Do Dark Pools Harm Price Discovery?*

Haoxiang Zhu
Graduate School of Business, Stanford University

October 7, 2011

Abstract

Dark pools are trading systems that do not display orders to the public market. Orders in dark pools are matched within the exchange bid-ask spread, without a guarantee of execution. Informed traders tend to cluster on one side of the market; therefore, when submitting orders to a dark pool, they face a higher risk that their orders will fail to be executed, compared with liquidity traders whose orders are relatively uncorrelated. Consequently, dark pools are more attractive to liquidity traders, whereas exchanges are more attractive to informed traders. Adding a dark pool tends to concentrate payoff-relevant information onto the exchange and, under natural conditions, improve price discovery. The results of this study have empirical implications and contribute to policy debates over equity market structure.

Keywords: dark pools, price discovery, liquidity, fragmentation, equity market structure

JEL Classifications: G12, G14, G18

*First version: November 2010. I am very grateful to Darrell Duffie, Jonathan Berk, John Beshears, Bradyn Breon-Drish, Peter DeMarzo, Thomas George, Steven Grenadier, Frank Hatheway, Dirk Jenter, Ron Kaniel, Arthur Korteweg, Ilan Kremer, Charles Lee, Han Lee, Ian Martin, Jim McLoughlin, Albert Menkveld, Stefan Nagel, Francisco Pérez-González, Paul Pfleiderer, Monika Piazzesi, Martin Schneider, Ken Singleton, Jeffrey Smith, Ilya Strebulaev, Mao Ye, Ruiling Zeng, Jeff Zwiebel, and seminar participants at Stanford University, joint Stanford-Berkeley student seminar, and WFA for helpful comments. All errors are my own. Corresponding address: Stanford Graduate School of Business, 655 Knight Way, Stanford, CA 94305-7298. Email: haoxiang.zhu@stanford.edu. Paper URL: http://ssrn.com/abstract=1712173

1 Introduction

Dark pools are trading systems that do not display their orders to the public markets. Some dark pools passively match buyers and sellers at prices derived from transparent exchanges. Other dark pools operate essentially as non-displayed limit order books that execute orders by price and time priority.

This paper investigates whether dark pools harm price discovery. Contrary to some misgivings expressed by regulators and market participants, I show that under natural conditions, adding a dark pool alongside an exchange improves price discovery.

As of September 2009, 32 dark pools in the United States accounted for 7.9% of consolidated trading volume, according to Securities and Exchange Commission (2010). Industry estimates from Tabb Group, a consultancy, and Rosenblatt Securities, a broker, attribute about 12% of U.S. equity trading volume to dark pools, as of mid 2011. The market shares of dark pools in Europe, Canada, and Asia are smaller but growing quickly (International Organization of Securities Commissions, 2010).

Dark pools have raised concerns among regulators and market participants. The Securities and Exchange Commission (2010), for example, has requested public comments on "the effect of undisplayed liquidity on order execution quality, the effect of undisplayed liquidity on public price discovery, and fair access to sources of undisplayed liquidity." The European Commission (2010) remarks that "[a]n increased use of dark pools . . . raise[s] regulatory concerns as it may ultimately affect the quality of the price discovery mechanism on the 'lit' markets." The International Organization of Securities Commissions (2011) similarly worries that "the development of dark pools and use of dark orders could inhibit price discovery if orders that otherwise might have been publicly displayed become dark." According to a recent survey conducted by the CFA Institute (2009), 71% of respondents believe that the operations of dark pools are "somewhat" or "very" problematic for price discovery. When speaking of non-displayed liquidity, SEC Commissioner Elisse Walter commented that "[b]ecause publicly displayed liquidity provides investors with a critical reference point for a stock, . . . there could be some truth to the criticism that every share that is crossed in the dark is a share that doesn't assist the market in determining an accurate price."[1]

My inquiry into dark pools builds on a simple model of strategic venue selection by informed and liquidity traders. Informed traders hope to profit from proprietary information regarding the value of the traded asset, whereas liquidity traders wish to meet their idiosyncratic liquidity needs. Both types of traders optimally choose between an exchange and a dark pool. The exchange displays a bid and an ask, and

[1] "Speech by SEC Commissioner: Opening Remarks Regarding Dark Pools," October 21, 2009.

guarantees to execute all submitted orders at the bid or the ask. The dark pool free-rides on exchange prices by matching orders within the exchange's bid and ask. Unlike the exchange, the dark pool has no market makers to absorb excess order flow and thus cannot guarantee execution. Sending an order to the dark pool, therefore, involves a tradeoff between potential price improvement and the risk of no execution.

Execution risk in the dark pool drives my results. Because matching in the dark pool depends on the availability of counterparties, some orders on the "heavier" side of the market—the side with more orders—fail to be executed. These unexecuted orders subsequently suffer a worse exchange price than that prevails just before the dark-pool crossing. Having correlated information and thus clustering on one side of the market, informed traders face a higher risk of failing to trade in the dark pool than liquidity traders. Anticipating this execution risk, proportionately more informed traders than liquidity traders submit their orders to the exchange, avoiding the dark pool. Under natural conditions, this self-selection lowers the noisiness of demand and supply on the exchange, improving price discovery.

The main intuition of my results does not hinge upon specific trading mechanisms used by dark pools. For example, a dark pool may execute orders at the midpoint of the exchange bid and ask, or operate as a non-displayed limit order book. Under both mechanisms, informed traders suffer a lower dark-pool execution probability than liquidity traders. Dark pools operating as limit order books, however, are relatively more attractive to informed traders, because informed traders in these venues can gain execution priority by submitting aggressive limit prices.

This study contributes to the policy debate on the architecture of equity markets by explaining how selective fragmentation of trading across transparent exchanges and dark pools can arise and can improve price discovery. This price-discovery benefit of dark pools differs from their "size discovery" function of handling large institutional orders.[2] My results further suggest that informed traders have even stronger incentives to trade on the exchange under a "trade-at" rule—trading venues not quoting the best price should either route incoming orders to venues quoting the best price or provide incoming orders with sufficiently large price improvements over the prevailing best price. The impact of a trade-at rule on price discovery complements previous fairness-motivated argument that displayed orders, which contribute to pre-trade transparency, should have strictly higher priority than non-displayed orders at the same price.[3]

[2]See, for example, Securities and Exchange Commission (2010) and Ready (2010) for discussions of the size-discovery function of dark pools.

[3]For example, the Joint CFTC-SEC Advisory Committee (2011) has noted: "Under current Regulation NMS routing rules, venues cannot 'trade through' a better price displayed on another market. Rather than route the order to the better price, however, a venue can retain and execute the order by matching the current best price even if it has not displayed a publicly accessible quote order at that price. While such a routing regime provides

The improvement in exchange-based price discovery promoted by dark pools need not coincide with higher liquidity or welfare. By definition, highly informative orders correspond to better price discovery, but also tend to worsen the adverse selection on the exchange, resulting in wider spreads and higher price impacts. I focus on price discovery and do not attempt to capture welfare effects, some of which would naturally materialize outside the setting of my model. For example, the effect of dark pools for welfare could naturally depend on how price information is used for production decisions, asset allocation, and capital formation.

To my knowledge, this paper is the first to show that adding a dark pool can improve price discovery. My finding stands in contrast to that of Ye (2009), who studies the venue choice of a large informed trader in the Kyle (1985) framework and concludes that adding a dark pool harms price discovery on the exchange. Ye (2009), however, assumes *exogenous* choices of trading venues by liquidity traders, whereas the *endogenous* venue choices of liquidity traders are critical to my results. Most other existing models of dark pools either fix the strategies of informed traders exogenously, as in Hendershott and Mendelson (2000), or do not consider the role of asymmetric information regarding the asset value, as in Degryse, Van Achter, and Wuyts (2009) and Buti, Rindi, and Werner (2010b). Kratz and Schöneborn (2010) derive the optimal liquidation strategy of a large trader, assuming exogenous prices and liquidity on the exchange and in the dark pool. In addition, going beyond the midpoint-matching mechanism, this study discovers that dark pools allowing more discretion in execution prices are more attractive to informed traders.

The focus of this paper on the fragmentation of order flow between an exchange and a dark pool differs from the focus of prior studies on competition among multiple markets. In exchange markets, for example, informed traders and liquidity traders tend to cluster by time (Admati and Pfleiderer, 1988) or by location (Pagano, 1989; Chowdhry and Nanda, 1991). As modeled here, however, informed traders cluster less with liquidity traders in the dark pool than on the exchange, because informed traders face higher execution risk in the dark pool. Related to the effect captured by my model, Easley, Keifer, and O'Hara (1996) suggest that the purchase of retail order flows ("cream-skimming") by regional exchanges results in higher order informativeness on the NYSE. In contrast with the mechanism of their paper, dark pools in my model rely on self-selection, rather than intermediaries, to separate, at least partially, informed traders from liquidity traders.

The findings of my investigation have several empirical implications. For example,

order execution at the current best displayed price, it does so at the expense of the limit order posting a best price which need not receive execution."

the model predicts that higher buy-versus-sell order imbalances tend to cause lower dark-pool activity; that more dark trading leads to wider spreads and higher price impacts on exchanges; that dark-pool market share is low when volatility, spread, or volume is low; that volume correlation across stocks is higher on exchanges than in dark pools; and that informed traders participate more actively in dark pools when asymmetric information is more severe or when the dark pool allows more discretion in execution prices. Section 6 discusses these implications as well as recent empirical evidence documented by Ready (2010), Buti, Rindi, and Werner (2010a), Ye (2010), Ray (2010), Nimalendran and Ray (2011), Degryse, de Jong, and van Kervel (2011), O'Hara and Ye (2011), and Weaver (2011), among others.

2 An Overview of Dark Pools

This section provides an overview of dark pools. I discuss why dark pools exist, how they operate, and what distinguishes them from each other. For concreteness, I tailor this discussion for the market structure and regulatory framework of the United States. Dark pools in Europe, Canada, and Asia operate similarly.

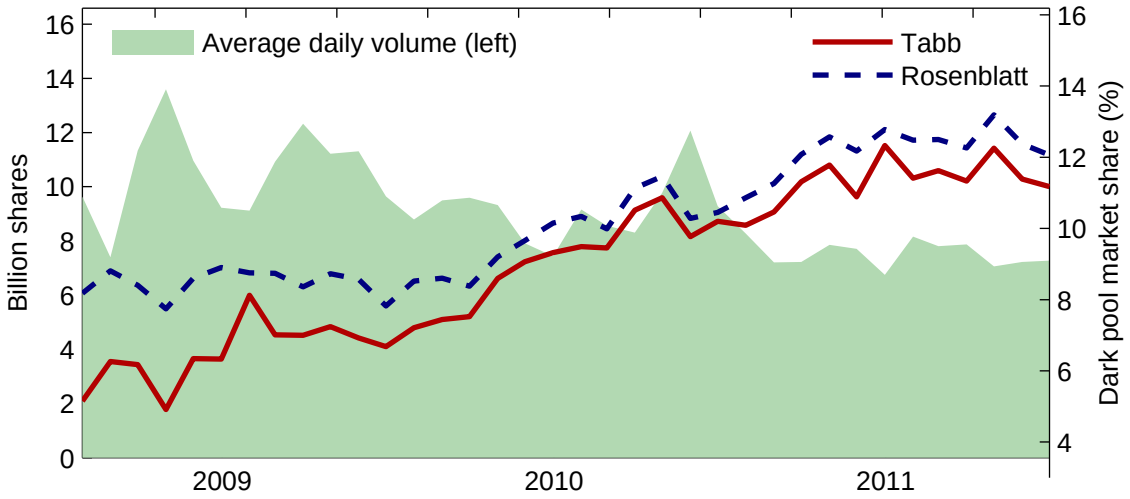
Before 2005, dark pools had low market share. Early dark pools were primarily used by institutions to trade large blocks of shares without revealing their intentions to the broad market, in order to avoid being front-run or "predated."[4] A watershed event for the U.S. equity market was the adoption in 2005 of Regulation National Market System, or Reg NMS (Securities and Exchange Commission, 2005), which abolished rules that had protected the manual quotation systems of incumbent exchanges. In doing so, Reg NMS encouraged newer and faster electronic trading centers to compete with the incumbents. Since Reg NMS came into effect, a wide variety of trading centers have been established. As of September 2009, the United States had about 10 exchanges, 5 electronic communication networks (ECNs), 32 dark pools, and over 200 broker-dealers (Securities and Exchange Commission, 2010). Exchanges and ECNs are referred to as transparent, or "lit," venues; dark pools and broker-dealer internalization are considered opaque, or "dark," venues. In Europe, the adoption in 2007 of the Markets in Financial Instruments Directive (MiFID) similarly led to increased competition and a fast expansion of equity trading centers.[5]

Figure 1 shows the consolidated volume of U.S. equity markets from July 2008 to June 2011, as well as the market share of dark pools during the same periods, estimated

[4] Predatory trading is modeled by Brunnermeier and Pedersen (2005) and Carlin, Lobo, and Viswanathan (2007).

[5] For example, according to CFA Institute (2009), European equity market had 92 regulated markets (exchanges), 129 "multilateral trading facilities" (MTFs), and 13 "systematic internalizers" as of Septempber 2010. For more discussion of MiFID and European equity market structure, see European Commission (2010).

Figure 1: U.S. equity trading volume and the market share of dark pools. The left axis plots the daily consolidated equity trading volume in the United States, estimated by Tabb Group. The right axis plots the market shares of dark pools as a percentage of the total consolidated volume, estimated by Tabb Group and Rosenblatt Securities.



by Tabb Group and Rosenblatt Securities. According to their data, the market share of dark pools roughly doubled from about 6.5% in 2008 to about 12% in 2011, whereas consolidated equity volume dropped persistently from about 10 billion shares per day in 2008 to about 7 billion shares per day in 2011. A notable exception to the decline in consolidated volume occurred around the "Flash Crash" of May 2010.

Dark pools have gained market share for reasons that go beyond recent regulations designed to encourage competition. Certain investors, such as institutions, simply need non-displayed venues to trade large blocks of shares without alarming the broad market. This need has increased in recent years as the order sizes and depths on exchanges have declined dramatically (Chordia, Roll, and Subrahmanyam, 2010). Further, dark pools attract investors by offering potential price improvements relative to the best prevailing bid and offer on exchanges. Finally, broker-dealers handling customer orders have strong incentives to set up their own dark pools, where they can better match customer orders internally, and therefore save trading fees that would otherwise be paid to exchanges and other trading centers.

At least two factors contribute to the lack of precise official data on dark-pool transactions in the United States. First, U.S. dark-pool trades are reported to "trade reporting facilities," or TRFs, which aggregate trades executed by all off-exchange venues—including dark pools, ECNs, and broker-dealer internalization—into a single category. Thus it is generally not possible to assign a TRF trade to a specific off-

exchange venue that executes the trade.[6] Second, dark pools often do not have their own identification numbers (MPID) for trade reporting. For example, a broker-dealer may report customer-to-customer trades in its dark pool *together with* the broker's own over-the-counter trades with institutions, all under the same MPID. Similarly, trades in an exchange-owned dark pool can be reported together with trades conducted on the exchange's open limit order book, all under the exchange's MPID. Because different trading mechanisms share the same MPID, knowing the MPID that executes a trade is insufficient to determine whether that trade occurred in a dark pool.[7]

Dark pools differ from each other in many ways. We can categorize them, roughly, into the three groups shown in the top panel of Table 1.

Dark pools in the first group match customer orders by acting as agents (as opposed to trading on their own accounts). In this group, transaction prices are typically derived from lit venues. These derived prices include the midpoint of the national best bid and offer (NBBO) and the volume-weighted average price (VWAP). Dark pools in this group include three block-crossing dark pools: ITG Posit, Liquidnet, and Pipeline.[8] Posit crosses orders a few times a day at scheduled clock times (up to some randomization), although in recent years it has also offered continuous crossing. Liquidnet is integrated into the order-management systems of institutional investors and alerts potential counterparties when a potential match is found. Pipeline maintains a flashboard of stock names in colors that change when serious trading interests are present. In addition to those three, Instinet is another agency broker that operates scheduled and continuous dark pools. Because Group-1 dark pools rely on lit venues to determine execution prices, they typically do not provide direct price discovery.

Within the second group, dark pools operate as continuous non-displayed limit order books, accepting market, limit, or "pegged" orders.[9] This group includes many of the dark pools owned by major broker-dealers, including Credit Suisse Crossfinder, Goldman Sachs Sigma X, Citi Match, Barclays LX, Morgan Stanley MS Pool, and UBS PIN. Unlike Group-1 dark pools that execute orders at the market midpoint or VWAP, Group-2 dark pools derive their own execution prices from the limit prices of submitted orders. Price discovery can therefore take place. Another difference is that Group-2 dark pools may contain proprietary order flows from the broker-dealers that

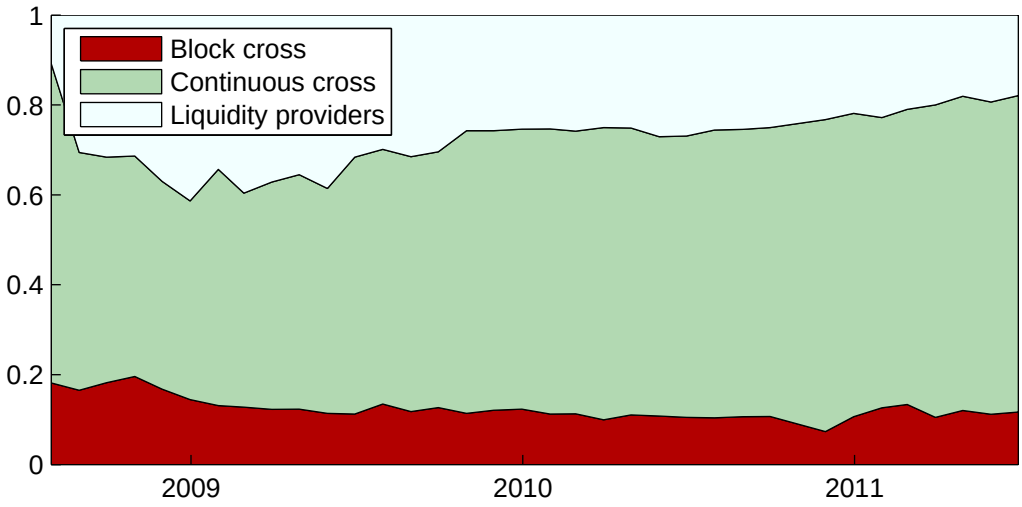
[6]The Securities and Exchange Commission (2009) has recently proposed a rule requiring that alternative trading systems (ATS), including dark pools, provide real-time disclosure of their identities on their trade reports.

[7]For example, Ye (2010) finds that only eight U.S. dark pools can be uniquely identified by MPIDs from their Rule 605 reports to the SEC. The majority of dark pools cannot.

[8]See also Ready (2010) for a discussion of these three dark pools.

[9]Pegged orders are limit orders with the limit price set relative to an observable market price, such as the bid, the offer, or the midpoint. As the market moves, the limit price of a pegged order moves accordingly. For example, a midpoint-pegged order has a limit price equal to the midpoint of the prevailing NBBO. A buy order pegged at the offer minus one cent has a limit price equal to the prevailing best offer minus one cent.

Figure 2: Market shares of three types of U.S. dark pools as fractions of total U.S. dark pool volume, estimated by Tabb Group. The three types are summarized in the middle panel of Table 1.



operate them. In this sense, these dark pools are not necessarily "agency only."

Dark pools in the third group act like fast electronic market makers that immediately accept or reject incoming orders. Examples include Getco and Knight. Like the second group, transaction prices on these platforms are not necessarily calculated from the national best bid and offer using a transparent rule. In contrast with dark pools in Groups 1 and 2, Group-3 dark pools typically trade on their own accounts as principals (as opposed to agents or marketplaces).

This three-way classification is similar to that of Tabb Group (2011), which classifies dark pools into block-cross platforms, continuous-cross platforms, and liquidity-provider platforms. The main features of these three groups are summarized in the middle panel of Table 1. Their respective market shares are plotted in Figure 2. As we can see, the market share of block-cross dark pools has declined from nearly 20% in 2008 to just above 10% in 2011. Continuous-cross dark pools have gained market share during the same period, from around 50% to around 70%. The market share of liquidity-providing dark pools increased to about 40% around 2009, but then declined to about 20% in mid 2011. Tabb Group's data, however, do not cover the entire universe of dark pools, and the components of each category can vary over time. For this reason, these statistics are noisy and should be interpreted with caution.

Dark pools are also commonly classified by their crossing frequencies and by how they find matching counterparties, as illustrated in the bottom panel of Table 1. Aside from mechanisms such as midpoint-matching and limit order books, advertisement is

Table 1: Dark pools by crossing mechanism. The top panel shows a classification by price discovery and order composition. The middle panel shows the classification of Tabb Group. The bottom panel shows another classification by the crossing frequencies and the methods of finding counterparties.

Classification 1

Types	Examples	Typical features
Matching at exchange prices	ITG Posit, Liquidnet, Pipeline, Instinet	Mostly owned by agency brokers and exchanges; typically execute orders at midpoint or VWAP, and customer-to-customer
Non-displayed limit order books	Credit Suisse Crossfinder, Goldman Sachs Sigma X, Citi Match, Barclays LX, Morgan Stanley MS Pool, UBS PIN	Most broker-dealer dark pools; may offer some price discovery and contain proprietary order flow
Electronic market makers	Getco and Knight	High-speed systems handling immediate-or-cancel orders; typically trade as principal

Classification 2

Types	Examples	Notes
Block cross	Liquidnet, BIDS, Instinet Cross	Similar to the first group of the top panel
Continuous cross	Credit Suisse Crossfinder, Goldman Sachs Sigma X, Barclays LX, Morgan Stanley MS Pool, LeveL, Deutsche Bank SuperX	Similar to the second group of the top panel; LeveL is owned by a consortium of broker-dealers
Liquidity provider	Getco and Knight	Same as the third group of the top panel

Classification 3

Types		Examples	Typical features
Scheduled		ITG POSIT Match, Instinet US Crossing	Cross at fixed clock times, with some randomization
Continuous	Matching	ITG POSIT Now, Instinet CBX, Direct Edge MidPoint Match	
	Advertized	Pipeline, POSIT Alert, Liquidnet	Electronic messages sent to potential matched counterparties
	Negotiated	Liquidnet	
	Internal	Credit Suisse Crossfinder, Goldman Sachs Sigma X, Knight Link, Getco	Owned by broker-dealers, run as non-displayed limit order books or electronic market makers

sometimes used to send selected information about orders resting in the dark pool to potential counterparties, in order to facilitate a match. In addition to the classifications summarized in Table 1, dark pools can be distinguished by their ownership structure. Today, most dark pools are owned by broker-dealers (with or without proprietary order flows). A small fraction is owned by consortiums of broker-dealers or exchanges. Dark pools also differ by their average trade size. According to Rosenblatt Securities (2011), two block-size dark pools (Liquidnet and Pipeline) have an order size of around 50,000 shares, which is larger than that of Posit (around 6,000 shares per order) and much larger than those of other broker dark pools (about 300 shares per order). This sharp contrast in order sizes can be attributed to the use of algorithms that split "parent" orders into smaller "children" orders, as observed by the Securities and Exchange Commission (2010).

Finally, there are two sources of non-displayed liquidity that are usually not referred to as dark pools. One is broker-dealer internalization, by which a broker-dealer handles customer orders as a principal or an agent (Securities and Exchange Commission, 2010). A crude way of distinguishing dark pools from broker-dealer internalization is that the former are often marketplaces that allow direct customer-to-customer trades, whereas the latter typically involves broker-dealers as intermediaries.[10] The other source of non-displayed liquidity is the use of hidden orders on exchanges. Examples include reserve ("iceberg") orders and pegged orders, which are limit orders that are partially or fully hidden from the public view.[11] For example, Nasdaq reports that more than 15% of its order flow is non-displayed.[12] In particular, midpoint-pegged orders on exchanges are similar to dark-pool orders waiting to be matched at the midpoint. Further discussion of dark pools and non-displayed liquidity is provided by Johnson (2010), Butler (2007), Carrie (2008), Securities and Exchange Commission (2010), European Commission (2010), CSA/IIROC (2009), and International Organization of Securities Commissions (2011).

[10]There are exceptions. For example, dark pools acting like electronic market makers (like Getco and Knight) also provide liquidity by trading on their own accounts. Nonetheless, they are highly automated systems and rely less on human intervention than, say, dealers arranging trades over the telephone.

[11]A reserve order consists of a displayed part, say 200 shares, and a hidden part, say 1,800 shares. Once the displayed part is executed, the same amount, taken from the hidden part, becomes displayed, until the entire order is executed or canceled. Pegged orders are often fully hidden. Typically, pegged orders and hidden portions of reserve orders have lower execution priority than displayed orders with the same limit price.

[12]See http://www.nasdaqtrader.com/Trader.aspx?id=DarkLiquidity

3 Modeling the Exchange and Dark Pool

This section presents a two-period model of trading-venue selection. Each trader chooses whether to trade on a transparent exchange or in a dark pool. The dark pool modeled in this section passively matches orders at the midpoint of the exchange's bid and ask. Section 4 models a dark pool that operates as a non-displayed limit order book. The order-book setting provides additional insights regarding the effect of the dark-pool crossing mechanism for price discovery. A dynamic equilibrium with sequential arrival of traders is characterized in Section 5. A glossary of key model variables can be found in Appendix C.

3.1 Markets and traders

There are two trading periods, denoted by $t = 1, 2$. At the end of period 2, an asset pays an uncertain dividend v that is equally likely to be $+\sigma$ or $-\sigma$. Thus $\sigma > 0$ is the volatility of the asset value. The asset value v is publicly revealed at the beginning of period 2.

Two trading venues operate in parallel: a lit exchange and a dark pool. The exchange is open in periods 1 and 2. On the exchange, a risk-neutral market maker sets competitive bid and ask prices. Market orders sent to the exchange arrive simultaneously. Exchange buy orders are executed at the ask; exchange sell orders are executed at the bid. The exchange here is thus similar to that modeled by Glosten and Milgrom (1985).[13] After period-1 orders are executed, the exchange announces its net order flow, the quantity of buy orders executed on the exchange less the quantity of sell orders executed on the exchange.

The dark pool executes (or "crosses") orders in period 1 and is closed in period 2. Closing the the dark pool in period 2 is without loss of generality, because once the dividend v is announced in period 2, exchange trading is costless. An order submitted to the dark pool is not observable to anyone but the order submitter. The execution price of dark-pool trades is the midpoint of the exchange bid and ask, also known simply as the "midpoint" or "mid-market" price. In the dark pool, orders on the "heavier side"—the buyers' side if buy orders exceed sell orders, and the sellers' side if sell orders exceed buy orders—are randomly selected for matching with those on the "lighter" side. For example, if the dark pool receives Q_B buy orders and $Q_S < Q_B$ sell orders, all of the same size, then Q_S of the Q_B buy orders are randomly selected, equally likely, to be executed against the Q_S sell orders at the mid-market price. Unmatched orders are

[13] As I describe shortly, the model of this section is not exactly the same as that of Glosten and Milgrom (1985), because orders here arrive in batches, instead of sequentially. Sequential arrival of orders is considered in Section 5.

returned to the order submitter at the end of period 1. As described in Section 2, this midpoint execution method is common in dark pools operated by agency brokers and exchanges. An alternative dark-pool mechanism, a non-displayed limit order book, is modeled in Section 4.

For-profit traders and liquidity traders, all risk-neutral, arrive at the beginning of period 1. There is a continuum of each type. Each trader is "infinitesimal" and can trade either one unit or zero unit of the asset. The mass of for-profit traders is a constant $\bar{\mu} > 0$. For-profit traders can acquire, at a cost, perfect information about v, and thus become informed traders. These information-acquisition costs are pairwise-independent,[14] with the cumulative distribution function $F : [0, \infty) \to [0, 1]$. After observing v, informed traders submit buy orders (in either venue) if $v = +\sigma$ and submit sell orders if $v = -\sigma$. For-profit traders who do not acquire the information do not trade. I let μ_I be the mass of informed traders; their signed trading interest is therefore $Y = \text{sign}(v) \cdot \mu_I$.

Liquidity buyers and liquidity sellers arrive at the market separately (not as netted). Each liquidity buyer already holds an undesired short position of one unit; each liquidity seller already holds an undesired long position of one unit. These undesired positions could, for example, be old hedges that have become useless. The mass Z^+ of liquidity buyers and the mass Z^- of liquidity sellers are modeled as

$$(Z^+, Z^-) = \begin{cases} (Z_0 + Z, Z_0), & \text{if } Z \geq 0, \\ (Z_0, Z_0 + |Z|), & \text{if } Z < 0, \end{cases} \tag{1}$$

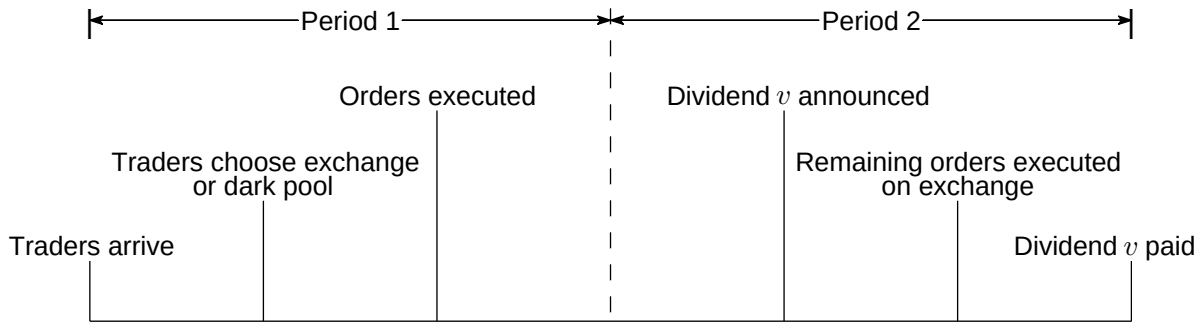
where Z_0 is the "balanced" part of the liquidity trading interests and Z is the "imbalanced" part. We assume that Z_0 and $|Z|$ have finite means, that Z^+ and Z^- have differentiable cumulative distribution functions, and that Z is distributed symmetrically with respect to zero. The symmetric distribution of Z guarantees that Z^+ and Z^- are identically distributed. The total expected mass of liquidity traders is

$$\mu_z \equiv 2\mathbb{E}(Z_0) + \mathbb{E}(|Z|). \tag{2}$$

Liquidity traders must hold collateral to support their undesired risky positions. For each liquidity trader, the minimum collateral requirement is the expected loss, conditional on a loss, of her undesired position. For example, a liquidity buyer who is already short one unit of the asset has a loss of σ if $v = \sigma$, and a gain of σ if $v = -\sigma$. The collateral requirement in this case is σ. For trader i, each unit of collateral has a

[14] Pairwise-independence here is in the "essential" sense of Sun (2006), who gives technical conditions for the exact law of large numbers on which I rely throughout.

Figure 3: Time line of the two-period model



funding cost of γ_i per period. A delay in trade is therefore costly. These funding costs $\{\gamma_i\}$ are pairwise-independently distributed across traders, with a twice-differentiable cumulative distribution function $G : [0, \Gamma) \to [0, 1]$, for some $\Gamma \in (1, \infty]$. Failing to trade in period 1, liquidity buyer i thus incurs a delay cost of

$$c_i = \gamma_i \mathbb{E}[\max(v, 0) \,|\, v > 0] = \gamma_i \sigma. \tag{3}$$

A like delay cost applies to liquidity sellers. We could alternatively interpret this delay cost as stemming from risk aversion or illiquidity. The key is that liquidity traders differ in their desires for immediacy, captured by the delay cost $c_i = \gamma_i \sigma$. The delay costs of informed traders, by contrast, stem from the loss of profitable trading opportunities after v is revealed in period 2.

Finally, random variables v, Z_0, Z, and the costs of information-acquisition and delay are all independent, and their probability distributions are common knowledge. Realizations of Y, Z^+ and Z^- are unobservable, with the exception that informed traders observe v, and hence know Y. Informed and liquidity traders cannot post limit orders on the exchange; they can trade only with the exchange market maker or by sending orders to the dark pool.

Figure 3 illustrates the sequence of actions in the two-period model.

3.2 Equilibrium

An equilibrium consists of the quoting strategy of the exchange market maker, the market participation strategies of for-profit traders, and the trading strategies of informed and liquidity traders. In equilibrium, the competitive market maker breaks even in expectation and all traders maximize their expected net profits.

Specifically, I let α_e and α_d be candidates for the equilibrium fractions of liquidity

traders who, in period 1, send orders to the exchange and to the dark pool, respectively. The remainder, $\alpha_0 = 1 - \alpha_e - \alpha_d$, choose not to submit orders in period 1 and delay trade to period 2. We let β be the period-1 fraction of informed traders who send orders to the dark pool. The remaining fraction $1 - \beta$ of informed traders trade on the exchange. (Obviously, informed traders never delay their trades as they will have lost their informational advantage by period 2.) Once the asset value v is revealed in period 2, all traders who have not traded in period 1—including those who deferred trading and those who failed to execute their orders in the dark pool—trade with the market maker at the unique period-2 equilibrium price of v.

I first derive the equilibrium exchange bid and ask, assuming equilibrium participation fractions $(\beta, \alpha_d, \alpha_e)$. Because of symmetry and the fact that the unconditional mean of v is zero, the midpoint of the market maker's bid and ask is zero. Therefore, the exchange ask is some $S > 0$, and the exchange bid is $-S$, where S is the exchange's effective spread, the absolute difference between the exchange transaction price and the midpoint. For simplicity, I refer to S as the "exchange spread." As in Glosten and Milgrom (1985), the exchange bid and ask are set before exchange orders arrive. Given the participation fractions $(\beta, \alpha_d, \alpha_e)$, the mass of informed traders on the exchange is $(1 - \beta)\mu_I$, and the expected mass of liquidity traders on the exchange is $\alpha_e \mathbb{E}(Z^+ + Z^-) = \alpha_e \mu_z$. Because the market maker breaks even in expectation, we have that

$$0 = -(1 - \beta)\mu_I(\sigma - S) + \alpha_e \mu_z S, \tag{4}$$

which implies that

$$S = \frac{(1 - \beta)\mu_I}{(1 - \beta)\mu_I + \alpha_e \mu_z}\, \sigma. \tag{5}$$

The dark pool crosses orders at the mid-market price of zero.

Next, I derive the equilibrium mass μ_I of informed traders. Given the value σ of information and the exchange spread S, the net profit of an informed trader is $\sigma - S$. The information-acquisition cost of the marginal for-profit trader, who is indifferent between paying for information or not, is also $\sigma - S$. Because all for-profit traders with lower information-acquisition costs become informed, the mass of informed traders in equilibrium is $\bar{\mu} F(\sigma - S)$, by the exact law of large numbers (Sun, 2006). We thus have

$$\mu_I = \bar{\mu} F\left(\sigma - S\right) = \bar{\mu} F\left(\frac{\alpha_e \mu_z}{(1 - \beta)\mu_I + \alpha_e \mu_z}\, \sigma\right). \tag{6}$$

For any fixed $\beta \geq 0$ and $\alpha_e > 0$, (6) has a unique solution $\mu_I \in (0, \bar{\mu})$.

Finally, I turn to the equilibrium trading strategies. Without loss of generality, I

focus on the strategies of buyers. In the main solution step, I calculate the expected payoffs of an informed buyer and a liquidity buyer, on the exchange and in the dark pool. The equilibrium is then naturally determined by conditions characterizing marginal traders who are indifferent between trading on the exchange and in the dark pool.

Suppose that $\alpha_d > 0$. Because informed buyers trade in the same direction, they have the dark-pool crossing probability of

$$r^- = \mathbb{E}\left[\min\left(1, \frac{\alpha_d Z^-}{\alpha_d Z^+ + \beta\mu_I}\right)\right], \tag{7}$$

where the denominator and the numerator in the fraction above are the masses of buyers and sellers in the dark pool, respectively. Liquidity buyers, on the other hand, do not observe v. If informed traders are buyers, then liquidity buyers have the crossing probability r^- in the dark pool. If, however, informed traders are sellers, then liquidity buyers have the crossing probability

$$r^+ = \mathbb{E}\left[\min\left(1, \frac{\alpha_d Z^- + \beta\mu_I}{\alpha_d Z^+}\right)\right]. \tag{8}$$

Obviously, for all $\beta > 0$, we have

$$1 > r^+ > r^- > 0. \tag{9}$$

Because liquidity buyers assign equal probabilities to the two events $\{v = +\sigma\}$ and $\{v = -\sigma\}$, their dark-pool crossing probability $(r^+ + r^-)/2$ is greater than informed traders' crossing probability r^-. In other words, correlated informed orders have a lower execution probability in the dark pool than relatively uncorrelated liquidity orders.

If the dark pool contains only liquidity orders (that is, $\beta = 0$), then any dark-pool buy order has the execution probability

$$\bar{r} = \mathbb{E}\left[\min\left(1, \frac{Z^-}{Z^+}\right)\right]. \tag{10}$$

For our purposes, $\bar{r}$ measures the degree to which liquidity orders are balanced. Perfectly balanced liquidity orders correspond to $\bar{r} = 1$. For $\alpha_d = 0$, I define $r^+ = r^- = 0$.

The expected profits of an informed buyer on the exchange and in the dark pool are, respectively,

$$W_e = \sigma - S, \tag{11}$$

$$W_d = r^-\sigma. \tag{12}$$

I denote by c the delay cost of a generic liquidity buyer. This buyer's net payoffs of deferring trade, trading on the exchange, and trading in the dark pool are, respectively,

$$X_0(c) = -c, \tag{13}$$

$$X_e = -S, \tag{14}$$

$$X_d(c) = -\frac{r^+ - r^-}{2}\,\sigma - c\left(1 - \frac{r^+ + r^-}{2}\right). \tag{15}$$

The terms on the right-hand side of (15) are the liquidity trader's adverse selection cost and delay cost in the dark pool, respectively. For $\beta > 0$, crossing in the dark pool implies a positive adverse selection cost, because execution is more likely if a liquidity trader is on the side of the market opposite to that of informed traders. For $\beta = 0$, this adverse-selection cost is zero.

From (11) and (14), $W_e - X_e = \sigma$. For all delay cost $c \leq \sigma$,

$$W_d - X_d(c) = \frac{r^+ + r^-}{2}\,\sigma + c\left(1 - \frac{r^+ + r^-}{2}\right) \leq \sigma = W_e - X_e. \tag{16}$$

That is, provided $c \leq \sigma$, the dark pool is more attractive to liquidity traders than to informed traders, relative to the exchange. In particular, (16) implies that a liquidity trader with a delay cost of σ (or a funding cost of $\gamma = 1$) behaves in the same way as an informed trader. In addition,

$$X_d(c) - X_0(c) = -\frac{r^+ - r^-}{2}\,\sigma + \frac{r^+ + r^-}{2}\,c. \tag{17}$$

So a liquidity trader with a funding cost of $\gamma = (r^+ - r^-)/(r^+ + r^-)$ is indifferent between deferring trade and trading in the dark pool.

To simplify the description of the equilibria, I further define $\hat{\mu}_I : [0, \infty) \to [0, \bar{\mu}]$ by

$$\hat{\mu}_I(s) = \bar{\mu} F\left(\frac{(1 - G(1))\mu_z}{\hat{\mu}_I(s) + (1 - G(1))\mu_z}\,s\right). \tag{18}$$

Given the value σ of information, $\hat{\mu}_I(\sigma)$ is the unique "knife-edge" mass of informed traders with the property that all informed traders and a fraction $1 - G(1)$ of liquidity traders send orders to the exchange.

Proposition 1. *The following results hold.*

1. *If*

$$\bar{r} \leq 1 - \frac{\hat{\mu}_I(\sigma)}{\hat{\mu}_I(\sigma) + (1 - G(1))\mu_z}, \tag{19}$$

then there exists an equilibrium $(\beta = 0, \alpha_d = \alpha_d^*, \alpha_e = 1-\alpha_d^*)$*, where* $\alpha_d^* \in (0, G(1)]$ *and* μ_I^* *solve*

$$G^{-1}(\alpha_d)(1-\bar{r}) = \frac{\mu_I}{\mu_I + (1-\alpha_d)\mu_z}, \tag{20}$$

$$\mu_I = \bar{\mu} F\left(\frac{(1-\alpha_d)\mu_z}{\mu_I + (1-\alpha_d)\mu_z}\,\sigma\right). \tag{21}$$

2. *If and only if*

$$\bar{r} > 1 - \frac{\hat{\mu}_I(\sigma)}{\hat{\mu}_I(\sigma) + (1-G(1))\mu_z}, \tag{22}$$

there exists an equilibrium $(\beta = \beta^*, \alpha_d = \alpha_d^*, \alpha_e = 1 - G(1))$*, where* β^**,* $\alpha_d^* \in (0, G(1)]$*, and* μ_I^* *solve*

$$r^- = 1 - \frac{(1-\beta)\mu_I}{(1-\beta)\mu_I + (1-G(1))\mu_z}, \tag{23}$$

$$\alpha_d = G(1) - G\left(\frac{r^+ - r^-}{r^+ + r^-}\right), \tag{24}$$

$$\mu_I = \bar{\mu} F\left(\frac{(1-G(1))\mu_z}{(1-\beta)\mu_I + (1-G(1))\mu_z}\,\sigma\right). \tag{25}$$

The proof of Proposition 1 is provided in Appendix B, but we outline its main intuition here. On one hand, for informed traders to avoid the dark pool ($\beta = 0$), the dark-pool execution probability $\bar{r}$ must be lower than the profit of informed traders on the exchange, as implied by (19). In this case, the marginal liquidity trader is indifferent between trading on the exchange and trading in the dark pool. The marginal for-profit trader is indifferent about whether to acquire the information.

On the other hand, for informed traders to participate in the dark pool, the maximum dark-pool execution probability $\bar{r}$ must be sufficiently high, as shown in (22). In this case, the equilibrium is determined by three indifference conditions. First, informed traders must be indifferent between trading in either venue, captured by (23). (If they all trade in the dark pool, then the exchange spread becomes 0.) By (16), a liquidity trader with a delay cost of σ is also indifferent between the two venues. Thus $\alpha_0 + \alpha_d = G(1)$ and $\alpha_e = 1 - G(1)$. The second indifference condition (24) then follows from (17). Here, the fraction α_0 of liquidity traders who delay trade must be strictly positive, because informed traders introduce adverse selection into the dark pool. The third condition (25) says that the marginal for-profit trader is indifferent about whether to acquire the information.

Similarly, we can characterize an equilibrium for a market structure in which only the exchange is operating and the dark pool is absent. This exchange-only equilibrium,

solved below, may also be interpreted as one in which a dark pool is open but no trader uses it.

Corollary 1. *With only an exchange and no dark pool, there exists an equilibrium in which* $\beta^* = \alpha_d^* = 0$*, and* μ_I^* *and* $\alpha_e^* \in (1 - G(1), 1)$ *solve*

$$\frac{\mu_I}{\mu_I + \alpha_e \mu_z} = G^{-1}(1 - \alpha_e) \tag{26}$$

$$\mu_I = \bar{\mu} F\left(\frac{\alpha_e \mu_z}{\mu_I + \alpha_e \mu_z}\sigma\right). \tag{27}$$

Equilibrium selection

The equilibria characterized in Proposition 1 need not be unique among all equilibria solving (20)-(21) and (23)-(25). For example, under the condition (19), both sides of (20) strictly increase in α_d. Similarly, both sides of (24) strictly increase in α_d, and both sides of (26) strictly decrease in α_e. Thus, given the absence of a single-crossing property, multiple equilibria may arise.[15]

I use stability as an equilibrium selection criterion, which allows me to compute the comparative statics of the selected equilibria. Among the equilibria characterized by Case 1 of Proposition 1, I select that with the smallest liquidity participation α_d^* in the dark pool among those with the property that, as α_d varies in the neighborhood of α_d^*, the left-hand side of (20) crosses the right-hand side from below. Under the conditions of Proposition 1, this equilibrium exists and is robust to small perturbations. If, for example, α_d^* is perturbed to $\alpha_d^* + \epsilon$ for sufficiently small $\epsilon > 0$, then the marginal liquidity trader has a higher cost in the dark pool than on the exchange, and therefore migrates out of the dark pool. Thus α_d is "pushed back" to α_d^* and the equilibrium is restored. There is a symmetric argument for a small downward perturbation to $\alpha_d^* - \epsilon$. By contrast, if there is an equilibrium in which, as α_d varies, the left-hand side of (20) crosses the right-hand side from above, this equilibrium would not be stable to local perturbations. Moreover, once α_d is determined in equilibrium, μ_I and β are uniquely determined, too, as shown in the proof of Proposition 1.

Similarly, among equilibria characterized by Case 2 of Proposition 1, I select the one with the smallest liquidity participation α_d^* in the dark pool among those with the property that, as α_d varies in the neighborhood of α_d^*, the left-hand side of (24) crosses the right-hand side from below. In a market without a dark pool (Corollary 1), I select the equilibrium with the largest liquidity participation α_e^* on the exchange among those

[15] One special condition that guarantees the uniqueness of the equilibrium in Case 1 of Proposition 1 is that the distribution function G of delay costs is linear. With a linear G, the condition (19) is also necessary for the existence of solutions to (20)-(21).

with the property that, as α_e varies in the neighborhood of α_e^*, the left-hand side of (26) crosses the right-hand side from below. By the argument given for Case 1 of Proposition 1, these selected equilibria exist and are stable.

3.3 Market characteristics and comparative statics

I now investigate properties of the equilibria characterized by Proposition 1. I aim to answer two questions:

1. In a market with a dark pool and an exchange, how do market characteristics vary with the value σ of private information?
2. Given a fixed value σ of private information, how does adding a dark pool affect market behavior?

Key to these questions are the equilibrium participation rates $(\beta, \alpha_d, \alpha_e)$ of informed and liquidity traders. The market characteristics that I analyze are exchange spread, non-execution probability, price discovery, trading volume, and the payoffs of each groups of traders. As we will see, these are closely related to each other. For example, better price discovery is naturally associated with a wider bid-ask spread, which in turn is associated with lower profits for informed traders and a lower total trading volume. The results presented below may also help interpret recent empirical evidence on dark pools and market fragmentation, as we discuss in Section 6.

By (18), increasing the value σ of information raises the knife-edge mass $\hat{\mu}(\sigma)$ of informed traders, which in turn tightens the condition (19) under which informed traders avoid the dark pool. Thus, there exists some unique volatility threshold $\bar{\sigma}$ at which (19) holds with an equality. For $\sigma \leq \bar{\sigma}$, I analyze the equilibrium in Case 1 of Proposition 1. For $\sigma > \bar{\sigma}$, I analyze the equilibrium in Case 2.

Proposition 2. *In the equilibrium of Proposition 1:*

1. *For $\sigma \leq \bar{\sigma}$, the dark-pool participation rate α_d of liquidity traders, the total mass μ_I of informed traders, and the scaled exchange spread S/σ are strictly increasing in σ. The exchange participation rate $\alpha_e = 1 - \alpha_d$ of liquidity traders is strictly decreasing in σ. Moreover, α_d, μ_I, and S are continuous and differentiable in σ.*
2. *For $\sigma > \bar{\sigma}$, all of μ_I, $\beta\mu_I$, r^+, and S/σ are strictly increasing in σ, whereas α_d and r^- are strictly decreasing in σ. Moreover, β, α_d, μ_I, S, r^+, and r^- are continuous and differentiable in σ.*

In the equilibrium of Corollary 1, μ_I and S/σ are strictly increasing in σ, whereas α_e is strictly decreasing in σ. Moreover, α_e, μ_I, and S are continuous and differentiable in σ.

Proof. See Appendix B. □

We can also compare the two equilibria of Proposition 1 and Corollary 1.

Proposition 3. *In the equilibria of Proposition 1 and Corollary 1:*

1. *For $\sigma \leq \bar{\sigma}$, adding a dark pool strictly reduces the exchange participation rate α_e of liquidity traders and the total mass μ_I of informed traders. Adding a dark pool strictly increases the exchange spread S and the total participation rate $\alpha_e + \alpha_d$ of liquidity traders in either venue.*

2. *For $\sigma > \bar{\sigma}$, adding a dark pool strictly reduces α_e. Moreover, adding a dark pool strictly increases the exchange spread S if and only if, in the equilibrium of Proposition 1,*

$$r^- < 1 - \frac{\mu_I}{\mu_I + (1 - G(1 - r^-))\mu_z}. \tag{28}$$

It is sufficient (but not necessary) for (28) that either

$$G''(\gamma) \leq 0 \text{ for all } 1 - \bar{r} \leq \gamma \leq 1 \text{ and } F(c) \to 1 \text{ for all } c > 0, \tag{29}$$

or

$$\bar{r} < 1 - \frac{\bar{\mu}}{\bar{\mu} + (1 - G(1))\mu_z}. \tag{30}$$

Proof. See Appendix B. □

We now discuss the intuition and implications of Proposition 2 and Proposition 3 through numerical examples.

3.3.1 Participation, spread, and non-execution risk

Figure 4 shows the equilibrium participation rates in the two venues. The top-left plot illustrates the participation rates in the dark pool. For a small value of information, specifically if $\sigma \leq \bar{\sigma}$, informed traders trade exclusively on the exchange because the exchange spread is smaller than the cost of non-execution risk in the dark pool. An increase in σ widens the exchange spread, encouraging more liquidity traders to migrate into the dark pool. For $\sigma > \bar{\sigma}$, informed traders use both venues. We observe that informed dark-pool participation rate β first increases in volatility σ and then decreases. The intuition for this non-monotonicity is as follows. Consider an increase in the value of information from σ to $\sigma + \epsilon$, for some $\epsilon > 0$. This higher value of information attracts additional informed traders. For a low β, the dark-pool execution risk stays relatively low, and these additional informed traders prefer to trade in the dark

pool, raising β. For sufficiently high β, however, informed orders cluster on one side of the dark pool and significantly reduce their execution probability. Thus these additional informed traders send orders to the exchange, reducing β. Nonetheless, the total quantity $\beta\mu_I$ of informed traders in the dark pool is strictly increasing in σ. Finally, because informed participation in the dark pool introduces adverse selection, liquidity traders with low delay costs migrate out of the dark pool, leading to a decline in their dark-pool participation rate α_d.

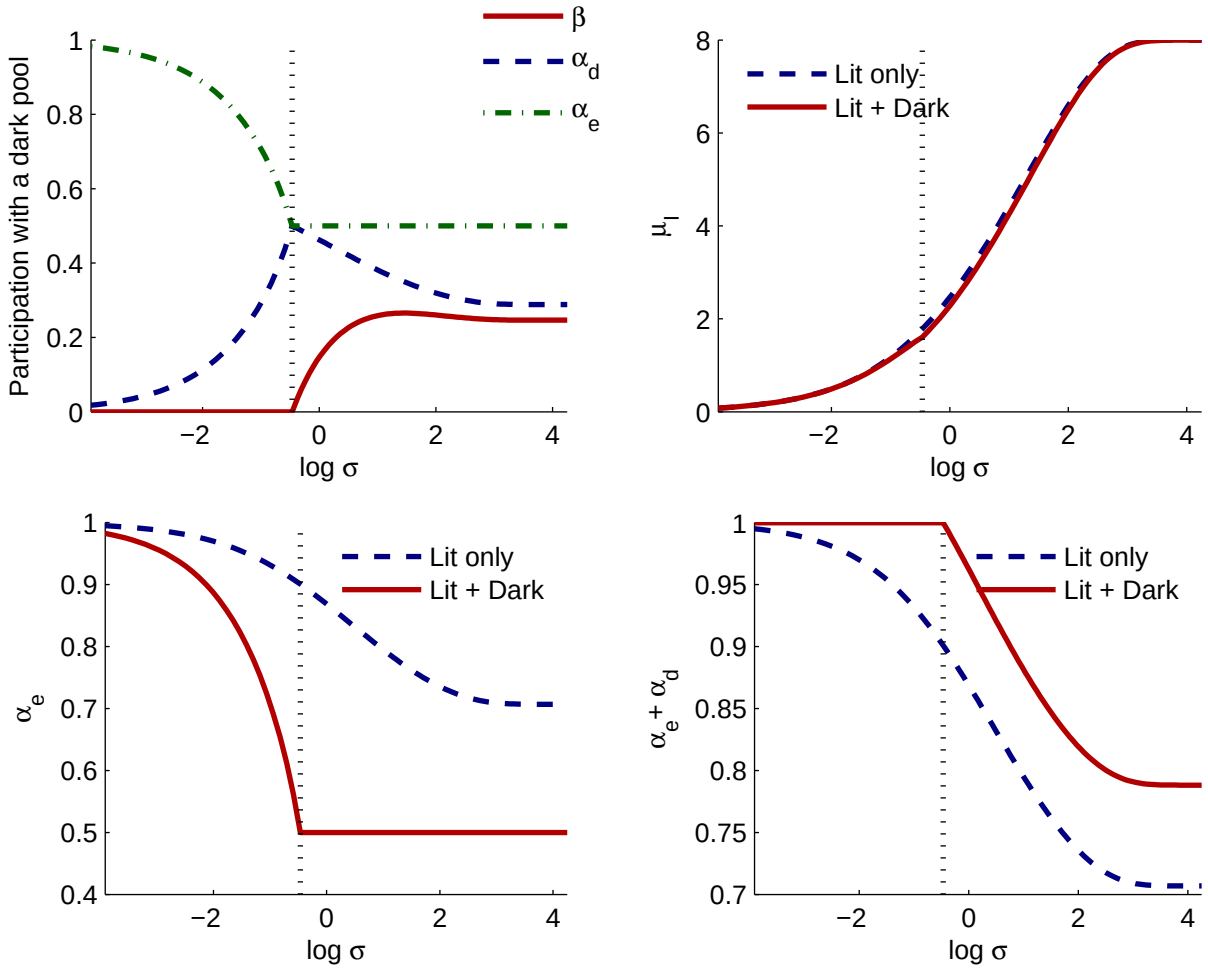
The remaining three plots of Figure 4 show the comparative statics of μ_I, α_e, and $\alpha_e + \alpha_d$ with respect to σ. For $\sigma \leq \bar{\sigma}$, adding a dark pool reduces the exchange participation rate α_e of liquidity traders, shown in the bottom-left plot, but increases their total participation rate $\alpha_e + \alpha_d$ to the highest level, 1, shown in the bottom-right plot. Intuitively, the dark pool offers a lower-cost trading service for those liquidity traders who consider the exchange spread "too expensive." These "latent" liquidity traders—traders who would never pay the exchange spread anyway—send their orders to the dark pool. With this "seed liquidity," the dark pool diverts additional liquidity traders off the exchange, widening the exchange spread. Consequently, a smaller mass μ_I of for-profit traders become informed, as shown in the top-right plot. For $\sigma \geq \bar{\sigma}$, α_e stays at a constant, $1 - G(1)$, which is smaller than the exchange participation rate of liquidity traders in a market without a dark pool. In this example, adding a dark pool widens the scaled exchange spread S/σ and reduces the informed trading interest μ_I.

Figure 5 plots the scaled exchange spread S/σ (shown on the left) and the dark-pool execution probabilities (shown on the right). Because a higher value σ of information encourages more for-profit traders to become informed, the scaled exchange spread S/σ increases in σ, whether a dark pool is present or not. For $\sigma \leq \bar{\sigma}$, adding a dark pool raises S/σ by diverting some liquidity traders, but none of the informed traders, off the exchange. Non-execution risk is unaffected. For $\sigma > \bar{\sigma}$, adding a dark pool in this example also increases the scaled spread S/σ because the dark pool diverts more liquidity traders than informed traders. The dark-pool execution probability $(r^+ + r^-)/2$ for liquidity traders is not monotone in σ. This non-monotonicity is natural because the crossing probabilities r^+ and r^- are non-linear in β, μ_I, and α_d.

3.3.2 Price discovery

Having analyzed participation rates, spread, and non-execution risk, I turn to price discovery, the extent to which the fundamental asset value can be inferred from publicly observable information, such as quoted prices and volume. In this model, the only public information is the period-1 net exchange order flow, $(1 - \beta)Y + \alpha_e Z$. Inference

Figure 4: Participation on the exchange and in the dark pool. The top left plot shows the equilibrium participation rates $(\beta, \alpha_d, \alpha_e)$ with a dark pool. The top right plot shows the equilibrium mass μ_I of informed traders, with or without a dark pool. The bottom left plot shows the equilibrium participation of liquidity traders on the exchange, with or without a dark pool. The bottom right plot shows the equilibrium participation of liquidity traders in either venue, with or without a dark pool. In all plots, the vertical dotted line indicates the threshold volatility $\bar{\sigma}$ at which the equilibrium of Proposition 1 changes from Case 1 to Case 2. Model parameters: $Z_0 = 2$, Z is a normal random variable with mean 0 and standard deviation $\sigma_z = 5$, $\bar{\mu} = 8$, $G(x) = x/2$ for $x \in [0, 2]$, and $F(x) = 1 - e^{-x/2}$ for $x \in [0, \infty)$.

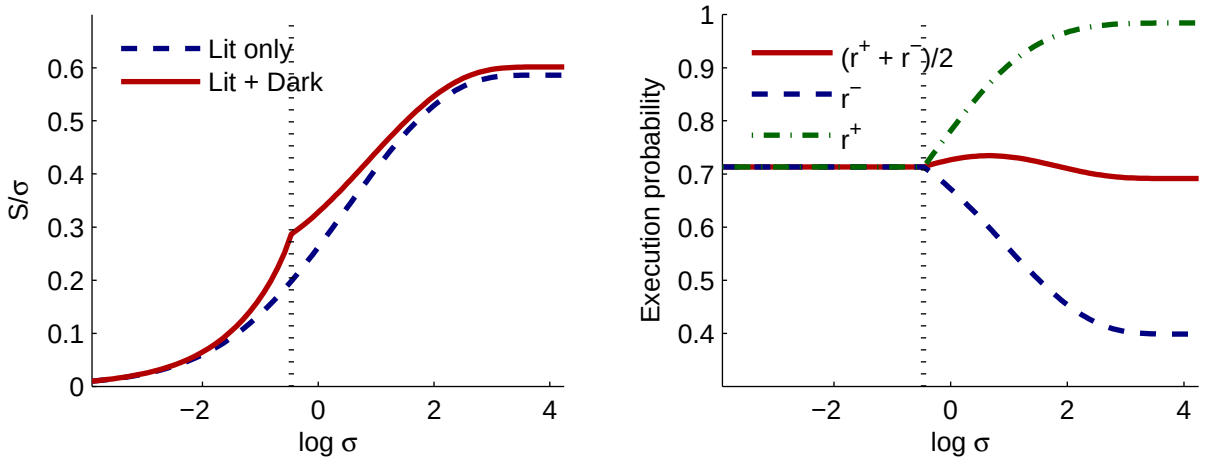


of the asset value v is captured by the likelihood ratio

$$R_1 = \log \frac{\mathbb{P}(v = +\sigma \,|\, (1-\beta)Y + \alpha_e Z)}{\mathbb{P}(v = -\sigma \,|\, (1-\beta)Y + \alpha_e Z)} = \log \frac{f_Z\left[\frac{1}{\alpha_e}((1-\beta)Y + \alpha_e Z - (1-\beta)\mu_I)\right]}{f_Z\left[\frac{1}{\alpha_e}((1-\beta)Y + \alpha_e Z + (1-\beta)\mu_I)\right]}, \tag{31}$$

where f_Z is the probability density function of the liquidity order imbalance Z. Without loss of generality, I condition on $v = +\sigma$. I consider price discovery to be unambiguously

Figure 5: Spread on the exchange and non-execution probability in the dark pool. The left plot shows the scaled exchange spread S/σ. The right plot shows the execution probabilities in the dark pool. As in Figure 4, the vertical dotted line indicates the threshold volatility $\bar{\sigma}$. Model parameters are those of Figure 4.



"better" if the probability distribution of R_1 is "higher" in the sense of first-order stochastic dominance. Complete revelation of $v = +\sigma$ corresponds to $R_1 = \infty$ almost surely.

Because the probability distribution of R_1 depends on the functional form of f_Z, for illustration I let the liquidity order imbalance Z be normally distributed, with mean zero and standard deviation σ_z. Normality of Z implies that

$$R_1 = \frac{2(1-\beta)\mu_I}{\alpha_e^2\sigma_z^2}[(1-\beta)Y + \alpha_e Z]. \tag{32}$$

Conditional on $Y = \mu_I$, R_1 is also normally distributed. Specially,

$$R_1 \sim N\left(2\left(\frac{(1-\beta)\mu_I}{\alpha_e\sigma_z}\right)^2, 4\left(\frac{(1-\beta)\mu_I}{\alpha_e\sigma_z}\right)^2\right) \sim N\left(2I(\beta,\alpha_e)^2, 4I(\beta,\alpha_e)^2\right), \tag{33}$$

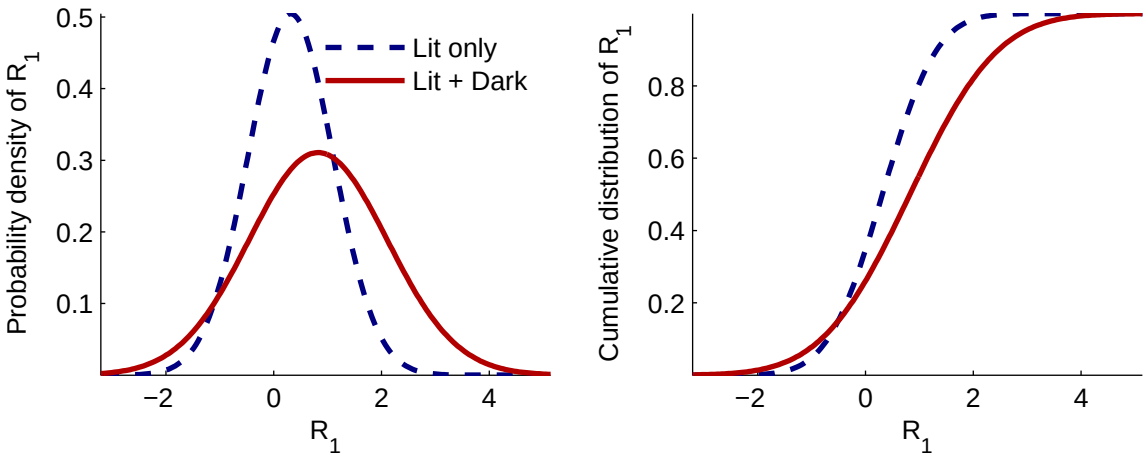
where

$$I(\beta,\alpha_e) \equiv \frac{(1-\beta)\mu_I}{\alpha_e\sigma_z} \tag{34}$$

is the "signal-to-noise" ratio, which is the mass of informed orders on the exchange ("signal") divided by the standard deviation of the imbalance of liquidity orders on the exchange ("noise"). Naturally, $I(\beta,\alpha_e)$ is increasing in the scaled exchange spread S/σ. A higher exchange spread corresponds not only to a higher mean of the log likelihood ratio R_1 but also to a higher variance.

Figure 6 plots the distribution function of R_1 with or without a dark pool. The

Figure 6: Distribution functions of R_1, with or without a dark pool. The true dividend is the threshold value $+\bar{\sigma}$ and other parameters are those of Figure 4.



value σ of information is set to be the threshold value $\bar{\sigma}$, so that $\beta = 0$ in the equilibria with a dark pool as well as the equilibria without a dark pool. By Proposition 3, adding a dark pool strictly increases the scaled spread S/σ and hence the signal-to-noise ratio $I(\beta, \alpha_e)$. With a dark pool, the conditional distribution of R_1 has a higher mean, but also a higher variance. For most realizations of R_1, and on average, adding a dark pool decreases the cumulative distribution of R_1 and leads to more precise inference of v. Nonetheless, for very low realizations of R_1, adding a dark pool increases the cumulative distribution of R_1, making extreme "false signals" more likely.

3.3.3 Volume and dark pool market share

I now calculate the trading volume handled by the exchange and the dark pool. The trading volume and the market share of the dark pool are direct empirical measures of dark-pool activity. I assume that once the dividend v is announced in period 2, informed traders who have not yet traded leave the market, because they will not be able to trade profitably. When calculating the exchange volume, I also include the transactions of liquidity traders in period 2. Thus the expected trading volumes in the dark pool, on the exchange, and in both venues are, respectively,

$$V_d = \beta\mu_I r^- + \alpha_d\mu_z\frac{r^+ + r^-}{2}, \tag{35}$$

$$V_e = (1-\beta)\mu_I + \alpha_e\mu_z + \alpha_d\mu_z\left(1 - \frac{r^+ + r^-}{2}\right) + \alpha_0\mu_z, \tag{36}$$

$$V = V_e + V_d = \mu_z + \mu_I(1 - \beta + \beta r^-). \tag{37}$$

By Proposition 2, these volumes are differentiable in the volatility σ in each of the two intervals $[0, \bar{\sigma}]$ and $(\bar{\sigma}, \infty)$.

For $\sigma \leq \bar{\sigma}$, the dark-pool volume, $V_d = \alpha_d \mu_z \bar{r}$, is increasing in the volatility σ, by Proposition 2. In particular, as $\sigma \to 0$, the dark-pool participation rate α_d of liquidity traders and the dark-pool market share V_d/V converge to zero. For a sufficiently small $\sigma < \bar{\sigma}$, therefore,

$$\frac{d(V_d/V)}{d\sigma} = \frac{d}{d\sigma}\left(\frac{\alpha_d \mu_z \bar{r}}{\mu_z + \mu_I}\right) = \frac{\mu_z \bar{r}}{\mu_z + \mu_I} \cdot \frac{d\alpha_d}{d\sigma} - \frac{\alpha_d \mu_z \bar{r}}{(\mu_z + \mu_I)^2} \cdot \frac{d\mu_I}{d\sigma} > 0,$$

where the inequality follows from the fact that $\lim_{\sigma \to 0} d\alpha_d/d\sigma > 0$ (shown in the proof of Proposition 2). That is, if volatility σ is low, the dark-pool market share V_d/V is also low, and is increasing in σ. For $\sigma \leq \bar{\sigma}$, because the total volume $V = \mu_z + \mu_I$ is increasing in σ, the dark-pool market share is increasing in the total volume. These results are illustrated in Figure 7.

Figure 7 further suggests that, as the volatility σ increases beyond $\bar{\sigma}$, the exchange volume V_e can increase substantially, but the dark-pool volume V_d may only increase mildly or even decline. Thus the dark-pool market share can decrease in volatility σ for sufficiently large σ, creating a hump-shaped relation between volatility and the dark-pool market share. The model also generates a similar relation between the total trading volume V and the dark-pool market share V_d/V, as shown on the right-hand plot of Figure 7. Although these hump-shaped patterns are not proved analytically here, they are consistent with the empirical findings of Ready (2010) and Ray (2010), and may help explain the mixed evidence regarding the relations between dark-pool market shares and market characteristics such as volatility, volume, and spread. I further discuss these questions in Section 6.

3.3.4 Informed traders' profits and liquidity traders' costs

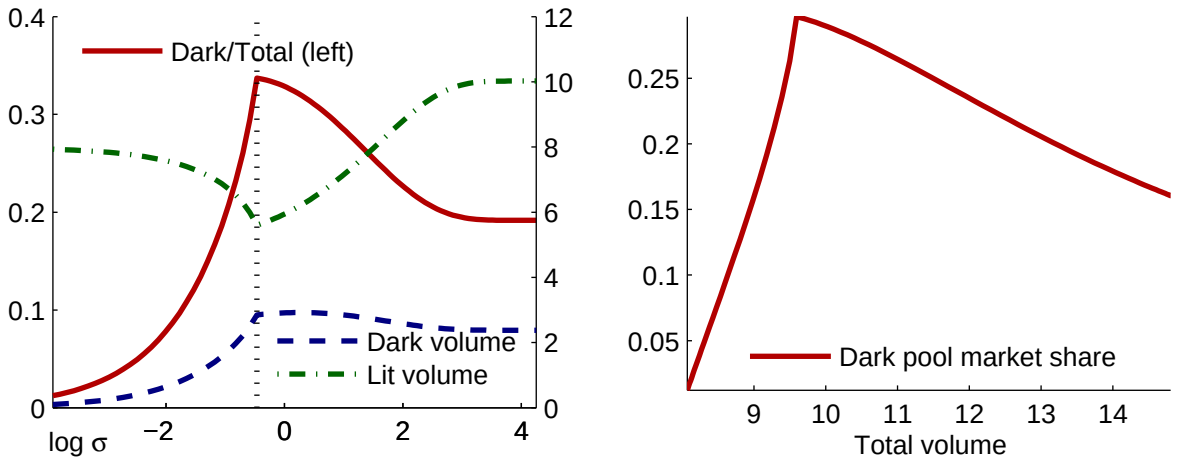
Finally, I calculate the expected net profits U_I of informed traders and the net expected costs $-U_L$ of liquidity traders. These calculations help us understand whether adding a dark pool benefits or harms these two groups of traders, at least in this model setting.

In equilibrium, the expected cost C_i of information acquisition and expected cost C_w of delay are, respectively,

$$C_i = \bar{\mu} \int_{x=0}^{\mu_I/\bar{\mu}} F^{-1}(x)\, dx, \tag{38}$$

$$C_w = \mu_z \sigma \left[\int_{x=0}^{\alpha_0} G^{-1}(x)\, dx + \left(1 - \frac{r^+ + r^-}{2}\right) \int_{x=\alpha_0}^{\alpha_0+\alpha_d} G^{-1}(x)\, dx \right]. \tag{39}$$

Figure 7: Expected trading volume on the exchange and in the dark pool. The left plot shows the volume in the two venues and the market share of the dark pool. The right plot shows the dark-pool market share as a function of the total volume. The vertical dotted line corresponds to the threshold volatility $\bar{\sigma}$. Parameters are those of Figure 4.



The net profit U_I and net cost $-U_L$ are, respectively,

$$U_I = \mu_I(\sigma - S) - C_i, \tag{40}$$

$$-U_L = \mu_z \alpha_e S + \mu_z \alpha_d \frac{r^+ - r^-}{2} \sigma + C_w. \tag{41}$$

The profit U_I and the cost $-U_L$ are differentiable in the volatility σ in each of the two intervals $[0, \bar{\sigma}]$ and $(\bar{\sigma}, \infty)$.

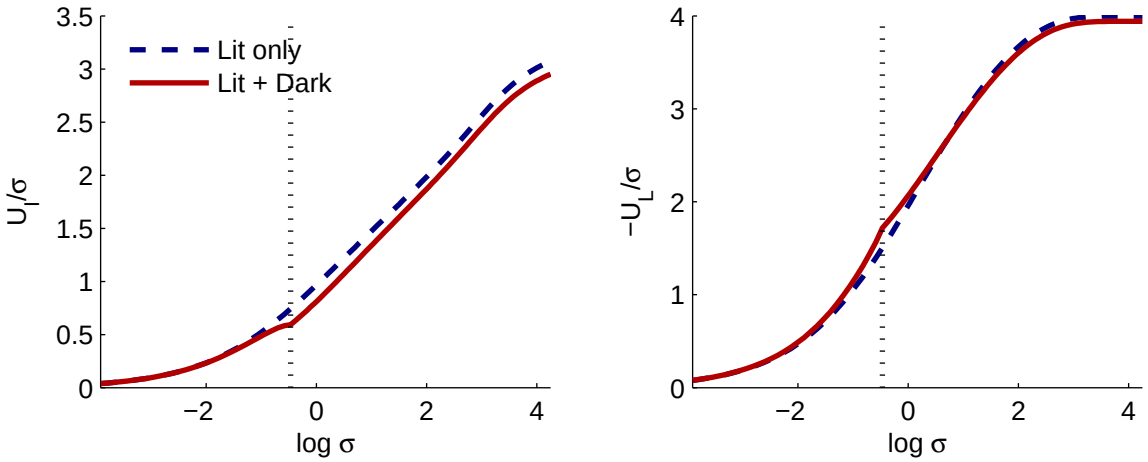
Figure 8 plots the comparative statics of U_I/σ and $-U_L/\sigma$ with respect to the value σ of information, with and without a dark pool. Naturally, a higher value of information implies higher profits for informed traders. For $\sigma \leq \bar{\sigma}$, for example,

$$\begin{aligned} \frac{d(U_I/\sigma)}{d\sigma} &= \left[\left(1 - \frac{S}{\sigma}\right) - \frac{1}{\sigma} F^{-1}\left(\frac{\mu_I}{\bar{\mu}}\right) - \mu_I \frac{\partial(1 - S/\sigma)}{\partial \mu_I}\right] \frac{d\mu_I}{d\sigma} + \frac{1}{\sigma^2} C_i \\ &= -\mu_I \frac{\partial(1 - S/\sigma)}{\partial \mu_I} \frac{d\mu_I}{d\sigma} + \frac{1}{\sigma^2} C_i > 0. \end{aligned}$$

In the above calculation, the second equality follows from (21), and the inequality follows from the facts that $d\mu_I/d\sigma > 0$ and that $\partial(1 - S/\sigma)/\partial\mu_I < 0$. The calculation of $d(U_I/\sigma)/d\sigma$ for $\sigma > \bar{\sigma}$ is similar. In this example, adding a dark pool widens the exchange spread and reduces the profits of informed traders.

Figure 8 suggests that adding a dark pool has a mixed impact on the expected costs of liquidity traders. For a sufficiently large value σ of information, adverse selection is so severe that the dark pool can reduce the expected transaction costs of liquidity traders

Figure 8: The expected profit U_I of informed traders (shown on the left) and the expected cost $-U_L$ of liquidity traders (shown on the right), both scaled by volatility σ. The vertical dotted line corresponds to the threshold volatility $\bar{\sigma}$. Parameters are those of Figure 4.



by potentially saving them the exchange spread. However, for $\sigma \leq \bar{\sigma}$, Proposition 3 implies that $d\alpha_d/d\bar{r} > 0$ and that $d\mu_I/d\bar{r} < 0$. So,

$$\frac{1}{\mu_z}\frac{d(-U_L/\sigma)}{d\bar{r}} = \left(G^{-1}(\alpha_d) - \frac{S}{\sigma}\right)\frac{d\alpha_d}{d\bar{r}} + (1-\alpha_d)\frac{d(S/\sigma)}{d\bar{r}} > 0.$$

That is, adding a dark pool, which is equivalent to an increase in $\bar{r}$, increases the total expected cost for liquidity traders. This seemingly counterintuitive result has a simple explanation. Although the dark pool reduces the trading costs for relatively patient liquidity traders, it raises the trading costs for relatively impatient ones. The net effect is that the latter dominates the former.

The impact of the dark pool on the trading costs for liquidity traders contrasts with the results of Foster, Gervais, and Ramaswamy (2007), who argue that adding a dark pool can provide a Pareto improvement over a market with only an exchange because the dark pool attracts latent liquidity traders. Their dark pool is "volume-conditional"—it executes orders only given a sufficiently large trading interest. Although the dark pool in both models attracts latent liquidity traders, the key difference is that liquidity traders ("hedgers") in their model have binomially distributed delay costs—high or low. Under certain parameter conditions, adding a dark pool in their model has no impact on the exchange spread, because hedgers with low delay costs never trade on the exchange even without a dark pool. In my model, by contrast, liquidity traders have a continuum of delay costs; migration of low-cost liquidity traders to the dark pool diverts additional liquidity traders with slightly higher delay costs, raising the exchange spread. The welfare implication of dark pools for liquidity traders,

therefore, can be complicated and depend on the characteristics of traders.

4 Dark Pools as Non-displayed Limit Order Books

So far we have studied a dark pool that crosses orders at the midpoint of the exchange bid and ask. In this section, I model a dark pool that operates as a non-displayed limit order book, where traders can specify their own limit prices. As described in Section 2, this order-book mechanism is typically used in dark pools operated by major broker-dealers, and transaction prices in these dark pools are generally determined by the submitted limit orders. Aside from confirming the basic intuition of Section 3 in this alternative setting, this section offers addition insights regarding the impact of dark-pool mechanisms on the participation incentives of informed traders.

Compared with midpoint-matching dark pools, dark pools operating as non-displayed limit order books allow more discretion in execution prices. Such price discretion, however, is limited by "best-execution" regulations. In the United States, for example, the Order Protection Rule, also known as the "trade through" rule, stipulates that transaction prices in any market center—including dark pools, ECN, and broker-dealer internalization—cannot be strictly worse than the prevailing national best bid and offer (NBBO).[16] For example, if the current NBBO is 10.00/10.50, then the transaction price in any market center must be in the interval $[10, 10.5]$. The trade-through rule thus imposes a boundary on the execution prices in dark pools.

Going beyond the trade-through rule, regulators have also proposed a stricter "trade-at" rule. Under a trade-at rule, market centers not already quoting the best price cannot execute incoming orders at the best price. Instead, they must either provide incoming orders with sufficiently large price improvements, or route them to venues that quote the best price. For example, the Joint CFTC-SEC Advisory Committee (2011) recommends that the SEC consider "its rule proposal requiring that internalized or preferenced orders only be executed at a price materially superior (e.g. 50 mils [0.5 cent] for most securities) to the quoted best bid or offer." With a trade-at rule, displayed orders that establish the market-wide best price have strictly higher execution priority than non-displayed orders at the same price. Execution prices in dark pools must then be sufficiently better than the best bid or offer on exchanges.

I now describe and solve a simple model of a dark pool that operates as a non-displayed limit order book.[17] The dark pool executes orders by price priority and

[16] In Europe, MiFID uses a decentralized best-execution rule, by which investment firms decide whether an execution works for the best interest of investors.

[17] This model of *non-displayed* limit order book differs from models of *displayed* limit order books. For models of the latter, see the survey of Parlour and Seppi (2008).

is modeled as a double auction. I model the effect of a trade-at rule, by assuming that transaction prices in the dark pool must be within the interval $[-xS, xS]$, where $S > 0$ is the exchange spread and $x \in [0, 1]$ captures the strictness of the trade-at rule. The trade-through rule currently applied in the United States corresponds to $x = 1$, indicating a mandatory price improvement of zero. A midpoint-matching mechanism corresponds to $x = 0$, indicating a price improvement of the entire effective spread S. With the exception of this trade-at rule, the model of this section is identical to that of Section 3. Because of intractability, I can only characterize equilibria in which all informed traders avoid the dark pool. Nonetheless, this equilibrium offers useful insight into how the trade-at rule affects the dark-pool participation of informed traders.

I start by fixing candidate equilibrium participation rates ($\beta = 0, \alpha_d, \alpha_e = 1 - \alpha_d$). I then calculate the execution price and execution probabilities in the dark pool, as well as the optimal limit prices chosen by liquidity traders. Finally, I derive incentive-compatibility conditions under which informed traders choose not to participate in the dark pool.

I let $y^+ : [-xS, xS] \to [0, \infty)$ be the aggregate *downward-sloping* demand schedule of liquidity buyers in the dark pool, and let $y^- : [-xS, xS] \to [0, \infty)$ be the aggregate *upward-sloping* supply schedule of liquidity sellers. For each p, $y^+(p)$ is the mass of liquidity buyers with a limit price of p or higher, and $y^-(p)$ is the mass of liquidity sellers with a limit price of p or lower. Because the dark pool crosses orders by price priority, its execution price p^* is

$$p^* = \begin{cases} xS, & \text{if } y^+(p) > y^-(p) \text{ for all } p \in [-xS, xS]. \\ -xS, & \text{if } y^+(p) < y^-(p) \text{ for all } p \in [-xS, xS]. \\ \{p : y^+(p) = y^-(p)\}, & \text{otherwise.} \end{cases} \tag{42}$$

I proceed under the conjecture that the set $\{p : y^+(p) = y^-(p)\}$ contains at most one element, in which case p^* of (42) is uniquely well-defined. I later verify this conjecture. Once p^* is determined, buy orders with limit prices above or equal to p^* are matched, at the price of p^*, with sell orders whose prices are at most p^*. If there is a positive mass of buy or sell orders at the price p^*, then traders setting the limit price p^* are rationed pro-rata, as before.

I now derive the optimal limit prices of liquidity traders in the dark pool, under the conjecture that the probability distribution of p^* has no atom in $(-xS, xS)$. This no-atom conjecture, verified later, implies that a liquidity trader quoting a price of $p \in (-xS, xS)$ has her order filled with certainty (is not rationed) if $p^* = p$. Thus a liquidity buyer who has a delay cost of $c \in [0, xS)$ and quotes a price of p in the dark

pool has the expected payoff (negative cost)

$$\begin{aligned} X_d(p;c) &= -\mathbb{E}\left[\mathbb{I}_{\{p\geq p^*\}}p^* + \mathbb{I}_{\{p<p^*\}}c\right] \\ &= -c - \int_{-xS}^{p} (p^* - c)\, dH(p^*), \end{aligned} \tag{43}$$

where $\mathbb{I}(\,\cdot\,)$ is the indicator function and $H(p^*)$ is the cumulative distribution function of p^*. Because there is no adverse selection in the dark pool, the execution cost for this liquidity buyer is either the payment p^* or the delay cost c. Conjecturing that $H(p^*)$ is differentiable with $H'(p^*) > 0$ for $p^* \in (-xS, xS)$, properties that are also verified later, we obtain

$$\frac{dX_d(p;c)}{dp} = -(p-c)H'(p). \tag{44}$$

Because (44) shows that the sensitivity of expected payoff to the limit price p is positive for $p < c$ and negative for $p > c$, the optimal limit price for the liquidity buyer is her delay cost c. Symmetrically, the optimal limit price for a liquidity seller with a delay cost of $c \in [0, xS)$ is $-c$. This "truth-telling" strategy is also ex-post optimal, in that no one wishes to deviate even after observing the execution price.[18]

The first-order condition (44) also implies that xS is the highest limit price in the dark pool, and that $-xS$ is the lowest limit price.[19] A liquidity buyer with a delay cost of $c > xS$ may wish to quote a limit price of c, but the trade-at rule forces her to quote xS. This "corner solution" suggests a strictly positive mass of liquidity buyers quoting the highest limit price xS, and a strictly positive mass of liquidity sellers quoting the lowest limit price $-xS$.

Let $y(p)$ be the downward-sloping demand schedule in the dark pool if $Z^+ = 1$. Because a limit price $p \in [0, xS)$ is submitted by the liquidity buyer with the delay cost p,

$$y(p) = \alpha_d - G\left(\frac{\max(0,p)}{\sigma}\right), \quad -xS < p < xS. \tag{45}$$

By symmetry, the liquidity buyers' demand schedule and the liquidity sellers' supply

[18] This strategy is reminiscent of the truth-telling strategy of MacAfee (1992), who considers a double auction with finitely many buyers and sellers. The double auction here has the institutional restriction that transaction prices are bounded by the trade-at rule.

[19] If the maximum limit price were smaller, say $p_0 < xS$, then a liquidity buyer with a delay cost of $p_0 + \epsilon$ for some small $\epsilon > 0$ would deviate to the dark pool and quote $p_0 + \epsilon$. This deviating buyer has an execution probability of 1 and pays at most $p_0 + \epsilon < xS \leq S$, which is better than execution on the exchange. The argument for the lowest limit price is symmetric.

schedule in the dark pool are, respectively,

$$y^+(p) = Z^+ y(p), \tag{46}$$

$$y^-(p) = Z^- y(-p). \tag{47}$$

Because the equation $y^+(p) = y^-(p)$ has at most one root, we have verified our earlier conjecture that the dark-pool execution price p^* is uniquely well-defined.

Given $y(p)$, the execution price p^* in the dark pool is

$$p^* = \begin{cases} +xS, & \text{if } [\alpha_d - G\left(\frac{xS}{\sigma}\right)]Z^+ \geq \alpha_d Z^-, \\ +\sigma G^{-1}\left[\alpha_d\left(1 - \frac{Z^-}{Z^+}\right)\right], & \text{if } \left[\alpha_d - G\left(\frac{xS}{\sigma}\right)\right] Z^+ < \alpha_d Z^- \leq \alpha_d Z^+, \\ -\sigma G^{-1}\left[\alpha_d\left(1 - \frac{Z^+}{Z^-}\right)\right], & \text{if } \left[\alpha_d - G\left(\frac{xS}{\sigma}\right)\right] Z^- < \alpha_d Z^+ \leq \alpha_d Z^-, \\ -xS, & \text{if } [\alpha_d - G\left(\frac{xS}{\sigma}\right)]Z^- \geq \alpha_d Z^+. \end{cases} \tag{48}$$

Because the mass Z^+ of liquidity buyers and the mass Z^- of liquidity sellers are identically distributed, the dark-pool execution price p^* has a mean of zero. By the differentiability of G and of the distribution function of Z^-/Z^+, $H(p^*)$ is continuous, differentiable, and strictly increasing on $(-xS, xS)$, as conjectured earlier.

What remains to be shown are the incentive-compatibility conditions of liquidity traders who set the limit price xS or $-xS$ in the dark pool, as well as the incentive-compatibility condition of informed traders who avoid the dark pool. A liquidity buyer quoting the limit price xS in the dark pool has an execution probability of

$$\bar{r}_x = \mathbb{E}\left[\min\left(1, \frac{\alpha_d Z^-}{(\alpha_d - G(xS/\sigma))Z^+}\right)\right], \tag{49}$$

and an expected payoff, given the delay cost c, of

$$X_d(xS; c) = -(1 - \bar{r}_x)(c - xS). \tag{50}$$

This expected payoff calculation follows from the fact that $\mathbb{E}(p^*) = 0$ and the fact that failing to cross in the dark pool incurs a delay cost of c but saves the payment xS.

Because informed traders avoid the dark pool with probability 1 in the conjectured equilibrium, an informed buyer who *deviates* to the dark pool also has the crossing probability $\bar{r}_x$. Moreover, in order to get the highest priority, this deviating informed trader sets the highest limit price xS. Her expected profit in the dark pool is thus

$$W_d = \sigma - (1 - \bar{r}_x)(\sigma - xS). \tag{51}$$

As before, for any delay cost $c \leq \sigma$,

$$W_d - X_d(xS; c) = \sigma \bar{r}_x + c(1 - \bar{r}_x) \leq \sigma = W_e - X_e. \tag{52}$$

That is, an informed buyer behaves in the same way as a liquidity buyer who has a delay cost of σ. For informed buyers to avoid the dark pool, the marginal liquidity buyer with a delay cost of σ must also weakly prefer the exchange. The argument for sellers is symmetric. Because the marginal liquidity trader indifferent between the two venues has a delay cost of $G^{-1}(\alpha_d)\sigma$, we look for an equilibrium in which $\alpha_d \leq G(1)$. In that equilibrium, $\beta = 0$.

Proposition 4. *In a market with an exchange and a dark pool that implements a double auction, there exists a unique threshold volatility $\bar{\sigma}(x) > 0$ with the property that, for any $\sigma \leq \bar{\sigma}(x)$, there exists an equilibrium $(\beta = 0, \alpha_d = \alpha_d^*, \alpha_e = 1 - \alpha_d^*)$, where $\alpha_d^* \in (0, G(1)]$ and μ_I^* solve*

$$\left[G^{-1}(\alpha_d) - \frac{xS}{\sigma}\right] \cdot (1 - \bar{r}_x) = \frac{\mu_I}{\mu_I + (1 - \alpha_d)\mu_z}, \tag{53}$$

$$\mu_I = \bar{\mu} F\left(\frac{(1 - \alpha_d)\mu_z}{\mu_I + (1 - \alpha_d)\mu_z}\,\sigma\right). \tag{54}$$

In this equilibrium with a fixed x:

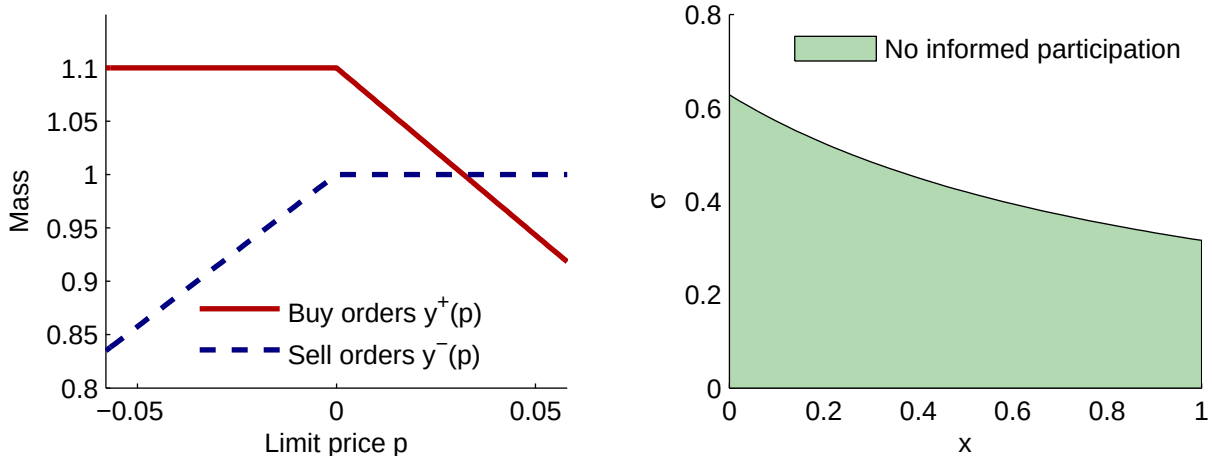
1. *The dark-pool execution price is given by (48).*
2. *If $c \in [0, xS)$, a liquidity buyer (resp. seller) with a delay cost of c quotes a limit price of c (resp. $-c$) in the dark pool. If $c \in [xS, G^{-1}(\alpha_d^*)\sigma]$, then a liquidity buyer (resp. seller) with a delay cost of c quotes a limit price of xS (resp. $-xS$) in the dark pool. Liquidity traders with delay costs higher than $G^{-1}(\alpha_d^*)\sigma$ trade on the exchange.*
3. *The dark-pool participation rate α_d of liquidity traders, the mass μ_I of informed traders, and the scaled exchange spread S/σ are all strictly increasing in the value σ of information.*

Moreover, for $x \in (0, 1)$, the volatility threshold $\bar{\sigma}(x)$ is strictly decreasing in x.

Proof. See Appendix B. □

The equilibrium of Proposition 4 with a double-auction dark pool is qualitatively similar to the equilibrium characterized in Case 1 of Proposition 1. It is determined by the marginal liquidity trader who is indifferent between the two venues, shown in (53), and the marginal for-profit trader who is indifferent about whether to acquire the

Figure 9: A dark pool as a non-displayed limit order book. The left-hand plot shows the limit orders in the dark pool, where $y^+(p)$ and $y^-(p)$ denotes the demand schedule and supply schedule, respectively. The right-hand plot shows the range of σ for which the equilibrium of Proposition 4 exists, that is, informed traders avoid the dark pool. Parameters: $Z_0 = 2$, Z is a normal random variable with mean 0 and standard deviation of 5, $\bar{\mu} = 8$, and $G(\gamma) = \gamma/2$ for $\gamma \in [0, 2]$. The left-hand plot also uses $x = 0.8$, $\sigma = \bar{\sigma}(0.8) = 0.35$, and realizations $Z^+ = 2.2$ and $Z^- = 2$.



information, shown in (54). If multiple equilibria exist, I select the equilibrium with the lowest α_d^* among those with the property that, as α_d varies in a neighborhood of α_d^*, the left-hand side of (53) crosses the right-hand side from below. The expression of $\bar{\sigma}(x)$ in equilibrium is provided in Appendix B.

The left-hand plot of Figure 9 shows the dark-pool orders in the equilibrium of Proposition 4, for $x = 0.8$. That is, the dark pool must provide a price improvement equal to 20% of the exchange spread S. In this example, about 85% of liquidity traders in the dark pool set the most aggressive limit price, $\pm xS$. The dark-pool transaction price in this case is about 0.03. The right-hand plot of Figure 9 shows that the volatility threshold $\bar{\sigma}(x)$ is strictly decreasing in x. With midpoint crossing ($x = 0$), informed traders avoid the dark pool if the value σ of information is lower than about 0.6. Under the current trade-through rule ($x = 1$), this volatility threshold is reduced to about 0.3.

In addition to confirming the basic intuition of Proposition 1, Proposition 4 offers new insights on the effect of a trade-at rule for dark-pool participation of informed traders. Because $\bar{\sigma}(x)$ is decreasing in x, the stricter is the trade-at rule, the less likely are informed traders to use the dark pool. The intuition is as follows. If an informed buyer deviates to the dark pool, she selects the most aggressive permissible limit price, xS, in order to maximize her execution probability. Although she will be rationed at the price xS, she competes only with those liquidity traders who have a delay cost of xS

or higher. The lower is x, the less scope there is for the informed trader to "step ahead of the queue" and gain execution priority. In particular, a midpoint dark pool with $x = 0$ has the greatest effectiveness in discouraging informed traders to participate.

The effect of the trade-at rule on informed participation in dark pools complements prior fairness-motivated arguments, which suggest that displayed orders should have strictly higher priority than non-displayed orders at the same price (Joint CFTC-SEC Advisory Committee, 2011). Proposition 4 predicts that implementing a trade-at rule is likely to reduce informed participation in dark pools. Proposition 4 also predicts that dark pools operating as limit order books are more likely to attract informed traders and impatient liquidity traders, compared to dark pools crossing at the midpoint. I further discuss these implications in Section 6.

5 Dynamic Trading

This short section generalizes the basic intuition of Section 3 to a dynamic market. Under natural conditions, all equilibria have the property that, after controlling for delay costs, an informed trader prefers the exchange to the dark pool, relative to a liquidity trader.

Time is discrete, $t \in \{1, 2, 3, \dots\}$. As before, an asset pays an uncertain dividend v that is $+\sigma$ or $-\sigma$ with equal probabilities. The dividend is announced at the beginning of period $T \geq 2$, where T is deterministic, and paid at the end of period T. The trading game ends immediately after the dividend payment.

In each period before the dividend payment, a new set of informed traders and liquidity traders arrive. To simplify the analysis, I drop endogenous information acquisition in this section. (Equivalently, it costs zero to acquire information.) The mass of informed traders arriving in period t, $\mu_I(t) > 0$, is deterministic. Informed traders observe the dividend v and trade in the corresponding direction. The mass of liquidity buyers and the mass of liquidity sellers arriving in period t are $Z^+(t) > 0$ and $Z^-(t) > 0$, respectively, with commonly known probability distributions. The public does not observe v or the realizations of $Z^+(t)$ or $Z^-(t)$.

As before, a lit exchange and a dark pool operate in parallel. In contrast to the model of Section 3, both venues are open in all periods. At the beginning of period t, the exchange market maker posts a bid price B_t and an ask price A_t. Any order sent to the exchange is immediately executed at the bid or the ask. After execution of exchange orders in each period, the market maker announces the exchange net order flow. The public information $\mathcal{F}_t$ at the beginning of period t consists of all exchange net order flows prior to period t. Thus the conditional distribution of asset value v at

the beginning of period t is represented by the likelihood ratio

$$R_t = \frac{\mathbb{P}_t(v = +\sigma)}{\mathbb{P}_t(v = -\sigma)}, \tag{55}$$

where $\mathbb{P}_t$ denotes the conditional probability based on $\mathcal{F}_t$. By construction, $R_0 = 1$. The public's conditional expectation of the asset value at the beginning of period t is therefore

$$V(R_t) = \sigma(\mathbb{P}_t(v = +\sigma) - \mathbb{P}_t(v = -\sigma)) = \frac{R_t - 1}{R_t + 1}\,\sigma. \tag{56}$$

The dark pool executes orders in each period, simultaneously with the execution of exchange orders. I assume that the dark pool implements a double auction with a trade-at rule, as in Section 4. Midpoint crossing, which offers a price improvement of the exchange spread, is a special case of this double auction. Liquidity traders are distinguished by their delay costs, as in Section 3. If a liquidity trader of cost type γ does not trade in period t, then she incurs a delay cost of $c(\gamma; R_t)$, which is strictly increasing in γ. A trader only incurs delay costs after she arrives.

I make the additional assumption that informed traders also incur positive delay costs, before they execute their orders. A type-γ informed trader incurs the delay cost $c(\gamma; R_t)$ in period t if she fails to execute her order in that period. Thus a type-γ informed buyer (resp. seller) and a type-γ liquidity buyer (resp. seller) differ only in their information about v.

I now fix a cost type $\gamma \geq 0$ and focus on the payoffs of buyers. For any (R_t, t), I let

$$W_e(R_t, t) = \sigma - A_t \tag{57}$$

$$X_e(R_t, t) = V(R_t) - A_t \tag{58}$$

be the payoffs of a type-γ informed buyer and a type-γ liquidity buyer, respectively, for trading immediately on the exchange. These payoffs do not depend on the cost type γ because exchange execution involves no delays. I let $W_d(R_t, t; \gamma)$ and $X_d(R_t, t; \gamma)$ be the corresponding continuation values of entering an order in the dark pool. Finally, I let $W(R_t, t; \gamma)$ and $X(R_t, t; \gamma)$ be the continuation values of the informed buyer and liquidity buyer, respectively, at the beginning of period t, before they make trading decisions. For $t = T$, $W(R_T, T; \gamma) = X(R_T, T; \gamma) = 0$. For $t < T$, the Bellman Principal implies that

$$W(R_t, t; \gamma) = \max\left[W_e(R_t, t), W_d(R_t, t), \mathbb{E}_t(W(R_{t+1}, t+1; \gamma))\right], \tag{59}$$

$$X(R_t, t; \gamma) = \max\left[X_e(R_t, t), X_d(R_t, t), \mathbb{E}_t(X(R_{t+1}, t+1; \gamma))\right], \tag{60}$$

where the three terms in the $\max(\cdot)$ operator represents a trader's three choices: sending her order to the exchange, sending her order to the dark pool, and delaying trade.

The following proposition shows equilibrium conditions under which, controlling for delay costs, the liquidity-versus-informed payoff difference $X_d(R_t, t; \gamma) - W_d(R_t, t; \gamma)$ in the dark pool is at least as high as the corresponding payoff difference $X_e(R_t, t) - W_e(R_t, t)$ on the exchange. It is in this "difference-in-difference" sense that the dark pool is more attractive to liquidity traders, and that the exchange is more attractive to informed traders.

Proposition 5. *In any equilibrium, if* $W_d(R_t, t; \gamma) \geq \mathbb{E}_t[W(R_{t+1}, t+1; \gamma)]$, *then*

$$X_d(R_t, t; \gamma) - W_d(R_t, t; \gamma) \geq X_e(R_t, t) - W_e(R_t, t). \tag{61}$$

Proof. See Appendix B. □

Proposition 5 identifies a restriction (61) that must be satisfied by all equilibria, provided that a informed buyer does not strictly prefer delaying trade to using the dark pool. The intuition for this result is as follows. Because the exchange guarantees to execute all buy orders at the same price A_t, the exchange payoff difference, $X_e(R_t, t) - W_e(R_t, t)$, reflects only the value of private information. The dark-pool payoff difference $X_d(R_t, t; \gamma) - W_d(R_t, t; \gamma)$, by contrast, reflects both the value of information and the execution risk. Compared with a liquidity buyer, an informed buyer in the dark pool is less likely to fill her order and, conditional on a trade, more likely to pay a higher price. This execution risk is costly for the informed buyer in equilibrium as long as she prefers dark-pool trading to waiting, as captured by $W_d(R_t, t; \gamma) \geq \mathbb{E}_t[W_d(R_{t+1}, t+1; \gamma)]$. I isolate this dark-pool execution risk from the value of information by taking the "difference-in-difference" of payoffs in (61).

Appendix A solves a dynamic equilibrium explicitly under a slightly different setting.

6 Empirical and Policy Implications

This section discusses some of the empirical and policy implications of these results, both in light of recent empirical evidence and in relation to the current policy debate on the effect of dark pools on price discovery and liquidity.

The discussion follows two organizing questions. First, what are the relations between dark-pool market share and observable market characteristics? Second, what are the impacts of dark-pool trading on price discovery and liquidity? For each question, I first state a number of empirical implications implied by the results of this paper and

then discuss them in the context of related empirical evidence. Some recent empirical papers are summarized in Table 2.

6.1 Determinants of dark-pool market share

Prediction 1. *All else equal, dark-pool market share is lower if the execution probability of dark-pool orders is lower.*

Prediction 2. *All else equal, if volatility (spread, total volume) is low, then dark-pool market share is low, and increasing in volatility (spread, total volume).*

Prediction 3. *All else equal, informed participation in dark pools is higher for higher-volatility stocks and days, and for dark pools that allow more discretion in execution prices.*

Prediction 4. *All else equal, dark-pool market share is lower for trading strategies relying on shorter-term information and strategies trading more stocks simultaneously.*

Prediction 1 follows from the results of Section 3 and Section 4. A lower execution probability, captured by a lower $\bar{r}$ or $\bar{r}_x$, discourages both types of traders from participating in the dark pool.[20] Ye (2010) constructs a proxy of execution probability in eight dark pools from their SEC Rule 605 reports.[21] He focuses on the relationship between non-execution probabilities and market characteristics such as price impacts and effective spreads. He does not, however, study how non-execution probability is related to the market share of dark pools. Using daily data collected by SIFMA from eleven anonymous dark pools in 2009, Buti, Rindi, and Werner (2010a) find that order imbalance as a percentage of total volume is negatively related to dark-pool market share. The absolute depth imbalance on lit venues, too, is negatively related to dark-pool market share. To the extent that a high order imbalance reduces dark-pool execution probability, and that depth imbalances are positively correlated with order imbalances, Prediction 1 is consistent with their findings.

Prediction 2 suggests that dark-pool market share is low if volatility, spread, or total volume is low, as we discuss in Section 3.3. Existing empirical evidence in this respect is mixed. Using transaction data from 2005 to 2007 in three block-crossing dark pools, Liquidnet, Posit, and Pipeline, Ready (2010) documents that stocks with

[20] This relation can be analytically proved for $\sigma < \bar{\sigma}$ in Proposition 1 and for $\sigma < \bar{\sigma}(x)$ in Proposition 4.

[21] Under Rule 605 of Reg NMS, market centers are required to make monthly electronic reports. These reports, according to FINRA, "include information about each market center's quality of execution on a stock-by-stock basis, including how market orders of various sizes are executed relative to the public quotes. These reports must also disclose information about effective spreads (the spreads actually paid by investors whose orders are routed to a particular market center). In addition, market centers must disclose the extent to which they provide executions at prices better than the public quotes to investors using limit orders."

Table 2: Some recent (academic) empirical studies on dark pools and market fragmentation. The first five papers focus on dark pools. The next three papers focus on market fragmentation (by dark and lit venues).

Paper	Sample trading venues	Sample period	Data source
Ready (2010)	Liquidnet, Posit, and Pipeline	07/2005 – 09/2007, monthly	Nasdaqtrader.com
Buti, Rindi, and Werner (2010a)	11 anonymous dark pools	2009, daily	Voluntary SIFMA survey
Ray (2010)	Liquidnet, Posit, and Pipeline	06/2005 – 06/2006, monthly	Nasdaqtrader.com
Ye (2010)	Goldman Sachs Sigma X, Level, Barclays ATS, ITG Posit Now, Instinet, Liquidnet, NYFIX Millennium, and BNY ConvergEx	01/2010 – 03/2010, monthly	SEC Rule 605 data
Nimalendran and Ray (2011)	One anonymous dark pool	06/2009 – 12/2009, intraday	
O'Hara and Ye (2011)	Trade reporting facilities	01/2008 – 06/2008, monthly	SEC Rule 605 data
Weaver (2011)	Trade reporting facilities	10/2010, intraday	TAQ
Degryse, de Jong, and van Kervel (2011)	European dark pools and OTC	2006 – 2009, intraday	Thomson Reuters/Markit Boat

the lowest spreads are less likely to trade in dark pools. Ready attributes this finding to soft-dollar arrangements: because institutions need to execute *some* orders through brokers (instead of dark pools), they send stocks with the lowest spreads to brokers and incur the lowest transaction costs. The model of this paper gives an alternative interpretation, namely that low-spread stocks are less costly to trade on exchanges than in dark pools, because dark-pool execution is subject to delays.[22] In contrast to Ready (2010), Buti, Rindi, and Werner (2010a) and Ye (2010) find that dark-pool market share is higher for lower-spread stocks and higher-volume stocks. In the time series, Buti, Rindi, and Werner (2010a) find that, fixing a stock, dark-pool market share is higher on days with wider quoted spreads.

A relatively consistent result across these studies is that dark-pool market share is lower when volatility is higher, but not all of them reach the same conclusion. Using the SIFMA sample of eleven dark pools, Buti, Rindi, and Werner (2010a) document a negative relation between dark-pool market share and volatility, as measured by the intraday range, the standard deviation of midpoint returns, and the average absolute returns. Similarly, Ye (2010) finds that the market shares of his sample of eight dark pools are lower when the standard deviations of stock returns are higher. Using the absolute monthly returns as a proxy of volatility, Ray (2010) reaches the same conclusion. In contrast to these studies, Ready (2010) estimates a structural model and finds that stocks with higher return volatilities are more likely to be routed to his sample of the three block-crossing dark pools.

One potential explanation for these different results is that dark-pool market share may not be monotone in volatility, spread, or volume in the first place, as we discuss in Section 3.3. Indeed, Ready (2010) finds that dark-pool market share first increases in the total volume and then decreases. Using a sub-sample of Ready's data, Ray (2010) finds a similar hump-shaped relation between dark-pool market share and the relative bid-ask spread. For this reason, a linear (or any monotone) empirical model could be difficult to interpret. These different results may also be driven by sample selection. For example, the Ready (2010) sample uses early data (2005-2007), Nasdaq stocks, and three dark pools, whereas the Buti, Rindi, and Werner (2010a) sample and the Ye (2010) sample cover more recent data (2009 and 2010, respectively), NYSE and Nasdaq stocks, and more dark pools (eleven and eight, respectively).

Prediction 3 proposes that dark pools with discretionary execution prices are more attractive to informed traders and to impatient liquidity traders than dark pools crossing at the midpoint. To my knowledge, this prediction is not yet tested in the data,

[22] Admittedly, small orders of this model are different from the large institutional orders in the data of Ready (2010), and I do not model brokers. However, to the extent that transaction costs on large orders and with brokers are proportional to the exchange spreads, the model of this paper can still be applicable.

perhaps due to the lack of transaction data that can be assigned to individual dark pools. For a given dark pool, Prediction 3 also suggests that dark-pool orders are more informative on average when information asymmetry is severe. Nimalendran and Ray (2011) provide some evidence for this, using data from an anonymous dark pool. They put each transaction in their data into one of three baskets, buy, sell, and unsigned, depending on whether the transaction price is above, below, or equal to the prevailing exchange midpoint. Then they construct a portfolio that buys the "buy basket" and sells the "sell basket." In a two-hour horizon, this strategy makes a profit if implemented on the most illiquid securities (those with the widest relative spread), but results in a loss if implemented on the most liquid securities. Their finding is consistent with the model's prediction that informed participation in dark pools, if any, is likely to concentrate on illiquid securities, for which the value of private information is likely to be high. Because informed traders tend to post aggressive limit prices and execute their orders at prices relatively far away from the exchange midpoint, the signed dark-pool transactions of illiquid stocks can contain price-relevant information. By contrast, aggressive buying and selling of the most liquid securities are more likely to come from impatient liquidity traders, and it may not be profitable to follow them.

Prediction 4 provides strategy-level implications on dark-pool activity. Strategies relying on shorter-term information have higher execution risks in dark pools, because information can become stale faster (see Appendix A). Related to this prediction, Ready (2010) finds that the usage of three block-crossing dark pools is lower for institutions with higher turnover, which is consistent with the notion that short-term strategies are best implemented in venues that guarantee execution. Because dark pools cannot guarantee simultaneous execution of multiple stocks, we also expect them to be less attractive to strategies tracking stock indices or those "arbitraging" perceived mispricing among similar securities. For these strategies, partial execution in dark pools can be particularly costly.

6.2 Effects of dark pools on price discovery and liquidity

Prediction 5. *All else equal, a higher dark-pool market share is associated with higher order informativeness on the exchange in the form of wider spreads and higher price impacts of trades.*

Prediction 6. *All else equal, a higher dark-pool market share increases the correlation of volumes across different stocks in lit exchanges. This cross-stock volume correlation is lower in dark pools than in lit exchanges.*

Prediction 7. *All else equal, dark-pool execution implies a positive adverse-selection cost in that shares bought in dark pools tend to have low short-term returns and that shares sold in dark pools tend to have high short-term returns. This cost, however, is lower than the exchange spread at the time of execution.*

Prediction 5 holds under the conditions stated in Proposition 3. For example, adding a dark pool increases the informativeness of exchange orders when the value of information is relatively low, when the mass of informed traders is relatively insensitive to the value of information, or when the dark-pool execution probability is relatively low. We observe that higher price discovery and higher liquidity need not imply each other, as they tend to have opposite effects for spreads and price impacts.

This price-discovery effect of dark pools is consistent with the findings of Degryse, de Jong, and van Kervel (2011), who examine the impact of dark trading, including trading in dark pools and over-the-counter, on characteristics of lit venues, including exchanges and "multilateral trading facilities,"[23] in Dutch equity markets. The market share of dark trading, they conclude, is associated with higher price impacts, higher quoted spreads, higher realized spreads, and smaller depths on lit markets. Using transaction data in an anonymous dark pool, Nimalendran and Ray (2011) find that following dark-pool transactions, bid-ask spreads tend to widen and price impacts tend to increase, especially if the relative bid-ask spreads are already high. By contrast, Buti, Rindi, and Werner (2010a) find in the SIFMA data that higher dark-pool trading is associated with lower spreads, higher depths, and lower volatility. These conflicting conclusions could be results of different data or empirical methods, and more analysis may be needed to unveil the nature of these disagreements. In early studies, where dark pools had much lower market shares, Gresse (2006) and Fong, Madhavan, and Swan (2004) conclude that dark-pool trading does not harm the quality of the London Stock Exchange and the Australian Stock Exchange, respectively. Due to endogeneity—dark-pool trading and market characteristics affect each other—we should interpret these results with caution.

Related to dark pools as non-displayed limit order books, Hendershott and Jones (2005) find that price discovery for exchange-traded funds (ETFs) on Island ECN, a limit-order market, worsened after Island stopped displaying its limit orders. Because Island ECN was the *dominant* market for affected ETFs, it differed from today's equity dark pools, which rely on exchanges prices as reference. Extending the Kyle (1985) framework, Boulatov and George (2010) model a non-displayed market by allowing informed market makers to hide their liquidity-supply schedules from the public view.

[23] Multilateral trading facility (MTF) is a type of transparent venues in Europe. An MTF is similar to an exchange.

They find that this opacity enhances competition among market makers, leading to more informative quotes. Since the non-displayed market of Boulatov and George (2010) has no competitors, their model is different from the two-venue model of this paper.

Because dark pools account for a substantial portion of total off-exchange trading, a related question is how off-exchange trading affects price discovery and liquidity. To answer this question, O'Hara and Ye (2011) use the market share of trade reporting facilities (TRFs) as a proxy of market fragmentation. Using a TRF sample of early 2008, they find that higher fragmentation is associated with faster execution, lower transaction costs, and more efficient prices (prices are closer to random walks). Using a more recent TRF sample of 2010, Weaver (2011) finds that stocks with higher levels of off-exchange trading tend to have wider spreads, higher price impacts, and higher volatilities. Weaver (2011) attributes the difference between his results and those of O'Hara and Ye (2011) to two *lit* venues, BATS and DirectEdge, which used to report to TRF in the O'Hara and Ye (2011) sample but later became registered exchanges and hence dropped out of the Weaver (2011) sample. In Dutch equity market, Degryse, de Jong, and van Kervel (2011) find that the fragmentation of lit venues has a non-monotone relation with liquidity. Moderate lit fragmentation increases depths, lowers spreads, and lowers price impacts, but too much lit fragmentation has the opposite effects. Together, these findings suggest that dark and lit fragmentation could have different impact on price discovery and liquidity.

Prediction 6 can be viewed as the mirror image of Prediction 4. Since dark pools are less attractive to strategies executing multiple stocks simultaneously, those strategies should have a higher concentration on lit venues than in dark pools. The volume correlation across stocks, consequently, should be higher on lit venues than in dark pools as well. Moreover, because simultaneous trading of multiple stocks is often executed by algorithms, we also expect algorithmic trading to have a higher concentration on lit exchanges than in dark pools.

Finally, Prediction 7 follows from the results of Section 3. If a liquidity trader is willing to participate in the dark pool despite the potential delay cost, the adverse-selection cost in the dark pool must be lower than the exchange spread. Using proprietary transaction data from institutional investors, Conrad, Johnson, and Wahal (2003) and Næs and Odegaard (2006) document lower trading costs in dark pools compared to traditional execution methods, such as brokers. Similar results have been documented more recently by Brandes and Domowitz (2010) in a selection of non-displayed venues in Europe. Domowitz, Finkelshteyn, and Yegerman (2009) find that a scheduled dark pool has lower transaction costs than a peer group of continuous dark pools. Nonetheless,

these studies do not measure the *ex-post* costs of adverse selection, that is, whether shares bought (sold) in dark pools have lower (higher) returns shortly after order execution. Næs and Odegaard (2006) provide anecdotal evidence that filled orders in a dark pool are subject to such adverse selection. A full test of Prediction 7 is likely to require detailed data on all orders submitted to a dark pool, not only executed orders.

Short-term adverse selection in dark pools can reduce the transaction quality of institutional investors, as discussed by Mittal (2008) and Saraiya and Mittal (2009). The Securities and Exchange Commission (2010), too, has noted that "[i]n theory, short-term price swings that hurt investors on one side of the market can benefit investors on the other side of the market. In practice, professional traders, who have the time and resources to monitor market dynamics closely, are far more likely than investors to be on the profitable side of short-term price swings." The SEC has further added that "[w]where the interests of long-term investors and short-term professional traders diverge, the Commission repeatedly has emphasized that its duty is to uphold the interests of long-term investors."

7 Concluding Remarks

In recent years, dark pools have become an important part of equity market structure. Although it is often suspected that dark pools harm price discovery, existing empirical studies have not found conclusive evidence supporting this view. This paper provides a potential explanation of this puzzle: it is the liquidity traders—not informed traders—who find dark pools more attractive. Under natural conditions, adding a dark pool reduces the noise in exchange orders and improves price discovery.

A few practically relevant respects of dark pools are not modeled in this paper. The first is size discovery: institutions trade large blocks of shares in dark pools without revealing their trading intentions to the broad markets. This size-discovery benefit of dark trading has been widely acknowledged by market participants and regulators, and today only a handful of dark pools specialize in such block-crossing services (Securities and Exchange Commission, 2010). A proper model of size discovery in dark pools could also be a model of competing execution methods, such as algorithms that probe different dark pools, algorithms that split orders in the lit markets,[24] and conventional OTC market makers.

Second, this paper does not consider communications in dark pools. For example, a dark pool may send an "indication of interest" (IOI), which contains selected order

[24]Models of splitting orders include those of Seppi (1990), Bernhardt and Hughson (1997), Bertsimas and Lo (1998), and Back and Baruch (2007).

information such as the ticker, to potential counterparties to facilitate a match. In this sense, these dark pools are not completely dark, as modeled by Buti, Rindi, and Werner (2010b). The Securities and Exchange Commission (2009) has proposed to treat actionable IOIs—IOIs containing the symbol, size, side, and price of an order—as quotes so that these IOIs should be disseminated to the broad market immediately.

Third, this paper assumes that investors have unfettered access to dark pools. In practice, fair access to dark pools is not warranted. For example, the U.S. regulation does not require dark pools to provide fair access unless the dark pool concerned reaches a 5% volume threshold. (Currently, no dark pool has reached this hurdle.) Whether investors suffer from the lack of fair access can depend on perspective. On the one hand, it seems plausible that the lack of fair access can reduce trading opportunities and the welfare of excluded traders. On the other hand, "some dark pools attempt to protect institutional trading interest by raising access barrier to the sell-side or certain hedge funds," observes SEC Deputy Director James Brigagliano.[25] Credit Suisse, a broker-dealer, has also opened a "Light Pool" ECN that aims to exclude "opportunistic" traders such as some high-frequency trading firms.[26]

Finally, this paper assumes that investors fully understand the operation mechanics of dark pools and their users. That is, although dark-pool orders are invisible, the trading rules of the dark pool are transparent. Innocuous in most contexts, this assumption may be worth further investigation for dark pools. For example, a Greenwich Associates survey of 64 active institutional users of dark pools reveals that, on many occasions, dark pools do not disclose sufficient information regarding what types of orders are accepted, how orders interact with each other, how customers' orders are routed, what anti-gaming controls are in place, whether customer orders are exposed to proprietary trading flows, and at what price orders are matched (Bennett, Colon, Feng, and Litwin, 2010). The International Organization of Securities Commissions (2010) also observes that "[l]ack of information about the operations of dark pools and dark orders may result in market participants making uninformed decisions regarding whether or how to trade within a dark pool or using a dark order." Opacity regarding dark-pool orders can help improve price discovery, but opacity regarding dark-pool operating mechanics can make it more difficult for investors and regulators to assess the impact of dark pools on price discovery and liquidity.

[25] "Keynote Speech to the National Organization of Investment Professionals," by James A. Brigagliano, April 19, 2010.

[26] See Nina Mehta, "Credit Suisse Plans Market for Long-Term Investors," *Bloomberg*, January 11, 2011.

Appendix

A Dynamic Trading with Stochastic Crossing

This appendix explicitly characterizes a family of equilibria in which informed traders do not participate in the dark pool. The setting of this section is different from Section 5 in that I assume continuous time and assume that event times follow Poisson processes. This Poisson assumption gives rise to tractable stationary equilibria.

Time is continuous, $t \geq 0$, and the market opens at time 0. As before, an asset pays an uncertain dividend v that is $+\sigma$ or $-\sigma$ with equal probabilities. The time of the dividend payment is exponentially distributed with mean $1/\lambda_F$, for $\lambda_F > 0$. Two types of risk-neutral traders—liquidity traders and informed traders—have independent Poisson arrivals with respective mean arrival rates of λ_L and λ_I. (Traders are thus "discrete.") Each trader can buy or sell one unit of the asset. As in Section 5, I do not consider endogenous information acquisition here. Upon arrival, an informed trader observes v perfectly. Liquidity traders, who are not informed regarding the dividend, arrive with an unwanted position in the asset whose size is either $+1$ or -1, equally likely and independent of all else.

As before, a lit exchange and a dark pool operate in parallel. A competitive and risk-neutral market maker on the exchange continually posts bid and ask prices for one unit of the asset, as in Glosten and Milgrom (1985). Any order sent to the exchange is immediately executed at the bid or the ask, and trade information is immediately disseminated to everyone. By competitive pricing, the bid price at any time t is the conditional expected asset value given the arrival of a new sell order at time t and given all public information up to, but before, time t. The ask price is set likewise. The market maker also maintains a public "mid-market" price that is the conditional expected asset value given all public information up to but before time t. Once an exchange order is executed, the market maker immediately updates her bid, ask, and mid-market prices.

The dark pool accepts orders continually, and an order sent to the dark pool is observable only by the order submitter. The dark pool executes orders at the mid-market price and at the event times of a Poisson process with intensity λ_C that is independent of all else. Allocation in the dark pool is pro-rata on the heavier side, as in Section 3. For analytical tractability, I assume that unmatched orders in the dark pool are immediately sent to the exchange market maker, who then executes these orders at the conditional expected asset value given all past public information and given the quantity and direction of unmatched orders from the dark pool.

As in Section 5, the conditional likelihood ratio of v at time t is

$$R_t = \frac{\mathbb{P}_t(v = +\sigma)}{\mathbb{P}_t(v = -\sigma)}, \tag{62}$$

where $\mathbb{P}_t$ denotes the market maker's conditional probability. By construction, $R_0 = 1$. The conditional expected asset value is, as in Section 5,

$$V(R_t) = \sigma(\mathbb{P}_t(v = +\sigma) - \mathbb{P}(v = -\sigma)) = \frac{R_t - 1}{R_t + 1}\,\sigma. \tag{63}$$

To calculate the bid and ask prices, I let λ_t be the time-t arrival intensity (conditional mean arrival rate) of traders of any type to the exchange, and let μ_t be the time-t conditional probability that an arriving exchange trader is informed. Then

$$q_t = \mu_t + (1 - \mu_t)0.5 = 0.5 + 0.5\mu_t \tag{64}$$

is the probability that an exchange trader arriving at t is "correct," that is, buying if $v = +\sigma$ and selling if $v = -\sigma$. The likelihood ratio

$$z_t = \frac{q_t}{1 - q_t} \tag{65}$$

then represents the informativeness of a time-t exchange order.[27] For example, if a buy order hits the market maker's bid at time t, then Bayes' Rule implies that

$$R_t = \frac{\mathbb{P}_t(v = +\sigma \,|\, Q = 1)}{\mathbb{P}_t(v = -\sigma \,|\, Q = 1)} = \frac{\mathbb{P}_t(Q = 1 \,|\, v = +\sigma)}{\mathbb{P}_t(Q = 1 \,|\, v = -\sigma)} \cdot \frac{\mathbb{P}_{t-}(v = +\sigma)}{\mathbb{P}_{t-}(v = -\sigma)} = R_{t-} z_t, \tag{66}$$

where $\mathbb{P}_{t-}$ denotes the market maker's probability conditional on all exchange transactions up to but before time t, and where $R_{t-} \equiv \lim_{s \uparrow t} R_s$. Similarly, if an exchange sell order arrives at time t, then

$$R_t = \frac{\mathbb{P}_t(v = +\sigma \,|\, Q = -1)}{\mathbb{P}_t(v = -\sigma \,|\, Q = -1)} = \frac{R_{t-}}{z_t}. \tag{67}$$

To break even, the market maker quotes a time-t bid price of $V(R_t z_t^{-1})$ and a time-t ask price of $V(R_t z_t)$. Because $V(\,\cdot\,)$ is nonlinear, $V(R_t)$ is generally not identical to the bid-ask midpoint, $(V(R_t z_t) + V(R_t z_t^{-1}))/2$. Nonetheless, for simplicity I refer to $V(R_t)$ as the "mid-market" price.

Liquidity traders must hold collateral equal to the expected loss on their unwanted

[27] In the equilibria characterized in this section, the information content of a buy order is equal to that of a sell order, so there is no need to specify them separately.

risky positions. With probability κ_j and independently of all else, an arriving liquidity trader incurs a cost of γ_j per unit of time for every unit of collateral support in her risky position, where $(\kappa_j)_{j=1}^J$ and $(\gamma_j)_{j=1}^J$ are commonly-known constants and satisfy

$$0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_{J-1} < \gamma_J, \tag{68}$$

$$\sum_{j=1}^{J} \kappa_j = 1. \tag{69}$$

Before executing her order, a liquidity buyer of type j incurs a flow cost of

$$c_t^j = \gamma_j \mathbb{E}_t[\max(0, v - V(R_t))] = \gamma_j \frac{R_t}{R_t+1} \cdot \left(1 - \frac{R_t - 1}{R_t + 1}\right) \sigma = \gamma_j \frac{2R_t}{(R_t+1)^2}\, \sigma. \tag{70}$$

A liquidity seller of type j has the same flow cost c_t^j because

$$\gamma_j \mathbb{E}_t[\max(0, V(R_t) - v)] = \gamma_j \frac{1}{R_t+1} \cdot \left(\frac{R_t - 1}{R_t + 1} - (-1)\right) \sigma = \gamma_j \frac{2R_t}{(R_t+1)^2}\, \sigma. \tag{71}$$

By independent splitting of Poisson processes, the arrival intensities of type-j liquidity buyers and type-j liquidity sellers are both $0.5\kappa_j\lambda_L$.

Without loss of generality, we focus on the strategies of informed buyers and liquidity buyers, whose payoffs are denoted $W(R_t)$ and $X(R_t)$, respectively. For simplicity, I use $\mathbb{E}_t^i[\,\cdot\,]$ as a shorthand for $\mathbb{E}_t[\,\cdot \,|\, v = \sigma]$, where the superscript "i" stands for "informed." Because I look for stationary equilibria, the payoffs $W(R_t)$ and $X(R_t)$ depend on the public information R_t but not on time t.

Proposition 6. *For fixed integer $M \in \{0, 1, 2, \ldots, J\}$, define*

$$z_e = \frac{\lambda_I + 0.5 \sum_{i=M}^{J} \kappa_i \lambda_L}{0.5 \sum_{i=M}^{J} \kappa_i \lambda_L}. \tag{72}$$

Under the conditions

$$\lambda_C < \frac{\sum_{i=M}^{J} \kappa_j \lambda_L}{2\lambda_I}\, \lambda_F, \tag{73}$$

$$\gamma_j < (\lambda_C + \lambda_F)\, \frac{z_e - 1}{z_e}, \quad 1 \leq j < M, \tag{74}$$

$$\gamma_j > (\lambda_C + \lambda_F)(z_e - 1) + \left(\lambda_I + \sum_{i=M}^{J} \kappa_i \lambda_L\right)\, \frac{(z_e - 1)^3}{z_e(\sqrt{z_e} + 1)^2}, \quad M \leq j \leq J, \tag{75}$$

there exists an equilibrium in which:

1. *Informed traders trade on the exchange immediately upon arrival.*
2. *Type-j liquidity traders, $M \leq j \leq J$, trade immediately on the exchange upon arrival.*
3. *Type-j liquidity traders, $1 \leq j < M$, enter orders in the dark pool. If the dark pool has not crossed by the time that the dividend is paid, they cancel their dark-pool orders and trade immediately on the exchange.*
4. *At time t, the market maker quotes a bid of $V(R_t z_e^{-1})$ and an ask of $V(R_t z_e)$. Moreover, immediately after a dark-pool crossing, the market maker executes all outstanding orders at a price of $V(R_t)$. Immediately after the dividend v is paid, the market maker executes all outstanding orders at the cum-dividend price of v.*

Proof. See Appendix B. □

A key step in the equilibrium solution of Proposition 6 is that informed traders expect the exchange price to move against them over time, but liquidity traders expect the exchange spread to narrow over time. Thus informed traders are relatively impatient, whereas liquidity traders are relatively patient. These different expectations of future prices, as formally stated in the following lemma, underlie the partial separation between informed traders and liquidity traders in the equilibria of Proposition 6.

Lemma 1. *Let Q be the direction of the next exchange order that arrives before the dividend payment, that is, $Q = 1$ denotes a buy order and $Q = -1$ denotes a sell order. Under the strategies stated in Proposition 6:*

- *The asset value is a martingale for liquidity traders and the public, in that*

$$V(R_t) = \mathbb{E}_t[V(R_t z_e^Q)]. \tag{76}$$

- *The exchange ask price is a submartingale for informed buyers, in that*

$$V(R_t z_e) < \mathbb{E}_t^i[V(R_t z_e^Q z_e)]. \tag{77}$$

- *The exchange ask price is a supermartingale for liquidity buyers, in that*

$$\mathbb{E}_t[V(R_t z_e^Q z_e)] = V(R_t z_e) - \frac{2R_t^2(z_e - 1)^3}{(R_t + 1)^2(R_t z_e + 1)(R_t z_e^2 + 1)}\,\sigma. \tag{78}$$

Proof. See Appendix B. □

In Proposition 6, z_e reflects the degree of information asymmetry on the exchange, because it is the ratio of the mean arrival rate of traders in the "correct" direction versus

the mean arrival rate of traders in the "wrong" direction. Proposition 6 says that an informed trader trades immediately on the exchange if the crossing frequency λ_C of the dark pool is sufficiently low relative to the risk that her private information becomes stale. A liquidity trader sends her order to the dark pool if and only if her delay cost γ is sufficiently low compared to the potential price improvement obtained by trading at the market midpoint. Moreover, because the exchange order informativeness z_e is increasing in M, the more liquidity traders trade in the dark pool, the more informative are exchange orders. This property is a dynamic analogue of the two-period equilibrium of Section 3.

We now briefly discuss the comparative statics of the equilibria, based on the tightness of the incentive constraints (73)-(75). First, a higher crossing frequency λ_C tightens (73), suggesting that informed traders are more likely to participate in the dark pool if the crossing frequency is higher. On the other hand, a higher λ_C relaxes (74) but tightens (75), making the dark pool more attractive to liquidity traders. As long as λ_C is sufficiently low, informed traders avoid in the dark pool. Second, a higher arrival rate λ_F of information relaxes (73), suggesting that informed traders are less likely to trade in the dark pool if they face a higher risk of losing their information advantage. By contrast, a higher λ_F makes the dark pool more attractive to liquidity traders by shortening their expected waiting time, as in (74)-(75). Third, a higher delay cost γ makes the dark pool less attractive to liquidity traders, without affecting the incentives of informed traders.

B Proofs

B.1 Proof of Proposition 1

It is clear that $\beta < 1$, as otherwise the exchange spread would be zero and informed traders would deviate to trade on the exchange. Thus in equilibrium either $\beta = 0$ or $0 < \beta < 1$.

We first look for an equilibrium in which $\beta = 0$. By (17), $\alpha_0 = 0$ and $\alpha_e = 1 - \alpha_d$. The indifference condition of the marginal liquidity trader is given by (20). For notational simplicity, we write the left-hand side of (20) as $-\tilde{X}_d(\alpha_d)$ and the right-hand side as $-\tilde{X}_e(\alpha_d)$. For each α_d, μ_I is uniquely determined by (21). We have

$$
\begin{aligned}
&-\tilde{X}_d(0) = 0 < -\tilde{X}_e(0), \\
&-\tilde{X}_d(G(1)) = 1 - \bar{r} \geq \frac{\hat{\mu}_I(\sigma)}{\hat{\mu}_I(\sigma) + (1 - G(1))\mu_z} = -\tilde{X}_e(G(1)),
\end{aligned}
$$

where the second inequality follows from (19), (21), and (18). So there exists a solution $\alpha_d^* \in (0, G(1)]$ that satisfies (20).

Now we look for an equilibrium in which $\beta > 0$, that is, informed traders are indifferent between the exchange and the dark pool. What remains to be shown is that the incentive-compatibility conditions (23)-(25) have a solution. For simplicity, we write the left-hand side of (23) as $\tilde{W}_d(\beta)$ and the right-hand side of (23) as $\tilde{W}_e(\beta)$. For each $\beta \geq 0$, μ_I is unique determined by (25) and is increasing in β. Under condition (22) and for each $\alpha_d > 0$,

$$\tilde{W}_d(0) = \bar{r} > 1 - \frac{\hat{\mu}_I(\sigma)}{\hat{\mu}_I(\sigma) + (1 - G(1))\mu_z} = \tilde{W}_e(0),$$
$$\tilde{W}_d(1) = r^- < 1 = \tilde{W}_e(1),$$

where the first inequality follows from (19), (21), and (18). So there exists a solution $\beta^* \in (0, 1)$ to (23), as a function of α_d. Because μ_I increases in β, we see that $\tilde{W}_d'(\beta) < 0$ and $\tilde{W}_e'(\beta) > 0$, holding α_d fixed. Thus the solution β^* to (23) is unique for each α_d.

Moreover, (23) implies that in equilibrium r^- is bounded away from 0. So there exists some $r_0 > 0$ such that $r^- > r_0$. So for sufficiently small $\alpha_d > 0$,

$$G(1) - G\left(\frac{r^+ - r^-}{r^+ + r^-}\right) > G(1) - G\left(\frac{1 - r_0}{1 + r_0}\right) > \alpha_d.$$

So there exists a solution $\alpha_d^* \in (0, G(1)]$ to (24). The equilibria characterized by (23)-(25) thus exist. To show that (22) is necessary for the existence of equilibria in which $\beta > 0$, suppose for contradiction that (22) does not hold. Then for all α_d and $\beta > 0$, $\tilde{W}_e(\beta) > \tilde{W}_e(0) \geq \tilde{W}_d(0) > \tilde{W}_d(\beta)$, which implies that all informed traders wish to deviate to the exchange, contradicting $\beta > 0$.

B.2 Proof of Proposition 2

Because β, α_d, α_e, μ, S, r^+ and r^- are implicitly defined by differentiable functions in each case of Proposition 1, they are continuous and differentiable in σ in each of the two intervals $[0, \bar{\sigma}]$ and $(\bar{\sigma}, \infty)$. At the volatility threshold $\sigma = \bar{\sigma}$, differentiability refers to right-differentiability in Case 1 of Proposition 1, and left-differentiability in Case 2.

Have a dark pool and $\sigma \leq \bar{\sigma}$

For $\sigma \leq \bar{\sigma}$, $\beta = 0$. Total differentiation of (20)-(21) with respect to σ yields

$$\underbrace{\left[\frac{dG^{-1}(\alpha_d)}{d\alpha_d}(1-\bar{r}) - \frac{\partial(S/\sigma)}{\partial\alpha_d}\right]}_{>0}\frac{d\alpha_d}{d\sigma} - \underbrace{\frac{\partial(S/\sigma)}{\partial\mu_I}}_{>0}\frac{d\mu_I}{d\sigma} = 0, \tag{79}$$

$$\underbrace{\left[1-\bar{\mu}F'(\sigma-S)\frac{\partial(\sigma-S)}{\partial\mu_I}\right]}_{>0}\frac{d\mu_I}{d\sigma} = \underbrace{\bar{\mu}F'(\sigma-S)\frac{\partial(\sigma-S)}{\partial\alpha_d}}_{<0}\frac{d\alpha_d}{d\sigma} + \underbrace{\bar{\mu}F'(\sigma-S)\left(1-\frac{S}{\sigma}\right)}_{>0}, \tag{80}$$

where the first term of (79) is positive because of equilibrium selection. If $d\alpha_d/d\sigma \leq 0$ at, say, some σ_0, then (80) implies that $d\mu_I/d\sigma > 0$ at σ_0. But then (79) cannot hold. Thus $d\alpha_d/d\sigma > 0$, $d\mu_I/d\sigma > 0$, and $d(S/\sigma)/d\sigma > 0$, by (20).

Have a dark pool and $\sigma > \bar{\sigma}$

Now suppose that $\sigma > \bar{\sigma}$. I denote by $r^{+'}$ and $r^{-'}$ the derivatives of r^+ and r^- with respect to $\beta\mu_I/\alpha_d$. We have $r^{+'} > 0$ and $r^{-'} < 0$. Total differentiation of (23)-(25) with respect to σ yields

$$\underbrace{\left(r^{-'}\frac{1}{\alpha_d} - \frac{\partial(1-S/\sigma)}{\partial(\beta\mu_I)}\right)}_{<0}\frac{d(\beta\mu_I)}{d\sigma} = \underbrace{\frac{\partial(1-S/\sigma)}{\partial\mu_I}}_{<0}\frac{d\mu_I}{d\sigma} + \underbrace{r^{-'}\frac{\beta\mu_I}{\alpha_d^2}}_{<0}\frac{d\alpha_d}{d\sigma}, \tag{81}$$

$$\underbrace{\left[1 - G'\left(\frac{r^+-r^-}{r^++r^-}\right)\frac{2(r^{+'}r^- - r^{-'}r^+)}{(r^++r^-)^2}\frac{\beta\mu_I}{\alpha_d^2}\right]}_{>0}\frac{d\alpha_d}{d\sigma} = -\underbrace{G'\left(\frac{r^+-r^-}{r^++r^-}\right)\frac{2(r^{+'}r^- - r^{-'}r^+)}{(r^++r^-)^2}\frac{1}{\alpha_d}}_{>0}\frac{d(\beta\mu_I)}{d\sigma}, \tag{82}$$

$$\underbrace{\left[1-\bar{\mu}F'(\sigma-S)\sigma\frac{\partial(1-S/\sigma)}{\partial\mu_I}\right]}_{>0}\frac{d\mu_I}{d\sigma} = \underbrace{\bar{\mu}F'(\sigma-S)\sigma\frac{\partial(1-S/\sigma)}{\partial(\beta\mu_I)}}_{>0}\frac{d(\beta\mu_I)}{d\sigma} + \underbrace{\bar{\mu}F'(\sigma-S)\left(1-\frac{S}{\sigma}\right)}_{>0}, \tag{83}$$

where the first term of (82) is positive because of equilibrium selection.

We can show that $d\alpha_d/d\sigma$ cannot switch signs in $[\bar{\sigma}, \infty)$. To see why, suppose otherwise, and $d\alpha_d/d\sigma$ switches signs at some σ_0. By continuity, at σ_0, $d\alpha_d/d\sigma = 0$. But (82) and (81) imply that $d(\beta\mu_I)/d\sigma = 0 = d\mu_I/d\sigma$ at σ_0 as well, which contradicts (83). Thus $d\alpha_d/d\sigma$ cannot switch signs in $[\bar{\sigma}, \infty)$; nor can it be zero.

At $\sigma = \bar{\sigma}$, $\beta = 0$ and $d\beta/d\sigma \geq 0$. Then by (82),

$$\frac{d(\beta\mu_I)}{d\sigma}\Big|_{\sigma=\bar{\sigma}} = \mu_I \frac{d\beta}{d\sigma}\Big|_{\sigma=\bar{\sigma}} \geq 0 \Longrightarrow \frac{d\alpha_d}{d\sigma}\Big|_{\sigma=\bar{\sigma}} \leq 0.$$

Because $d\alpha_d/d\sigma$ cannot be zero, it must be strictly negative for all $\sigma \in [\bar{\sigma}, \infty)$. By (82)-(83), for all $\sigma \in [\bar{\sigma}, \infty)$, $\beta\mu_I$ and μ_I are both strictly increasing in σ. Then (23) implies that

$$\frac{d(S/\sigma)}{d\sigma} = -\frac{dr^-}{d\sigma} = -r^{-\prime}\frac{d}{d\sigma}\left(\frac{\beta\mu_I}{\alpha_d}\right) > 0.$$

The spread itself, $S = \sigma \cdot (S/\sigma)$, obviously increases in σ as well. Finally,

$$\frac{dr^+}{d\sigma} = r^{+\prime}\frac{d}{d\sigma}\left(\frac{\beta\mu_I}{\alpha_d}\right) > 0, \quad \frac{dr^-}{d\sigma} = r^{-\prime}\frac{d}{d\sigma}\left(\frac{\beta\mu_I}{\alpha_d}\right) < 0.$$

No dark pool

The comparative statics for Corollary 1 are similar to that for the first case of Proposition 1 and are omitted.

B.3 Proof of Proposition 3

Have a dark pool and $\sigma \leq \bar{\sigma}$

For $\sigma \leq \bar{\sigma}$, adding a dark pool is equivalent to increasing $\bar{r}$. Total differentiation of (20)-(21) with respect to $\bar{r}$ yields

$$\underbrace{\left[(1-\bar{r})\frac{\partial G^{-1}(\alpha_d)}{\partial \alpha_d} - \frac{\partial(S/\sigma)}{\partial \alpha_d}\right]}_{>0}\frac{d\alpha_d}{d\bar{r}} = G^{-1}(\alpha_d) + \underbrace{\frac{\partial(S/\sigma)}{\partial \mu_I}}_{>0}\frac{d\mu_I}{d\bar{r}}, \tag{84}$$

$$\underbrace{\left[1 - \bar{\mu}F'(\sigma - S)\frac{\partial}{\partial \mu_I}(\sigma - S)\right]}_{>0}\frac{d\mu_I}{d\bar{r}} = \underbrace{\bar{\mu}F'(\sigma - S)\frac{\partial(\sigma - S)}{\partial \alpha_d}}_{<0}\frac{d\alpha_d}{d\bar{r}}, \tag{85}$$

where the first term on the left-hand side of (84) is positive because of the equilibrium selection. If $d\alpha_d/d\bar{r} \leq 0$ at any σ_0, then (85) implies that $d\mu_I/d\bar{r} \geq 0$ at σ_0. But that contradicts (84). Thus $d\alpha_d/d\bar{r} > 0$ and $d\mu_I/d\bar{r} < 0$. Adding a dark pool, which is equivalent to an increase in $\bar{r}$, raises α_d and reduces $\alpha_e = 1 - \alpha_d$. The total participation rate of liquidity traders in either the dark pool or the exchange is $\alpha_d + \alpha_e = 1$, higher than a market without a dark pool. Moreover, by (21), a lower μ_I implies a wider spread S on the exchange.

Have a dark pool and $\sigma > \bar{\sigma}$

Now suppose that $\sigma > \bar{\sigma}$. In a market with a dark pool, $\alpha_e = 1 - G(1)$, a constant. Substituting it into (26) and we have

$$\frac{\mu_I}{\mu_I + (1 - G(1))\mu_z} < 1.$$

So the equilibrium α_e without a dark pool resides in the interval $(1 - G(1), 1)$. That is, adding a dark pool reduces α_e.

Moreover, adding a dark pool increases the exchange spread if and only if α_e in the equilibrium of Corollary 1 is larger than $(1-G(1))/(1-\beta)$, where $\beta > 0$ is determined in Proposition 1. By the equilibrium selection rule and by (23),

$$\alpha_e > \frac{1 - G(1)}{1 - \beta} \Longleftrightarrow G^{-1}\left(1 - \frac{1 - G(1)}{1 - \beta}\right) > \frac{\mu_I}{\mu_I + \mu_z(1 - G(1))/(1 - \beta)} = 1 - r^-, \tag{86}$$

where the μ_I is given by

$$\mu_I = \bar{\mu} F\left(\frac{(1 - G(1))\mu_z}{(1 - \beta)\mu_I + (1 - G(1))\mu_z}\right).$$

We rearrange (86) and obtain

$$\beta < \frac{G(1) - G(1 - r^-)}{1 - G(1 - r^-)}.$$

On the other hand, because the left-hand side of (23) is decreasing in β and the right-hand side is increasing in β, the above condition is equivalent to (28).

As $F(c) \to 1$ for all $c > 0$, (25) implies that $\mu_I \to \bar{\mu}$, a constant. Holding $\mu_I = \bar{\mu}$ fixed, we now show that if $G''(1 - r^-) \leq 0$, then (28) holds for all $r^- \in [0, \bar{r}]$. At $r^- = \bar{r}$, we have $\sigma = \bar{\sigma}$ and (28) holds by the definition of $\bar{\sigma}$. At $r^- = 0$, (28) also holds trivially. Take the first and second derivatives of the right-hand side of (28) with respect to r^- and we obtain

$$\begin{aligned}
\frac{d[rhs(28)]}{dr^-} &= \frac{\bar{\mu}\mu_z G'(1 - r^-)}{[\bar{\mu} + (1 - G(1 - r^-))\mu_z]^2} > 0, \\
\frac{d^2[rhs(28)]}{d(r^-)^2} &= \bar{\mu}\mu_z \frac{G''(1 - r^-)[\bar{\mu} + (1 - G(1 - r^-))\mu_z] - 2\mu_z[G'(1 - r^-)]^2}{[\bar{\mu} + (1 - G(1 - r^-))\mu_z]^3} < 0.
\end{aligned}$$

Thus the right-hand side of (28) is concave and (28) holds for all $r^- \in [0, \bar{r}]$.

If the condition (30) holds, we have

$$r^- \leq \bar{r} < 1 - \frac{\bar{\mu}}{\bar{\mu} + (1 - G(1))\mu_z} \leq 1 - \frac{\mu_I}{\mu_I + (1 - G(1 - r^-))\mu_z}.$$

B.4 Proof of Proposition 4

If informed traders do not participate in the dark pool, an equilibrium is determined by a marginal liquidity trader who is indifferent between the dark pool and the exchange. Given α_d, this liquidity trader has a delay cost of $G^{-1}(\alpha_d)\sigma$. So we must have $X_d(xS; G^{-1}(\alpha_d)) = -S$, or (53). We want to find an equilibrium in which $\alpha_d^* \leq G(1)$.

We first calculate the comparative statics, assuming the existence of an equilibrium, and then show conditions under which the stated equilibrium exists. Total differentiation of (53) and (54) with respect to σ yields

$$\underbrace{\left(\frac{\partial[lhs(53)]}{\partial \alpha_d} - \frac{\partial[rhs(53)]}{\partial \alpha_d}\right)}_{>0} \frac{d\alpha_d}{d\sigma} + \underbrace{\left(\frac{\partial[lhs(53)]}{\partial \mu_I} - \frac{\partial[rhs(53)]}{\partial \mu_I}\right)}_{<0} \frac{d\mu_I}{d\sigma} = 0, \tag{87}$$

$$\underbrace{\left(1 - \frac{\partial[rhs(54)]}{\partial \mu_I}\right)}_{>0} \frac{d\mu_I}{d\sigma} = \underbrace{\frac{\partial[rhs(54)]}{\partial \alpha_d}}_{<0} \frac{d\alpha_d}{d\sigma} + \underbrace{\bar{\mu} F'(\sigma - S)\left(1 - \frac{S}{\sigma}\right)}_{>0}. \tag{88}$$

As before, if $d\alpha_d/d\sigma \leq 0$ at some σ_0, then (87) implies that $d\mu_I/d\sigma \leq 0$ at σ_0 as well. But this contradicts (88). Thus the comparative statics with respect to σ follow. And given the equilibrium, the dark-pool execution price p^* and the optimal limit prices follow from calculations done in the text.

Now I characterize the condition for the existence of an equilibrium and the threshold volatility $\bar{\sigma}(x)$. For $x \in [0, 1]$, I define $\bar{K}(x)$ implicitly by

$$(1 - x\bar{K}(x))\left\{1 - \mathbb{E}\left[\min\left(1, \frac{G(1)Z^-}{[G(1) - G(x\bar{K}(x))]Z^+}\right)\right]\right\} = \bar{K}(x). \tag{89}$$

This $\bar{K}(x)$ is uniquely well-defined because the left-hand side of (89) is decreasing in $\bar{K}(x)$ and the right-hand side is strictly increasing in $\bar{K}(x)$. Moreover, total differentiation of (89) with respect to x yields

$$\underbrace{\left(\frac{\partial[lhs(89)]}{\partial \bar{K}(x)} - 1\right)}_{<0} \bar{K}'(x) + \underbrace{\frac{\partial[lhs(89)]}{\partial x}}_{<0} = 0.$$

So we have $\bar{K}'(x) < 0$.

On the other hand, given $\bar{K}(x)$, I define $\mu_I^*(x)$ by

$$\frac{\mu_I^*(x)}{\mu_I^*(x) + (1 - G(1))\mu_z} = \bar{K}(x),$$

and define $\bar{\sigma}(x)$ by

$$\mu_I^*(x) = \bar{\mu} F\left(\frac{(1 - G(1))\mu_z}{\mu_I^*(x) + (1 - G(1))\mu_z}\,\bar{\sigma}(x)\right).$$

Because $\mu_I^*(x)$ is strictly increasing in $\bar{K}(x)$ and because $\bar{\sigma}(x)$ is strictly increasing in $\mu_I^*(x)$, $\bar{\sigma}(x)$ is strictly increasing in $\bar{K}(x)$. Because $\bar{K}'(x) < 0$, $\bar{\sigma}'(x) < 0$.

What remains to be shown is that, for $\sigma \leq \bar{\sigma}(x)$, an equilibrium characterized by Proposition 4 exists. Clearly, once α_d is determined, μ_I is uniquely determined by (54). For sufficiently small α_d, the left-hand side of (53) is negative, whereas the right-hand side is strictly positive. For $\alpha_d = G(1)$, (54) implies that

$$\mu_I = \bar{\mu} F\left(\frac{(1 - G(1))\mu_z}{\mu_I + (1 - G(1))\mu_z}\,\sigma\right),$$

which is no larger than $\mu_I^*(x)$. Thus

$$K \equiv \frac{\mu_I}{\mu_I + (1 - G(1))\mu_z} = \frac{S}{\sigma}\Big|_{\alpha_d = G(1)} \leq \bar{K}(x),$$

and, by the definition of $\bar{K}(x)$,

$$(1 - xK)\left\{1 - \mathbb{E}\left[\min\left(1, \frac{G(1)Z^-}{[G(1) - G(xK)]Z^+}\right)\right]\right\} > K.$$

That is, at $\alpha_d = G(1)$, the left-hand side of (53) is weakly higher than the right-hand side. Therefore, there exists a solution $\alpha_d^* \in (0, G(1))$ to (53) and an equilibrium exists.

B.5 Proof of Proposition 5

Suppose that $W_d(R_t, t; \gamma) \geq \mathbb{E}_t[W(R_{t+1}, t+1; \gamma)]$ in an equilibrium. I denote by $\hat{X}(R_t, t; \gamma)$ the "auxiliary payoff" of a type-γ liquidity buyer who "imitates" the strategy of a type-γ informed buyer. That is, the imitating buyer behaves as if $v = +\sigma$. Clearly, such imitation is suboptimal for the liquidity buyer, so $X(R_t, t; \gamma) \geq \hat{X}(R_t, t; \gamma)$ for all R_t, t, and γ. For notional simplicity, in the calculations below I suppress the cost type γ and likelihood ratio R_t as function arguments.

Suppose that the informed buyer enters an order in the dark pool at time t. The

imitating liquidity buyer does the same, by construction. Let $\hat{X}_d^+$ be the dark-pool payoff of the imitating buyer conditional on $v = +\sigma$, and let $\hat{X}_d^-$ be the dark-pool payoff of the imitating buyer conditional on $v = -\sigma$. I define $\hat{X}_e^+$ and $\hat{X}_e^-$ similarly. Then we have

$$\begin{aligned}\hat{X}_d(t) - W_d(t) &= \frac{R_t}{R_t+1}\hat{X}_d^+(t) + \frac{1}{R_t+1}\hat{X}_d^-(t) - W_d(t) \\ &= \frac{1}{R_t+1}\left(\hat{X}_d^-(t) - \hat{X}_d^+(t)\right),\end{aligned}$$

where the last equality follows from the fact that, conditional on the true dividend, the expected payoff W_d of the informed buyer is the same as the payoff $\hat{X}_d^+$ of the imitating liquidity buyer. If we can show that $\hat{X}_d^+(t) - \hat{X}_d^-(t) \leq 2\sigma$, then

$$X_d(t) - W_d(t) \geq \hat{X}_d(t) - W_d(t) \geq -\frac{2\sigma}{R_t+1} = X_e(R_t, t) - W_e(R_t, t).$$

Now I prove that $\hat{X}_d^+(t) - \hat{X}_d^-(t) \leq 2\sigma$. I denote by $\mathcal{C}$ the event that the imitating buyer's order is crossed in the dark pool, and let

$$\begin{aligned}k_t^+ &\equiv \mathbb{P}_t[\mathcal{C} \,|\, v = +\sigma], \\ k_t^- &\equiv \mathbb{P}_t[\mathcal{C} \,|\, v = -\sigma],\end{aligned}$$

be the crossing probabilities of the imitating buyer in the dark pool in period t, conditional on $v = +\sigma$ and $v = -\sigma$, respectively. Then we have

$$\begin{aligned}\hat{X}_d^+(t) &= k_t^+[+\sigma - \mathbb{E}_t(p^* \,|\, \mathcal{C}, v = +\sigma)] + (1 - k_t^+)\mathbb{E}_t(\hat{X}^+(t+1)), \\ \hat{X}_d^-(t) &= k_t^-[-\sigma - \mathbb{E}_t(p^* \,|\, \mathcal{C}, v = -\sigma)] + (1 - k_t^-)\mathbb{E}_t(\hat{X}^-(t+1)),\end{aligned}$$

where p^* is the execution price in the dark pool in period t, and $\hat{X}^+$ and $\hat{X}^-$ are the imitating buyer's payoffs conditional on $v = +\sigma$ and $v = -\sigma$, respectively. Because informed buyers have either a zero or positive mass in the dark pool in period t, we have

$$\begin{aligned}&k_t^+ \leq k_t^-, \\ &\mathbb{E}_t(p^* \,|\, \mathcal{C}, v = +\sigma) \geq \mathbb{E}_t(p^* \,|\, \mathcal{C}, v = -\sigma).\end{aligned}$$

Because $W_d(t) \geq \mathbb{E}_t[W(t+1)]$, we have $\hat{X}_d^+(t) \geq \mathbb{E}_t(\hat{X}^+(t+1))$. Thus

$$\begin{aligned}\hat{X}_d^+(t) - \mathbb{E}_t(\hat{X}^+(t+1)) &= k_t^+[\sigma - \mathbb{E}_t(p^* \,|\, \mathcal{C}, v = +\sigma) - \mathbb{E}_t(\hat{X}^+(t+1))] \\ &\leq k_t^-[\sigma - \mathbb{E}_t(p^* \,|\, \mathcal{C}, v = -\sigma) - \mathbb{E}_t(\hat{X}^+(t+1))],\end{aligned}$$

which implies that

$$\hat{X}_d^+(t) - \hat{X}_d^-(t) \leq 2\sigma k_t^- + (1 - k_t^-)\mathbb{E}_t[\hat{X}^+(t+1) - \hat{X}^-(t+1)]. \tag{90}$$

I now prove that $\hat{X}_d^+(t) - \hat{X}_d^-(t) \leq 2\sigma$ and that $\hat{X}^+(t) - \hat{X}^-(t) \leq 2\sigma$, by backward induction. We use the fact that, for all $t < T$, $X_e^+(t) - X_e^-(t) = 2\sigma$. Because v is revealed in period T, $\hat{X}^+(T) = \hat{X}^-(T) = 0$. By (90), $\hat{X}_d^+(T-1) - \hat{X}_d^-(T-1) \leq 2\sigma$. Because the venue choice of the imitating liquidity buyer does not depend on realizations of v,

$$\begin{aligned}&X^+(T-1) - X^-(T-1) \\ &= \max\left[X_e^+(T-1) - X_e^-(T-1), X_d^+(T-1) - X_d^-(T-1), \mathbb{E}_{T-1}(X^+(T) - X^-(T))\right] \leq 2\sigma.\end{aligned}$$

For the induction step, suppose that $\hat{X}^+(t+1) - \hat{X}^-(t+1) \leq 2\sigma$. Then (90) implies that $X_d^+(t) - X_d^-(t) \leq 2\sigma$. Thus

$$X^+(t) - X^-(t) = \max\left[X_e^+(t) - X_e^-(t), X_d^+(t) - X_d^-(t), \mathbb{E}_t(X^+(t+1) - X^-(t+1))\right] \leq 2\sigma,$$

which completes the proof.

B.6 Proof of Lemma 1

Given the public information R_t, the probability that $v = +\sigma$ is $R_t/(R_t+1)$. Let q be implicitly defined by $z_e = q/(1-q)$. That is, q is the probability that an arriving exchange order is in the same direction as informed orders. Under a liquidity trader's belief, the probability that the next exchange order is a buy order is

$$\frac{R_t}{R_t+1}q + \frac{1}{R_t+1}(1-q) = (1-q)\frac{R_t z_e + 1}{R_t+1}.$$

Similarly, the probability that the next exchange order is a sell order is

$$\frac{R_t}{R_t+1}(1-q) + \frac{1}{R_t+1}q = q\frac{R_t z_e^{-1} + 1}{R_t+1}.$$

We can verify that identity

$$\begin{aligned}
&(1-q)\frac{R_t z_e+1}{R_t+1}V(R_t z_e)+q\frac{R_t z_e^{-1}+1}{R_t+1}V(R_t z_e^{-1})\\
=&(1-q)\frac{R_t z_e+1}{R_t+1}\left(1-\frac{2}{R_t z_e+1}\right)\sigma+q\frac{R_t z_e^{-1}+1}{R_t+1}\left(1-\frac{2}{R_t z_e^{-1}+1}\right)\sigma\\
=&\left(1-\frac{2}{R_t+1}\right)\sigma, \qquad (91)
\end{aligned}$$

which implies $\mathbb{E}_t[V(R_t z_e^Q)] = V(R_t)$, that is, the expected mid-market price is a martingale for liquidity traders.

Note that the identity (91) holds for all R_t, including the informed trader's likelihood ratio ∞. Using (91) again, we have that

$$\begin{aligned}
V(R_t z_e) &= \left(1-\frac{2}{R_t z_e+1}\right)\sigma\\
&= (1-q)\frac{R_t z_e z_e+1}{R_t z_e+1}\left(1-\frac{2}{R_t z_e z_e+1}\right)\sigma+q\frac{R_t z_e z_e^{-1}+1}{R_t z_e+1}\left(1-\frac{2}{R_t z_e z_e^{-1}+1}\right)\sigma\\
&< (1-q)z_e\left(1-\frac{2}{R_t z_e z_e+1}\right)\sigma+qz_e^{-1}\left(1-\frac{2}{R_t z_e z_e^{-1}+1}\right)\sigma\\
&= \mathbb{E}_t^i[V(R_t z_e^Q z_e)].
\end{aligned}$$

That is, the exchange ask price is a submartingale for informed buyers.

Finally, direct calculation gives

$$\begin{aligned}
\mathbb{E}_t[V(R_t z_e^Q z_e)] &= V(R_t z_e)-\frac{2R_t^2(z_e-1)^3}{(R_t+1)^2(R_t z_e+1)(R_t z_e^2+1)}\sigma,\\
\mathbb{E}_t[V(R_t z_e^Q z_e^{-1})] &= V(R_t z_e^{-1})+\frac{2R_t^2(1-z_e^{-1})^3}{(R_t+1)^2(R_t z_e^{-1}+1)(R_t z_e^{-2}+1)}\sigma.
\end{aligned}$$

That is, for liquidity traders, the exchange ask price is a supermartingale and the exchange bid price is a submartingale.

B.7 Proof of Proposition 6

I prove Proposition 6 by direct verification. The quoting strategy of the market maker simply follows from risk neutrality and zero profit.

Under the proposed equilibrium strategy, the total arrival intensity of exchange orders is $\lambda_t = \lambda_I + \sum_{i=M}^{J}\kappa_i\lambda_L$. The Hamilton-Jacobi-Bellman (HJB) equation of an

informed buyer is

$$W(R_t) = \max\left[\sigma - V(R_t z_e), \frac{\lambda_t \mathbb{E}_t^i[W(R_t z_e^Q)] + \lambda_C(\sigma - V(R_t))}{\lambda_t + \lambda_C + \lambda_F}\right],$$

where the profit of immediate trading on exchange is $\sigma - V(R_t z_e)$ and the expected profit of trading in the dark pool is the weighted sum of:

- $\mathbb{E}_t^i[W(R_t z_e^Q)]$, the expected profit if the next event is the arrival of an exchange order.
- $\sigma - V(R_t)$, the profit if the next event is a dark-pool cross.
- 0, the profit if the next event is the dividend payment.

To verify that $W(R_t) = \sigma - V(R_t z_e)$, it is sufficient to verify that, for all t and all realizations of random variable R_t,

$$\sigma - V(R_t z_e) > \frac{\lambda_t \mathbb{E}_t^i[\sigma - V(R_t z_e^Q z_e)] + \lambda_C(\sigma - V(R_t))}{\lambda_t + \lambda_C + \lambda_F}. \tag{92}$$

By Lemma 1, the expected profit for informed buyers to trade on the exchange is a supermartingale, that is, $\sigma - V(R_t z_e) > \mathbb{E}_t^i[\sigma - V(R_t z_e^Q z_e)]$. Thus a sufficient condition for (92) is

$$\lambda_C + \lambda_F > \sup_{R \in (0,\infty)} \left\{\lambda_C \frac{\sigma - V(R)}{\sigma - V(R z_e)}\right\} = z_e \lambda_C,$$

which simplifies to (73).

Now we turn to a type-j liquidity buyer, whose HJB equation is

$$X(R_t) = \max\left[-(V(R_t z_e) - V(R_t)), \frac{\lambda_t \mathbb{E}_t[X(R_t z_e^Q)] - c_t^j}{\lambda_t + \lambda_C + \lambda_F}\right].$$

Her cost of liquidation $C(R_t)$ satisfies the HJB equation

$$C(R_t) = -X(R_t) = \min\left[V(R_t z_e) - V(R_t), \frac{\lambda_t \mathbb{E}_t[C(R_t z_e^Q)] + c_t^j}{\lambda_t + \lambda_C + \lambda_F}\right].$$

There are two cases, depending on j.

If $1 \le j < M$, to verify that $C(R_t) < V(R_t z_e) - V(R_t)$, it suffices to verify

$$V(R_t z_e) - V(R_t) > \frac{\lambda_t \mathbb{E}_t[V(R_t z_e^Q z_e) - V(R_t z_e^Q)] + c_t^j}{\lambda_t + \lambda_C + \lambda_F}, \tag{93}$$

where $C(R_t z_e^Q)$ is replaced by the higher cost of $V(R_t z_e^Q z_e) - V(R_t z_e^Q)$, as implied by

the conjectured equilibrium. Because the ask spread is a supermartingale for liquidity buyers (Lemma 1), a sufficient condition for (93) is

$$\lambda_C + \lambda_F > \sup_{R\in(0,\infty)} \left\{ \frac{c_t^j}{V(Rz_e) - V(R)} \right\} = \sup_{R\in(0,\infty)} \left\{ \frac{\gamma_j \cdot 2R/(R+1)^2 \cdot \sigma}{V(Rz_e) - V(R)} \right\} = \gamma_j \frac{z_e}{z_e - 1},$$

which simplifies to (74).

If $M \le j \le J$, to verify that $C(R_t) = V(R_t z_e) - V(R_t)$, it suffices to verify

$$V(R_t z_e) - V(R_t) < \frac{\lambda_t \mathbb{E}_t[V(R_t z_e^Q z_e) - V(R_t z_e^Q)] + c_t^j}{\lambda_t + \lambda_C + \lambda_F},$$

that is, by Lemma 1,

$$(\lambda_C + \lambda_F)[V(R_t z_e) - V(R_t)] < -\lambda_t \frac{2R_t^2(z_e-1)^3}{(R_t+1)^2(R_t z_e + 1)(R_t z_e^2 + 1)} \sigma + \gamma_j \frac{2R_t}{(R_t+1)^2} \sigma, \tag{94}$$

A sufficient condition for (94) is

$$\begin{aligned} \gamma_j &> \sup_{R\in(0,\infty)} \left\{ (\lambda_C + \lambda_F) \frac{(R+1)(z_e-1)}{Rz_e + 1} \right\} + \sup_{R\in(0,\infty)} \left\{ \lambda_t \frac{R(z_e-1)^3}{(Rz_e+1)(Rz_e^2+1)} \right\} \\ &= (\lambda_C + \lambda_F)(z_e - 1) + \lambda_t \frac{(z_e-1)^3}{z_e(\sqrt{z_e}+1)^2}. \end{aligned}$$

The argument for sellers is symmetric and yields the same parameter conditions.

C List of Model Variables

This appendix summarizes key variables used in Section 3 and Section 4.

Variable	Description
	Variables introduced in Section 3
v, σ	Asset value v is either $+\sigma$ or $-\sigma$, for $\sigma > 0$
$\bar{\mu}$, μ_I	Total masses of for-profit traders and informed traders
F	Cumulative distribution function (c.d.f.) of information-acquisition cost
Y	Signed informed trading interests: $Y = \text{sign}(v) \cdot \mu_I$
Z^+, Z^-	Masses of liquidity buyers and liquidity sellers, respectively
Z_0, Z	Balanced and imbalanced parts of liquidity trading interests, respectively
μ_z	Total expected mass of liquidity trading interests: $\mu_z = 2\mathbb{E}(Z_0) + E\|Z\|$
c, γ, G	Delay cost of liquidity traders $c = \sigma\gamma$, and γ has c.d.f. G
α_e, α_d, α_0	Fractions of liquidity traders who trade on the exchange, trade in the dark pool, and defer trading, respectively
β	Fraction of informed traders who trade in the dark pool
S	Exchange (effective) spread; bid is $-S$ and ask is S
$\bar{r}$	Dark-pool crossing probability if no informed traders go to the dark pool
r^-, r^+	Dark-pool crossing probabilities conditional on informed traders being on the same and opposite side, respectively
$\bar{\sigma}$	Maximum volatility for which informed traders avoid the dark pool
$\hat{\mu}_I(\sigma)$	Knife-edge mass of informed traders, defined by (18)
W_e, W_d	Expected profits of an informed buyer on the exchange and in the dark pool
$X_0(c)$, X_e, $X_d(c)$	Payoff of a liquidity buyer with a delay cost of c who defers trading, trades on the exchange, and trades in the dark pool, respectively
R_1	Period-1 log likelihood ratio of $\{v = +\sigma\}$ versus $\{v = -\sigma\}$
$I(\beta, \alpha)$	Signal-to-noise ratio of period-1 exchange order flow
V_d, V_e, V	Expected volumes in the dark pool, on the exchange, and both, respectively
U_I, $-U_L$	Informed traders' total profits and liquidity traders' total costs, respectively
C_i, C_w	Total costs of information acquisition and delay, respectively
	Variables introduced in Section 4
x	Strictness of trade-at rule; maximum (minimum) dark-pool price is xS ($-xS$)
y	Aggregate demand schedule in the dark pool if $Z^+ = 1$
y^+, y^-	Aggregate dark-pool demand and supply schedules, respectively
p^*, H	Dark-pool transaction price p^* has a c.d.f. of H
$X_d(p; c)$	Dark-pool payoff of a liquidity buyer with the limit price p and delay cost c
$\bar{r}_x$	Dark-pool crossing probability of a liquidity buyer with the limit price xS
$\bar{\sigma}(x)$	Maximum volatility for which informed traders avoid the dark pool

References

Admati, A. R., P. Pfleiderer, 1988. A Theory of Intraday Patterns: Volume and Price Variability. Review of Financial Studies 1(1), 3–40.

Back, K., S. Baruch, 2007. Working Orders in Limit Order Markets and Floor Exchanges. Journal of Finance 62(4), 1589–1621.

Bennett, J., J. Colon, J. Feng, J. Litwin, 2010. Ten Questions Every Institution Should Ask Their Dark Pool Providers. Journal of Trading 5(4), 10–13.

Bernhardt, D., E. Hughson, 1997. Splitting Orders. Review of Financial Studies 10(1), 69–101.

Bertsimas, D., A. W. Lo, 1998. Optimal control of execution costs. Journal of Financial Markets 1, 1–50.

Boulatov, A., T. J. George, 2010. Securities Trading when Liquidity Providers are Informed. Working paper University of Houston.

Brandes, Y., I. Domowitz, 2010. Alternative Trading Systems in Europe: Trading Performance by European Venues Post-MiFID. Journal of Trading 5(3), 17–30.

Brunnermeier, M. K., L. H. Pedersen, 2005. Predatory Trading. Journal of Finance 60(4), 1825–1863.

Buti, S., B. Rindi, I. M. Werner, 2010a. Diving into Dark Pools. Working paper Fisher College of Business, Ohio State University.

——— , 2010b. Dynamic Dark Pool Trading Strategies in Limit Order Markets. Working paper.

Butler, G., 2007. Liquidity Aggregation: What Institutional Investors Need to Know. Journal of Trading 2(2), 108–113.

Carlin, B. I., M. S. Lobo, S. Viswanathan, 2007. Episodic Liquidity Crisis: Cooperative and Predatory Trading. Journal of Finance 62(5), 2235–2274.

Carrie, C., 2008. Illuminating the New Dark Influence on Trading and U.S. Market Structure. Journal of Trading 3(1), 40–55.

CFA Institute, 2009. Market Microstructure: The Impact of Fragmentation under the Markets in Financial Instruments Directive. Technical report CFA Institute.

Chordia, T., R. Roll, A. Subrahmanyam, 2010. Recent Trends in Trading Activity. Working Paper.

Chowdhry, B., V. Nanda, 1991. Multimarket Trading and Market Liquidity. Review of Financial Studies 4(3), 483–511.

Conrad, J., K. M. Johnson, S. Wahal, 2003. Institutional trading and alternative trading systems. Journal of Financial Economics 70, 99–134.

CSA/IIROC, 2009. Dark Pools, Dark Orders, and Other Development in Market Structure in Canada. Consulation paper 23-404 Canadian Securities Administration and Investment Industry Regulatory Organization of Canada.

Degryse, H., F. de Jong, V. van Kervel, 2011. The Impact of Dark and Visible Fragmentation on Market Quality. Working paper.

Degryse, H., M. Van Achter, G. Wuyts, 2009. Dynamic Order Submission Strategies with Competition between a Dealer Market and a Crossing Network. Journal of Financial Economics 91, 319–338.

Domowitz, I., I. Finkelshteyn, H. Yegerman, 2009. Cul de Sacs and Highways, An Optical Tour of Dark Pool Trading Performance. Journal of Trading 4(1), 16–22.

Easley, D., N. M. Keifer, M. O'Hara, 1996. Cream-Skimming or Profit-Sharing? The Curious Role of Purchased Order Flow. Journal of Finance 51(3), 811–833.

European Commission, 2010. Review of the Markets in Financial Instruments Directive (MiFID). Public consultation European Commission.

Fong, K., A. Madhavan, P. L. Swan, 2004. Upstairs, Downstairs: Does the Upstairs Market Hurt the Downstairs?. Working paper.

Foster, D., S. Gervais, K. Ramaswamy, 2007. The Benefits of Volume-Conditional Order-Crossing. Working paper.

Glosten, L. R., P. R. Milgrom, 1985. Bid, Ask and Transaction Prices in a Specialist Market with Heterogenously Informed Traders. Journal of Financial Economics 14, 71–100.

Gresse, C., 2006. The Effect of Crossing-Network Trading on Dealer Market's Bid-Ask Spreads. European Financial Management 12, 143–160.

Hendershott, T., C. M. Jones, 2005. Island Goes Dark: Transparency, Fragmentation, and Regulation. Review of Financial Studies 18(3), 743–793.

Hendershott, T., H. Mendelson, 2000. Crossing Networks and Dealer Marekts: Competition and Performance. Journal of Finance 55(5), 2071–2115.

International Organization of Securities Commissions, 2010. Issues Raised by Dark Liquidity. Consultation report Technical Committee of the International Organization of Securities Commissions.

——— , 2011. Principles for Dark Liquidity. Final report Technical Committee of the International Organization of Securities Commissions.

Johnson, B., 2010. Algorithmic Trading & DMA: An introduction to direct access trading strategies. 4Myeloma Press, London, .

Joint CFTC-SEC Advisory Committee, 2011. Recommendations regarding Regulatory Responses to the Market Event of May 6, 2010. Summary report.

Kratz, P., T. Schöneborn, 2010. Optimal Liquidation in Dark Pools. Working paper.

Kyle, A. S., 1985. Continuous Auctions and Insider Trading. Econometrica 53(6), 1315–1335.

MacAfee, R. P., 1992. A Dominant Strategy Double Auction. Journal of Economic Theory 56, 434–450.

Mittal, H., 2008. Are You Playing in a Toxic Dark Pool? A Guide to Preventing Information Leakage. The Journal of Trading 3(3), 20–33.

Næs, R., B. A. Odegaard, 2006. Equity trading by institutional investors: To cross or not to cross?. Journal of Financial Markets 9, 79–99.

Nimalendran, M., S. Ray, 2011. Informed Trading in Dark Pools. Working paper University of Florida.

O'Hara, M., M. Ye, 2011. Is Market Fragmentation Harming Market Quality?. Journal of Financial Economics 100(3), 459–474.

Pagano, M., 1989. Trading Volume and Asset Liquidity. Quarterly Journal of Economics 104(2), 255–274.

Parlour, C. A., D. J. Seppi, 2008. Limit Order Markets: A Survey. Chapter 3, Handbook of Financial Intermediation and Banking.

Ray, S., 2010. A Match in the Dark: Understanding Crossing Network Liquidity. Working paper University of Florida.

Ready, M. J., 2010. Determinants of Volume in Dark Pools. Working paper University of Wisconsin-Madison.

Rosenblatt Securities, 2011. Let there be light. Market structure analysis.

Saraiya, N., H. Mittal, 2009. Understanding and Avoiding Adverse Selection in Dark Pools. Working paper Investment Technology Group.

Securities and Exchange Commission, 2005. Regulation NMS. Federal Register 70(124), 37496–37644.

——— , 2009. Regulation of Non-Public Trading Interests; Proposed Rule. Federal Register 74(224), 61207–61238.

——— , 2010. Concept Release on Equity Market Structure; Proposed Rule. Federal Register 75(13), 3593–3614.

Seppi, D., 1990. Equilibrium Block Trading and Asymmetric Information. Journal of Finance 45(1), 73–94.

Sun, Y., 2006. The exact law of large numbers via Fubini extension and characterization of insurable risks. Journal of Economic Theory 126, 31–69.

Tabb Group, 2011. Liquidity Matrix. Technical report.

Weaver, D., 2011. Off-Exchange Reporting and Market Quality in a Fragmented Market Structure. Working paper.

Ye, M., 2009. A Glimpse into the Dark: Price Formation, Transaction Costs and Market Share of the Crossing Network. Working paper.

——— , 2010. Non-execution and Market Share of Crossing Networks. Working paper University of Illinois.