\\
H\left(\delta, \chi, \kappa, T\right) &\equiv \tfrac{1}{\kappa-\chi}\left\{\delta^{\chi-\kappa}N\left[q\left(\delta, \chi, T\right)\right] - \delta_B^{\chi-\kappa}e^{\frac{1}{2}\left[(\kappa+a)^2 - (\chi+a)^2\right]\sigma^2 T}N\left[q\left(\delta, \kappa, T\right)\right]\right\},
\end{aligned}$$

[16] We obtain closed form solution for E via a variation of coefficients solution method that is applicable to linear ODEs, a technique shown in more detail in Milbradt (forthcoming).

where $q(\cdot,\cdot,\cdot)$ is given in Proposition 1.

We can also calculate the total levered firm value at time 0 with fundamental δ_0. Following LT96,[17] we assume that at time 0 the firm is issuing new bonds to H type investors only with a uniform distribution of maturities on $[0,T]$.[18] Given the results established above, the levered firm value $TV_0(\delta_0,T;\delta_B)$ is

$$\begin{aligned} & TV_0(\delta_0,T;\delta_B) \\ = \; & E(\delta_0;\delta_B)+\frac{1}{T}\int_0^T D_H(\delta_0,\tau;\delta_B)\,d\tau \\ = \; & E(\delta_0;\delta_B)+[P_{01},P_{02}]\left(\left[\begin{array}{c} A_1 \\ A_2 \end{array}\right]+\left[\begin{array}{c} B_1\left(\frac{1-e^{-\hat{r}_1T}}{\hat{r}_1T}-I_1(\delta_0,T)\right) \\ B_2\left(\frac{1-e^{-\hat{r}_2T}}{\hat{r}_2T}-I_2(\delta_0,T)\right) \end{array}\right]+\left[\begin{array}{c} C_1J_1(\delta_0,T) \\ C_2J_2(\delta_0,T) \end{array}\right]\right) \end{aligned} \quad (9)$$

where

$$\begin{aligned} I_j(\delta,T) &= \frac{1}{\hat{r}_jT}\left[G_j(\delta,T)-e^{-\hat{r}_jT}F(\delta,T)\right], \\ J_j(\delta,T) &= \frac{1}{(\eta_{1j}+a)\,\sigma\sqrt{T}}\sum_{i=1}^{2}(-1)^i\left(\frac{\delta}{\delta_B}\right)^{\eta_{ij}}N\left[q(\delta,\eta_{ij},T)\right]q(\delta,\eta_{ij},T). \end{aligned}$$

3.3 Endogenous Default Boundary

So far we have taken the default boundary δ_B as given. We now use the standard smooth pasting condition $E_\delta(\delta_B;\delta_B)=0$ to determine the optimal δ_B that is chosen by equity holders. The following proposition gives the closed-form solution for the endogenous default boundary δ_B.

Proposition 3 *The endogenous default boundary δ_B^* is given by*

$$\delta_B^*(T) = (r-\mu)\left[\kappa_2-1+\frac{1}{T}\sum_{j=1}^{2}P_{0j}\hat{\alpha}_jh_{G_j}\right]^{-1}\left(-\kappa_2K_0+\frac{h_F}{T}\sum_{j=1}^{2}P_{0j}B_je^{-\hat{r}_jT}+\frac{1}{T}\sum_{j=1}^{2}P_{0j}A_jh_{G_j}\right),$$

[17] The reader should note the difference that we have one unit measure of bonds, whereas LT96 expand the measure of bonds according to maturity.

[18] The following closed form only holds if all initial bonds are issued uniformly and evenly across τ and H and L types. The Appendix presents the integral expression that would result if H and L proportions vary with τ.

Firm Characteristics			Illiquid Secondary Market		
Parameter	Interpretation	Value	Parameter	Interpretation	Value
r	Discount rate	8%	$\overline{r}$	Liq. shock discount rate	9%
σ	Volatility	18%	ξ	Intensity of liquidity shock	4/10
μ	Drift	6%	λ	Intensity to meet dealers	1/10
π	Tax shield	27%	β	Bargaining power of investors	1/2
p	Principal	1	α_H	Recovery value H type	6/10
c	Coupon	8	α_L	Recovery value L type	3/10
T	Bond maturity	2			

Table 1: Benchmark parameters.

where $\hat{\boldsymbol{\alpha}} \equiv \mathbf{P}^{-1}\boldsymbol{\alpha}$*,* P_{0j} *is defined in the previous proposition and*

$$\begin{aligned} h_F &\equiv -\tfrac{2}{\sigma^2}\textstyle\sum_{i=1}^{2}\frac{1}{\kappa_1-\gamma_i}\left\{N\left[-\left(\gamma_i+a\right)\sigma\sqrt{T}\right]-e^{rT}N\left[-\left(\kappa_1+a\right)\sigma\sqrt{T}\right]\right\}, \\ h_{G_j} &\equiv -\tfrac{2}{\sigma^2}\textstyle\sum_{i=1}^{2}\frac{1}{\kappa_1-\eta_{ij}}\left\{N\left[-\left(\eta_{ij}+a\right)\sigma\sqrt{T}\right]-e^{(r-\hat{r}_j)T}N\left[-\left(\kappa_1+a\right)\sigma\sqrt{T}\right]\right\}. \end{aligned}$$

One can easily verify that $\lim_{T\to\infty}\frac{\delta_B^*}{r-\mu}=\lim_{T\to\infty}V_B^*=\frac{\kappa_2(1-\pi)c}{\kappa_2-1}$, which is the optimal bankruptcy boundary obtained in Leland (1994). When debt maturity T goes to zero, we can further show its finite limit, which for $\alpha=\alpha_L=\alpha_H$ converges to $\lim_{T\to 0}\frac{\delta_B^*}{r-\mu}=\lim_{T\to\infty}V_B^*=\frac{p}{\alpha}$, also the boundary derived in Leland (1994).

4 Endogenous Liquidity and Feedback Effects

Given the results derived in Section 3, we now discuss the model implications. Section 4.1 analyzes the endogenous transaction cost that depends on both firm fundamental and time-to-maturity. Based on endogenous liquidity, Section 4.2 shows that the endogenous liquidity implies an interesting feedback effect between fundamental and liquidity for corporate bonds. Table 1 gives the the baseline parameters that we use for illustration in this section.

4.1 Endogenous Liquidity

Our paper incorporates micro-structure trading frictions into a Leland-type corporate finance structural model, which allows us to derive an endogenous bid-ask spread that depends on both the firm

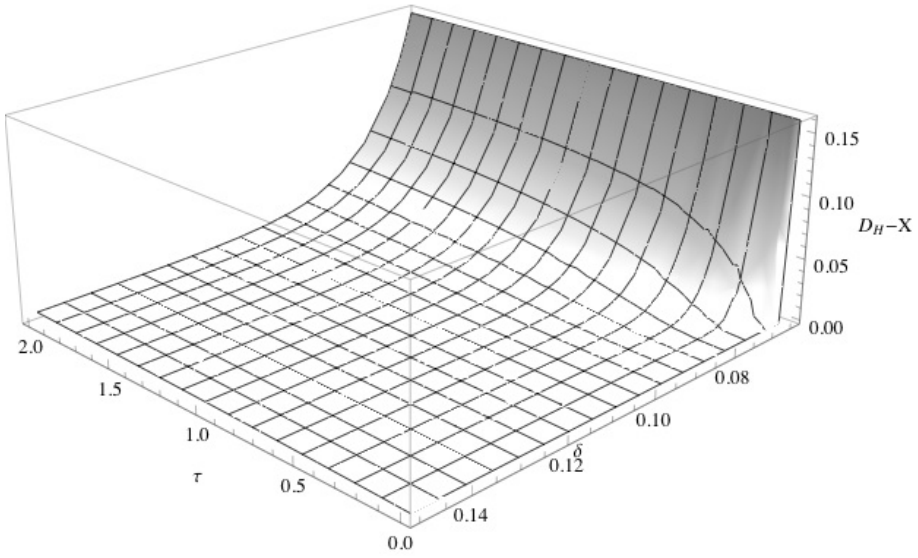


Figure 3: **Dollar bid-ask spread** $D_H - X$ with $T = 2$ as a function of both time-to-maturity τ and fundamental cash-flow δ. The bankruptcy boundary is $\delta_B^*(2) \approx .064$.

fundamental, $\frac{\delta}{\delta_B}$, and the bond's time-to-maturity, τ, via the surplus (and thus also L types outside option) dynamics.

4.1.1 Endogenous bid-ask spread

The (dollar) bid-ask spread is simply the difference between the bid price $X(\delta, \tau)$ and the ask price $D_H(\delta, \tau)$:

$$(1 - \beta) S(\delta, \tau) = D_H(\delta, \tau) - X(\delta, \tau). \tag{10}$$

We see that the dollar bid-ask spread is just a constant positive multiple of the surplus S for any $\beta < 1$. In the following proofs, we thus concentrate on the behavior of S. We plot the bid-ask spread in Figure 3 as a function of both distance-to-default (that is state dynamics) and time-to-maturity (that is time dynamics). Note that the highest time-to-maturity is just the maturity for newly issued bonds, which in the figure is $T = 2$. The distance-to-default is captured by the difference between the current firm fundamental δ from the endogenous bankruptcy boundary $\delta_B^*(2) \approx .064$

Time-to-maturity First, let us fix the firm fundamental and study the bid-ask spread when we vary the time-to-maturity. Figure 3 shows that the endogenous bid-ask spread is lower for shorter

time-to-maturities. Formally, we have the following proposition.

Proposition 4 *Under sufficient conditions provided in the Appendix, we have* $S_\tau(\delta, \tau) < 0$*, i.e. the bid-ask spread is smaller for bonds with shorter time-to-maturity.*

The intuition for this result is simple. Because a shorter time-to-maturity delivers the full principal back to L type investors sooner, this enhances L type investors' bargaining position and reduces the rent extracted by dealers, thereby resulting in a smaller bid-ask spread. In fact, it is easy to show analytically that the bid-ask spread vanishes as time-to-maturity goes towards 0, i.e.,

$$\lim_{\tau \to 0} S(\delta, \tau) = 0.$$

Intuitively, if the bond's face value is almost immediately demandable from the firm, L type investors gain little value from trade with dealers, and as a result the bid-ask spread vanishes. This indicates that short-term debt provides liquidity for bond investors, and we will discuss the role of liquidity provision in more detail in Section 5.3.

Distance-to-default Second, let us fix the time-to-maturity $\tau > 0$ and then investigate the bid-ask spread by varying the distance-to-default (i.e., $\delta - \delta_B$). As shown in Figure 3, the bid-ask spread rises when the firm fundamental deteriorates towards the bankruptcy boundary δ_B, which is consistent with the empirical regularity in Edwards, Harris, and Piwowar (2007), Bao, Pan, and Wang (forthcoming), and Figure 1. Formally, we have the following proposition.

Proposition 5 *Under sufficient conditions provided in the Appendix, we have* $S_\delta(\delta, \tau) < 0$*, i.e. the bid-ask spread is smaller for bonds with higher firm fundamental.*

Recall that, in Section 2.4, we assumed, motivated by the empirical facts, that the secondary market for defaulted debt is less liquid, and bond investors need to wait some time before they receive the cash pay-out of $\frac{\alpha \delta_B}{r - \mu}$. It is easy to show that as the firm fundamental converges towards

δ_B, for any bonds that still have time-to-maturity left, i.e. $\tau > 0$, we have

$$\lim_{\delta \to \delta_B} S(\delta, \tau) = (\alpha_H - \alpha_L) \frac{\delta_B}{r - \mu} > 0, \tag{11}$$

Here, we assume that the post-default illiquidity $\alpha_H - \alpha_L$ is sufficiently high, especially relative to the bid-ask spread for default-free bonds.[19] As a a result, the endogenous bid-ask spread rises when the cash flow rate δ deteriorates and the firm is closer to bankruptcy.

Finally, note that for ease of analytical derivations we have focused on dollar bid-ask spread $S(\delta, \tau)$. Another commonly used illiquidity measure is the effective percentage bid-ask spread $\Delta(\delta, \tau)$, which is defined as the dollar bid-ask spread $S(\delta, \tau)$ divided by the mid point of transaction prices (bid price X and ask price D_H):

$$\Delta(\delta, \tau) = \frac{(1-\beta)\left[D_H(\delta, \tau) - D_L(\delta, \tau)\right]}{\frac{1}{2}X(\delta, \tau) + \frac{1}{2}D_H(\delta, \tau)} = \frac{2(1-\beta) S(\delta, \tau)}{(1+\beta) S(\delta, \tau) + D_L(\delta, \tau)},$$

which shares the same qualitative properties as $S(\delta, \tau)$. A bond's value decreases naturally as the firm is closer to default and thus tends to shrink the overall scale $S(\delta, \tau)$. This complicates our proof of the derivatives of the dollar bid-ask spread $S(\delta, \tau)$. In contrast, the percentage bid-ask spread $\Delta(\delta, \tau)$ is free of this artificial (negative) force. To the extent that percentage transaction cost is the more empirically relevant illiquidity measure, the sufficient conditions in Proposition 5 are much stronger than necessary and our theoretical results are more general than it appears.

[19] The intuition is quite simple: When $\delta = \infty$, so that bonds are risk-free, we have

$$\begin{aligned}
\begin{bmatrix} D_H(\infty, \tau) \\ D_L(\infty, \tau) \end{bmatrix} &= \mathbf{A}^{-1}\mathbf{c} + \exp(-\mathbf{A}\tau)\left(\mathbf{p} - \mathbf{A}^{-1}\mathbf{c}\right) \\
&= \frac{c}{(r+\xi_1)(\overline{r}+\lambda\beta) - \xi_1\lambda\beta} \begin{bmatrix} \overline{r}+\xi_1+\lambda\beta \\ r+\xi_1+\lambda\beta \end{bmatrix} + \exp(-\mathbf{A}\tau) \begin{bmatrix} p - \frac{c(\overline{r}+\xi_1+\lambda\beta)}{(r+\xi_1)(\overline{r}+\lambda\beta)-\xi_1\lambda\beta} \\ p - \frac{c(r+\xi_1+\lambda\beta)}{(r+\xi_1)(\overline{r}+\lambda\beta)-\xi_1\lambda\beta} \end{bmatrix}
\end{aligned}$$

Together with $S_\tau(\delta, \tau) < 0$, we know that S reaches a maximum when $\tau = T$. The most important part of the proof is that $S(\delta_B, \tau) - \lim_{\delta \to \infty} S(\delta, \tau) < 0$. That is, a necessary condition is that the bid-ask spread of the default-free bond is below that of the defaulted bond. Unfortunately we are unable to show the sufficiency of this condition due to the complex nature of the functions involved, and in the proof of Proposition 5 we impose stronger sufficient conditions.

Interaction between time-to-maturity and distance-to-default Interestingly, Eq. (11) shows that at $\delta = \delta_B$ we have a completely flat bid-ask spread across all maturities excluding only $\tau = 0$. Let us now switch to the percentage bid-ask spread $\Delta(\delta, \tau)$ as this will be the one used for our empirical predictions. We now investigate the impact of the interaction between time-to-maturity and distance-to-default on the endogenous bid-ask spread. Similar to $S(\delta, \tau)$, $\Delta(\delta, \tau)$ is increasing with τ for $\delta > \delta_B$ as shorter maturity provides better liquidity. However, we also know from (11) that, as we approach the bankruptcy boundary δ_B, $\Delta(\delta, \tau)$ becomes independent of $\tau > 0$. Thus, when the firm edges closer and closer to default, the slope of $\Delta(\delta, \tau)$ with respect to time-to-maturity τ for any $\tau > 0$ has to become flatter and flatter. In Figure 4, we observe that the slope on time-to-maturity increases with distance to default. In other words, for financially healthy firms, the difference between the bid-ask spreads of long-term bond and short-term bond is greater than that of firms in imminent danger of bankruptcy.

This property is intuitive. Default, by forcing firms to enter lengthy bankruptcy proceeding that puts all debt holders of equal seniority on equal footing, eliminates debt difference due to maturities. For financially healthy firms, default is remote, and therefore the time-to-maturity has a positive and significant impact on the bid-ask spread. However, when default is imminent, although the bid-ask spreads for both long-term and short-term bonds soar, their difference diminishes as it is more likely that the stated time-to-maturity eventually becomes irrelevant. This intuition, which only relies on the fact that maturity plays no role in bankruptcy, holds generally, although we cannot provide rigorous proofs for this property.

Empirical predictions The results discussed in Section 4.1.1 has the following testable predictions regarding the relation between the corporate bond's bid-ask spread and the bond's time-to-maturity and the firm's distance-to-default. We envision the following regression specification:

$$\Delta_{i,t} = b_0 + \underset{+}{b_{Maturity}} \cdot Maturity_{i,t} + \underset{+}{b_{CDS}} \cdot CDS_{i,t} + \underset{-}{b_{Maturity*CDS}} \cdot Maturity_{i,t} \times CDS_{i,t}. \quad (12)$$

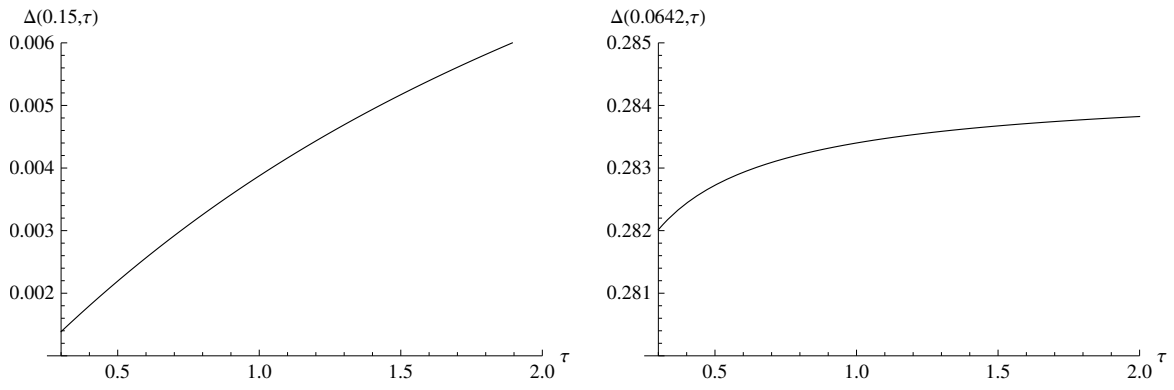


Figure 4: **Bid-ask spread** $S(\delta, \tau)$ with $T = 2$ as a function of time-to-maturity τ for financially healthy firms with $\delta = .15$ (left-hand panel) and for financially distressed firms with $\delta = .0642$ (right-hand panel).

As shown, our model predicts a positive $b_{Maturity}$, i.e., bonds with longer time-to-maturity should have a higher bid-ask spread. Further, the model predicts a positive b_{CDS}, i.e., the bond that is closer to default should have a higher bid-ask spread as well. These two predictions conform with the empirical findings in Edwards, Harris, and Piwowar (2007), and Bao, Pan, and Wang (forthcoming). Finally, Figure 4 implies that $b_{Maturity*CDS} < 0$, i.e., the difference between the bid-ask spreads of long-term and short-term bonds in financially healthy firms is greater than that of financially distressed firms. As just explained, this new testable prediction is intuitive and awaiting future empirical research.

4.1.2 Delay to trade and instantaneous transaction costs

In this section we connect our theoretical framework with the one used in AM86 and HX11. There are two important distinctions. First, in AM86 and HX11, there is no delay to trade for investors hit by liquidity shocks as they are forced to sell their holdings to some dealer immediately; second, both AM86 and HX11 feature an *exogenously* given constant proportional transaction cost. In contrast, our model has a secondary market modeled as a search market, which features a delay to trade and endogenous bargaining.

First, we can establish an equivalent *instantaneous* transaction cost $k(\delta, \tau)$ that would make the creditor indifferent to selling immediately or using the time consuming search market. We use

the following simple equivalence condition at the time of the shock to establish the appropriate transaction cost:

$$[1-k(\delta,\tau)]\,D_H(\delta,\tau)=D_L(\delta,\tau)\Leftrightarrow k(\delta,\tau)=1-\frac{D_L(\delta,\tau)}{D_H(\delta,\tau)}, \tag{13}$$

The interpretation is the following. When hit by a liquidity shock, the bond value transitions to $D_L(\delta,\tau)$ in our model. This would be equivalent to being able to sell debt immediately for an after-cost price $[1-k(\delta,\tau)]\,D_H(\delta,\tau)$. It is easy to show that

$$\lim_{\tau\to 0}k(\delta,\tau) \;=\; 0,\ \text{and}\ \lim_{\delta\to\delta_B}k(\delta,\tau) \;=\; 1-\tfrac{\alpha_L}{\alpha_H}>0\ ,$$

i.e. $k(\delta,\tau)$ vanishes for ultra short-term bonds (or bonds close to maturity), and approaches a positive constant for bonds close to default. The pattern of the implied instantaneous transaction cost $k(\delta,\tau)$ in (13) is similar to the bid-ask spread $S(\delta,\tau)$ depicted in Figure 4.

Although the implied instantaneous transaction cost $k(\delta,\tau)$ admits a simple interpretation, it is not directly comparable to the exogenous transaction cost k in the benchmark model of HX11. Rather, we will define $k_{implied}(\delta,\tau)$ as the hypothetical constant transaction cost that equates the debt price derived in HX11 to the newly issued price $D_H(\delta,T)$ derived in our model, i.e.

$$D_H(\delta,T)=D(\delta,T;k_{implied},\text{HX11_Model}).$$

This measure will be used in Section (4.2.1) to emphasize role of endogenous liquidity. Clearly, $k_{implied}$ varies with the firm fundamental δ, which is in sharp contrast with the assumption of constant transaction cost in HX11.

4.2 Feedback Loop between Fundamental and Liquidity

Linking the secondary market liquidity endogenously to firm fundamental is the key feature that distinguishes our paper from HX11, as this endogenous effect allows us to study the new feed-

back loop in which the deterioration of firm fundamental, via worsening liquidity of the secondary bond market, edges the firm even closer to default, which in turn leads to further deterioration in secondary market liquidity.

This result is in the spirit of Brunnermeier and Pedersen (2009). We observe that the rollover operation of the firm is a funding operation. Thus, secondary market liquidity feeds back into the funding liquidity of the firm, which here is the ease of raising outside money against the promise of a fixed payment in the future. This funding liquidity in turn affects the market liquidity via its impact on the default decision of the firm. When funding liquidity is low and therefore the firm is close to default, raising money is difficult or costly, which in turn affects the market liquidity of the secondary market.

4.2.1 Endogenous liquidity, rollover losses, and endogenous default

The combination of endogenous secondary market liquidity and endogenous default decision taken by equity holders are the building blocks for the positive feedback loop between fundamental and liquidity. For illustration, we contrast our model with HX11 who assume an exogenous transaction cost whenever H investors are hit by liquidity shocks and have to sell the bond holding *immediately*.[20] In both our paper and HX11, equity holders make endogenous default decision; however, in our paper the bond market liquidity (bid-ask spread) endogenously worsens when the firm is closer to default. Interestingly, relative to the HX11 benchmark, this endogenous pro-cyclical secondary bond market liquidity drives equity holders to default earlier.

To understand the mechanism, consider the rollover losses borne by equity holders as a function of firm cash flow rate δ. The dashed line in the left panel of Figure 5 graphs the benchmark rollover losses implied by HX11 who assume a constant (proportional) transaction cost k (with a value 1.2% in this example). In HX11, the (absolute value) of rollover losses $\frac{1}{T}\left[D\left(\delta, T\right) - p\right]$ rises when the firm fundamental deteriorates, simply because forward looking bond investors adjust the market price of newly issued bonds downward when the firm is closer to default. This force is already

[20]The HX11 benchmark has discount rate r, coupon c, principal p, recovery value α_H, and liquidity shock intensity ξ given in Table 1. Additionally, we assume a proportional transaction cost of $k = 1.2\%$.

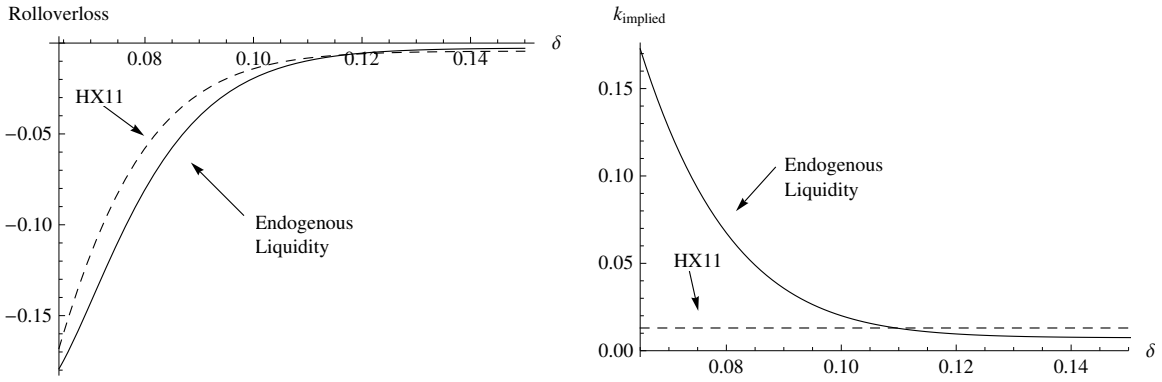


Figure 5: Left panel: **Rollover loss** $\frac{1}{T}\left[D_H\left(\delta,T\right)-p\right]$ as a function of fundamental value δ for main model (solid line) and the HX11 model (dashed line) with $k_{HX11}=1.2\%$. Right panel: **Implied transaction cost** $k_{implied}$ for HX11 model (dashed line), and for main model with fully optimal bankruptcy boundary (solid line).

present in LT96.

In our model, the endogenous secondary market liquidity further amplifies the rollover losses. In Figure 3 we have seen that the bid-ask spread $\Delta\left(\delta,\tau\right)$ decreases with the firm's cash flow rate δ, suggesting that the secondary market liquidity dries up exactly when the firm fundamental deteriorates. The right panel of Figure 5 graphs the implied endogenous transaction cost $k_{implied}$ (solid line) of our model against the transaction cost in HX11 (dashed line) under the assumption $k=1.2\%$. As mentioned in Section (4.1.2), $k_{implied}$ is the hypothetical constant transaction cost that equates the debt price derived in HX11 to our newly issued price $D_H\left(\delta,T\right)$.[21] As expected, the implied transaction cost $k_{implied}$ goes up with a lower δ, indicating a worsening secondary market liquidity for firms with lower fundamentals.[22]

Therefore, relative to HX11 with constant secondary market liquidity, the endogenous search market depresses the bond market price $D_H\left(\delta,T\right)$ further for low fundamental states. Put differently, equity holders have to absorb heavier rollover losses exactly in bad times. This result is shown in the solid line in the left panel in Figure 5; relative to HX11, equity holders' rollover loss in our model is more sensitive to the firm cash flow state δ. The pro-cyclical secondary market

[21]The reader can think of the derivation of $k_{implied}$ as an exercise very much like deriving the implied volatility w.r.t. the Black-Scholes formula for any given or observed option price. It is thus different from the instantaneous transaction cost $k\left(\delta,\tau\right)$ as it takes into account the possible changes of $k\left(\delta,\tau\right)$ over the life of the bond.

[22]Again, drawing parallels to empirical option pricing, there exists a "smirk" in $k_{implied}$ w.r.t. δ.

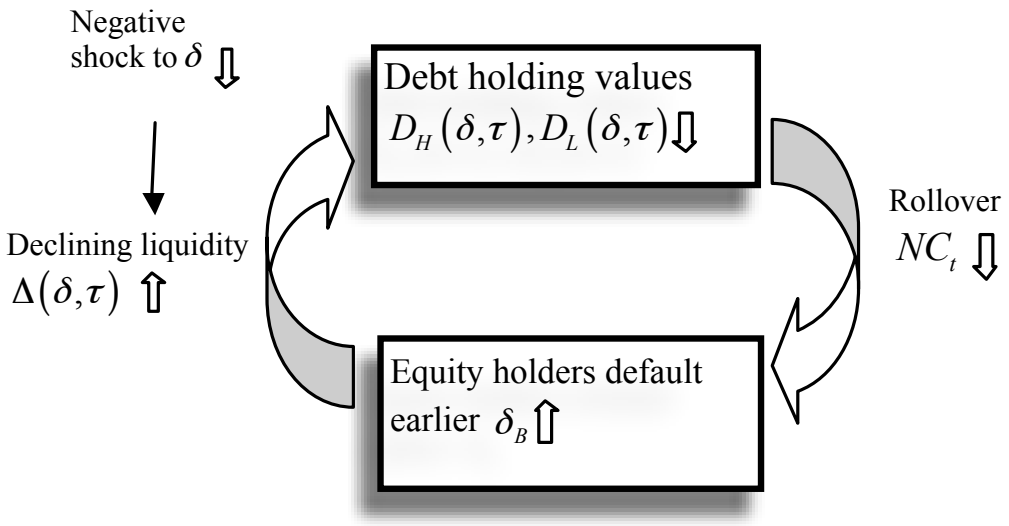


Figure 6: **Feedback loop** between secondary market liquidity and equity holders' default decision.

liquidity significantly reduces the equity holders' option value of serving the debt especially in bad times, and consequently they default earlier.

4.2.2 Positive feedback between fundamental and liquidity

The above discussion implies an important positive feedback loop between firm fundamental and secondary market liquidity for corporate bonds, which is illustrated in Figure 6. For investors of defaultable corporate bonds, the bond fundamental can be measured as the firm's distance to default, i.e., $\delta - \delta_B$. Imagine a *negative* shock to firm cash flow rate δ. Since this negative shock brings the firm closer to default, this constitutes a pure-fundamental driven negative shock to bond investors and lowers the holding values of D_H and D_L. This force is already present in LT96 and HX11.

Now *relative to HX11* which has fixed secondary market liquidity (or transaction costs), our model features a new endogenous loop. A shock to δ not only lowers debt values and worsens the net cash-flow to equity holders NC_t in (3), but the lower distance-to-default also affects secondary market liquidity. The lower distance to default implies that default is more likely, which worsens the

L types' outside option when bargaining with a dealer (as default leads to drawn out bankruptcy court decisions and an even less liquid secondary bond market). Consequently, the effective transaction costs for the L type go up and the secondary market becomes more illiquid—as indicated by the left large arrow in Figure 6, the bid-ask spread $\Delta(\delta, \tau)$ goes up. This is the pro-cyclicality of liquidity we already discussed above.

Rational H type bondholders will thus value bonds less as they expect to face a less liquid secondary market when they are hit by liquidity shocks and trade with dealers, and D_H will decline. As shown in Figure 6, the worsening liquidity in the secondary market lowers both the investors' holding values D_H and D_L, and consequently a lower primary market bond issuing price D_H relative to an environment with constant market liquidity.

The lower bond prices now feed back to the equity holders' default decision via the rollover channel, as the arrow on the right of Figure 6 indicates. Because maturing bonds still require the same promised principal payment p, equity holders are absorbing heavier rollover losses (i.e. net cash flow NC_t in (3) goes down), as we have seen in the left panel of Figure 5 in Section 4.2.1. Consequently, equity holders default earlier relative to an environment with a constant market liquidity. That is, they default at a higher threshold δ_B.

The higher default threshold now translates into a shorter distance to default $\delta - \delta_B$, and therefore a lower effective fundamental for bond investors. This, of course, has a direct impact on bond prices (not shown in the figure) as bond default has become more likely. Additionally, there is now an important indirect liquidity effect via the secondary market—as shown on the left-hand side in Figure 6, the higher default boundary now *further* worsens market liquidity via the declining outside option of the L type investors. The loop repeats as the lower liquidity now again lowers effective bond prices.

The positive feedback loop between a worse asset fundamental (i.e., a lower distance-to-default, $\frac{\delta}{\delta_B}$) and a worse secondary market liquidity is the novel economic channel present in our model. In equilibrium, equity holders default much earlier in our model with endogenous secondary market liquidity, compared to the model in HX11 with exogenous constant liquidity. This positive feedback

loop can have significant impact on observed corporate bond spreads (and equivalently ask prices $D_H(\delta, \tau)$) as bond investors rationally anticipate pro-cyclical liquidity and the default decision of equity holders.

4.2.3 Credit spreads

The bond credit spread is the spread between the corporate bond and the risk-free rate r. The bond yield is typically computed as the equivalent return on a bond conditional on it being held to maturity without default or re-trading. Given a bond of value $D(\delta, \tau)$, the bond yield y is defined as the solution to the following equation:

$$D(\delta, \tau) = \frac{c}{y}(1 - e^{-y\tau}) + pe^{-y\tau} \tag{14}$$

where the right-hand side is the price of a bond with a constant coupon payment c over time and a principal payment p at the bond maturity, conditional on no default or trading before maturity. Here, we use the ask price $D_H(\delta, \tau)$ derived in Proposition 1 as our bond price for the left-hand side of equation (14).

In Figure 7 we plot the credit spread $y - r$ of the benchmark bond at $\tau = T$ as a function of δ, under various economic settings. The dashed line gives the credit spread under HX11, who fix the bond market liquidity at an exogenous constant. The dot-dashed line is the hypothetical bond credit spread which takes into account the endogenous secondary market liquidity only, but fixes the equity holders' default boundary at $\delta_B = \delta_B^{HX11}$, the optimal default boundary in the HX11 model. Because bonds become more illiquid for lower distance-to-default, the observed credit spread rises sharply relative to the HX11 benchmark as δ deteriorates. Finally, the solid line gives the bond credit spread predicted by our model, which takes into account the full feedback loop between the endogenous market liquidity and the endogenous default decision. Because equity holders respond to worsening secondary market liquidity by defaulting earlier, the observed credit spread rises even faster when the firm fundamental deteriorates, but provides a lower credit spread than the HX11

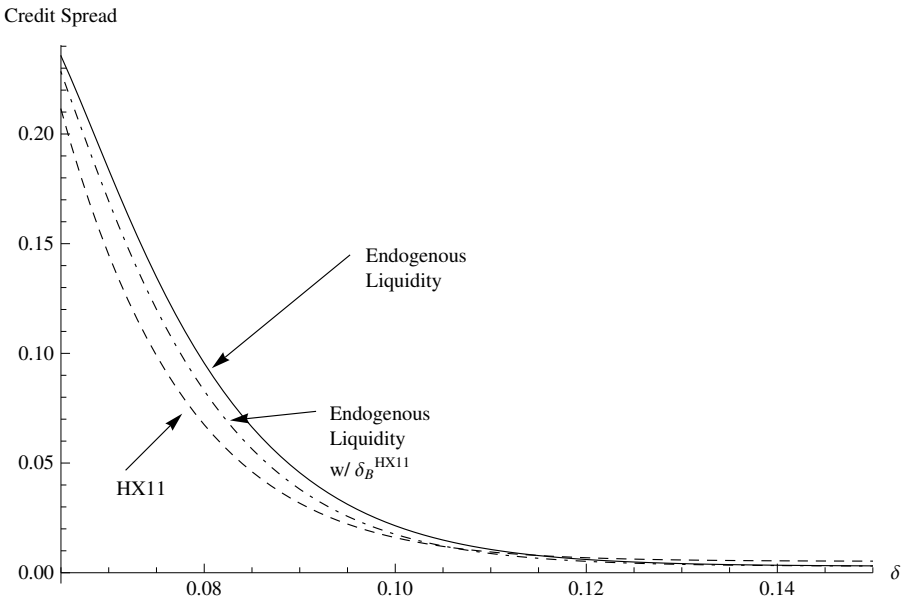


Figure 7: **Credit spread** $y - r$ as a function of fundamental cash-flow δ for main model (solid line), for the HX11 model with $k_{HX11} = 1.2\%$ (dashed line) and the off-equilibrium model of imposing the bankruptcy boundary δ_B^{HX11} that is optimal in the HX11 model in the search-based model (dot-dashed line).

benchmark when the distance to default gets large.

5 Extensions and Discussions

5.1 Endogenous Investment

So far we focused on the feedback loop between the liquidity and fundamental of corporate bonds. The endogenous equity holders' default decision enters directly into the fundamental value of corporate bonds, which is mainly captured by the firm's distance-to-default. Generally, this mechanism should apply to the firm level, i.e., a feedback loop between the firm fundamental and the firm's (debt) financing liquidity, if the firm fundamental is affected by endogenous investment decision.

Indeed, one may broadly interpret default as a form of disinvestment. Although in our model the cash flow rate δ evolves as an exogenous stochastic process in (1), the total firm value (including both equity and debt values) in (9) depends on the equity holders' default decision. In this section we push this idea further to consider a simple extension where equity holders make an initial endogenous investment decision in our base model; a full-blown model with dynamic investment

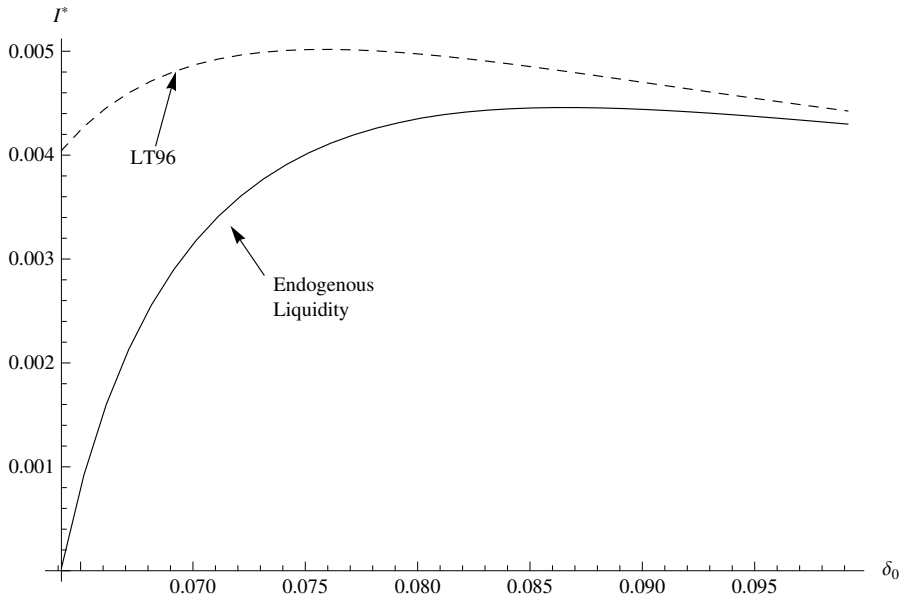


Figure 8: **Optimal initial investment** I as a function of initial cash-flow δ_0 for $\phi = \frac{18}{(r-\mu)^2}$ for the LT96 model (dashed line) and for the main model (solid line).

opportunities is interesting for future research.

Suppose that at date-0 the equity holders in the firm can invest to improve the firm fundamental δ_0, where the investment technology is such that investing $I > 0$ can increase the initial cash flow δ_0 to $\delta_0 + I$ (and thus the progression of all δ thereafter via equation (1)), at a quadratic cost of $\frac{\phi}{2}I^2$. For simplicity, assume that equity holders bear the initial investment outlay I. Because the equity value only depends on the post-investment cash flow $\delta_0 + I$, at date 0 equity holders are solving

$$\max_{I>0} \; E(\delta_0 + I) - \frac{\phi}{2}I^2,$$

where $E(\delta_0 + I)$ is given in (8) with the optimal $\delta_B = \delta_B^*(T)$. The above problem gives the endogenous date-0 investment I^*, and we are interested in the impact of endogenous secondary market liquidity on the firm investment.

As emphasized in Diamond and He (2011), in this setting equity holders' investment decision suffers the classic debt overhang problem coined in Myers (1977). Our model features an additional feedback effect from the secondary market liquidity to the firm fundamental through the

equity holders' investment incentives. The mechanism is similar to the one illustrated in Figure 6, where we consider a hypothetical negative shock to the initial firm fundamental cash flow δ_0. The lower fundamental worsens the firm's financing liquidity immediately, leading to higher endogenous transaction costs in the secondary market for its existing bonds and lower primary market prices for its newly issued bonds. As explained, relative to models without endogenous liquidity, e.g., LT96, this pro-cyclical financing liquidity force amplifies the equity holders' rollover losses in bad times which pushes equity holders to default earlier. Moreover, in our extension, earlier default feeds back to a lower endogenous initial investment taken by equity holders, therefore an even lower firm fundamental. This will adversely affect the the firm's financing liquidity, which lowers the initial investment even further, and so forth. The extra positive feedback effect is illustrated in the wedge between the investment implied by our model and and that of LT96 in Figure 8.

5.2 Liquidity Premium and Default Premium

It has been widely recognized that the credit spread of corporate bonds not only reflects a default premium determined by the firm's credit risk, but also a liquidity premium due to the illiquidity of the secondary debt market, e.g., Longstaff, Mithal, and Neis (2005), and Chen, Lesmond, and Wei (2007). However, both academics and policy makers tend to treat the default premium and liquidity premium as independent, and thus ignore interactions between them. For instance, it is common practice to decompose firms' credit spreads into independent liquidity-premium and default-premium components and then assessing their quantitative contributions, e.g., Longstaff, Mithal, and Neis (2005), Beber, Brandt, and Kavajecz (2009), and Schwarz (2010).

The positive feedback derived in our model implies a rich interaction between the liquidity and default premia for corporate bonds, which challenges this approach. In fact, based on reasonable calibrations, HX11 have demonstrated that an exogenous rise of the liquidity premium (say, bond investors become more likely to suffer liquidity shocks) will lead to a sizable increase in default premium,[23] due to the endogenous earlier default by equity holders.

[23]For instance, HX11 show that if an unexpected shock causes liquidity premium to increase by 100 bps, default

Similar to HX11, this force of exogenous liquidity shocks from the investors' side (say a surge in the liquidity shock intensity ξ) is also present in our model. But our paper goes further, highlighting two new economic mechanisms arising by endogenizing secondary market liquidity. First, even with exogenous negative liquidity shocks, because equity holders default earlier facing a less liquid corporate bond market, this leads to a lower distance-to-default for the firm, and therefore a further worsening of secondary bond market liquidity. This amplification effect may help in future quantitative exercises when assessing the impact of surging aggregate liquidity shocks. Second and perhaps more importantly, we show that the origin of shock to the liquidity premium of corporate bonds can in fact be the deterioration of the firm fundamental itself. Thus, both default premium and liquidity premium are inter-dependent with each other, and the positive feedback loop will further amplify and reinforce both premia in a nontrivial way.

One important implication of our model is that even far away from the default boundary, i.e. for high $\frac{\delta}{\delta_B}$, there is still a non-negligible credit-spread that arises because of the uninsurability of each individual agent's liquidity shock and the search-friction inherent in the secondary market. Figure 7 illustrates, as even to the far right of the graph, there remains some positive credit spread. This insight can help resolve a weak point in the structural credit literature: as Huang and Huang (2003) point out, structural credit models have difficulty explaining the AAA credit spread in a satisfactory manner once calibrated to historic default probabilities and asset prices. Our model offers a resolution via the presence of illiquidity that is uninsurable on the agent level and thus does not affect the translation of the real probabilities to the risk-neutral probilities, i.e. the pricing kernel. A careful calibration exercise therefore is a priority for future work.

In sum, our model suggests the importance of structural models in the empirical study of corporate bonds where the liquidity component and default component are necessarily intertwined. We believe the theoretical results obtained in our model are helpful for future research on this regard.

premium of a firm with speculative grade B rating and 1 year debt maturity (a financial firm) would rise by 70 bps, which contributes to 41% of the total credit spread increase.

5.3 Optimal Debt Maturity

Beyond the feedback loop between fundamental and liquidity, our model features a natural trade-off between liquidity provision and equity-debt conflict of interest, which allows us to derive the optimal debt maturity (given the stationary maturity structure).

5.3.1 Liquidity provision: the bright side of short maturity

Section 4.1 has shown that bonds with shorter maturity have a more liquid secondary market, suggesting the role of liquidity provision for short-term debt. For illustration, we compare the newly issued debt value D_H to the hypothetical LT96 debt valuation D^{LT96} without liquidity shocks. As shown in Figure 9, this efficiency gain decreases with the debt maturity.[24]

The efficiency gain due to short-term maturity arises from two channels:

First, debt holders who are hit by liquidity shocks become inefficient holders of bonds, and due to trading frictions the inefficient holding lasts for a while. Reducing maturity alleviates this inefficiency, because of the firm's advantage in the primary market: whenever debt matures, the firm moves debt from inefficient L investors to efficient H investors by new issuance. As detailed in the Appendix, the steady-state proportion of L types if the firm is able to issue only to H types is

$$\mu_L(T) = \frac{\xi}{\lambda+\xi} - \underbrace{\frac{\xi\left[1-e^{-T(\lambda+\xi)}\right]}{T(\lambda+\xi)^2}}_{Allocative\,efficiency},$$

with $\lim_{T\to\infty}\mu_L(T) = \frac{\xi}{\lambda+\xi}$ and $\lim_{T\to 0}\mu_L(T) = 0$. Thus, the second term in the above equation is the *allocative efficiency gain* of shortening the aggregate maturity T, arising from substituting the firm's superior primary market liquidity for the debt holders inferior secondary market liquidity.

Second, a shorter maturity reduces the rent extracted by dealers in the secondary market, thus leading to a *bargaining efficiency gain*. Intuitively, a shorter maturity, by allowing L investors to receive principal payment earlier, raises their outside option of waiting during bargaining and in turn lowers the dealer's rent. Given that λ was assumed as an exogenous parameter, this effect is

[24]The LT96 benchmark has discount rate r, coupon c, principal p and recovery value α_H.

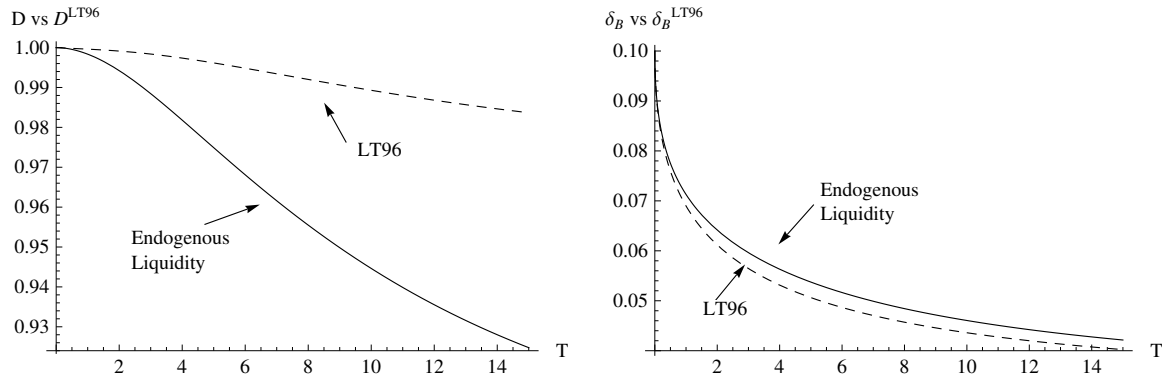


Figure 9: Left panel: **Debt values** for high type in main model (solid line) and without frictions in the LT96 model (dashed line) as a function of initial maturity T. Right panel: **Bankruptcy boundary** $\delta_B(T)$ in main model (solid line) and without frictions in the LT96 model (dashed line) as a function of initial maturity T.

independent of the first effect discussed above.

5.3.2 Earlier default: the dark side of short maturity

On the other hand, as first shown in LT96 (and formally proven in HX11), shorter debt maturity in an LT96 style model always exacerbates the conflict of interest between equity holders and debt holders, leading to earlier default and thus greater dead-weight bankruptcy cost. In other words, the optimal maturity in LT96 and HX11 is $T^* = \infty$, so that debt should always be a consol bond. Consider the equity holders' rollover losses $\frac{1}{T}[D_H(\delta, T) - P]$ given a low firm fundamental δ, which is difference between the market price $D_H(\delta, T)$ for newly issued bonds and the principal payment of P, and modulated by the rollover frequency $\frac{1}{T}$. Although short-term debt has a greater market price, the higher rolling over frequency leads to heavier rollover losses. In words, as the debt maturity T shrinks, equity holders are bearing heavier rollover losses in the states with low firm fundamental. As a result, as shown in the right panel of Figure 9, equity holders default earlier if the firm is using shorter maturity debt.

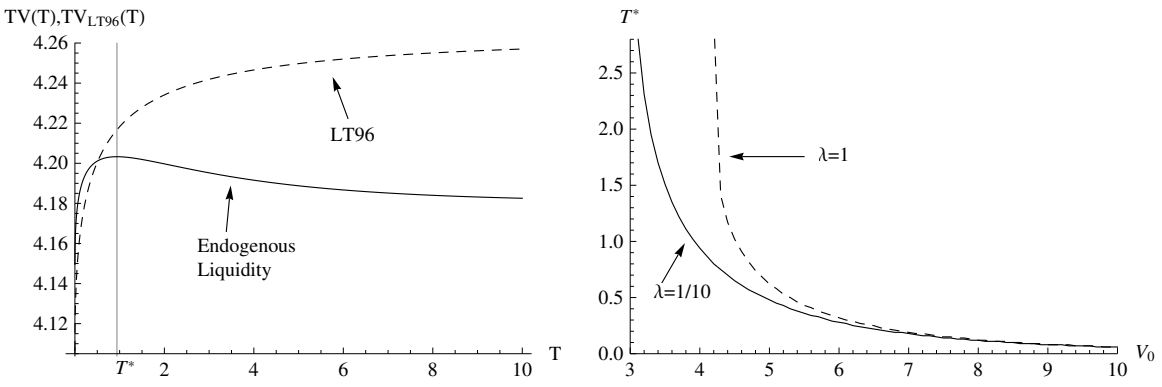


Figure 10: Left panel: **Total firm value** in the main model (solid line) and without frictions in the LT 96 model (dashed line). Right panel: **Optimal maturity** T^* in the main model as a function of unlevered firm value $V_0 = \frac{\delta_0}{r-\mu}$ for different levels of search frictions, $\lambda = .1$ (solid line) and $\lambda = 1$ (dashed line). Both lines at some point jump to $T^* = \infty$ for small enough finite V_0.

5.3.3 Optimal Debt Maturity

The above trade-off between liquidity provision and debt-equity conflict of interest naturally leads to an endogenous optimal maturity structure. In Figure 10 we plot in the left-hand panel the ex ante levered firm value $TV(\delta_0)$ given in (9) for both our model (solid line) and the LT96 benchmark model (dashed line) as a function of the debt maturity T for an initial unlevered value $V(\delta_0) = 4$. The hump shape of levered firm value suggests that we can find an interior solution for the optimal maturity structure, which is just less than 1 year in this case. In contrast, the total firm value in the LT96 case is monotonically increasing in debt maturity.

As explained, we can loosely interpret the initial leverage as the distance of the initial unlevered firm value $V_0 \equiv \frac{\delta_0}{r-\mu}$ to the face value p. In the right panel of Figure 10 we draw the optimal maturity T^* as a function of $V_0 = \frac{\delta_0}{r-\mu}$ (for different levels of δ_0, holding all other parameters constant), which is inversely related to "initial leverage." The solid line depicts the optimal maturity for a secondary market with low intermediation, i.e., $\lambda = 1/10$, whereas the dashed line depicts the optimal maturity for a secondary market with high intermediation, i.e., $\lambda = 1$. We observe that when initial leverage is low (high), bankruptcy becomes more (less) remote, and the effect of liquidity provision (bankruptcy cost) dominates, resulting in a shorter (longer) optimal debt

maturity. Additionally, we see that for the poorly intermediated market with $\lambda = 0.1$, the firm provides liquidity to its debt holders through short optimal maturity. In contrast, for a better intermediated market with $\lambda = 1$, the firm's optimal maturity shifts out uniformly, and jumps to infinity for initial unlevered firm values $V_0 < 4.2$. In other words, a better functioning secondary market reduces the need to provide liquidity via short maturity and thus alleviates the bankruptcy pressure generated by the short debt structure.

5.4 Further discussion

5.4.1 How about put provisions?

The firm could, instead of providing liquidity via maturity, allow bondholders with liquidity shocks to put back their bonds at the face value p. This seemingly perfect solution suffers two important drawbacks. First, if the firm cannot distinguish who was hit by a liquidity shock, whenever $D_H < p$ everyone will put back their debt at the same time. In fact, the put provision is akin to making bonds demand deposits and we are at traditional models of bank runs with potential bad run equilibrium. Second, even if the liquidity shock is observable, there will be an additional flow term $\xi [D_H - p] dt$ as L investors are putting back their bonds to the firm every instant. This additional refinancing losses may influence the bankruptcy boundary in an adverse way and destroy the liquidity thus provided. The full implications of expanded bond contract terms (beyond the choice of initial maturity T covered in this paper) is left for future work.

6 Conclusion

We study the endogenous liquidity of defaultable bonds in a search-based OTC markets, together with the endogenous default decision by equity holders from the firm side. By solving a system of PDEs, we derive the endogenous secondary market liquidity jointly with the debt valuations, equity valuations, and endogenous default policy, in closed-form. The fundamentals of corporate bonds, which is mainly driven by the firm's distance-to-default, affects the endogenous liquidity

of corporate bonds. And, through the rollover channel in which equity holders are absorbing refinancing losses, worsening liquidity of corporate bonds significantly hurts the equity holders' option value of keeping the firm alive. As a result, illiquidity of secondary corporate bond market feeds back to the fundamental of corporate bonds by edging the firm closer to bankruptcy. With the aid of recent empirical techniques, we hope our fully solved structural model can pave the way of bringing more structural approach in the empirical study of the impact of liquidity on corporate bonds.

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A Appendix

First, let us call $r_H \equiv r$, $r_L \equiv \overline{r}$, $\xi_H \equiv \xi$ and $\xi_L \equiv \lambda\beta$, and $\tilde{\mu} = \mu - \frac{\sigma^2}{2}$. Second, define the log-transform $\tilde{\delta} = \log(\delta)$ so that $d\tilde{\delta} = \tilde{\mu}dt + \sigma dZ$. Third, for brevity we use the notation $D' \equiv \frac{\partial D}{\partial v}$ and $\dot{D} \equiv \frac{\partial D}{\partial \tau}$. We will, with abuse of notation, write $q\left(\tilde{\delta}, ...\right)$ to mean $\frac{\tilde{\delta}_B - \tilde{\delta} + \cdots}{\cdots}$.

A.1 2x2 matrix formulas

As the 2x2 specification is frequently used in the text, we present the results here in compact form. Suppose

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then $\mathbf{A} = \mathbf{P}\hat{\mathbf{D}}\mathbf{P}^{-1}$ where

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ \mathbf{P} &= \begin{bmatrix} 1 & \frac{b}{\hat{r}_2 - a} \\ \frac{c}{\hat{r}_1 - d} & 1 \end{bmatrix} \\ \hat{\mathbf{D}} &= \begin{bmatrix} \hat{r}_1 & 0 \\ 0 & \hat{r}_2 \end{bmatrix}, \end{aligned}$$

where of course alternative versions of $\mathbf{P}$ can be chosen. However, to show convergence to frictionless markets we chose this form of $\mathbf{P}$ as it allows convergence to an upper triangular form. The roots

$$\begin{aligned} \hat{r}_{1/2} &= \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} \\ &= \frac{a+d \pm \sqrt{(a-d)^2 + 4bc}}{2} \end{aligned}$$

solve $\det[\mathbf{A} - \rho\mathbf{I}] = 0$, i.e. $\hat{r}_{1/2}$ are the roots of the characteristic polynomial

$$g(\hat{r}) = (a - \hat{r})(d - \hat{r}) - bc = \hat{r}^2 - (a+d)\hat{r} + (ad - bc).$$

If $a > 0$ and $d > 0$ and $b < 0$ and $c < 0$ as well as $(ad - bc) > 0$, then both roots $\hat{r}_{1/2} > 0$.

Identifying $a = r_H + \xi_H, b = -\xi_H, c = -\xi_L, d = r_L + \xi_L$, we have

$$\hat{r}_i = \frac{r_H + r_L + \xi_H + \xi_L - (-1)^i \sqrt{[(r_H + \xi_H) - (r_L + \xi_L)]^2 + 4\xi_H\xi_L}}{2}.$$

We can also derive bounds on $\hat{r}_i$ by noting the following results:

$$\begin{aligned} g(r_H) &= \xi_H(r_L - r_H) > 0 \\ g(r_L) &= -\xi_L(r_L - r_H) < 0 \\ g(r_H + \xi_H) &= -\xi_H\xi_L < 0 \\ g(r_L + \xi_L) &= -\xi_H\xi_L < 0 \\ g(r_H + \xi_H + \xi_L) &= -\xi_L(r_L - r_H) < 0 \\ g(r_L + \xi_H + \xi_L) &= \xi_H(r_L - r_H) > 0 \end{aligned}$$

so that we know that

$$\begin{aligned} r_H &< \hat{r}_1 < \min\{r + \xi_H, r_L\} \\ \max\{r_H + \xi_H + \xi_L, r_L + \xi_L\} &< \hat{r}_2 < r_L + \xi_H + \xi_L. \end{aligned}$$

It is easy to show that as $\xi_H \to 0$, $\hat{r}_1 = r_L + \xi_L$ and $\hat{r}_2 = r_H$, and $\lim_{b\to 0}\mathbf{P} = \begin{bmatrix} 0 & 1 \\ \cdot & \cdot \end{bmatrix}$, so that D_H converges towards the LT96 solution.

Next, consider $\lambda \to \infty$ such that $\xi_L \to \infty$, that is, what happens when the market becomes very liquid. Note

that we can rewrite the characteristic polynomial as

$$g(\hat{r}) = \xi_L \left[(r_H + \xi_H - \hat{r}) \left(\frac{r_L}{\xi_L} + 1 - \frac{\hat{r}}{\xi_L} \right) - \xi_H \right]$$

Suppose now that $\hat{r}$ is finite. Then we know that the square bracket, as $\xi_L \to \infty$, becomes

$$(r_H + \xi_H - \hat{r}) - \xi_H = 0$$

so that $\hat{r}_2 = r_H > 0$. Thus, as both roots are positive, we must have that the second root $\hat{r}_1 \to \infty$. The diagonal decomposition becomes unstable, in that $\lim_{\lambda \to \infty} \mathbf{P} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

Finally, for $r = r_H = r_L$ we can show that $\mathbf{P}^{-1}\mathbf{1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so that $\hat{\mathbf{c}} = \begin{bmatrix} c \\ 0 \end{bmatrix}$, and for $\alpha = \alpha_H = \alpha_L$ we have $\hat{\boldsymbol{\alpha}} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$.

A.2 Proofs of Section 3

A.2.1 Debt

Proof of Proposition 1.

Applying the log transform $\tilde{\delta} = \log(\delta)$ to the system of PDEs we are left with a linear system of PDEs:

$$\begin{aligned} \begin{bmatrix} r_H + \xi_H & -\xi_H \\ -\xi_L & r_L + \xi_L \end{bmatrix} \begin{bmatrix} d_H \\ d_L \end{bmatrix} &= \begin{bmatrix} c \\ \rho c \end{bmatrix} + \tilde{\mu} \begin{bmatrix} d_H \\ d_L \end{bmatrix}' + \frac{\sigma^2}{2} \begin{bmatrix} d_H \\ d_L \end{bmatrix}'' - \begin{bmatrix} \dot{d_H} \\ d_L \end{bmatrix} \\ \iff \mathbf{A} \times \mathbf{d} &= \mathbf{c} + \tilde{\mu}\mathbf{d}' + \frac{\sigma^2}{2}\mathbf{d}'' - \dot{\mathbf{d}} \end{aligned}$$

Here we allow for general changes to the coupon payment c by premultiplying by a parameter $\rho \leq 1$ to acknowledge that there might be linear holding costs above and beyond the higher discount rate. In the paper, we have $\rho = 1$. Let us decompose $\mathbf{A} = \mathbf{P}\hat{\mathbf{D}}\mathbf{P}^{-1}$ where $\hat{\mathbf{D}}$ is a diagonal matrix with its diagonal elements the eigenvalues of $\mathbf{A}$ and $\mathbf{P}$ is a matrix of the respective stacked eigenvectors. The resulting eigenvalues are defined

$$g(\hat{r}) = (r_H + \xi_H - \hat{r})(r_L + \xi_L - \hat{r}) - \xi_L \xi_H = 0$$

and $g(r_H) = \xi_H (r_L - r_H) > 0$ and $g(r_L) = -\xi_L (r_L - r_H) < 0$. We thus have $\hat{r}_i = \frac{r+\xi+\overline{r}+\lambda\beta \pm \sqrt{[(r+\xi)-(\overline{r}+\lambda\beta)]^2 + 4\xi\lambda\beta}}{2}$.

Premultiplying the system by $\mathbf{P}^{-1}$ and noting that $\mathbf{P}^{-1}\mathbf{A} = \hat{\mathbf{D}}\mathbf{P}^{-1}$ we have a delinked system PDEs with a common bankruptcy boundary $\tilde{\delta}_B \equiv \log(\delta_B)$ and payout boundary $t = 0$

$$\begin{aligned} \hat{\mathbf{D}}\mathbf{P}^{-1}\mathbf{d} &= \mathbf{P}^{-1}\mathbf{c} + \tilde{\mu}\mathbf{P}^{-1}\mathbf{d}' + \frac{\sigma^2}{2}\mathbf{P}^{-1}\mathbf{d}'' - \mathbf{P}^{-1}\dot{\mathbf{d}} \\ \iff \hat{\mathbf{D}}\mathbf{y} &= \widehat{\mathbf{c}} + \tilde{\mu}\mathbf{y}' + \frac{\sigma^2}{2}\mathbf{y}'' - \dot{\mathbf{y}} \end{aligned}$$

where $\mathbf{y} = \mathbf{P}^{-1}\mathbf{d}$ and $\hat{\mathbf{c}} = \mathbf{P}^{-1}\mathbf{c}$. The rows of the system are now delinked, and we are left with two PDEs of the form

$$\hat{r}_i y_i = \widehat{c}_i + \tilde{\mu} y_i' + \frac{\sigma^2}{2} y_i'' - \dot{y}_i$$

with given boundary conditions at $t = 0$ and $\tilde{\delta} = \tilde{\delta}_B$, whose solutions are known from LT96. The decomposition works because the boundaries are the same across rows. The solution takes the form

$$\begin{aligned} y_i &= A_i + B_i e^{-\hat{r}_i t}(1 - F_i) + C_i G_i \\ F_j\left(\tilde{\delta}, t\right) &= \sum_{i=1}^{2} e^{\left(\tilde{\delta} - \tilde{\delta}_B\right)\gamma_{ij}} N\left[q\left(\tilde{\delta}, \gamma_{ij}, t\right)\right] \\ G_j\left(\tilde{\delta}, t\right) &= \sum_{i=1}^{2} e^{\left(\tilde{\delta} - \tilde{\delta}_B\right)\eta_{ij}} N\left[q\left(\tilde{\delta}, \eta_{ij}, t\right)\right] \end{aligned}$$

where

$$q\left(\tilde{\delta},\chi,t\right)=\frac{\tilde{\delta}_B-\tilde{\delta}-(\chi+a)\cdot\sigma^2 t}{\sigma\sqrt{t}}$$

and constants

$$\begin{aligned}A_i &= \frac{\hat{c}_i}{\hat{r}_i}\\ B_i &= \left(\hat{p}_i-\frac{\hat{c}_i}{\hat{r}_i}\right)\\ C_i &= \left(\hat{\alpha}_i\frac{e^{\tilde{\delta}_B}}{r-\mu}-\frac{\hat{c}_i}{\hat{r}_i}\right)\end{aligned}$$

and some yet to be determined parameters γ_{ij},η_{ij}. Note that $\lim_{t\to 0} q\left(\tilde{\delta},\chi,t\right)=\lim_{t\to 0}\frac{\tilde{\delta}_B-\tilde{\delta}}{\sigma\sqrt{t}}=-\infty$ as $\tilde{\delta}_B<\tilde{\delta}$, so $N\left[q\left(\tilde{\delta},\chi,0\right)\right]=0$ for all i and $\tilde{\delta}>\tilde{\delta}_B$. Further note that $\lim_{\tilde{\delta}\to\infty} q\left(\tilde{\delta},\chi,t\right)=-\infty$, so $\lim_{\tilde{\delta}\to\infty} N\left[q\left(\tilde{\delta},\chi,t\right)\right]=0$. Substituting the candidate solution y_i into the PDE with $A_i=\widehat{\frac{c_i}{\hat{r}_i}}, B_i=\hat{p}_i-\frac{\hat{c}_i}{\hat{r}_i}, C_i=\hat{\alpha}_i\frac{\exp\left(\tilde{\delta}_B\right)}{r-\mu}-\frac{\hat{c}_i}{\hat{r}_i}$, we see that

$$\begin{aligned}b_i e^{-\hat{r}_i t}\left[\hat{r}_i\left(1-F_i\right)+\tilde{\mu}F_i'+\frac{\sigma^2}{2}F''-\left[\hat{r}_i\left(1-F_i\right)+\dot{F}_i\right]\right]&\\ +c_i\left[\hat{r}_i G_i-\tilde{\mu}G_i'-\frac{\sigma^2}{2}G_i''+\dot{G}_i\right] &= 0\\ \iff b_i e^{-\hat{r}_i t}\left[\tilde{\mu}F'+\frac{\sigma^2}{2}F''-\dot{F}\right]&\\ +c_i\left[\hat{r}_i G_i-\tilde{\mu}G_i'-\frac{\sigma^2}{2}G_i''+\dot{G}_i\right] &= 0\end{aligned}$$

We see that both $\dot{F}_i$ and $\dot{G}_i$ have no term $N(\cdot)$. As q is linear in $\tilde{\delta}$, we have $q''=0$ (where $q'=q_{\tilde{\delta}}$ and $\dot{q}=q_t$). We thus have, for F,

$$\begin{aligned}N\left[q\left(\tilde{\delta},\gamma,t\right)\right]\left[\tilde{\mu}\gamma+\frac{\sigma^2}{2}\gamma^2\right]&\\ +\phi\left[q\left(v,\gamma,t\right)\right]\left[\tilde{\mu}q'+\frac{\sigma^2}{2}\left[2\gamma q'-q\left(q'\right)^2\right]-\dot{q}\right] &= 0\end{aligned}$$

So the roots for F_i are $\gamma_1=0=-a+a$ and $\gamma_2=-\frac{2\tilde{\mu}}{\sigma^2}=-a-a$ where $a\equiv\frac{\tilde{\mu}}{\sigma^2}$. We see that this is independent of i, that is, it is independent of what row of $\mathbf{y}$ we picked, as $\hat{r}_i$ is cancelled out. Further, for G, we have

$$\begin{aligned}N\left[q\left(v,\eta,t\right)\right]\left[\tilde{\mu}\eta+\frac{\sigma^2}{2}\eta^2-\hat{r}_i\right]&\\ +\phi\left[q\left(v,\eta,t\right)\right]\left[\tilde{\mu}q'+\frac{\sigma^2}{2}\left[2\eta q'-q\left(q'\right)^2\right]-\dot{q}\right] &= 0\end{aligned}$$

so the roots for G_i are $\eta_{i1}=\frac{-\tilde{\mu}+\sqrt{\tilde{\mu}^2+2\sigma^2\hat{r}_i}}{\sigma^2}=-a+\frac{\sqrt{\tilde{\mu}^2+2\sigma^2\hat{r}_i}}{\sigma^2}$ and $\eta_{i2}=-a-\frac{\sqrt{\tilde{\mu}^2+2\sigma^2\hat{r}_i}}{\sigma^2}$. Simply plugging in the functional form of q results in the term in square brackets in the second row to vanish.

For the boundary condition, we have

$$\begin{aligned}\mathbf{y}\left(v,0\right) &= \mathbf{P}^{-1}\mathbf{1}\cdot p=\hat{\mathbf{p}}\\ \mathbf{y}\left(v_B,t\right) &= \mathbf{P}^{-1}\boldsymbol{\alpha}\frac{\exp\left(\tilde{\delta}_B\right)}{r-\mu}=\hat{\boldsymbol{\alpha}}\frac{\exp\left(\tilde{\delta}_B\right)}{r-\mu}\end{aligned}$$

which defines the remaining parameters of the solution.

As a last step, we retranslate the system back into the original debt functions by premultiplying by $\mathbf{P}$ and noting that $F(v,t)=F_i(v,t)=F_{-i}(v,t)$ by the symmetry of the γ's, and by rewriting it in terms of δ instead of $\tilde{\delta}$. ■

A.2.2 Equity

Proof of Proposition 2.

Equity has the following ODE where for notational ease we define $m = \frac{1}{T}$

$$rE = \exp\left(\tilde{\delta}\right) - (1-\pi)\,c + \tilde{\mu}E' + \frac{\sigma^2}{2}E'' + m\left[D_H\left(\tilde{\delta}, T\right) - p\right]$$

The term in square brackets is the cash-flow term that arises out of rollover of debt (while keeping coupon, principal and maturity stationary), a term first pointed out by LT96. We will establish the (closed-form) solution in several steps.

First, the homogenous solutions to the ODE are $M\left(\tilde{\delta}\right) = e^{\kappa_1\tilde{\delta}}$ and $U\left(\tilde{\delta}\right) = e^{\kappa_2\tilde{\delta}}$ where

$$\frac{\sigma^2}{2}\kappa^2 + \tilde{\mu}\kappa - r = 0$$

so that

$$\kappa_{1/2} = \frac{-\tilde{\mu} \pm \sqrt{\tilde{\mu}^2 + 2\sigma^2 r}}{\sigma^2} = -a \pm \frac{\sqrt{\tilde{\mu}^2 + 2\sigma^2 r}}{\sigma^2}$$

and $\kappa_1 > 1 > 0 > \kappa_2$. As the debt term D_H is bounded, to impose the condition that equity does not grow orders of magnitude faster than the unlevered value of the firm $V = \frac{e^{\tilde{\delta}}}{r-\mu}$ we need $\lim_{\tilde{\delta}\to\infty}\left|K_1\frac{e^{\kappa_1\tilde{\delta}}}{e^{\tilde{\delta}}}\right| < \infty$. The only solution is then $K_1 = 0$.We are thus left with κ_2 and the coefficient K on $e^{\kappa_2\tilde{\delta}}$.

Next, let us establish the Wronskian

$$\begin{aligned} Wr(s) &= M(s)U'(s) - M'(s)U(s) \\ &= -(\kappa_1 - \kappa_2)\exp\{(\kappa_1+\kappa_2)s\} \\ &= -\Delta\kappa \cdot M(s)U(s) \end{aligned}$$

Then, by the variation of coefficient solutions to linear ODEs, we have

$$\begin{aligned} g_p(x|l) &= \frac{2}{\sigma^2}\int_x^l part(s)\frac{M(s)U(x) - M(x)U(s)}{Wr(s)}ds \\ &= \frac{2}{\sigma^2}\int_x^l part(s)\frac{e^{-\kappa_2 s}e^{\kappa_2 x} - e^{\kappa_1 x}e^{-\kappa_1 s}}{-\Delta\kappa}ds \\ g_p'(x|l) &= \frac{2}{\sigma^2}\int_x^l part(s)\frac{M(s)U'(x) - M'(x)U(s)}{Wr(s)}ds \\ &= \frac{2}{\sigma^2}\int_x^l part(s)\frac{\kappa_2 M(s)U(x) - \kappa_1 M(x)U(s)}{Wr(s)}ds \\ g_p''(x|l) &= \frac{2}{\sigma^2}\int_x^l part(s)\frac{\kappa_2^2 M(s)U(x) - \kappa_1^2 M(x)U(s)}{Wr(s)}ds - \frac{2}{\sigma^2}part(x) \end{aligned}$$

for an arbitrary limit $l \in (v_B, \infty)$. Let us take $l \to \infty$ and $g_p(x) \equiv g_p(x|\infty)$. We see that $g_p(x)$ and $g_p'(x)$ (and so forth) consists of a finite sum of integrals of the form $\int_x^\infty e^{cst\cdot s}N[q(s,\chi,T)]\,ds$ where cst is a constant.

Second, let us briefly establish two auxiliary results. First, let us note that for $aa > 0$ we have

$$aa\int_x^\infty \phi(-aa\cdot s + bb)\,ds = \int_{-\infty}^{-aa\cdot x + bb}\phi(y)\,dy = N[-aa\cdot x + bb]$$

by simple change of variables. Second, note that

$$\begin{aligned} e^{cst\cdot x}\phi(-aa\cdot x + bb) &= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left[(-aa\cdot x + bb)^2 - 2cst\cdot x\right]\right\} \\ &= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left[\left(-aa\cdot x + bb + \frac{cst}{aa}\right)^2 + bb^2 - \left(bb + \frac{cst}{aa}\right)^2\right]\right\} \\ &= \phi\left(-aa\cdot x + bb + \frac{cst}{aa}\right)e^{\frac{cst}{aa}\left(bb + \frac{1}{2}\frac{cst}{aa}\right)} \end{aligned}$$

by a simple completion of the square. Now, we can solve the integral in question via integration by parts:

$$\begin{aligned}
&\int_x^\infty e^{cst\cdot s} N\left[-aa\cdot s+bb\right] ds \\
= \quad & \frac{e^{cst\cdot s}}{cst} N\left[-aa\cdot s+bb\right]\Big|_{s=x}^{\infty} + \frac{1}{cst}\left[aa\cdot\int_x^\infty e^{cst\cdot s}\phi\left(-aa\cdot s+bb\right) ds\right] \\
= \quad & -\frac{e^{cst\cdot x}}{cst} N\left[-aa\cdot x+bb\right] + \frac{1}{cst}\left[a\int_x^\infty \phi\left(-aa\cdot s+bb+\frac{cst}{aa}\right) ds\right] e^{\frac{cst}{aa}\left(bb+\frac{1}{2}\frac{cst}{aa}\right)} \\
= \quad & -\frac{e^{cst\cdot x}}{cst} N\left[-aa\cdot x+bb\right] + \frac{1}{cst} N\left[-aa\cdot x+bb+\frac{cst}{aa}\right] e^{\frac{cst}{aa}\left(bb+\frac{1}{2}\frac{cst}{aa}\right)}
\end{aligned}$$

where we used the fact that $\lim_{x\to\infty} N\left[-aa\cdot x+bb\right] e^{cst\cdot x}=0$ for any constant cst.[25]

Next, note that $D_i\left(\tilde{\delta},t\right)=...+...e^{\left(\tilde{\delta}-\tilde{\delta}_B\right)\chi}N\left[q\left(\tilde{\delta},\chi,t\right)\right]+...$ for some χ, so that we are essentially facing integrals

$$\begin{aligned}
&\frac{2}{\sigma}\int_x^\infty e^{\left(s-\tilde{\delta}_B\right)\chi} N\left[q\left(s,\chi,t\right)\right]\frac{M\left(s\right)U\left(x\right)}{Wr\left(s\right)} ds \\
= \quad & \frac{2}{\sigma}\frac{1}{-\Delta\kappa} e^{\kappa_2 x} e^{-\tilde{\delta}_B\chi}\int_x^\infty e^{(\chi-\kappa_2)s} N\left[q\left(s,\chi,t\right)\right] ds \\
= \quad & \frac{2}{\sigma}\frac{1}{-\Delta\kappa} e^{\kappa_2 x} e^{-\tilde{\delta}_B\chi}\frac{1}{\chi-\kappa_2} \\
& \times\left[-e^{(\chi-\kappa_2)} N\left[q\left(x,\chi,t\right)\right] + N\left[q\left(x,\kappa_2,t\right)\right] e^{(\chi-\kappa_2)\left\{\tilde{\delta}_B-\frac{1}{2}\left[(\kappa+a)^2-(\chi+a)^2\right]\sigma^2 T\right\}}\right]
\end{aligned}$$

Here, we used $cst=(\chi-\kappa_2)$, $aa=\frac{1}{\sigma\sqrt{T}}$, $b=\frac{\tilde{\delta}_B-(\chi+a)\sigma^2 T}{\sigma\sqrt{T}}$, $q\left(x,\chi,t\right)+\left(\chi-\kappa\right)\sigma\sqrt{t}=q\left(x,\kappa,t\right)$ and the fact that

$$\begin{aligned}
\left(\chi-\kappa\right)(-)\left[\chi+a-\frac{1}{2}\left(\chi-\kappa\right)\right] &= \left(\chi-\kappa\right)(-)\left[\frac{1}{2}\chi+\frac{1}{2}a+\frac{1}{2}\kappa+\frac{1}{2}a\right] \\
&= \frac{1}{2}\left[\left(\kappa+a\right)^2-\left(\chi+a\right)^2\right]
\end{aligned}$$

where we note that the last term is independent of if we pick the larger or smaller root, as both κ and all possible χ are centered around $-a$. Lastly, we note that $\frac{2}{\sigma}\int_x^\infty e^{\left(s-\tilde{\delta}_B\right)\chi} N\left[q\left(s,\chi,t\right)\right]\frac{M(x)U(s)}{Wr(s)} ds$ has the same form of solution only with κ_1 replacing κ_2. Define

$$\begin{aligned}
H\left(x,\chi,\kappa,T\right) &\equiv \int_x^\infty e^{(\chi-\kappa)\cdot s} N\left[q\left(s,\chi,T\right)\right] ds \\
&= -\frac{1}{cst}\left\{e^{cst\cdot x} N\left[q\left(x,\chi,T\right)\right] - e^{cst\cdot\tilde{\delta}_B}\exp\left\{-cst\left(\chi+a-\frac{1}{2}cst\right)\sigma^2 T\right\} N\left[q\left(x,\chi,T\right)+cst\cdot\sigma\sqrt{T}\right]\right\} \\
&= \frac{1}{\kappa-\chi}\left\{e^{(\chi-\kappa)x} N\left[q\left(x,\chi,T\right)\right] - e^{(\chi-\kappa)\tilde{\delta}_B} e^{\frac{1}{2}\left[(\kappa+a)^2-(\chi+a)^2\right]\sigma^2 T} N\left[q\left(x,\kappa,T\right)\right]\right\}
\end{aligned}$$

[25] If $cst>0$, a simple application of L'Hopital's rule is sufficient to establish the result:

$$\lim_{x\to\infty}\frac{N\left[-aa\cdot x+b\right]}{e^{-cst\cdot x}} = \frac{"0"}{"0"} = \lim_{x\to\infty}\frac{-aa\cdot\phi\left(-aa\cdot x+b\right)}{-cst\cdot e^{-cst\cdot x}} = 0$$

as ϕ is of negative exponential quadratic form. However, for numerical purposes, we observe that this function can become very large before converging to zero. This observation also allows us to note that the integrals $\int_x^\infty e^{cst\cdot s} N\left[-aa\cdot s+b\right] ds$ are everywhere bounded for $x\geq 0$, justifying our result that $K_1=0$.

The solution to the particular part for F then is

$$\begin{aligned} g_F(x) &\equiv \frac{2}{\sigma^2}\int_x^\infty F(s)\frac{M(s)U(x)-M(x)U(s)}{Wr(s)}ds \\ &= \frac{1}{-\Delta\kappa}\frac{2}{\sigma^2}\sum_{i=1}^{2}\left\{e^{\kappa_2 x}e^{-\gamma_i\tilde{\delta}_B}H(x,\gamma_i,\kappa_2,T)-e^{\kappa_1 x}e^{-\gamma_i\tilde{\delta}_B}H(x,\gamma_i,\kappa_1,T)\right\} \\ g'_F(x) &\equiv \frac{2}{\sigma^2}\int_x^\infty F(s)\frac{\kappa_2 M(s)U(x)-\kappa_1 M(x)U(s)}{Wr(s)}ds \\ &= \frac{1}{-\Delta\kappa}\frac{2}{\sigma^2}\sum_{i=1}^{2}\left\{\kappa_2 e^{\kappa_2 x}e^{-\gamma_i\tilde{\delta}_B}H(x,\gamma_i,\kappa_2,T)-\kappa_1 e^{\kappa_1 x}e^{-\gamma_i\tilde{\delta}_B}H(x,\gamma_i,\kappa_1,T)\right\} \end{aligned}$$

and the solution to the particular part for G_j is

$$\begin{aligned} g_{G_j}(x) &\equiv \frac{2}{\sigma^2}\int_x^\infty G_j(s)\frac{M(s)U(x)-M(x)U(s)}{Wr(s)}ds \\ &= \frac{1}{-\Delta\kappa}\frac{2}{\sigma^2}\sum_{i=1}^{2}\left\{e^{\kappa_2 x}e^{-\eta_{ji}\tilde{\delta}_B}H(x,\eta_{ji},\kappa_2,T)-e^{\kappa_1 x}e^{-\eta_{ji}\tilde{\delta}_B}H(x,\eta_{ji},\kappa_1,T)\right\} \\ g'_{G_j}(x) &\equiv \frac{2}{\sigma^2}\int_x^\infty G_j(s)\frac{\kappa_2 M(s)U(x)-\kappa_1 M(x)U(s)}{Wr(s)}ds \\ &= \frac{1}{-\Delta\kappa}\frac{2}{\sigma^2}\sum_{i=1}^{2}\left\{\kappa_2 e^{\kappa_2 x}e^{-\eta_{ji}\tilde{\delta}_B}H(x,\eta_{ji},\kappa_2,T)-\kappa_1 e^{\kappa_1 x}e^{-\eta_{ji}\tilde{\delta}_B}H(x,\eta_{ji},\kappa_1,T)\right\} \end{aligned}$$

Plugging in $x=\tilde{\delta}_B$, and noting that $q\left(\tilde{\delta}_B,\chi,t\right)=-(\chi+a)\sigma\sqrt{t}$, we make the important observation that

$$e^{\kappa\tilde{\delta}_B}e^{-\chi\tilde{\delta}_B}H\left(\tilde{\delta}_B,\chi,\kappa,T\right)=\frac{1}{\kappa-\chi}\left\{N\left[-(\chi+a)\sigma\sqrt{T}\right]-e^{\frac{1}{2}\left[(\kappa+a)^2-(\chi+a)^2\right]\sigma^2T}N\left[-(\kappa+a)\sigma\sqrt{T}\right]\right\}$$

is independent of $\tilde{\delta}_B$. We thus conclude that for any particular part $g_p\left(\tilde{\delta}_B\right)$, of the form given above, and its derivative $g'_p\left(\tilde{\delta}_B\right)$ are **independent** of $\tilde{\delta}_B$ besides $C\left(\tilde{\delta}_B\right)$ containing $e^{\tilde{\delta}_B}$. Also note that for $\chi=\{\gamma_1,\gamma_2\}$ we have

$$e^{\frac{1}{2}\left[(\kappa+a)^2-(\gamma+a)^2\right]\sigma^2T}=e^{rT}$$

and for $\chi=\{\eta_{i1},\eta_{i2}\}$ we have

$$e^{\frac{1}{2}\left[(\kappa+a)^2-\left(\eta_{ij}+a\right)^2\right]\sigma^2T}=e^{(r-\hat{r}_i)T}$$

Total equity is now easily written out to be

$$\begin{aligned} E\left(\tilde{\delta}\right) &= Ke^{\kappa_2\left(\tilde{\delta}-\tilde{\delta}_B\right)}+\frac{e^{\tilde{\delta}}}{r-\mu}+K_0+g_p\left(\tilde{\delta}\right) \\ &= Ke^{\kappa_2\left(\tilde{\delta}-\tilde{\delta}_B\right)}+\frac{e^{\tilde{\delta}}}{r-\mu}+K_0-m\left(P_{11}B_1e^{-\hat{r}_1T}+P_{12}B_2e^{-\hat{r}_2T}\right)g_F\left(\tilde{\delta}\right)+P_{11}mC_1\left(\tilde{\delta}_B\right)g_{G_1}\left(\tilde{\delta}\right)+P_{12}mC_2\left(\tilde{\delta}_B\right)g_{G_2}\left(\tilde{\delta}\right) \end{aligned}$$

where we scaled K by $e^{-\kappa_2\tilde{\delta}_B}$. The constant term K_0 is

$$K_0=\frac{1}{r}\left\{-(1-\pi)c+m\left[A_1+A_2+\sum_j P_{1j}B_ie^{-\hat{r}_jT}-p\right]\right\}$$

The constant K is derived by setting

$$\begin{aligned} 0 = E\left(\tilde{\delta}_B\right) &= K + \frac{e^{\tilde{\delta}_B}}{r-\mu} + K_0 - m\left(\sum_j P_{1j} B_i e^{-\hat{r}_j T}\right) g_F\left(\tilde{\delta}_B\right) \\ &\quad + m\sum_{j=1}^{2} C_j\left(\tilde{\delta}_B\right) g_{G_j}\left(\tilde{\delta}_B\right) \\ \iff K\left(\tilde{\delta}_B\right) &= -\left[\frac{e^{\tilde{\delta}_B}}{r-\mu} + K_0 - m\left(\sum_j P_{1j} B_i e^{-\hat{r}_j T}\right) g_F\left(\tilde{\delta}_B\right) + m\sum_{j=1}^{2} C_j\left(\tilde{\delta}_B\right) g_{G_j}\left(\tilde{\delta}_B\right)\right] \end{aligned}$$

The term in brackets only features linear combinations of constants independent of $\tilde{\delta}_B$. ■

Proof of Proposition 3.

The optimal $\delta_B = e^{\tilde{\delta}_B}$ is now easily derived. Plugging in $K\left(\tilde{\delta}_B\right)$ into the smooth pasting condition $E'\left(\tilde{\delta}_B\right) = 0$, we can derive $\delta_B = e^{\tilde{\delta}_B}$ in closed form:

$$\begin{aligned} 0 &= E'\left(\tilde{\delta}_B\right) \\ &= K\left(\tilde{\delta}_B\right)\kappa_2 + \frac{e^{\tilde{\delta}_B}}{r-\mu} - m\left(B_1 e^{-\hat{r}_1 T} + P_{12} B_2 e^{-\hat{r}_2 T}\right) g'_F\left(\tilde{\delta}_B\right) + m\sum_{j=1}^{2} P_{1j} C_j\left(\tilde{\delta}_B\right) g'_{G_j}\left(\tilde{\delta}_B\right) \\ &= \kappa_2\left[-\frac{e^{\tilde{\delta}_B}}{r-\mu} - K_0 + m\left(B_1 e^{-\hat{r}_1 T} + P_{12} B_2 e^{-\hat{r}_2 T}\right) g_F\left(\tilde{\delta}_B\right) - m\sum_{j=1}^{2} P_{1j}\left(\hat{\alpha}_j \frac{e^{\tilde{\delta}_B}}{r-\mu} - A_j\right) g_{G_j}\left(\tilde{\delta}_B\right)\right] \\ &\quad + \frac{e^{\tilde{\delta}_B}}{r-\mu} - m\left(B_1 e^{-\hat{r}_1 T} + P_{12} B_2 e^{-\hat{r}_2 T}\right) g'_F\left(\tilde{\delta}_B\right) + m\sum_{j=1}^{2} P_{1j}\left(\hat{\alpha}_j \frac{e^{\tilde{\delta}_B}}{r-\mu} - A_j\right) g'_{G_j}\left(\tilde{\delta}_B\right) \\ &= -\frac{e^{\tilde{\delta}_B}}{r-\mu}\left[\kappa_2 - 1 + m\sum_{j=1}^{2} P_{1j}\hat{\alpha}_j\left\{\kappa_2 g_{G_j}\left(\tilde{\delta}_B\right) - g'_{G_j}\left(\tilde{\delta}_B\right)\right\}\right] \\ &\quad - \kappa_2 K_0 + m\left(B_1 e^{-\hat{r}_1 T} + P_{12} B_2 e^{-\hat{r}_2 T}\right)\left\{\kappa_2 g_F\left(\tilde{\delta}_B\right) - g'_F\left(\tilde{\delta}_B\right)\right\} + m\sum_{j=1}^{2} P_{1j} A_j\left\{\kappa_2 g_{G_j}\left(\tilde{\delta}_B\right) - g'_{G1}\left(\tilde{\delta}_B\right)\right\} \end{aligned}$$

which yields

$$\begin{aligned} \delta_B = e^{\tilde{\delta}_B} &= (r-\mu) \times \left[\kappa_2 - 1 + m\sum_{j=1}^{2} P_{1j}\hat{\alpha}_j\left\{\kappa_2 g_{G_j}\left(\tilde{\delta}_B\right) - g'_{G_j}\left(\tilde{\delta}_B\right)\right\}\right]^{-1} \\ &\quad \times \left[\begin{array}{c} -\kappa_2 K_0 + m\left(B_1 e^{-\hat{r}_1 T} + P_{12} B_2 e^{-\hat{r}_2 T}\right)\left\{\kappa_2 g_F\left(\tilde{\delta}_B\right) - g'_F\left(\tilde{\delta}_B\right)\right\} \\ + m\sum_{j=1}^{2} P_{1j} A_j\left\{\kappa_2 g_{G_j}\left(\tilde{\delta}_B\right) - g'_{G_j}\left(\tilde{\delta}_B\right)\right\} \end{array}\right] \end{aligned}$$

where we note that the right hand side is independent of $\tilde{\delta}_B$ by previous results. We can simplify further by noting that each of the terms in curly brackets can be written as

$$\begin{aligned} &\kappa_2 g_F\left(\tilde{\delta}_B\right) - g'_F\left(\tilde{\delta}_B\right) \\ &= \kappa_2 \frac{2}{\sigma^2}\int_{\tilde{\delta}_B}^{\infty} F(s)\frac{M(s)\,U\left(\tilde{\delta}_B\right) - M\left(\tilde{\delta}_B\right)U(s)}{Wr\left(\tilde{\delta}_B\right)}ds - \frac{2}{\sigma^2}\int_{\tilde{\delta}_B}^{\infty} F(s)\frac{\kappa_2 M(s)\,U\left(\tilde{\delta}_B\right) - \kappa_1 M\left(\tilde{\delta}_B\right)U(s)}{Wr\left(\tilde{\delta}_B\right)}ds \\ &= \frac{2}{\sigma^2}\int_{\tilde{\delta}_B}^{\infty} F(s)\frac{(\kappa_1-\kappa_2)\,M\left(\tilde{\delta}_B\right)U(s)}{Wr\left(\tilde{\delta}_B\right)}ds \\ &= -\frac{2}{\sigma^2}\sum_{i=1}^{2} e^{(\kappa_1-\gamma_i)\tilde{\delta}_B} H\left(\tilde{\delta}_B, \gamma_i, \kappa_1, T\right) \\ &= -\frac{2}{\sigma^2}\sum_{i=1}^{2}\frac{1}{\kappa_1-\gamma_i}\left\{N\left[-(\gamma_i+a)\sigma\sqrt{T}\right] - e^{\frac{1}{2}\left[(\kappa_1+a)^2-(\gamma_i+a)^2\right]\sigma^2 T} N\left[-(\kappa_1+a)\sigma\sqrt{T}\right]\right\} \end{aligned}$$

We thus established a closed form, albeit quite complex, for the optimal $\tilde{\delta}_B$.

The limit $\lim_{T\to\infty}\delta_B$ can be easily derived by noting that the normal distributions either converge to 0 or 1, so the only difficulty remaining is the term $e^{\frac{1}{2}\left[(\kappa_1+a)^2-(\gamma_i+a)^2\right]\sigma^2 T}$. Let us establish a series of results:

First, we note that in addition to $e^{\frac{1}{2}\left[(\kappa_1+a)^2-(\gamma_i+a)^2\right]\sigma^2 T}=e^{r_H T}$, we have

$$e^{\frac{1}{2}\left[(\kappa_1+a)^2-\left(\eta_{ji}+a\right)^2\right]\sigma^2 T}=e^{\left(r_H-\hat{r}_j\right)T}$$

and since we established that $\hat{r}_j>r_H$ we note that this term is converging to zero.

Second, we note that

$$\begin{aligned}
\lim_{T\to\infty}\frac{N\left[-\left(\kappa_1+a\right)\sigma\sqrt{T}\right]}{e^{-r_H T}} &= \frac{\text{"0"}}{\text{"0"}}=\lim_{T\to\infty}\frac{\left(N\left[-\left(\kappa_1+a\right)\sigma\sqrt{T}\right]\right)'}{\left(e^{-r_H T}\right)'}\\
&= \lim_{T\to\infty}\frac{\left(\kappa_1+a\right)\sigma}{2r_H\sqrt{T}}\exp\left\{-\frac{1}{2}\left(\kappa_1+a\right)^2\sigma^2 T+r_H T\right\}\\
&= \lim_{T\to\infty}\frac{\left(\kappa_1+a\right)\sigma}{2r_H\sqrt{T}}\exp\left\{-T\left[\frac{\tilde{\mu}^2}{2\sigma^2}+r_H-r_H\right]\right\}\\
&= \lim_{T\to\infty}\frac{\left(\kappa_1+a\right)\sigma}{2r_H\sqrt{T}}\exp\left\{-\frac{\tilde{\mu}^2}{2\sigma^2}T\right\}=0
\end{aligned}$$

where we used the fact that $\left(\kappa_1+a\right)^2=\frac{\tilde{\mu}^2+2\sigma^2 r_H}{\sigma^4}$. Thus, all terms involving functions g vanish and no complication arises from premultiplying by $m=\frac{1}{T}$, and we are left with

$$\lim_{T\to\infty}\frac{\delta_B}{r-\mu}=\lim_{T\to\infty}V_B=\lim_{T\to\infty}\frac{-\kappa_2 K_0\left(T\right)}{\kappa_2-1}=\frac{\kappa_2\left(1-\pi\right)c}{\kappa_2-1}$$

which is the same result as in Leland (1994) once we identify (in Leland's notation) $x=-\kappa_2$, so that $\lim_{T\to\infty}V_B=\frac{(1-\pi)\frac{c}{r}x}{x+1}$. In the infinite maturity limit, the equity holders care about the illiquidity they impose on bondholders via the valuation spread between H and L only at the beginning when issuing bonds, but since there is no rollover their default decision is not affected by bond market illiquidity for a given level of aggregate face value and coupon.

Next, let us investigate $T\to 0$, which essentially renders the secondary bond market completely liquid. But of course there is a large effect of $T\to 0$ on the bankruptcy decision of the equity holders. Using L'Hopital's rule, we need to investigate

$$\lim_{T\to 0}\frac{1}{T}\left[\kappa_2 g_F\left(v_B\right)-g_F'\left(v_B\right)\right]$$

We see that two terms that exactly give $\kappa_i-\chi$ explode at the rate $\frac{1}{\sqrt{T}}$, so that in the limit we have

$$\lim_{T\to\infty}\frac{\delta_B}{r-\mu}=\lim_{T\to 0}V_B=\frac{\sum_{j=1}^{2}P_{0j}\left(B_j+A_j\right)}{\sum_{j=1}^{2}P_{0j}\hat{\alpha}_j}=\frac{p\left[\begin{array}{cc}P_{01} & P_{02}\end{array}\right]\mathbf{P^{-1}1}}{\left[\begin{array}{cc}P_{01} & P_{02}\end{array}\right]\mathbf{P}^{-1}\boldsymbol{\alpha}}$$

If $\alpha=\alpha_H=\alpha_L$, we are back to the L96 solution of $V_B=\frac{p}{\alpha}$. ■

A.3 Proofs of Section 4

Recall that debt values are given by

$$\begin{aligned}
\left[\begin{array}{c}D_H\left(\delta,\tau\right)\\ D_L\left(\delta,\tau\right)\end{array}\right] &= \mathbf{P}\left[\begin{array}{c}A_1+B_1 e^{-\hat{r}_1\tau}\left[1-F\left(\delta,\tau\right)\right]+C_1 G_1\left(\delta,\tau\right)\\ A_2+B_2 e^{-\hat{r}_2\tau}\left[1-F\left(\delta,\tau\right)\right]+C_2 G_2\left(\delta,\tau\right)\end{array}\right]\\
&= \mathbf{P}\left[\begin{array}{c}A_1\\ A_2\end{array}\right]+\left[1-F\left(\delta,\tau\right)\right]\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\mathbf{P}^{-1}\mathbf{P}\left[\begin{array}{c}B_1\\ B_2\end{array}\right]+\mathbf{P}\left[\begin{array}{cc}G_1\left(\delta,\tau\right) & 0\\ 0 & G_2\left(\delta,\tau\right)\end{array}\right]\mathbf{P}^{-1}\mathbf{P}\left[\begin{array}{c}C_1\\ C_2\end{array}\right]\\
&= \mathbf{P}\left[\begin{array}{c}A_1\\ A_2\end{array}\right]+\left[1-F\left(\delta,\tau\right)\right]\exp\left(-\mathbf{A}\tau\right)\mathbf{P}\left[\begin{array}{c}B_1\\ B_2\end{array}\right]+\mathbf{P}\left[\begin{array}{cc}G_1\left(\delta,\tau\right) & 0\\ 0 & G_2\left(\delta,\tau\right)\end{array}\right]\mathbf{P}^{-1}\mathbf{P}\left[\begin{array}{c}C_1\\ C_2\end{array}\right]
\end{aligned}$$

Here, by defining $a \equiv \frac{\mu-\frac{\sigma^2}{2}}{\sigma^2}$, $\gamma_1 \equiv 0$, $\gamma_2 \equiv -2a$, $\eta_{i1,2} \equiv -a \pm \frac{\sqrt{a^2\sigma^4+2\sigma^2\hat{r}_i}}{\sigma^2}$, and $q(\delta,\chi,t) \equiv \frac{\log(\delta_B)-\log(\delta)-(\chi+a)\cdot\sigma^2 t}{\sigma\sqrt{t}}$, the constants in (6) are given by:

$$\left[\begin{array}{c} A_1 \\ A_2 \end{array}\right] \equiv c\hat{\mathbf{D}}^{-1}\mathbf{P}^{-1}\mathbf{1}, \quad \left[\begin{array}{c} B_1 \\ B_2 \end{array}\right] \equiv p\mathbf{P}^{-1}\mathbf{1} - c\hat{\mathbf{D}}^{-1}\mathbf{P}^{-1}\mathbf{1}, \quad \left[\begin{array}{c} C_1 \\ C_2 \end{array}\right] \equiv \frac{\delta_B}{r-\mu}\mathbf{P}^{-1}\boldsymbol{\alpha} - c\hat{\mathbf{D}}^{-1}\mathbf{P}^{-1}\mathbf{1}$$

$$\mathbf{P}\left[\begin{array}{c} A_1 \\ A_2 \end{array}\right] \equiv c\mathbf{A}^{-1}\mathbf{1}, \qquad \mathbf{P}\left[\begin{array}{c} B_1 \\ B_2 \end{array}\right] \equiv p\mathbf{1} - c\mathbf{A}^{-1}\mathbf{1}, \qquad \mathbf{P}\left[\begin{array}{c} C_1 \\ C_2 \end{array}\right] \equiv \frac{\delta_B}{r-\mu}\boldsymbol{\alpha} - c\mathbf{A}^{-1}\mathbf{1}$$

and the functions F and G are given by

$$F(\delta,\tau) \equiv \sum_{i=1}^{2}\left(\frac{\delta}{\delta_B}\right)^{\gamma_i} N\left[q(\delta,\gamma_i,\tau)\right], \; G_j(\delta,\tau) \equiv \sum_{i=1}^{2}\left(\frac{\delta}{\delta_B}\right)^{\eta_{ij}} N\left[q(\delta,\eta_{ij},\tau)\right],$$

where $N(x)$ is the cumulative distribution function for a standard normal distribution.

Define $\boldsymbol{\omega} \equiv [1,-1]\mathbf{A} = \left[\begin{array}{c} (r_H+\xi_H+\xi_L) \\ -(r_L+\xi_H+\xi_L) \end{array}\right]^{\top}$ and $S \equiv D_H - D_L = [1,-1]\left[\begin{array}{c} D_H \\ D_L \end{array}\right]$. We will also write the shorthand $\sqrt{\cdot}$ for $\sqrt{[(r+\xi)-(\overline{r}+\lambda\beta)]^2+4\xi\lambda\beta}$ and note that $\hat{r}_1 - \sqrt{\cdot} = \hat{r}_2 > 0$.

A.3.1 Time-to-maturity τ derivative

Proof of Proposition 4.

The derivative w.r.t. τ is easily established: First, we note that $q_\tau(\delta,\chi,\tau) = \frac{\log(\delta)-\log(\delta_B)-(\chi+a)\sigma^2\tau}{\sigma\sqrt{\tau}}\frac{1}{2\tau}$, so δ and δ_B have reversed signs. Then, we have

$$\left[\begin{array}{c} \dot{D_H}(\delta,\tau) \\ D_L(\delta,\tau) \end{array}\right] = \mathbf{P}\left[\begin{array}{c} -\hat{r}_1 B_1 e^{-\hat{r}_1\tau}\left[1-F(\delta,\tau)\right] - B_1 e^{-\hat{r}_1\tau}\dot{F}(\delta,\tau) + C_1\dot{G}_1(\delta,\tau) \\ -\hat{r}_2 B_2 e^{-\hat{r}_2\tau}\left[1-F(\delta,\tau)\right] - B_2 e^{-\hat{r}_2\tau}\dot{F}(\delta,\tau) + C_2\dot{G}_2(\delta,\tau) \end{array}\right]$$

and the derivatives of the auxiliary functions are

$$\begin{aligned}
\dot{F}(\delta,\tau) &= \sum_{i=1}^{2}\left(\frac{\delta}{\delta_B}\right)^{\gamma_i}\phi\left[q(\delta,\gamma_i,\tau)\right]q_\tau(\delta,\gamma_i,\tau) \\
&= \phi\left[q(\delta,0,\tau)\right]\sum_{i=1}^{2} q_\tau(\delta,\gamma_i,\tau) \\
&= \phi\left[q(\delta,0,\tau)\right]\frac{\log\left(\frac{\delta}{\delta_B}\right)}{\sigma\tau^{3/2}} > 0 \\
\dot{G}_j(\delta,\tau) &= \sum_{i=1}^{2}\left(\frac{\delta}{\delta_B}\right)^{\eta_{ij}}\phi\left[q(\delta,\eta_{ij},\tau)\right]q_\tau(\delta,\eta_{ij},\tau) \\
&= \phi\left[q(\delta,0,\tau)\right]e^{-\hat{r}_j\tau}\sum_{i=1}^{2} q_\tau(\delta,\eta_{ij},\tau) \\
&= \phi\left[q(\delta,0,\tau)\right]e^{-\hat{r}_j\tau}\frac{\log\left(\frac{\delta}{\delta_B}\right)}{\sigma\tau^{3/2}} \\
&= e^{-\hat{r}_j\tau}\dot{F}(\delta,\tau) > 0
\end{aligned}$$

where we used

$$\begin{aligned}
\left(\frac{\delta}{\delta_B}\right)^{\gamma_i}\phi\left[q(\delta,\gamma_i,\tau)\right] &= \phi\left[q(\delta,0,\tau)\right] \\
\left(\frac{\delta}{\delta_B}\right)^{\eta_{ij}}\phi\left[q(\delta,\eta_{ij},\tau)\right] &= \phi\left[q(\delta,0,\tau)\right]e^{-\hat{r}_j\tau}
\end{aligned}$$

This is easily derived:

$$\begin{aligned}
\left(\frac{\delta}{\delta_B}\right)^{\gamma_i}\phi\left[q\left(\delta,\gamma_i,\tau\right)\right] &= e^{-\gamma_i\left(\tilde{\delta}_B-\tilde{\delta}\right)}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{\tilde{\delta}_B-\tilde{\delta}-\left(\gamma_i+a\right)\sigma^2 t}{\sigma\sqrt{t}}\right]^2} \\
&= \exp\left\{-\gamma_i\left(\tilde{\delta}_B-\tilde{\delta}\right)\right\}\frac{1}{\sqrt{2\pi}}\exp\left\{-\left[\frac{\left(\tilde{\delta}_B-\tilde{\delta}\right)^2}{2\sigma^2 t}-2\frac{\left(\gamma_i+a\right)\sigma^2 t\left(\tilde{\delta}_B-\tilde{\delta}\right)}{2\sigma^2 t}+\frac{\left[\left(\gamma_i+a\right)\sigma^2 t\right]^2}{2\sigma^2 t}\right]\right\} \\
&= \frac{1}{\sqrt{2\pi}}\exp\left\{-\left[\frac{\left(\tilde{\delta}_B-\tilde{\delta}\right)^2}{2\sigma^2 t}-2\frac{a\left(\tilde{\delta}_B-\tilde{\delta}\right)\sigma^2 t}{2\sigma^2 t}+\frac{\left(\gamma_i+a\right)^2\left(\sigma^2 t\right)^2}{2\sigma^2 t}\right]\right\} \\
&= \frac{1}{\sqrt{2\pi}}\exp\left\{-\left[\frac{\left(\tilde{\delta}_B-\tilde{\delta}\right)^2}{2\sigma^2 t}-2\frac{a\left(\tilde{\delta}_B-\tilde{\delta}\right)\sigma^2 t}{2\sigma^2 t}+\frac{a^2\left(\sigma^2 t\right)^2}{2\sigma^2 t}+\frac{\left(2\gamma_i a+\gamma_i^2\right)\left(\sigma^2 t\right)^2}{2\sigma^2 t}\right]\right\} \\
&= \phi\left[q\left(\delta,0,\tau\right)\right]\exp\left\{-\frac{\left(2\gamma_i a+\gamma_i^2\right)\sigma^2 t}{2}\right\}
\end{aligned}$$

and we finally note that $\tilde{\mu}\gamma+\frac{\sigma^2}{2}\gamma^2=0 \iff \frac{2\tilde{\mu}}{\sigma^2}\gamma+\gamma^2=0 \iff 2\gamma a+\gamma^2=0$ which gives the result in conjunction with the fact that $(\gamma_i+a)+(\gamma_{-i}+a)=0$ as they are complementary roots centered around $-a$. Plugging in, we have

$$\begin{aligned}
\left[\begin{array}{c}\dot{D_H}\left(\delta,\tau\right)\\ D_L\left(\delta,\tau\right)\end{array}\right] &= \mathbf{P}\left[\begin{array}{c}-\hat{r}_1 B_1 e^{-\hat{r}_1\tau}\left[1-F\left(\delta,\tau\right)\right]+\left(C_1-B_1\right)e^{-\hat{r}_1\tau}\dot{F}\left(\delta,\tau\right)\\ -\hat{r}_2 B_2 e^{-\hat{r}_2\tau}\left[1-F\left(\delta,\tau\right)\right]+\left(C_2-B_2\right)e^{-\hat{r}_2\tau}\dot{F}\left(\delta,\tau\right)\end{array}\right] \\
&= \mathbf{P}\left[\begin{array}{cc}e^{-\hat{r}_1\tau} & 0\\ 0 & e^{-\hat{r}_2\tau}\end{array}\right]\left[\begin{array}{c}-\hat{r}_1 B_1\left[1-F\left(\delta,\tau\right)\right]+\left(C_1-B_1\right)\dot{F}\left(\delta,\tau\right)\\ -\hat{r}_2 B_2\left[1-F\left(\delta,\tau\right)\right]+\left(C_2-B_2\right)\dot{F}\left(\delta,\tau\right)\end{array}\right] \\
&= \mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\left[\begin{array}{c}-\hat{r}_1 B_1\left[1-F\left(\delta,\tau\right)\right]+\left(C_1-B_1\right)\dot{F}\left(\delta,\tau\right)\\ -\hat{r}_2 B_2\left[1-F\left(\delta,\tau\right)\right]+\left(C_2-B_2\right)\dot{F}\left(\delta,\tau\right)\end{array}\right] \\
&= \mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\left(-\left[1-F\left(\delta,\tau\right)\right]\hat{\mathbf{D}}\left[\begin{array}{c}B_1\\ B_2\end{array}\right]+\dot{F}\left(\delta,\tau\right)\left[\begin{array}{c}C_1-B_1\\ C_2-B_2\end{array}\right]\right) \\
&= \exp\left(-\mathbf{A}\tau\right)\left(-\left[1-F\left(\delta,\tau\right)\right]\mathbf{A}\mathbf{P}\left[\begin{array}{c}B_1\\ B_2\end{array}\right]+\dot{F}\left(\delta,\tau\right)\mathbf{P}\left[\begin{array}{c}C_1-B_1\\ C_2-B_2\end{array}\right]\right)
\end{aligned}$$

where we used the fact that $\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)=\exp\left(-\mathbf{A}\tau\right)\mathbf{P}$ and $\mathbf{P}\hat{\mathbf{D}}=\mathbf{A}\mathbf{P}$. Premultiplying by the difference vector $[1,-1]$ and plugging in the definitions of $\mathbf{A}, B_i, C_i$, we have

$$\begin{aligned}
\dot{S}\left(\delta,\tau\right) &= [1,-1]\left[\begin{array}{c}\dot{D_H}\left(\delta,\tau\right)\\ D_L\left(\delta,\tau\right)\end{array}\right] \\
&= [1,-1]\exp\left(-\mathbf{A}\tau\right)\left\{\left[1-F\left(\delta,\tau\right)\right]\left[\begin{array}{c}c-pr_H\\ c-pr_L\end{array}\right]+\dot{F}\left(\delta,\tau\right)\left[\begin{array}{c}\frac{\delta_B}{r-\mu}\alpha_H-p\\ \frac{\delta_B}{r-\mu}\alpha_L-p\end{array}\right]\right\}
\end{aligned}$$

Let us derive a formula for a general vector $\left[\begin{array}{c}x\\ y\end{array}\right]$:

$$\begin{aligned}
[1,-1]\exp\left(-\mathbf{A}\tau\right)\left[\begin{array}{c}x\\ y\end{array}\right] &= \frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\times\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)\left[x\left(r_L-r_H-\xi_H-\xi_L\right)-y\left(r_H-r_L-\xi_L-\xi_H\right)\right]+\sqrt{\cdot}\left(1+e^{\tau\sqrt{\cdot}}\right)\left[x-y\right]\right\} \\
&= \frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\times\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)\left(\left[r_L,-r_H\right]\left[\begin{array}{c}x\\ y\end{array}\right]-\boldsymbol{\omega}\left[\begin{array}{c}x\\ y\end{array}\right]\right)+\sqrt{\cdot}\left(1+e^{\tau\sqrt{\cdot}}\right)[1,-1]\left[\begin{array}{c}x\\ y\end{array}\right]\right\} \\
&= \frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\times\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)\left(\left[r_L,-r_H\right]-\boldsymbol{\omega}+\sqrt{\cdot}\left[1,-1\right]\right)\left[\begin{array}{c}x\\ y\end{array}\right]+2\sqrt{\cdot}\left[1,-1\right]\left[\begin{array}{c}x\\ y\end{array}\right]\right\}
\end{aligned}$$

When $x>y$, it is clear that for $\tau=0$, we have $[1,-1]\exp\left(-\mathbf{A}\cdot 0\right)\left[\begin{array}{c}x\\ y\end{array}\right]=(x-y)>0$. Further, if it is to hold for

any τ, we need

$$\left(e^{\tau\sqrt{\cdot}}-1\right)\left([r_L,-r_H]\begin{bmatrix}x\\y\end{bmatrix}-\boldsymbol{\omega}\begin{bmatrix}x\\y\end{bmatrix}+\sqrt{\cdot}\,[1,-1]\begin{bmatrix}x\\y\end{bmatrix}\right)\geq 0$$

Our derivation of $\dot{S}$ has two terms of this form, multiplied by $[1-F]>0$ and $\dot{F}>0$. To ensure positivity, this implies conditions on $p,c,r_H,r_L,\alpha_H,\alpha_L,\delta_B$ once we identify $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}c-pr_H\\c-pr_L\end{bmatrix}$ and $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{\delta_B}{r-\mu}\alpha_H-p\\\frac{\delta_B}{r-\mu}\alpha_L-p\end{bmatrix}$. Thus, we have the following two conditions for these two cases:

$$\begin{aligned}
\begin{bmatrix}c-pr_H\\c-pr_L\end{bmatrix}:-(r_L-r_H)\left[p\left(r_H+r_L+\xi_H+\xi_L-\sqrt{\cdot}\right)-2c\right] &> 0\\
\iff (r_L-r_H)\,2\left[c-p\hat{r}_2\right] &> 0\\
\iff w_1\equiv c-p\hat{r}_2 &> 0\\
\begin{bmatrix}\frac{\delta_B}{r-\mu}\alpha_H-p\\\frac{\delta_B}{r-\mu}\alpha_L-p\end{bmatrix}:V_B\left[\alpha_L\left(r_L-r_H+\xi_H+\xi_L-\sqrt{\cdot}\right)-\alpha_H\left(r_H-r_L+\xi_H+\xi_L-\sqrt{\cdot}\right)\right]+2p\left(r_H-r_L\right) &> 0\\
\iff V_B\left[\alpha_L\left(-2r_H+2\hat{r}_2\right)-\alpha_H\left(-2r_L+2\hat{r}_2\right)\right]+2p\left(r_H-r_L\right) &> 0\\
\iff w_2\equiv V_B\left[\alpha_L\left(\hat{r}_2-r_H\right)+\alpha_H\left(r_L-\hat{r}_2\right)\right]-p\left(r_L-r_H\right) &> 0
\end{aligned}$$

where we note that $r_H<\hat{r}_2<r_L$. So we need sufficiently high $c>p\hat{r}_2$ and also sufficiently high α_L,α_H in the face of a large discount differential r_L-r_H. We thus have proved the following proposition. Thus, under the sufficient conditions

$$\begin{aligned}
w_1\equiv c-p\hat{r}_2 &\geq 0\\
w_2\equiv V_B\left[\alpha_L\left(\hat{r}_2-r_H\right)+\alpha_H\left(r_L-\hat{r}_2\right)\right]-p\left(r_L-r_H\right) &\geq 0,
\end{aligned}$$

we have $S_\tau(\delta,\tau)<0$, i.e. the bid-ask spread $(1-\beta)S(\delta,\tau)$ is smaller for bonds with shorter time-to-maturity. ■

If either of these conditions are not satisfied, then we can still find τ_{cp} and/or τ_{apr} such that

$$\begin{aligned}
[1,-1]\exp\left(-\mathbf{A}\tau_{cpr}\right)\begin{bmatrix}c-pr_H\\c-pr_L\end{bmatrix} &= 0\\
[1,-1]\exp\left(-\mathbf{A}\tau_{ap}\right)\begin{bmatrix}V_B\alpha_H-p\\V_B\alpha_L-p\end{bmatrix} &= 0
\end{aligned}$$

and where τ_{xy} is given by

$$\tau_{xy}=\frac{1}{\sqrt{\cdot}}\log\left[1-\frac{2\sqrt{\cdot}\,(x-y)}{\left([r_L,-r_H]-\boldsymbol{\omega}-\sqrt{\cdot}\,[1,-1]\right)\begin{bmatrix}x\\y\end{bmatrix}}\right]$$

Then a sufficient (but of course not necessary) condition for $\dot{S}>0$ is $\tau\leq\min\{\tau_{cpr},\tau_{ap}\}$, where $\tau_{xy}=\infty$ if the positivity condition holds.

We conclude with the observation that

$$\begin{aligned}
\dot{S}(\delta,0) &= [1-F(\delta,0)]\,p\left(r_L-r_H\right)+\lim_{\tau\to 0}\dot{F}(\delta,0)\,\frac{\delta_B}{r-\mu}\left(\alpha_H-\alpha_L\right)\\
&= p\left(r_L-r_H\right)>0
\end{aligned}$$

A.3.2 Proof of $S'<0$ via the system of PDEs and LHS

Proof of Proposition 5.

First, note that when we subtract the second line from the first line of the differential equation we have

$$\begin{aligned}
[1,-1]\begin{bmatrix} r_H+\xi_H & -\xi_H \\ -\xi_L & r_L+\xi_L \end{bmatrix}\begin{bmatrix} D_H \\ D_L \end{bmatrix} &= [1,-1]\left(\begin{bmatrix} c \\ c \end{bmatrix}+\tilde{\mu}\delta\begin{bmatrix} D_H \\ D_L \end{bmatrix}'+\frac{\sigma^2}{2}\delta^2\begin{bmatrix} D_H \\ D_L \end{bmatrix}''-\begin{bmatrix} \dot{D_H} \\ D_L \end{bmatrix}\right) \\
\iff \boldsymbol{\omega}\begin{bmatrix} D_H \\ D_L \end{bmatrix}+\dot{S} &= \tilde{\mu}S'+\frac{\sigma^2}{2}S'' \\
\iff LHS &= \tilde{\mu}S'+\frac{\sigma^2}{2}S''
\end{aligned}$$

where

$$\boldsymbol{\omega} \equiv [r_H+\xi_H+\xi_L, -(r_L+\xi_L+\xi_H)].$$

Let us first establish a limit of $LHS(\delta,\tau)$:

$$\begin{aligned}
\lim_{\tau\to 0} LHS(\delta,\tau) &= \boldsymbol{\omega}\begin{bmatrix} D_H(\delta,0) \\ D_L(\delta,0) \end{bmatrix}+\lim_{\tau\to 0}\dot{S}(\delta,\tau) \\
&= -p(r_L-r_H)+p(r_L-r_H) \\
&= 0
\end{aligned}$$

Outline of the proof:

1. Show that $L\dot{H}S$ as a function of τ only changes sign once.
2. Show, when τ is small, that LHS increases, that is
$$L\dot{H}S(\delta,\tau) > 0$$
3. Show that $LHS(\delta,\infty) \geq 0$.
4. Show that
$$S(\delta_B,\tau) - \lim_{\delta\to\infty} S(\delta,\tau) > 0$$

Then we are done: (1.) implies that the can at most be one local extrema. By (2.), we know that there is a local maximum in LHS in terms of τ, i.e., LHS has to go up and then down again to approach from above the value in (3.), which is zero or something positive. Finally, (4.) gives us a contradiction if ever $S' > 0$. First, by continuity of the expectation, we have that $S' < 0$ for some part of the state space (δ_B,∞), as otherwise the surplus couldn't be less at ∞ than at 0. Suppose now that there is an interval on which $S' < 0$. This means that there exist a local maximum with $S' = 0 > S''$. But this would imply $LHS = \tilde{\mu}S' + \frac{\sigma^2}{2}S'' < 0$, a contradiction. Thus, $S' > 0$ everywhere.

Step 1: Recall that

$$\begin{aligned}
\begin{bmatrix} \dot{D_H}(\delta,\tau) \\ D_L(\delta,\tau) \end{bmatrix} &= \exp(-\mathbf{A}\tau)\mathbf{P}\begin{bmatrix} -\hat{r}_1 B_1[1-F(\delta,\tau)]+(C_1-B_1)\dot{F}(\delta,\tau) \\ -\hat{r}_2 B_2[1-F(\delta,\tau)]+(C_2-B_2)\dot{F}(\delta,\tau) \end{bmatrix} \\
&= \exp(-\mathbf{A}\tau)\mathbf{P}\left(-[1-F(\delta,\tau)]\hat{\mathbf{D}}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}+\dot{F}(\delta,\tau)\begin{bmatrix} C_1-B_1 \\ C_2-B_2 \end{bmatrix}\right)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\left[\begin{array}{c} \ddot{D_H}(\delta,\tau) \\ D_L(\delta,\tau) \end{array}\right] &= \exp(-\mathbf{A}\tau)(-\mathbf{A})\mathbf{P}\left(-[1-F(\delta,\tau)]\hat{\mathbf{D}}\left[\begin{array}{c} B_1 \\ B_2 \end{array}\right] + \dot{F}(\delta,\tau)\left[\begin{array}{c} C_1-B_1 \\ C_2-B_2 \end{array}\right]\right) \\
&\quad + \exp(-\mathbf{A}\tau)\mathbf{P}\left(\dot{F}(\delta,\tau)\hat{\mathbf{D}}\left[\begin{array}{c} B_1 \\ B_2 \end{array}\right] + \ddot{F}(\delta,\tau)\left[\begin{array}{c} C_1-B_1 \\ C_2-B_2 \end{array}\right]\right) \\
&= \exp(-\mathbf{A}\tau)\left([1-F(\delta,\tau)]\mathbf{A}^2\mathbf{P}\left[\begin{array}{c} B_1 \\ B_2 \end{array}\right] - \dot{F}(\delta,\tau)\mathbf{A}\mathbf{P}\left[\begin{array}{c} C_1-B_1 \\ C_2-B_2 \end{array}\right]\right) \\
&\quad + \exp(-\mathbf{A}\tau)\left(\dot{F}(\delta,\tau)\mathbf{A}\mathbf{P}\left[\begin{array}{c} B_1 \\ B_2 \end{array}\right] + \dot{F}(\delta,\tau)(...)\mathbf{P}\left[\begin{array}{c} C_1-B_1 \\ C_2-B_2 \end{array}\right]\right) \\
&= \exp(-\mathbf{A}\tau)\left([1-F(\delta,\tau)]\mathbf{A}^2\mathbf{P}\left[\begin{array}{c} B_1 \\ B_2 \end{array}\right]\right) \\
&\quad + \exp(-\mathbf{A}\tau)\dot{F}(\delta,\tau)\left(\mathbf{A}\mathbf{P}\left[\begin{array}{c} 2B_1-C_1 \\ 2B_2-C_2 \end{array}\right] + (...)\mathbf{I}\mathbf{P}\left[\begin{array}{c} C_1-B_1 \\ C_2-B_2 \end{array}\right]\right)
\end{aligned}$$

where we used the fact that $\mathbf{AP} = \mathbf{P}\hat{\mathbf{D}}$ and $\mathbf{A}\exp(-\mathbf{A}\tau) = \mathbf{P}\hat{\mathbf{D}}\mathbf{P}^{-1}\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\mathbf{P}^{-1} = \mathbf{P}\hat{\mathbf{D}}\exp\left(-\hat{\mathbf{D}}\tau\right)\mathbf{P}^{-1} = \mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\hat{\mathbf{D}}\mathbf{P}^{-1} = \exp(-\mathbf{A}\tau)\mathbf{A}$ as diagonal matrices of the same order commute.

Thus, if we can show that $L\dot{H}S > 0$ for any $\delta > \delta_B$ we are done. Note that

$$\begin{aligned}
\frac{\partial^2 S(\delta,\tau)}{\partial\tau^2} = \ddot{S} &= [1,-1](-\mathbf{A})\left[\begin{array}{c} \dot{D_H}(\delta,\tau) \\ D_L(\delta,\tau) \end{array}\right] + [1,-1]\exp(-\mathbf{A}\tau)\left\{-\dot{F}(\delta,\tau)\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right] + \ddot{F}(\delta,\tau)\left[\begin{array}{c} \frac{\delta_B}{r-\mu}\alpha_H - p \\ \frac{\delta_B}{r-\mu}\alpha_L - p \end{array}\right]\right\} \\
&= [1,-1]\exp(-\mathbf{A}\tau)\left\{-\mathbf{A}\left([1-F(\delta,\tau)]\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right] + \dot{F}(\delta,\tau)\left[\begin{array}{c} \frac{\delta_B}{r-\mu}\alpha_H - p \\ \frac{\delta_B}{r-\mu}\alpha_L - p \end{array}\right]\right) - \dot{F}(\delta,\tau)\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right] + \ddot{F}(\delta,\tau)\right. \\
&= -\boldsymbol{\omega}\left[\begin{array}{c} \dot{D_H}(\delta,\tau) \\ D_L(\delta,\tau) \end{array}\right] + [1,-1]\exp(-\mathbf{A}\tau)\left\{-\dot{F}(\delta,\tau)\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right] + \ddot{F}(\delta,\tau)\left[\begin{array}{c} \frac{\delta_B}{r-\mu}\alpha_H - p \\ \frac{\delta_B}{r-\mu}\alpha_L - p \end{array}\right]\right\}
\end{aligned}$$

where we used the fat that $\mathbf{A}\exp(-\mathbf{A}\tau) = \mathbf{P}\hat{\mathbf{D}}\mathbf{P}^{-1}\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\mathbf{P}^{-1} = \mathbf{P}\hat{\mathbf{D}}\exp\left(-\hat{\mathbf{D}}\tau\right)\mathbf{P}^{-1} = \mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\hat{\mathbf{D}}\mathbf{P}^{-1} = \exp(-\mathbf{A}\tau)\mathbf{A}$ as diagonal matrices of the same order commute.

We realize that the $\boldsymbol{\omega}\left[\begin{array}{c} \dot{D_H}(\delta,\tau) \\ D_L(\delta,\tau) \end{array}\right]$ parts cancel out in $L\dot{H}S$, and we are left with

$$L\dot{H}S(\delta,\tau) = [1,-1]\exp(-\mathbf{A}\tau)\left\{-\dot{F}(\delta,\tau)\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right] + \ddot{F}(\delta,\tau)\left[\begin{array}{c} \frac{\delta_B}{r-\mu}\alpha_H - p \\ \frac{\delta_B}{r-\mu}\alpha_L - p \end{array}\right]\right\}$$

Further note that with $\dot{F}(\delta,\tau) = \phi[q(\delta,0,\tau)]\frac{\log\left(\frac{\delta}{\delta_B}\right)}{\sigma\tau^{3/2}}$, $q_\tau(\delta,0,\tau) = \frac{\log\left(\frac{\delta}{\delta_B}\right)-a\sigma^2\tau}{2\sigma\tau^{3/2}}$, and $\phi'(x) = -x\phi(x)$, we have

$$\begin{aligned}
\ddot{F}(\delta,\tau) &= \phi'[q(\delta,0,\tau)]q_\tau(\delta,0,\tau)\frac{\log\left(\frac{\delta}{\delta_B}\right)}{\sigma\tau^{3/2}} + \phi[q(\delta,0,\tau)]\frac{\log\left(\frac{\delta}{\delta_B}\right)}{\sigma\tau^{3/2}}\left(-\frac{3}{2\tau}\right) \\
&= \dot{F}(\delta,\tau)\left[-q(\delta,0,\tau)q_\tau(\delta,0,\tau) - \frac{3}{2\tau}\right] \\
&= \dot{F}(\delta,\tau)\left[-\frac{-\log\left(\frac{\delta}{\delta_B}\right)-a\sigma^2\tau}{\sigma\sqrt{\tau}}\cdot\frac{\log\left(\frac{\delta}{\delta_B}\right)-a\sigma^2\tau}{2\sigma\tau^{3/2}} - \frac{3}{2\tau}\right] \\
&= \dot{F}(\delta,\tau)\left[\frac{\log\left(\frac{\delta}{\delta_B}\right)^2 - a^2\left(\sigma^2\right)^2\tau^2}{2\sigma^2\tau^2} - \frac{3}{2\tau}\right] \\
&= \dot{F}(\delta,\tau)\left[\frac{\log\left(\frac{\delta}{\delta_B}\right)^2}{\sigma^2\tau^2} - \frac{a^2\sigma^2}{2} - \frac{3}{2\tau}\right]
\end{aligned}$$

so that

$$L\dot{H}S\left(\delta,\tau\right)=\dot{F}\left(\delta,\tau\right)\left[1,-1\right]\exp\left(-\mathbf{A}\tau\right)\left\{\left(\frac{\log\left(\frac{\delta}{\delta_B}\right)^2}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right)\left[\begin{array}{c}\frac{\delta_B}{r-\mu}\alpha_H-p\\ \frac{\delta_B}{r-\mu}\alpha_L-p\end{array}\right]-\left[\begin{array}{c}c-pr_H\\ c-pr_L\end{array}\right]\right\}$$

Let us now write out this term in more detail. First, note that

$$\begin{aligned}\left[1,-1\right]\exp\left(-\mathbf{A}\tau\right)\left[\begin{array}{c}V_B\alpha_H-p\\ V_B\alpha_L-p\end{array}\right] &= \frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\times\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2+2\sqrt{\cdot}V_B\left(\alpha_H-\alpha_L\right)\right\}\\ \left[1,-1\right]\exp\left(-\mathbf{A}\tau\right)\left[\begin{array}{c}c-pr_H\\ c-pr_L\end{array}\right] &= \frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\times\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_1+2\sqrt{\cdot}p\left(r_L-r_H\right)\right\}\end{aligned}$$

Then, let $x\equiv\log\left(\frac{\delta}{\delta_B}\right)^2\in(0,\infty)$, to simplify to

$$L\dot{H}S=\dot{F}\times\frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\left[\left(\frac{x}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right)\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2+2\sqrt{\cdot}V_B\left(\alpha_H-\alpha_L\right)\right\}-\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_1+2\sqrt{\cdot}p\left(r_L-r_H\right)\right\}\right]$$

As $\dot{F}\times\frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}>0$, we know that the term $[\cdot]$ determines the sign of $L\dot{H}S$. Writing it out, we have

$$\begin{aligned}&\left[\left(\frac{x}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right)\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2+2\sqrt{\cdot}V_B\left(\alpha_H-\alpha_L\right)\right\}-\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_1+2\sqrt{\cdot}p\left(r_L-r_H\right)\right\}\right]\\ =\ &\left(e^{\tau\sqrt{\cdot}}-1\right)\left[\left(\frac{x}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right)w_2-w_1\right]+2\sqrt{\cdot}\left[V_B\left(\alpha_H-\alpha_L\right)-p\left(r_L-r_H\right)\right]\end{aligned}$$

We note that $\lim_{\tau\to0}\frac{e^{\tau\sqrt{\cdot}}-1}{\tau}=\frac{\text{"0"}}{\text{"0"}}=\sqrt{\cdot}>0$, so that $\lim_{\tau\to\infty}\frac{e^{\tau\sqrt{\cdot}}-1}{\tau^2}=\infty$. Thus, at τ in the vicinity of 0, the sign of the term is determined by w_2. Next, when $\tau\to\infty$, we have the sign being determined by $-\frac{a^2\sigma^2}{2}w_2-w_1<0$.

Multiplying out $w_2\left(e^{\tau\sqrt{\cdot}}-1\right)>0$, and defining $Q_1\left(x,\tau\right)=\left(\frac{x}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right)$, we have

$$\begin{aligned}Q\left(x,\tau\right) &= Q_1\left(x,\tau\right)-\frac{w_1}{w_2}+\frac{2\sqrt{\cdot}\left[V_B\left(\alpha_H-\alpha_L\right)-p\left(r_L-r_H\right)\right]}{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2}\\ &= Q_1\left(x,\tau\right)-\frac{\left(e^{\tau\sqrt{\cdot}}-1\right)w_1-2\sqrt{\cdot}\left[V_B\left(\alpha_H-\alpha_L\right)-p\left(r_L-r_H\right)\right]}{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2}\\ &= Q_1\left(x,\tau\right)-\frac{\left(e^{\tau\sqrt{\cdot}}-1\right)w_1-w_3}{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2}\\ &= Q_2\left(x,\tau\right)-Q_2\left(\tau\right)\end{aligned}$$

where

$$w_3=2\sqrt{\cdot}\left[V_B\left(\alpha_H-\alpha_L\right)-p\left(r_L-r_H\right)\right].$$

Note that $Q_1\left(x,\tau\right)$ changes sign only once. Then, we know that

$$\begin{aligned}\dot{Q}_2\left(\tau\right) &= \frac{\sqrt{\cdot}e^{\tau\sqrt{\cdot}}w_1\left(e^{\tau\sqrt{\cdot}}-1\right)w_2-\left[\left(e^{\tau\sqrt{\cdot}}-1\right)w_1-w_3\right]\sqrt{\cdot}e^{\tau\sqrt{\cdot}}w_2}{\left(\cdot\right)^2}\\ &= \frac{w_2w_3\sqrt{\cdot}e^{\tau\sqrt{\cdot}}}{\left(\cdot\right)^2}\end{aligned}$$

Thus, if $w_2w_3>0$, then $\dot{Q}_2\left(\tau\right)>0$ and we know that $Q\left(x,\tau\right)$ is composed of a part that crosses from positive to negative as τ increase ($Q_1\left(x,\tau\right)$) and of a part that is monotonically decreasing as τ increases ($-Q_2\left(\tau\right)$).

Step 2: From the derivation above, we know that for τ in the vicinity of 0, the sign of the $L\dot{H}S$ is determined by w_2. Next, when $\tau\to\infty$, we have the sign being determined by $-\frac{a^2\sigma^2}{2}w_2-w_1<0$.

Step 3: Note that

$$LHS\left(\delta,\infty\right)=\boldsymbol{\omega}\mathbf{P}\left[\begin{array}{cc}\left(\frac{\delta}{\delta_B}\right)^{\eta_{12}} & 0\\ 0 & \left(\frac{\delta}{\delta_B}\right)^{\eta_{22}}\end{array}\right]\mathbf{P}^{-1}\mathbf{P}\left[\begin{array}{c}C_1\\ C_2\end{array}\right]$$

with $\eta_{12}<\eta_{22}<0$, so that $0<X_1=\left(\frac{\delta}{\delta_B}\right)^{\eta_{12}}<\left(\frac{\delta}{\delta_B}\right)^{\eta_{22}}=X_2$. Note that for $\delta\to\delta_B$, the LHS becomes

$$\lim_{\delta\to\delta_B}LHS\left(\delta,\infty\right)=-\alpha_L\left(r_L-r_H\right)+\left(\alpha_H-\alpha_L\right)\frac{r_H+\xi_H+\xi_L}{2}$$

$$\begin{aligned}\mathbf{P}\left[\begin{array}{c}C_1\\ C_2\end{array}\right] &= \frac{\delta_B}{r-\mu}\boldsymbol{\alpha}-c\mathbf{A}^{-1}\mathbf{1}\\ \boldsymbol{\omega}\mathbf{P}\left[\begin{array}{cc}X_1 & 0\\ 0 & X_2\end{array}\right]\mathbf{P}^{-1}\boldsymbol{\alpha} &= \boldsymbol{\omega}\mathbf{P}\left[\begin{array}{cc}X_1 & 0\\ 0 & X_2\end{array}\right]\mathbf{P}^{-1}\left[\begin{array}{c}\alpha_H\\ \alpha_L\end{array}\right]\\ &= \frac{\alpha_L\left(r_L-r_H\right)\left[\hat{r}_1\left(X_2-X_1\right)-X_2\sqrt{\cdot}\right]}{\sqrt{\cdot}}\\ &\quad+\left(\alpha_H-\alpha_L\right)\left(X_2-X_1\right)\frac{\left[\left(r_H+\xi_H\right)\left(r_L-r_H-\xi_H\right)-\xi_L\left(r_L+\xi_L+2\xi_H\right)\right]}{2\sqrt{\cdot}}\\ &\quad+\left(\alpha_H-\alpha_L\right)\left(X_1+X_2\right)\frac{r_H+\xi_H+\xi_L}{2}\\ \boldsymbol{\omega}\mathbf{P}\left[\begin{array}{cc}X_1 & 0\\ 0 & X_2\end{array}\right]\mathbf{P}^{-1}\mathbf{A}^{-1}\mathbf{1} &= \frac{\left(r_L-r_H\right)\left(X_2-X_1\right)}{\sqrt{\cdot}}\end{aligned}$$

Let $\epsilon=\eta_{22}-\eta_{12}>0$, and $x=\log\left(\frac{\delta}{\delta_B}\right)$. Then sufficient condition for $LHS\left(\delta,\infty\right)>0$ for all δ is

$$\begin{aligned}\boldsymbol{\omega}\mathbf{P}\left[\begin{array}{cc}-\epsilon\exp\left(-\epsilon x\right) & 0\\ 0 & 0\end{array}\right]\mathbf{P}^{-1}\mathbf{P}\left[\begin{array}{c}C_1\\ C_2\end{array}\right] &= V_B\frac{\alpha_L\left(r_L-r_H\right)\hat{r}_1}{\sqrt{\cdot}}-c\frac{\left(r_L-r_H\right)}{\sqrt{\cdot}}-V_B\left(\alpha_H-\alpha_L\right)\frac{r_H+\xi_H+\xi_L}{2}\\ &\quad+V_B\left(\alpha_H-\alpha_L\right)\frac{\left[\left(r_H+\xi_H\right)\left(r_L-r_H-\xi_H\right)-\xi_L\left(r_L+\xi_L+2\xi_H\right)\right]}{2\sqrt{\cdot}}\\ &> 0\end{aligned}$$

as well as

$$-\alpha_L\left(r_L-r_H\right)+\left(\alpha_H-\alpha_L\right)\frac{r_H+\xi_H+\xi_L}{2}>0$$

Step 4: We have

$$\begin{aligned}S\left(\delta_B,\tau\right) &= \frac{\delta_B}{r-\mu}\left(\alpha_H-\alpha_L\right)\\ \lim_{\delta\to\infty}S\left(\delta,\tau\right) &= \left[1,-1\right]\left[c\mathbf{A}^{-1}\mathbf{1}+\exp\left(-\mathbf{A}\tau\right)\left(p\mathbf{1}-c\mathbf{A}^{-1}\mathbf{1}\right)\right]\end{aligned}$$

Under our assumption that $S_\tau\left(\delta,\tau\right)>0$, we know that the highest $S\left(\delta,\tau\right)$ is at $\tau=\infty$. Noting

$$\begin{aligned}\left[1,-1\right]\mathbf{A}^{-1}\mathbf{1} &= \frac{r_L-r_H}{\left(r_H+\xi_H\right)\left(r_L+\xi_L\right)-\xi_H\xi_L}\\ \left[1,-1\right]\exp\left(-\mathbf{A}\tau\right)\mathbf{1} &= \frac{\left(r_L-r_H\right)e^{-\hat{r}_1\tau}\left(e^{\sqrt{\cdot}\tau}-1\right)}{\sqrt{\cdot}}\\ \left[1,-1\right]\exp\left(-\mathbf{A}\tau\right)\mathbf{A}^{-1}\mathbf{1} &= -\frac{\left(r_L-r_H\right)e^{-\hat{r}_1\tau}\left[\hat{r}_1\left(e^{\sqrt{\cdot}\tau}-1\right)+\sqrt{\cdot}\right]}{\sqrt{\cdot}\left[\left(r_H+\xi_H\right)\left(r_L+\xi_L\right)-\xi_H\xi_L\right]}\end{aligned}$$

we have

$$\begin{aligned}S\left(\delta_B,\tau\right)-\lim_{\delta\to\infty}S\left(\delta,\tau\right) &> \lim_{\tau\to\infty}\left\{S\left(\delta_B,\tau\right)-\lim_{\delta\to\infty}S\left(\delta,\tau\right)\right\}\\ &= \left(\alpha_H-\alpha_L\right)\frac{\delta_B}{r-\mu}-\left(r_L-r_H\right)\frac{c}{\left[\left(r_H+\xi_H\right)\left(r_L+\xi_L\right)-\xi_H\xi_L\right]}>0\end{aligned}$$

for appropriate parameter restrictions.

Taken together, we established parameter restrictions that result in $S_\delta(\delta,\tau)<0$. ■

Looser sufficiency conditions can be established for $S_\delta(\delta,\tau)$ in the vicinity of $\tau=0$ or $\delta=\delta_B$. We omit these proofs for brevity.

A.4 The steady-state distribution of types

We now derive the cross-sectional (w.r.t. τ) steady-state distribution of L types. Let $p_H(t,\tau)$ be the proportion at time t of H types of maturity τ. Then we have

$$\frac{\partial p_H(t,\tau)}{\partial t}-\frac{\partial p_H(t,\tau)}{\partial \tau}=\lambda p_L(t,\tau)-\xi p_H(t,\tau)$$

as when time advances, maturity shrinks. To impose a steady-state, we note that $\frac{\partial p_H(t,\tau)}{\partial t}=0$ and that $p_H(t,T)=1$, i.e., at any time t, due to the firm being able to issue to only H types, the proportion of H types with the longest maturity T is always 1. Further note that $p_H+p_L=1$, so that in the end we have

$$\begin{aligned}
-\frac{\partial p_H(\tau)}{\partial \tau} &= \lambda p_L(t,\tau)-\xi p_H(t,\tau)\\
p_H(\tau) &= \frac{\lambda+\xi e^{(\tau-T)(\lambda+\xi)}}{\lambda+\xi}\\
p_L(\tau) &= \frac{\xi}{\lambda+\xi}\left[1-e^{(\tau-T)(\lambda+\xi)}\right]
\end{aligned}$$

Let $\tilde{p}_i(\tau)\equiv\frac{p_i(\tau)}{T}$ so that $\int_0^T \tilde{p}_H(\tau)+\tilde{p}_L(\tau)\,d\tau=1$ and we are appropriately adjusting for the amount of outstanding bonds (of measure 1). The steady state mass of H and L types then is

$$\begin{aligned}
\mu_H(T) &= \int_0^T \tilde{p}_H(\tau)\,d\tau\\
&= \frac{\lambda}{\lambda+\xi}+\frac{\xi\left(1-e^{-T(\lambda+\xi)}\right)}{T(\lambda+\xi)^2}\\
\mu_L(T) &= \frac{\xi}{\lambda+\xi}-\frac{\xi\left(1-e^{-T(\lambda+\xi)}\right)}{T(\lambda+\xi)^2}
\end{aligned}$$

and we note that $\lim_{T\to 0}\mu_H(T)=1$ and $\lim_{T\to 0}\mu_L(T)=0$, as well as $\lim_{T\to\infty}\mu_H(T)=\frac{\lambda}{\lambda+\xi}$ and $\lim_{T\to\infty}\mu_L(T)=\frac{\xi}{\lambda+\xi}$.

The steady-state total value of the firm, given by the simple sum of the equity holders and creditors value functions, is thus

$$TV_{ss}(\delta_0,T;\delta_B)=E(\delta_0;\delta_B)+\int_0^T\left[\tilde{p}_H(\tau)D_H(\delta_0,\tau;\delta_B)+\tilde{p}_L(\tau)D_L(\delta_0,\tau;\delta_B)\right]d\tau.$$

A similar expression can be established for the steady-state value function of dealers, which requires solving an ODE for the value of intermediating bonds of only maturity τ (which can be solved in closed form), and then integrating with respect to $\tilde{p}_L(\tau)$ (which, as TV_{ss} above, cannot to our knowledge be integrated out in closed-form).