Observable Long-Run Ambiguity and Long-Run Risk

Maxim Ulrich
Columbia University[‡]

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Abstract

This paper derives and estimates a general equilibrium model for the real and nominal term structure of U.S. government bonds which takes into account that investors are confronted with a set of multiple long-run risk models. The paper accounts for model misspecification doubts about long-run GDP risk and about long-run inflation risk. We find that an increase in macro uncertainty leads to a steepening in TIPS and nominal yields. Increased uncertainty about the long-run GDP model generates a steeper slope in TIPS yields than the inflation uncertainty counterpart. But on the other hand, we find that the term premium in TIPS and nominal bond yields is dominated by model uncertainty about long-run inflation. The estimated robustness preference for ambiguity about long-run inflation is 6.6 and 1.2 for long-run GDP ambiguity.

Keywords: Multiple prior, Ambiguity aversion, Equilibrium, Long-run risk, Macro-finance, Term structure

JEL: E43, E44, G12

[‡] Columbia Business School, 3022 Broadway, New York NY 10027. Email: mu2167@columbia.edu. Homepage: www.columbia.edu\ $\sim mu2167$.
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1 Introduction

Data evidence argues that investors are confronted with a set of multiple long-run risk models. Long-run risk models are characterized by GDP growth and inflation dynamics that have time-varying first moments. We provide data evidence that investors observe a set of multiple, potentially correct, long-run risk models. Recent models in the literature fall short in addressing how the observable set of multiple long-run risk models affects marginal utility of investors. We close this gap in the literature by deriving and estimating a term structure model with in real-time observable set of potentially correct long-run GDP risk- and long-run inflation risk models. The term structure information provides direct evidence of how different sources of model uncertainty affect investors' marginal utility for different time horizons.

The empirical set of potentially correct long-run risk models coincides with the observable amount of model estimation risk. The amount of model estimation risk is substantially higher for the long-run GDP risk component, compared to the long-run inflation risk component. Said differently, the set of potential long-run GDP risk models is higher than the set of potential long-run inflation risk models. The relative size of both sets, measured in terms of the amount of model estimation risk, peaked to factor ten in the mid 1980s.

Recent research has followed this empirical characteristic by nearly exclusively studying asset pricing with long-run GDP ambiguity. Bansal and Shaliastovich (2010) call that measure "confidence risk" and show that together with Epstein and Zin (1989) preferences that uncertainty helps to account for sharp equity return fluctuations. Buraschi and Jiltsov (2006) use the same measure of GDP growth model uncertainty to characterize the amount of model uncertainty (disagreement) that two investors face in the equity option market. Cagetti, Hansen, Sargent, and Williams (2002) study in a production economy

how a concern about ambiguity in long-run GDP growth affects equity prices.[1] A similar direction can be found in the term structure literature. Gagliardini, Porchia, and Trojani (2009) argue that ambiguity about long-run GDP risk is the driver for premia in the Government bond market.[2] Similarly, Kleshecheslski and Vincent (2009) argue that model uncertainty about short-run GDP risk generates an upward sloping yield curve. Ulrich (2010) is the first paper who focuses on model uncertainty about inflation and who attributes the upward sloping nominal yield curve to inflation ambiguity. It is unsatisfactory that these studies do not include both sources of ambiguity into an analysis. They leave open the question of whether it is indeed model uncertainty about long-run GDP growth or model uncertainty about long-run inflation that account for the upward sloping nominal yield curve.

In this paper we present a general equilibrium model that generates insights regarding that question. Our model features observable long-run risk and accounts for the observable amount of model estimation risk that investors face with regard to long-run GDP growth and long-run inflation. The model is analytically very tractable and allows to account for risk and model uncertainty in a very convenient way. This tractability allows an empirical implementation that is not complex at all.

We find that the higher amount of observable long-run GDP ambiguity does not automatically coincide with a higher equilibrium impact on marginal utility. In particular, our maximum likelihood estimation which takes a rich panel of macro and bond yield data into account finds that the equilibrium model uncertainty premium for long-run GDP am-

[1] Cagetti, Hansen, Sargent, and Williams (2002) study model uncertainty about aggregate productivity, but this translates directly into model uncertainty about expected GDP growth.

[2] Similar to Cagetti, Hansen, Sargent, and Williams (2002), Gagliardini, Porchia, and Trojani (2009) argue in terms of model uncertainty about the productivity, but in equilibrium this translates into model uncertainty about the expected growth rate of GDP.

biguity is substantially smaller than the model uncertainty premium for long-run inflation ambiguity. We find that most of the nominal equilibrium term spread is paid because of model uncertainty about the long-run inflation model.

We also find that both model uncertainty premiums have different qualitative implications on real and nominal bonds. While periods of increased uncertainty lead to a steepening of the TIPS and nominal yield curve, it is the increase in the observed set of potential long-run GDP models which makes the TIPS yield curve slope stronger upwards. This highlights the finding that both sources of model uncertainty affect the term structure of real and nominal bonds qualitatively and quantitatively differently.

The rest of the paper is structured as follows. In the next section we set up the model, discuss the risk and model uncertainty dynamics of the economy, discuss the preferences of the agent, and solve for real and nominal equilibrium bond yields. Section 3 compares our model with the literature. Section 4 specifies the data and econometric method that we use for estimating the model. In section 5 we present the empirical findings. Section 6 concludes.

2 Model Setup

2.1 Domain

Time is continuous and varies over $t \in [0, ..., \infty)$. Real and nominal macroeconomic risk is represented by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q^0)$, where Q^0 stands for the reference macroeconomic model for the economy. For intuitive reasons it is useful to treat the solution of the reference model under Q^0 as the solution to the rational expectations model. All expectations in the reference model are taken under Q^0. We denote these expectations as $E[.]$ instead of $E^{Q^0}[.]$. The probability measure for the robust economy

will be determined endogenously in equilibrium. We will denote that measure as Q.

2.2 Economy Setup

2.2.1 Long-Run Risk

We follow previous research like Bansal and Yaron (2004), Brennan and Xia (2002), Lettau and Wachter (2009) and Piazzesi and Schneider (2006) and assume a dividend process, $d\ln Y$, and an inflation process, $d\ln p$, with stochastic expected growth and stochastic variance:[3]

$$d\ln Y_t = (g_0 + z_t)dt + \sqrt{\sigma_{0g} + \sigma_{1g}u_t}dW_t^Y, \tag{2.1}$$

$$d\ln p_t = (p_0 + w_t)dt + \sqrt{\sigma_{0g} + \sigma_{1g}u_t}\rho_{pg}dW^Y + \sqrt{\sigma_{0p} + \sigma_{1p}v_t}dW^p. \tag{2.2}$$

Following the long-run risk literature, one can think of z as the long-run GDP risk component, u as the short-run GDP risk component and w as the long-run inflation risk component and v as the short-run inflation risk component. All Brownian motions are orthogonal to each other. Having $\rho_{pg} \neq 0$ allows the model to capture a stochastic inflation risk premium. We let the data identify that parameter.

We group all state variables that characterize macroeconomic long-run and short-run risk into a state vector $\bar{X}$, where $\bar{X}_t = [u_t\, v_t\, w_t\, z_t]'$. Macroeconomic risk follows a continuous-time $AR(1)$ process with pairwise orthogonal innovations. Under the reference measure that means

$$d\bar{X} = \kappa\bar{X}dt + \Sigma dW, \tag{2.3}$$

where κ is a diagonal 4×1 matrix which captures the $AR(1)$ nature of macroeconomic risk, Σ is a 4×4 volatility matrix and dW collects the pairwise orthogonal innovations to

[3]We work with an endowment economy and with a representative agent. In these models, GDP and consumption coincide in equilibrium. We prefer to work with GDP directly.

macroeconomic risk. The volatility matrix takes the following form

$$\Sigma := \begin{pmatrix} \sigma_u & 0 & 0 & 0 \\ 0 & \sigma_v & 0 & 0 \\ 0 & 0 & \sigma_w & 0 \\ 0 & 0 & \sigma_{2z} & \sigma_{1z} \end{pmatrix}, \tag{2.4}$$

where σ_{2z} captures the correlation between expected GDP growth and expected inflation. In our sample, that correlation is -0.32.

2.2.2 Long-Run Ambiguity

As a novelty, we assume that the investor observes a multiple set of potentially correct long-run risk models. That means that the investor has several models for w and z on the table which all seem equally plausible to her. We do not assume that our investor is able to form a "correct" prior about all these models. This prevents her from applying Bayesian techniques to learn about long-run risk. As a result, the investor has no prior on which model within the set is right. The agent wants instead to find the long-run risk model within that set which minimizes her continuation utility. Since the investor has a unique reference model for GDP growth and inflation, she only needs to find endogenously the optimal amount of long-run ambiguity that she should take into account when doing investment decisions. We call the optimal amount of distortion in the long-run GDP risk component h_z, while the counterpart for long-run inflation ambiguity is called h_w.

The endogenous amount of long-run ambiguity coincides with the market price of ambiguity in the multiple prior literature or with the market price of one unit model uncertainty in the robust control literature.[4] We derive the long-run ambiguity premium

[4]Compare Epstein and Wang (1994), Epstein and Schneider (2003), Chen and Epstein (2002), Epstein and Miao (2003), and Drechsler (2009) for the former and Anderson, Hansen, and Sargent (2003), Cagetti, Hansen, Sargent, and Williams (2002), Hansen and Sargent (2007), Hansen and Sargent (2005), Hansen, Sargent, Turmuhambetova, and Williams (2005), Maenhout (2004), and Maenhout (2006) for the latter.

$h = (h_w, h_z)'$ in the next section. Note that h is known to the investor in every point in time and in every state of the world.

According to equation (2.3), we can state the long-run GDP risk model under the robust probability measure Q:

$$dz = \kappa_z z dt + \sigma_{1z}(dW^{z,h} + h_z dt) + \sigma_{2z}(dW^{w,h} + h_w dt). \tag{2.5}$$

Equation (2.5) and equation (2.3) state that an ambiguity averse investor who is confronted with a set of multiple long-run GDP risk models does not fully trust her reference model in equation (2.3). Instead, she adjusts the shocks to long-run GDP risk and long-run inflation risk in equation (2.3) by the corresponding market price of long-run ambiguity. This coincides with h_z for long-run GDP ambiguity and h_w for long-run inflation ambiguity, respectively.

The long-run inflation risk model under the robust probability measure Q follows analogously:

$$dw = \kappa_w w dt + \sigma_w(dW^{w,h} + h_w dt). \tag{2.6}$$

2.3 Preferences

The risk attitude of our investor is described by an agent with log utility preferences over the uncertain dividend stream. In addition, our investor is ambiguity averse with regard to the observed set of multiple long-run risk models. Compared to a rational expectations equilibrium, our investor solves a min-max optimization problem. First, the investor maximizes her utility with regard to the optimal consumption policy for all priors in her set of potential models. Second, the investor chooses from the set of multiple priors the prior which minimizes her life-time expected utility. We abstract from the first step, because we work with a standard endowment economy. This leaves us with the second step only. The solution to the minimization problem provides us with the market prices of long-run ambiguity.

The dynamic minimization problem is given by

$$\min_{Z\in Z(UB)} E^Z\left[\int_t^\infty e^{-\rho(s-t)}\ln Y_s ds|\mathcal{F}_t\right] \tag{2.7}$$

$$s.t. \tag{2.8}$$

$$d\ln Y_t = (g_0+g_1 z_t)dt + \sqrt{\sigma_{0g}+\sigma_{1g}u_t}dW_t^Y \tag{2.9}$$

$$dz_t = \kappa_z z_t dt + \sigma_{1z}(dW^{z,h}+h_z(t)dt) + \sigma_{2z}(dW^{w,h}+h_w(t)dt) \tag{2.10}$$

$$Z(UB) = \{Z\in\mathcal{Z}: \frac{1}{2}h_z^2(t)\leq UB_z(t)\}\cap\{Z\in\mathcal{Z}: \frac{1}{2}h_w^2(t)\leq UB_w(t)\}, \tag{2.11}$$

where ρ is the subjective time discount factor of the investor. The set $\mathcal{Z}$ is a well defined set of probability measures which are absolutely continuous with regard to the benchmark measure Q^0. For an observed two dimensional upper boundary UB, $Z(UB)$ contains all absolutely continuous macroeconomic models that fulfill the two entropy constraints.[5] The first entropy constraint requires that the amount of long-run GDP ambiguity must be weakly smaller than the observed upper boundary of potential long-run GDP risk models. The second entropy constraint requires that the amount of long-run inflation ambiguity must be weakly smaller than the observed upper boundary of potential long-run inflation risk models.

The novel contribution of our model is that we allow the agent to observe both sets of multiple long-run risk models. Said differently, our investor observes in each point in time and in each state of the world a set of potential long-run GDP risk models and a set of potential long-run inflation risk models. The investor chooses from that set the reference long-run risk model. One can think of that model as being the best description of the macroeconomic data. The amount of estimation risk that the investor faces in determining the reference model is captured by the cross-sectional variance of all potentially correct models with regard to the reference model. The investor knows that the distance of all potentially correct models is weakly smaller than a constant times the observed amount

[5] Chen and Epstein (2002) contain a detailed analysis of the required conditions.

of estimation risk:

$$UB_z(t) := A_z\eta_z^2(t) \qquad UB_w(t) := A_w\eta_w^2(t), \tag{2.12}$$

where η_z^2 and η_w^2 are the amount of stochastic real-time model estimation risk and A_z and A_w are positive scaling parameters. One can think of these constant parameters as determining the size of the confidence interval within which the true model lies.

Our investor is able to measure in real-time the amount of model estimation risk. She therefore treats these variables as additional observable macro variables. We assume the square-root of the amount of estimation risk follows a continuous-time $AR(1)$ process:

$$d\begin{pmatrix}\eta_w(t)\\ \eta_z(t)\end{pmatrix} = \left(\begin{pmatrix}a_{\eta_w}\\ a_{\eta_z}\end{pmatrix} + \begin{pmatrix}\kappa_{\eta_w} & 0\\ \kappa_{\eta_{zw}} & \kappa_{\eta_z}\end{pmatrix}\begin{pmatrix}\eta_w(t)\\ \eta_z(t)\end{pmatrix}\right)dt + \begin{pmatrix}\sigma_{\eta_w} & 0\\ 0 & \sigma_{\eta_z}\end{pmatrix} d\begin{pmatrix}W^{\eta_w}\\ W^{\eta_z}\end{pmatrix}, \tag{2.13}$$

where the Brownian shocks are pairwise orthogonal to all Brownian shocks in the economy. The scalar κ_{zw} captures a potential feedback effect from the long-run inflation ambiguity to long-run GDP ambiguity.

The solution to the minimization problem is well known in the literature[6]:

$$h_z(t) = -\sqrt{2 \cdot UB_z(t)} \tag{2.14}$$

$$h_w(t) = \sqrt{2 \cdot UB_w(t)}. \tag{2.15}$$

The first equation states that the absolute size of the long-run GDP ambiguity premium increases monotonically with the amount of its potential models. In ambiguous times, where the set of potential long-run GDP risk models increases, the investor *reduces* her expected prospects for the real growth rate of GDP by less than one to one.[7] With a

[6] Compare Hansen and Sargent (2007), Hansen, Sargent, Turmuhambetova, and Williams (2005), Chen and Epstein (2002) and others.

[7] This concave relationship arises because shocks to long-run GDP risk are Gaussian.

similar interpretation, in the second equation, when the set of potential long-run inflation risk models increases in ambiguous times, the investor *increases* her robust growth rate for inflation by less than one to one.

Plugging the parametric form of the upper boundary into the equilibrium condition reveals that the optimal amount of protection against long-run model risk is linear in the cross-sectional standard deviation of all potentially correct models:

$$h_z(t) = m_z \eta_z(t) \tag{2.16}$$

$$h_w(t) = m_w \eta_w(t), \tag{2.17}$$

where m_z is defined as $m_z := -\sqrt{2A_z} \in R^-$ and m_w is $m_w := \sqrt{2A_w} \in R^+$. The intuition of this equilibrium outcome follows the basic intuition of confidence intervals in classical statistics. The higher the confidence interval of the estimated long-run risk model, the more model uncertainty is in the economy. For an ambiguity averse investor it becomes optimal to require a premium for model misspecification doubts whose market price is a linear function of the amount of model estimation risk. The scaling parameters m_z and m_w are investor specific and measure the amount of standard deviations by which the investor perturbs her reference long-run risk model.[8] One can therefore think of m_z and m_w as model uncertainty preference parameters. The higher the parameters the bigger the distance between the estimated reference model and the endogenously selected worst-case model. This coincides with a higher premium for model uncertainty.

An application of Ito's lemma reveals that the market prices of model uncertainty

[8]The intuition of model estimation risk is related to the notion of "confidence risk" in Bansal and Shaliastovich (2010) and amount of macro uncertainty in David and Veronesi (2008) and the amount investor's model disagreement in Buraschi and Jiltsov (2006).

follow a Markov diffusion:

$$dh_z(t) = \left(m_z a_{\eta_z} + \kappa_{\eta_z} h_z(t) + \kappa_{\eta_{zw}} \frac{m_z}{m_w} h_w(t) \right) dt + m_z \sigma_{\eta_z} dW^{\eta_z} \tag{2.18}$$

$$dh_w(t) = \left(m_w a_{\eta_w} + \kappa_{\eta_w} h_w(t) \right) dt + m_w \sigma_{\eta_w} dW^{\eta_w}. \tag{2.19}$$

2.4 Government Bond Yields

The appendix shows that the equilibrium term structure of inflation-protected default-free bond yields is affine in long-run and short-run GDP risk and the market prices of long-run ambiguity. That means that any real yield, y^r, with time to maturity τ can be written as:

$$y_t^r(\tau) = -\frac{1}{\tau} \left(A^r(\tau) + B^{r\,\prime}(\tau) S(t) \right), \; S(t) = (u(t)\, z(t)\, h_w(t)\, h_z(t))', \tag{2.20}$$

where A^r and B^r are deterministic functions of the underlying economy.

By a similar argument, one derives that the equilibrium term structure of nominal default-free bond yields is affine in long-run and short-run GDP risk, long-run and short-run inflation risk and the market prices of long-run ambiguity:

$$y_t^{\$}(\tau) = -\frac{1}{\tau} \left(A^{\$}(\tau) + B^{\$\,\prime}(\tau) \bar{X}(t)' \right), \tag{2.21}$$

where $A^{\$}$ and $B^{\$}$ are deterministic functions of the underlying economy.

3 Related Literature

Our paper builds and extends several strands of the literature. Bansal and Yaron (2004) have proposed a macro-finance general equilibrium model where expected and unpredictable consumption growth are stochastic. Bansal and Yaron (2004) show that, a difficult to detect predictable component in expected consumption growth can in conjunction with Epstein and Zin (1989) preferences account for the equity premium, the low risk-free

rate and volatile stock markets. These promising results have generated a huge literature on refining the basic set-up of Bansal and Yaron (2004). Following the tradition of Lucas (1978), and Breeeden (1979), Mehra and Prescott (1985) most of these refinements have focused on equity related assets, such as stocks, stock options.

A smaller class of papers has used a version of Bansal and Yaron (2004) to study the implications for real and nominal bonds. Piazzesi and Schneider (2006) explain the upward sloping nominal yield curve with an Epstein and Zin (1989) investor who learns about the expected growth rate of inflation and consumption. In their benchmark model, relative risk aversion of 59 and IES of 1 amplifies the inflation risk premium which is induced through a negative correlation of expected GDP growth and expected inflation. Bansal and Shaliastovich (2009) work with an Epstein and Zin (1989) investor who has relative risk aversion of ten and IES of 1.5. Their model matches the nominal term premium through an amplified inflation risk premium, which arises in equilibrium because unpredicted changes in realized- and expected consumption growth correlate negatively with realized- and expected inflation.

Our paper is different in several aspects. First, we relax the assumption that the investor knows the unique prior for the long-run risk model. In our economy, investors have model misspecification doubts about the expected GDP growth rate and about the expected inflation rate. An ambiguity averse investor selects the worst-case long-run GDP risk and long-run inflation risk model from the observed set of potentially correct models. This wish for robustness induces a model uncertainty premium that accounts for the positive nominal term premium. Second, we work with a log utility model, which induces low risk aversion and low IES. Our model shows that a logarithmic utility model does not prevent the derivation of a meaningful macro-finance term structure model, because (i), interest rates are primarily affected by the IES and not so much by the degree of

risk aversion, (ii) Vissing-Jorgensen (2002) finds that the IES of bond holders is around $0.8-1$, (iii) the nominal term premium of approximately 1.5% is far lower than an equity premium of 6%, (iv) the term premium also depends on inflation and inflation premiums and (v) we account for long-run ambiguity which helps to explain the nominal term premium. Our model requires only a small amount of model uncertainty. The estimate for model uncertainty aversion about long-run inflation is 6.6 and the model uncertainty aversion about long-run GDP is 1.2.

There are only a few papers that study model uncertainty about long-run macro risk. In Hansen and Sargent (2009) the investor is uncertain whether consumption follows an i.i.d process or whether it follows a long-run risk model as in Bansal and Yaron (2004). Drechsler (2009) introduces ambiguity aversion into a long-run risk set-up and shows that model uncertainty can help to account for the variance premium and option skew of equity index options. Drechsler (2009) follows the ambiguity aversion literature which treats the set of potential models as a latent process.[9] Our model adds to this literature by treating the set of potential models as observable. The investor in our model observes in each point in time and in each state of the world the set of potential long-run risk models. We identify the set of potential long-run risk models by the dispersion in expected GDP growth and expected inflation forecasts, which are published by the Survey of Professional Forecasters.

Our paper adds also insights to the term structure literature. The calibration in Gagliardini, Porchia, and Trojani (2009) as well as the estimation of Kleshecheslski and Vincent (2009) support the view that ambiguity about GDP growth accounts for the upward sloping yield curve in U.S. data. This contradicts the finding of Ulrich (2010) who finds that ambiguity about inflation accounts for the upward sloping nominal yield curve.

[9]Early contributions to that literature include Epstein and Wang (1994), Epstein and Schneider (2003), Chen and Epstein (2002), Epstein and Miao (2003).

While each paper has its own valid motivation to focus its ambiguity analysis on GDP growth only or on inflation only, these studies open up the question of how an economy looks like if the investor is faced simultaneously by model uncertainty about GDP growth and by model uncertainty about inflation. Our model closes this gap. We derive a model which accounts for both sources of model uncertainty and we decompose the term spread of real and nominal bond yields into GDP ambiguity and inflation ambiguity components.

4 Data and Econometric Methodology

The goal of this section is to briefly describe the data and the econometric methodology used for estimating the equilibrium model. The empirical exercise shows how long-run ambiguity and long-run risk affect the term structure of nominal and real bonds. The intuition of previous research is that long-run inflation risk is much more important for explaining variations in nominal bond yields than fluctuations in long-run GDP risk [compare Ang, Bekaert, and Wei (2008), Gürkaynak, Sack, and Swanson (2005) and others.]. We analyze whether this intuition carries over to long-run ambiguity.

4.1 Data

In the model, the investor observes a set of potentially correct models. The data counterpart is the Survey of Professional Forecasters. Macro-econometricians from business and academic institutions as well as Wall Street econometricians are asked to provide quarterly forecasts for GDP growth and inflation. These professionals do not enclose the model that they use for these macro variables.[10] The outcomes of these different models differ sometimes stronger and sometimes less strong from each other. The investor does not know the identity of these forecasters and forecasting institutions. From the investor's perspective, she observes several potentially correct models and has to decide how to use

[10] The usual amount of forecasters lies between 30 and 40.

that information when pricing bonds.

The findings of Ang, Bekaert, and Wei (2007) suggests that the best out-of-sample forecasts for long-run inflation risk is achieved by using the median forecast from the Survey of Professional Forecasters. Our investor takes this into account by setting her reference long-run risk model to the median forecast. In particular, z coincides with the demeaned median forecast of GDP growth and g_0 coincides with its unconditional expectation. In the same way, w, coincides with the demeaned forecast for inflation and p_0 coincides with its unconditional expectation.

Our investor worries about the other potentially correct models which do not coincide with the median forecast. According to the model, the investor determines the amount of model estimation risk by estimating the cross-sectional standard deviation among the forecasters. The investor requires a market price of long-run ambiguity that is given by equation (2.16) and (2.17), where η coincides with the estimated model estimation risk. We find the constants m_z and m_w via the estimation.

Realized GDP growth, the median forecast for long-run GDP risk and the amount of model estimation risk are plotted in Figure 1. The long-run GDP component is an unbiased estimate for realized GDP growth and it is more persistent than GDP growth. The R^2 for a predictive regression of realized GDP growth on the reference long-run GDP risk model is 9% for a quarterly forecast horizon. The amount of model estimation risk is strongly time-varying, peaking during the Savings and Loan Crisis in the mid 1980s and tightening during the period of the Great Moderation.

The amount of model estimation risk for long-run inflation is lower than the long-run GDP counterpart. This coincides with a higher R^2 of 65% when regressing realized inflation on the median of the quarterly SPF inflation forecast. Graphically this is shown in

Figure 2. Figure 3 shows graphically that investor's face substantially more model estimation risk for the long-run GDP component compared to the long-run inflation component. During the Savings and Loan Crisis of the mid 1980s, model estimation risk for former outweighed the latter by factor ten.

We test our model with a rich panel of macro and bond yield data. Our macro variables include realized GDP growth, realized inflation and the Federal Funds rate. We downloaded that data from the St. Louis FRED data base. We use continuously compounded nominal U.S. government bond yields of maturities 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 years. The nominal bond yields are taken from the Board of Governors of the Federal Reserve System. We use continuously compounded yields of U.S. Treasury Inflation-Protected Securities (TIPS) of maturities 5, 6, 7, 8, 9, and 10 years, for the time horizon first quarter of 2003 to second quarter of 2009. This data is provided by the Board of Governors of the Federal Reserve System. All data, except TIPS, start in the first quarter of 1972 and end in the second quarter of 2009. We use TIPS data from the first quarter of 2003 onwards. We estimate the model by Quasi Maximum Likelihood and recover the latent short-run risk states u and v by inverting the affine yield relation for the one year and ten year nominal bond yield.

5 Empirical Findings

We find that the endogenously selected worst-case long-run risk model is very close to the reference long-run risk model. In a statistical sense, both models cannot be distinguished from each other. The estimated detection-error-probability is 47.7%. In other words, if an econometrician were asked whether the macro and financial data in our sample was generated by the reference long-run risk model or by the worst-case long-run risk model, she would select the wrong model with an unconditional probability of 47.7%. The mirror image of this statistic is that the time-series behavior of inflation and GDP growth in both models move so closely to each other that one cannot distinguish between both models.

Figure 4 shows graphically that the time-series for the reference GDP growth process is very close to the endogenously selected worst-case GDP growth model. Figure 5 shows the analog for the time-series of the reference inflation model and the worst-case inflation model.

Our model provides an excellent fit to real and nominal bonds. Table 1 shows that the average mean pricing error across all eight fitted nominal yields is only 2 basis points per quarter. The mean quarterly estimation error has been 4.4 basis points for GDP growth and 0.7 basis points for inflation. Figure 10 shows that the model fit for the entire sample and several subsamples is excellent. We have estimated the model only for the entire sample and contrasted also the model implied yield curve during the period of the Monetary Policy Experimentation and the Great Moderation with the empirically observed counterpart. The resulting yield curve fit is very good. The excellent fit during the Monetary Policy Experimentation is especially noteworthy. Structural bond models have difficulties in explaining the slightly upward sloping nominal yield curve in that period, because realized inflation and expected inflation were strongly mean-reverting. Our model accounts for that by a strongly downward sloping term structure of inflation expectations (green dotted solid line). Our model argues that the long-run ambiguity premium, especially long-run inflation ambiguity premium, has been strongly upward sloping, offsetting the downward sloping term structure of inflation. Said differently, although inflation seemed to mean-revert, investors at that time were faced with a relatively high and volatile amount of model estimation risk. That increased the set of potential long-run risk models. The resulting term structure of the ambiguity premium has been steep. We find that the inflation risk premium has remained positive but close to zero, even during the volatile period of the Monetary Policy Experimentation.

5.1 The Cross-Section of Long-Run Ambiguity

Figure 6 presents the two channels through which model uncertainty aversion with regard to the long-run risk components affects the cross-section of the term structure. The model implied TIPS yield curve is affine in the state and can be written as

$$y_t^r(\tau) = \alpha^r(\tau) + \beta_u^r(\tau)u(t) + \beta_z^r(\tau)z(t) + \\ + \beta_{h_w}^r(\tau)h_w(t) + \beta_{h_z}^r(\tau)h_z(t). \tag{5.1}$$

The last two beta loadings capture how the TIPS yield curve changes if the market price of model uncertainty changes. As shown in equation (2.17) and (2.16) this market price depends linearly on the size of the set of potential models and on the investor specific model uncertainty parameters m_w and m_z, respectively. The latter model uncertainty parameters, together with the steady state values of η, affect also the α^r loadings. Both channels together characterize the effect that model uncertainty has on the term structure of TIPS yields.

The upper panel of Figure 6 shows the model implied α for different ambiguity constellations. Intuitively, the graph shows that if there was no model uncertainty about the long-run risk model, the TIPS yield curve would be flat, at 2.18%, in steady state. This holds because the steady state values for the macro risk factors are all zero, which makes the corresponding α to be the steady state TIPS yield curve in absence of model uncertainty. Allowing for model uncertainty about the long-run GDP model lowers the α marginally across all maturities. Introducing model uncertainty about the long-run inflation model lowers long-term TIPS yields more than it lowers short-term yields. All else equal, this leads to a downward sloping TIPS yield curve. It is a combination of the higher estimated preference parameter m_w and steady state amount of model estimation risk a_{η_w} together with the h_w beta loading which gives inflation ambiguity a stronger effect on α.

The lower panel of Figure 6 shows how the cross-section of TIPS yields changes if the set of potential long-run risk models increases. Short-term TIPS yields drop more if the set of potential long-run GDP models increases. The TIPS yields drop because the expected GDP growth rate under the worst-case scenario is lower than under the reference scenario. Due to mean reversion in z and h, the effect of ambiguity dies out for longer duration TIPS.

The effect of ambiguity is different for the cross-section of nominal bonds. Similarly to TIPS yields, the model predicts an affine relationship for nominal yields:

$$y_t^{\$}(\tau) = \alpha^{\$}(\tau) + \beta_u^{\$}(\tau)u(t) + \beta_v^{\$}v(t) + \beta_w^{\$}(\tau)w(t) + \beta_z^{\$}(\tau)z(t) + \\ + \beta_{h_w}^{\$}(\tau)h_w(t) + \beta_{h_z}^{\$}(\tau)h_z(t). \tag{5.2}$$

As the upper panel of Figure 7 shows, the steady state nominal yield curve would be downward sloping if there was no model uncertainty. Being confronted with a set of several plausible long-run inflation risk models makes even the steady state nominal yield curve strongly upward sloping. The intuition for that result is that even when time goes to infinity, there will always remain several models in the set of potential long-run inflation models. Since the worst-case inflation forecast is higher than the reference forecast the corresponding nominal yield curve slopes upwards, because the investor cautiously expects a higher steady state nominal growth rate than under the reference model.

5.2 The Time-Series of Long-Run Ambiguity

For a one year nominal bond yield we find a small and time-varying ambiguity premium for long-run GDP ambiguity. That premium is negative, because an ambiguity averse investor lowers her expected path of future expected GDP growth, which ceteris paribus lowers real and nominal interest rates. The amount by which the one year nominal interest rate is lowered coincides with the long-run GDP ambiguity premium. The second panel

of Figure 8 shows that this premium is time-varying around minus one basis point. The upper panel shows that the long-run inflation ambiguity premium on a one year nominal bond fluctuates around 20 basis points. That premium is positive because an ambiguity averse investor in our model increases her worst-case inflation forecast slightly, in order to ceteris paribus lower the anticipated worst-case GDP path. The increase in the long-run inflation component coincides with the long-run inflation ambiguity premium. The ambiguity premium is small for short-term nominal bonds. The lower panel of Figure 8 shows that the overall long-run ambiguity premium in a one year nominal bond yield is small compared to the magnitude of that yield and its statistical standard deviation.

The longer the duration of the bond, the higher the absolute value of the long-run ambiguity premium. The time-series pattern does not change, because it is inherited from the observed set of model estimation risk. Figure 9 shows that the ambiguity premium for a ten year investment horizon is dominated by model misspecification doubts about long-run inflation. That premium fluctuates around 2% and has remained relatively stable since the 1991/1992 recession. The long-run GDP ambiguity premium is close to zero. The lower panel shows graphically, that although one is tempted to argue that the ten year nominal bond yield with ambiguity (black solid line) looks different than the implied ten year nominal bond yield if we shut down the long-run ambiguity channel (red dotted line), one can hardly distinguish both models since they lie well within a standard 95% empirical confidence interval (blue dotted line).

5.3 Relative Importance of Long-Run GDP Ambiguity vs. Long-Run Inflation Ambiguity

As shown in Figure 3, when looking only at the amount of model estimation risk in the data, one would conclude that there is more model uncertainty with regard to the long-

run GDP risk model than there is model uncertainty about the long-run inflation risk model. Such a conclusion would be supported by findings of Gagliardini, Porchia, and Trojani (2009) and Kleshecheslski and Vincent (2009) who argue that model uncertainty with regard to the GDP process is the main contributor to the model uncertainty premium.

The conclusion changes if we use a maximum likelihood estimation that takes also real and nominal bond yields into account. In such a specification we conclude that although there is more model uncertainty about long-run GDP risk in the data, the resulting ambiguity premium is substantially smaller than the ambiguity premium for long-run inflation risk. This result is confirmed by Ulrich (2010), who assumes that the central bank creates model uncertainty about future inflation and who finds that this model uncertainty can account for the positive slope of the nominal yield curve.

Our analysis allows us to analyze why investors require a substantially larger inflation ambiguity premium, although the Survey of Professional Forecasters tells us that there is more model uncertainty about the long-run GDP risk model. Our findings indicate that inflation ambiguity dominates the long-run GDP counterpart by factor 216. Our estimates suggest that the dominance is based on three drivers. First, an ambiguity averse investor who is uncertain about the true long-run inflation model cautiously increases her inflation forecast by an endogenous amount. Our estimate for the half-life of expected inflation indicates that it takes 15.8 years for the worst-case inflation forecast to reduce by fifty percent. Being confronted with GDP ambiguity is less costly in terms of time because the estimated half-life of an increase in the robust long-run GDP component is only 1.3 years. Intuitively, this means that in terms of half-life, getting the inflation model wrong is 12 times more costly. Second, we have estimated that investor's preference for protection against a misspecification of long-run inflation risk is approximately six times larger than the preference for robustness against long-run GDP risk. In particular, we

have estimated $m_w = 6.6$ and $m_z = -1.1$. Third, the estimated volatility of long-run inflation risk is approximately three times higher than the volatility of long-run GDP risk. Multiplying these effects ($12 \cdot 6 \cdot 3 = 216$) shows that the model uncertainty premium for being confronted with a set of multiple long-run risk models is approximately 200 times larger than the long-run GDP ambiguity premium counterpart.

The qualitative implications of both ambiguity premiums is different. As Figure 7 and 6 have shown, the factor loadings on both uncertainties is upward sloping, relating an increase in the observed amount of model uncertainty to a steepening in the real and nominal yield curve. We have also seen that the slope becomes steeper if the set of long-run GDP models goes up. Ambiguity about the long-run GDP risk models helps therefore to explain the upward sloping yield curve in TIPS. Such a model misspecification doubt reduces the short-end of the yield curve stronger than it does it for the long-end of the yield curve.

Ambiguity about the long-run inflation risk model helps to explain the upward sloping nominal yield curve. The last argument is mainly rooted in the higher persistence of long-run inflation compared to long-run GDP growth. The main reason for inflation ambiguity to dominate long-term nominal bond yields is consistent with Ang, Bekaert, and Wei (2008), Gürkaynak, Sack, and Swanson (2005), and Ulrich (2010), who all show the importance of bond implied inflation expectations for explaining the behavior long-term nominal bonds, compared to short-term nominal bonds.

6 Conclusion

We develop an equilibrium model which accounts for misspecification doubts about the long-run GDP risk model and the long-run inflation risk model. We connect the set of potentially correct long-run risk models to the observable amount of model estimation risk.

We use the cross-sectional variance of the mean forecast in the Survey of Professional Forecasters for inflation and GDP growth to measure the amount of model estimation risk that investors are confronted with. Our representative agent takes that measure of model uncertainty into account when pricing financial assets.

We find that our measure for the set of potentially correct long-run risk models is larger for long-run GDP risk, compared to long-run inflation risk. All else equal, this leads to the intuition that the marginal utility of investors is more affected by long-run GDP ambiguity. We find the opposite is true. We have estimated the model with a rich panel of macro and bond yield data and find that marginal utility of investors is dominated by long-run inflation ambiguity, and not by long-run GDP ambiguity. The main reason for this finding is based on the empirical properties of long-run GDP risk and long-run inflation risk. Our estimates show that long-run inflation risk is more persistent and more volatile. The former is especially important, because an ambiguity averse investor increases her worst-case inflation forecast and it takes 12 times longer for that forecast to reduce by fifty percent, compared to a misspecification of the long-run GDP growth component. In addition, investors seem to seek more protection against a potentially misspecified long-run inflation shock, compared to a potentially misspecified long-run GDP shock.

Our estimated term structure model argues that both source of ambiguity are important to capture different aspects of bond yields. Accounting for the set of multiple long-run GDP models helps to qualitatively account for a steepening of the TIPS yield curve in times of higher long-run GDP ambiguity. Model misspecification doubts about long-run inflation risk is essential to capture the upward sloping yield curve in U.S. nominal bonds.

Our analysis concludes that a model with log utility and observable amount of model

misspecification doubts about long-run GDP risk and long-run inflation risk is very well able to explain the real and nominal U.S. yield curve. This finding has important implications for future research. It establishes the finding that nominal uncertainty might be even more important for understanding investor's marginal utility than measures of real uncertainty.

A Appendix:

A.1 Term Structure of Inflation-Indexed Bonds:

The equilibrium price of an inflation-indexed zero-coupon bond $B_t(\tau)$ with time to maturity τ equals the inflation ambiguity adjusted conditional expected value of the intertemporal marginal rate of consumption substitution:

$$B_t(\tau) = e^{-\rho\,\tau} E_t^Q \left[\frac{u_Y(Y_{t+\tau})}{u_Y(Y_t)} \right]. \tag{A.1}$$

Plugging in the log utility function together with the consumption and inflation process and defining $\kappa_t \equiv \rho t + \ln(Y_t)$ yields

$$B_t(\tau) = \frac{1}{\exp(-\kappa_t)} E_t^Q \left[\exp(-\kappa_{t+\tau}) \right]. \tag{A.2}$$

The no-arbitrage price at time t of a zero-coupon bond maturing in $t+\tau$ solves the stochastic problem in (A.2). To get a closed-form solution we apply Feynman-Kac's Theorem and solve the dual parabolic PDE:

$$\frac{\partial B(\cdot,\tau)}{\partial \tau} = \mathcal{A} B(\cdot,\tau) \tag{A.3}$$

$$s.t. \quad B(\cdot,0) = 1, \tag{A.4}$$

where $B(\cdot,\tau) \equiv B(\kappa_t, u_t, z_t, h_w(t), h_z(t); \tau)$ and $\mathcal{A}$ represents the second-order differential operator applied to function $B(\cdot,\tau)$. Define $\phi(\kappa_t, u_t, z_t, h_w(t), h_z(t); \tau)$ to be the solution of the stochastic problem:

$$\phi(\kappa_t, u_t, z_t, h_w(t), h_z(t); \tau) = E_t^Q \left[\exp(-\kappa_{t+\tau}) \right]. \tag{A.5}$$

Since our economy has logarithmic preferences with an affine consumption process, we guess that the solution has the form:

$$\phi(\kappa_t, u_t, z_t, h_w(t), h_z(t); \tau) = e^{-\kappa_t} Z(\tau)\, e^{b_u(\tau) u_t + b_z(\tau) z_t + b_{h_w}(\tau) h_w(t) + b_{h_z}(\tau) h_z(t)}. \tag{A.6}$$

If (A.6) solves the stochastic problem than it also solves the PDE

$$\frac{\partial \phi(\cdot,\tau)}{\partial \tau} = \mathcal{A}\phi(\cdot,\tau) \tag{A.7}$$

$$s.t. \quad \lim_{\tau \downarrow 0} \phi(\cdot,\tau) = \exp(-\kappa_t), \tag{A.8}$$

where $\phi(\cdot,\tau) \equiv \phi(\kappa_t, u_t, z_t, h_w(t), h_z(t); \tau)$. Solving the pde gives the result.

A.2 Nominal Term Structure:

The equilibrium price of a nominal zero-coupon bond $N_t(\tau)$ with time to maturity τ equals the inflation ambiguity adjusted conditional expected value of the intertemporal marginal rate of consumption substitution times the real payoff at maturity:

$$N_t(\tau) = e^{-\rho\,\tau} E_t^Q \left[\frac{u_Y(Y_{t+\tau})}{u_Y(Y_t)} \frac{p_t}{p_{t+\tau}} \right]. \tag{A.9}$$

Plugging in the log utility function together with the consumption and inflation process and defining $\kappa_t \equiv \rho t + \ln(Y_t)$ yields

$$N_t(\tau) = \frac{1}{\frac{\exp(-\kappa_t)}{p_t}} E_t^Q \left[\frac{\exp(-\kappa_{t+\tau})}{p_{t+\tau}} \right]. \tag{A.10}$$

The no-arbitrage price at time t of a zero-coupon bond maturing in $t+\tau$ solves the stochastic problem in (A.10). To get a closed-form solution we apply Feynman-Kac's Theorem and solve the dual parabolic PDE:

$$\frac{\partial N(\cdot,\tau)}{\partial \tau} = \mathcal{A}N(\cdot,\tau) \tag{A.11}$$

$$s.t. \quad N(\cdot,0) = 1, \tag{A.12}$$

where $N(\cdot,\tau) \equiv N(\kappa_t, p_t, u_t, v_t, w_t, z_t, h_w(t), h_z(t); \tau)$ and $\mathcal{A}$ represents the second-order differential operator applied to function $N(\cdot,\tau)$. Define $\phi(\kappa_t, p_t, u_t, v_t, w_t, z_t, h_w(t), h_z(t); \tau)$ to be the solution of the stochastic problem:

$$\phi(\kappa_t, p_t, u_t, v_t, w_t, z_t, h_w(t), h_z(t); \tau) = E_t^Q \left[\frac{\exp(-\kappa_{t+\tau})}{p_{t+\tau}} \right]. \tag{A.13}$$

Since our economy has logarithmic preferences with an affine consumption and inflation process, we guess that the solution has the form:

$$\phi(\kappa_t, p_t, u_t, v_t, w_t, h_w(t), h_z(t); \tau) = \frac{e^{-\kappa_t} Z(\tau)\, e^{b_u(\tau)u_t + b_v(\tau)v_t + b_w(\tau)w_t + b_z(\tau)z_t + b_{h_w}(\tau)h_w(t) + b_{h_z}(\tau)h_z(t)}}{p_t}. \tag{A.14}$$

If (A.14) solves the stochastic problem than it also solves the PDE

$$\frac{\partial \phi(\cdot, \tau)}{\partial \tau} = \mathcal{A}\phi(\cdot, \tau) \tag{A.15}$$

$$s.t. \quad \lim_{\tau \downarrow 0} \phi(\cdot, \tau) = \frac{\exp(-\kappa_t)}{p_t}, \tag{A.16}$$

where $\phi(\cdot, \tau) \equiv \phi(\kappa_t, p_t, u_t, v_t, w_t, z_t, h_w(t), h_z(t); \tau)$. Solving the pde gives the result.

A.3 Detection Error Probability:

The derivation of the detection-error probabilities follows directly from Maenhout (2006). We sketch the main steps in the following.

$$\epsilon_T(m_w, m_z) = \frac{1}{2}\left(Pr(\ln Q_T > \ln Q_T^0 | Q^0, \mathcal{F}_0) + Pr(\ln Q_T^0 > \ln Q_T | Q, \mathcal{F}_0)\right) \tag{A.17}$$

$$= \frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \left(Re\left(\frac{\phi^Q(w, 0, T)}{iw}\right) - Re\left(\frac{\phi(w, 0, T)}{iw}\right)\right) dw \tag{A.18}$$

where $\phi(.)$ is defined as $\phi(w, 0, T) := E\left[e^{i \cdot w \cdot \xi_{1,T}} | \mathcal{F}_0\right]$ and $\phi^Q(.)$ is defined as $\phi^Q(w, 0, T) := E^Q\left[e^{i \cdot w \cdot \xi_{1,T}} | \mathcal{F}_0\right]$ and $\xi_{1,T} = \ln \frac{dQ_T}{dQ_T^0}$.

Applying Feynman-Kac theorem to ϕ^Q and ϕ reveals that they are an exponentially quadratic function in the amount of ambiguity distortion h_t:

$$\phi^Q(w, t, T) = z_t^{iw+1} e^{G(\tau) + E(\tau)h_w(t) + F(\tau)h_z(t) + \frac{K(\tau)}{2}h_w^2(t) + \frac{M(\tau)}{2}h_z^2(t) + N(\tau)h_w(t)h_z(t)} \tag{A.19}$$

$$\phi(w, t, T) = z_t^{iw} e^{\hat{G}(\tau) + \hat{E}(\tau)h_w(t) + \hat{F}(\tau)h_z(t) + \frac{\hat{K}(\tau)}{2}h_w^2(t) + \frac{\hat{M}(\tau)}{2}h_z^2(t) + N(\tau)h_w(t)h_z(t)} \tag{A.20}$$

$$z_T := e^{\xi_{1,T}}, \tag{A.21}$$

where $G(\tau), E(\tau), F(\tau), K(\tau), M(\tau), N(\tau), \hat{G}(\tau), \hat{E}(\tau), \hat{F}(\tau), \hat{K}(\tau), \hat{M}(\tau), \hat{N}(\tau)$ are deterministic solutions to standard complex valued Riccati equations.

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Table 1: PARAMETER ESTIMATES (Standard Errors $\times 1.0e-8$)

Panel A: State Variables

	Drift, κ		Volatility, σ		a
u	-0.109 (26.9)		0.004 (0.04)		0 (fixed)
v	-0.0086 (7.4)		0.0038 (0.2)		0 (fixed)
w	-0.011 (15.2)		0.0095 (30.3)		0 (fixed)
z	-0.132 (3851.5)		0.0031 (11.9)	-0.0014 (0.05)	0 (fixed)
η_w	-0.489 (99.1)		0.0047 (0.0007)		0.0027 (0.04)
η_z	-2.6 (3082866.1)	0.128 (14150.7)	0.0076(0.6)		0.00066 (0.73)

Panel B: Growth and Inflation

c_0	0.0065 (fixed)
c_1	1.0 (fixed)
p_0	0.0096 (fixed)
p_1	1.0 (fixed)
σ_{0c}	0.000035 (0.0012)
σ_{1c}	1.945 (526.7)
σ_{0p}	0.00019 (0.0012)
σ_{1p}	0.982 (14288.4)
ρ_{pc}	0.892 (135.6)
ρ	0.001 (fixed)
m_w	6.57 (1509.9)
m_z	-1.17 (40582.0)

Table 2: Yield Curve, in %, per quarter

$y^{\$}$

maturity	R	4	8	12	16	20	24	28	32	36	40
data	1.6	1.5822	1.6391	1.6819	1.7159	1.7453	1.7717	1.795	1.8154	1.8327	1.8484
model	1.54	1.5822	1.5705	1.6141	1.6596	1.7056	1.7508	1.7964	1.8417	1.8875	1.8484

y^{r}

maturity	20	24	28	32	36	40
data	0.3782	0.4109	0.4400	0.4650	0.4859	0.5014
model	0.5526	0.5464	0.5403	0.5347	0.53	0.528

Figure 1: **GDP Growth, Long-Run GDP Risk, Long-Run GDP Ambiguity, 1972.I - 2009.II**
This figure presents quarterly GDP growth and its ex-ante expected value together with the empirical amount of model estimation risk. The green solid line presents realized quarterly GDP growth. The black dotted line presents the ex-ante expected GDP growth as measured by the median GDP growth forecast of the Survey of Professional Forecasters (SPF). The two solid blue lines presents a 3 standard deviation confidence interval for the precision of the median forecast. The standard deviation is based on the cross-sectional standard deviation of the forecasted SPF models. All values are annualized an in percent.

The ex-ante expected GDP growth rate is our measure of long-run GDP risk. The cross-sectional standard deviation of potentially correct forecast models is our real-time measure for long-run GDP ambiguity.

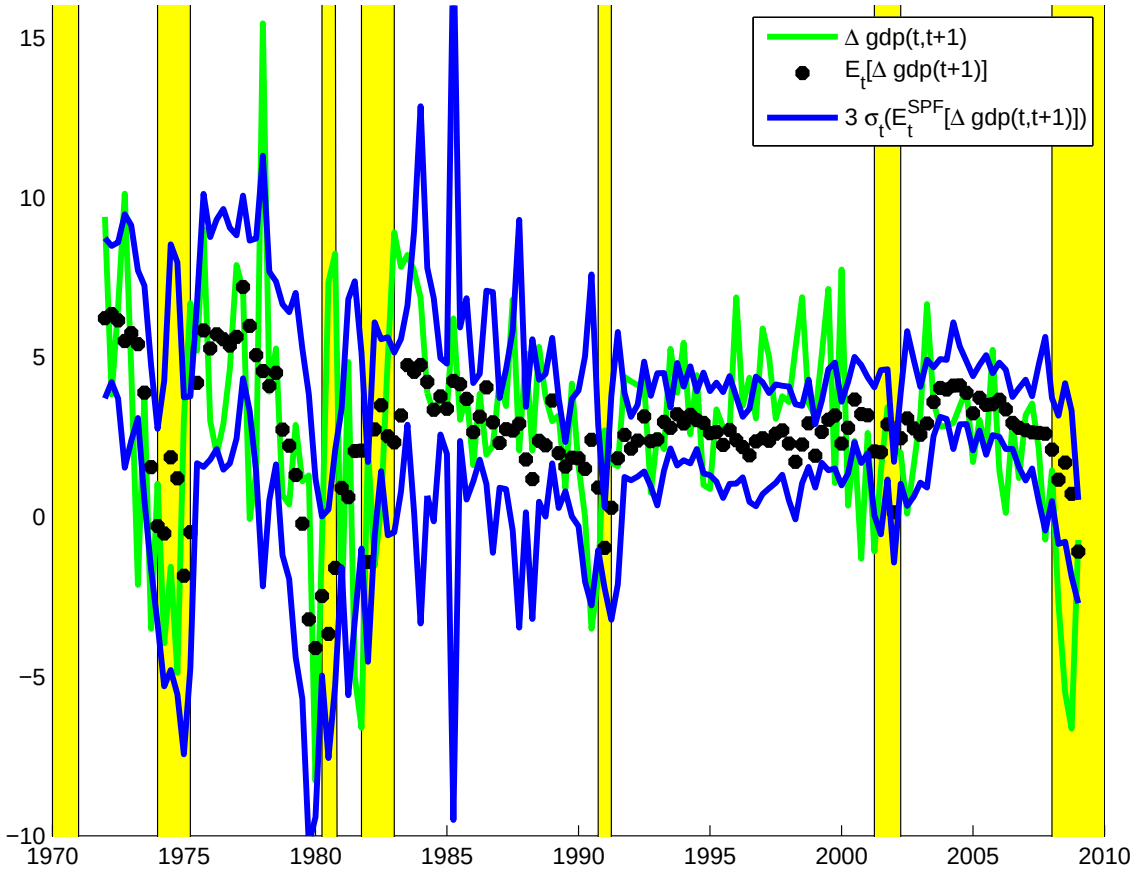


Figure 2: **Inflation, Long-Run Inflation Risk, Long-Run Inflation Ambiguity, 1972.I - 2009.II**
This figure presents quarterly inflation and its ex-ante expected value together with the empirical amount of model estimation risk. The red solid line presents realized quarterly inflation. The black dotted line presents the ex-ante expected inflation as measured by the median inflation forecast of the Survey of Professional Forecasters (SPF). The two solid blue lines presents a 3 standard deviation confidence interval for the precision of that median forecast. The standard deviation is based on the cross-sectional standard deviation of the forecasted SPF models. All values are annualized an in percent.

The ex-ante expected inflation rate is our measure of long-run inflation risk. The cross-sectional standard deviation of potentially correct forecast models is our real-time measure for long-run inflation ambiguity.

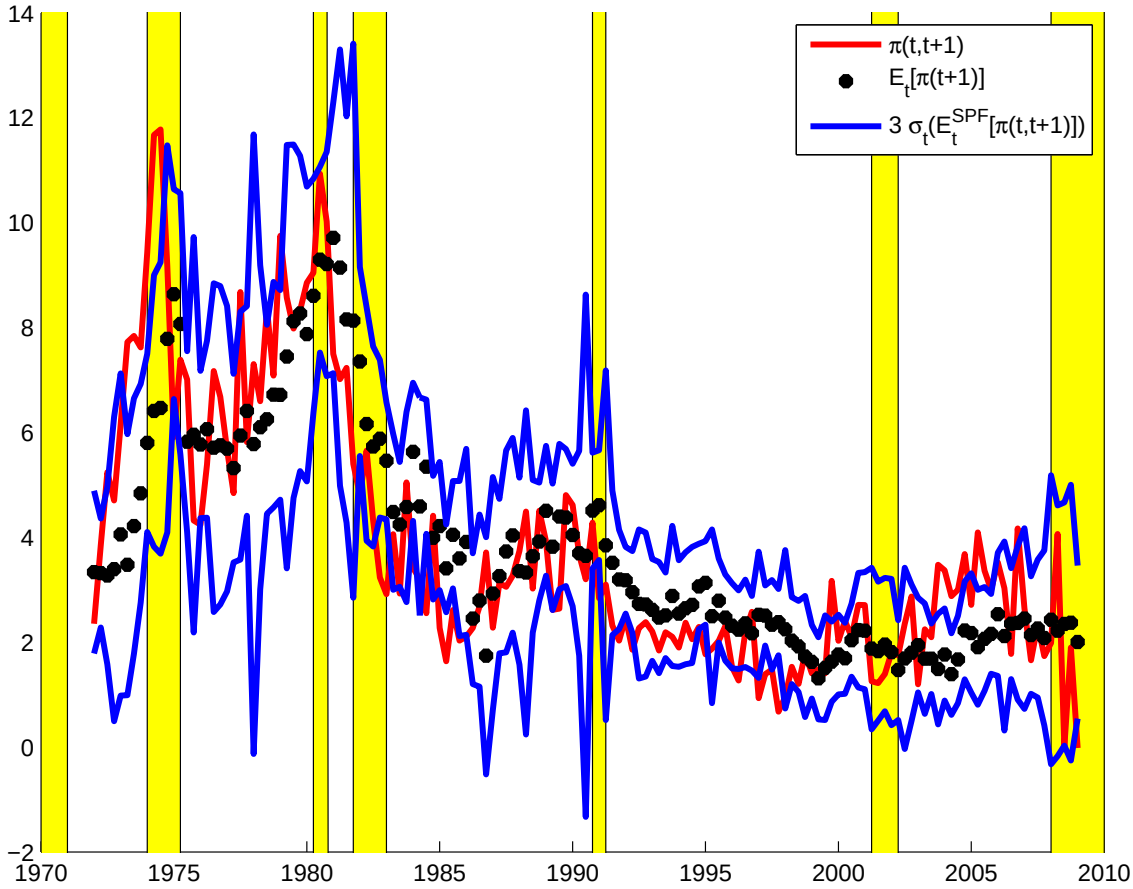


Figure 3: **Observable Long-Run Ambiguity, 1972.I - 2009.II**
This figure plots the quarterly amount of observable long-run ambiguity. It coincides with the amount of model estimation risk. That measure of model uncertainty coincides with the cross-sectional standard deviation of forecasted long-run risk models. The forecasts are taken from the Survey of Professional Forecasters. The green solid line coincides with the amount of model estimation risk for the long-run GDP risk model. The red solid line represents the amount of model estimation risk for the long-run inflation risk model.

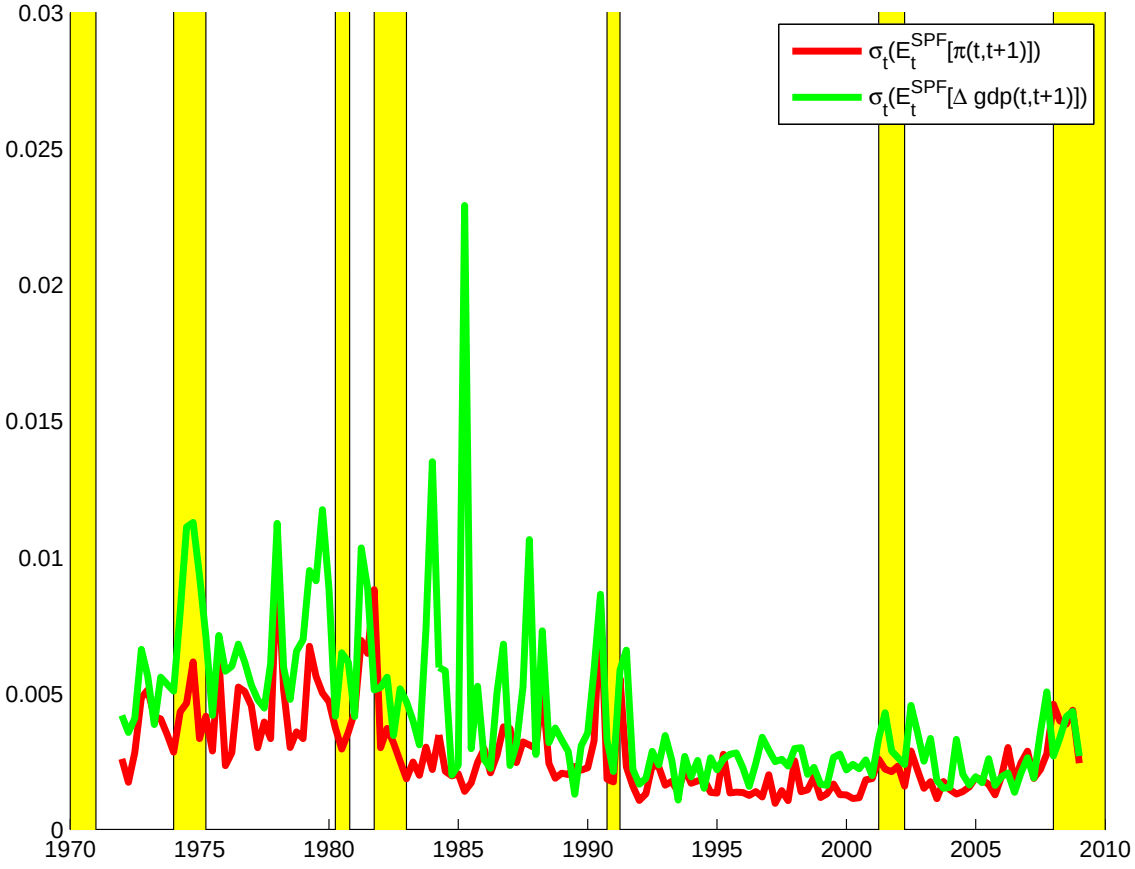


Figure 4: **Empirical- vs. Worst-Case Long-Run GDP Risk Model, 1972.I - 2009.II**
This figure plots the reference model (empirical) for long-run GDP risk (red star) together with the worst-case long-run GDP risk model (blue dotted). The reference model coincides with the median forecast of the Survey of Professional Forecasters. The worst-case model is endogenously determined as an equilibrium outcome.

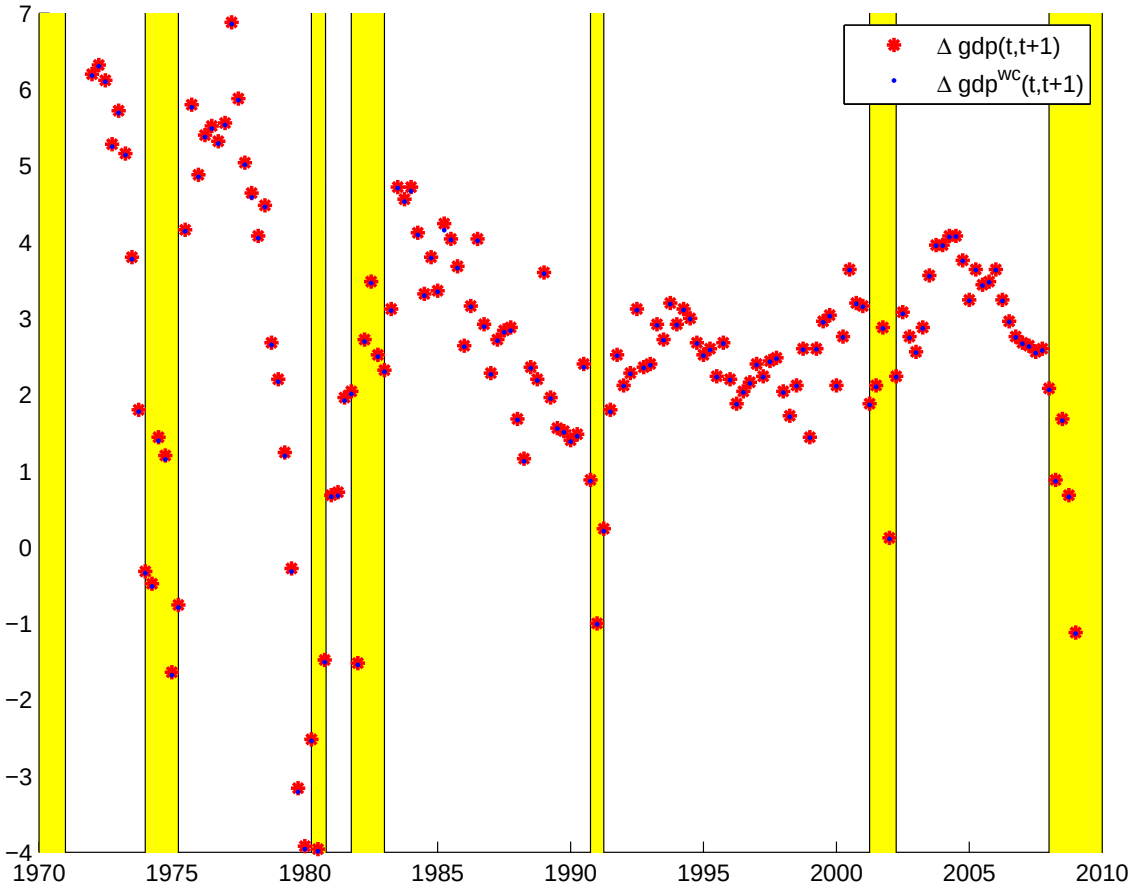


Figure 5: **Empirical- vs. Worst-Case Long-Run Inflation Risk Model, 1972.I - 2009.II**
This figure plots the reference model (empirical) for long-run inflation risk (red star) together with the worst-case long-run inflation risk model (blue dotted). The reference model coincides with the median forecast of the Survey of Professional Forecasters. The worst-case model is endogenously determined as an equilibrium outcome.

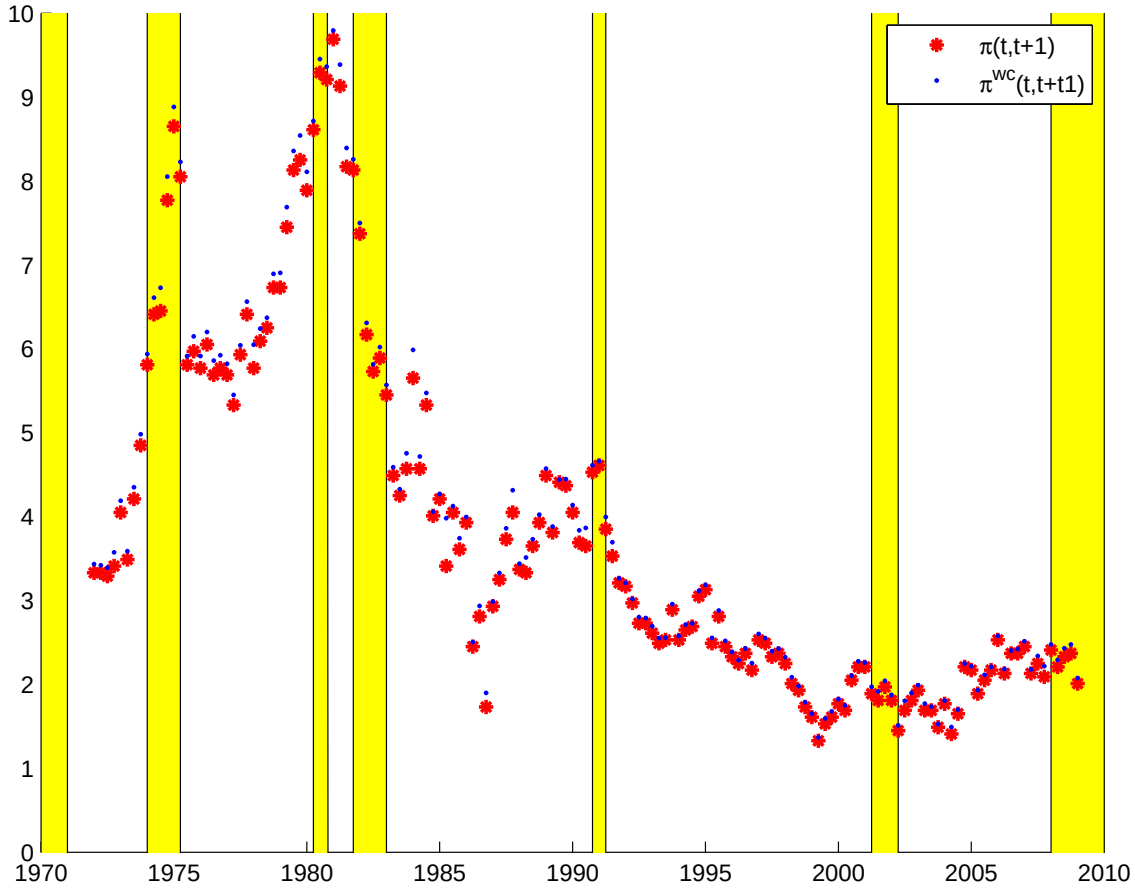


Figure 6: **Long-Term and Short-Term Ambiguity in TIPS Yields**
This graph decomposes the channel through which ambiguity affects TIPS yields. The cross-section is characterized by $y_t^r(\tau) = \alpha^r(\tau) + ...\beta_{h_w}^r(\tau)h_w(t) + \beta_{h_z}^r(\tau)h_z(t)$. Ambiguity enters through α^r and $\beta_{h_w}^r, \beta_{h_z}^r$. The upper panel plots the model implied α^r for different ambiguity specifications, where $\alpha^{r,NoAmbig}$ refers to the case of no ambiguity at all, $\alpha^{r,zAmbig}$ stands for the case where only long-run GDP ambiguity exists and $\alpha^{r,wAmbig}$ summarizes the case if only long-run inflation ambiguity exists. The lower panel presents the factor loadings on both long-run uncertainty measures.

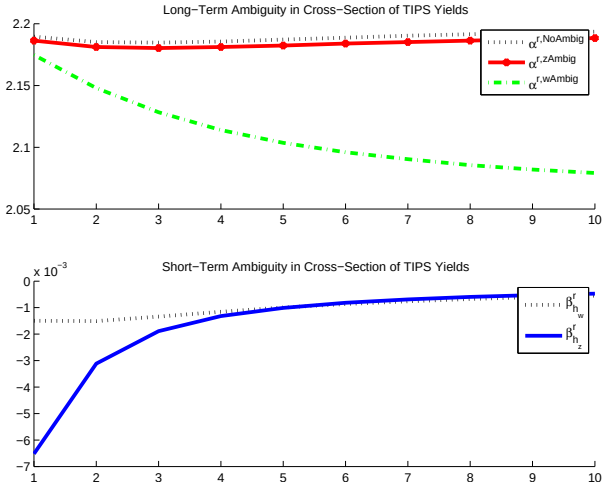


Figure 7: **Long-Term and Short-Term Ambiguity in Nominal Yields**
This graph decomposes the channel through which ambiguity affects nominal yields. The cross-section is characterized by $y_t^{\$}(\tau) = \alpha^{\$}(\tau) + ...\beta_{h_w}^{\$}(\tau)h_w(t) + \beta_{h_z}^{\$}(\tau)h_z(t)$. $\alpha^{\$,NoAmbig}$ refers to the case of no ambiguity at all, $\alpha^{\$,zAmbig}$ stands for the case where only long-run GDP ambiguity exists and $\alpha^{\$,wAmbig}$ summarizes the case if only long-run inflation ambiguity exists.

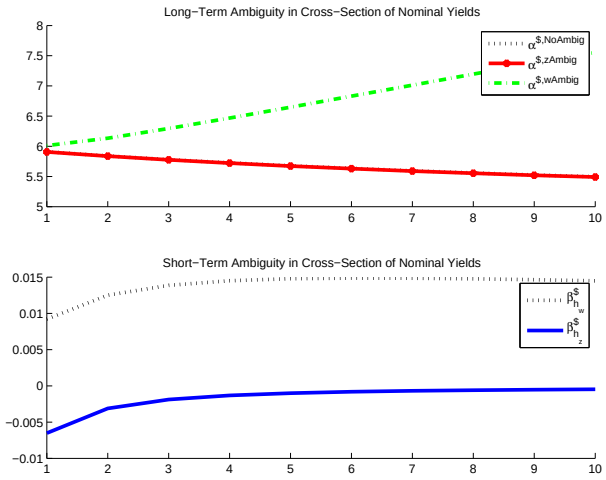


Figure 8: **Premium for Long-Run Ambiguity in Short-Term Yield, 1972.I - 2009.II**
The black solid line corresponds to the one year nominal bond yield, as observed in the data. The dashed blue line summarizes its empirical 95% confidence interval. The red dotted line plots the model implied one year nominal yield under the reference model. The reference and worst-case model are so close that one cannot distinguish both bond yield time-series.

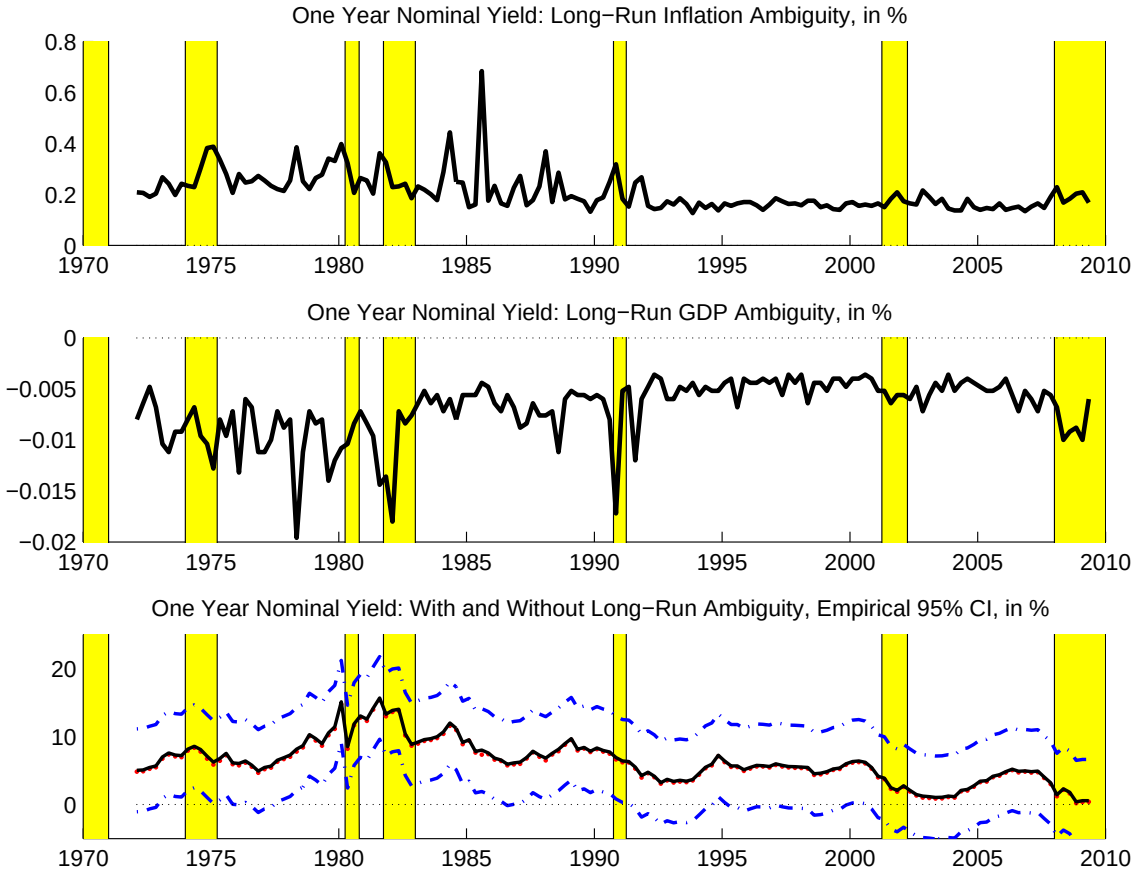


Figure 9: **Premium for Long-Run Ambiguity in Long-Term Nominal Yield, 1972.I - 2009.II**
The black solid line corresponds to the ten year nominal bond yield, as observed in the data. The dashed blue line summarizes its empirical 95% confidence interval. The red dotted line plots the model implied ten year nominal yield under the reference model. The reference and worst-case model are so close that one cannot distinguish both bond yield time-series.

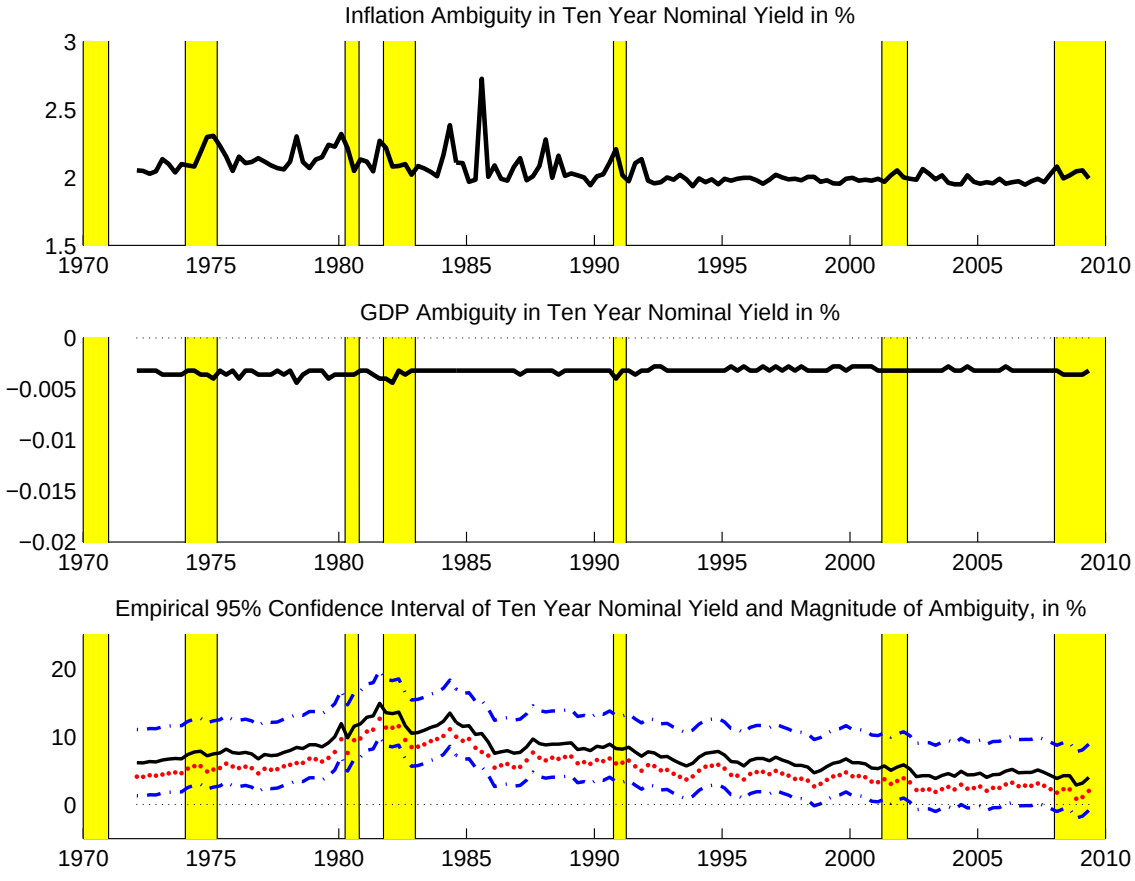


Figure 10: **Decomposing Nominal Yield Curve**
The solid black line represents the model implied average nominal yield curve. The yellow . − . line shows the average nominal yield curve in the data. The green −∗ line represents the inflation forecast (empirical measure). The red .− graph represents the ambiguity premium (GDP and inflation). The pink ∗ graph represents the estimated TIPS yield curve. The solid blue *star* line represents the inflation risk premium.

The model is estimated over the entire sample 1972.I - 2009.II. The estimated parameters and states are held fixed for the different sub-samples.

