Quiet Bubbles*

Harrison Hong† David Sraer‡

First Draft: November 1, 2010

This Draft: February 21, 2011

Abstract

Classic speculative bubbles are loud—price is high and so are price volatility and share turnover. The credit bubble of 2003-2007 is quiet — price is high but price volatility and share turnover are low. We develop a model, based on investor disagreement and short-sales constraints, that explains why credit bubbles are quieter than equity ones. Since debt up-side pay-offs are bounded, debt is less sensitive to disagreement about underlying asset value than equity and hence has a smaller resale option and lower price volatility and turnover. Leverage and skewed priors also lead to quiet credit bubbles. A few extreme optimists buy the supply using leverage, resulting in high prices but little price volatility and turnover since other traders' valuations are unlikely to exceed the extreme optimists' over time. Our theory provides a first taxonomy of bubbles and offers a rationale for why banks and regulators missed the credit bubble—it was quiet!

*Hong acknowledges support from the National Science Foundation through grant SES-0850404. We are deeply appreciative to Jeremy Stein for very insightful comments. We also thank Eric Silvergold and seminar participants at the MIT Sloan School of Management, Bendheim Center for Finance 10th anniversary conference, the 40th anniversary conference of Micro-Foundations for Macroeconomics at Columbia University, Casa Des Gasas in Rio De Janeiro, Australasian Banking and Finance Conference, Society for Financial Management Conference at National Taiwan University for many helpful comments. This paper was previously circulated under the title "A Taxonomy of Bubbles".

†Princeton University and NBER (e-mail: hhong@princeton.edu)

‡Princeton University (e-mail: dsraer@princeton.edu)

1. Introduction

Many commentators point to a bubble in credit markets from 2003 to 2007, particularly in the AAA and AA tranches of the sub-prime mortgage collateralized default obligations (CDOs), as the culprit behind the great financial crisis of 2008 to 2009. However, the credit bubble lacked many of the features that characterize classic episodes. These episodes, such as the internet bubble, are typically loud—characterized by high price, high price volatility, and high trading volume or share turnover as investors purchase in anticipation of capital gains.[1] In contrast, the credit bubble is quiet—characterized by high price but low price volatility and low share turnover. We argue for a taxonomy that distinguishes between equity and credit bubbles.

We begin by making the case that this credit bubble is quiet and hence fundamentally different from classic speculative equity ones. By all accounts, sub-prime mortgage CDOs experience little price volatility between 2003 until the onset of the financial crisis in mid-2007. In Figure 1, we plot the prices of the AAA and AA tranches of the sub-prime CDOs. The ABX price index only starts trading in January of 2007, very close to the start of the crisis. Nonetheless, one can see that, in the months between January 2007 until mid-2007, the AAA and AA series are marked by high price and low price volatility. Price volatility only jumps at the beginning of the crisis in mid-2007. This stands in contrast to the behavior of dot-com stock prices—the price volatility of some internet stocks during 1996-2000 (the period *before* the collapse of the internet bubble) exceed 100% per annum, more than three times the typical level of stocks.

Another way to see the quietness is to look at the prices of the credit default swaps for the financial companies that had exposure to these sub-prime mortgage CDOs. This is shown in Figure 2. The price of insurance for the default of these companies as reflected in the spreads of these credit default swaps is extremely low and not very volatile during the

[1] See Hong and Stein (2007) for a review of stylized facts about classic bubbles and Ofek and Richardson (2003) for a focus on the dot-com bubble.

years before the crisis. One million dollars of insurance against default cost a buyer only a few thousand dollars of premium each year. This price jumps at the start of the crisis, at about the same time as when price volatility increases for the AAA and AA tranches of the sub-prime mortgage CDOs.

The low price volatility coincided with little share turnover or re-selling of the sub-prime mortgage CDOs before the crisis. Since CDOs are traded over-the-counter, exact numbers on turnover are hard to come by. But anecdotal evidence suggests extremely low trading volume in this market particularly in light of the large amounts of issuance of these securities. Issuance totals around $100 billion dollars per quarter during the few years before the crisis but most of these credits are held by buyers for the interest that they generate. To try to capture this low trading volume associated with the credit bubble, in Figure 3, we plot the average monthly share turnover for financial stocks. Turnover for finance firms is low and only jumps at the onset of the crisis as does turnover of sub-prime mortgage CDOs according to anecdotal accounts (see, e.g., Michael Lewis (2010)). This stands in contrast to the explosive growth in turnover that coincided with the internet boom in Figure 4—obtained from Hong and Stein (2007). As shown in this figure, the turnover of internet stocks and run-up in valuations dwarfs those of the rest of the market.

In this paper, we develop a theory that explains why credit bubbles are quieter than equity ones. Our model builds on earlier work on the disagreement approach to asset pricing to generate loud equity bubbles. In these models, short-sales constraints and disagreement among investors lead to over-pricing in a static setting (Miller (1977) and Chen et al. (2002)) and a re-sale option in which investors value the potential to re-sell to someone with a higher valuation in a dynamic setting (Harrison and Kreps (1978) and Scheinkman and Xiong (2003)).[2] Dispersion and revision of beliefs among investors lead to high price, high price

[2]See Hong and Stein (2007) for a more extensive review of the disagreement approach to the modeling of bubbles. There are other approaches. For the possibility of rational bubbles, see Blanchard and Watson (1983), Tirole (1985), Santos and Woodford (1997), and Allen et al. (1993). For an agency approach, Allen and Gorton (1993) and Allen and Gale (2000). For other behavioral aproaches, see Delong et al. (1990) and Abreu and Brunnermeier (2003).

volatility and high share turnover.

We show in this same framework why debt bubbles are quieter than equity ones. We consider the pricing of a security with an arbitrary concave payoff over an underlying asset or fundamental, of which debt, ranging from safe to risky, are special cases. Investors have disagreement over the underlying asset value. This assumption differs from what has been considered in the literature, which has just focused on equity as opposed to a richer set of instruments including debt. Distributions of belief regarding the fundamental can be potentially right skewed. Two separate mechanisms emerge, which can generate quiet bubbles, in particular in credit.

Our first mechanism does not rely on the shape of the distribution of beliefs. It simply emanates from the bounded up-side pay-off of debt claims. When the underlying fundamental of the economy is strong, debt payoff becomes more concave (one can think about the debt payoff almost as a risk-free asset) and hence its value becomes very insensitive to beliefs. Thus, to the extent there is a bubble on a credit asset, the resale option is very limited, and as a consequence, both price volatility and turnover are low.[3] In other words, the bubble is quiet. Conversely, when the fundamental in the economy worsens, credit becomes riskier – closer to an equity claim – and beliefs start having a stronger influence on valuation. Therefore, the resale option grows and both price volatility and trading volume increase. This analysis also suggests that a debt bubble can only be significant when there is cheap leverage available to investors.

Our second mechanism then highlights additionally the role of leverage and skewness of priors in making credit bubbles quiet. When there is little leverage, increasing skewness of priors while holding fixed dispersion of beliefs actually decreases the amount of over-pricing generated by short-sales constraints. However, if the extreme optimists have access to leverage, then the results can be reversed and prices are actually even higher when the

[3]This point is related to Gorton and Pennacchi (1990) who points out that when the economy is hit by a negative shock, a debt claim becomes more sensitive to private information – in our setting, debt becomes more sensitive to beliefs and hence to disagreement.

distribution of priors is skewed than if it is not. Moreover, if there is enough leverage and skewed investors own the credit, then there is also little trading and price volatility in steady state because it is unlikely that fluctuations in other traders' valuations can exceed the extreme optimists' over time. Investors in our model still value an asset for the option to re-sell their shares to potentially more optimistic agents in the future. But the size of this re-sale option is small relative to cash-flows valuation when there is excessive leverage and skewed priors. Note that this second mechanism relies again on the up-side bounded-ness of debt pay-offs. For investors with very optimistic beliefs (i.e. skew guys), the re-sale option is limited since debt is essentially risk-free from their perspective and there is little trading among the extreme optimists.

We believe both mechanisms have important empirical counterparts. The first mechanism is consistent with the time-series of the crisis. The crisis starts with a negative shock to the fundamental value of the economy. Thus, beliefs (and disagreement) becomes much more important in the valuation of debt contracts. This in turn triggers an increase in both price volatility and turnover and the bubble stops being "quiet". This time-series behavior has been documented in the previous figures.

The second mechanism can be seen in accounts of the sub-prime mortgage CDO bubble during the years of 2003-2008 as being a quiet one. Big banks' exposures to these structured credits are responsible for their demise. Michael Lewis (2010) provides a detailed account of the history of this market. We make the case based on his account and other sources that an example of the extreme optimist with deep pockets in our model is AIG-FG – the financial markets group of AIG. AIG-FG insured 50 billion dollars worth of sub-prime AAA tranches between 2002 and 2005 at extremely low prices, years during which the issuance in the market was still relatively low, on the order of 100 billion dollars a year. Hence the bubble in CDOs was quiet because AIG-FP ended up being the skewed extreme optimist buyer of a sizeable fraction of the supply of these securities. We also provide compelling evidence for the importance of short-sales constraints in this market and how the loosening

of these constraints after 2006 precipitated the collapse of the market, consistent with the premise of our model.

Our model builds on two key elements—the boundedness of debt pay-offs and leverage with skewed priors. The boundedness of debt-payoffs and the related insensitivity of the pay-offs of highly rated debt to belief differences distinguishes this analysis from earlier work. Our use of priors is similar to Morris (1996) who showed that small differences in priors can lead long-run divergences in the Harrison and Kreps (1978) setting that we also consider. We interpret our priors assumption as in Morris (1996) who argues that it applies in settings where there is a financial innovation or a new company and hence there is room for disagreement. The sub-prime mortgage-backed CDOs are new financial instruments and hence the persistence effect that he identifies also holds in our setting.[4]

Our analysis takes the cost of leverage as exogenously decreasing with the efficiency of the banking sector. Our main contribution here is to lay out a taxonomy of bubbles as defined by price volatility and share turnover and to provide a theory of the conditions that would lead to one type versus another regardless of what drives the cost of credit. Geanakoplos (2010) shows that leverage effects in this disagreement setting can persist in general equilibrium though disagreement mutes the amount of lending on the part of pessimists to optimists. Simsek (2010) shows that the amount of lending is further mediated by the nature of the disagreement—disagreement about good states leads to more equilibrium leverage while disagreement about bad states leads to less. While our effects will still hold in a general equilibrium setting, they are likely to be amplified by institutional considerations such as AIG-FP being able to insure such large quantities of sub-prime mortgage-backed securities which are beyond the scope of general equilibrium models.

Our theory has a number of implications. First, it offers a unified approach to explain both "classic" bubbles such as the dot-com and so called the recent so-called credit bubble.[5]

[4]An alternative and more behavioral interpretation of our model is that overconfident investors who overreacted to information as in Odean (1999) and Daniel et al. (1998).

[5]Our paper focuses on mispricing of the mortgages as opposed to the homes since evidence from Mian and Sufi (2009) indicate that the run-up in home prices was due to the cheap subprime home loans and that

Second, it offers a new and parsimonious explanation for how a financial crisis of this size could emerge despite our recent experience with bubbles. Anecdotal accounts of the crisis invariably point to how the crisis and the credit bubble caught everyone by surprise.

Our unified theory offers a rationale for why smart investment banks and regulators failed to see the bubble on time. The low price volatility made these instruments appear safer in contrast to the high price volatility of dot-com stocks which made traders and regulator more aware of their dangers. Interestingly, our theory also predicts that riskier tranches of CDOs trade more like equity and hence have more price volatility and turnover, which is also consistent with the evidence. These securities were also less responsible for the failure of the investment banks which were mostly caught with the higher rated AAA tranches of the sub-prime mortgage CDOs. Other theories for the crisis based on agency or financial innovation cannot explain why banks did not get caught during the dot-com bubble since they had the same agency problems and innovative products to price. We discuss this point more in the conclusion.

Third, a flip side of this analysis is that regulators with hindsight believe that they missed the bubble because of the shadow banking economy. Recent regulation on bringing the trading of these securities onto exchanges and the Office of Financial Research's bubble detection unit are driven by this perspective. Our work suggests that the quietness of the bubble would have made it difficult to detect and that regulation needs to distinguish between different types of bubbles.

Fourth, the role of leverage in amplifying shocks has been understood and is clearly critical in melt-down or fire-sale part of the crisis (see Kiyotaki and Moore (1997) and Shleifer and Vishny (1997)). Our analysis shows that leverage can lower price volatility when there is an asset price bubble. Our analysis also echoes Hyman Minsky's warnings of bubbles fueled by credit. All in all, this paper can be thought of as offering an explanation for why bubbles

households took out refinancing from the equity of their homes for consumption as opposed to the purchase of additional homes, suggesting that there was limited speculation among households in the physical homes themselves. Khandani et al. (2010) make a similar point that the financial crisis was due to the refinancing of home loans as opposed to the speculative aspects described in the media.

fueled by credit are more dangerous – because they are quiet.

Finally, our theory is also consistent with the literature analyzing the destabilizing role of institutions in financial markets. This literature has pointed mostly toward constraints as the culprit for demand shocks (see, e.g., Vayanos and Gromb (2010)). Our theory offers a novel approach – namely that that outlier beliefs in institutional settings are more likely to be amplified with access to leverage.

Our paper proceeds as follows. We present the model in Section 2. We provide evidence to support our model in Section 3. We explain why the bubble came out of the blue in Section 4 and draw some lessons for regulatory reform. We conclude in Section 5.

2. Model

Our model has three dates $t = 0$, 1, and 2. There are two assets in the economy. A risk-free asset offers a risk-free rate each period which we normalize to zero. A risky debt contract with a face value of D has the following pay-offs at time t given by

$$m_t = \text{Max}[0, \text{Min}(D, \tilde{F}_t)], \tag{1}$$

where

$$\tilde{F}_t = F + \epsilon_t \tag{2}$$

for $t = 1, 2$ and the ϵ_t's are $i.i.d.$ with a standard normal distribution $\Phi(\cdot)$. We think of $\tilde{F}_t$ as the underlying asset value which determines the pay-offs of the risky debt. We assume that the $t = 1$ (interim) cash-flow and the payments of the date-0 and date-1 transaction costs all occur on the terminal date. This assumption is made so we do not have to keep track of the interim wealth of the investors. There is an initial supply Q of this risky asset.

Competitive agents hold heterogenous priors relative to the mean of the underlying asset.

Let $\beta \in [0, \frac{1}{2}]$ and define $y(\beta)$ and $x(\beta)$ as

$$\begin{cases} y(\beta) = \sigma\sqrt{\dfrac{\beta}{\frac{1}{2}-\beta}} \\ x(\beta) = \sigma\sqrt{\dfrac{\frac{1}{2}-\beta}{\beta}} \end{cases}. \tag{3}$$

Agents believe at $t = 0$ that the underlying asset process is actually

$$\tilde{V}_t = V + \epsilon_t, \tag{4}$$

where V has the following distribution function across the population of agents:

$$V = \begin{cases} F + \sigma \text{ with probability } \dfrac{1}{4} \\ F - \sigma \text{ with probability } \dfrac{1}{4} \\ F + x(\beta) \text{ with probability } \beta \\ F - y(\beta) \text{ with probability } \dfrac{1}{2} - \beta \end{cases}. \tag{5}$$

The average prior across the agents is F (i.e. $\mathbb{E}[\tilde{V}] = F$). The variance of the priors distribution is σ^2 (i.e. $\mathbb{E}[(\tilde{V} - F)^2] = \sigma^2$). The skewness coefficient (adjusted by variance) of the priors distribution is:

$$S(\beta) = \left(\left(\frac{1}{2} - \beta \right) \sqrt{\frac{\frac{1}{2}-\beta}{\beta}} - \beta\sqrt{\frac{\beta}{\frac{1}{2}-\beta}} \right). \tag{6}$$

It goes from $-\infty$ (when $\beta = \frac{1}{2}$) to ∞ (when $\beta = 0$) and is strictly decreasing with β. As β increases the mass of additional optimistic buyers increases but their beliefs decrease (i.e. they become less optimistic) so that eventually the distribution is less and less right skewed. When $\beta = \frac{1}{4}$ the distribution is symmetric and simply equal to a binomial distribution with two mass points $(-\sigma, \sigma)$ that occur with equal probability.

At $t = 1$, only the agents with $V = \sigma$ will receive a binomial shock to their belief given by

$$V_1 = \begin{cases} \sigma + \eta \text{ with probability } \frac{1}{2} \\ \sigma - \eta \text{ with probability } \frac{1}{2} \end{cases}. \tag{7}$$

The revision of beliefs of the σ-agents is the main shock that influences the determination of asset price at $t = 1$. The expected payoff of an agent with belief v regarding m_t is given by

$$\pi(v) = E[m_t|v] = \int_{-v}^{D-v} (v + \epsilon_t)\phi(\epsilon_t)d\epsilon_t + D\left(1 - \Phi\left(D - v\right)\right). \tag{8}$$

Our analysis below applies to any (weakly) concave expected pay-off function, which would include equity as well as other structured credits.

Agents are risk-neutral and are endowed with zero liquid wealth but large illiquid wealth W (which becomes liquid and is perfectly pledgeable at date 2). To be able to trade, these agents need to access a credit market that is imperfectly competitive. The discount rate is 0 but banks charge a positive interest rate, which we call $\frac{1}{\lambda}$. In particular, because illiquid assets are large, the loans made by the financial sector are risk-free. Thus, a perfectly competitive credit market is the limiting case where $\lambda \to \infty$. So λ measures the efficiency of the financial sector. Let

$$\mu = \frac{1}{1 + \lambda^{-1}} \in [0, 1]. \tag{9}$$

μ is increasing with the efficiency of the credit market. Investors also face quadratic trading costs given by

$$c(\Delta n_t) = \frac{(n_t - n_{t-1})^2}{2\gamma}, \tag{10}$$

where n_t is the shares held by an agent at time t. The parameter γ captures the severity of the trading costs—the higher is γ the lower the trading costs. Note that $n_{-1} = 0$ for all agents, i.e. agents are not endowed with any risky asset. Investors are also short sales constrained.

Let P_1 be the price of the asset at $t = 1$. Then at $t = 1$, for an investor with belief V_1 and prior V, her optimization problem is given by:

$$\max_{n_1(V_1)} \left\{ n_1(V_1)\pi(V_1) - \frac{1}{\mu}\left((n_1(V_1) - n_0(V))P_1 + \frac{(n_1(V_1) - n_0(V))^2}{2\gamma} \right) \right\}. \tag{11}$$

subject to the short-sales constraint that

$$n_1(V_1) \geq 0, \tag{12}$$

and given her inherited position $n_0(V)$ from $t = 0$. If $n_1(V_1) - n_0(V)$ is positive, an agent borrows $(n_1(V_1) - n_0(V))P_1 + \frac{(n_1(V_1)-n_0(V))^2}{2\gamma}$ to buy additional shares $n_1(V_1) - n_0(V)$. If $n_1(V_1) - n_0(V)$ is negative, an agent gets to lend at the rate λ^{-1} on the sales. The trading cost is symmetric (buying and selling costs are similar) and only affects the number of shares one purchases or sells, and not the entire position (i.e. $n_1 - n_0$ vs. n_1). Let $J(n_0; V)$ be the value function of agent-V's optimization program. $J(n_0; V)$ is driven in part by the possibility of the re-sale of the asset bought at $t = 0$ at a higher price to the σ-agents that get a draw on their belief at $t = 1$.

Let P_0 be the price of the asset at $t = 0$. Then at $t = 0$, agents with prior V have the following optimization program:

$$\max_{n_0(V)} \left\{ n_0(V)\pi(V) - \frac{1}{\mu}\left(n_0(V)P_0 + \frac{n_0(V)^2}{2\gamma} \right) + J(n_0(V); V) \right\}. \tag{13}$$

subject to the constraint at

$$n_0(V) \geq 0. \tag{14}$$

The equilibrium prices in both periods will be determined by the usual market clearing conditions.

We will focus our analysis below on the set of parameters where $\mu\gamma$ is big relative to Q. Bigger values of $\mu\gamma$ correspond to cheaper credit and lower trading costs. When these

parameters are large, it is more likely that only optimistic agents will be long the market. Assuming $\mu\gamma$ is big enough and Q is small enough, the $-\sigma$ and $-y$ agents do not participate. They are there for us to calibrate the model. The economics of our model can be transparently seen using the σ and x agents.

Our goal here is to construct a dynamic equilibrium with the following features that match a quiet credit bubble. First, we want to obtain a high price (i.e. high P_0 and P_1 relative to the fundamental F). A sufficient condition is by having only the x-agents, the most optimistic, be long at $t = 0$. Since there are short-sales constraints, there will be over-pricing in equilibrium. Indeed, when only the x-agents are long the asset, price will be even higher than if the distribution of priors were symmetric.

Second, we want to have an equilibrium with low price volatility between $t = 0$ and $t = 1$. Price volatility is defined simply by

$$\sigma_P = |P_1(-\eta) - P_1(\eta)|. \tag{15}$$

Price only varies at $t = 1$ due to the shock to the σ-agents' belief. The extent to which this shock gets materialized in prices determines price volatility in our model. Hence, we can simply use the absolute value of the difference of prices across the two states of η and $-\eta$ as a measure of price volatility. The reason for this is that our model is essentially a CARA (constant absolute risk aversion) set-up. So share returns, defined as $P_1 - P_0$, and share return volatility $\text{var}(P_1 - P_0)$ are the natural objects of analysis. This reduces to the definition of σ_P.

Third, we want to have low share turnover. Since the equilibrium we consider will have the x-agents long all the shares at $t = 0$, expected turnover is defined simply as

$$T = \mathbb{E}|n_1(x) - n_0(x)|, \tag{16}$$

where the x-agents' holding at $t = 1$ depend on the shock to the belief of the σ-agents.

The time $t = 1$ pricing and trading will be determined by the draw of the belief revision of the σ-agents. A positive η means the σ-agents become more optimistic than x and buy shares from the x-agents. These shocks to beliefs will not impact price volatility and turnover for two reasons. The first is the upside boundedness of the the debt contract. Price volatility and trading depends on $\pi(x)$ compared to $\pi(\sigma + \eta)$ and since π is bounded above for debt, even if $\sigma + \eta$ is higher than x, this will not translate to price volatility and trading since their assessments of the debt pay-off will not differ even if they have big disagreement on fundamentals, i.e. $\sigma + \eta$ is much bigger than x. The critical parameter here is D. The lower is D, the more debt is risk-free and the quieter it is. This first mechanism does not rely on skewness. One can think of $x = \sigma + \epsilon$ in this symmetric model benchmark of our general model. The second is that skewed priors and the more leverage these investors have because it becomes less likely that the revision of beliefs of other traders' (i.e. the σ agents) can exceed the valuation of the x-agents.

We develop this equilibrium in two steps. First, we consider a static version of our model in which is there is only one cash-flow and no possibility of re-selling to develop intuition for how to construct an equilibrium with high price. Second, we consider a dynamic setting in which agents at $t = 0$ not only value the cash-flows but also the possibility of re-selling their shares and hence their demand at $t = 0$ depends on the value of this re-sale option.

2.1. Static Setting

We first consider a static version of our model to show how skewed priors and excessive leverage can lead to high prices independent of the dispersion of opinion. While dispersion of opinion, σ, always increases price because of short-sales constraints, increasing skewness does not lead to the same result holding fixed σ. Higher skewness only leads to higher prices when leverage is sufficiently cheap, i.e. when μ is sufficiently large. In other words, it is possible to construct a high price equilibrium with skewed priors but only if there is excessive leverage. Leverage in other words plays a special role in a skewed priors setting

that it would not in a dispersed or symmetric priors setting.

To derive these results, we first quickly restate the static optimization program of an agent with prior V (with zero endowed wealth):

$$\max_{n(V)} n(V)\pi(V) - \frac{1}{\mu}\left(n(V)P + \frac{n(V)^2}{2\gamma}\right) \tag{17}$$

subject to $n(V) \geq 0$. The agent borrows $n(V)P + \frac{n(V)^2}{2\gamma}$ to purchase $n(V)$ shares of the assets, gets the expected payoff from these assets $(n(V)\pi(V))$ and has to reimburse $\frac{1}{\mu}\left(n(V)P + \frac{n(V)^2}{2\gamma}\right)$ to the banks. The optimal demand is then given by

$$n(V) = \gamma\left(\mu\pi(V) - P\right). \tag{18}$$

The following theorem characterizes the equilibrium.

Theorem 1. *Assume $\mu \geq \frac{4Q}{\gamma(\pi(\sigma)-\pi(0))}$. There is $\bar{\theta} > 0$ such that:*

- *If $\theta = \mu\gamma > \bar{\theta}$, the equilibrium price function is given as follows for $x \in [\sigma, \infty)$.*

  1. *There is $[\underline{x}, \bar{x}] \in [\sigma, \infty]$ such that only x agents are long and the equilibrium price is*

  $$P^*(x) = \mu\pi(x) - \frac{Q}{\gamma\beta(x)} = \mu\pi(x) - \frac{2Q}{\gamma}\left(1 + \frac{x^2}{\sigma^2}\right). \tag{19}$$

  2. *For $x \in [\sigma, \underline{x}] \cup [\bar{x}, \infty[$, both x and σ agents are long and the equilibrium price is*

  $$P^*(x) = \frac{\mu\left(\frac{1}{4}\pi(\sigma) + \beta(x)\pi(x)\right) - \frac{Q}{\gamma}}{\frac{1}{4} + \beta(x)}. \tag{20}$$

- *For $\theta < \bar{\theta}$, both x and σ agents are long and the equilibrium price is given by (20). Agents $-\sigma$ and $-y$ will not be long in equilibrium and price is higher than fundamental value.*

The condition that $\mu \geq \frac{4Q}{\gamma(\pi(\sigma)-\pi(0))}$ is sufficient to ensure that agents $-y$ are never long the market due to short-sales constraints. Intuitively, when there is enough leverage relative

to supply divided by the cost of trading (i.e. μ large relative to $\frac{Q}{\gamma}$), the buying power of the σ agents is enough to clear the market without the participation of the less optimistic agents. Moreover, since the x agents are the most optimistic, they will always be long. The issue is whether the demand from these x agents is enough to absorb the supply Q at high enough a price without needing the σ agents to participate. It turns out that without enough leverage, this is not possible. But when leverage is large enough, then we can have an equilibrium where only the x agents are long the risky asset, at least for intermediate values of x.

With this price function, we can show the following relationships between the skewness of the belief distribution, the cost of leverage and the equilibrium price.

Proposition 1. *When $\theta = \mu\gamma < \frac{2Q}{\sigma\pi'(\sigma)}$, more skewness in the distribution of priors decreases the equilibrium asset price. When the cost of leverage is sufficiently low, i.e. $\theta > \frac{2Q}{\sigma\pi'(\sigma)}$, there is an inverted-U shape relationship between positive skewness of the prior's distribution and the asset's equilibrium price.*

In contrast, increasing σ always leads to more over-pricing.

The following graph (Figure 5) summarizes the main results from Proposition 1. θ_0 corresponds to low leverage. θ_1 corresponds to moderate leverage. θ_2 corresponds to high leverage. The inflection points for the moderate and high leverage scenarios are denoted $\tilde{X}(\theta_1)$ and $\tilde{X}(\theta_2)$. The intuition for Proposition 1 and Figure 5 is as follows. In our model, more skewness leads the most optimistic buyer to be even more optimistic, which increases its valuation for the asset and hence (1) its direct demand for the asset ($\pi(x)$ is increasing with x) and (2) its demand for leverage. Both effects tend to increase prices.

However, more skewness also means that the weight of this more optimistic buyer in the population decreases, which decreases its demand. When there is no leverage, because π is concave, the increased demand effect is always dominated by the lower weight effect so that the price unambiguously decreases with skewness. As the cost of leverage decreases, the buying power of the most optimistic buyer increases which counteract the lower weight effect. When skewness grows very large, because π is concave, the larger demand/larger

leverage effect always ends up being dominated by the weight effect so that the price always eventually comes down as skewness increases.

These results stand in stark contrast to the comparative static with σ which always leads to more over-pricing. The reason is that increasing σ increases the valuations of all optimistic buyers without affecting their relative weight. These results are valid for any financial contract with an increasing, weakly concave payoff function $\pi()$. In particular, this entails both the case of debt and equity contracts.

Finally, we point out that skewness and leverage can lead to even more over-pricing than under a symmetric priors distribution and that decreasing the cost of leverage allows skewed priors to dominate the market and makes it easier for us to construct an equilibrium in which only the skewed investors are long the market.

Proposition 2. *The asset's equilibrium price can be larger with a positively skewed distribution than under a symmetric distribution of priors. A decrease in the cost of leverage leads to (1) higher prices (more over-pricing) overall and (2) prices that weakly increase over a wider range of skewness.*

In other words, we can achieve an equilibrium with high price and skewed priors independent of the dispersion of opinion as measured by σ.

2.2. Dynamic Setting

Having developed the intuition for the effect of skewness and leverage on pricing, we turn to analyzing the equilibrium of our dynamic model. Our analysis in the static setting demonstrates how to get high price with skewed priors. Now, we want to show that in a dynamic setting we can simultaneously generate high price, low price volatility and low share turnover. Here, the upside boundedness of debt and the skewed priors and leverage generates two distinct mechanisms for getting quiet bubbles.

The following theorem characterizes the equilibrium of interest in which only the x-agents are long the asset at $t = 0$. From our analysis in the static setting, when $\mu\gamma$ is large enough

relative to Q, this tends to imply that only x-agents own all the shares at $t = 0$ since their skewed beliefs with leverage give them enough buying power. The main intuition from this analysis carry through in our dynamic setting with resale options for σ-agents.

Theorem 2. *Denote by x^2 the unique x such that: $\pi(x) = \pi(\sigma+\eta) - \frac{\beta(x)+\frac{1}{4}}{\beta(x)}\frac{4Q}{\theta}$ and denote $\bar{\theta}$ the unique $\theta > 0$ such that $\theta = \frac{Q}{\pi(\sigma+\eta)-\pi(\sigma)}\left(\frac{5}{2} + 6\beta(x^2)\right)$. For $\theta \geq \bar{\theta}$, there exists $x_1^4 < x^2 < \sigma + \eta < x_2^1$ such that for $x \in [x_1^4, x_2^1]$, only the x-agents are long at date-0. Over this range of skewness, the equilibrium can be characterized in the following way:*

Case 1: For $x \in [\sigma + \eta, x_2^1]$, only the x-agents are long, both at date 0 and date 1:

$$P_1(-\eta) = \mu\pi(x),\ P_1(\eta) = \mu\pi(x) \text{ and } P_0 = 2\mu\pi(x) - \frac{Q}{\gamma\beta(x)}. \tag{21}$$

Case 2: For $x \in [x^2, \sigma + \eta]$, only the x-agents are long at date 0 and 1; the σ agents are long at date 1 if and only if they become more optimistic (i.e. $\tilde{\eta} = +\eta$):

$$P_1(-\eta) = \mu\pi(x),\ P_1(\eta) = \mu\left\{\frac{\frac{1}{4}\pi(\sigma+\eta) + \beta(x)\pi(x)}{\frac{1}{4} + \beta(x)}\right\} \text{ and } P_0 = 2\mu\pi(x) - \frac{Q}{\gamma\beta(x)}. \tag{22}$$

Case 3: For $x \in [x_1^4, x^2]$, only the x-agents are long at date 0; conditional on the positive belief shock ($\tilde{\eta} = +\eta$), only the σ-agents are long; conditional on the negative belief shock ($\tilde{\eta} = -\eta$), only the x-agents are long:

$$P_1(-\eta) = \mu\pi(x),\ P_1(\eta) = \mu\pi(\sigma+\eta) - \frac{4Q}{\gamma},\ \text{and } P_0 = \frac{3}{2}\mu\pi(x) + \frac{1}{2}\mu\pi(\sigma+\eta) - \frac{2Q}{\gamma\beta(x)}\left(\beta(x) + \frac{3}{4}\right). \tag{23}$$

The equilibrium price at $t = 0$ is above fundamental value.

The intuition for the equilibrium is as follows. Case 1 corresponds to x being large and priors being very positively skewed. As a result, the positive innovation in the beliefs of the σ agent at time 1 (i.e. $\sigma + \eta$) does not exceed x. Hence there is no trading and price $P_1(\eta) = \mu\pi(x)$ so that agent-x continues to hold all the supply. Notice that the date-1 price here does not depend on γ since no trading cost is incurred in the interim period by agent-x

because he does not trade. When the innovation of the σ agent at $t = 1$ is negative, i.e. $V_1 = \sigma - \eta$, $V_1 < x$ and hence x continues to hold all the shares at $t = 1$ and there is no trading. So $P_1(-\eta) = P_1(\eta) = \mu\pi(x)$.

The price at time $t = 0$ is determined as in the manner discussed in the static setting of the previous section. The price here depends on $2\mu\pi(x)$ net the supply-adjusted trading cost. Compared to the solution in the static setting, the only difference is that the fundamental valuation of the x-agent is now $2\mu\pi(x)$ instead of $\mu\pi(x)$ since there are 2 cashflows to value in the dynamic setting as opposed to only one in the static setting.

Case 2 corresponds to x being moderately large such that $\sigma+\eta > x$ and yet both x and σ agents are in the market at $t = 1$ when there is a positive innovation in beliefs, i.e. $\tilde{\eta} = -\eta$. In this case, i.e. with the positive innovation to the σ belief at $t = 1$, the date-1 equilibrium is simply the weighted average of x's and $\sigma+\eta$'s beliefs. The price in the negative innovation state continues to be $\mu\pi(x)$. Despite the different equilibrium structure, P_0 remains the same as in Case 1. The reason is that even with the positive belief shock ($\tilde{\eta} = \eta$), the marginal value of investing one dollar at date 0 for the x agents is $\mu\pi(x)$. Indeed, when investing an additional dollar, the x agents anticipate that in the case of a positive shock ($\tilde{\eta} = \eta$), he will be able to sell this additional share at a price P_1 at date 1. He also anticipates a larger cost of adjusting his position at date 1, larger by $\frac{n^1-n^0}{\gamma}$ so that the value of this additional dollar invested in the asset is $P_1 + \frac{n^1-n^0}{\gamma}$. Now it turns out that the marginal cost of investing an additional dollar in the asset at date 1 is also: $P_1 + \frac{n^1-n^0}{\gamma}$ and that this optimally equated to the marginal benefit of investing an additional dollar in period 1, i.e. $\mu\pi(x)$. Thus, the marginal value of investing one dollar at date 0 for the x agents is $\mu\pi(x)$ and eventually the price in the second equilibrium is similar to that in the first equilibrium.

Case 3 corresponds to x being small relative to $\sigma+\eta$ and as such when there is a positive innovation to σ-agents' belief at $t = 1$, x-agents sell all their shares to the agents with belief $\sigma + \eta$. Price at time $t = 1$ given a positive η, $P_1(\eta)$, depends on the belief of σ-agent and on the avalaible supply Q. The price given a negative η draw continues to be $\mu\pi(x)$.

Importantly, this case corresponds to a re-sale option for the x-agents since the short-sales constraints bind. The price at $t = 0$ incorporates this resell option:

$$P_0 = \underbrace{2\mu\pi(x) - \frac{3Q}{2\gamma\beta(x)}}_{\text{fundamental value for } x\text{-agents}} + \underbrace{\frac{1}{2}\left(P_1(\eta) - \mu\pi(x)\right)}_{\text{resale option}}$$

As we move from Case 3 to Case 2 to Case 1, we can think of this as a comparative static on x in which x is getting larger and priors are more skewed.

In the proofs of Theorem 2 and Proposition 3 in the Appendix, we derive the equilibrium prices, price volatility and share turnover for the other cases in which both the x and σ-investors are long the asset at $t = 0$. These cases are the following.

- Case 4: Both x and σ are long at $t = 0$ and only x-agents are long at $t = 1$ irrespective of $\tilde{\eta}$.
- Case 5: Both x and σ are long at $t = 0$ and only x-agents are long at $t = 1$ when $\tilde{\eta} = -\eta$ and both agents are long when $\tilde{\eta} = \eta$.
- Case 6: Both x and σ are long at $t = 0$ and only σ-agents are long when $\tilde{\eta} = \eta$.
- Case 7: Both x and σ are long at $t = 0$ and both are long at $t = 1$ irrespective of $\tilde{\eta}$.

In cases 4-7, x is either too large so that the skewed investors do not have enough buying power at date-0 to absorb all supply or too small so that the skewed investors are not sufficiently optimistic to absorb all the date-0 supply. These cases encompass instances familiar from the traditional set-up of re-sell option bubbles with low leverage in which both x and σ agents are in. In other words, we have now extended the intuition from the static setting in which x-agents are more likely to be the only agents long when x takes on intermediate values and θ is large.

2.3. Mechanism 1: Debt Less Disagreement-Sensitive than Equity

Our analysis then offers two rationales for why debt bubbles are quieter than equity ones or why bubbles in AAA debt are quieter than bubbles in risky or junk debt. The first intuition for the "quietness" of debt bubbles in our model can be seen by considering different levels of risk for the debt contract. Note first that for valuing a debt contract, only the belief on the probability of default of the contract is relevant. This is because conditional on no-default, the payoff is bounded by D and thus insensitive to belief on the distribution of payoffs above D. Thus, when D is low, there is very little scope for disagreement, and hence less scope for volatility and turnover. When D becomes larger, agents start disagreeing more on the probability of default so that the impact of beliefs on the expected value of the asset becomes larger – in the extreme, when D grows to infinity, beliefs become relevant for the entire payoff distribution and the scope for disagreement is maximum. Thus, for larger D, both volatility and turnover becomes larger.

We capture these insights in the following proposition:

Proposition 3. *Consider the Cases 1-3 in Theorem 2. An increase in the riskiness of debt (i.e. an increase in D) leads to (1) higher over-valuation (defined as P_0 minus the price in the no-short-sales constraints setting), (2) higher volatility, and (3) higher turnover.*

Proof. Consider Case 1-3. In Case 1, volatility and turnover are 0 and hence independent of D. In Case 2, both volatility and turnover are a function of $\pi(\sigma+\eta)-\pi(x)$ for $x \leq \sigma+\eta$. (See Proof of Theorem 2.) Clearly, $\frac{\partial \pi}{\partial D}(\sigma+\eta)-\frac{\partial \pi}{\partial D}(x) = \Phi(D-x)-\Phi(D-(\sigma+\eta)) > 0$. Thus, both turnover and volatility will increase with D. In Case 3, turnover is independent of D, but volatility is also a function of $\pi(\sigma+\eta)-\pi(x)$ and is thus strictly increasing with D as well.

Note also that $\frac{\partial x_2^1}{\partial D} > 0$ as $\frac{\partial \psi^1}{\partial D} = (\Phi(D-\sigma)-\Phi(D-x)) > 0$ and $\frac{\partial \psi^1}{\partial D}(x_2^1) < 0$ by definition. Similarly, $\frac{\partial x_4^1}{\partial D} < 0$ Thus, we also conclude that with a riskier debt contract, the equilibrium is more likely to feature only the x-agents long at date 0. This is intuitive as with a larger

D, beliefs have a stronger influence on the valuation of the asset, and hence the optimistic skewed agents have an increase willingness to buy the asset relative to the less optimistic agents. It then follows that the greater the mis-pricing for larger D.

□

This regarding how the up-side bounded-ness of debt pay-offs makes bubbles quiet is general since it works for any value of x. For formally, to show the generality of the first mechanism regarding the upside boundedness of debt pay-offs, we show that it holds even in a symmetric model of dispersed priors. Recall that by the symmetric model, we mean here the model where $x = \sigma + \epsilon$ with an arbitrarily small ϵ. This is necessary to think about turnover, as in a perfectly symmetric model where $x = \sigma$, it is indeterminate who hold share at date-0.

Proposition 4. *The same results in Proposition 3 apply in a symmetric setting without non-skewed priors. In other words, bubbles in higher-rated or safer debt is quieter than in lower-rated or riskier debt or equity.*

2.4. Mechanism 2: Leverage and Skewed Priors Also Makes Credit Bubbles Quieter

The second reason derived from the equilibrium analysis above is that since debt pay-off are bounded while equity is not, it is harder to get a debt bubble started. As a result, a debt bubble can only emerge when there is cheap leverage available to investors. We now show that the combination of leverage and skewed priors leads to quiet credit bubbles. Importantly, our results rely again on the upside boundeness of credit.

The following proposition then states a second mechanism for quietness which is that price volatility and expected turnover are lower as x increases.

Proposition 5. *Let θ be greater than $\bar{\theta}$ and $x \in [x_1^4, x_2^1]$. Price volatility and expected turnover are both weakly decreasing with x, i.e. price volatility and expected turnover are*

decreasing as one moves from Case 3 to Case 2 to Case 1.

In Case 1, corresponding to high skew, there is no turnover as x continues to hold onto all their shares irrespective of the draw of η. There is also no price volatility since the price is the same irrespective of the draw of η. In Case 2, corresponding to moderate skew, there is positive expected turnover. There is strictly positive turnover when there is a positive η draw as the x-agents sell some of their shares to $\sigma + \eta$-agents. There is no turnover when the η draw is negative as the x-agents keep all their shares. And there is price volatility as the $P_1(\eta)$ differs from $P_1(-\eta)$. Finally, in Case 3, expected turnover is at its maximum level $\frac{Q}{2}$: x-agents sell all their shares Q to $\sigma + \eta$ agents when there is a positive η draw. Also price volatility is higher in Case 3 than in Case 2 since $P_1(\eta)$ differs more from $P_1(-\eta)$ (as, conditional on the positive shock η, the σ-agents become more optimistic in Case 3 than in Case 2.

In sum, we have shown that when credit is cheap enough then at $t = 0$ only the skewed investors are long the asset and a larger x means a lower chance of a crossing in which $\sigma + \eta$ can be larger than x and hence there will be less expected turnover and price volatility. As such, we have constructed a dynamic equilibrium that fulfills the properties of a quiet bubble: high price but low price volatility and low share turnover.

In particular, we can compare the price volatility and turnover for two equilibria—the one with skewed priors, Case 1, to a symmetric model of dispersed priors. Note that by the symmetric model, we mean here the model where $x = \sigma + \epsilon$ with an arbitrarily small ϵ. This is necessary to think about turnover, as in a perfectly symmetric model where $x = \sigma$, it is indeterminate who hold share at date-0. We can show that price volatility and turnover will be higher in the model with skewed priors than one of symmetric dispersed priors holding fixed the same price level.

Let $\theta > \bar{\theta}$ and $x \in [\sigma+\eta, x_2^1]$. The equilibrium price at $t = 0$ is then given by $2\mu\pi(x) - \frac{Q}{\gamma\beta(x)}$. In particular, it is easily shown that for any $x \in [\sigma + \eta, x_2^1]$, this price is strictly greater than

the price in the symmetric model, i.e. $2\mu\pi(\sigma) - \frac{4Q}{\gamma}$.[6] Let σ' be the dispersion such that: $2\mu\pi(\sigma') - \frac{4Q}{\gamma} = P_0(\sigma, x)$. Then it is easily shown that $\mathbb{T}(\sigma', \sigma') > \mathbb{T}(\sigma, x)$ and $\sigma_P(\sigma', \sigma') > \sigma_P(\sigma, x)$. In other words, in a model with dispersion only, for a given bubble size (i.e. a given price), price volatility and turnover are strictly larger.[7]

We now show that this mechanism crucially relies on credit and that it is not true in general for equity. To see this, what if we had $x - \epsilon$ agents, couldn't they start trading with the x agents? As such, do the results in Proposition 5 depend on this discrete distance assumption between x and $\sigma + \eta$? Here, the upside bounded-ness of debt payoffs emerges as being crucial again for our results. To the extent that the x and $x - \epsilon$ agents own the assets. Since they are extremely optimistic, they view the credit as being risk-free and hence the resale option is much lower for them than for agents who are less optimistic, i.e. the σ agents. For instance, imagine now that $x - \epsilon$ agents get belief shocks by some η amount. But their upside is limited since $\pi(x - \epsilon + \eta)$ and $\pi(x)$ are not very different from each other when x is large relative to D. This is the same debt bounded upside mechanism. As such, there will naturally be less trading in the credit among the extremely optimistic agents and hence trading from the crossing of the $\sigma + \eta$ and x is the key to loudness of the bubble. But leverage but delivering the shares to the x agents hands makes the bubble quieter as a result. This is not true however for equity since there will always be trading between the $x - \epsilon$ and x agents.

In our model, we feature skewed priors x and comparative statics with respect to x as opposed to σ, in contrast to models of loud bubbles, for additionally empirically motivated reasons. First, there is conflicting evidence on whether or not priors or beliefs among investors were dispersed regarding the sub-prime mortgage CDOs. Because these securities are relatively new, there is likely to be dispersed priors regarding their value. Indeed, there is much anecdotal evidence suggesting that there are many investors who are pessimistic

[6] We know that $P_0(x) = 2\mu\pi(x) - \frac{Q}{\gamma\beta(x)}$ is strictly concave and $P_0(x_2^1) > 2\mu\pi(\sigma) - \frac{4Q}{\gamma}$ and $P_0(\sigma + \eta) > 2\mu\pi(\sigma) - \frac{4Q}{\gamma}$

[7] This is essentially comparing Case 1 to Case 6 in the Appendix.

regarding their payoffs, while others, notably many German Banks, are optimistic (see, e.g., Michael Lewis (2010)). In other words, σ in the verbiage of our model may be high as is in the case for the dot-com stocks. Moreover, many previously pessimistic banks such as UBS changed their minds and bought large amounts of these securities even as the market was collapsing in 2007, suggesting that the shock η in our model may also be high.

A standard measure of investor disagreement is the standard deviation of security analysts' forecasts about the performance of a risky asset. Unfortunately, there are no direct measures of security analyst opinions on the performance of sub-prime mortgage CDOs. We look at the dispersion of security analysts' earnings forecasts for the stocks of the financial and homebuilder companies that potentially had exposures to these structured credits instead. There is little difference in the dispersion of beliefs between finance and home builder stocks compared to the rest of the market during the recent sample period. When the financial crisis started, dispersion of analyst opinions for these firms jump compared to the rest of the market. In contrast, the dispersion of analyst forecasts is much higher for dot-com stocks than the rest of the market during the internet bubble period. So one might make a case based on this evidence that perhaps dispersion may not have been too high.

Our model does not rely on taking a particular stand on this issue. It is likely as in all new markets that sub-prime mortgage CDO investors may have had significant divergence of opinion. Our point is that leverage and skew makes the extreme optimists' belief dominate the market and moreover makes the bubble a quiet one.

2.5. Loosening Short-Sales Constraints

Up to this point, our model does not allow for short-selling. We can introduce risk-averse arbitrageurs who can short-sell as in Chen et al. (2002). The effect would be to dampen the over-pricing in the market the more risk-tolerance these arbitrageurs have. A similar effect to loosening short-sales constraints is to increase supply through more issuance of these securities. Indeed, Hong et al. (2006) point out that the re-sale option is highly sensitive to

issuance.

More interestingly, since we consider a debt contract, our model allows for the endogenous determination of the cut-off value D at which point debt does not get paid in full—i.e. credit quality. We can consider short-sellers who can choose what cut-off value of D they would like to short-sell. We can show that the more skewed are the investors, the higher a cut-off D the short-sellers will choose. In other words, in a world of disagreement, investors will pick a larger D so as to best express their disagreement.

3. Evidence from the Sub-prime Mortgage CDO Market

In this section, we provide evidence from the sub-prime mortgage CDO market to support our model. We have already described the evidence on the quietness of credit, especially of the higher-rated tranches of the sub-prime CDOs, compared to equity or the riskier tranches of the CDOs as well as some time series evidence from the figures in the Introduction. These directly support our first mechanism. We now focus on two other key stylized facts to support our model, especially the second quietness mechanism. First, Michael Lewis (2010) provides evidence that AIG-FP, the trading division of AIG responsible for insuring sub-prime CDOs, maybe the skewed x-agent in our model and as a result might have influenced prices and issuance in this market. Second, we provide evidence that short-sales constraints were tightly binding until around 2006 when synthetic mezzanine ABS CDOs allowed hedge funds to short sub-prime CDOs, thereby leading to the implosion of the credit bubble.

We begin with a summary of Michael Lewis (2010)'s extremely detailed timeline of the events surrounding the sub-prime mortgage CDO market's rise between 2003 to 2007 and its implosion in 2008.

The Timeline of Sub-prime Mortgage CDO Bubble

- Before 2005, AIG-FP was de-facto the main optimistic buyer of sub-prime mortgages because it offered extremely cheap insurance for buyers of AAA tranches of these CDOs.

- In early years before 2005, AIG-FP ended up with a $50 billion position in the CDO market, which was one-fourth of the total annual issuance then.

- AIG FP stopped insuring new mortgages after 2005, though they maintained insurance on old ones.

- After 2005, other banks including German Banks and UBS that were previously pessimistic started buying.

- After 2006, shorting becomes possible in the market through synthetic shorts on so-called mezzanine tranches (the worst quality sub-prime names) and the introduction of ABX indices.

- Market begins to collapse in 2007.

Michael Lewis (2010) makes the case, and convincingly from our perspective, that the market for sub-prime mortgage CDOs might not have taken off between 2003 and 2005 without the extremely cheap insurance offered by AIG-FP, which was then the largest insurance company in the world and one of the few companies with a AAA-rating and perceived invulnerable balance sheet. AIG-FP's extremely low insurance to companies like Goldman who underwrote the mortgages effectively makes them the optimistically skewed x-agent in our model since AIG naturally had access to extreme leverage by virtue of them being able to write insurance contracts.

For instance, Michael Lewis (2010) writes, "Stage Two, beginning at the end of 2004, was to replace the student loans and the auto loans and the rest with bigger piles consisting of nothing but U.S. subprime mortgage loans...The consumer loan piles that Wall Street firms, led by Goldman Sachs, asked AIG-FP to insure went from being 2 percent subprime

mortgages to being 95 percent subprime mortgages. In a matter of months, AIG-FP in effect bought $50 billion in triple B rated subprime mortgage by insuring them against default."

Why did AIG-FP take on such a large position? They did not think home prices could fall—i.e. our skewed priors assumption. Michael Lewis (2010) writes, "Confronted with the new fact that his company was effectively long $50 billion in triple-B rated subprime mortgage bonds, masquerading as triple A-rated diversified pools of consumer loans—Cassano at first sought to rationalize it. He clearly thought that any money he received for selling default insurance on highly rated bonds was free money. For the bonds to default, he now said, U.S. home prices had to fall and Joe Cassano didn't believe house prices could ever fall everywhere in the country at once. After all, Moody's and S&P had rated this stuff triple-A!" Indeed, AIG FP continued to keep their insurance contracts even after 2005. In sum, anecdotal evidence very directly points to the central role of skewed priors and leverage in influencing the sub-prime mortgage CDO bubble.

In Figure 6, we plot a figure shown from Stein (2010), who argues that the huge growth in issuance in the non-traditional CDO market was an important sign that a bubble might have taken hold here compared to traditional structured products. Indeed, this plot of issuance activity between 2000 and 2009 shows that activity really jumped in 2004 when AIG begins to insure significant amounts of the the sub-prime names as described in Michael Lewis (2010).

In Figure 7, we plot issuance of synthetic mezzanine ABS CDOs which is how hedge funds such as John Paulson's finally were able to short the sub-prime mortgage CDOs. Notice that there was very little shorting in this market until the end as issuances of this type of CDO are not sizeable until 2006. In other words, short-sales constraints were binding tightly until around 2007, consistent with the premise of our model. The collapse coincided with a large supply of these securities in 2007, similar to what happened during the dot-com period.

4. Narratives of the Financial Crisis

Finally, our theory complements recent academic work on the financial crisis. An important narrative is that a wave of money from China needed a safe place to park and in light of the lack of sovereign debt, Wall Street created AAA securities as a new parking place for this money. An unexpected fall in home prices made these securities toxic. A model which delivers such a story of neglected risks and financial innovation is Gennaioli et al. (2010). Our narrative complements theirs in showing that a quiet bubble took hold in this midst of this safe asset creation. Empirical evidence pointing to certain finance companies doing better than others (e.g. JP Morgan versus Lehman or AIG) along with the very high prices of credit are hard to reconcile entirely with the safe parking place narrative. Another narrative is that of agency and risk-shifting as in Allen and Gale (2000), which can also deliver asset price bubbles and would seem vindicated by the bailouts of the big banks. Our model complements this analysis in pointing out how leverage might have led to low price volatility and turnover which made the credit bubble harder to detect.

One issue with these explanations is that in reading accounts of the financial crisis, there is a unifying theme of how so many, from regulators to very sophisticated professionals, could have missed the credit bubble building up in the system. Indeed, this is an especially large conundrum when one considers that sophisticated finance companies such as Goldman Sachs were able to survive and indeed thrive through many prior speculative episodes, including the the dot-com boom and bust. So why did companies like Goldman Sachs get caught this time?

Our analysis suggests that the reason everyone missed it was that this credit bubble, in contrast to most speculative episodes, was quiet. Most sophisticated finance firms have elaborate risk management systems, such as relying on Value-at-Risk models, that automatically rein in risk-taking when volatility of their positions rise. Since the credit bubble was quiet, their risk management systems did not ring alarm bells until it was too late. Hence, our model provides an explanation to the conundrum found in many accounts of why everyone

missed it this time.

Our analysis also has implications for thinking about the Dodd-Frank financial reform package. This legislation establishes, among many things, an Office of Financial Research and a Financial Stability Oversight Committee that are meant to detect the next bubble. The premise of the Office of Financial Research is that by forcing finance firms to disclose their positions, regulators will be in a better position to detect the next speculative episode. Our model suggests that the quietness of the credit bubble would have made it difficult to detect regardless. Hence, regulators should distinguish between loud versus quiet bubbles as the signs are very different.

5. Conclusion

With the onset of the financial crisis, the term "bubble" is being used, from our perspective too liberally, to describe any type of potential mis-pricings in the market ranging from equities to housing and credit. The term speculative bubble has traditionally been used to connote the idea that investors purchase an asset knowing that the price is high because they anticipate capital gains. The classic speculative episodes such as the recent dot-com bubble fit this definition and usually come with high price, high price volatility and high turnover. We argue that the credit bubble in sub-prime mortgage CDOs is fundamentally different as it was more quiet—price is high but price volatility and turnover are low.

We offer a theory that generates a taxonomy of bubbles that distinguishes between loud equity bubbles and quiet credit bubbles. This theory builds on the platform of disagreement and short-sales constraints that are key to getting loud bubbles. We show that credit bubbles are quieter than equity ones for two distinct reasons. The first is that the up-side boundedness of debt pay-offs means debt instruments (especially higher rated ones) are less disagreement-sensitive than lower rated credit or equity. As a result, a bubbles is characterized by lower price volatility and turnover. The second is that sufficiently skewed priors and leverage can

generate a quiet bubble. Leverage also makes credit bubbles quiet because it allows extreme skewed optimists to buy all the shares and there is little trading and price volatility as the likelihood of other investors having a more optimistic belief than these skewed optimists are low. Our analysis hence suggests naturally that debt is more susceptible to quiet bubbles compared to equity. Future work elaborating on this taxonomy and providing other historical evidence would be very valuable.

A. Appendix

Proof of Theorem 1 and Propositions 1 and 2.

Consider first the case where only agents with belief x are long the asset. The market clearing condition is then:

$$\beta(x)\gamma\left(\mu\pi(x) - P\right) = Q. \tag{24}$$

So that the equilibrium price is given by:

$$P_1(x) = \mu\pi(x) - \frac{Q}{\gamma\beta(x)} = \mu\pi(x) - \frac{2Q}{\gamma}\left(1 + \frac{x^2}{\sigma^2}\right) \tag{25}$$

Notice that P_1 is concave in x provided that π is concave. This is an equilibrium if and only if $P_1(x) \geq \mu\pi(\sigma)$ (otherwise the agents with belief σ also want to be long the asset). Call $\tilde{x}$ such that: $\mu\pi'(\tilde{x}) = \frac{4Q\tilde{x}}{\gamma\sigma^2}$, i.e. $P_1'(\tilde{x}) = 0$. There exists an $x \in \mathbb{R}_+$ such that $P_1(x) \geq \mu\pi(\sigma)$ if and only if $P_1(\tilde{x}) \geq \mu\pi(\sigma)$.

This last condition can be rewritten:

$$\pi(\tilde{x}) - \pi(\sigma) - \frac{2Q}{\mu\gamma}\left(1 + \frac{\tilde{x}^2}{\sigma^2}\right) \geq 0 \Leftrightarrow \psi(\tilde{x}(\gamma\mu), \gamma\mu) \geq 0$$

By definition of $\tilde{x}$, $\frac{\partial\psi}{\partial\tilde{x}} = 0$. Therefore: $\frac{d\psi}{d(\gamma\mu)} = \frac{\partial\psi}{\partial(\gamma\mu)} = \frac{2Q}{(\gamma\mu)^2}\left(1 + \frac{\tilde{x}^2}{\sigma^2}\right) > 0$

Thus there exists a unique $\bar{\theta} > 0$ such that for $\gamma\mu > \bar{\theta}$, $P_1(\tilde{x}) > \mu\pi(\sigma)$ and $P_1(\tilde{x}) = \mu\pi(\sigma)$ for $\gamma\mu = \bar{\theta}$.

We want to show that for $\gamma\mu \geq \bar{\theta}$, $\tilde{x} > \sigma$. Assume this is not the case, i.e. $\tilde{x} \leq \sigma$. $\tilde{x}$ is a strictly increasing function of $\gamma\mu$ as π is concave and $\lim_{\gamma\mu\to\infty}\tilde{x} = \infty$. Thus, there is $\hat{l} > \bar{\theta}$ such that if $\gamma\mu = \hat{l}$ then $\tilde{x} = \sigma$. But $P_1(\sigma) < \mu\pi(\sigma)$ and the definition of $\bar{\theta}$ is that for $\gamma > \bar{\theta}$, $P_1(\tilde{x}) > \mu\pi(\sigma)$. A contradiction. Thus, for $\gamma\mu \geq \bar{\theta}$, $\tilde{x} \geq \sigma$.

Thus, for $\gamma\mu > \bar{\theta}$, there exists $\sigma < \underline{x} < \bar{x}$ such that $P_1(x) > \mu\pi(\sigma)$ if and only if $x \in]\underline{x}, \bar{x}[$ and P_1 is strictly increasing over $]\underline{x}, \tilde{x}[$ and strictly decreasing over $]\tilde{x}, \bar{x}[$.

For $\gamma\mu > \bar{\theta}$, $\underline{x}$ is defined as: $P_1(x) = \mu\pi(\sigma)$ and $P_1'(x) > 0$. $\bar{x}$ is defined as: $P_1(x) = \mu\pi(\sigma)$ and $P_1'(x) < 0$. $\underline{x}$ is strictly decreasing with $\gamma\mu$:

$$\frac{\partial\underline{x}}{\partial(\gamma\mu)}\underbrace{\frac{\partial P_1}{\partial x}}_{>0} + \underbrace{\frac{\partial P_1}{\partial(\gamma\mu)}}_{>0} = 0.$$

Similarly, $\bar{x}$ is strictly increasing with $\gamma\mu$.

When $\gamma\mu > \bar{\theta}$ but $x \in [\sigma, \underline{x}[\cup]\bar{x}, \infty[$, then the equilibrium entails both σ agents and x agents long the asset. The market clearing condition is then simply:

$$\frac{1}{4}\gamma\left(\mu\pi(\sigma)-P\right)+\beta(x)\gamma\left(\mu\pi(x)-P\right)=Q. \tag{26}$$

And the equilibrium price is simply:

$$P_2(x)=\frac{\frac{1}{4}\mu\pi(\sigma)+\beta(x)\mu\pi(x)-\frac{Q}{\gamma}}{\frac{1}{4}+\beta(x)} \tag{27}$$

Notice that thanks to the assumption that $\mu \geq \frac{4Q}{\gamma(\pi(\sigma)-\pi(0))}$, $P_2 \geq \mu\pi(0)$. Indeed:

$$P_2(x) \geq \mu\pi(\sigma)-\frac{4Q}{\gamma} \geq \mu\pi(0).$$

As a consequence: $P_2 \geq \mu\pi(-y)$. Similarly, it is easily shown that for $x \in [\sigma, \underline{x}[\cup[\bar{x}, \infty[$, $P_2 < \mu\pi(\sigma)$. Assume this is not the case, $P_2 \geq \mu\pi(\sigma)$. Notice that: $P_2 = P_1 + \frac{1}{4}\frac{\mu\pi(\sigma)-P_2}{\beta(x)}$, so that $P_2 \leq P_1$. But we know that for $x \in [\sigma, \underline{x}[\cup[\bar{x}, \infty[$, $P_1(x) < \mu\pi(\sigma)$ so that $P_2(x) < \mu\pi(\sigma)$, a contradiction. Thus $P_2 < \mu\pi(\sigma)$ and the equilibrium over $[\sigma, \underline{x}[\cup[\bar{x}, \infty[$ is indeed defined by x and σ agents being long the asset and P_2 being the equilibrium price.

We show that P_2 is concave.

$$P_2'(x)=\frac{\mu(\beta(x)\pi'(x)+\beta'(x)\pi(x))-\beta'(x)P_2(x)}{\frac{1}{4}+\beta(x)}$$

$P_2'(x) > 0 \Leftrightarrow P_2(x) > \mu(\frac{\beta(x)}{\beta'(x)}\pi'(x)+\beta'(x)\pi(x))$
Call $\psi(x) = \frac{\beta(x)}{\beta'(x)}\pi'(x)+\pi(x) = \pi(x) - \frac{1}{2}\left(x+\frac{\sigma^2}{x}\right)\pi'(x)$
$\psi'(x) = \frac{1}{2}\left(1+\frac{\sigma^2}{x^2}\right)\pi'(x) - \frac{1}{2}\left(x+\frac{\sigma^2}{x}\right)\pi''(x) > 0$
Thus: $P_2'(x) > 0 \Leftrightarrow P_2(x) > \mu\psi(x)$ where ψ is strictly increasing.
We now use the fact that: $P_2 = P_1 + \frac{1}{4}\frac{\mu\pi(\sigma)-P_2}{\beta(x)}$. We have that:

$$\left(1+\frac{1}{4\beta(x)}\right)P_2' = P_1' - \frac{\beta'(x)}{4(\beta(x))^2}\left(\mu\pi(\sigma)-P_2\right)$$

Because $P_2(\bar{x}) = \mu\pi(\sigma)$ and $P_1'(\bar{x}) < 0$, we have that $P_2'(\bar{x}) < 0$. Similarly, $P_2'(\underline{x}) > 0$.

Thus, there is $\hat{x} \in [\underline{x}, \bar{x}]$ such that $P_2'(\hat{x}) = 0$. There can be only one x such that $P_2'(x) = 0$. Indeed, assume that $\exists x > \hat{x}$ such that $P_2'(x) = 0$. Call $\hat{x}'$ the smallest such x. In the right vicinity of $\hat{x}$, $P_2 < \psi(x)$ (as ψ is strictly increasing and P_2 derivative is 0 at $\hat{x}$). Thus P_2 is strictly decreasing in the right vicinity of $\hat{x}$ and $P_2'(x) < 0$ for all $x \in]\hat{x}, \hat{x}'[$. Thus $P_2(\hat{x}) > P_2(\hat{x}')$. But $P_2(\hat{x}) = \psi(\hat{x}) < \psi(\hat{x}') = P_2(\hat{x}')$. A contradiction. As a consequence P_2 is concave, strictly increasing over $[\sigma, \underline{x}[$ and strictly decreasing over $[\bar{x}, \infty[$.

Consider finally the case where $\mu\gamma < \bar{\theta}$. The equilibrium price is then P_2 for all $x \geq \sigma$. We still

have that: $P_2'(x) > 0 \Leftrightarrow P_2(x) > \mu\psi(x)$. FIrst, we know that because π is concave, for all $x_0 \geq \sigma$, $\pi(x) \leq \pi'(x_0)(x - x_0) + \pi(x_0)$. Thus:

$$\forall x \geq x_0, \quad 0 \leq \frac{\pi(x)}{x^2} \leq \frac{\pi'(x_0)(x - x_0)}{x^2} + \frac{\pi(x_0)}{x^2} \rightarrow_{x\rightarrow\infty} 0$$

As a consequence: $\lim_{x\rightarrow\infty} \beta(x)\pi(x) = 0$ and $\lim_{x\rightarrow\infty} P_2(x) = \mu\pi(\sigma) - \frac{4Q}{\gamma}$.

π is increasing and concave, and it thus admits a limit $\bar{\pi}$ when $x \rightarrow \infty$.

Consider first the case where $\bar{\pi} = \infty$. Because π is concave, $\psi(x) \geq \frac{\pi(x)}{2} - \frac{\sigma^2}{2x}\pi'(x)$. But: $0 \leq \frac{\pi'(x)}{x} \leq \frac{\pi(x)}{x^2} \rightarrow_{x\rightarrow\infty} 0$, so that: $\psi(x) \rightarrow \infty$ when $x \rightarrow \infty$.

Consider now the case where $\bar{\pi} < \infty$. This implies that $\lim_{x\rightarrow\infty} \pi'(x) = 0$ (as $0 \leq \pi'(x) \leq \pi(x)/x$). Using again the concavity of π:

$$\forall x \geq \sigma, \quad x\pi'(x) \leq \pi(x) - \pi(\sigma) + \sigma\pi'(x)$$

So that: $\psi(x) \geq \pi(x) - \frac{1}{2}\left(\pi(x) - \pi(\sigma) + \sigma\pi'(x)\right) - \frac{\sigma^2}{2x}\pi'(x)$. ψ is an increasing, bounded and positive function, so it admits a finite positive limits and $\lim_{x\rightarrow\infty} \psi(x) \geq \frac{\bar{\pi}+\pi(\sigma)}{2} \geq \pi(\sigma)$.

Thus, in both cases (π has a finite or infinite limit), $\lim_{x\rightarrow\infty} \mu\psi(x) \geq \mu\pi(\sigma) > \lim x \rightarrow \infty P_2(x)$. Thus, P_2 is strictly decreasing when $x \rightarrow \infty$.

Two cases arise:

1. $P_2(\sigma) > \mu\psi(\sigma) \Leftrightarrow \mu\sigma\pi'(\sigma) > \frac{2Q}{\gamma}$. In that case, P_2 is increasing and then strictly decreasing with x. In particular, there exists a unique $\tilde{x} > \sigma$ such that $P_2'(x) = 0$. (By Rolle's Theorem, there exists at least one $x > \sigma$ with $P_2'(x) = 0$. Call $\tilde{x}$ the smallest such x. In the right vicinity of $\tilde{x}$, ψ is strictly decreasing. If there is another $\tilde{x}'$ with $P_2'(\tilde{x}') = 0$, then P_2 is strictly decreasing over $]\tilde{x}, \tilde{x}'[$ but $P_2(\tilde{x}) = \psi(\tilde{x}) < \psi(\tilde{x}') = P_2(\tilde{x}')$ which is a contradiction.

2. $\mu\sigma\pi'(\sigma) \leq \frac{2Q}{\gamma}$. Then P_2 is strictly decreasing with x. (simply apply the previous proof to show there cannot be one x with $P_2'(x) > 0$.)

Statements in Propositions 1 and 2 follow easily from these properties of P_1 and P_2. QED

Proof of Theorem 2 and Proposition 5.

Analysis of date-1 equilibrium: We first analyze the date-1 equilibrium. Let $n_0(V)$ the shares held by an investor with initial belief V at $t = 0$.The investor with belief V's program at $t = 1$ is simply:

$$\max_{n_1} n_1\pi(V) - \frac{1}{\mu}\left((n_1 - n_0)P_1 + \frac{(n_1 - n_0)^2}{2\gamma}\right) \tag{28}$$

The solution to the previous program is simply:

$$n_1 = n_0 + \gamma\left(\mu\pi(V) - P_1\right) \text{ if } n_1 \geq 0 \text{ and 0 else.} \tag{29}$$

Note: $\phi(x,\tilde{\eta}) = \frac{\pi(x)-\pi(\sigma+\tilde{\eta})}{\beta(x)+\frac{1}{4}}$, with $\tilde{\eta} \in \{-\eta, \eta\}$. We also define $P_1(x,\tilde{\eta})$ to be the equilibrium price at date 1 for skewness x and with a belief shock to the σ agent $\tilde{\eta} \in \{-\eta, \eta\}$.

At date 1, three cases arise:

1. Both investors are long (i.e. $n_1^x > 0$ and $n_1^{\sigma+\tilde{\eta}} > 0$). This implies that the demand of both investors is given by:

$$\begin{cases} n_1^x(\tilde{\eta}) = n_0^x + \gamma\left(\mu\pi(x) - P_1\right) \\ n_1^\sigma(\tilde{\eta}) = n_0^\sigma + \gamma\left(\mu\pi(\sigma+\tilde{\eta}) - P_1\right) \end{cases}$$

The date-1 market clearing condition imposes that:

$$\beta(x)n_1^x(\tilde{\eta}) + \frac{1}{4}n_1^\sigma(\tilde{\eta}) = Q$$

But the date-0 market clearing condition is: $\beta(x)n_0^x + \frac{1}{4}n_0^\sigma = Q$, so that the date-1 market clearing condition becomes:

$$P_1(x,\tilde{\eta}) = \mu\frac{\beta(x)\pi(x) + \frac{\pi(\sigma+\tilde{\eta})}{4}}{\beta(x)+\frac{1}{4}}$$

Thus, the date-1 equilibrium has both agents long if and only if:

$$-\frac{4n_0^x}{\gamma\mu} < \phi(x,\tilde{\eta}) < \frac{n_0^\sigma}{\gamma\mu\beta(x)}$$

2. Only the x investors are long (i.e. $n_1^x(\tilde{\eta}) > 0$ and $n_1^\sigma(\tilde{\eta}) = 0$).

This implies that the demand of investors is simply:

$$\begin{cases} n_1^x(\tilde{\eta}) = n_0^x + \gamma\left(\mu\pi(x) - P_1\right) \\ n_1^\sigma(\tilde{\eta}) = 0 \end{cases}$$

The date-1 market clearing condition is just:

$$\beta(x)n_1^x(\tilde{\eta}) = Q = \beta(x)n_0^x + \gamma\beta(x)\left(\mu\pi(x) - P_1\right) = Q - \frac{n_0^\sigma}{4} + \gamma\beta(x)\left(\mu\pi(x) - P_1\right),$$

where the third equality comes from the date-0 market clearing condition.

This provides the date-1 equilibrium price: $P_1(x,\tilde{\eta}) = \mu\pi(x) - \frac{n_0^\sigma}{4\gamma\beta(x)}$.

The equilibrium has only x agents long iff the demand of σ agents is indeed 0, i.e.:

$$n_0^\sigma + \gamma\left(\mu\pi(\sigma+\tilde{\eta}) - P_1(x,\tilde{\eta})\right) < 0 \Leftrightarrow \phi(x,\tilde{\eta}) > \frac{n_0^\sigma}{\gamma\mu\beta(x)}$$

3. Only the $\sigma+\tilde{\eta}$ investors are long (i.e. $n_1^x(\tilde{\eta}) = 0$ and $n_1^\sigma(\tilde{\eta}) > 0$).

   This implies that the demand of investors is simply:

$$\begin{cases} n_1^x(\tilde{\eta}) = 0 \\ n_1^\sigma(\tilde{\eta}) = n_0^\sigma + \gamma\left(\mu\pi(\sigma+\tilde{\eta}) - P_1\right) \end{cases}$$

   The date-1 market clearing condition is just:

$$\frac{n_1^\sigma(\tilde{\eta})}{4} = Q = \frac{n_0^\sigma}{4} + \frac{\gamma}{4}\left(\mu\pi(\sigma+\tilde{\eta}) - P_1\right) = Q - \beta(x)n_0^x + \frac{\gamma}{4}\left(\mu\pi(\sigma+\tilde{\eta}) - P_1\right),$$

   where the third equality comes from the date-0 market clearing condition.

   This provides the date-1 equilibrium price: $P_1(x,\tilde{\eta}) = \mu\pi(\sigma+\tilde{\eta}) - \frac{\beta(x)n_0^x}{\frac{\gamma}{4}}$.

   The equilibrium has only x agents long iff the demand of x agents is indeed 0, i.e.:

$$n_0^x + \gamma\left(\mu\pi(x) - P_1(x,\tilde{\eta})\right) < 0 \Leftrightarrow \phi(x,\tilde{\eta}) < -\frac{4n_0^x}{\gamma\mu}$$

Analysis of date-0 equilibrium:

We now turn to the date-0 investment problem. The date-0 program of the σ investor is given by:

$$\max_{n_0^\sigma} n_0^\sigma\pi(\sigma) - \frac{1}{\mu}\left(n_0^\sigma P_0 + \frac{(n_0^\sigma)^2}{2\gamma}\right) + \mathbb{E}_{\tilde{\eta}}\left[\max_{n_1} n_1\pi(\sigma+\tilde{\eta}) - \frac{1}{\mu}\left((n_1 - n_0^\sigma)P_1(x,\tilde{\eta}) + \frac{(n_1-n_0^\sigma)^2}{2\gamma}\right)\right]$$

Similarly, the date-0 program of the x investor is given by:

$$\max_{n_0^x} n_0^x\pi(x) - \frac{1}{\mu}\left(n_0^x P_0 + \frac{(n_0^x)^2}{2\gamma}\right) + \mathbb{E}_{\tilde{\eta}}\left[\max_{n_1} n_1\pi(x) - \frac{1}{\mu}\left((n_1 - n_0^x)P_1(x,\tilde{\eta}) + \frac{(n_1-n_0^x)^2}{2\gamma}\right)\right]$$

We now consider the various possible cases and find the conditions under which they exist.

1. We first consider Case 1 where only the x agents are long, both at date 0 and 1, and irrespective of the $\tilde{\eta}$ shock. Formally, this is defined by $n_0^\sigma = 0$, $n_1^\sigma(-\eta) = 0$ and $n_1^\sigma(+\eta) = 0$.

   Because $x \geq \sigma > \sigma - \eta$, $\phi(x, -\eta) > 0 = \frac{n_0^\sigma}{\gamma\mu\beta(x)}$ so that only the x investors can be long at date 1 following the negative shock. Formally, $n_0^\sigma = 0 \Rightarrow n_1^\sigma(-\eta) = 0$.

   If the σ agent is not long at date-1, then: $n_1^\sigma(+\eta) = 0 \Leftrightarrow \phi(x, \eta) > 0 \Leftrightarrow x > \sigma + \eta$.

   Because at date-1, the σ agent would be reselling any shares acquired at date-0, her date-0 program is simply:

$$\max_{n_0^\sigma} n_0^\sigma \pi(\sigma) - \frac{1}{\mu}\left(n_0^\sigma P_0 + \frac{(n_0^\sigma)^2}{2\gamma}\right) + \frac{1}{\mu}\left(n_0^\sigma \mathbb{E}\left[P_1(x, \tilde{\eta})\right] - \frac{(n_0^\sigma)^2}{2\gamma}\right)$$

   So that her optimal demand at date-0 is simply: $n_0^\sigma = \max\left(0, \frac{\gamma}{2}\left(\mu\pi(\sigma) + \mathbb{E}\left[P_1(x, \tilde{\eta})\right] - P_0\right)\right)$ We now from the analysis above that in this equilibrium, the date-1 price are simply given by: $P_1(x, \eta) = P_1(x, -\eta) = \mu\pi(x)$. Thus the condition for the σ investors not to be long at date-0 is simply: $P_0 > \mu\left(\pi(\sigma) + \pi(x)\right)$.

   The date-0 program of the x agents is given by:

$$\max_{n_0^x} n_0^x \pi(x) - \frac{1}{\mu}\left(n_0^x P_0 + \frac{(n_0^x)^2}{2\gamma}\right) + \mathbb{E}_{\tilde{\eta}}\left[\max_{n_1} n_1 \pi(x) - \frac{1}{\mu}\left((n_1 - n_0^x)P_1(x, \tilde{\eta}) + \frac{(n_1 - n_0^x)^2}{2\gamma}\right)\right]$$

   Because in this equilibrium the x agents are long at date-1 for any $\tilde{\eta}$, we can simply apply the envelop theorem to their date-1 utility in the previous program and find the following date-0 first order condition:

$$n_0^x - \frac{1}{\mu}\left(P_0 + \frac{n_0^x}{\gamma}\right) + \frac{1}{\mu}\mathbb{E}_{\tilde{\eta}}\left[P_1(x, \tilde{\eta}) + \frac{n_1^x(\tilde{\eta}) - n_0^x}{\gamma}\right] = 0$$

   The analysis above showed that in the date-1 equilibrium where only the x agents are long, $n_1^x(\tilde{\eta}) = n_0^x + \gamma\left(\mu\pi(x) - P_1(x, \tilde{\eta})\right)$, so that in this case: $n_1^x(\tilde{\eta}) - n_0^x = 0$.

   This eventually leads to the following first order condition for the x investor:

$$n_0^x = \gamma\left(2\mu\pi(x) - P_0\right)$$

   The date-0 market clearing condition imposes that:

$$P_0 = 2\mu\pi(x) - \frac{Q}{\gamma\beta(x)}$$

Thus, the condition for the σ investors to stay out of the market at date 0 is simply that $n_0^\sigma = 0$, which can be written as:

$$\mu\left(\pi(x) - \pi(\sigma)\right) > \frac{Q}{\gamma\beta(x)}$$

Eventually, noting $\psi^1(x) = \pi(x) - \pi(\sigma) - \frac{Q}{\gamma\mu\beta(x)}$, this equilibrium exists if and only if:

$$\begin{cases} \pi(x) > \pi(\sigma + \eta) \\ \psi^1(x) > 0 \end{cases}$$

Note that $\psi^1(x) = \pi(x) - \pi(\sigma) - \frac{Q}{\gamma\mu\beta(x)}$ is a strictly concave function of x.

In this equilibrium, expected turnover is $\mathbb{T} = 0$ as the x agents buy all the shares at date 0 and keep them. Price volatility, which is defined here as $\sigma_P = |P_1(\eta) - P_1(-\eta)|$ is also equal to 0, as the date-1 price is in both cases equal to $\mu\pi(x)$.

2. We now consider Case 2where only the x investors are long at date-0, only the x investors are long at date-1 following a $-\eta$ shock and both investors are long following a $+\eta$ shock. Formally, this equilibrium is defined as: $n_0^\sigma = 0$, $n_0^x > 0$, $n_1^\sigma(-\eta) = 0$, $n_1^x(-\eta) > 0$, $n_1^\sigma(+\eta) > 0$ and $n_1^x(+\eta) > 0$.

   As in the previous case, we have that $x \geq \sigma > \sigma - \eta$, so that $\phi(x, -\eta) > 0 = \frac{n_0^\sigma}{\gamma\mu\beta(x)}$, i.e. $n_0^\sigma = 0 \Rightarrow n_1^\sigma(-\eta) = 0$. Again, if the σ agents are not long at date-0, only the x agents can be long at date-1 when $\tilde{\eta} = -\eta$.

   If both agents are long at date-1 when $\tilde{\eta} = +\eta$ (i.e. $n_1^\sigma(+\eta) > 0$ and $n_1^x(+\eta) > 0$), our analysis of the date-1 equilibrium implies that $0 > \phi(x, \eta) > -\frac{4Q}{\gamma\mu\beta(x)}$ (as $\beta(x)n_0^x = Q$). These two conditions can simply be rewritten as:

$$x < \sigma + \eta \text{ and } \pi(x) > \pi(\sigma + \eta) - \frac{\beta(x) + \frac{1}{4}}{\beta(x)}\frac{4Q}{\gamma\mu}$$

   We now turn to the condition under which the σ agents do not want to be long at date-0. In this equilibrium, their date-0 program is simply:

$$\begin{aligned} &\max_{n_0^\sigma} n_0^\sigma \pi(\sigma) - \tfrac{1}{\mu}\left(n_0^\sigma P_0 + \tfrac{(n_0^\sigma)^2}{2\gamma}\right) + \tfrac{1}{2\mu}\left(n_0^\sigma P_1(x, -\eta) - \tfrac{(n_0^\sigma)^2}{2\gamma}\right) \\ &+\tfrac{1}{2}\left(\max_{n_1}\left\{n_1\pi(\sigma + \eta) - \tfrac{1}{\mu}\left((n_1 - n_0^\sigma)P_1(x, \eta) + \tfrac{(n_1 - n_0^\sigma)^2}{2\gamma}\right)\right\}\right) \end{aligned}$$

As before, we notice, using the envelop theorem to the date-1 program that admits an interior solution, that:

$$\frac{\partial\left(\max_{n_1}\left\{n_1\pi(\sigma+\eta)-\frac{1}{\mu}\left((n_1-n_0^\sigma)P_1(x,\eta)+\frac{(n_1-n_0^\sigma)^2}{2\gamma}\right)\right\}\right)}{\partial n_0^\sigma}=\pi(\sigma+\eta)$$

Thus, the date 0 program of the σ investor, if it has an interior solution, admits the following first order condition:

$$0=\pi(\sigma)-\frac{1}{\mu}\left(P_0+\frac{n_0^\sigma}{\gamma}\right)+\frac{1}{2}\left(P_1(x,-\eta)-\frac{n_0^\sigma}{\gamma}\right)+\frac{1}{2}\mu\pi(\sigma+\eta)$$

So that their date-0 holdings are defined by:

$$\begin{aligned}n_0^\sigma &= \max\left(0,\frac{2\gamma}{3}\left(\mu\pi(\sigma)+\frac{1}{2}\left(P_1(x,-\eta)+\mu\pi(\sigma+\eta)\right)-P_0\right)\right)\\ &= \max\left(0,\frac{2\gamma}{3}\left(\mu\pi(\sigma)+\frac{1}{2}\left(\mu\pi(x)+\mu\pi(\sigma+\eta)\right)-P_0\right)\right)\end{aligned}$$

Similar to the analysis developped in the first case, the date 0 program of the x investor admits the following first order condition:

$$n_0^x=\gamma\left(2\mu\pi(x)-P_0\right)$$

The date-0 market clearing condition imposes that:

$$P_0=2\mu\pi(x)-\frac{Q}{\gamma\beta(x)}$$

Eventually, the condition for the σ investors to stay out of the market at date 0 is simply that their date-0 holding be non-positive, i.e.:

$$\mu\left(\frac{3}{2}\pi(x)-\pi(\sigma)-\frac{1}{2}\pi(\sigma+\eta)\right)>\frac{Q}{\gamma\beta(x)}$$

Noting $\psi^2(x)=\pi(x)-\pi(\sigma+\eta)+\frac{\beta(x)+\frac{1}{4}}{\beta(x)}\frac{4Q}{\gamma\mu}$ and $\psi^3(x)=\pi(x)-\frac{2}{3}\pi(\sigma)-\frac{1}{3}\pi(\sigma+\eta)-\frac{2}{3}\frac{Q}{\gamma\mu\beta(x)}$, we see that this equilibrium exists if and only if:

$$\begin{cases}x<\sigma+\eta\\ \psi^2(x)>0\\ \psi^3(x)>0\end{cases}$$

Note that ψ^2 is a strictly increasing function of x and ψ^3 is a strictly concave function of x.

In this equilibrium, expected turnover is

$$\mathbb{T} = \frac{1}{2} \times \frac{1}{4} n_1^\sigma(+\eta) = \frac{\mu\gamma}{8} \frac{\beta(x)}{\beta(x) + \frac{1}{4}} \left(\pi(\sigma + \eta) - \pi(x)\right)$$

Price volatility is simply:

$$\sigma_P = |P_1(\eta) - P_1(-\eta)| = \frac{\mu}{4} \frac{\pi(\sigma + \eta) - \pi(x)}{\beta(x) + \frac{1}{4}}$$

3. The third case, Case 3, we consider is one where only the x agents are long at date 0, only the x agents are long at date 1 when $\tilde{\eta} = -\eta$ and only the σ agents are long at date 1 when $\tilde{\eta} = \eta$. Formally: $n_0^\sigma = 0$, $n_0^x > 0$, $n_1^\sigma(-\eta) = 0$, $n_1^x(-\eta) > 0$, $n_1^\sigma(+\eta) > 0$ and $n_1^x(+\eta) = 0$.

   As in the last two previous case, we have that $x \geq \sigma > \sigma - \eta$, so that $\phi(x, -\eta) > 0 = \frac{n_0^\sigma}{\gamma\mu\beta(x)}$, i.e. $n_0^\sigma = 0 \Rightarrow n_1^\sigma(-\eta) = 0$. Again, if the σ agents are not long at date-0, only the x agents can be long at date-1 when $\tilde{\eta} = -\eta$.

   We know from our analysis of the date-1 equilibrium that $n_1^\sigma(+\eta) > 0$ and $n_1^x(+\eta) = 0 \Leftrightarrow \phi(x, \eta) < -\frac{4Q}{\gamma\mu\beta(x)} \Leftrightarrow \pi(x) < \pi(\sigma + \eta) - \frac{\beta(x) + \frac{1}{4}}{\beta(x)} \frac{4Q}{\gamma\mu} \Leftrightarrow \psi^2(x) < 0$.

   We now turn to the condition under which the σ agents do not want to be long at date-0. The first order condition of their date-0 maximization problem is similar to the previous case (their date-1 utility maximization has an interior solution for $\tilde{\eta} = \eta$ and they resell all their shares if $\tilde{\eta} = -\eta$. Because in this equilibrium, the date-1 price conditional on $\tilde{\eta} = -\eta$ is also: $P_1(x, -\eta) = \mu\pi(x)$, the date-0 demand for shares of σ agents is exactly similar to that of the previous case, i.e.:

$$n_0^\sigma = \max\left(0, \frac{2\gamma}{3}\left(\mu\pi(\sigma) + \frac{1}{2}\left(\mu\pi(x) + \mu\pi(\sigma + \eta)\right) - P_0\right)\right)$$

   The date 0 program of the x investor can be rewritten as:

$$\begin{array}{l} \max_{n_0^x} n_0^x \pi(x) - \frac{1}{\mu}\left(n_0^x P_0 + \frac{(n_0^x)^2}{2\gamma}\right) + \frac{1}{2}\frac{1}{\mu}\left(n_0^x P_1(x, \eta) - \frac{(n_0^x)^2}{2\gamma}\right) \\ + \frac{1}{2}\left(\max_{n_1}\left\{n_1\pi(x) - \frac{1}{\mu}\left((n_1 - n_0^x)P_1(x, -\eta) + \frac{(n_1 - n_0^x)^2}{2\gamma}\right)\right\}\right) \end{array}$$

   It is again easy to see, using the envelop theorem, that:

$$\frac{\partial\left(\max_{n_1}\left\{n_1\pi(x) - \frac{1}{\mu}\left((n_1 - n_0^x)P_1(x, -\eta) + \frac{(n_1 - n_0^x)^2}{2\gamma}\right)\right\}\right)}{\partial n_0^x} = \pi(x)$$

Using the fact that in this equilibrium, $P_1(x,\eta) = \pi(\sigma+\eta) - \frac{4Q}{\gamma}$ (see the analysis of the date-1 equilibrium), the date-0 program of the x investors admit the following first-order condition:

$$n_0^x = \gamma\left(\mu\pi(x) + \frac{\mu}{3}\pi(\sigma+\eta) - \frac{4Q}{3\gamma} - \frac{2}{3}P_0\right)$$

The date-0 market clearing condition imposes that:

$$P_0 = \frac{3}{2}\mu\pi(x) + \frac{1}{2}\mu(\pi(\sigma+\eta) - \frac{Q}{\gamma\beta(x)}\left(\frac{3}{2} + 2\beta(x)\right)$$

Eventually, the σ investors stay out of the market at date 0 if their desired holdings at date-0 are non-positive, i.e.:

$$\pi(x) - \pi(\sigma) > \frac{Q}{\mu\gamma\beta(x)}\left(\frac{3}{2} + 2\beta(x)\right)$$

Noting $\psi^4(x) = \pi(x) - \pi(\sigma) - \frac{Q}{\gamma\mu\beta(x)}\left(\frac{3}{2} + 2\beta(x)\right)$, we see that this equilibrium exists if and only if:

$$\begin{cases} \psi^2(x) < 0 \\ \psi^4(x) > 0 \end{cases}$$

Note that ψ^4 is a strictly concave function of x.

In this equilibrium, expected turnover is

$$\mathbb{T} = \frac{1}{2} \times \frac{1}{4} n_1^\sigma(+\eta) = \frac{Q}{2}$$

Price volatility is simply:

$$\sigma_P = |P_1(\eta) - P_1(-\eta)| = \mu\left(\pi(\sigma+\eta) - \pi(x)\right)$$

4. The fourth Case, Case 4, we consider is one where both agents are long at date 0, and only the x agents are long at date 1 irrespective of $\tilde{\eta}$. Formally: $n_0^\sigma > 0$, $n_0^x > 0$, $n_1^\sigma(-\eta) = 0$, $n_1^x(-\eta) > 0$, $n_1^\sigma(+\eta) = 0$ and $n_1^x(+\eta) > 0$.

   We know from the analysis of date-1 equilibrium that if the σ agents are not long when $\tilde{\eta} = +\eta$, then they are not long for $\tilde{\eta} = -\eta$, as $\phi(x,-\eta) > \phi(x,\eta)$. Thus, the defining condition for the structure of

the equilibrium at date 1 is:

$$n_1^\sigma(+\eta) = 0 \Leftrightarrow \phi(x,\eta) > \frac{n_0^\sigma}{\gamma\mu\beta(x)}$$

We now turn to the condition under which the σ investors are long at date-0. As in our first equilibrium, the date-0 program of the σ investor can be rewritten as:

$$\max_{n_0^\sigma} n_0^\sigma \pi(\sigma) - \frac{1}{\mu}\left(n_0^\sigma P_0 + \frac{(n_0^\sigma)^2}{2\gamma}\right) + \frac{1}{\mu}\left(n_0^\sigma \mathbb{E}\left[P_1(x,\tilde{\eta})\right] - \frac{(n_0^\sigma)^2}{2\gamma}\right)$$

So that the date-0 holdings of the σ agents are given by:

$$n_0^\sigma = \max\left(0, \frac{\gamma}{2}\left(\mu\pi(\sigma) + \mathbb{E}\left[P_1(x,\tilde{\eta})\right] - P_0\right)\right)$$

We proved in our analysis of the date-1 equilibrium that when only the x agents are long at date 1, prices are given by: $P_1(x,\eta) = P_1(x,-\eta) = \mu\pi(x) - \frac{n_0^\sigma}{4\gamma\beta(x)}$.

Thus, the σ investors demand at date-0 can be rewritten:

$$n_0^\sigma = \max\left(0, \gamma\frac{\beta(x)}{2\beta(x)+\frac{1}{4}}\left(\mu\pi(\sigma) + \mu\pi(x) - P_0\right)\right)$$

As in our first case, Case 1, x agents demand at date-0 is defined by:

$$n_0^x = \gamma\left(2\mu\pi(x) - P_0\right)$$

This comes from the fact that in this equilibrium, at date-1, the x agents have interior solutions for both $\tilde{\eta} = \eta$ and $\tilde{\eta} = -\eta$. Thus, we can apply the envelop theorem to the x-agents date-1 utility, which gives a utility of $\pi(x)$.

The date-0 market clearing condition imposes that:

$$\frac{1}{4}\frac{\beta(x)}{2\beta(x)+\frac{1}{4}}\left(\mu\pi(\sigma) + \mu\pi(x) - P_0\right) + \beta(x)\left(2\mu\pi(x) - P_0\right) = \frac{Q}{\gamma}$$

Which provides us with the equilibrium price:

$$P_0(x) = \frac{\frac{1}{8}}{\beta(x)+\frac{1}{4}}\mu\pi(\sigma) + \frac{2\beta(x)+\frac{3}{8}}{\beta(x)+\frac{1}{4}}\mu\pi(x) - \frac{Q}{\gamma\beta(x)}\frac{\beta(x)+\frac{1}{8}}{\beta(x)+\frac{1}{4}}$$

The σ agents are long at date-0 if and only if their date-0 holdings are strictly positive, i.e.:

$$n_0^\sigma > 0 \Leftrightarrow P_0(x) < \mu(\pi(\sigma) + \pi(x)) \Leftrightarrow \frac{Q}{\mu\gamma\beta(x)} > \pi(x) - \pi(\sigma) \Leftrightarrow \psi^1(x) < 0$$

We can also compute the date-0 holdings of the σ agents n_0^σ:

$$n_0^\sigma = \frac{\gamma}{2}\frac{\beta(x)}{\beta(x) + \frac{1}{4}}\left(\mu(\pi(\sigma) - \pi(x)) + \frac{Q}{\gamma\beta(x)}\right)$$

So that the condition for the σ agents not to be long at date 1 when the shock $\tilde{\eta}$ is $+\eta$ can be rewritten as:

$$\phi(x, \eta) > \frac{n_0^\sigma}{\gamma\mu\beta(x)} \Leftrightarrow \pi(x) > \frac{2}{3}\pi(\sigma + \eta) + \frac{\pi(\sigma)}{3} + \frac{Q}{3\mu\gamma\beta(x)}$$

Noting $\psi^5(x) = \pi(x) - \frac{2}{3}\pi(\sigma + \eta) - \frac{\pi(\sigma)}{3} - \frac{Q}{3\mu\gamma\beta(x)}$, this equilibrium exists if and only if:

$$\begin{cases} \psi^1(x) < 0 \\ \psi^5(x) > 0 \end{cases}$$

Note that ψ^5 is a strictly concave function.

In this equilibrium, expected turnover is

$$\mathbb{T} = \frac{1}{4}n_0^\sigma = \frac{\mu\gamma}{8}\frac{\beta(x)}{\beta(x) + \frac{1}{4}}\left(\pi(\sigma) - \pi(x) + \frac{Q}{\mu\gamma\beta(x)}\right) = -\frac{\mu\gamma}{8}\frac{\beta(x)}{\beta(x) + \frac{1}{4}}\psi^1(x)$$

Price volatility is simply:

$$\sigma_P = |P_1(\eta) - P_1(-\eta)| = 0$$

5. The fifth case, Case 5, we consider is one where both agents are long at date 0, only the x agents are long at date 1 when $\tilde{\eta} = -\eta$ and both agents are long when $\tilde{\eta} = \eta$. Formally: $n_0^\sigma > 0$, $n_0^x > 0$, $n_1^\sigma(-\eta) = 0$, $n_1^x(-\eta) > 0$, $n_1^\sigma(+\eta) > 0$ and $n_1^x(+\eta) > 0$.

   First, the fact that the σ agents hold no asset at date 1 when the shock is $\tilde{\eta} = -\eta$ implies that:

$$\phi(x, -\eta) > \frac{n_0^\sigma}{\gamma\mu\beta(x)}$$

Second, the fact that both agents are long at date 1 when $\tilde{\eta} = \eta$ implies that:

$$\frac{n_0^\sigma}{\gamma\mu\beta(x)} > \phi(x,\eta) > -\frac{4n_0^x}{\gamma\mu}$$

We now consider the condition under which the σ agents are long at date 0. The first order condition that defines their date-0 holding is similar to that of equilibrium 2 (where the date 1 equilibrium structure is similar), i.e. :

$$0 = \pi(\sigma) - \frac{1}{\mu}\left(P_0 + \frac{n_0^\sigma}{\gamma}\right) + \frac{1}{2}\left(P_1(x,-\eta) - \frac{n_0^\sigma}{\gamma}\right) + \frac{1}{2}\mu\pi(\sigma+\eta)$$

So that their date-0 equilibrium holdings are given by:

$$\begin{aligned} n_0^\sigma &= \max\left(0, \frac{2}{3}\gamma\left(\mu\pi(\sigma) + \frac{1}{2}\left(P_1(x,-\eta) + \mu\pi(\sigma+\eta)\right) - P_0\right)\right) \\ &= \max\left(0, \frac{2}{3}\gamma\left(\mu\pi(\sigma) + \frac{1}{2}\left(\mu\pi(x) - \frac{n_0^\sigma}{4\gamma\beta(x)} + \mu\pi(\sigma+\eta)\right) - P_0\right)\right) \\ &= \max\left(0, \gamma\frac{\beta(x)}{\frac{3}{2}\beta(x)+\frac{1}{8}}\left(\mu\pi(\sigma) + \frac{1}{2}\left(\mu\pi(x) + \mu\pi(\sigma+\eta)\right) - P_0\right)\right), \end{aligned}$$

where the second equality comes from the date-1 price of the asset when $\tilde{\eta} = -\eta$.

Because the x investors are long at every date for all shocks, their date-0 demand is again simply given by:

$$n_0^x = \gamma\left(2\mu\pi(x) - P_0\right)$$

The date-0 market clearing condition imposes that:

$$\begin{aligned} \frac{Q}{\gamma\beta(x)} &= 2\mu\pi(x) - P_0 + \frac{\frac{1}{4}}{\beta(x)+\frac{1}{4}}\left(\mu\pi(\sigma) + \frac{1}{2}\left(\mu\pi(x) + \mu\pi(\sigma+\eta)\right) - P_0\right) \\ \Leftrightarrow \qquad P_0 &= \frac{\mu}{\beta(x)+\frac{1}{4}}\left((2\beta(x)+\tfrac{1}{4})\pi(x) + \tfrac{1}{12}\pi(\sigma+\eta) + \tfrac{1}{6}\pi(\sigma)\right) - \frac{\beta+\frac{1}{12}}{\beta+\frac{1}{4}}\frac{Q}{\gamma\beta} \end{aligned}$$

So one can compute the date-0 demand of σ investors:

$$n_0^\sigma = \frac{2}{3}\frac{\mu\gamma\beta(x)}{\beta(x)+\frac{1}{4}}\left(\pi(\sigma) + \frac{\pi(\sigma+\eta)}{2} - \frac{3}{2}\pi(x) + \frac{Q}{\gamma\mu\beta(x)}\right)$$

And:

$$n_0^\sigma > 0 \Leftrightarrow \frac{2}{3}\pi(\sigma) + \frac{\pi(\sigma+\eta)}{3} + \frac{2Q}{3\gamma\mu\beta(x)} > \pi(x) \Leftrightarrow \psi^3(x) < 0$$

And the date-0 demand of x investors simply rewrite:

$$n_0^x = \gamma\mu \frac{\frac{1}{4}}{\beta(x)+\frac{1}{4}} \left(\pi(x) - \frac{2}{3}\pi(\sigma) - \frac{1}{3}\pi(\sigma+\eta) + \frac{4Q}{\gamma\mu\beta(x)} \left(\beta(x) + \frac{1}{12} \right) \right)$$

We can now rewrite the conditions under which the date-1 equilibrium exists, i.e.:

$$\begin{cases} \phi(x,-\eta) > \dfrac{n_0^\sigma}{\gamma\mu\beta(x)} \Leftrightarrow \pi(x) > \dfrac{\pi(\sigma-\eta)}{2} + \dfrac{1}{3}\pi(\sigma) + \dfrac{1}{6}\pi(\sigma+\eta) + \dfrac{1}{3}\dfrac{Q}{\gamma\mu\beta(x)} \Leftrightarrow \psi^6(x) > 0 \\ \phi(x,\eta) < \dfrac{n_0^\sigma}{\gamma\mu\beta(x)} \Leftrightarrow \pi(x) < \dfrac{1}{3}\pi(\sigma) + \dfrac{2}{3}\pi(\sigma+\eta) + \dfrac{1}{3}\dfrac{Q}{\gamma\mu\beta(x)} \Leftrightarrow \psi^5(x) < 0 \\ \phi(x,\eta) > -\dfrac{4n_0^x}{\gamma\mu} \Leftrightarrow \pi(x) > \dfrac{2}{3}\pi(\sigma+\eta) + \dfrac{1}{3}\pi(\sigma) - \dfrac{2Q}{\gamma\mu\beta(x)} \left(\beta(x) + \dfrac{1}{12} \right) \Leftrightarrow \psi^7(x) > 0 \end{cases}$$

with

$$\psi^6(x) = \pi(x) - \frac{\pi(\sigma-\eta)}{2} - \frac{1}{3}\pi(\sigma) - \frac{1}{6}\pi(\sigma+\eta) - \frac{1}{3}\frac{Q}{\gamma\mu\beta(x)}$$

and:

$$\psi^7(x) = \pi(x) - \frac{2}{3}\pi(\sigma+\eta) - \frac{1}{3}\pi(\sigma) + \frac{2Q}{\gamma\mu\beta(x)} \left(\beta(x) + \frac{1}{12} \right)$$

Note that ψ^6 is a strictly concave function of x and ψ^7 is a strictly increasing function of x.

Finally, this equilibrium exists if and only if::

$$\begin{cases} \psi^6(x) > 0 \\ \psi^5(x) < 0 \\ \psi^7(x) > 0 \\ \psi^3(x) < 0 \end{cases}$$

In this equilibrium, expected turnover is

$$\mathbb{T} = \frac{1}{2} \times \frac{1}{4} n_0^\sigma + \frac{1}{2} \times \left(\frac{1}{4} |n_0^\sigma - n_1^\sigma(\eta)| \right)$$

We know that $n_1^\sigma(\eta) = n_0^\sigma + \gamma\left(\mu\pi(\sigma+\eta) - P_1(\eta)\right)$ so that:

$$n_1^\sigma(\eta) - n_0^\sigma = \gamma\mu \frac{\beta(x)}{\beta(x)+\frac{1}{4}} \left(\pi(\sigma+\eta) - \pi(x) \right)$$

Assume first that $x > \sigma + \eta$, then expected turnover is given by:

$$\mathbb{T} = \frac{\gamma\mu}{12} \frac{\beta(x)}{\beta(x)+\frac{1}{4}} \left(\pi(\sigma) - \pi(\sigma+\eta) + \frac{Q}{\mu\gamma\beta(x)} \right)$$

Assume now that $x < \sigma + \eta$, then expected turnover is given by:

$$\mathbb{T} = -\frac{\gamma\mu}{4}\frac{\beta(x)}{\beta(x)+\frac{1}{4}}\left(\pi(x) - \frac{2}{3}\pi(\sigma+\eta) - \frac{\pi(\sigma)}{3} - \frac{Q}{3\mu\gamma\beta(x)}\right) = -\frac{\gamma\mu}{4}\frac{\beta(x)}{\beta(x)+\frac{1}{4}}\psi^5(x)$$

Price volatility is simply:

$$\sigma_P = |P_1(\eta) - P_1(-\eta)| = -\frac{\mu}{2\left(\beta(x)+\frac{1}{4}\right)}\psi^5(x)$$

6. The sixth case, Case 6, we consider is one where both agents are long at date 0, only the x agents are long at date 1 when $\tilde{\eta} = -\eta$ and only the σ agents are long when $\tilde{\eta} = -\eta$. Formally: $n_0^\sigma > 0$, $n_0^x > 0$, $n_1^\sigma(-\eta) = 0$, $n_1^x(-\eta) > 0$, $n_1^\sigma(+\eta) > 0$ and $n_1^x(+\eta) = 0$.

   We know from the analysis of the date-1 equilibrium that if only the x agents are long when $\tilde{\eta} = -\eta$ then:

$$\phi(x, -\eta) > \frac{n_0^\sigma}{\gamma\mu\beta(x)}$$

   Similarly, if only the σ agents are long when $\tilde{\eta} = -\eta$, then:

$$\phi(x, \eta) < -\frac{4n_0^x}{\gamma\mu}$$

   The program for the σ agents in this equilibrium is similar to that of equilibrium 3 (except that the date-1 price now depends on the date-0 holdings of the σ agents). It thus yields a similar first order condition:

$$0 = \pi(\sigma) - \frac{1}{\mu}\left(P_0 + \frac{n_0^\sigma}{\gamma}\right) + \frac{1}{2}\frac{1}{\mu}\left(P_1(x, -\eta) - \frac{n_0^\sigma}{\gamma}\right) + \frac{1}{2}\pi(\sigma+\eta)$$

   Note that in this equilibrium when $\tilde{\eta} = -\eta$: $P_1(x, -\eta) = \mu\pi(x) - \frac{n_0^\sigma}{4\gamma\beta(x)}$. So that the date-0 holdings of agents σ can be written as:

$$n_0^\sigma = \max\left(0, \gamma\frac{\beta(x)}{\frac{3}{2}\beta(x)+\frac{1}{8}}\left(\mu\left(\pi(\sigma) + \frac{1}{2}\pi(x) + \frac{1}{2}\pi(\sigma+\eta)\right) - \frac{P_0}{\mu}\right)\right)$$

   Similarly, the program for the x agents is similar to that of equilibrium 3, so that the first order condition of this program is simply:

$$\frac{3}{2}\frac{n_0^x}{\mu\gamma} = \frac{3}{2}\pi(x) + \frac{1}{2\mu}P_1(x, \eta) - \frac{P_0}{\mu}$$

   Our analysis of the date-1 equilibrium price shows that when only σ agents are long, the equilibrium

price is: $P_1(x,\eta) = \mu\pi(\sigma+\eta) - \frac{4\beta(x)n_0^x}{\gamma}$. So that the date-0 demand for the asset of the x agent is simply:

$$n_0^x = \max\left(0, \gamma\frac{1}{\frac{3}{2}+2\beta(x)}\left(\mu\left(\frac{3}{2}\pi(x)+\frac{1}{2}\pi(\sigma+\eta)\right)-\frac{P_0}{\mu}\right)\right)$$

The date-0 market clearing condition gives us the date-0 asset price:

$$P_0(x) = \mu\left(\frac{1}{2}\pi(\sigma+\eta) + \frac{\frac{3}{4}+\beta(x)}{\beta(x)+\frac{1}{4}}\frac{\pi(\sigma)}{4} + \frac{\frac{3}{16}+\frac{5}{4}\beta(x)}{\beta(x)+\frac{1}{4}}\pi(x) - \frac{\left(\frac{3}{4}+\beta(x)\right)\left(\frac{3}{2}\beta(x)+\frac{1}{8}\right)}{\beta(x)+\frac{1}{4}}\frac{Q}{\mu\gamma\beta(x)}\right)$$

And we can express the date-0 demand of both investors types as a function of the primitives:

$$\begin{cases} n_0^\sigma = \dfrac{\mu\gamma\beta(x)}{2(\beta(x)+\frac{1}{4})}\left(\pi(\sigma)-\pi(x)+\dfrac{Q}{\mu\gamma\beta(x)}\left(2\beta(x)+\dfrac{3}{2}\right)\right) \\ n_0^x = \dfrac{\mu\gamma}{8(\beta(x)+\frac{1}{4})}\left(\pi(x)-\pi(\sigma)+\dfrac{6Q}{\mu\gamma\beta(x)}\left(\beta(x)+\dfrac{1}{12}\right)\right) \end{cases}$$

We thus have the σ investors long at date-0 if and only if:

$$n_0^\sigma > 0 \Leftrightarrow \pi(\sigma) + \frac{Q}{\mu\gamma\beta(x)}\left(2\beta(x)+\frac{3}{2}\right) > \pi(x) \Leftrightarrow \psi^4(x) < 0$$

The expressions for the date-0 demand in asset also allow us to rewrite the conditions under which the date-1 equilibrium structure holds:

$$\begin{cases} \phi(x,-\eta) > \dfrac{n_0^\sigma}{\gamma\mu\beta(x)} \Leftrightarrow \pi(x) > \dfrac{2}{3}\pi(\sigma-\eta)+\dfrac{1}{3}\pi(\sigma)+\dfrac{1}{3}\dfrac{Q}{\mu\gamma\beta(x)}(2\beta(x)+\dfrac{3}{2}) \Leftrightarrow \psi^8(x) > 0 \\ \phi(x,\eta) < -\dfrac{4n_0^x}{\gamma\mu} \Leftrightarrow \pi(x) < \dfrac{2}{3}\pi(\sigma+\eta)+\dfrac{1}{3}\pi(\sigma)-\dfrac{2Q}{\mu\gamma\beta(x)}(\beta(x)+\dfrac{1}{12}) \Leftrightarrow \psi^7(x) < 0 \end{cases}$$

with $\psi^8(x) = \pi(x) - \frac{2}{3}\pi(\sigma-\eta) - \frac{1}{3}\pi(\sigma) - \frac{1}{3}\frac{Q}{\mu\gamma\beta(x)}(2\beta(x)+\frac{3}{2})$. Note that ψ^8 is a strictly concave function of x.

Finally, this equilibrium exists if and only if:

$$\begin{cases} \psi^8(x) > 0 \\ \psi^7(x) < 0 \\ \psi^4(x) < 0 \end{cases}$$

Expected turnover in this equilibrium is given by:

$$\mathbb{T} = \frac{1}{2} \times \frac{1}{4} n_0^\sigma + \frac{1}{2} \times \beta(x) n_0^x = \frac{Q}{2}$$

And price volatility is simply:

$$\sigma_P = \frac{3}{2}\mu \left| \pi(x) - \frac{2}{3}\pi(\sigma + \eta) - \frac{1}{3}\pi(\sigma) + \frac{1}{2}\frac{Q}{\mu\gamma\beta(x)} \left(\beta(x) - \frac{1}{4} \right) \right|$$

7. The last case, Case 7, we consider is one where both agents are long at date 0, and both agents are long at date 1 irrespective of $\tilde{\eta}$. Formally: $n_0^\sigma > 0$, $n_0^x > 0$, $n_1^\sigma(-\eta) > 0$, $n_1^x(-\eta) > 0$, $n_1^\sigma(+\eta) > 0$ and $n_1^x(+\eta) > 0$.

   The two relevant conditions for this equilibrium structure of date-1 are:

$$\begin{cases} \phi(x, -\eta) < \dfrac{n_0^\sigma}{\gamma\mu\beta(x)} \\ \phi(x, \eta) > -\dfrac{4n_0^x}{\gamma\mu} \end{cases}$$

   Denote $2\hat{\pi}(\sigma) = \pi(\sigma) + \frac{1}{2}(\pi(\sigma - \eta) + \pi(\sigma + \eta)) < 2\pi(\sigma)$ thanks to the concavity of π.

   Because both investors have strictly positive holdings for every state of nature, their date-0 demand is simply given by:

$$\begin{cases} n_0^\sigma = \gamma(2\mu\hat{\pi}(\sigma) - P_0) \\ n_0^x = \gamma(2\mu\pi(x) - P_0) \end{cases}$$

   The date-0 market clearing condition imposes that:

$$P_0 = \frac{\mu}{\beta(x) + \frac{1}{4}} \left(2\beta(x)\pi(x) + \frac{1}{2}\hat{\pi}(\sigma) - \frac{Q}{\gamma\mu} \right)$$

   So that the date-0 demand for the asset can be rewritten as:

$$\begin{cases} n_0^\sigma = \dfrac{\mu\gamma\beta(x)}{\beta(x) + \frac{1}{4}} \left(2\hat{\pi}(\sigma) - 2\pi(x) + \dfrac{Q}{\mu\gamma\beta(x)} \right) \\ n_0^x = \dfrac{\mu\gamma}{4(\beta(x) + \frac{1}{4})} \left(2\pi(x) - 2\hat{\pi}(\sigma) + \dfrac{4Q}{\mu\gamma} \right) \end{cases}$$

   Thus, the σ investors are long at date-0 if and only if:

$$n_0^\sigma > 0 \Leftrightarrow \hat{\pi}(\sigma) + \frac{Q}{2\mu\gamma\beta(x)} > \pi(x) \Leftrightarrow \psi^9(x) < 0$$

with $\psi^9(x) = \pi(x) - \hat{\pi}(\sigma) - \frac{Q}{2\mu\gamma\beta(x)}$ is a strictly concave function of x.

We can also rewrite the conditions under which the date-1 equilibrium structure holds:

$$\begin{cases} \phi(x,-\eta) < \dfrac{n_0^\sigma}{\gamma\mu\beta(x)} \Leftrightarrow \pi(x) < \dfrac{2}{3}\hat{\pi}(\sigma) + \dfrac{1}{3}\pi(\sigma-\eta) + \dfrac{Q}{3\mu\gamma\beta(x)} \Leftrightarrow \psi^9(x) < 0 \\ \phi(x,\eta) > -\dfrac{4n_0^x}{\gamma\mu} \Leftrightarrow \pi(x) \geq \dfrac{\pi(\sigma+\eta)}{3} + \dfrac{2\hat{\pi}(\sigma)}{3} - \dfrac{4Q}{3\mu\gamma} \Leftrightarrow \psi^{10}(x) > 0 \end{cases}$$

with $\psi^{10}(x) = \pi(x) - \frac{\pi(\sigma+\eta)}{3} - \frac{\hat{\pi}(\sigma)}{3} + \frac{4Q}{3\mu\gamma}$. Note that $\psi^{10}(x)$ is a strictly increasing function of x.

Finally, this equilibrium exists if and only:

$$\begin{cases} \psi^9(x) < 0 \\ \psi^6(x) < 0 \\ \psi^{10}(x) > 0 \end{cases}$$

Turnover in this equilibrium is simply given by:

$$\begin{aligned} \mathbb{T} &= \frac{1}{2} \times \frac{1}{4} \left| n_0^\sigma - n_1^\sigma(-\eta) \right| + \frac{1}{2} \times \frac{1}{4} \left| n_0^\sigma - n_1^\sigma(\eta) \right| \\ &= \frac{\mu\gamma}{8} \frac{\beta(x)}{\beta(x) + \frac{1}{4}} \left(\pi(\sigma+\eta) - \pi(\sigma-\eta) \right) \end{aligned}$$

And price volatility is also simply given by:

$$\sigma_P = \frac{\mu}{4} \frac{\pi(\sigma+\eta) - \pi(\sigma-\eta)}{\beta(x) + \frac{1}{4}}$$

Equilibrium structure: quiet bubble occurs for intermediate values of x

We now turn to the proof of the date-0 equilibrium structure as a function of x. To simplify the exposition of the analysis, we make the following simplifying assumption. Let x^2 be the unique x such that $\psi^2(x^2) = 0$. Then we assume that $\theta = \mu\gamma$ is large enough so that:

$$\pi(\sigma+\eta) \geq \pi(\sigma) + \frac{Q}{\mu\gamma\beta(x^2)} \left(\frac{5}{2} + 6\beta(x^2) \right) \tag{30}$$

Note that this implies also that:

$$\pi(\sigma+\eta) - \pi(\sigma) > \frac{Q}{\mu\gamma\beta(\sigma+\eta)} \tag{31}$$

This last condition states that leverage is large enough so that even when $\sigma + \eta = x$ (i.e. when the σ

investors receive the η shock they are as optimistic as the x investors), only the x investors are long both at date-0 and date-1.

Note that because β is decreasing with x, this implies: $\pi(\sigma+\eta)-\pi(\sigma) > \frac{4Q}{\mu\gamma}$ Note that because π is concave, this implies that: $\pi(\sigma)-\pi(\sigma-\eta) > \frac{Q}{\mu\gamma\beta(\sigma+\eta)} > \frac{4Q}{\mu\gamma}$

We start from Case 1. Thanks to assumption 30, it exists for $x=\sigma+\eta$ as assumption 30 is equivalent to $\psi^1(\sigma+\eta)>0$. Because $\psi^1(x)$ is a strictly concave function of x strictly positive at $x=\sigma+\eta$ and equal to $-\infty$ at $x=-\infty$, there is $x_2^1 > \sigma+\eta$ such that equilibrium 1 exists over $[\sigma+\eta, x_2^1]$ and $\psi^1(x_2^1)=0$ and $\frac{\partial\psi^1}{\partial x}(x_2^1)<0$.

Consider now Case 4. We know that for all $x \geq x_2^1$, $\psi^1(x)<0$. For $x=x_2^1$, we have: $\psi^5(x_2^1) = \pi(\sigma)-\pi(\sigma+\eta)+\frac{Q}{\mu\gamma\beta(x^1)} = \pi(x^1)-\pi(\sigma+\eta)>0$. So equilibrium 4 exists for $x=x_2^1$. Because $\psi^5(x)$ is a strictly concave function of x, strictly positive for $x=x_2^1$ and equal to $-\infty$ at $x=-\infty$, this implies that there is $x_2^5 > x_2^1$ such that equilibrium 4 exists over $[x_2^1, x_2^5]$ and $\psi^5(x_2^5)=0$ and $\frac{\partial\psi^5}{\partial x}(x_2^5)<0$.

Consider now Case 5. We know that for all $x \geq x_2^5$, $\psi^5(x)<0$. We also know that for all $x \geq x_2^5 > \sigma+\eta$, $\psi^7(x)>0$. For $x=x_2^5$, $\psi^3(x_2^5) = -\pi(x_2^5)+\pi(\sigma+\eta)<0$. $\psi^3(x)$ is concave, strictly negative for $x=x_2^5$ and strictly positive for $x=\sigma+\eta$. Thus it is strictly negative for all $x \geq x_2^5$. Finally, $2\psi^6(x_2^5) = \pi(\sigma+\eta)-\pi(\sigma-\eta)>0$. Note that $\psi^6(x)$ is a strictly concave function, strictly positive at $x=x_2^5$ and that goes to $-\infty$ as $x=\infty$. Thus, there is a unique $x_2^6 > x_2^5$ such that equilibrium 5 exists over $[x_2^5, x_2^6]$.

Consider now Case 7. By the definition of x_2^6, we know that for all $x \geq x_2^6$, $\psi^6(x)<0$. For $x \geq x_2^6 > x+\eta$, $\psi^{10}(x)>0$. Finally, $2\psi^9(x_2^6) = -\pi(x_2^6)+\pi(\sigma-\eta)<0$. Note that $\psi^9(\sigma+\eta) = \frac{\pi(\sigma+\eta)-\pi(\sigma)}{2} + \frac{\pi(\sigma)-\frac{1}{2}\pi(\sigma+\eta)-\frac{1}{2}\pi(\sigma-\eta)}{2} - \frac{Q}{2\mu\gamma\beta(\sigma+\eta)} > 0$ thanks to assumption 30 and to the assumption that π is concave. Thus: ψ^9 is a strictly concave function, strictly positive at $x=\sigma+\eta$ and strictly negative for $x=x_2^6$. Thus it is strictly negative for all $x \geq x_2^6$. Thus, equilibrium 7 is the equilibrium over $[x_2^6, \infty]$.

Consider now Case 2. Clearly, $\psi^2(\sigma+\eta)>0$ and by assumption: $\psi^3(\sigma+\eta)>0$. Thus, equilibrium 2 exists for $x=\sigma+\eta$. $\psi^2(x)$ is a strictly increasing function of x, which is strictly positive for $x=\sigma+\eta$. Thus, there is $x^2 < \sigma+\eta$ such that for all $x \in [x^2, \sigma+\eta]$, this function is positive and $\psi^2(x^2)=0$. ψ^3 is a concave function, strictly positive at $x=\sigma+\eta$ and strictly negative at $x=\sigma$. Thus, there is $x_1^3 \in]\sigma, \sigma+\eta[$ such that this function is positive on $[x_1^3, \sigma+\eta]$ and $\psi^{3'}(x_1^3)>0$. Thanks to assumption 30, it is easily shown that $x^2 > x_1^3$. Thus, equilibrium 2 will occur over $[x^2, \sigma+\eta]$

Consider now Case 3. By the definition of x^2, for all $x \leq x^2$, $\psi^2(x)<0$. For $x=x^2$, $\psi^4(x^2) = \pi(\sigma+\eta)-\pi(\sigma)-\frac{Q}{\mu\gamma\beta(x^2}\left(6\beta(x^2)+\frac{5}{2}\right)>0$, thus equilibrium 3 exists for $x=x^2$. ψ^4 is a strictly concave function of x, strictly positive for $x=x^2$ and strictly negative for $x=\sigma$. Thus there is a unique $x_1^4 \in [\sigma, x_1^4]$ such that equilibrium 3 exists over $[x_1^4, x^2]$.

Finallym consider Case 6. By the definition of x_1^4, we know that for all $x \leq x_1^4$, $\psi^4(x)<0$. Consider

the function ψ^8. It is a strictly concave function of x. It is equal to $\frac{2}{3}\left(\pi(x_1^4)-\pi(\sigma-\eta)\right)>0$ for $x=x_1^4$ and $\frac{2}{3}\left(\pi(\sigma)-\pi(\sigma-\eta)-\frac{4Q}{\mu\gamma}\right)>0$ for $x=\sigma$. Thus it is strictly positive over $[\sigma,x_1^4]$. Finally, the condition that $x^2>x_3^1$ implies that $\pi(\sigma+\eta)-\pi(\sigma)>\frac{Q}{\mu\gamma\beta(x^2)}\left(6\beta(x^2)+\frac{5}{2}\right)$. But $\frac{3}{2}\psi^7(x_1^4)=\pi(\sigma)-\pi(\sigma+\eta)+\frac{Q}{\mu\gamma\beta(x_1^4)}\left(6\beta(x_1^4)+\frac{5}{2}\right)<\pi(\sigma)-\pi(\sigma+\eta)+\frac{Q}{\mu\gamma\beta(x^2)}\left(6\beta(x^2)+\frac{5}{2}\right)<0$ as β is decreasing. The function ψ^7 is a strictly increasing function of x and it is strictly negative for $x=x^4$, thus it is strictly negative for all $x<x^4$. Eventually, equilibrium 6 exists over $[\sigma,x^4]$.

Thus, under assumption 30, the structure of the equilibrium is (with x increasing from σ to ∞):

$$\text{Case 6} \rightarrow \text{Case 3} \rightarrow \text{Case 2} \rightarrow \text{Case 1} \rightarrow \text{Case 4} \rightarrow \text{Case 5} \rightarrow \text{Case 7}$$

Properties of the equilibrium

We now derive the property of the equilibrium pricing function as x increases from σ to ∞ under assumption 30. The equilibrium structure is: $6\rightarrow 3\rightarrow 2\rightarrow 1\rightarrow 4\rightarrow 5\rightarrow 7$. We also focus on the portion of the equilibrium where only the x agents are long at date 0, i.e. for $x\in[x_1^4,x_2^1]$. Over this range of x, the equilibrium pricing function is given by: $P(x)=P^0(x)=2\mu\pi(x)-\frac{Q}{\mu\gamma\beta(x)}$ for $x\in[x^2,x_2^1]$ and by $P(x)=\frac{3}{2}\mu\pi(x)+\frac{1}{2}\mu(\pi(\sigma+\eta)-\frac{Q}{\gamma\beta(x)}\left(\frac{3}{2}+2\beta(x)\right)=P^0(x)+\underbrace{\left(-\frac{\mu}{2}\psi^2(x)\right)}_{\text{resale option}}$ for $x\in[x_1^4,x^2]$.

Note that both these functions are concave. We now prove that $P'(x)>0$ when $x=(x^2)^-$. $P'((x^2)^-)=\frac{3}{2}\mu\left(\pi'(x^2)-\frac{4Q}{\mu\gamma}\frac{x^2}{\sigma^2}\right)$

Because π is convex and $x^2<\sigma+\eta$, we know that: $\pi'(x^2)\geq\frac{\pi(\sigma+\eta)-\pi(x^2)}{\sigma+\eta-x^2}=\frac{\beta(x^2)+\frac{1}{4}}{\beta(x^2)}\frac{4Q}{\mu\gamma}\frac{1}{\sigma+\eta-x^2}$. Thus[8]:

$$\begin{aligned}
P'((x^2)^-) &\geq \frac{3}{2}\frac{4Q}{\gamma}\left(\frac{\beta(x_2)+\frac{1}{4}}{\beta(x_2)}\frac{1}{\sigma+\eta-x_2}-\frac{x_2}{\sigma^2}\right)\\
&\geq \frac{3}{2}\frac{4Q}{\gamma}\frac{1}{\sigma^2(\sigma+\eta-x_2)}\left(\frac{\beta(x_2)+\frac{1}{4}}{\beta(x_2)}\sigma^2-x_2(\sigma+\eta-x_2)\right)\\
&\geq \frac{3}{2}\frac{4Q}{\gamma}\frac{1}{\sigma^2(\sigma+\eta-x_2)}\left(2\sigma^2-\frac{(\sigma+\eta)^2)}{4}\right)\\
&\geq \frac{3}{2}\frac{4Q}{\gamma}\frac{1}{\sigma^2(\sigma+\eta-x_2)}\left(2\sigma^2-\sigma^2\right)\\
&> 0
\end{aligned}$$

Clearly, $P'(x^2)^+>0$ as: $2\mu\pi'(x)+\frac{Q}{\gamma(\beta(x))^2}\beta'(x)>\frac{3}{2}\mu\left(\pi'(x_2)-\frac{4Q}{\mu\gamma}\frac{x_2}{\sigma^2}\right)>0$.

Because P^0 is strictly concave, we thus conclude that over the range of equilibrium where only x agents are long at 0, the equilibrium price is either strictly increasing or increasing and then decreasing with x.

[8]The third inequality comes from the fact that $\frac{\beta(x)}{\beta(x)+\frac{1}{4}}>1$ and $x(\sigma+\eta-x)\leq\frac{(\sigma+\eta)^2}{4}$

Turnover and skewness

We now turn to turnover. It is $\frac{Q}{2}$ for $x \in [x_1^4, x^2]$, $\frac{\mu\gamma}{8}\frac{\beta(x)}{\beta(x)+\frac{1}{4}}(\pi(\sigma+\eta)-\pi(x))$ for $x \in [x^2, \sigma+\eta]$ and 0 for $x \in [\sigma+\eta, x_1^1]$. It is thus constant over $[x_1^4, x^2]$ and $[x^2, \sigma+\eta]$ and strictly decreasing over $[x^2, \sigma+\eta]$.

Volatility and skewness

We finally turn to price volatility. It is $\mu(\pi(\sigma+\eta)-\pi(x))$ for $x \in [x_1^4, x^2]$, $\frac{\mu}{4}\frac{\pi(\sigma+\eta)-\pi(x)}{\beta(x)+\frac{1}{4}}$ for $x \in [x^2, \sigma+\eta]$ and 0 for $x \in [\sigma+\eta, x_1^1]$. It is thus strictly decreasing over $[x_1^4, x^2]$. We show that it is also strictly decreasing over $[x^2, \sigma+\eta]$.

Using π concavity, we can bound the derivative of σ_P for equilibrium 2 in the following way:

$$\begin{aligned}\left(\beta(x)+\frac{1}{4}\right)\sigma_P'(x) &= \frac{1}{4}\mu\left((\pi(\sigma+\eta)-\pi(x))\frac{(-\beta'(x))}{\beta(x)+\frac{1}{4}}-\pi'(x)\right)\\ &\leq \frac{1}{4}\mu\pi'(x)\left((\sigma+\eta-x)\frac{(-\beta'(x))}{\beta(x)+\frac{1}{4}}-1\right)\end{aligned}$$

But:

$$\frac{(-\beta'(x))}{\beta(x)+\frac{1}{4}} = \frac{x}{\frac{3}{4}\sigma^2+\frac{1}{4}x^2}\frac{1}{1+\left(\frac{x}{\sigma}\right)^2} < \frac{x}{\frac{3}{4}\sigma^2+\frac{1}{4}x^2}$$

Thus:

$$\left(\beta(x)+\frac{1}{4}\right)\sigma_P'(x) \leq \frac{1}{4}\mu\pi'(x)\frac{1}{\frac{3}{4}\sigma^2+\frac{1}{4}x^2}\left(\underbrace{(\sigma+\eta-x)x-\frac{3}{4}\sigma^2+\frac{1}{4}x^2}_{=\rho(x)}\right)$$

ρ is a quadratic that reaches its maximum for $x = \frac{2}{5}(\sigma+\eta) < \sigma$. It is thus strictly decreasing over $[x^2, \sigma+\eta]$ and in particular over this range: $\rho(x) < \rho(\sigma) = \sigma(\sigma+\eta)-2\sigma^2 \leq 0$. Thus, σ_P' is strictly decreasing over $[x^2, \sigma+\eta]$.

Finally, volatility is constant and equal to 0 for equilibrium 1.

Proof of Proposition 4

The symmetric equilibrium (when $x=\sigma$) can be described in the following way:

1. When $\frac{4Q}{\mu\gamma} < \pi(\sigma+\eta)-\pi(\sigma)$, then Case 6 prevails. Indeed, $\psi^4(\sigma)$ is always < 0, $\psi^7(\sigma) < 0$ thanks to the assumption and because of π concavity, $\pi(\sigma)-\pi(\sigma-\eta) > \pi(\sigma+\eta)-\pi(\sigma) > \frac{4Q}{\mu\gamma}$ so that $\psi^8(\sigma) > 0$ and Case 6 is the equilibrium.

2. When $2\pi(\sigma)-\frac{3}{2}\pi(\sigma-\eta)-\frac{1}{2}\pi(\sigma+\eta) > \frac{4Q}{\mu\gamma} > \pi(\sigma+\eta)-\pi(\sigma)$, then Case 5 prevails. Indeed, $\psi^5(\sigma)$ and $\psi^3(\sigma)$ are always strictly negative and our two assumptions ensure that $\psi^6(\sigma) > 0$ and $\psi^7(\sigma) > 0$.

3. Finally, when $2\pi(\sigma) - \frac{3}{2}\pi(\sigma-\eta) - \frac{1}{2}\pi(\sigma+\eta) < \frac{4Q}{\mu\gamma}$, then $\psi^6(\sigma) < 0$. It turns out that $\psi^9(\sigma) < 0$ if and only if $\psi^6(\sigma) < 0$. Finally, $\psi^{10}(\sigma) > 0$ as $2\pi(\sigma) > \pi(\sigma-\eta) + \pi(\sigma+\eta)$. Thus, in this case, Case 7 prevails.

We start by looking at mis-pricing, i.e. deviations of date-0 price relative to the price when there is no short sales constraints. Note that in this symmetric case ($x = \sigma$), this price is equal to:

$$P_0^{unc.} = \mu\left(\frac{\pi(\sigma)}{2} + \frac{\pi(\sigma+\eta)}{4} + \frac{\pi(\sigma-\eta)}{4} + \pi(-\sigma) - \frac{Q}{\mu\gamma}\right)$$

Consider first Case 6. The price at date 0 in this case is given by:

$$P_0 = \mu\left(\frac{3}{2}\pi(\sigma) + \frac{1}{2}\pi(\sigma+\eta) - \frac{Q}{\mu\gamma}\right)$$

So that:

$$P_0 - P_0^{unc.} = \mu\left(\pi(\sigma) - \pi(-\sigma) + \frac{\pi(\sigma+\eta) - \pi(\sigma-\eta)}{4}\right)$$

And this expression is clearly increasing with D. Again, this comes from the intuition that the re-sale option increases with D, i.e. when the debt contract becomes more equity-like, which increases the amount of mispricing.

Consider now Case 5. The price when $x = \sigma$ is then given by: $P_0 = \mu\left(\frac{5}{3}\pi(\sigma) + \frac{1}{3}\pi(\sigma+\eta) - \frac{8Q}{3\mu\gamma}\right)$, so that:

$$P_0 - P_0^{unc.} = \mu\left(\pi(\sigma) - \pi(-\sigma) + \frac{\pi(\sigma+\eta) - \pi(\sigma-\eta)}{12} + \frac{\pi(\sigma) - \pi(-\sigma)}{6} - \frac{5}{3}\frac{Q}{\mu\gamma}\right)$$

This expression increases with D as well.

Consider finally Case 7. The price when $x = \sigma$ is then given by:

$$P_0 = \mu\left(\frac{3}{2}\pi(\sigma) + \frac{1}{4}\pi(\sigma+\eta) + \frac{1}{4}\pi(\sigma-\eta) - 2\frac{Q}{\mu\gamma}\right)$$

, so that:

$$P_0 - P_0^{unc.} = \mu\left(\pi(\sigma) - \pi(-\sigma) - \frac{Q}{\mu\gamma}\right)$$

This is again increasing with D.

So that eventually, mispricing is increasing with the riskiness of the debt for all level of leverage.

Intuitively, when the supply is low and leverage is high, there is high turnover and high volatility, as traders have sufficient buying power to buy all the shares from the less optimistic traders. When supply and

leverage are intermediate, the $\sigma+\eta$-agents have not enough buying power to acquire all the shares from the σ-agents, but the σ agents can buy all the shares from the $\sigma-\eta$-agents. Note that this asymmetry comes from the concavity of the payoff function. Finally, when supply is large and leverage is low, agents cannot absorb much supply and hence everyone is long at date-1, irrespective of the date-1 distribution of beliefs.

Note that turnover is $\frac{Q}{2}$ in Case 6, $\frac{\gamma\mu}{12}\left(\pi(\sigma+\eta)-\pi(\sigma)+\frac{2Q}{\mu\gamma}\right)$ in Case 5, and $\frac{\mu\gamma}{16}(\pi(\sigma+\eta)-\pi(\sigma))$ in Case 7. As a consequence, turnover is increasing with D (strictly for Case 5 and 7, weakly for Case 6), as $\pi(\sigma+\eta)-\pi(\sigma)$ is strictly decreasing with D.

Similarly, volatility is $\mu(\pi(\sigma+\eta)-\pi(\sigma)$ in Case 6, $\frac{\gamma\mu}{6}\left(\pi(\sigma+\eta)-\pi(\sigma)+\frac{2Q}{\mu\gamma}\right)$ in Case 5, and $\frac{\mu}{2}(\pi(\sigma+\eta)-\pi(\sigma))$ in Case 7. As a consequence, volatility is strictly increasing with D, as $\pi(\sigma+\eta)-\pi(\sigma)$ is strictly decreasing with D.

References

Abreu, Dilip and Markus K. Brunnermeier. "Bubbles and Crashes." *Econometrica* 71-1 (January 2003): 173–204.

Allen, Franklin and David Gale. "Bubbles and Crises." *Economic Journal* 110 2000): 236–255.

__ and Gary Gorton. "Churning Bubbles." *Review of Economic Studies* 60-4 (October 1993): 813–36.

__ , Stephen Morris, and Andrew Postlewaite. "Finite Bubbles with Short Sale Constraints and Asymmetric Information." *Journal of Economic Theory* 61 1993): 206–229.

Blanchard, Olivier J. and Mark W. Watson. "Bubbles, Rational Expectations and Financial Markets." NBER Working Papers: National Bureau of Economic Research, Inc, June 1983.

Chen, Joseph, Harrison Hong, and Jeremy C. Stein. "Breadth of ownership and stock returns." *Journal of Financial Economics* 66-2-3 2002): 171–205.

Daniel, Kent, David Hirshleifer, and Avanidhar Subrahmanyam. "Investor Psychology and Security Market Under- and Overreactions." *Journal of Finance* 53-6 (December 1998): 1839–1885.

Delong, J Bradford, Andrei Shleifer, Lawrence Summers, and Mark Waldmann. "Positive Feedback Investment Strategies and Destabilizing Rational Speculation." *Journal of Finance* 45-2 (June 1990): 379–95.

Geanakoplos, John. "The Leverage Cycle." in "NBER Macroeconomics Annual 2009, Volume 24" NBER Chapters, National Bureau of Economic Research, Inc, (april-jun 2010).

Gennaioli, Nicola, Andrei Shleifer, and Robert Vishny. "Neglected Risks, Financial Innovation and Financial Fragility." Working Papers: Fondazione Eni Enrico Mattei, September 2010.

Gorton, Gary and George Pennacchi. "Financial Intermediaries and Liquidity Creation." *Journal of Finance* 45-1 (March 1990): 49–71.

Harrison, J Michael and David M Kreps. "Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations." *The Quarterly Journal of Economics* 92-2 (May 1978): 323–36.

Hong, Harrison and Jeremy C. Stein. "Disagreement and the Stock Market." *Journal of Economic Perspectives* 21-2 (Spring 2007): 109–128.

__ , Jose Scheinkman, and Wei Xiong. "Asset Float and Speculative Bubbles." *Journal of Finance* 61-3 (06 2006): 1073–1117.

Khandani, Amir E., Andrew Lo, and Robert Merton. "Systemic Risk and the Refinancing Ratchet Effect." Manuscript, September 2010.

Kiyotaki, Nobuhiro and John Moore. "Credit Cycles." *Journal of Political Economy* 105-2 (April 1997): 211–48.

Mian, Atif and Amir Sufi. "The Consequences of Mortgage Credit Expansion: Evidence from the US Mortgage Default Crisis." *Quarterly Journal of Economics* 124-4 2009): 1449–1496.

Michael Lewis. *The Big Short: Inside The Doomsday Machine*, New York, New York: W.W. Norton and Company, 2010).

Miller, Edward M. "Risk, Uncertainty, and Divergence of Opinion." *Journal of Finance* 32-4 (September 1977): 1151–68.

Morris, Stephen. "Speculative Investor Behavior and Learning." *The Quarterly Journal of Economics* 111-4 (November 1996): 1111–33.

Odean, Terrance. "Do Investors Trade Too Much?" *American Economic Review* 89-5 (December 1999): 1279–1298.

Ofek, Eli and Matthew Richardson. "DotCom Mania: The Rise and Fall of Internet Stock Prices." *Journal of Finance* 58-3 (06 2003): 1113–1138.

Santos, Manuel S. and Michael Woodford. "Rational Asset Pricing Bubbles." *Econometrica* 65-1 (January 1997): 19–58.

Scheinkman, Jose A. and Wei Xiong. "Overconfidence and Speculative Bubbles." *Journal of Political Economy* 111-6 (December 2003): 1183–1219.

Shleifer, Andrei and Robert W Vishny. "The Limits of Arbitrage." *Journal of Finance* 52-1 (March 1997): 35–55.

Simsek, Alp. "When Optimists Need Credit: Asymmetric Disciplining of Optimism and Implications for Asset Prices." Manuscript, May 2010.

Stein, Jeremy C. "Securitization, Shadow Banking and Fragility." *Daedalus* 2010).

Tirole, Jean. "Asset Bubbles and Overlapping Generations." *Econometrica* 53-6 (November 1985): 1499–1528.

Vayanos, Dimitri and Denis Gromb. "Limits of Arbitrage: The State of the Theory." FMG Discussion Papers dp650: Financial Markets Group, March 2010.

Figure 1: ABX Prices

The figure plots the ABX 7-1 Prices for various credit tranches including AAA, AA, A, BBB, and BBB-.

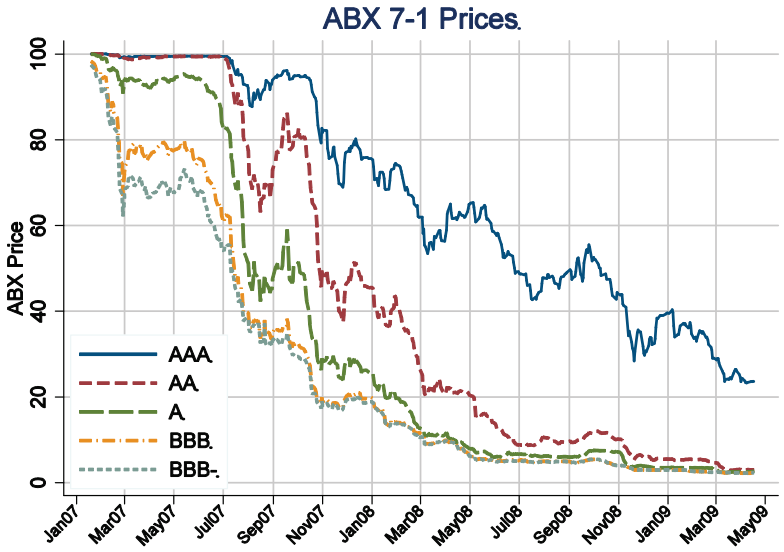
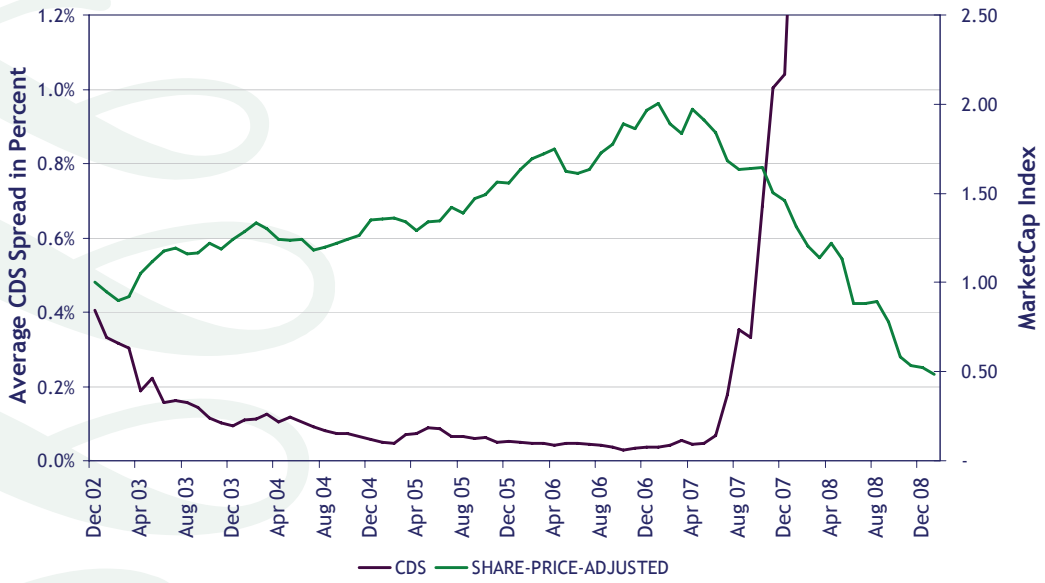


Figure 2: CDS Prices of Basket of Finance Companies

The figure plots the average CDS prices for a basket of large finance companies between December 2002 and December 2008.

Financial firms' CDS and share prices



Source: *Moody's KMV, FSA Calculations*

Firms included: Ambac, Aviva, Banco Santander, Barclays, Berkshire Hathaway, Bradford & Bingley, Citigroup, Deutsche Bank, Fortis, HBOS, Lehman Brothers, Merrill Lynch, Morgan Stanley, National Australia Bank, Royal Bank of Scotland and UBS.

CDS series peaks at 6.54% in September 2008.

12

Figure 3: Monthly Share Turnover of Financial Stocks

The figure plots the average monthly share turnover of financial stocks.

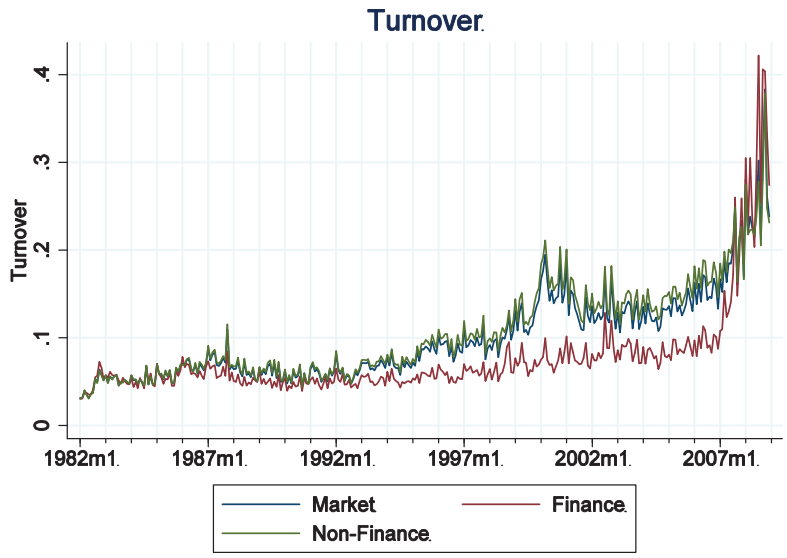


Figure 4: Monthly Share Turnover of Internet Stocks

The figure plots the average monthly share turnover of internet stocks compared to the rest of the market.

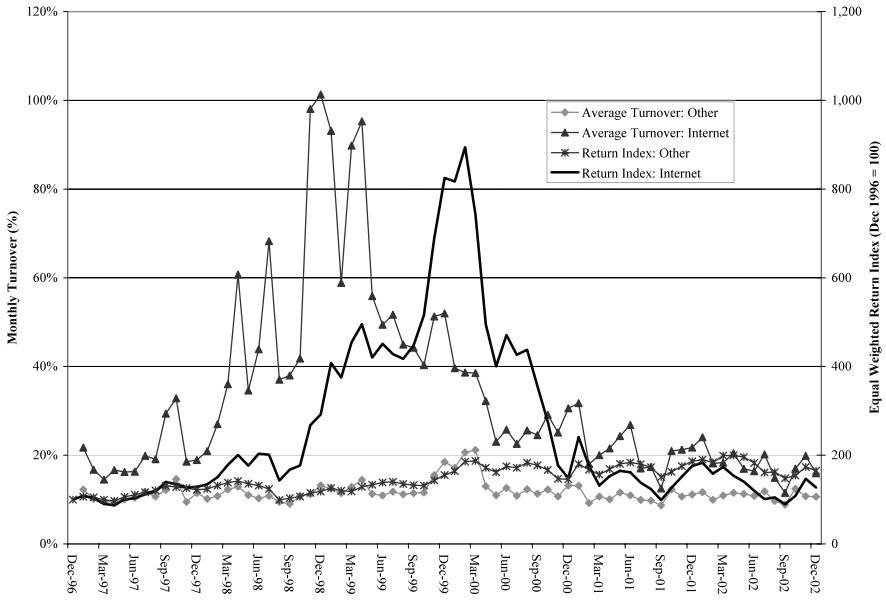


Figure 5: Equilibrium price and skewness in the distribution of priors

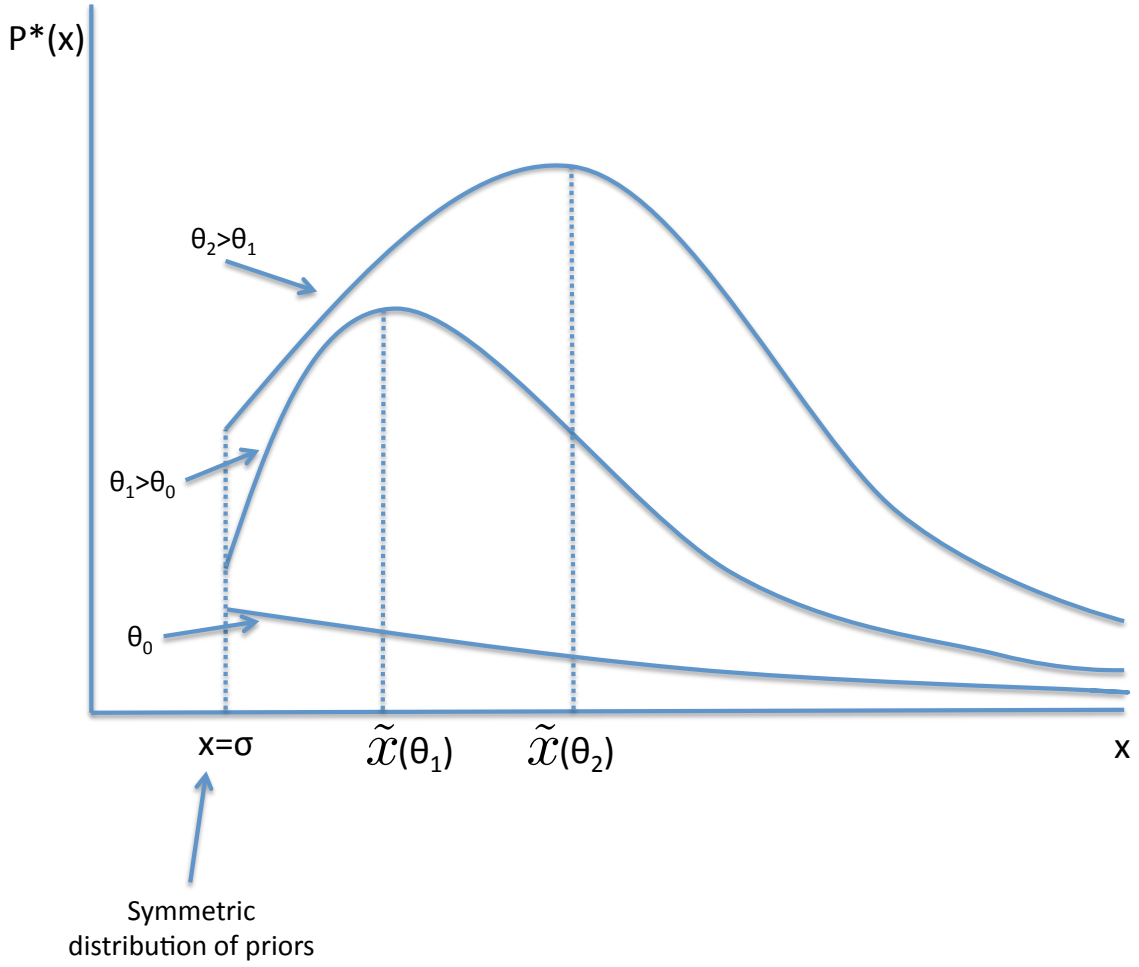


Figure 6: Traditional and Non-Traditional Issuance of Asset-Backed Securities (Quarterly)

The figure plots the issuance of traditional and non-traditional asset-backed securities by quarter.

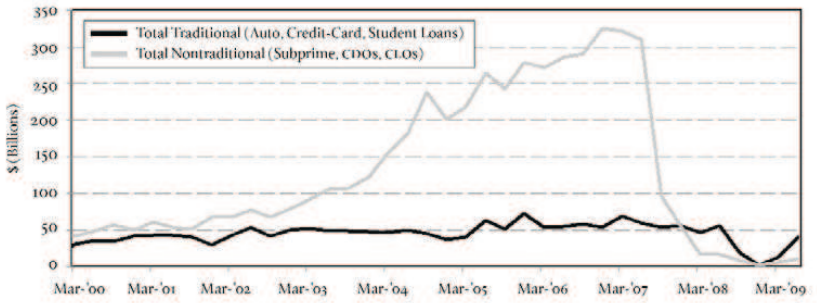


Figure 7: Synthetic Mezzanine ABS CDO Issuance

The figure plots the issuance of synthetic mezzanine ABS CDOs by year.

