NBER WORKING PAPER SERIES

OPTIMAL POPULATION GROWTH IN THE DIAMOND OLG MODEL

Laurence M. Ball

Working Paper 35823
http://www.nber.org/papers/w35823

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
September 2026

I am grateful for outstanding research assistance from Srisha Murthy Nippani and for suggestions from N. Gregory Mankiw and David Romer. I used AI extensively in deriving this paper's results, but I wrote and checked the final proofs presented here. The views expressed herein are those of the author and do not necessarily reflect the views of the National Bureau of Economic Research.

NBER working papers are circulated for discussion and comment purposes. They have not been peer-reviewed or been subject to the review by the NBER Board of Directors that accompanies official NBER publications.

© 2026 by Laurence M. Ball. All rights reserved. Short sections of text, not to exceed two paragraphs, may be quoted without explicit permission provided that full credit, including © notice, is given to the source.

Optimal Population Growth in the Diamond OLG Model
Laurence M. Ball
NBER Working Paper No. 35823
September 2026
JEL No. H55, J11, O41

ABSTRACT

This paper studies the effects of population growth on welfare in an OLG model, in which lower population growth raises output per worker but also reduces the ratio of workers to retirees. Under fairly mild restrictions on the model's parameters, there exists an optimal population-growth rate—one that maximizes steady-state welfare. This growth rate may well be negative: it may be optimal for the population to shrink over time. The introduction of a pay-as-you-go social security system reduces the optimal population-growth rate.

Laurence M. Ball
Johns Hopkins University
Department of Economics
and NBER
lball@jhu.edu

1 Introduction

In the late 20th century, economists worried that high rates of population growth were a threat to prosperity. An intellectual foundation for this view was the neoclassical growth model (Solow, 1956), in which population growth reduces capital accumulation and therefore output per worker. Concern about population growth led governments to adopt policies to curb it, most famously China's one-child policy.

In the 21st century, population growth has slowed in most of the world and there has been a sharp shift in thinking: economists now worry about the costs of *low* population growth. A major concern is that this development raises the number of retirees relative to the productive workforce, which is especially problematic in the presence of a pay-as-you-go social security system.[1]

Is population growth generally good or bad for an economy? Is there some optimal rate of population growth, with welfare losses from growth rates either above or below that level? If so, what factors determine optimal population growth?

The costs and benefits of population growth are analyzed in a number of recent studies, such as Kotschy and Bloom (2023) and Weil (2026). This paper contributes to this literature by studying the effects of population growth in Diamond's (1965) overlapping-generations model. This canonical growth model is a natural one to consider because it captures both the capital-eroding effect of high population growth and the adverse demographic effect of lower growth. The model is also a standard tool for studying economies with social security systems.

The Diamond model is highly stylized; it lacks the quantitative realism of many studies of population growth. The virtues of the Diamond model are its analytical tractability, transparency, and microeconomic foundations, which enable a rigorous welfare analysis.

[1] For 20th century worries about rapid population growth, see the 1977 speech by World Bank President McNamara, who says that "population growth is the gravest issue the world faces" except for the threat of nuclear war. For recent worries about low population growth, see the IMF's analysis of the "silver economy" in its *World Economic Outlook* (2025).

Just as this simple model has helped us understand optimal saving and the effects of fiscal policy, it can clarify the costs and benefits of population growth.[2]

Section 2 of this paper presents a standard version of the Diamond model without a social security system, and Section 3 asks how population growth affects welfare in the model's steady state. An optimal population-growth rate does not always exist, but one does exist under fairly mild restrictions on parameters (in particular, on the capital intensity of the production function and the substitutability of consumption when young and old). Under these conditions, there is a closed-form solution for optimal population growth. For some reasonable calibrations of the model, this growth rate is negative: it is optimal for the population to shrink over time.

Section 4 adds a social security system to the model. The main result here is that the presence of social security reduces the optimal population-growth rate. This result may seem surprising, because social security creates a cost of lower population growth: it reduces the benefit to retirees for a given tax on workers. However, social security also has an opposite effect: it increases the marginal value of capital, which raises the gain from increasing capital through lower population growth. The higher marginal value of capital arises partly because social security reduces the level of capital—the crowding-out effect—and partly through a more subtle effect for a given level of capital. In the model, the resulting effect on optimal population growth outweighs the effect through the size of the social security benefit.

Section 5 concludes the paper. I discuss how the analysis fits into the broader debate about population growth and outline some ways in which the model might be extended. I also introduce a companion paper (Ball, 2026), which goes beyond this paper's steady-state analysis to study transitions between regimes with higher and lower population growth.

[2] A recent paper with some similarities to this one is Abel and Panageas (2025). That paper introduces immigration of foreigners into an OLG model and derives the optimal level of immigration. Future work might consider a model in which population growth depends on both immigration and the fertility of natives.

2 The Model

My specification of the Diamond model mostly follows the Romer textbook (2019, chapter 2). However, I specialize the model by assuming a Cobb-Douglas functional form for the production function, which yields analytical results that we can interpret. The Appendix presents the more general but messier case of a CES production function.

Assumptions

The model is set in discrete time. In each period t, a cohort of L_t people is born and lives for two periods, t and $t+1$. The size of a cohort grows at a gross rate N: $L_t = NL_{t-1}$. When young at t, an individual inelastically supplies a unit of labor, earns a wage, and divides this income between current consumption and saving. Each unit of saving produces a unit of capital, which the individual rents out at $t+1$ to finance consumption when old. Capital depreciates fully after one period.[3]

A person born in period t has the utility function:

$$U_t = \frac{(C_t^y)^{1-\theta}}{1-\theta} + \beta\frac{(C_{t+1}^o)^{1-\theta}}{1-\theta}, \qquad \beta > 0, \quad \theta > 0, \tag{1}$$

where C_t^y is the person's consumption when young at t and C_{t+1}^o is their consumption when old at $t+1$. The parameter β is a discount factor, and θ is the coefficient of relative risk aversion; the intertemporal elasticity of substitution in consumption, which we will denote by η, is $\eta = 1/\theta$. This utility function is a standard assumption because it implies balanced growth.

Firms hire workers, rent capital, and produce output with a Cobb-Douglas production function with labor-augmenting technical progress. Individual firms' production

[3]The original Diamond paper and the Romer text assume that capital does not depreciate. The assumption of full depreciation makes the model more tractable and is also more realistic when a period is half a lifetime.

functions can be aggregated to yield

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}, \qquad 0 < \alpha < 1, \tag{2}$$

where Y is aggregate output, K is capital, and A is the effectiveness of labor, which grows at rate G: $A_t = GA_{t-1}$. This assumption implies that output per effective worker, $y_t = Y_t/(A_t L_t)$, is a function of capital per effective worker, $k_t = K_t/(A_t L_t)$:

$$y_t = f(k_t) = k_t^{\alpha}. \tag{3}$$

Factor markets are competitive, so labor and capital are paid their marginal products. The rental price of capital is $R_t = f'(k_t) = \alpha k_t^{\alpha-1}$ and the wage per effective worker is $w_t = f(k_t) - f'(k_t)k_t = (1-\alpha)k_t^{\alpha}$. Someone born at t supplies A_t units of effective labor and earns a total of $A_t w_t$. Firms' profits are zero so it does not matter who owns them.

Capital Accumulation

A person born at t chooses their saving when young to maximize lifetime utility. Let s_t be the fraction of earnings that is saved, so a person's saving is $s_t A_t w_t$. Consumption when young equals the earnings that are not saved, and consumption when old is the return on saving:

$$C_t^y = (1-s_t)A_t w_t; \tag{4}$$

$$C_{t+1}^o = R_{t+1} s_t A_t w_t. \tag{5}$$

Substituting these expressions into the utility function and maximizing with respect to s_t yields

$$s_t = \frac{\beta^{\eta}}{\beta^{\eta} + R_{t+1}^{1-\eta}}, \tag{6}$$

where again $\eta = 1/\theta$. Saving per effective worker equals $s_t w_t$ and total saving is $s_t w_t A_t L_t$.

Total capital at $t+1$ equals total saving at t, and the number of effective workers at $t+1$ is $A_{t+1}L_{t+1}$. Thus capital per effective worker at $t+1$ is $k_{t+1} = s_t w_t A_t L_t/(A_{t+1}L_{t+1}) = s_t w_t/(NG)$. Using equation (6) and the facts that $w_t = (1-\alpha)k_t^{\alpha}$ and $R_{t+1} = \alpha k_{t+1}^{\alpha-1}$, we obtain

$$k_{t+1} = \frac{(1-\alpha)k_t^{\alpha}\beta^{\eta}}{NG(\beta^{\eta} + \alpha^{1-\eta}k_{t+1}^{(\alpha-1)(1-\eta)})}, \tag{7}$$

which implicitly defines k_{t+1} as a function of k_t. Setting $k_{t+1} = k_t$ defines a steady-state level of capital per effective worker, which I denote by k without a time subscript. The condition defining k simplifies to

$$k^{1-\alpha} + \alpha^{1-\eta}\beta^{-\eta}k^{\eta(1-\alpha)} = \frac{1-\alpha}{NG}. \tag{8}$$

One can show that this equation defines a unique, stable steady state with $k > 0$.

Effects of Population Growth

Holding constant the other parameters of the model, steady state capital is a function of population growth: $k = k(N)$. A key feature of the model is that $k(N)$ is strictly decreasing:

$$dk/dN < 0. \tag{9}$$

This is a standard result in neoclassical growth models. Faster labor-force growth reduces k because it raises break-even investment—the investment needed to maintain a given level of capital per effective worker.

Because $k(N)$ is strictly decreasing, the expressions for w_t and R_t imply

$$dw/dN < 0; \tag{10}$$

$$dR/dN > 0, \tag{11}$$

where w and R are the steady-state wage and rental price of capital. Because higher population growth makes capital scarcer, it reduces the real wage and raises the rental price of capital. Since a lower wage reduces an individual's utility and a higher return on saving raises utility, the net effect on welfare is not obvious.

3 Optimal Population Growth

This section derives the relationship between the population-growth rate N and welfare in the model's steady state and asks whether there is an optimal level of N.

A Measure of Welfare

Let $c_t^y = C_t^y/A_t$ and $c_{t+1}^o = C_{t+1}^o/A_t$, where again C_t^y and C_{t+1}^o are consumption when young and old of someone born at t. In steady state, c_t^y and c_{t+1}^o are constant at some levels c^y and c^o, and C_t^y and C_{t+1}^o grow with the exogenous variable A_t. Using these facts and equation (1), the steady-state utility of someone born at t is $(A_t^{1-\theta})U$, where

$$U = \frac{(c^y)^{1-\theta}}{1-\theta} + \beta\frac{(c^o)^{1-\theta}}{1-\theta}. \tag{12}$$

I measure steady-state welfare by U because a higher U raises utility for all generations. A population-growth rate N^* is optimal if it maximizes U given the other parameters of the model.[4]

The Main Result

The following Proposition summarizes the results of this section:

Proposition 1: In the absence of social security, an optimal population-growth rate

[4]I follow traditional welfare economics in equating welfare with the utility of an individual. Some recent work argues that true economic welfare also depends on the number of individuals or total utility in the economy (e.g., Adhami et al., 2024; Spears and Geruso, 2025). Considering this alternative view is beyond the scope of this paper.

N^* exists if and only if $\alpha < 1/2$ and $\eta < 1$. Under those conditions, N^* is defined by

$$f'(k(N^*)) = N^*G, \tag{13}$$

which implies

$$N^* = (\frac{1}{G})\beta^{\eta/(1-\eta)}(\frac{1-2\alpha}{\alpha})^{1/(1-\eta)}. \tag{14}$$

Proposition 1 is proved in the Appendix. Here I sketch the argument and discuss the different parts of the Proposition.

An Indirect Utility Function

To begin, notice that equations (4) and (5) imply $c^y = (1-s)w$ and $c^o = swR$, where s is the steady-state level of the saving rate s_t. Letting $Q = wR$, consumption levels and hence utility are determined by w, Q, and s. An individual takes w and Q as exogenous and chooses s optimally, which implies an indirect utility function of the form

$$U = V(w, Q). \tag{15}$$

We can interpret w as the highest possible c^y (the level when $s = 0$) and Q as the highest possible c^o (the level when $s = 1$).

Both w and $Q = wR$ are determined by k, the steady-state level of capital per effective worker. We can therefore write

$$U = V(w(k), Q(k)). \tag{16}$$

Finally, recall that steady state capital depends on population growth: $k = k(N)$.

With this set-up, the effect of population growth on welfare is

$$dV/dN = (dV/dk)(dk/dN). \tag{17}$$

Using equation (16) to expand dV/dk,

$$dV/dN = \underset{(+)}{[V_w}\underset{(+)}{(dw/dk)} + \underset{(+)}{V_Q}\underset{(?)}{(dQ/dk)}]\underset{(-)}{(dk/dN)}, \tag{18}$$

where V_w and V_Q are the partial derivatives of $V(\cdot)$ and we show the signs of various terms. V_w and V_Q are positive (utility is increasing in both w and wR) and dk/dN is negative. dw/dk is positive: a higher capital stock raises the wage. dQ/dk is ambiguous because $Q = wR$ and a higher k raises w but reduces R.

If N^* exists, it satisfies the first-order condition that $dV/dN = 0$ at $N = N^*$. Since $dk/dN < 0$, equation (17) implies that this condition reduces to $dV/dk = 0$ at $k = k(N^*)$.

The Restrictions on α and η

We can now understand the conditions needed for N^* to exist:

$\alpha < 1/2$: The production function implies $w = (1-\alpha)k^\alpha$ and $R = \alpha k^{\alpha-1}$, which together imply $Q = wR = \alpha(1-\alpha)k^{2\alpha-1}$. If $\alpha \geq 1/2$, that is, if the production function is capital-intensive, then Q is non-decreasing in k. In this case, a higher k raises utility through its effect on w and either raises utility or has no effect through Q. As a result, utility is strictly increasing in k. Since $dk/dN < 0$, utility is strictly decreasing in N and no $N^* > 0$ maximizes welfare.

$\eta < 1$: This means that consumption when young and when old are not very substitutable. To understand why this condition is needed, note that, for $\alpha < 1/2$, $k \to \infty$ implies $w \to \infty$ and $Q \to 0$. In turn, this behavior of w and Q implies $c^y \to \infty$ and $c^o \to 0$ for any saving rate s between zero and one. The opposite case of $k \to 0$ implies $w \to 0$ and $Q \to \infty$, and hence $c^y \to 0$ and $c^o \to \infty$. In sum, either $k \to 0$ or $k \to \infty$ causes consumption to approach 0 in one period of life and ∞ in the other. In both cases, average consumption over the two periods approaches ∞.

If consumption is highly substitutable across periods, the effect of $k \to 0$ or $k \to \infty$ on

average consumption means that utility grows without bound in either of these extreme cases. These cases arise as $N \to \infty$ or $N \to 0$, so once again no $N^* > 0$ maximizes utility. On the other hand, if consumption when young and when old are not very substitutable, then utility suffers if these two consumption levels differ greatly, which occurs for extreme values of N and k. For $\eta < 1$, the benefit from smoothing consumption across the two periods of life is great enough to imply an N^* between 0 and ∞.

For many purposes, the assumption of $\eta = 1$—logarithmic utility—is a convenient simplification of the Diamond model. For this application, however, this simplification is not innocuous: it rules out the existence of an optimal population growth rate.

The restrictions that $\alpha < 1/2$ and $\eta < 1$ do not seem very strong. For the U.S., a typical calibration of α is $1/3$, based on capital's share of income. In the literature on consumption, most estimates of the substitution elasticity η are less than one (or equivalently, the coefficient of relative risk aversion θ is greater than one). While estimates of η vary considerably, one meta-analysis suggests a consensus estimate of $\eta = 1/3$ (Havranek, 2015).

The Optimality Condition

The condition defining N^* is derived in the Appendix. I derive the components of dV/dk (the terms inside the brackets in equation (18)) for a given N and $k = k(N)$ and set $dV/dk = 0$, which leads to the simple condition $f'(k(N)) = NG$ stated in Proposition 1. Note that $f'(k) = R = \alpha k^{\alpha-1}$. Under the assumptions that $\alpha < 1/2$ and $\eta < 1$, this condition defines a unique N^* that maximizes utility.

The condition $f'(k) = NG$ is a familiar one: it defines the "golden-rule" level of capital in neoclassical growth models (Phelps, 1961). However, the condition has a somewhat different meaning from usual. The golden-rule k is usually derived assuming that N is a parameter fixed at an arbitrary level. The golden rule maximizes steady state consumption per effective worker for the given N. Here, the familiar condition defines an optimal N^* and a corresponding $k(N^*)$.

Setting $N = N^*$ does *not* maximize steady-state consumption per effective worker. Consumption per effective worker is $f(k(N))–NGk(N)$ (output minus investment), and the N that maximizes this expression is defined by the first-order condition $[f'(k(N))–NG]\cdot(dk/dN) – Gk(N) = 0$; note the $Gk(N)$ term that does not appear in the condition defining N^*. One can show that the welfare-maximizing N^* always exceeds the N that maximizes consumption per effective worker. This makes sense because, for a given level of consumption per worker, a higher N has a benefit: it raises the ratio of workers to retirees and therefore raises consumption per person.[5]

*The Solution for N**

The final step is to combine the optimality condition we have just discussed with the condition defining $k(N)$ (equation (8)). This leads to the explicit solution for N^* in Proposition 1.

The solution for N^* tells us how the model's parameters affect optimal population growth. For $\alpha < 1/2$ and $\eta < 1$,

$$\partial N^*/\partial G < 0 \tag{19}$$

$$\partial N^*/\partial \alpha < 0 \tag{20}$$

$$\partial N^*/\partial \beta > 0 \tag{21}$$

$$sign(\partial N^*/\partial \eta) = sign(\beta - \alpha - 2\alpha\beta). \tag{22}$$

N^* depends inversely on G. (Equivalently, there is an optimal growth rate for total output NG.) N^* is decreasing in the capital intensity of the production function and increasing in the discount factor. The effect of the substitution elasticity η is ambiguous and depends

[5]This intuition might suggest that the optimal N is the level that maximizes consumption per person, accounting for effects on both consumption per worker and workers per person. But that is also not the case: one can show that N^* can be either higher or lower than the N that maximizes consumption per person. Welfare depends not only on average consumption per person but also on the division of consumption between the young and old and the sizes of those two cohorts, which depend on N in somewhat complex ways.

on α and β.

For common calibrations of the parameters, N^* is less than one: it is optimal for the population to shrink over time. In particular, if $\alpha = 1/3$, sufficient conditions for $N^* < 1$ are $\beta < 1$ and $G \geq 1$: there is some discounting of future utility and technology does not regress over time.

As a specific numerical example, suppose $\alpha = 1/3$ and $\eta = 1/3$. To calibrate G and β, let us interpret a period in the model as 30 years: a person works for 30 years and then lives off capital income for 30 years. Suppose the growth rate of productivity is 1 percent per year and the annual discount factor is 0.99. Compounding over 30 years yields $G = 1.35$ and $\beta = 0.74$. Substituting these parameter values into equation (14) yields $N^* = 0.64$. This level of N implies that the population falls by about 1.5 percent per year.

4 The Implications of Social Security

In discussing the effects of low population growth, economists often emphasize the strain on pay-as-you-go social security systems from a high ratio of retirees to workers. This section adds a social security system to the model and asks how it affects the optimal level of population growth.

Introducing Social Security

Assume the government levies a tax on the young which is a fraction τ of their earnings, where $0 \leq \tau < 1$, and uses the proceeds to pay social security benefits to the old. Since there are N young workers per old retiree, the benefit to each old person is a fraction $N\tau$ of a current worker's earnings. Under this policy, the expressions for consumption of

someone born in period t become

$$C_t^y = (1-\tau-s_t)A_t w_t; \tag{23}$$

$$C_{t+1}^o = R_{t+1}s_t A_t w_t + \tau N G A_t w_{t+1}, \tag{24}$$

where s_t is saving as a fraction of pre-tax earnings and I use the fact that a worker's earnings at t+1 are $A_{t+1}w_{t+1} = GA_t w_{t+1}$. Consumption when young is the part of earnings $A_t w_t$ that is neither taxed away nor saved, and consumption when old is the return on saving plus the social security benefit at $t+1$.

As before, we can derive a person's optimal choice of s_t by maximizing utility given the equations for C_t^y and C_{t+1}^o:

$$s_t = \frac{(1-\tau)\beta^\eta - \tau N G R_{t+1}^{-\eta}(w_{t+1}/w_t)}{R_{t+1}^{1-\eta} + \beta^\eta} \tag{25}$$

assuming that the s_t so defined is positive (as it is in steady state). Using the fact that $k_{t+1} = s_t w_t/(NG)$, we can again determine a function $k_{t+1}(k_t)$ and then set $k_{t+1} = k_t = k$. This leads to

$$k^{1-\alpha} + \alpha^{1-\eta}\beta^{-\eta}k^{\eta(1-\alpha)} = \frac{1-\alpha}{NG}(1-\tau) - (1-\alpha)\tau\alpha^{-\eta}\beta^{-\eta}k^{\eta(1-\alpha)}, \tag{26}$$

which defines a unique, stable steady state with $k > 0$.

In this version of the model, steady-state capital per effective worker depends on both population growth and the social security tax rate: $k = k(N,\tau)$. One can show that k is strictly decreasing in both N and τ. The effect of τ is a standard result in OLG models: a social security system crowds out capital.

The Main Result with Social Security

The following Proposition generalizes Proposition 1:

Proposition 2: In the presence of social security, an optimal population-growth rate N^* again exists if and only if $\alpha < 1/2$ and $\eta < 1$. Under those conditions, N^* is defined by:

$$[f'(k(N^*,\tau)) - N^*G](\partial k/\partial N) + \frac{\tau G}{\alpha(1-\tau)}k(N^*,\tau) = 0. \tag{27}$$

Proposition 2 is proved in the Appendix. It is possible to derive a closed-form solution for N^* with social security, but it is too complex to yield any economic insight. Here I explore the factors that determine N^* and how N^* is affected by the tax rate τ.

The Optimality Condition

Introducing social security complicates the model because both τ and N appear in the equations for C_t^y and C_{t+1}^o ((23) and (24)). This means that τ and N directly affect consumption possibilities for given levels of w and Q. The indirect utility function becomes

$$U = V(w, Q, N, \tau). \tag{28}$$

The partial derivative V_N is positive because higher population growth raises the social security benefit received by the old.

Once again $w(k)$ and $Q(k)$ are determined by the production function, and here $k = k(N,\tau)$. The effect of N on welfare is

$$dV/dN = [V_w(dw/dk) + V_Q(dQ/dk)](\partial k/\partial N) + V_N. \tag{29}$$

The Appendix shows that the first order condition $dV/dN = 0$ leads to the condition for N^* in Proposition 2. Notice that $\partial k/\partial N$ no longer drops out of that condition because it does not multiply all the terms in dV/dN.

In equation (27), the second term on the left is positive and $\partial k/\partial N$ is negative. These facts imply that $f'(k(N^*,\tau)) > N^*G$: N^* is no longer determined by a version of the golden

rule for capital accumulation.

The Effect of Social Security on N^*

The next Proposition tells us how a social security system affects optimal population growth:

Proposition 3: $dN^*/d\tau < 0$.

The optimal growth rate N^* is strictly decreasing in the tax rate τ, so it is lower with social security ($\tau > 0$) than without social security ($\tau = 0$).

Proposition 3 is proved in the Appendix; here I explore the economics behind the result. For simplicity, this discussion considers a small social security system, whose effect is determined by $dN^*/d\tau$ evaluated at $\tau = 0$. We will see that social security raises N^* through one channel but reduces it through another, stronger channel.

To begin, notice that the levels of N and τ determine all the variables in the welfare function $V(\cdot)$ either directly or through their effects on $w(k)$ and $Q(k)$. We can therefore write welfare as

$$U(N,\tau) \;=\; V(w(k(N,\tau)),Q(k(N,\tau)),N,\tau). \tag{30}$$

It will also prove useful to write welfare as

$$U(N,\tau) \;= \hat{V}(k(N,\tau),N,\tau), \tag{31}$$

where the effect of k on $\hat{V}$ is its total effect on welfare through both $w(k)$ and $Q(k)$.

The first-order condition for N^* can be written as $U_N = 0$. Differentiating this condition with respect to τ leads to

$$dN^*/d\tau \;=\; -\frac{U_{N\tau}}{U_{NN}}, \tag{32}$$

where $U_{N\tau}$ and U_{NN} are second derivatives of $U(\cdot)$ evaluated at $N = N^*$. By the second-

order condition for N^*, $U_{NN} < 0$, which implies

$$sign(dN^*/d\tau) \;=\; sign(U_{N\tau}). \tag{33}$$

To analyze the cross-derivative $U_{N\tau}$, we start with the first derivative U_N, which from (31) is

$$U_N \;=\; \hat{V}_k(\partial k/\partial N) + \hat{V}_N. \tag{34}$$

This equation breaks U_N into two parts: an effect of N that operates through k and a direct effect for a given k.

The cross-derivative $U_{N\tau}$ is the derivative of U_N with respect to τ. As shown in the Appendix, taking the derivative of (34) and evaluating it at $\tau = 0$ (which eliminates some terms) leads to

$$U_{N\tau}|_{\tau=0} \;=\; \underset{(+)}{(\partial \hat{V}_k/\partial \tau)}\underset{(-)}{(\partial k/\partial N)} \;+\; \underset{(+)}{\hat{V}_{N\tau}}. \tag{35}$$

We see that $U_{N\tau}$ is the sum of two terms, one negative (the product of a positive and a negative) and one positive. Each of these terms has an economic interpretation.

The positive term $\hat{V}_{N\tau}$ is the effect of social security on $\hat{V}_N$, the direct benefit from higher population growth. $\hat{V}_N$ is zero when $\tau = 0$: as shown above, with no social security N does not affect welfare for a given k. For $\tau > 0$, $\hat{V}_N$ becomes positive because a higher N raises the social security received by the old. By creating this benefit from a higher N, social security tends to raise U_N and N^*.

The negative term in equation (35) reflects the fact that social security raises the marginal value of capital ($\partial \hat{V}_k/\partial \tau > 0$) and higher N reduces capital ($\partial k/\partial N < 0$), which together imply a more negative effect of N on welfare. This effect of social security tends to reduce U_N and N^*.

This effect is somewhat complex because a rise in τ raises $\hat{V}_k$ through two channels,

one direct and one that works through k:

$$\partial \hat{V}_k/\partial \tau \;=\; \underset{(-)}{\hat{V}_{kk}}\underset{(-)}{(\partial k/\partial \tau)} \;+\; \underset{(+)}{\hat{V}_{k\tau}}. \tag{36}$$

We see here that social security crowds out capital ($\partial k/\partial \tau < 0$) and lower capital raises the marginal value of capital ($\hat{V}_{kk} < 0$). In addition, as shown in the Appendix, social security raises the marginal value of capital at a given level of capital ($\hat{V}_{k\tau} > 0$). The reason for this last effect is rather subtle: social security reduces the reliance of retirees on capital income, which mitigates their loss when higher capital reduces the return $R(k)$, and thereby raises the net benefit from higher capital.

At first glance, the opposite effects in equation (35) suggest that the sign of $U_{N\tau}$, and hence of $dN^*/d\tau$, is ambiguous. The Appendix shows, however, that the negative term in (35) is larger in absolute value than the positive term, so $U_{N\tau}$ is negative.

A Numerical Example

We can determine N^* for specific parameter values by combining the condition defining the steady state k and the condition defining N^*, equations (26) and (27), which are two equations in N^* and k. In (27), the expression for $\partial k/\partial N$ is determined by differentiating (26).[6]

As an example, assume as before that $\alpha = 1/3$, $\eta = 1/3$, $G = 1.35$, and $\beta = 0.74$. With no social security system ($\tau = 0$), these parameters imply $N^* = 0.64$. With a social security system with a 10 percent tax on wages ($\tau = 0.1$), $N^* = 0.46$. This N^* implies that the population falls by 2.5 percent per year if a period is 30 years.

[6]Differentiation yields $\partial k/\partial N = -\frac{1-\tau}{GN^2[k^{-\alpha}+\eta(\alpha\beta)^{-\eta}(\alpha+\tau-\alpha\tau)k^{\eta(1-\alpha)-1}]}$.

5 Discussion

This paper asks how population growth affects steady-state welfare in the Diamond OLG model. A welfare-maximizing growth rate exists under non-trivial but plausible restrictions on the model's parameters. I derive this optimal growth rate and find that it may be negative: it may well be optimal for the population to shrink over time. The presence of a social security system reduces the optimal growth rate.

Many economists have studied the costs and benefits of population growth. Important work over the years includes Cutler et al. (1990), Lee at al. (2014), Kotschy and Bloom (2023), and Weil (2026). This research emphasizes the same issues as this paper—the effects of population growth on demographic structure, capital intensity, and social security programs. Generally the authors analyze detailed age distributions that evolve under realistic assumptions about fertility and mortality. This paper's stylized model is a step backward in quantitative realism, but it allows a rigorous welfare analysis, one that accounts for interactions between different effects of population growth and for the adjustment of saving behavior.

Future research should extend the model in this paper, retaining its microfoundations but bringing it closer to the realism of other work. One could, for example, introduce a richer demographic structure, as in Auerbach and Kotlikoff's (1987) OLG model with 55-period lives. One could include a period of childhood before a person's working years and consider changes in mortality and retirement ages as well as fertility. For practical policy applications, it would be important to introduce heterogeneity across people in labor income, saving behavior, and reliance on social security benefits.

Of course a complete analysis of optimal population growth must also consider factors ignored by neoclassical growth models, such as possible effects of population on technological progress (e.g., Kremer, 1993) and environmental externalities (e.g., McKibben, 1998).

This paper analyzes welfare in the steady state of the Diamond model. A companion

paper (Ball, 2026) considers the transition from one steady state to another if population growth falls over time, as it has recently in many countries. In the model, even if population growth is initially above the optimal level and the slowdown raises steady state welfare, not all generations gain; in particular, the cohort that is old at the start of the transition loses because slower labor-force growth reduces its capital income. The companion paper derives the effects of the population slowdown on every generation and considers possible policy responses that redistribute consumption from the winners to the losers.

Appendix

Proof of Proposition 1

Proposition 1 concerns the optimal population growth rate N^* in the model without social security. As explained in the text, steady-state welfare in that case can be written as

$$V(w(k(N)), Q(k(N))). \tag{37}$$

I establish the different parts of the Proposition by examining the first and second derivatives of $V(\cdot)$ with respect to N.

The First Order Condition for N^*: From equation (18) in the text, the first derivative is $dV/dN = [V_w w'(k) + V_Q Q'(k)](dk/dN)$, where $w'(k) = dw/dk$ and $Q'(k) = dQ/dk$. I derive dV/dN by evaluating the terms in this equation. In doing so, I hold constant an individual's saving rate s, which is justified by the envelope theorem because s is chosen to maximize V. The analysis uses an individual's first order condition for optimal saving: $(C^y)^{-1/\eta} = R\beta(C^o)^{-1/\eta}$. It also uses the condition that $k = (sw)/(NG)$ in steady state.

The equations for $U(\cdot)$ and for $C_t^y = A_t c_t^y$ (equations (12) and (4)) imply

$$V_w = (\partial U/\partial c^y)(dc^y/dw) = (c^y)^{-1/\eta}(1-s). \tag{38}$$

Notice that w affects welfare only through c^y because c^o is not influenced by w for a given Q.

Similarly, equations (12) and (5) imply

$$V_Q = (\partial U/\partial c^o)(dc^o/dQ) = \beta(c^o)^{-1/\eta}s = \frac{(c^y)^{-1/\eta}}{R}s, \tag{39}$$

where the last equality uses an individual's first order condition.

Substituting these expressions for V_w and V_Q into the equation for dV/dN yields

$$dV/dN = (c^y)^{-1/\eta}[(1-s)w'(k) + (s/R(k))Q'(k)](dk/dN). \tag{40}$$

We next use the steady-state relation $k = (sw)/(NG)$ and the fact that $Q'(k) = w(k)R'(k) + w'(k)R(k)$. Substituting these relations into equation (40) leads after algebra to

$$dV/dN = (c^y)^{-1/\eta}[w'(k) + \frac{kNGR'(k)}{R(k)}](dk/dN). \tag{41}$$

The wage and rental price of capital are $w(k) = f(k)–kf'(k)$ and $R(k) = f'(k)$. Differentiating these functions leads to the result that $w'(k) = -kR'(k)$, and substituting into (41) yields

$$dV/dN = (c^y)^{-1/\eta}w'(k)[1 - (NG)/R(k)](dk/dN). \tag{42}$$

Since both $(c^y)^{-1/\eta}w'(k)$ and dk/dN are non-zero, the first-order condition $dV/dN = 0$ reduces to

$$\frac{NG}{R(k)} = 1, \tag{43}$$

which is equivalent to the condition $f'(k) = NG$ stated in Proposition 1.

As discussed in the text, this condition cannot be satisfied for $\alpha \geq 1/2$ because dV/dk is always positive in that case. Since there is no optimal N^* for $\alpha \geq 1/2$, I assume $\alpha < 1/2$ in the rest of this analysis. For the moment I also assume $\eta \neq 1$; I return below to the knife-edge case of $\eta = 1$.

The Solution for N^*: To derive an explicit solution for N^*, substitute $f'(k) = \alpha k^{\alpha-1}$ into the condition $f'(k) = NG$, which leads to

$$k = (\frac{\alpha}{NG})^{1/(1-\alpha)}. \tag{44}$$

Substituting this expression for k into equation (8), which defines the steady-state k, and

solving for N yields the expression for N^* in the Proposition. For $\alpha < 1/2$ and $\eta \neq 1$, this N^* satisfies the first order condition for optimality.

The Second Order Condition: The second order condition for N^* is $d^2V/dN^2 < 0$ at $N = N^*$. To derive this second derivative, start with the expression for dV/dN in equation (40). Substituting in the expressions for $R(k)$, $w'(k)$, and $Q'(k)$ from the Cobb-Douglas production function leads after simplification to

$$dV/dN = (c^y)^{-1/\eta}(1-\alpha)k^{\alpha-1}(dk/dN)[\alpha - s + \alpha s]. \tag{45}$$

Differentiating this equation with respect to N and then using the first-order condition $dV/dN = 0$ (which implies $\alpha - s + \alpha s = 0$) yields

$$d^2V/dN^2 = (c^y)^{-1/\eta}(1-\alpha)k^{\alpha-1}(dk/dN)(\alpha - 1)(ds/dN). \tag{46}$$

Since dk/dN and $(\alpha - 1)$ are both negative,

$$sign(d^2V/dN^2) = sign(ds/dN). \tag{47}$$

The saving rate s is $s(R(k(N)))$, with $s(R)$ given by equation (6). Therefore,

$$ds/dN = (ds/dR)(dR/dk)(dk/dN). \tag{48}$$

Since dR/dk and dk/dN are both negative,

$$sign(ds/dN) = sign(ds/dR) = sign(\eta - 1), \tag{49}$$

where the second equality follows from equation (6).

Together, these results imply that the second order condition for N^* is satisfied if and only if $\alpha < 1/2$ and $\eta < 1$. If $\alpha < 1/2$ and $\eta > 1$, the first order condition $R = NG$ determines

a level of N that *minimizes* welfare.

The Case of $\eta = 1$: In the well-known case of $\eta = 1$ (log utility), the steady state condition (8) leads to an explicit solution for k, which in turn implies

$$R = \frac{\alpha(1+\beta)NG}{\beta(1-\alpha)}. \tag{50}$$

Here, R is proportional to NG. The first order condition $R = NG$ becomes

$$\frac{\alpha(1+\beta)}{\beta(1-\alpha)} = 1, \tag{51}$$

which does not depend on N. For combinations of α and β that satisfy this condition, the first order condition is satisfied for all N, and otherwise it is satisfied for no N. In either case an optimal N does not exist.

Proof of Proposition 2

Proposition 2 says that, with a social security system, an optimal N^* exists if and only if $\alpha < 1/2$ and $\eta < 1$, and is determined by equation (27). To prove the Proposition, I first show that the first order condition for an optimum implies (27). I then show that for $\alpha < 1/2$ and $\eta < 1$, (27) has a unique solution that defines an optimum, and finally that if either of these parameter restrictions fails, (27) does not determine an optimum.

Derivation of Equation (27): The derivation of (27) parallels the analysis of the model without social security. With social security, welfare is $V(w, Q, N, \tau)$ and differentiating yields

$$dV/dN = [V_w w'(k) + V_Q Q'(k)](dk/dN) + V_N. \tag{52}$$

Using equations (23)-(24), one can derive:

$$V_w = (c^y)^{-1/\eta}(1-\tau-s) + \beta(c^o)^{-1/\eta}\tau NG; \tag{53}$$

$$V_Q = s\beta(c^o)^{-1/\eta}; \tag{54}$$

$$V_N = \beta(c^o)^{-1/\eta}\tau Gw, \tag{55}$$

where the equations for V_w and V_N include effects on social security taxes and benefits.

Once again, we can rewrite dV/dN using an individual's first order condition, the steady state condition $sw = kNG$, and the expressions for $w'(k)$ and $Q'(k)$ implied by the production function. Those facts lead to the following steps, which are generalizations of equations (40)-(42):

$$\frac{dV}{dN} = (c^y)^{-1/\eta}\left\{\left[1-\tau-s+\frac{\tau NG}{R(k)}\right]w'(k) + \frac{s}{R(k)}Q'(k)\right\}\frac{\partial k}{\partial N} + (c^y)^{-1/\eta}\frac{\tau Gw}{R(k)}. \tag{56}$$

$$\frac{dV}{dN} = (c^y)^{-1/\eta}\left[(1-\tau)w'(k) + \frac{\tau NG}{R(k)}w'(k) + \frac{NGkR'(k)}{R(k)}\right]\frac{\partial k}{\partial N} + (c^y)^{-1/\eta}\frac{\tau Gw}{R(k)}. \tag{57}$$

$$\frac{dV}{dN} = (c^y)^{-1/\eta}(1-\tau)w'(k)\left[1-\frac{NG}{R(k)}\right]\frac{\partial k}{\partial N} + (c^y)^{-1/\eta}\frac{\tau Gw}{R(k)}. \tag{58}$$

Setting $dV/dN = 0$ and using $w(k)$ and $R(k)$ for the Cobb-Douglas production function yields equation (27).

A Unique N^* for $\alpha < 1/2$ and $\eta < 1$: I next show that, under the parameter restrictions in the Proposition, equation (27) has a unique solution for N^*.

To simplify what follows, let $Z(k) = (\alpha\beta)^{-\eta}(\alpha+\tau-\alpha\tau)k^{(1-\alpha)(\eta-1)}$. I show that the optimality condition (27) and the steady-state condition (26) imply a unique positive solution for Z. We can invert the definition of Z to get a unique k, which determines a unique N^* through the steady-state condition.

Using the definition of Z, we can rearrange (26) to obtain

$$k^{1-\alpha}(1+Z) = \frac{(1-\alpha)(1-\tau)}{NG}. \tag{59}$$

Using this equation, we can also write the expression for $\partial k/\partial N$ in footnote 6 as

$$\partial k/\partial N \;=\; -\frac{(1+Z)k}{(1-\alpha)N(1+\eta Z)}. \tag{60}$$

In the optimality condition (27), we can use equation (59) to eliminate NG, substitute (60) for $\partial k/\partial N$, and replace $f'(k)$ with $\alpha k^{\alpha-1}$. Using considerable algebra, we can then write the condition that determines N^* as a quadratic equation in Z, with k eliminated as a separate term:

$$\phi_2 Z^2 \;+\; \phi_1 Z \;+\; \phi_0 \;=\; 0, \tag{61}$$

where

$$\phi_2 = \alpha^2;\; \phi_1 = 2\alpha^2 - \alpha(1-\alpha)(1-\tau) - \tau\eta(1-\alpha)^2;\; \phi_0 = (2\alpha-1)[\alpha + \tau(1-\alpha)]. \tag{62}$$

The coefficient ϕ_2 on Z^2 is positive and the constant term ϕ_0 is negative under the assumption that $\alpha < 1/2$. These facts imply that (61) has one positive solution for Z and one negative solution. The unique positive solution implies a unique positive N^* that satisfies the optimality condition (27).

The Solution for N^* is an Optimum: For $\alpha < 1/2$ and $\eta < 1$, we can show that welfare approaches $-\infty$ as either $N \to \infty$ or $N \to 0$, because consumption approaches 0 in one period of life. This limiting behavior and the fact that the first order condition for N^* has a unique solution imply that this solution is an optimum.

For $N \to \infty$, the steady state condition implies that $k \to 0$ and therefore $c^y \to 0$. For $\eta < 1$, that implies that utility approaches $-\infty$.

With social security, $c^o = sQ(k) + \tau NGw(k)$ (capital income plus the social security benefit). As $N \to 0$, $k \to \infty$, and with $\alpha < 1/2$ the capital income term approaches zero. Using $w(k) = (1-\alpha)k^\alpha$ and the steady state relation between N and k, the second term can

be written as a function of k:

$$\tau N G w(k) \;=\; \frac{\tau(1-\alpha)^2(1-\tau)}{k^{1-2\alpha} + (\alpha\beta)^{-\eta}[\alpha + \tau(1-\alpha)]k^{\eta(1-\alpha)-\alpha}}. \tag{63}$$

As $N \to 0$ and $k \to \infty$, this expression approaches 0. Since both terms in the equation for c^o approach 0 as $N \to 0$, c^o approaches zero and welfare approaches $-\infty$.

$\eta < 1$ and $\alpha < 1/2$ Are Necessary: To see why the restriction on η is necessary, recall that $N \to 0$ implies that k, w, and c^y approach infinity. For $\eta > 1$, this implies that utility approaches ∞ regardless of c^o. This limiting behavior means there is no positive N^* that maximizes utility.

If $\eta = 1$ (log utility), one can derive a closed-form solution for $k(N, \tau)$ and use it to simplify the optimality condition (27). It turns out that the result is the same as equation (51), the condition with $\eta = 1$ and no social security, which is satisfied for either all N or no N. This result and the one for $\eta > 1$ imply that $\eta < 1$ is necessary for an optimal N^*.

Finally, to see that $\alpha < 1/2$ is also necessary, return to equation (61), the quadratic equation for Z. If $\alpha \geq 1/2$, then the coefficient on Z^2 is positive, the coefficient on Z is positive, and the constant term is non-negative. These facts imply that there is no positive solution for Z and hence no solution for N^*.

Proof of Proposition 3

Proposition 3 concerns the effect of the tax rate τ on N^*. Letting Z^* denote the positive root of equation (61), we can use the definition of $Z(k)$ and the steady-state condition as written in (59) to write N^* as a function of Z^* and parameters, including τ:

$$N^* \;=\; \frac{(1-\alpha)(1-\tau)}{G}\Big[\frac{Z^*}{(\alpha\beta)^{-\eta}(\alpha + \tau - \alpha\tau)}\Big]^{1/(1-\eta)}\Big[\frac{1}{1+Z^*}\Big]. \tag{64}$$

For this proof, it is convenient to define the terms $\Omega = (\alpha\beta)^{-\eta}(\alpha + \tau - \alpha\tau)$ and $X^* = Z^*/\Omega$

and write N^* as

$$N^* = \frac{(1-\alpha)(1-\tau)(X^*)^{1/(1-\eta)}}{G(1+\Omega X^*)}. \tag{65}$$

In equation (65), the tax rate τ affects N^* directly and through Ω and X^*. Notice that $d\Omega/d\tau = (\alpha\beta)^{-\eta}(1-\alpha)$. Taking logs and differentiating leads to

$$d\ln(N^*)/d\tau = -\frac{1}{1-\tau} - \frac{X^*(\alpha\beta)^{-\eta}(1-\alpha)}{1+\Omega X^*} + (dX^*/d\tau)[\frac{1}{(1-\eta)X^*} - \frac{\Omega}{1+\Omega X^*}]. \tag{66}$$

In this equation, the first two terms on the right side are negative, and in the third term the expression in brackets is positive for $\eta < 1$. Consequently, to prove that $dln(N^*)/d\tau < 0$, and therefore $dN^*/d\tau < 0$, it suffices to show that $dX^*/d\tau < 0$.

The variable X^* is determined by the definition $X^* = Z^*/\Omega$ and the solution for Z^*, which is given in terms of ϕ_2, ϕ_1, and ϕ_0 by the quadratic equation. If we substitute in the ϕ's in (62), a considerable amount of algebra leads to

$$X^* = \frac{2(1-2\alpha)(\alpha\beta)^{\eta}}{\Delta}, \tag{67}$$

where

$$\Delta = \Lambda + \Sigma, \tag{68}$$

with $\Sigma = 3\alpha^2 - \alpha + \tau(1-\alpha)\{\alpha - \eta(1-\alpha)\}$ and $\Lambda = [\Sigma^2 + 4\alpha^2(1-2\alpha)\{\alpha + \tau(1-\alpha)\}]^{1/2}$.

The numerator in equation (67) is a positive constant for $\alpha < 1/2$, so $sign(dX^*/d\tau) = -sign(d\Delta/d\tau)$ and we need to show $d\Delta/d\tau > 0$. Differentiating (68) and collecting terms leads after more algebra to

$$\frac{d\Delta}{d\tau} = \frac{(1-\alpha)[\{\alpha - \eta(1-\alpha)\}\Delta + 2\alpha^2(1-2\alpha)]}{\Lambda}. \tag{69}$$

To determine the sign of this expression, it is convenient to define

$$J(\tau) = (1-\alpha)\{[\alpha - \eta(1-\alpha)]\Delta(\tau) + 2\alpha^2(1-2\alpha)\}, \tag{70}$$

so we can write

$$\frac{d\Delta}{d\tau} = \frac{1}{\Lambda}J(\tau). \tag{71}$$

The term $1/\Lambda$ is positive, so $sign(d\Delta/d\tau) = sign(J(\tau))$. To prove $d\Delta/d\tau > 0$ and thereby complete the proof of Proposition 3, we need to show $J(\tau) > 0$. We do so by first showing that $J(\tau)$ is positive at $\tau = 0$ and then showing that it has the same sign for all τ.

At $\tau = 0$, equation (68) simplifies to $\Delta = 2\alpha^2$. Substituting that fact into (70) yields $J(0) = 2\alpha^2(1-\alpha)^2(1-\eta)$, so $J(0) > 0$ under our restriction on η.

To see that $J(\tau)$ has the same sign for all τ, we differentiate (70):

$$J'(\tau) = (1-\alpha)\{\alpha - \eta(1-\alpha)\}\frac{d\Delta}{d\tau}. \tag{72}$$

Substituting (71) into (72) and recalling that Λ depends on τ:

$$J'(\tau) = \frac{(1-\alpha)\{\alpha - \eta(1-\alpha)\}}{\Lambda(\tau)}J(\tau). \tag{73}$$

Thus $J(\tau)$ satisfies a differential equation of the form $J'(\tau) = h(\tau)J(\tau)$, where the function $h(\cdot)$ is continuous. Given this fact, a standard result for ordinary differential equations (the Picard-Lindelof Theorem) implies that $J(\tau)$ has the same sign for all τ.

Derivation of $U_{N\tau}$ and Its Sign

Here I analyze the cross-derivative $U_{N\tau}$, which determines the sign of $dN^*/d\tau$, at $\tau = 0$. When evaluating derivatives at $\tau = 0$, I also set $N = N^*(0)$ and $k = k(N^*(0), 0)$.

Derivation of (35): Differentiating equation (34) for U_N with respect to τ yields

$$U_{N\tau} = \hat{V}_k(\partial^2 k/\partial N \partial \tau) + (\partial k/\partial N)(\partial \hat{V}_k/\partial \tau) + \hat{V}_{Nk}(\partial k/\partial \tau) + \hat{V}_{N\tau}. \tag{74}$$

Two of the derivatives in this equation are zero at $\tau = 0$:

• $\hat{V}_k = 0$ from the first order condition for N^* with no social security.

• $\hat{V}_{Nk} = 0$. To see this, note that $\hat{V}_N = V_N$ (the effect of N holding constant k is the same as the effect holding constant $w(k)$ and $Q(k)$). Differentiating V_N in equation (55) with respect to k yields

$$\hat{V}_{Nk} = \beta\tau G[\partial((c^o)^{-1/\eta}w)/\partial k], \tag{75}$$

which equals zero at $\tau = 0$.

Setting these two derivatives equal to zero in (74) yields equation (35) in the text.

Derivatives of $k(N,\tau)$ and $\hat{V}(k,N,\tau)$: We obtain the derivatives of $k(\cdot)$ by differentiating the steady-state condition (26). At $\tau = 0$,

$$\partial k/\partial N = -\frac{Gk^{2-\alpha}}{\alpha(\alpha + \eta(1-2\alpha))} < 0; \tag{76}$$

$$\partial k/\partial \tau = -\frac{(1-\alpha)k}{\alpha(\alpha + \eta(1-2\alpha))} < 0, \tag{77}$$

where the signs follow from the assumption that $\alpha < 1/2$.

To obtain $\hat{V}_{N\tau}$, we differentiate $\hat{V}_N = V_N$ with respect to τ and use the consumer's first order condition and the Cobb-Douglas expressions for $w(k)$ and $R(k)$. At $\tau = 0$,

$$V_{N\tau} = \frac{(1-\alpha)(c^y)^{-1/\eta}Gk}{\alpha} > 0. \tag{78}$$

Finally, to derive the other second derivatives of $\hat{V}(\cdot)$, we first determine $\hat{V}_k = V_w w'(k) + V_Q Q'(k)$. Using equations (53)-(54) for V_w and V_Q, the consumer's first order condition,

and the definition $Q = Rw$, we can derive:

$$\hat{V}_k \;=\; (c^y)^{-1/\eta}[(1-\tau+\frac{\tau NG}{R(k)})w'(k)+\frac{sw(k)}{R(k)}R'(k)], \tag{79}$$

where s is the saving rate and $s(k,N,\tau)$ is given by equation (25) with $R_{t+1} = R(k)$ and $w_{t+1}/w_t = 1$. Differentiating (79), using the Cobb-Douglas expressions for $w(k)$ and $R(k)$, and evaluating at $\tau = 0$ yields:

$$\hat{V}_{kk} \;=\; -(c^y)^{-1/\eta}\alpha(1-\alpha)(1-\eta)(1-2\alpha)k^{\alpha-2} \;<\; 0; \tag{80}$$

$$\hat{V}_{k\tau} \;=\; (c^y)^{-1/\eta}(1-\alpha)^2k^{\alpha-1} \;>\; 0, \tag{81}$$

where I have used results that, at $\tau = 0$, $\partial s/\partial\tau = -1$ and $\partial s/\partial k = [\alpha(1-\eta)(1-2\alpha)]/[(1-\alpha)k]$.

Equation (81) captures a subtle effect of the tax rate τ on the marginal value of capital $\hat{V}_k$. To understand this effect, notice that there are two terms inside the brackets in (79), the equation for $\hat{V}_k$. The first of these terms is positive and reflects the effect of capital on the wage ($w'(k) > 0$); the second is negative and reflects the effect on the return on capital ($R'(k) < 0$). If τ rises, the response of the positive term is proportional to $NG/R(k) - 1$, which is zero at $\tau = 0$. In contrast, a higher τ reduces the magnitude of the negative term, because $\partial s/\partial\tau < 0$: social security reduces saving. Lower saving reduces an individual's loss from a fall in $R(k)$, and thereby increases the net benefit of higher capital.

The Sign of $U_{N\tau}$: We can compare the two terms in equation (35) by plugging in the various derivatives in (76)-(81). The ratio of the negative term in (35) to the positive term simplifies to

$$\frac{(\partial\hat{V}_k/\partial\tau)(\partial k/\partial N)}{\hat{V}_{N\tau}} = -(\frac{1-\alpha}{\alpha+\eta(1-2\alpha)})^2. \tag{82}$$

For $\eta < 1$ and $\alpha < 1/2$, the absolute value of this ratio is greater than one. Thus the absolute value of the negative term in (35) is larger than the positive term and $U_{N\tau} < 0$.

Generalizing the Production Function

Here I generalize the model by assuming a CES production function. In this case, output per effective worker is

$$y = f(k) = [\alpha k^{\rho} + (1-\alpha)]^{1/\rho}, \qquad \rho < 1. \tag{83}$$

The elasticity of substitution between capital and labor is $\sigma = 1/(1-\rho)$. The Cobb-Douglas production function is the special case in which $\sigma = 1$ ($\rho \to 0$).

It is challenging to analyze the model with this production function, because its dynamics may be pathological: $k(t+1)$ may not be a single-valued function of $k(t)$, and there may be no steady state with $k > 0$ or multiple steady states with varying stability properties (see Romer, chapter 2). Below I put a lower bound on the elasticity of substitution σ that makes the model sufficiently well-behaved for our purposes. Under this restriction, there is at most one stable steady state, and in evaluating welfare for a given N I assume the economy is in that steady state if it exists. I assume that an N is not optimal if there is no stable steady state with $k > 0$ (in which case k converges to zero from any initial level).

Here I assume there is no social security system. The main result is

Proposition A1: In the absence of social security, with a CES production function, sufficient conditions for an optimal population growth rate N^* to exist are $\eta \leq 1$ and $\underline{\sigma} < \sigma < 1$, where $\underline{\sigma}$ is a lower bound between zero and one. Under these conditions, N^* is determined by the steady state condition

$$\frac{kNG}{w(k)} = \frac{\beta^{\eta}}{\beta^{\eta} + (R(k))^{1-\eta}}, \tag{84}$$

where $w(k) = (1-\alpha)[\alpha k^{\rho} + (1-\alpha)]^{(1-\rho)/\rho}$ and $R(k) = \alpha k^{\rho-1}[\alpha k^{\rho} + (1-\alpha)]^{(1-\rho)/\rho}$, and the condition

$$f'(k) = NG. \tag{85}$$

In this Proposition, the conditions for N^* to exist are sufficient but not necessary;

full necessary and sufficient conditions are too complex to interpret. One can derive a condition defining $\underline{\sigma}$, the lower bound on the elasticity of substitution, but it is also complex. When all the model's parameters except σ are set at their levels in previous numerical examples, $\underline{\sigma} = 0.54$. Notice that, for $\underline{\sigma} < \sigma < 1$, the restrictions on the other parameters are a bit weaker than they are in the Cobb-Douglas case: no restriction on α is needed, and N^* exists for $\eta = 1$ (log utility).

I will sketch the reasoning behind this Proposition but eschew a detailed proof. As before, the steady state condition is derived from equation (6) for the saving rate and the condition $sw = NGk$, and the factor prices are implied by the CES production function. The optimality condition $f'(k) = NG$ is the same as in the Cobb-Douglas case; notice that the derivation of that condition above does not use the Cobb-Douglas assumption.[7]

The parameter restrictions in the Proposition ensure that equations (84)-(85) define a unique N^* that is a global optimum. As mentioned above, the lower bound on σ rules out pathological dynamics. The condition $\sigma < 1$ (less substitutability of capital and labor than in the Cobb-Douglas case) implies that, as k approaches either 0 or ∞, one of $w(k)$ and $R(k)$ approaches 0 and the other approaches a finite limit, which in turn implies that either c^y or c^o approaches 0 and utility approaches $-\infty$ for any $\eta \leq 1$ (including log utility). These results for extreme values of k ensure the existence of an optimal k and hence an optimal N.

Combining Propositions 1 and A1, sufficient conditions for N^* to exist are $\eta < 1$ and either $\underline{\sigma} < \sigma < 1$ or $\sigma = 1$ and $\alpha < 1/2$. For the US economy, a meta-analysis of 77 studies concludes that σ is likely between 0.45 and 0.87 (Knoblach et al., 2020). With this range, the restriction that $\sigma \leq 1$, like the restriction on η, is non-trivial but not especially strong. The restriction that $\sigma > \underline{\sigma}$ is arguably stronger, as $\underline{\sigma} = 0.54$ for the usual values of the other parameters.

[7] Another nuance: For $\underline{\sigma} < \sigma < 1$, one can show that the model has no steady state for N above some bound N' and two steady states for $N < N'$, an unstable one and a stable one with a higher k. However, when the steady state condition is combined with the optimality condition $f'(k) = NG$, these equations determine a unique (N, k) pair with $N < N'$ and k at its level in the stable steady state.

The effect of σ on optimal population growth, $\partial N^*/\partial\sigma$, is ambiguous, with its sign depending in complicated ways on most of the model's parameters. As a numerical example, I set all the parameters except σ at their usual levels and compute N^* for various σ's. In this example, N^* falls monotonically as σ falls, from 0.64 at $\sigma = 1$ to 0.33 as $\sigma \to \underline{\sigma}$. For $\sigma = 0.66$, the midpoint of the range in Knoblach et al., $N^* = 0.35$. If a period is 30 years, this value of N^* means that the population falls by 3.4 percent per year.

References

Abel, Andrew B., and Stavros Panageas, "Optimal Immigration and Long-Run Growth," manuscript, 2025.

Adhami, Mohamad, Mark Bils, Charles I. Jones, and Peter J. Klenow, "Population and Welfare: Measuring Growth when Life is Worth Living," NBER Working Paper, 2024.

Auerbach, Alan J. and Laurence J. Kotlikoff, *Dynamic Fiscal Policy*, Cambridge University Press, 1987.

Ball, Laurence, "Welfare and Fiscal Policy During a Slowdown in Population Growth," in progress, 2026.

Cutler, David M., James M. Poterba, Louise M. Sheiner, and Lawrence H. Summers, "An Aging Society: Opportunity or Challenge?", *Brookings Papers on Economic Activity*, 1990.

Diamond, Peter A., "National Debt in a Neoclassical Growth Model," *American Economic Review*, 1965.

Havranek, Tomas, "Measuring Intertemporal Substitution: The Importance of Method Choices and Selective Reporting," *Journal of the European Economic Association*, 2015.

International Monetary Fund, "The Rise of the Silver Economy: Global Implications of Population Aging," *World Economic Outlook*, 2025.

Knoblach, Michael, Martin Roessler, and Patrick Zwerschke, "The Elasticity of Substitution Between Capital and Labour in the US Economy: A Meta-Regression Analysis," *Oxford Bulletin of Economics and Statistics*, 2020.

Kotschy, Rainer and David E. Bloom, "Population Aging and Economic Growth: From Demographic Dividend to Demographic Drag?", NBER Working Paper 31585, 2023.

Kremer, Michael, "Population Growth and Technological Change: One Million B.C. to 1990," *Quarterly Journal of Economics*, 1993.

Lee, Ronald, et al., "Is Low Fertility Really a Problem? Population Aging, Dependency, and Consumption," *Science*, 2014.

McKibben, Bill, *Maybe One: A Personal and Environmental Argument for Single Child Families*, Simon and Schuster, 1998.

McNamara, Robert S., Address to the Massachusetts Institute of Technology, 1977.

Phelps, Edmund, "The Golden Rule of Accumulation: A Fable for Growthmen," *American Economic Review*, 1961.

Romer, David, *Advanced Macroeconomics*, 5th ed., McGraw Hill, 2019.

Solow, Robert M., "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, 1956.

Spears, Dean, and Michael Geruso, *After the Spike: Population, Progress, and the Case for People*, Simon and Schuster, 2025.

Weil, David N., "How Much Would Continued Low Fertility Affect the US Standard of Living?", *Journal of Economic Perspectives*, Winter 2026.