Imported Inflation: Production Networks, Currency Invoicing, and Monetary Policy

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We study optimal monetary policy after foreign price shocks in a dynamic small open economy with production networks and heterogeneous currency invoicing. We derive a second-order welfare loss and the allocation under commitment. Heterogeneous invoicing generally prevents stabilization of a domestic-value-added inflation index from closing the output gap. The monetary response depends jointly on the imported-input centrality of the shocked sector and the currency in which the affected imports are priced. In a forty-four-sector calibration, an imported-energy shock produces similar impact consumer price inflation under optimal policy and a standard Taylor rule, but the Taylor rule induces a larger interest-rate increase and output-gap contraction. A four-region quantitative application shows that targeting core rather than headline inflation captures most of the stabilization gain from optimizing the policy rule after an imported-energy shock. For an economy indirectly exposed to bilateral tariffs, the corresponding gain is negligible. The results connect monetary stabilization to the sectoral origin of imported inflation and its transmission through domestic production.

Keywords: monetary policy, imported inflation, production networks, currency invoicing, energy shocks, optimal simple rules, tariffs.
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1. Introduction

A rise in foreign prices presents a familiar problem for monetary policy. It raises domestic inflation and reduces purchasing power, but its source lies outside domestic demand. Tightening can restrain the resulting inflation, at a cost to activity. How much should the central bank tighten, if anything? In an economy that imports both final goods and intermediate inputs, the answer depends on which foreign prices have risen and on how their increase reaches domestic producers. The increase in the consumer price index (CPI) alone is therefore not enough to determine the appropriate monetary response.

This paper studies the joint role of the sectoral origin of foreign price shocks and currency invoicing in optimal monetary policy. We use a dynamic small open economy (SOE) with domestic and international input–output (IO) linkages, heterogeneous price rigidity, and heterogeneous import invoicing. A foreign input used by many domestic industries transmits its price increase through their marginal costs. The exchange rate can alter this transmission, but its effect depends on the currency in which each imported good is priced. Production links and invoicing thus determine both the disturbance the central bank faces and the consequences of its response.

Our closest theoretical benchmark is Qiu *et al.* (2026), who characterize monetary policy and the inflation weights that close the output gap in a static SOE with production networks. Aguilar *et al.* (2026) study the dynamic transmission of foreign price disturbances and the associated monetary-policy trade-offs. We build on these analyses by characterizing the welfare criterion and optimal allocation under heterogeneous import invoicing. The additional margin matters for policy: an exchange-rate movement changes import costs unevenly across sectors, so a domestic inflation index that aligns with the aggregate markup distortion need no longer stabilize the output gap. Our contribution concerns this interaction between the production network and currency pricing, and its implications for stabilization after shocks to different foreign sectors.

The model is a New Keynesian economy in which firms use labor and domestic and imported intermediates, and prices adjust at sector-specific frequencies. We obtain a tractable SOE from a general multi-country environment by imposing logarithmic utility, Cobb–Douglas expenditure and production shares, and complete international financial markets. Imports can be priced in the producer's currency (PCP) or the local currency (LCP). The positive model accommodates import tariffs, foreign price wedges, and tariffs on Home exports. The welfare analysis considers foreign price wedges under an efficient, tariff-free natural benchmark.

The analytical results have two parts. First, we derive the sectoral Phillips curves and the domestic-value-added Divine Coincidence Inflation (DCI) index. Its weights combine sectoral sales, inverse price flexibility, and domestic production content. Under the pricing

restrictions that eliminate the exchange-rate residual, stabilizing this index closes the output gap. Heterogeneous invoicing generally leaves a residual in its Phillips curve: the exchange rate affects imported costs in a way that the output gap alone does not summarize. The index remains well defined, but its stabilization is no longer sufficient to close the gap.

Second, we derive a second-order welfare loss and the optimal allocation under commitment. In addition to the output gap and domestic sectoral inflation, welfare depends on gaps in domestic prices and sectoral terms of trade. These relative-price gaps distort the allocation of consumption and intermediate inputs. Monetary policy must therefore weigh the effects of demand and exchange-rate adjustment on sectoral inflation against their effects on resource allocation. The resulting conditions characterize this trade-off without replacing it by a choice of aggregate inflation weights. We also solve the discretionary problem, in which inherited price gaps continue to matter but the central bank cannot commit to future actions.

We use a calibration with forty-four sectors to examine how this policy problem varies with the origin of the shock. Sectoral counterfactuals separate consumption shares, domestic-input links, imported-input links, price rigidity, and invoicing. Imported-input centrality—the domestic sales-weighted use of the shocked foreign input—organizes the largest differences in aggregate responses. A small share in household expenditure does not make a foreign good unimportant for policy if it is used throughout domestic production. Invoicing adds substantial variation to this relationship. Under the Taylor rule, heterogeneous invoicing nearly doubles the cross-sector standard deviation of eight-quarter cumulative CPI inflation responses to equal-sized foreign price shocks when imported-input shares are the only heterogeneous real dimension.

Energy provides a useful illustration. Mining of coal, crude petroleum, and natural gas is the high-centrality outlier in the sectoral experiments. A price wedge in this sector produces nearly the same impact increase in headline inflation under commitment and a standard CPI Taylor rule. The Taylor rule nevertheless raises the interest rate more and generates a larger contraction in the output gap. The difference concerns the domestic adjustment to the imported price increase: the affected input enters the costs of many firms whose prices are sticky. Optimal policy accounts for those downstream distortions as well as aggregate inflation. Under discretion, the impact output-gap and interest-rate responses lie between commitment and the Taylor rule, while impact CPI inflation remains similar.

A quantitative application illustrates these mechanisms in a four-region economy comprising the Euro Area (EA), the United States, China, and the rest of the world, with forty-four sectors in each region. It adds bilateral production links, wage rigidity, energy-specific substitution, and dominant-currency pricing. We compare the baseline Taylor rule with an

optimal simple rule (OSR) chosen under an aggregate stabilization criterion.[1] For imported energy, the optimized rule reduces this loss by 24.7%. Changing the baseline rule's target from headline to core inflation, with its coefficients unchanged, reduces it by 22.8%, recovering most of the gain. For bilateral U.S.–China tariffs, the EA is a third party and its exposure is much smaller: optimizing the rule improves the stated criterion by only 0.01%. The tariff spillover nearly disappears under uniform producer-currency pricing. These applications show why the origin of imported inflation and the currency of the affected trade flows matter for the policy comparison.

Related literature. Our analysis belongs to the literature on monetary policy in production networks. La'O and Tahbaz-Salehi (2022) and Rubbo (2023) characterize the aggregation of sectoral distortions and optimal policy in closed economies; Pasten *et al.* (2020) and Ghassibe (2021) show how input links and price rigidity shape monetary transmission. Our SOE retains this sectoral accounting while allowing currency pricing to alter the effects of monetary policy through imported inputs.

Currency pricing is central to open-economy monetary policy. Gopinath *et al.* (2020) document the role of dollar pricing in international transmission, and Egorov and Mukhin (2023) characterize optimal policy under dominant-currency pricing, including environments with intermediate inputs. Our analysis takes invoicing as a sectoral primitive and examines how the location of a foreign disturbance changes the stabilization problem when the exchange rate passes through to some imported prices but not others.

The applications also connect to work on energy shocks and tariffs. Bodenstein *et al.* (2008) establish that reacting aggressively to headline inflation after an energy shock can generate unnecessary volatility, while Bodenstein *et al.* (2012) emphasize that the source of the oil-price movement matters. We examine this distinction in a network where energy enters the costs of sticky-price domestic producers. Recent tariff studies analyze the policy response in economies imposing or directly facing protection, including Lindé and Pescatori (2019); Bergin and Corsetti (2023, 2025); Werning *et al.* (2025); Bianchi and Coulibaly (2025). Our tariff application instead considers a third-country central bank, whose exposure operates through international production links, relative prices, and invoicing.

Section 2 presents the general environment. Section 3 develops the tractable SOE and its equilibrium conditions. Section 4 derives optimal policy, reports the sectoral experiments, and presents the quantitative application. Section 5 concludes. The appendices contain the proofs, discretionary-policy results, and the full quantitative model and experiments.

[1] In the tractable SOE, the aggregate-loss rule captures 97.4% of the welfare improvement achieved by commitment relative to the baseline Taylor rule for imported-energy wedges. The quantitative criterion uses output growth in place of the SOE output gap.

2. General Model

We consider a world economy composed of K countries, indexed by k. Each country comprises I production sectors, possibly heterogeneous within and between countries and linked to one another through domestic and international input–output relationships. Within each sector, a unit mass of monopolistically competitive firms indexed by $f \in (0,1)$ produces differentiated varieties using labor and intermediate goods supplied by domestic and foreign sectors. Nominal price rigidity is heterogeneous across sectors and countries.

2.1. Households

There is a representative household in each country k that derives utility from consumption and disutility from labor according to the following per-period utility function:

$$U_t = U(C_{k,t}) - N_{k,t}^{1+\varphi}/(1+\varphi), \tag{1}$$

where $N_{k,t}$ is aggregate labor supplied by the household, $U(C_{k,t})$ is a constant-relative-risk-aversion utility function over aggregate consumption, $C_{k,t}$, and φ denotes the inverse of the Frisch elasticity.

Aggregate consumption is defined as a constant-returns-to-scale (CRS) composite of sectoral consumption, each of which is itself a CRS aggregation of country-specific consumption goods. This nested structure is summarized by

$$C_{k,t} = \mathcal{C}_k\left(\{C_{ki,t}\}_{i=1}^{I}\right) \quad \text{and} \quad C_{ki,t} = \mathcal{C}_{ki}\left(\{C_{kl\,i,t}\}_{l=1}^{K}\right), \tag{2}$$

where $C_{ki,t}$ is consumption of sector i at time t, and $C_{kl\,i,t}$ is country k's household's consumption of the good produced by industry i in country l. Finally, we let $C_{kl\,i,t}$ be a Dixit and Stiglitz (1977) aggregator over differentiated goods produced by firms in sector i in country l:

$$C_{kl\,i,t} = \left(\int_0^1 C_{kl\,if,t}^{(\epsilon_{ki}^p-1)/\epsilon_{ki}^p} df\right)^{\epsilon_{ki}^p/(\epsilon_{ki}^p-1)} \tag{3}$$

where ϵ_{ki}^p denotes the sectoral constant elasticity of substitution between good varieties.

The household faces the following per-period budget constraint:

$$P_{k,t}^C C_{k,t} + \frac{B_{k,t}}{1+i_{k,t}} + \mathcal{E}_{k,t}^K \sum_{h\in\mathcal{H}} Q_t^h B_{k,t}^h \leq W_{k,t}N_{k,t} + \mathcal{E}_{k,t}^K \sum_{h\in\mathcal{H}} (Q_t^h + D_t^h) B_{k,t-1}^h + B_{k,t-1} + \Pi_{k,t} - T_{k,t} \tag{4}$$

where $W_{k,t}$ is the nominal wage received by the household, $\Pi_{k,t} = \sum_{i=1}^{I} \Pi_{ki,t}$ are profits of domestic firms, and $T_{k,t}$ is a lump-sum tax or transfer. $P^C_{k,t}$ is the consumer price index in country k, implied by the consumption aggregators (2).

There are two sets of assets in the economy. The first of them, $B_{k,t}$, is a one-period nominal bond only traded domestically that pays a gross nominal interest rate $1 + i_{k,t}$, set by the monetary authority of country k. The second one is a set $\mathcal{H}$ of bonds traded internationally at price Q^h_t, with dividend payouts D^h_t.[2] These bonds are denominated in the currency of country K, and hence $\mathcal{E}^K_{k,t}$ is the bilateral nominal exchange rate between country k and country K.

Under the previous notation, we have the following allocation of consumption across sectors, countries, and differentiated goods:

(5)
$$\frac{\partial \mathcal{C}_{k,t}}{\partial C_{ki,t}} = \frac{P^C_{ki,t}}{P^C_{k,t}}, \quad \frac{\partial \mathcal{C}_{ki}}{\partial C_{kl\,i,t}} = \frac{(1+\tau_{kl\,i,t})(1+\tau^w_{kl\,i,t})P^k_{kl\,i,t}}{P^C_{ki,t}}, \quad \text{and} \quad C_{kl\,if,t} = \left(\frac{P^k_{kl\,if,t}}{P^k_{kl\,i,t}}\right)^{-\epsilon^p_{ki}} C_{kl\,i,t},$$

where $P^C_{ki,t}$ denotes the consumer price index of sector i faced by the household in country k, and $P^k_{kl\,i,t} \equiv \left[\int_0^1 (P^k_{kl\,if,t})^{1-\epsilon^p_{ki}} df\right]^{1/(1-\epsilon^p_{ki})}$, where $P^m_{kl\,i,t}$ is the price of a good produced in sector i and country l, purchased by country k, in the currency of country m.[3]

The terms $\tau_{kl\,i,t}$ and $\tau^w_{kl\,i,t}$ are exogenous wedges between the price set by industry i in country l when exporting to country k, $P^k_{kl\,i,t}$, and the gross price paid by agents in country k. They differ only in the incidence of the associated revenue. The revenue from the tariff $\tau_{kl\,i,t}$ accrues to the importing government, which rebates it lump sum to domestic households. The revenue from the price wedge $\tau^w_{kl\,i,t}$ accrues instead to the exporting government, so the wedge generates a direct international transfer: a wealth loss for the importing country and a gain for the exporting country.

Finally, the first-order condition for households' savings in domestic bonds results in the following Euler equation, $U'(C_{k,t}) = \beta \mathbb{E}_t \left[U'(C_{k,t+1})(1+i_{k,t})/(1+\pi^C_{k,t+1})\right]$, where $1+\pi^C_{k,t+1} \equiv P^C_{k,t+1}/P^C_{k,t}$ is the gross consumer price inflation in country k, and β denotes the discount factor.

[2] This specification follows Egorov and Mukhin (2023). Note if $\mathcal{S}$ is the set of possible states of nature, we have that financial markets are complete if $\mathcal{H} = \mathcal{S}$. On the other hand, by setting $\mathcal{H} = 1$ we have the usual specification of incomplete financial markets with a single bond traded internationally.

[3] Under the CRS assumptions, we have that $P^C_{k,t} C_{k,t} = \sum_{i=1}^{I} P^C_{ki,t} C_{ki,t}$ and $P^C_{ki,t} C_{ki,t} = \sum_{l=1}^{K} (1+\tau_{kl\,i,t}) P^k_{kl\,i,t} C_{kl\,i,t}$.

2.2. Firms

There are I industries in each economy, indexed by $i \in \{1, 2, ..., I\}$, and within each industry there is a unit mass of firms, indexed by $f \in (0, 1)$.

Production. Each firm f, in sector i and country k, produces a differentiated good $Y_{kif,t}$ with a CRS production function F_{ki} that combines labor $N_{kif,t}$ and a basket of intermediate goods $X_{kif,t}$:

$$Y_{kif,t} = F_{ki}\left(N_{kif,t}, X_{kif,t}\right). \tag{6}$$

Industry-level output is a CES composite of the production of variety f,

$$Y_{ki,t} = \left[\int_0^1 Y_{kif,t}^{(\epsilon_{ki}^p - 1)/\epsilon_{ki}^p} df\right]^{\epsilon_{ki}^p/(\epsilon_{ki}^p - 1)} \tag{7}$$

with an implied sectoral price index

$$P_{ki,t} = \left[\int_0^1 P_{kif,t}^{1-\epsilon_{ki}^p} df\right]^{1/(1-\epsilon_{ki}^p)} \tag{8}$$

The bundle of intermediate goods $X_{kif,t}$ is defined similar to the household's consumption basket, given by a CRS aggregator of sectoral intermediate goods, which are themselves defined as a CRS aggregator of country-specific intermediate goods:

$$X_{kif,t} = \mathcal{X}_{ki}\left(\left\{X_{kijf,t}\right\}_{j=1}^{I}\right) \quad \text{and} \quad X_{kijf,t} = \mathcal{X}_{kij}\left(\left\{X_{klijf,t}\right\}_{l=1}^{K}\right), \tag{9}$$

where $X_{kijf,t}$ is firm's f demand in sector i of country k for goods produced in sector j, and $X_{klijf,t}$ is firm's f demand in sector i of country k for goods produced in sector j in country l. $X_{klijf,t}$ is itself a Dixit-Stiglitz aggregator over differentiated goods produced by firms in sector j in country l: $X_{klijf,t} = \left[\int_0^1 X_{klijff',t}^{(\epsilon_{kj}^p - 1)/\epsilon_{kj}^p} df'\right]^{\epsilon_{kj}^p/(\epsilon_{kj}^p - 1)}$.

Cost minimization by firms delivers the following first-order conditions for labor and intermediate goods demands, $W_{k,t} = \text{MC}_{ki,t}\partial F_{ki}/\partial N_{kif,t}$ and $P_{kij,t}^X = \text{MC}_{ki,t}\partial F_{ki}/\partial X_{kijf,t}$, and

the allocation of intermediate goods demand across sectors and countries:

(10)
$$\frac{\partial \mathcal{X}_{ki}}{\partial X_{kij,t}} = \frac{P^X_{kij,t}}{P^X_{ki,t}}, \quad \frac{\partial \mathcal{X}_{kij}}{\partial X_{klij,t}} = \frac{(1+\tau_{klj,t})(1+\tau^w_{klj,t})P^k_{klj,t}}{P^X_{kij,t}}, \quad X_{klijff',t} = \left(\frac{P^k_{ljf',t}}{P^k_{klj,t}}\right)^{-\epsilon^p_{kj}} X_{klijf,t}.$$

Here, $P^X_{ki,t}$ denotes the price index of the intermediate input bundle $X_{kif,t}$ faced by firms in sector i in country k, and $P^X_{kij,t}$ is the price index of the sectoral intermediate input $X_{kijf,t}$ faced by firms in sector i in country k. As in the case of households, the prices faced by domestic industries are subject to tariffs and price wedges.

Nominal marginal costs in sector i of country k are denoted by $\text{MC}_{ki,t}$. Under CRS, all firms in a given sector choose the same combination of inputs, and $\text{MC}_{ki,t}$ is common across firms and given by:

(11)
$$\text{MC}_{ki,t} = \min_{\{N_{ki,t}, X_{klij,t}\}} W_{ki,t}N_{ki,t} + \sum_{j=1}^{I}\sum_{l=1}^{K}(1+\tau_{klj,t})(1+\tau^w_{klj,t})P^k_{klj,t}X_{klij,t} \quad \text{s.t. } F_{ki}(N_{ki,t}, X_{ki,t}) \geq 1.$$

Price Setting. Firms set prices in a staggered manner (Calvo 1983). Specifically, firms in sector i of country k can reset their price with probability $1-\theta^p_{ki}$ each period. We allow for a country- and sector-specific probability of price adjustment. A price that is not reset remains fixed in its currency of invoicing until the next adjustment opportunity.

We entertain three possibilities for the pricing decisions to foreign markets: producer currency pricing (PCP), local currency pricing (LCP), or dominant currency pricing (DCP). Under PCP, the firm sets its export price in its own currency, so the selling prices to the domestic and foreign markets coincide, and the price of a good in sector i produced in country k and purchased by agents in country l, in currency units of country m, is given by

(12)
$$P^m_{lki,t} = \mathcal{E}^k_{m,t}P_{ki,t}.$$

A firm that resets its price chooses its optimal selling price to the domestic market, $\overline{P}_{ki,t}$, by solving the following problem:

(13)
$$\max_{\overline{P}_{ki,t}} \mathbb{E}_t \sum_{s=0}^{\infty} \text{SDF}_{t,t+s}(\theta^p_{ki})^s \left(\overline{P}_{ki,t} - (1-\tau^p_{ki})\text{MC}_{ki,t+s}\right)\mathcal{D}_{ki,t+s},$$

where $\text{SDF}_{t,t+s} = \beta^s U'(C_{k,t+s})/U'(C_{k,t})$ is the stochastic discount factor between periods t and $t+s$, and $\mathcal{D}_{ki,t+s}$ denotes the demand for the firm's good from domestic agents given in equations (5) and (10). We further allow for having a country-sector specific production

subsidy τ^p_{ki}.[4]

Under LCP, the firm sets its export price to country l in the currency of country l, denoted by $\overline{P}_{l\,ki,t}$,

$$P^m_{l\,ki,t} = \mathcal{E}^l_{m,t} P_{l\,ki,t}, \tag{14}$$

solving the problem:

$$\max_{\overline{P}_{l\,ki,t}} \mathbb{E}_t \sum_{s=0}^{\infty} \text{SDF}_{t,t+s} (\theta^p_{ki})^s \left(\mathcal{E}^l_{k,t+s} \overline{P}_{l\,ki,t} - (1-\tau^p_{ki}) \text{MC}_{ki,t+s} \right) \mathcal{D}_{l\,ki,t+s}, \tag{15}$$

where $\mathcal{D}_{l\,ki,t+s}$ is the demand from agents from country l for the firm's production.

Finally, under DCP, the firm sets a common US-dollar export price, denoted by $\overline{P}^{\text{US}}_{ki,t}$, for its dollar-invoiced foreign sales,

$$P^m_{l\,ki,t} = \mathcal{E}^{\text{US}}_{m,t} P^{\text{US}}_{ki,t}, \tag{16}$$

solving the following problem:

$$\max_{\overline{P}^{\text{US}}_{ki,t}} \mathbb{E}_t \sum_{s=0}^{\infty} \text{SDF}_{t,t+s} (\theta^p_{ki})^s \left(\mathcal{E}^{\text{US}}_{k,t+s} \overline{P}^{\text{US}}_{ki,t} - (1-\tau^p_{ki}) \text{MC}_{ki,t+s} \right) \mathcal{D}^{\text{US}}_{ki,t+s}. \tag{17}$$

where $\mathcal{D}^{\text{US}}_{ki,t+s}$ aggregates the destination-specific demand for country k, sector i, goods that are invoiced in US dollars.

Profits in sector i of country k are given by export- and domestic-market revenues converted into the producer currency, net of costs, $\Pi_{ki,t} = \sum_{l=1}^{K} \mathcal{E}^m_{k,t} P^m_{l\,ki,t} D_{l\,ki,t} - \text{MC}_{ki,t} Y_{ki,t}$, and aggregate profits rebated to households are $\Pi_{k,t} = \sum_{i=1}^{I} \Pi_{ki,t}$.

2.3. Government

The government is composed of a fiscal authority and a central bank. The fiscal authority issues domestic debt $B_{k,t}$, provides production subsidies to firms, and collects lump-sum taxes $T_{k,t}$ from the household in order to balance its budget constraint: $\frac{B_{k,t}}{1+i_{k,t}} + T_{k,t} + \sum_{l\neq k}^{K} \sum_{i=1}^{I} \left(\tau_{kl\,i,t} P^k_{kl\,i,t} C_{kl\,i,t} + \sum_{j=1}^{I} \tau_{kl\,i,t} P^k_{kl\,i,t} X_{kl\,ji,t} \right) = B_{k,t-1} + \sum_{i=1}^{I} \tau^p_{k,i} \text{MC}_{ki,t} Y_{ki,t}$, where $Y_{ki,t}$ denotes the production of sector i in country k.

The central bank sets the nominal interest rate, $i_{k,t}$. Since the specific monetary policy rule

[4] Production subsidies will be useful to derive our analytical results in section 3, as they allow us to eliminate steady-state distortions arising from monopolistic competition (Galí 2015).

is inconsequential for our analytical results in Section 3, we defer its definition to Section E.

2.4. Market Clearing and GDP

Market clearing in the goods market requires that the quantity produced of each good matches the quantity demanded at home and abroad, either for direct consumption or intermediate use. That is,

$$Y_{ki,t} = \sum_{l=1}^{K} C_{l\,ki,t} + \sum_{l=1}^{K}\sum_{j=1}^{I} X_{l\,kji,t}. \tag{18}$$

Market clearing in the labor market requires that the aggregate labor supplied matches the sum of labor demand across sectors, for each country. That is,

$$N_{k,t} = \sum_{i=1}^{I} N_{ki,t}. \tag{19}$$

Finally, the aggregate resource constraint of the economy requires that the net foreign position of country k equals its trade balance:

$$\mathcal{E}^{K}_{k,t}\sum_{h\in\mathcal{H}} Q^{h}_{t} B^{h}_{k,t} - \mathcal{E}^{K}_{k,t}\sum_{h\in\mathcal{H}} (Q^{h}_{t} + D^{h}_{t}) B^{h}_{k,t-1} = P^{\text{EXP}}_{k,t}\text{EXP}_{k,t} - P^{\text{IMP}}_{k,t}\text{IMP}_{k,t} \tag{20}$$

where the first term in the right-hand side of (20) is the total nominal exports of country k,

$$P^{\text{EXP}}_{k,t}\text{EXP}_{k,t} = \sum_{l\neq k}^{K}\sum_{i=1}^{I}\left(P^{k}_{l\,ki,t} C_{l\,ki,t} + \sum_{j=1}^{I} P^{k}_{l\,ki,t} X_{l\,kji,t}\right). \tag{21}$$

The second term in (20) denotes the total nominal imports,

$$P^{\text{IMP}}_{k,t}\text{IMP}_{k,t} = \sum_{l\neq k}^{K}\sum_{i\in I}\left(P^{k}_{kl\,i,t} C_{kl\,i,t} + \sum_{j=1}^{I} P^{k}_{kl\,i,t} X_{kl\,ji,t}\right). \tag{22}$$

Nominal GDP is defined as the sum of total household consumption and nominal net exports,

$$\mathcal{Y}_{k,t} = P^{C}_{k,t} C_{k,t} + P^{\text{EXP}}_{k,t}\text{EXP}_{k,t} - P^{\text{IMP}}_{k,t}\text{IMP}_{k,t} \tag{23}$$

Defining the GDP deflator, $P^{Y}_{k,t}$, as the ratio between nominal GDP measured using time–t

prices and nominal GDP measured using steady-state prices

$$P^Y_{k,t} = \frac{P^C_{k,t} C_{k,t} + P^{\text{EXP}}_{k,t} \text{EXP}_{k,t} - P^{\text{IMP}}_{k,t} \text{IMP}_{k,t}}{P^C_k C_{k,t} + P^{\text{EXP}}_k \text{EXP}_{k,t} - P^{\text{IMP}}_k \text{IMP}_{k,t}},$$

we have that real GDP is given by:

$$Y_{k,t} = \mathcal{Y}_{k,t} / P^Y_{k,t}. \tag{24}$$

3. The Tractable Case: a Small Open Economy

This section specializes the general model of Section 2 to an SOE that can be characterized analytically. The assumptions below render the model tractable and nest familiar benchmarks in the literature, making transparent the mechanisms that openness and production networks add to the monetary-policy problem.

3.1. Assumptions

First, we assume that there are only two countries, $K = 2$: Home, the SOE, and the foreign country, which we refer to as the rest of the world (ROW). To ease notation, foreign variables are denoted with an asterisk. Under the SOE assumption, aggregate foreign variables remain constant at their steady-state values. Furthermore, we focus on the case where preferences and technology are symmetric across countries.

Second, we make the following functional assumptions about preferences, wage rigidities, and technology. Regarding preferences, we assume that utility (1) is logarithmic in consumption:

$$U(C_t) = \log C_t. \tag{25}$$

We furthermore assume that there are no nominal wage rigidities, leading to the following first-order condition for labor supply:

$$N^{\varphi}_{k,t} = U'(C_{k,t}) W_{k,t} / P^C_{k,t}. \tag{26}$$

In addition, we assume that all consumption (2) and intermediate-goods (9) aggregators

are Cobb–Douglas (Cobb and Douglas 1928):

$$\mathcal{C}(\{C_{i,t}\}_{i=1}^{I}) = \prod_{i=1}^{I} C_{i,t}^{\beta_i} \quad \text{with} \quad \sum_{i=1}^{I} \beta_i = 1, \quad \text{and} \quad \mathcal{C}_i(\{C_{i,t}^H, C_{i,t}^F\}) = (C_{i,t}^H)^{1-\zeta_i}(C_{i,t}^F)^{\zeta_i} \tag{27}$$

$$\mathcal{X}_i(\{X_{ij,t}\}_{j=1}^{I}) = \prod_{j=1}^{I} X_{ij,t}^{\nu_{ij}} \quad \text{with} \quad \sum_{j=1}^{I} \nu_{ij} = 1 \,\forall i, \quad \text{and} \quad \mathcal{X}_{ij}(\{X_{ij,t}^H, X_{ij,t}^F\}) = (X_{ij,t}^H)^{1-\zeta_{ij}}(X_{ij,t}^F)^{\zeta_{ij}}. \tag{28}$$

Within each origin, goods are aggregated via a Dixit and Stiglitz (1977) CES aggregator with elasticity ϵ_i^p:

$$C_{i,t}^H = \left(\int_0^1 (C_{if,t}^H)^{(\epsilon_i^p - 1)/\epsilon_i^p} \, df \right)^{\epsilon_i^p/(\epsilon_i^p - 1)}. \tag{29}$$

Furthermore, the elasticity of substitution between labor and intermediate goods in the production function (6) is unitary:

$$Y_{if,t} = N_{if,t}^{\alpha_i} X_{if,t}^{1-\alpha_i}. \tag{30}$$

Third, we assume that international financial markets are complete. This, together with log utility (25), implies the following risk-sharing condition holds:

$$C_t = Q_t C_t^*, \tag{31}$$

where Q_t is the real exchange rate given by:

$$Q_t = \mathcal{E}_t P_t^{C,*} / P_t^C. \tag{32}$$

Fourth, regarding pricing, SOE exports to the ROW are invoiced in the producer's currency (PCP), while SOE imports from the ROW may be invoiced, sector by sector, either in the producer's currency or in the SOE's currency (LCP). For imports invoiced under PCP, the law of one price holds:

$$P_{i,t}^F = \mathcal{E}_t P_{i,t}^*, \tag{33}$$

where $P_{i,t}^F$ is the price of sectoral good i produced in the ROW, denominated in domestic currency, and $P_{i,t}^*$ is the same good's price denominated in the ROW currency. For imports

invoiced under LCP,

$$P^F_{i,t} = P^*_{i,t}, \tag{34}$$

where $P^*_{i,t}$ is set in the SOE currency.

We assume a tariff-free, zero-inflation efficient steady state, supported by fixed fiscal instruments and lump-sum financing.[5]

Three wedges coexist, with the same ad valorem convention as in the general model: $\tau^M_{i,t}$ is the import tariff collected by Home and rebated to its household, $\tau^W_{i,t}$ is the price wedge collected abroad, and $\tau^X_{i,t}$ is the tariff levied by the ROW on Home exports. For the first-order equilibrium conditions, we can write $\boldsymbol{\tau}^F_t \equiv \boldsymbol{\tau}^M_t + \boldsymbol{\tau}^W_t$. The price paid by a domestic buyer is $(1+\tau^M_{i,t})(1+\tau^W_{i,t})P^F_{i,t}$, while domestic tariff receipts are $\mathfrak{T}^M_t = \sum_i \tau^M_{i,t}(1+\tau^W_{i,t})P^F_{i,t}M_{i,t}$, where $M_{i,t}$ includes imports for final consumption and intermediate use. The foreign payment is $(1+\tau^W_{i,t})P^F_{i,t}M_{i,t}$ in domestic currency.

Each wedge follows its own process:

$$\boldsymbol{\tau}^j_t = \boldsymbol{\rho}_j \boldsymbol{\tau}^j_{t-1} + \boldsymbol{\varepsilon}^j_t, \qquad j \in \{M, W, X\}. \tag{35}$$

3.2. Log-Linearized Equilibrium Conditions

We log-linearize around the symmetric zero-inflation steady state; lowercase variables denote deviations from it, and boldface denotes vectors and matrices. Appendix A.2 gives the derivations.

We first obtain the set of sectoral Phillips curves in the domestic economy by log-linearizing the first-order conditions associated with (13):

$$\boldsymbol{\pi}^H_t = \boldsymbol{\kappa}(\mathbf{mc}_t - \boldsymbol{p}^H_t) + \beta \mathbb{E}_t \boldsymbol{\pi}^H_{t+1}, \tag{36}$$

where $\boldsymbol{\pi}^H_t = [\pi^H_{1,t}, \ldots, \pi^H_{I,t}]^\intercal = \boldsymbol{p}^H_t - \boldsymbol{p}^H_{t-1}$ is an $I \times 1$ vector containing the domestic sectoral inflation rates, $\boldsymbol{p}^H_t = [p^H_{1,t}, \ldots, p^H_{I,t}]^\intercal$ is an $I \times 1$ vector of domestic sectoral prices, $\mathbf{mc}_t = [\mathrm{mc}_{1,t}, \ldots, \mathrm{mc}_{I,t}]^\intercal$ is an $I \times 1$ vector of sectoral nominal marginal costs, and $\boldsymbol{\kappa} = \mathrm{diag}\,(\kappa_1, \ldots, \kappa_I)$ is an $I \times I$ diagonal matrix containing the slopes of the sectoral Phillips curves, with $\kappa_i = (1-\beta\theta_i)(1-\theta_i)/\theta_i$.

Price rigidity determines the direct slopes in (36); production links operate through

[5] For the welfare analysis, we also maintain efficiency of the flexible-price allocation under foreign price-wedge shocks in the absence of tariffs. The steady-state correction is fixed across policy experiments.

marginal costs:

$$\mathbf{mc}_t = \boldsymbol{\alpha} w_t + \boldsymbol{\Omega}_H \boldsymbol{p}_t^H + \boldsymbol{\Omega}_F \left(\boldsymbol{\tau}_t^F + \boldsymbol{p}_t^F \right) = \boldsymbol{\alpha} w_t + \boldsymbol{\Omega} \boldsymbol{p}_t^H + \boldsymbol{\Omega}_F \boldsymbol{s}_t, \tag{37}$$

where $\boldsymbol{\alpha}$ collects labor cost shares, $\boldsymbol{\Omega}_H$ and $\boldsymbol{\Omega}_F$ the domestic and imported input shares, and $\boldsymbol{\Omega} = \boldsymbol{\Omega}_H + \boldsymbol{\Omega}_F$. The vector $\boldsymbol{p}_t^F$ contains foreign prices in Home currency, with $p_{i,t}^F = (1 - \mathbb{1}_i) e_t + p_{i,t}^*$; $\mathbb{1}_i = 1$ for LCP imports and zero for PCP imports. Appendix A.1 gives the share definitions. Sectoral ToT include the total import-price wedge:

$$\boldsymbol{s}_t = \boldsymbol{\tau}_t^F + \boldsymbol{p}_t^F - \boldsymbol{p}_t^H. \tag{38}$$

The i-th column of $\boldsymbol{\Omega}_F$ measures direct exposure to an import-price shock in sector i; $\boldsymbol{\Omega}_H$ propagates the resulting domestic-price changes. Exchange-rate and wage responses add general-equilibrium effects.

Labor supply implies

$$w_t - p_{C,t} = c_t + \varphi n_t = c_t + \varphi y_t, \tag{39}$$

where $n_t = y_t$ in real value added. The corresponding GDP identity is

$$y_t = (1 + \varphi \boldsymbol{\lambda}^\intercal \boldsymbol{\Omega}_F \mathbf{1})^{-1} \left[c_t + (\boldsymbol{\beta}_F^\intercal + \boldsymbol{\lambda}^\intercal \boldsymbol{\Omega}_F)(\boldsymbol{s}_t - \boldsymbol{\tau}_t^X) - \boldsymbol{\lambda}^\intercal \mathrm{diag}\,(\boldsymbol{\Omega}_F \mathbf{1})\, \boldsymbol{n}_t \right], \tag{40}$$

where $\boldsymbol{n}_t = [n_{1,t}, \ldots, n_{I,t}]^\intercal$ is an $I \times 1$ vector of sectoral employment levels, $\boldsymbol{\beta}_F = \left[P_i^F C_i^F / (P^C C), \ldots, P_I^F C_I^F / (P^C C) \right]^\intercal = [\beta_1 \zeta_1, \ldots, \beta_I \zeta_I]^\intercal$ denotes the $I \times 1$ vector of foreign consumption shares, $\boldsymbol{\lambda} = \left[P_i^H Y_i / (P^Y Y), \ldots, P_I^H Y_I / (P^Y Y) \right]^\intercal$ denotes the $I \times 1$ vector of Domar weights and satisfies $\boldsymbol{\lambda}^\intercal = \boldsymbol{\beta}^\intercal (\boldsymbol{I} - \boldsymbol{\Omega})^{-1}$, $\boldsymbol{\lambda}^\intercal \boldsymbol{\Omega}_F \mathbf{1} = \sum_{i=1}^I \sum_{j=1}^I P_i X_{ji}^F / (P^Y Y)$ denotes the share of imported intermediate goods over GDP, and the diagonal matrix $\mathrm{diag}\,(\boldsymbol{\Omega}_F \mathbf{1})$ collects, for each sector j, the total cost share of imported intermediate inputs. The consumption share vector $\boldsymbol{\beta} = \begin{bmatrix} \beta_1 & \beta_2 & \ldots & \beta_I \end{bmatrix}^\intercal = \boldsymbol{\beta}_H + \boldsymbol{\beta}_F$ contains elements $\beta_i = P_i C_i / (P_C C)$, and $\boldsymbol{\beta}_H = \left[P_i^H C_i^H / (P_C C), \ldots, P_I^H C_I^H / (P_C C) \right]^\intercal = \left[\beta_1 (1 - \zeta_1), \ldots, \beta_I (1 - \zeta_I) \right]^\intercal$ denotes the $I \times 1$ vector of domestic consumption shares.

The relative price $\boldsymbol{s}_t - \boldsymbol{\tau}_t^X$ shifts foreign demand and, through imports, the domestic trade balance. Import invoicing determines which prices also respond to depreciation: PCP prices do, whereas LCP prices are fixed in Home currency on impact. The prefactor in (40) captures the feedback from imported inputs to wages and aggregate activity.

Equation (40) can be further simplified. As shown in Appendix A.2, using the goods market-clearing conditions (18) we obtain the following expression for the vector of sectoral

employment:

$$\boldsymbol{n}_t = \mathcal{N}_s \boldsymbol{s}_t - \mathcal{N}_\tau(\boldsymbol{\tau}_t^F - \mathbb{1}e_t) - \mathcal{N}_x \boldsymbol{\tau}_t^X + (\mathcal{N}_y - \mathbf{1})\varphi y_t, \tag{41}$$

where $\mathcal{N}_s = \left(\boldsymbol{I} - \Lambda^{-1}\Omega_H^\top \Lambda\right)^{-1}(\boldsymbol{I} - \Omega_H)$ and $\mathcal{N}_\tau = \left(\boldsymbol{I} - \Lambda^{-1}\Omega_H^\top \Lambda\right)^{-1}(\boldsymbol{I} - \Omega)$ are $I \times I$ matrices that summarize the elasticity of sectoral employment to sectoral ToT and to the tariff shocks, with $\Lambda = \text{diag}(\boldsymbol{\lambda})$, while $\mathcal{N}_y = \left(\boldsymbol{I} - \Lambda^{-1}\Omega_H^\top \Lambda\right)^{-1}\boldsymbol{\alpha}$ is a $I \times 1$ vector, and $\mathbf{1}$ is a $I \times 1$ vector of ones. The export-demand transmission matrix $\mathcal{N}_x \equiv (\boldsymbol{I} - \Lambda^{-1}\Omega_H^\top \Lambda)^{-1}\Lambda^{-1}\,\text{diag}(\beta_F + \Omega_F^\top \boldsymbol{\lambda})$ summarizes the elasticity of sectoral employment to export tariffs. In (41), $\mathcal{N}_s$ captures expenditure switching, $\mathcal{N}_\tau$ the import-price channel, and $\mathcal{N}_x$ foreign export demand. The last term records the wage response to aggregate activity; without domestic input links, $\mathcal{N}_y = \boldsymbol{\alpha}$.

Introducing (41) into (40) we obtain:

$$\begin{aligned} y_t = (1 + \varphi \boldsymbol{\lambda}^\intercal \text{diag}\left(\Omega_F \mathbf{1}\right) \mathcal{N}_y)^{-1} \Big\{ & c_t + \left[\beta_F^\intercal + \boldsymbol{\lambda}^\intercal \left(\Omega_F - \text{diag}\left(\Omega_F \mathbf{1}\right) \mathcal{N}_s\right)\right] \boldsymbol{s}_t + \boldsymbol{\lambda}^\intercal \text{diag}\left(\Omega_F \mathbf{1}\right) \mathcal{N}_\tau (\boldsymbol{\tau}_t^F - \mathbb{1}e_t) \\ & - (\beta_F^\intercal + \boldsymbol{\lambda}^\intercal \Omega_F)\boldsymbol{\tau}_t^X + \boldsymbol{\lambda}^\intercal \,\text{diag}(\Omega_F \mathbf{1})\, \mathcal{N}_x \boldsymbol{\tau}_t^X \Big\}. \end{aligned} \tag{42}$$

The multiplier in (42) incorporates the wage and imported-input feedback. Its export-tariff terms combine the direct demand reduction with the induced employment adjustment. These direct shock terms drop out of the gap relation below.

Flexible-price equilibrium. We next consider the natural allocation, denoted by a superscript n, and give its derivation in Appendix A.2.[6]

In the flexible-price equilibrium, where price–cost ratios remain at their reference values ($p_{i,t}^{H,n} = \text{mc}_{i,t}^n$) and nominal wages remain constant, the domestic prices adjust as $\boldsymbol{p}_t^{H,n} = \Omega\, \boldsymbol{p}_t^{H,n} + \Omega_F \boldsymbol{s}_t^n = (\boldsymbol{I} - \Omega)^{-1}\,\Omega_F \boldsymbol{s}_t^n = (\boldsymbol{I} - \Omega_H)^{-1}\,\Omega_F \left(\boldsymbol{\tau}_t^F + \boldsymbol{p}_t^{F,n}\right)$. Since the domestic-currency price of foreign goods is $\boldsymbol{p}_t^{F,n} = (\mathbf{1} - \mathbb{1})e_t^n$, with foreign prices constant, we can write:

$$\boldsymbol{p}_t^{H,n} = (\boldsymbol{I} - \Omega_H)^{-1}\Omega_F \left[\boldsymbol{\tau}_t^F + (\mathbf{1} - \mathbb{1})e_t^n\right]. \tag{43}$$

Equation (43) shows how, in a flexible-price equilibrium, domestic prices adjust in response to the total import wedges $\boldsymbol{\tau}_t^F$ and, through the natural exchange rate e_t^n, to the sectoral export tariffs $\boldsymbol{\tau}_t^X = [\tau_{1,t}^X, \ldots, \tau_{I,t}^X]^\intercal$.

The total import wedge shifts natural domestic prices through direct imported-input

[6] Under tariff shocks this allocation need not be efficient; the welfare analysis in Section 4 maintains efficiency only for tariff-free foreign-wedge shocks.

exposure, Ω_F, and through the domestic Leontief inverse, $(\boldsymbol{I}-\Omega_H)^{-1}$. Under PCP, export-tariff-induced movements in the natural exchange rate also enter foreign prices in domestic currency.

Given the response of domestic prices in the natural equilibrium (43), the response of the ToT is $\boldsymbol{s}_t^n = \boldsymbol{\tau}_t^F + \boldsymbol{p}_t^{F,n} - \boldsymbol{p}_t^{H,n} = \left[\boldsymbol{I} - (\boldsymbol{I}-\Omega_H)^{-1}\Omega_F\right]\left[\boldsymbol{\tau}_t^F + (\mathbf{1}-\mathbb{1})e_t^n\right]$.

Finally, Appendix A.2 also shows that, under these flexible-price adjustments, GDP in the flexible-price equilibrium remains unaffected by import tariff shocks, and is only affected by the export tariffs that the ROW levies on the SOE's exports:

(44)
$$y_t^n = -\frac{1}{1+\varphi}\,\ell_X^\intercal \boldsymbol{\tau}_t^X, \qquad e_t^n = -\varphi\, y_t^n = \frac{\varphi}{1+\varphi}\,\ell_X^\intercal \boldsymbol{\tau}_t^X, \qquad \ell_X \equiv \operatorname{diag}\left[(\boldsymbol{I}-\Omega_H)^{-1}\boldsymbol{\alpha}\right]\left(\boldsymbol{\beta}_F + \Omega_F^\intercal \boldsymbol{\lambda}\right).$$

The vector ℓ_X measures the exposure of domestic value added to each export tariff. An export tariff on sector i lowers foreign demand in proportion to $(\boldsymbol{\beta}_F + \Omega_F^\intercal\boldsymbol{\lambda})_i$. Only part of the associated gross-output decline is domestic value added: $[(\boldsymbol{I}-\Omega_H)^{-1}\boldsymbol{\alpha}]_i$ is the direct and indirect labor content of sector-i output, while the remainder is imported-input content. The resulting decline in natural activity depreciates the natural exchange rate, $e_t^n = -\varphi\, y_t^n$, transmitting export tariffs to natural prices under PCP.[7]

Equilibrium conditions in gaps. In what follows we express the equilibrium conditions in gaps relative to the flexible-price allocation, $\widetilde{x}_t \equiv x_t - x_t^n$. Log-linearized risk sharing gives $\widetilde{c}_t = \widetilde{e}_t - \widetilde{p}_t^C$, while the CPI gap is $\widetilde{p}_t^C = \boldsymbol{\beta}^\top \widetilde{\boldsymbol{p}}_t^H + \boldsymbol{\beta}_F^\top \widetilde{\boldsymbol{s}}_t$. Writing (42) in gaps yields

(45)
$$\begin{aligned} \widetilde{c}_t &= \widetilde{e}_t - \widetilde{p}_t^C, \qquad \widetilde{p}_t^C = \boldsymbol{\beta}^\top \widetilde{\boldsymbol{p}}_t^H + \boldsymbol{\beta}_F^\top \widetilde{\boldsymbol{s}}_t, \\ D_y \widetilde{y}_t &= \widetilde{c}_t + \left[\boldsymbol{\beta}_F^\top + \boldsymbol{\lambda}^\top\left(\Omega_F - \operatorname{diag}(\Omega_F\mathbf{1})\mathcal{N}_s\right)\right]\widetilde{\boldsymbol{s}}_t - \boldsymbol{\lambda}^\top \operatorname{diag}(\Omega_F\mathbf{1})\mathcal{N}_\tau \mathbb{1}\widetilde{e}_t. \end{aligned}$$

Here $D_y \equiv 1 + \varphi\boldsymbol{\lambda}^\top \operatorname{diag}(\Omega_F\mathbf{1})\mathcal{N}_y$ is the inverse general-equilibrium multiplier in (42). The coefficient on the ToT gaps combines expenditure switching and the induced demand for imported inputs. The last term captures the imported-input demand channel associated with LCP; it vanishes under PCP. Combining the three conditions gives

(46)
$$D_y \widetilde{y}_t = \left[1 - \boldsymbol{\lambda}^\top \operatorname{diag}(\Omega_F\mathbf{1})\mathcal{N}_\tau \mathbb{1}\right]\widetilde{e}_t - \boldsymbol{\beta}^\top \widetilde{\boldsymbol{p}}_t^H + \boldsymbol{\lambda}^\top\left[\Omega_F - \operatorname{diag}(\Omega_F\mathbf{1})\mathcal{N}_s\right]\widetilde{\boldsymbol{s}}_t.$$

Because financial markets are complete, (46) ties the exchange-rate gap to the output gap for given domestic price gaps and ToT gaps.

[7] For a common export tariff, $\boldsymbol{\tau}_t^X = \mathbf{1}\tau_t^X$, $\mathbf{1}^\top\ell_X$ is the domestic value-added content of exports relative to GDP and $\partial y_t^n/\partial\tau_t^X = -\mathbf{1}^\top\ell_X/(1+\varphi)$. With heterogeneous tariffs, $\ell_X^\top\boldsymbol{\tau}_t^X$ is the corresponding exposure-weighted shock.

3.3. Sectoral Phillips Curves

Domestic and foreign production links determine how international price shocks reach sectoral inflation.

One-Sector Small Open Economy. With one sector and intermediate inputs, the Phillips curve becomes (Appendix A.2)

$$\pi_t^H = \kappa(1+\varphi)(1-\omega_H)\widetilde{y}_t + \beta\mathbb{E}_t\pi_{t+1}^H, \tag{47}$$

where $\widetilde{y}_t$ is the gap from flexible-price output and ω_H is the domestic-intermediate cost share. Closing the gap stabilizes domestic inflation; domestic intermediates flatten the curve, while openness does not change its slope in this log-utility, unit-elastic benchmark.[8]

Multi-sector Small Open Economy. In our baseline SOE with production networks and multiple sectors, combining equations (36), (37), (39), and (42), the domestic sectoral Phillips curves can be written as[9]

$$\boldsymbol{\pi}_t^H = \mathcal{B}\left[1+\varphi+\varphi\boldsymbol{\lambda}^\intercal \operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\mathcal{N}_y\right]\widetilde{y}_t + \boldsymbol{\Psi}\widetilde{\boldsymbol{s}}_t + \boldsymbol{\Gamma}\mathbb{1}\widetilde{e}_t - \mathcal{V}\boldsymbol{\chi}_t + \beta(\boldsymbol{I}-\mathcal{V})\mathbb{E}_t\boldsymbol{\pi}_{t+1}^H, \tag{48}$$

where $\widetilde{\boldsymbol{s}}_t = \begin{bmatrix}\widetilde{s}_{1,t} & \dots & \widetilde{s}_{I,t}\end{bmatrix}^\intercal$ denotes the vector of sectoral ToT gaps and $\widetilde{e}_t \equiv e_t - e_t^n$ denotes the nominal exchange-rate gap. The inefficient-price residual is

$$\boldsymbol{\chi}_t \equiv \boldsymbol{p}_{t-1}^H - \boldsymbol{p}_t^{H,n} = \widetilde{\boldsymbol{p}}_{t-1}^H - \Delta\boldsymbol{p}_t^{H,n}. \tag{49}$$

It collects lagged domestic price gaps net of the change in natural prices. Substituting (44) into (43) and taking first differences gives $\Delta\boldsymbol{p}_t^{H,n}$ in terms of the foreign shocks. Hence sectoral inflation satisfies

$$\boldsymbol{\pi}_t^H = \widetilde{\boldsymbol{p}}_t^H - \widetilde{\boldsymbol{p}}_{t-1}^H + \Delta\boldsymbol{p}_t^{H,n} = \widetilde{\boldsymbol{p}}_t^H - \widetilde{\boldsymbol{p}}_{t-1}^H + (\boldsymbol{I}-\boldsymbol{\Omega}_H)^{-1}\boldsymbol{\Omega}_F\left[\Delta\boldsymbol{\tau}_t^F + (\mathbf{1}-\mathbb{1})\frac{\varphi}{1+\varphi}\boldsymbol{\ell}_X^\intercal\Delta\boldsymbol{\tau}_t^X\right]. \tag{50}$$

The derivations of the matrices $\mathcal{B} = \boldsymbol{\Delta}(\boldsymbol{I}-\boldsymbol{\Omega}\boldsymbol{\Delta})^{-1}\boldsymbol{\alpha}/[1-\boldsymbol{\beta}^\intercal\boldsymbol{\Delta}(\boldsymbol{I}-\boldsymbol{\Omega}\boldsymbol{\Delta})^{-1}\boldsymbol{\alpha}]$ and $\mathcal{V} = \left\{\boldsymbol{\Delta}(\boldsymbol{I}-\boldsymbol{\Omega}\boldsymbol{\Delta})^{-1} - \mathcal{B}[\boldsymbol{\lambda}^\intercal - \boldsymbol{\beta}^\intercal\boldsymbol{\Delta}(\boldsymbol{I}-\boldsymbol{\Omega}\boldsymbol{\Delta})^{-1}]\right\}(\boldsymbol{I}-\boldsymbol{\Omega})$ are equivalent to those in the closed-economy version of the model in Rubbo (2023), with $\boldsymbol{\Delta} = (\boldsymbol{I}+\boldsymbol{\kappa})^{-1}\boldsymbol{\kappa}$. Finally, $\boldsymbol{\Psi} \equiv \boldsymbol{\Delta}(\boldsymbol{I}-\boldsymbol{\Omega}\boldsymbol{\Delta})^{-1}\boldsymbol{\Omega}_F + \mathcal{B}\boldsymbol{\beta}^\intercal\boldsymbol{\Delta}(\boldsymbol{I}-\boldsymbol{\Omega}\boldsymbol{\Delta})^{-1}\boldsymbol{\Omega}_F + \mathcal{B}\boldsymbol{\lambda}^\intercal\left[\operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\mathcal{N}_s - \boldsymbol{\Omega}_F\right]$ denotes the slope of the Phillips curve with respect to the ToT gaps, and $\boldsymbol{\Gamma} \equiv \mathcal{B}\boldsymbol{\lambda}^\top\operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\mathcal{N}_\tau$ denotes the slope with respect

[8] The quantitative model relaxes these functional assumptions (Appendix E).

[9] We relegate the derivation to Appendix A.2.

to the nominal exchange-rate gap that arises from LCP pricing.

Unlike the one-sector case, closing $\widetilde{y}_t$ need not stabilize every $\pi^H_{i,t}$: sectoral ToT gaps and χ_t remain in (48).

Slope with respect to the output gap. The coefficient on $\widetilde{y}_t$ in (48) is $\mathcal{B}[1+\varphi+\varphi\boldsymbol{\lambda}^\top \operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\mathcal{N}_y]$. The first factor transmits wage changes through domestic suppliers and price setting, as in Rubbo (2023). The second accounts for imported inputs: producing additional value added changes import demand and hence the labor required to support it. Proposition A1 establishes the slope comparison with a closed economy.

Transmission through ToT gaps. The matrix $\boldsymbol{\Psi}$ has three components. Imported-input exposure $\boldsymbol{\Omega}_F$ passes a relative-price change directly into marginal costs; the rigidity-adjusted domestic Leontief inverse propagates it to downstream prices. The induced CPI movement changes nominal wages and adds a common cost component. Finally, expenditure switching changes sectoral employment and import demand, with a further wage effect. This last channel vanishes in the one-sector limit (Proposition A2). Appendix A.3 gives the decomposition and signs.

The "as if" supply shock. The residual χ_t combines lagged domestic price gaps with the change in natural prices. The latter transmits foreign price disturbances through imported-input exposure $\boldsymbol{\Omega}_F$ and domestic production links $(\boldsymbol{I}-\boldsymbol{\Omega}_H)^{-1}$. The matrix $\mathcal{V}$ maps this residual into current inflation through marginal costs and the induced real-wage adjustment, extending the domestic-shock mechanism of Rubbo (2023) to an open economy.

The Divine Coincidence Inflation Index. The divine coincidence is the one-sector result that stabilizing inflation also closes the output gap absent cost-push shocks (Blanchard and Galí 2007). With sectoral heterogeneity, the relevant inflation measure becomes an index (Rubbo 2023). In the SOE, imported inputs add domestic production content to its weights. The following proposition extends the open-economy indexes of Qiu *et al.* (2026) and Aguilar *et al.* (2026) to heterogeneous invoicing.

PROPOSITION 1. *Let* $d_i \equiv [(\boldsymbol{I}-\boldsymbol{\Omega}_H)^{-1}\boldsymbol{\alpha}]_i$ *and* $S \equiv \sum_{j=1}^{I}(\lambda_j/\kappa_j)d_j$. *The domestic-value-added Divine Coincidence Inflation (DCI) index*

$$\pi^{\mathrm{DC}}_t = S^{-1}\sum_{i=1}^{I}\frac{\lambda_i}{\kappa_i}d_i\pi^H_{i,t} \tag{51}$$

satisfies

$$\pi_t^{\mathrm{DC}} = \frac{1+\varphi}{S}\widetilde{y}_t - \frac{R_e}{S}\widetilde{e}_t + \beta\mathbb{E}_t\pi_{t+1}^{\mathrm{DC}}, \tag{52}$$

where

$$R_e \equiv \boldsymbol{\beta}^{\mathsf{T}}\mathbb{1} + \boldsymbol{\lambda}^{\mathsf{T}}\operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\mathcal{N}_\tau\mathbb{1} - \left\{\boldsymbol{\beta}^{\mathsf{T}} + \boldsymbol{\lambda}^{\mathsf{T}}\left[\boldsymbol{\Omega}_F - \operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\mathcal{N}_s\right]\right\}(\boldsymbol{I}-\boldsymbol{\Omega}_H)^{-1}(\boldsymbol{I}-\boldsymbol{\Omega})\mathbb{1}. \tag{53}$$

The strict Divine Coincidence holds if and only if $R_e = 0$. Otherwise, heterogeneous invoicing leaves an exchange-rate residual.

PROOF. See Appendix A.3. □

The DCI remains the unique constant-weight domestic inflation index aligned with the output-gap markup aggregate. Heterogeneous invoicing therefore prevents its stabilization from closing the output gap.

4. Optimal Monetary Policy

This section characterizes optimal monetary policy in the tractable SOE for foreign price-wedge shocks. We first derive the welfare criterion and optimal allocation under commitment. We then compare that allocation with a Taylor rule and use sectoral counterfactuals to identify the disturbances for which the choice of policy is consequential. Tariff shocks remain in the positive model; the welfare and commitment exercises below set $\boldsymbol{\tau}_t^M = \boldsymbol{\tau}_t^X = \mathbf{0}$.

4.1. The Welfare Criterion

Home import tariffs make the price paid by buyers exceed the marginal foreign payment borne by the country. Foreign export tariffs change Home's receipts from sales abroad. Either intervention can separate the flexible-price allocation from the efficient benchmark used in the welfare expansion. We maintain an efficient natural allocation for the tariff-free foreign-wedge experiment and restrict the following welfare result to that experiment.

PROPOSITION 2. *When $\boldsymbol{\tau}_t^M = \boldsymbol{\tau}_t^X = \mathbf{0}$, the second-order welfare-based loss function in the SOE and IO economy is given by*

$$L_t = \frac{1}{2}(1+\varphi)\widetilde{y}_t^2 + \frac{1}{2}(\boldsymbol{\pi}_t^H)^{\top}\left(\boldsymbol{\Lambda}\mathbf{E}^{p}\boldsymbol{\kappa}^{-1}\right)\boldsymbol{\pi}_t^H + \frac{1}{2}\begin{pmatrix}\widetilde{\mathbf{p}}_t^H\\ \widetilde{\mathbf{s}}_t\end{pmatrix}^{\top}\mathbf{H}\begin{pmatrix}\widetilde{\mathbf{p}}_t^H\\ \widetilde{\mathbf{s}}_t\end{pmatrix}, \tag{54}$$

where $\mathbf{E}^p = \text{diag}(\epsilon_1^p, \ldots, \epsilon_I^p)$ *and* $\mathbf{H} = \mathbf{H}(\mathbb{1})$ *is a* $2I \times 2I$ *positive semidefinite matrix encoding the welfare cost of cross-sector misallocation.*

PROOF. See Appendix B. □

The output gap term $\frac{1+\varphi}{2}\,\widetilde{y}_t^2$ penalizes deviations of aggregate output from its efficient level. The coefficient $1+\varphi$ reflects the slope of labor supply: when φ is large (inelastic labor supply), a given output gap requires a larger deviation of the real wage from its efficient level, amplifying the welfare cost. This term is identical to the one-sector case of Gali and Monacelli (2005).

The inflation term $1/2\,(\boldsymbol{\pi}_t^H)^\top\big(\boldsymbol{\Lambda}\mathbf{E}^p\boldsymbol{\kappa}^{-1}\big)\boldsymbol{\pi}_t^H = 1/2\sum_i \lambda_i(\epsilon_i^p/\kappa_i)(\pi_{i,t}^H)^2$ penalizes within-sector price dispersion, expressed as squared inflation. The weight on sector i's inflation reflects three forces. First, the elasticity ϵ_i^p captures that sectors with higher substitutability between varieties have larger welfare costs from a given level of price dispersion, because demand reallocates more aggressively toward cheaper varieties. Second, the inverse slope $1/\kappa_i$ captures that sectors with stickier prices (lower κ_i) incur a higher cost per unit of inflation, because a given inflation rate implies a larger accumulated stock of price dispersion. These two channels are present in the one-sector benchmark; the third is specific to the network economy: the Domar weight λ_i captures the macroeconomic importance of sector i (sales as a share of GDP).

The final term collects the welfare costs associated with cross-sector relative-price misallocation. To make its economic content transparent, partition $\mathbf{H}$ as

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_{pp} & \mathbf{H}_{ps} \\ \mathbf{H}_{sp} & \mathbf{H}_{ss} \end{pmatrix}, \qquad \mathbf{H}_{sp} = \mathbf{H}_{ps}^\top. \tag{55}$$

The blocks $\mathbf{H}_{pp}$, $\mathbf{H}_{ss}$, and $\mathbf{H}_{ps}$ respectively capture inefficient domestic-price dispersion, sectoral ToT dispersion, and their interaction.

Appendix B constructs these blocks from the mappings between price gaps and consumption and input-demand distortions.[10]

4.2. Optimal Policy under Commitment

We take the (per-period) welfare loss in the LQ approximation to be (54), with $\boldsymbol{W}_\pi \equiv \boldsymbol{\Lambda}\mathbf{E}^p\boldsymbol{\kappa}^{-1}$, and make use of the partition (55).

[10]The one-sector closed-economy and SOE limits have $\mathbf{H} = \mathbf{0}$, while the multi-sector closed-economy limit retains only $\mathbf{H}_{pp}$.

Ramsey Problem under Commitment. The planner chooses $\{\widetilde{y}_t, \boldsymbol{\pi}_t^H, \widetilde{\mathbf{p}}_t^H, \widetilde{\boldsymbol{s}}_t, \widetilde{e}_t\}_{t\geq 0}$ to minimize $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t L_t$ subject to the sectoral price dynamics (50), the sectoral Phillips curves (48), and the structural definition of the sectoral ToT gap,

$$\widetilde{\mathbf{s}}_t = (\mathbf{1} - \mathbb{1})\widetilde{e}_t - \widetilde{\mathbf{p}}_t^H, \tag{56}$$

and the goods-market restriction (46). Two features of the problem deserve emphasis. First, the inefficient-price residual χ_t in (49) is not exogenous: it contains last period's domestic price gaps, which are themselves chosen by the planner. Under commitment, the planner therefore internalizes that the price gaps it implements today shift next period's Phillips curves. Second, because financial markets are complete, the exchange-rate gap and the output gap are not independent instruments. Given domestic price gaps and ToT gaps, (46) ties $\widetilde{e}_t$ to $\widetilde{y}_t$, so that the single policy instrument, the nominal interest rate, implements the allocation through the Euler equation.

Let $\boldsymbol{\eta}_t \in \mathbb{R}^I$ be the multiplier on the sectoral Phillips curves (48), $\boldsymbol{\nu}_t \in \mathbb{R}^I$ the multiplier on the price dynamics (50), $\boldsymbol{\xi}_t \in \mathbb{R}^I$ the multiplier on the ToT-definition constraint, and $\vartheta_t \in \mathbb{R}$ the multiplier on the goods-market restriction (46). The Lagrangian is

$$\begin{aligned}\mathcal{L}_t = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ & L_t + \boldsymbol{\eta}_t^{\top}\Big[\boldsymbol{\pi}_t^H - \mathbf{b}_y \widetilde{y}_t - \boldsymbol{\Psi}\widetilde{\boldsymbol{s}}_t - \boldsymbol{\Gamma}\mathbb{1}\widetilde{e}_t + \mathcal{V}\boldsymbol{\chi}_t - \beta(\boldsymbol{I} - \mathcal{V})\mathbb{E}_t \boldsymbol{\pi}_{t+1}^H\Big] \\ & + \boldsymbol{\nu}_t^{\top}\left(\widetilde{\mathbf{p}}_t^H - \widetilde{\mathbf{p}}_{t-1}^H + \Delta \boldsymbol{p}_t^{H,n} - \boldsymbol{\pi}_t^H\right) + \boldsymbol{\xi}_t^{\top}\left(\widetilde{\mathbf{s}}_t - (\mathbf{1} - \mathbb{1})\widetilde{e}_t + \widetilde{\mathbf{p}}_t^H\right) \\ & + \vartheta_t \Bigg\{ D_y \widetilde{y}_t - \Big[1 - \boldsymbol{\lambda}^{\top} \operatorname{diag}(\boldsymbol{\Omega}_F \mathbf{1})\mathcal{N}_{\tau}\mathbb{1}\Big]\widetilde{e}_t + \boldsymbol{\beta}^{\intercal}\widetilde{\mathbf{p}}_t^H + \left\{\boldsymbol{\beta}_F - \Big[\boldsymbol{\beta}_F + (\boldsymbol{\Omega}_F - \operatorname{diag}(\boldsymbol{\Omega}_F \mathbf{1})\mathcal{N}_s)^{\top}\boldsymbol{\lambda}\Big]\right\}^{\intercal} \widetilde{\mathbf{s}}_t \Bigg\} \Bigg\},\end{aligned}$$

where $\mathbf{b}_y \equiv \mathcal{B}\left(1 + \varphi + \varphi \boldsymbol{\lambda}^{\top} \operatorname{diag}(\boldsymbol{\Omega}_F \mathbf{1})\, \mathcal{N}_y\right)$ denotes the slopes of the Phillips curves, and $\boldsymbol{\eta}_{-1} = \mathbf{0}$.

The FOC with respect to the output gap yields

$$\widetilde{y}_t = \frac{1}{1+\varphi}\left(\mathbf{b}_y^{\top}\boldsymbol{\eta}_t - D_y \vartheta_t\right). \tag{57}$$

Differentiating with respect to sectoral inflation $\boldsymbol{\pi}_t^H$ yields

$$\boldsymbol{\eta}_t = \boldsymbol{\nu}_t - \boldsymbol{W}_{\pi}\boldsymbol{\pi}_t^H + (\boldsymbol{I} - \mathcal{V})^{\top}\boldsymbol{\eta}_{t-1}. \tag{58}$$

Because $\widetilde{\mathbf{p}}_t^H$ enters the loss, the price dynamics at t and $t+1$, the ToT definition, the goods-market restriction, and next period's inefficient-price residual χ_{t+1}, the first-order

condition with respect to the domestic price-level gaps is

$$\boldsymbol{\nu}_t = \beta \mathbb{E}_t \boldsymbol{\nu}_{t+1} - \beta \boldsymbol{\mathcal{V}}^\top \mathbb{E}_t \boldsymbol{\eta}_{t+1} - H_{pp} \widetilde{\mathbf{p}}_t^H - H_{ps} \widetilde{\boldsymbol{s}}_t - \boldsymbol{\xi}_t - \boldsymbol{\beta}\, \vartheta_t. \tag{59}$$

Differentiating with respect to the ToT $\widetilde{\boldsymbol{s}}_t$ gives the open-economy wedge condition

$$\Psi^\top \boldsymbol{\eta}_t = H_{sp} \widetilde{\mathbf{p}}_t^H + H_{ss} \widetilde{\boldsymbol{s}}_t + \boldsymbol{\xi}_t + \left\{ \boldsymbol{\beta}_F - \left[\boldsymbol{\beta}_F + (\boldsymbol{\Omega}_F - \operatorname{diag}(\boldsymbol{\Omega}_F \mathbf{1}) \mathcal{N}_s)^\top \boldsymbol{\lambda} \right] \right\} \vartheta_t. \tag{60}$$

Finally, the exchange rate FOC gives

$$(\boldsymbol{\Gamma} \mathbb{1})^\top \boldsymbol{\eta}_t + (\mathbf{1} - \mathbb{1})^\top \boldsymbol{\xi}_t + \left(1 - \boldsymbol{\lambda}^\top \operatorname{diag}(\boldsymbol{\Omega}_F \mathbf{1}) \mathcal{N}_\tau \mathbb{1} \right) \vartheta_t = 0. \tag{61}$$

Characterization. The Ramsey allocation under commitment is characterized by (50), (48), (56), and (46), together with the optimality system (57)-(61).

Interpretation of multipliers. The vector multiplier $\boldsymbol{\eta}_t$ is the shadow value of relaxing the system of sectoral Phillips curves (48): it measures the marginal welfare benefit of reducing sectoral inflation pressures. Equation (57) implies that the aggregate output gap is used to affect the collection of sectoral inflation rates through their slope vector $\mathbf{b}_y$, net of the cost of moving the exchange rate that the goods-market restriction requires.

The multiplier $\boldsymbol{\nu}_t$ is the shadow value of domestic price-level gaps $\widetilde{\mathbf{p}}_t^H$, and it captures that the planner cares about the *stock* of cross-sector misallocation, not only about the *flow* of inflation. The term $-\beta \boldsymbol{\mathcal{V}}^\top \mathbb{E}_t \boldsymbol{\eta}_{t+1}$ in (59) is specific to multi-sector economies: today's price-level gaps are tomorrow's lagged prices in $\boldsymbol{\chi}_{t+1}$, so relative-price gaps left in place today feed next period's cost-push pressure. The term is absent in the one-sector economy, where $\boldsymbol{\mathcal{V}} = \mathbf{0}$, and it vanishes along uniform movements of all price gaps because $\boldsymbol{\mathcal{V}} \mathbf{1} = \mathbf{0}$ (Rubbo 2023).

The multiplier $\boldsymbol{\xi}_t$ is the shadow value of relaxing the structural relation between the sectoral ToT gap, domestic price gaps, and the nominal exchange rate. It captures that movements in $\widetilde{e}_t$ affect the economy not only through the LCP term in the Phillips curve, but also through their effect on import relative prices and hence on $\widetilde{\mathbf{s}}_t$. Finally, ϑ_t is the shadow value of the goods-market restriction (46). It measures the welfare cost of the fact that, with complete markets and a single instrument, the central bank cannot move the output gap without moving the exchange-rate gap, and hence the ToT gaps and the CPI.

The optimality conditions couple welfare-weighted sectoral inflation, output, domestic price-level gaps, and international relative prices. Appendix D makes the inflation–output–ToT trade-off transparent by first considering a static economy. It then derives a target criterion under commitment, showing why optimal policy depends on earlier price adjustments as well as current conditions.

4.3. The Taylor-Rule Benchmark

We benchmark our optimal monetary policy results to a counterfactual in which the central bank follows a Taylor rule, setting the nominal interest rate according to

$$i_t = \rho_i i_{t-1} + (1-\rho_i)\left(\phi_\pi \pi_t^C + \phi_y \widetilde{y}_t + \phi_{\Delta\pi}\Delta\pi_t^C + \phi_{\Delta y}\Delta\widetilde{y}_t\right), \tag{62}$$

where $\Delta x_t \equiv x_t - x_{t-1}$, and $\pi_t^C = \widetilde{p}_t^C - \widetilde{p}_{t-1}^C + \Delta p_t^{C,n}$ is CPI inflation. Here $\widetilde{p}_t^C$ is the CPI gap in (45), and $p_t^{C,n} = \boldsymbol{\beta}^\intercal \boldsymbol{p}_t^{H,n} + \boldsymbol{\beta}_F^\intercal \boldsymbol{s}_t^n$ is the natural CPI, with natural domestic prices (43) and natural ToT $\boldsymbol{s}_t^n$ from Section 3. Like domestic inflation, CPI inflation includes the change in natural prices, not only the change in the gap. The coefficient ρ_i captures interest-rate smoothing, ϕ_π is the response to CPI inflation, and ϕ_y is the response to the output gap, while $\phi_{\Delta\pi}$ and $\phi_{\Delta y}$ denote the responses to inflation growth and output-gap growth, respectively.

Under optimal monetary policy, the policy rate is recovered from the consumption Euler equation in gap form,

$$\widetilde{c}_t = \mathbb{E}_t\widetilde{c}_{t+1} - \left(i_t - \mathbb{E}_t\pi_{t+1}^C - r_t^n\right), \tag{63}$$

where $\widetilde{c}_t \equiv c_t - c_t^n$ is the consumption gap and r_t^n is the natural real interest rate, given by

$$r_t^n = \mathbf{a}_M^c(\boldsymbol{\rho}_M - \mathbf{I})\boldsymbol{\tau}_t^M + \mathbf{a}_M^c(\boldsymbol{\rho}_W - \mathbf{I})\boldsymbol{\tau}_t^W + \mathbf{a}_X^c(\boldsymbol{\rho}_X - \mathbf{I})\boldsymbol{\tau}_t^X. \tag{64}$$

The coefficient vectors $\mathbf{a}_M^c$ and $\mathbf{a}_X^c$, derived in Appendix C, map tariff wedges into natural consumption; $\boldsymbol{\rho}_M$ and $\boldsymbol{\rho}_X$ collect the sectoral persistence parameters. The foreign wedge has its own persistence matrix $\boldsymbol{\rho}_W$ and the same private-price loading as $\boldsymbol{\tau}^M$. Appendix C reports the natural-consumption coefficients for the positive block.

4.4. Calibration and Counterfactual Design

The quarterly SOE calibration contains $I = 44$ sectors and preserves their relative exposure to final demand, imported inputs, domestic input-output linkages, nominal rigidities, and currency invoicing.

Consumption and Input-output Shares. The sectoral consumption shares, domestic and imported input-output matrices, and labor shares are constructed from the World Input-Output Database (ICIO, OECD) after aggregating the Euro Area into the home economy and the rest of the world into the foreign economy. Appendix G.1 reports the sectoral objects.

Standard parameters. The remaining preference and nominal parameters follow standard values in the monetary policy and business-cycle literature. We set $\beta = 0.99$, corresponding to a quarterly steady-state real interest rate close to one percent, and $\varphi = 1$, in line with the macro labor-supply evidence summarized by Chetty *et al.* (2011). The elasticity across varieties is set to $\epsilon_i^p = 11$, implying a ten percent steady-state markup.

Staggered Price-Setting. Price rigidities are sector-specific and calibrated from the Euro Area price-rigidity evidence in Dhyne *et al.* (2005); Gautier *et al.* (2024).

Currency Invoicing. The import-invoicing calibration follows the empirical evidence that a large share of international trade is priced in a small set of vehicle currencies, and that dominant-currency pricing generates high pass-through from the dominant exchange rate into import prices (Boz *et al.* 2020; Gopinath *et al.* 2020). The SOE model uses a binary invoicing block. We therefore classify sectors as either LCP-like, when import prices are local-currency-priced in the short run, or PCP-like, when import prices are foreign-currency-sensitive. The latter category includes sectors whose empirical pricing is closer to producer-currency pricing as well as sectors that are closer to dominant- or vehicle-currency pricing, because both cases expose domestic-currency import prices to an external exchange rate in the binary model. Commodities, energy, tradable manufacturing, machinery and equipment, and international transport are assigned to the PCP-like group; local services, retail and distribution, construction, real estate, public services, and other consumer-facing services are assigned to the LCP-like group. Table G.2 reports the sector-by-sector assignment, and Figures G.1–G.2 summarize the associated sectoral heterogeneity.

Shock Processes. The three foreign wedges are persistent, with $\rho_i^M = \rho_i^W = \rho_i^X = 0.99$ for all sectors. Reported responses scale innovations to a 10 log-point increase in the relevant wedge.

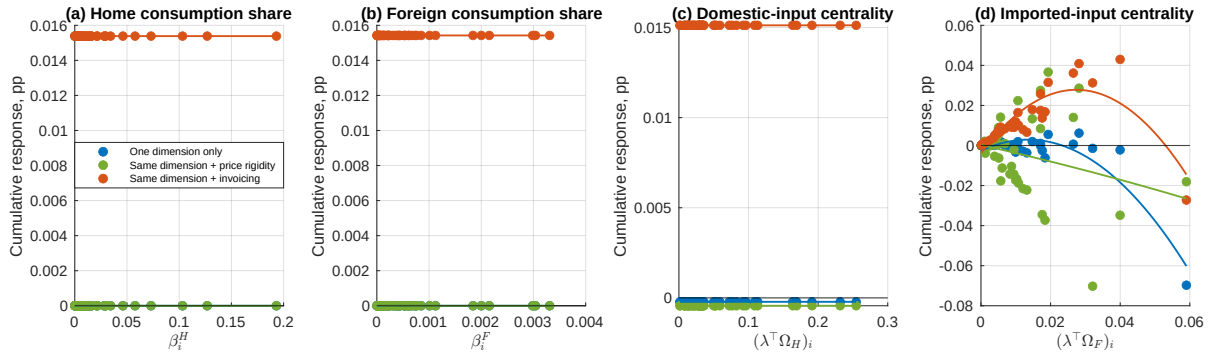
Monetary Policy. The Taylor rule uses standard coefficients, $\rho_i = 0.80$, $\phi_\pi = 1.50$, $\phi_y = 0.125$, $\phi_{\Delta\pi} = 0$ and $\phi_{\Delta y} = 0$, and targets CPI inflation in the baseline.

Counterfactual Economies. To isolate mechanisms, we begin with an economy in which only one real dimension is heterogeneous: home final-consumption shares, foreign final-consumption shares, the domestic IO block, or the imported IO block. We then combine that dimension separately with heterogeneous price rigidity and with heterogeneous invoicing.

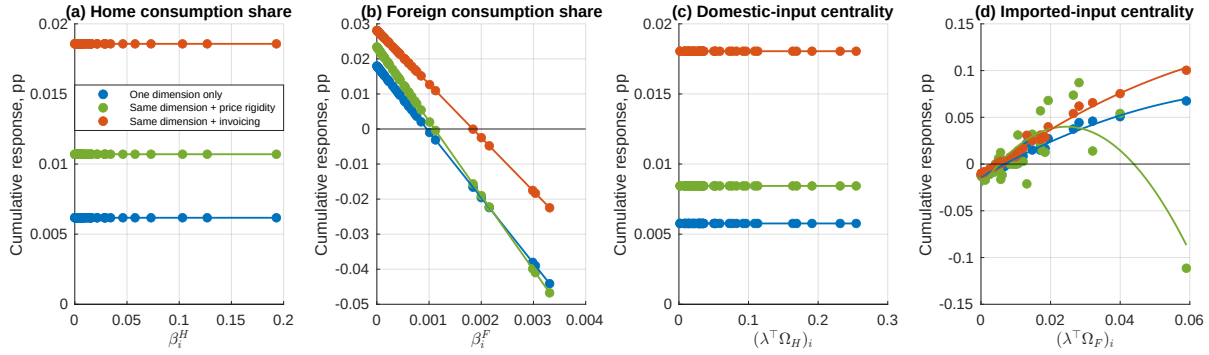
4.5. Sectoral Exposure and Policy Responses

Figure 1 reports eight-quarter cumulative responses to a 10 log-point foreign price wedge applied to one sector at a time. The horizontal axes vary Home and foreign final-consumption shares and domestic- and imported-input centrality. Each panel compares heterogeneity in the indicated characteristic alone with counterfactuals that also vary price rigidity or currency invoicing. The first four panels show the output gap and CPI inflation under commitment and the Taylor rule; the final two show the implementing policy rates.[11]

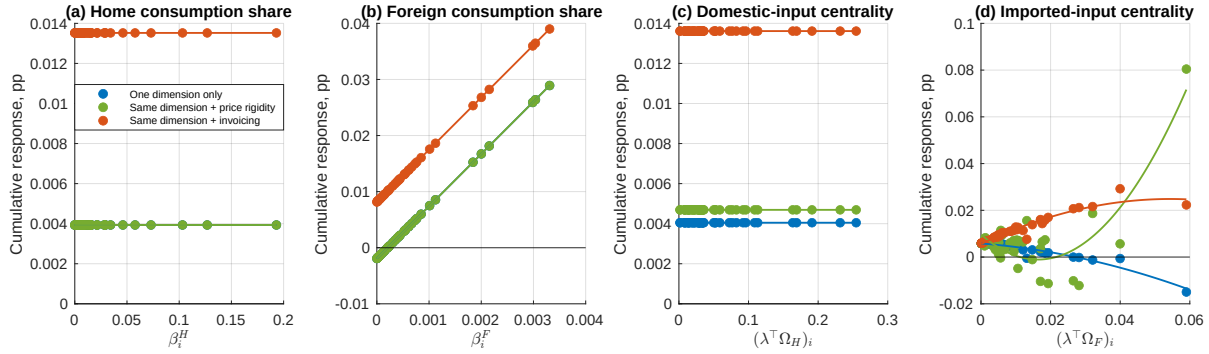
[11] Appendix D.3 repeats these sectoral and aggregate comparisons under discretion. Imported-input centrality remains prominent; for an *Energy* wedge, the impact output-gap and policy-rate responses under discretion lie between commitment and the Taylor rule, while impact CPI inflation is nearly identical across regimes.



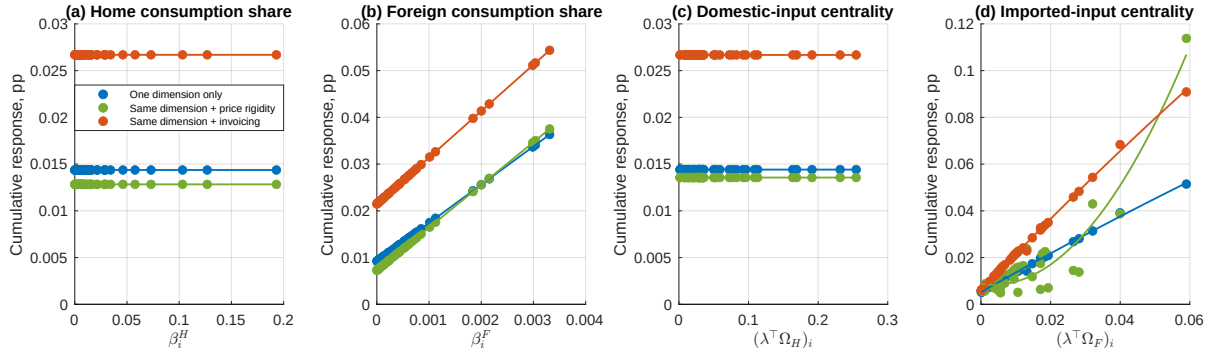
A. Output gap, commitment



B. Output gap, Taylor rule

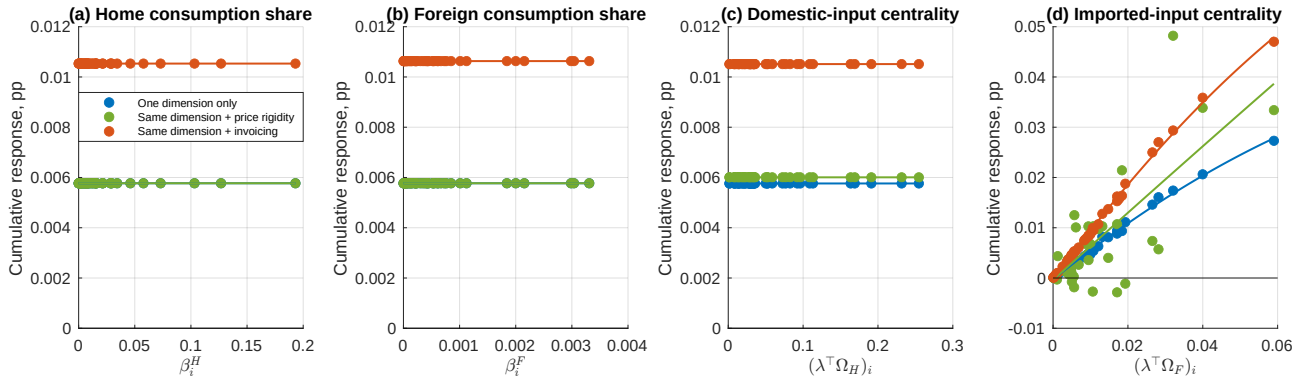


C. CPI inflation, commitment

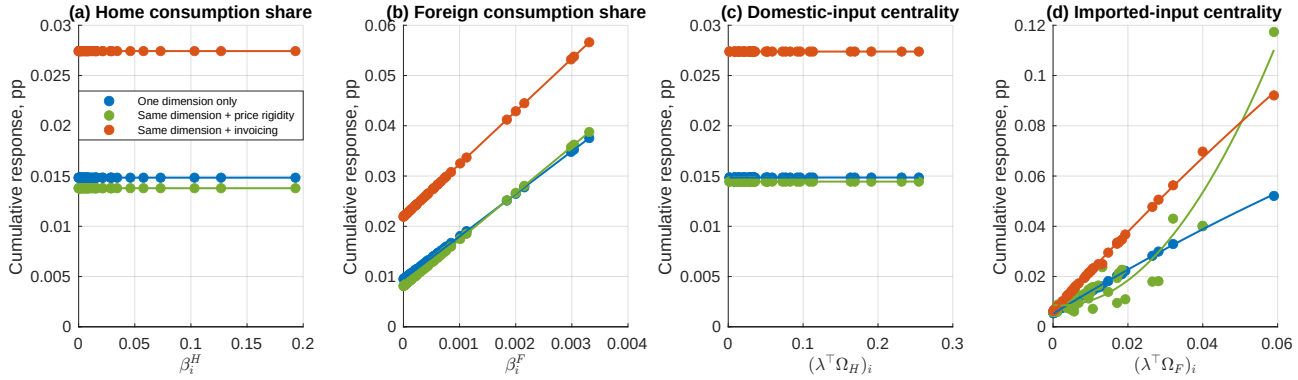


D. CPI inflation, Taylor rule

FIGURE 1. Cumulative responses to sectoral foreign price-wedge shocks. Each subfigure contains four sectoral characteristics. Responses accumulate over eight quarters after a 10 log-point shock. The counterfactuals vary the horizontal-axis characteristic alone, jointly with price rigidity, and jointly with currency invoicing. Panels (a)–(d) show commitment and Taylor-rule outcomes under the tariff-free welfare benchmark.



E. Policy rate, commitment



F. Policy rate, Taylor rule

FIGURE 1. Cumulative responses to sectoral foreign price-wedge shocks, continued.

The response varies most with the domestic use of the shocked imported input: the i-th column of Ω_F measures direct exposure, while Ω_H transmits it to downstream producers. Invoicing changes the force of that exposure. When imported-input shares are the only heterogeneous real dimension, adding heterogeneous invoicing raises the cross-sector standard deviation of eight-quarter cumulative Taylor-rule CPI responses from 0.009 to 0.017 percentage points. This interaction between foreign-input centrality and currency pricing is absent from closed-economy network accounts of inflation (Rubbo 2023). The high-centrality outlier near 0.06 is sector 3—mining of coal, crude petroleum, and natural gas, henceforth *Energy*. Its position reflects upstream input use rather than a large share in household consumption.

Figure 2 reports the Taylor-rule response minus the commitment response to the same shocks. The same exposure measures organize this policy difference, while heterogeneous invoicing changes how the exchange rate moves sectoral import prices. The policy implication goes beyond inflation transmission: an aggregate CPI rule does not respond to the sectoral inflation dispersion and relative-price misallocation weighted in (54).

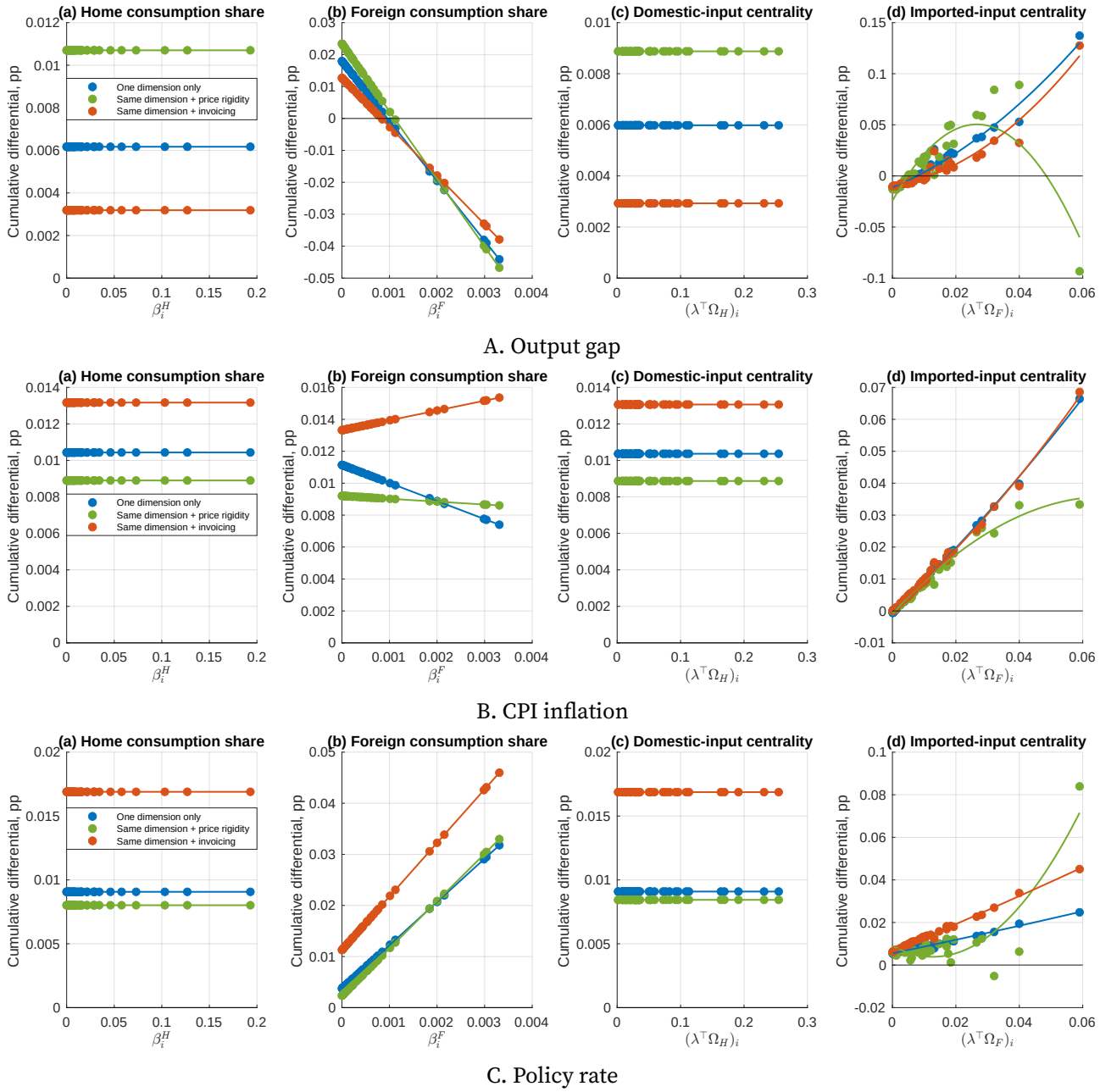
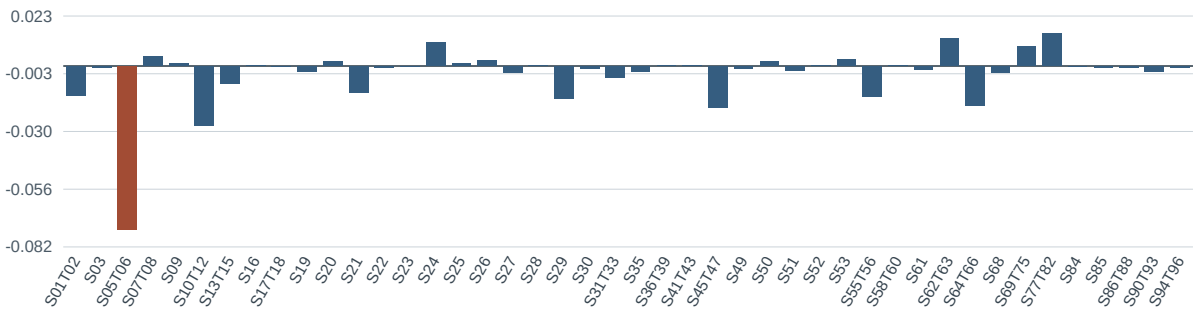
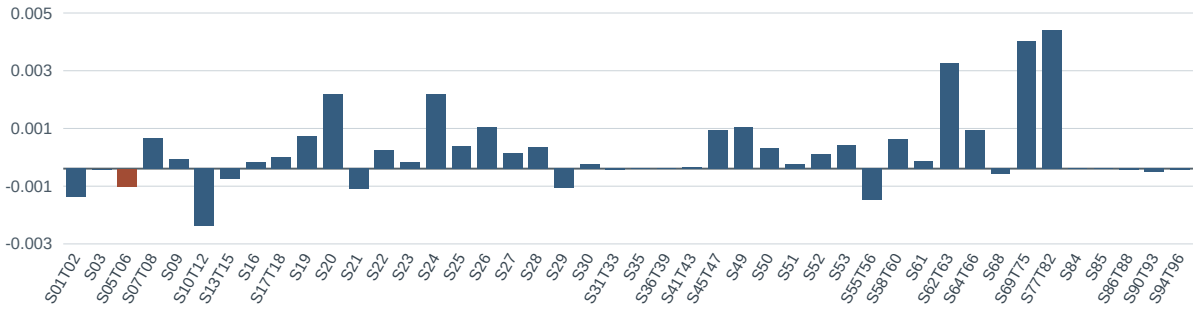


FIGURE 2. Taylor-rule minus commitment-policy cumulative responses after sectoral foreign price-wedge shocks. Shock size, sectoral characteristics, counterfactual heterogeneity, and eight-quarter horizon match Figure 1.

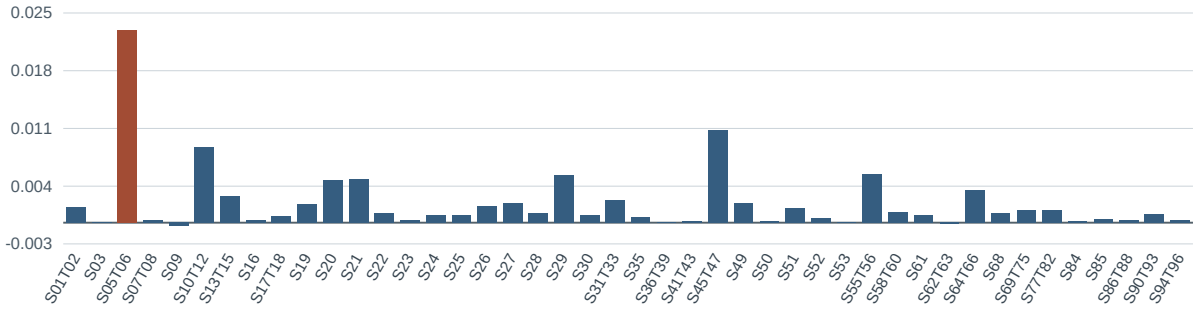
The cumulative scatter plots suppress sector identities and timing. Figure 3 shows the impact policy difference for each sector in the fully heterogeneous economy; *Energy* stands out in the output-gap and policy-rate panels. The sectoral labels are NACE Rev. 2 codes, with *Energy* highlighted.



A. Output gap



B. CPI inflation



C. Policy rate

FIGURE 3. Sectoral impact differences after foreign price-wedge shocks in the fully heterogeneous SOE. The bars report Taylor-rule minus commitment responses, in percentage points, to a 10 log-point wedge in each NACE sector. The highlighted sector is *Energy* (S05T06).

The imported-input centrality of *Energy* motivates the aggregate experiment in Figure 4. Unlike cumulative responses, the dynamic paths separate the imported first-round price increase from its subsequent domestic propagation and show the policy rate implementing each allocation. Headline inflation rises by nearly the same amount on impact under both policies. The Taylor rule raises the nominal rate more and generates a deeper output contraction; commitment limits propagation through domestic sticky-price users of energy.

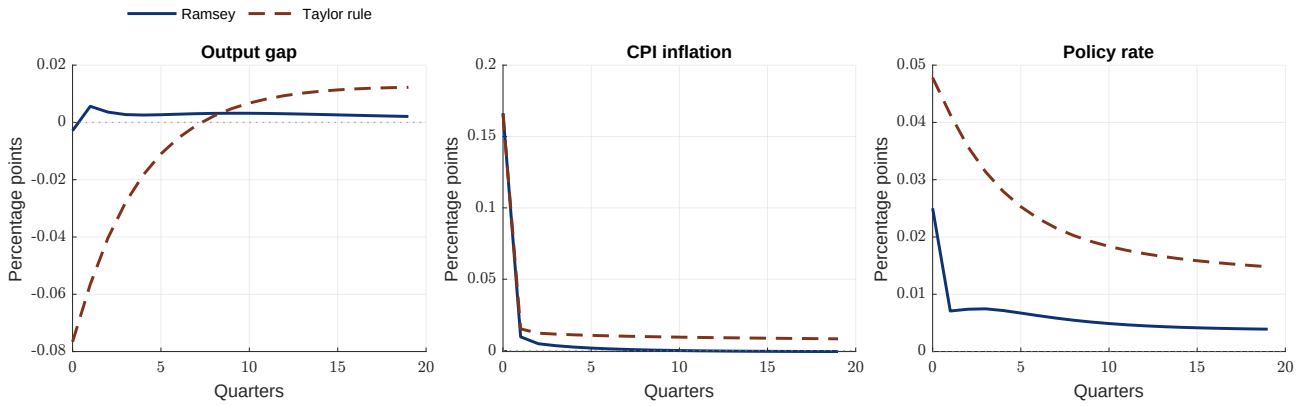


FIGURE 4. Responses to a 10 log-point foreign price wedge in *Energy* in the fully heterogeneous SOE. The commitment line minimizes (54) under the tariff-free welfare benchmark; the Taylor-rule line uses (62).

4.6. Quantitative Implications

The tractable SOE isolates the roles of input linkages, price rigidity, and invoicing. We now report how these channels operate in the calibrated multi-country model; Appendix E presents its equations, calibration, and additional experiments. The Euro Area (EA) is a bystander to the bilateral U.S.–China tariff shocks but directly imports the energy input. We compare the baseline Taylor rule with an optimal simple rule (OSR), which chooses the coefficients of the same instrument rule to minimize a quadratic criterion in EA output growth, CPI inflation, and changes in the policy rate. This operational criterion is not a household-welfare measure.[12] The tariff OSR has $(\rho_r, \phi_\pi, \phi_{\Delta y}) = (0.92, 14.87, 1.34)$; the energy OSR instead has (0.96, 6.84, 6.30). Appendix E.4 reports the full coefficient table.

Figure 5 compares the rules after a homogeneous U.S. tariff on Chinese goods, the reciprocal Chinese tariff, and a foreign energy-price wedge. Each innovation is normalized to 10 log points. The U.S. tariff has the larger EA spillover because the affected Chinese inputs are more central in the EA production network and their prices respond under dollar invoicing. Both tariff experiments produce much smaller EA responses than the energy shock.

The optimized tariff rule reacts more strongly to inflation, yet its gain in the stated quadratic criterion is only 0.01% in the stochastic tariff environment. Thus a visible change in coefficients need not translate into a meaningful stabilization gain for a third party. The energy shock raises EA headline inflation on impact by about 0.17 percentage points under either rule: the imported first round is largely beyond domestic monetary control. The baseline rule raises rates more sharply and contracts output growth more; the energy OSR reduces

[12] In the tractable SOE, the aggregate-loss rule captures 97.4% of the welfare improvement achieved by commitment relative to the baseline Taylor rule for imported-energy wedges (Appendix E.3). This motivates its use as a practical stabilization benchmark in the quantitative model, where the real-activity target is output growth.

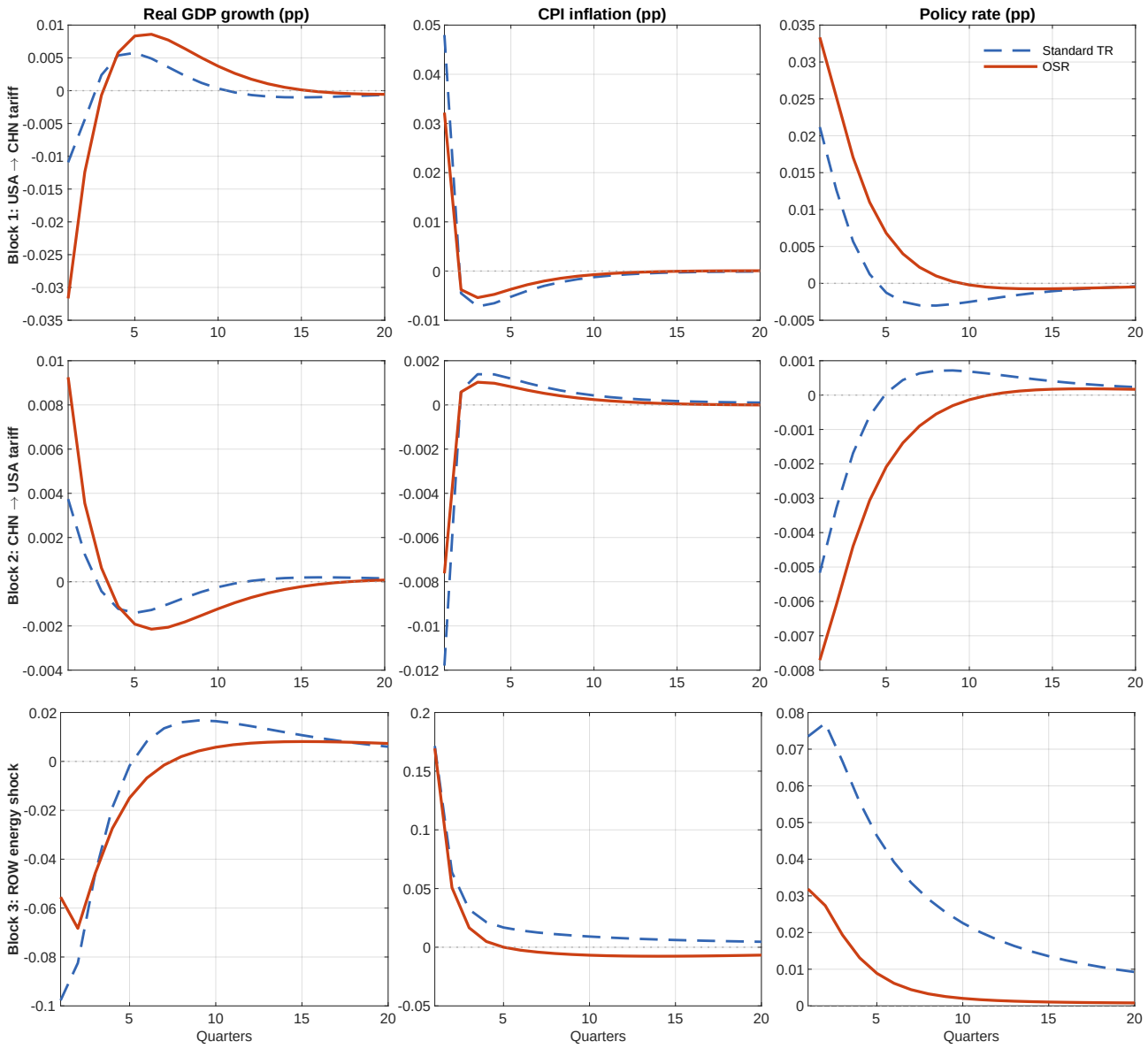


FIGURE 5. Euro Area responses to tariff and energy shocks under the baseline Taylor rule (dashed) and the optimal simple rule (solid). Top row: U.S. tariff on China; middle row: Chinese tariff on the United States; bottom row: rest-of-world energy-price shock. Columns: real GDP growth, CPI inflation, and the policy rate, in percentage points. Each shock is scaled to a 10 log-point increase.

the stabilization loss by 24.7%.

Most of this gain is available without reoptimizing any coefficients. When the baseline Taylor rule targets core rather than headline CPI, its loss falls by 22.8% under the same criterion, recovering about nine-tenths of the OSR gain. Core targeting looks through the imported first round; the remaining difference reflects the OSR's stronger response to domestic propagation through the production network. Appendix E.4 reports the three-rule comparison. Under the observed 2025 tariff schedule, the EA inflation peak is about 0.12

percentage points under the baseline rule and 0.08 under the tariff OSR; the corresponding paths and invoicing counterfactual are also reported there.

As a robustness check on the energy OSR, we allow a contemporaneous monetary perturbation and credible announcements of future perturbations, following Barnichon and Mesters (2023). With announcements extending eight quarters ahead, the optimal policy perturbation (OPP) lowers conditional discounted loss by a further 8.6% relative to the OSR. Under this common evaluation criterion, the OSR captures about four-fifths of the total improvement attained by OSR plus OPP relative to the baseline Taylor rule. The additional gain ranges from 8.5% to 9.1% across announcement horizons of four to sixteen quarters. For the benchmark OSR coefficients, most of the measured stabilization gain is therefore retained when announced policy adjustments are available, with economically meaningful scope for further improvement.[13] Appendix E.5 reports the construction, policy paths, and loss decomposition.

5. Conclusion

The monetary response to an increase in foreign prices depends on where the disturbance enters domestic production and on the currency in which imports are priced. An imported input with a small consumption share can create a substantial stabilization problem if many domestic producers use it. Exchange-rate adjustment changes that problem unevenly when invoicing differs across sectors.

The tractable SOE makes these considerations explicit. The domestic-value-added inflation index that aligns with aggregate markup distortions remains well defined under heterogeneous invoicing, but an exchange-rate residual generally prevents its stabilization from closing the output gap. The welfare criterion also penalizes domestic-price and terms-of-trade gaps, which distort consumption and input use. Optimal policy must account for these sectoral allocation effects as well as aggregate activity and inflation. The contribution is to characterize how the production network and invoicing jointly shape this policy trade-off.

The sectoral experiments show why this distinction matters. Imported-input centrality organizes the largest differences in responses, and heterogeneous invoicing adds substantial dispersion. Imported energy is particularly central to domestic production. After an imported-energy wedge, the impact increase in headline inflation is similar under commitment and a CPI Taylor rule, yet the Taylor rule induces a larger interest-rate increase and output-gap contraction. Similar aggregate inflation can therefore conceal different costs of stabilization.

[13] Both paths are evaluated with $\beta = 0.99$ and the same period-loss weights. The existing OSR coefficients were optimized for the unconditional-variance criterion, whereas OPP optimizes the conditional discounted response. The comparison evaluates the benchmark rule; the gap can reflect both policy-path flexibility and the difference in optimization criteria.

The quantitative application gives an implementable counterpart. Targeting core rather than headline inflation captures most of the stabilization gain from optimizing the rule after an imported-energy shock. The same gain is negligible for the EA as a third party to bilateral U.S.–China tariffs, and its exposure depends strongly on dollar pricing. These results favor identifying the domestic propagation of a foreign disturbance before choosing the monetary response. Extending the welfare accounting to the full multi-country economy would allow the quantitative rules to be assessed on the same basis as the SOE commitment allocation.

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Appendix

Appendix A. Inspecting the Mechanisms

In this section, we derive the equilibrium conditions under the simplified version of the model.

A.1. Production Networks and Inflation

The input-share matrices used in the main text are

$$\Omega_H = (\omega_{ij}^H)_{i,j=1}^I, \qquad \Omega_F = (\omega_{ij}^F)_{i,j=1}^I,$$

where $\omega_{ij}^H = P_j X_{ij}^H/(P_i Y_i) = (1-\alpha_i)\nu_{ij}(1-\zeta_{ij})$ and $\omega_{ij}^F = P_j X_{ij}^F/(P_i Y_i) = (1-\alpha_i)\nu_{ij}\zeta_{ij}$.

The nominal marginal cost faced by sector i is given by

$$\text{mc}_{i,t} = \alpha_i w_t + (1-\alpha_i)\, p^X_{i,t}, \tag{A.1}$$

where α_i is the steady-state ratio of labor costs to sales in sector i, and $1-\alpha_i = P^X_i X_i/(P_i Y_i)$ denotes the steady-state ratio of input costs to sales in sector i. The price index of intermediate goods faced by sector i is given by:

$$p^X_{i,t} = \sum_{j=1}^{I} \nu_{ij} \left[(1-\zeta_{ij})\, p^H_{j,t} + \zeta_{ij}(\tau^F_{j,t} + p^F_{j,t})\right] = \sum_{j=1}^{I} \nu_{ij} \left(p^H_{j,t} + \zeta_{ij} s_{j,t}\right), \tag{A.2}$$

where $s_{i,t} = \tau^F_i + p^F_{i,t} - p^H_{i,t}$ denotes the gross terms of trade (including the tariff) of the SOE with respect to the ROW, $\nu_{ij} = P_{ij}X_{ij}/(P^X_i X_i)$ denotes the purchases of sector i from sector j as a share of total intermediate goods' purchases of sector i, and $1-\zeta_{ij} = (P_j X^H_{ij})/(P_{ij}X_{ij})$ denotes the purchases of sector i from Home's sector j over total intermediate goods' purchases of sector i from sector j.

Introducing condition (A.2) into the marginal cost equation (A.1), and stacking over sectors, we obtain:

$$\mathbf{mc}_t = \boldsymbol{\alpha} w_t + \Omega_H \mathbf{p}^H_t + \Omega_F \left(\boldsymbol{\tau}^F_t + \mathbf{p}^F_t\right) = \boldsymbol{\alpha} w_t + \Omega_H \mathbf{p}^H_t + \Omega_F \left[\boldsymbol{\tau}^F_t + (\mathbf{1}-\mathbb{1})e_t + \mathbf{p}^*_t\right] = \boldsymbol{\alpha} w_t + \Omega \mathbf{p}^H_t + \Omega_F \boldsymbol{s}_t. \tag{A.3}$$

where $\mathbf{p}^F_t = (\mathbf{1}-\mathbb{1})e_t + \mathbf{p}^*_t$ depending on the currency invoicing regime, where $\mathbf{1}$ is a $I \times 1$ vector of ones and $\mathbb{1}$ is a $I \times 1$ vector of indicator functions $\mathbb{1}_i$

We now seek to obtain the Phillips curve in price terms. We assume that domestic sales are invoiced in the domestic currency, and thus the Phillips curve can be written as

$$\pi^H_{i,t} = \kappa_i \left(\text{mc}_{i,t} - p^H_{i,t}\right) + \beta \mathbb{E}_t \pi^H_{i,t+1},$$

where $\pi^H_{i,t} = p^H_{i,t} - p^H_{i,t-1}$ denotes the inflation rate of a good in sector i in country H sold domestically. We can write the sectoral Phillips curves in matrix form as (36)

$$\begin{aligned}
\boldsymbol{\pi}^H_t &= \boldsymbol{\kappa}\left(\mathbf{mc}_t - \mathbf{p}^H_t\right) + \beta \mathbb{E}_t \boldsymbol{\pi}^H_{t+1} = \boldsymbol{\kappa}\left[\mathbf{mc}_t - \left(\boldsymbol{\pi}^H_t + \mathbf{p}^H_{t-1}\right)\right] + \beta \mathbb{E}_t \boldsymbol{\pi}^H_{t+1} \\
&= (\boldsymbol{I} + \boldsymbol{\kappa})^{-1}\boldsymbol{\kappa}\left(\mathbf{mc}_t - \mathbf{p}^H_{t-1}\right) + (\boldsymbol{I} + \boldsymbol{\kappa})^{-1}\beta \mathbb{E}_t \boldsymbol{\pi}^H_{t+1} \\
&= \boldsymbol{\Delta}\left[\boldsymbol{\alpha} w_t + \Omega \mathbf{p}^H_t + \Omega_F \boldsymbol{s}_t - \mathbf{p}^H_{t-1}\right] + (\boldsymbol{I} - \boldsymbol{\Delta})\beta \mathbb{E}_t \boldsymbol{\pi}^H_{t+1} \\
&= \boldsymbol{\Delta}\left[\boldsymbol{\alpha} w_t + \Omega_F \boldsymbol{s}_t + \Omega \boldsymbol{\pi}^H_t - (\boldsymbol{I} - \Omega)\mathbf{p}^H_{t-1}\right] + (\boldsymbol{I} - \boldsymbol{\Delta})\beta \mathbb{E}_t \boldsymbol{\pi}^H_{t+1}
\end{aligned}$$

$$= \boldsymbol{\Delta}\left[\boldsymbol{\alpha} w_t + \boldsymbol{\Omega}_F \boldsymbol{s}_t + \boldsymbol{\Omega}\boldsymbol{\pi}_t^H + \boldsymbol{\Omega}\mathbf{p}_{t-1}^H - \mathbf{p}_{t-1}^H\right] + (\boldsymbol{I} - \boldsymbol{\Delta})\beta\mathbb{E}_t\boldsymbol{\pi}_{t+1}^H$$

$$(\boldsymbol{I} - \boldsymbol{\Delta}\boldsymbol{\Omega})\boldsymbol{\pi}_t^H = \boldsymbol{\Delta}\left[\boldsymbol{\alpha} w_t + \boldsymbol{\Omega}_F \boldsymbol{s}_t - (\boldsymbol{I} - \boldsymbol{\Omega})\mathbf{p}_{t-1}^H\right] + (\boldsymbol{I} - \boldsymbol{\Delta})\beta\mathbb{E}_t\boldsymbol{\pi}_{t+1}^H \tag{A.4}$$

where we have introduced nominal marginal costs (A.3), and the different objects are defined in section 3.3.

A.2. Production Networks and Monetary Policy

We now seek to write the sectoral Phillips curves in gaps from the flexible-price equilibrium. Under no nominal wage rigidities, the labor supply condition is given by:

$$w_t - p_t^C = c_t + \varphi n_t \tag{A.5}$$

where p_t^C is the consumption price index given by:

$$p_t^C = \sum_{i=1}^{I} \beta_i \left[(1-\zeta_i)\, p_{i,t}^H + \zeta_i(\tau_{i,t}^F + p_{i,t}^F)\right] = \sum_{i=1}^{I} \beta_i \left[p_{i,t}^H + \zeta_i s_{i,t}\right] = \boldsymbol{\beta}^\intercal \mathbf{p}_t^H + \boldsymbol{\beta}_F^\intercal \boldsymbol{s}_t \tag{A.6}$$

where $\beta_i = P_i C_i/(P_C C)$ denotes the total consumption share of sector i, and $1-\zeta_i = (P_i C_i^H)/(P_i C_i)$ denotes the home share within sector i. Thus $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_I]^\top$ is the vector of steady-state sectoral consumption shares and $\boldsymbol{\beta}_F = [\beta_1\zeta_1, \ldots, \beta_I\zeta_I]^\top$ is the vector of imported final-consumption shares.

The risk-sharing condition (under complete markets and constant rest of the world variables, i.e. $c_t^* = 0$) is given by:

$$c_t = q_t + c_t^* = e_t - p_t^C. \tag{A.7}$$

Furthermore, the definition of nominal GDP is given by:

$$P_t^Y Y_t = P_t^C C_t + \sum_{i=1}^{I}\left(P_{i,t}^{H,*} C_{i,t}^{H,*} - P_{i,t}^F C_{i,t}^F\right) + \sum_{i=1}^{I}\sum_{j=1}^{I}\left(P_{i,t}^{H,*} X_{ji,t}^{H,*} - P_{i,t}^F X_{ji,t}^F\right) \tag{A.8}$$

where P_t^Y is the GDP deflator, $C_{i,t}^{H,*}$ and $X_{ij,t}^{H,*}$ denote Home's sectoral exports to households and firms in ROW, and $C_{i,t}^F$ and $X_{ij,t}^F$ are Home's sectoral imports from households and firms.

We start by finding an expression for *real* GDP. In order to do so, we first linearize the

definition of nominal GDP (A.8):

$$p_t^Y + y_t = p_t^C + c_t + \sum_{i=1}^{I} \beta_i \zeta_i \left(p_{i,t}^{H,*} + c_{i,t}^{H,*} - p_{i,t}^F - c_{i,t}^F \right) + \sum_{i=1}^{I} \sum_{j=1}^{I} \lambda_j (1-\alpha_j) \nu_{ji} \zeta_{ji} \left(p_{i,t}^{H,*} + x_{ji,t}^{H,*} - p_{i,t}^F - x_{ji,t}^F \right)$$

Next, note that we have the following expression for the GDP deflator:

$$\begin{aligned} p_t^Y &= p_t^C + \sum_{i=1}^{I} \beta_i \zeta_i \left(p_{i,t}^{H,*} - p_{i,t}^F \right) + \sum_{i=1}^{I} \sum_{j=1}^{I} \lambda_j (1-\alpha_j) \nu_{ji} \zeta_{ji} \left(p_{i,t}^{H,*} - p_{i,t}^F \right) \\ &= p_t^C + \sum_{i=1}^{I} \left(\beta_i \zeta_i + \sum_{j=1}^{I} \lambda_j (1-\alpha_j) \nu_{ji} \zeta_{ji} \right) \left(p_{i,t}^{H,*} - p_{i,t}^F \right) = p_t^C - (\boldsymbol{\beta}_F^\intercal + \boldsymbol{\lambda}^\intercal \boldsymbol{\Omega}_F)(\mathbf{p}_t^F - \mathbf{p}_t^{H,*}) \end{aligned}$$

Hence we have that real GDP is given by:

$$y_t = c_t + \sum_{i=1}^{I} \beta_i \zeta_i \left(c_{i,t}^{H,*} - c_{i,t}^F \right) + \sum_{i=1}^{I} \sum_{j=1}^{I} \lambda_j \nu_{ji} (1-\alpha_j) \zeta_{ji} \left(x_{ji,t}^{H,*} - x_{ji,t}^F \right). \tag{A.9}$$

Consider first the term $\sum_{i=1}^{I} \beta_i \zeta_i \left(c_{i,t}^{H,*} - c_{i,t}^F \right)$. Under PCP, we have

$$\begin{aligned} c_{i,t}^{H,*} &= p_{i,t}^F - \tau_{i,t}^X - p_{i,t}^H + c_{i,t}^* = e_t + p_{i,t}^* - \tau_{i,t}^X - p_{i,t}^H + c_i^* = e_t + p_{i,t}^* - \tau_{i,t}^X - p_{i,t}^H + (e_t + p_t^{C,*}) - (e_t + p_{i,t}^*) + c_t^* \\ &= -\tau_{i,t}^X - p_{i,t}^H + (e_t + p_t^{C,*}) + c_t - q_t = -\tau_{i,t}^X - p_{i,t}^H + p_t^C + c_t, \end{aligned}$$

where we have made use of the risk-sharing condition (A.7), the relation $p_{i,t}^* + c_{i,t}^* = p_t^{C,*} + c_t^*$, and the logarithmic transformation of (32). Equivalently, under LCP, we have that

$$\begin{aligned} c_{i,t}^{H,*} &= p_{i,t}^* - \tau_{i,t}^X - p_{i,t}^{H^*} + c_{i,t}^* = p_{i,t}^* - \tau_{i,t}^X - p_{i,t}^{H^*} + c_i^* = p_{i,t}^* - \tau_{i,t}^X - p_{i,t}^{H^*} + p_t^{C,*} - p_{i,t}^* + c_t^* \\ &= -\tau_{i,t}^X - p_{i,t}^{H^*} + p_t^{C,*} + c_t - q_t = -e_t - \tau_{i,t}^X - p_{i,t}^{H^*} + p_t^C + c_t = -\tau_{i,t}^X - p_{i,t}^H + p_t^C + c_t \end{aligned}$$

so that the allocation under both invoicing schemes is identical. In addition, under PCP note that we have

$$c_{i,t}^F = p_{i,t} - \tau_{i,t}^F - p_{i,t}^F + c_{i,t} = p_{i,t} - \tau_{i,t}^F - p_{i,t}^F + (p_t^C - p_{i,t}) + c_t = p_t^C + c_t - \tau_{i,t}^F - p_{i,t}^F = p_t^C + c_t - \tau_{i,t}^F - e_t - p_{i,t}^*,$$

and, under LCP,

$$c_{i,t}^F = p_{i,t} - \tau_{i,t}^F - p_{i,t}^* + c_{i,t} = p_{i,t} - \tau_{i,t}^F - p_{i,t}^* + (p_t^C - p_{i,t}) + c_t = p_t^C + c_t - \tau_{i,t}^F - p_{i,t}^*$$

Therefore, we have that

$$\sum_{i=1}^{I} \beta_i \zeta_i \left(c_{i,t}^{H,*} - c_{i,t}^{F} \right) = \sum_{i=1}^{I} \beta_i \zeta_i \left(\tau_{i,t}^{F} + p_{i,t}^{F} - \tau_{i,t}^{X} - p_{i,t}^{H} \right) = \boldsymbol{\beta}_F^{\intercal} (\boldsymbol{s}_t - \boldsymbol{\tau}_t^{X}), \tag{A.10}$$

Next, we work with the term $x_{ji,t}^{H,*} - x_{ji,t}^{F}$. Under PCP, we have that

$$\begin{aligned} x_{ji,t}^{H,*} &= p_{i,t}^{F} - \tau_{i,t}^{X} - p_{i,t}^{H} + x_{ji,t}^{*} = e_t + p_{i,t}^{*} - \tau_{i,t}^{X} - p_{i,t}^{H} + x_{ji,t}^{*} = e_t + p_{i,t}^{*} - \tau_{i,t}^{X} - p_{i,t}^{H} + (e_t + p_{j}^{X,*}) - (e_t + p_{i,t}^{*}) + \\ &= -\tau_{i,t}^{X} - p_{i,t}^{H} + e_t + p_t^{C,*} + (w_t^{*} - p_t^{C,*}) + n_{j,t}^{*} = -\tau_{i,t}^{X} - p_{i,t}^{H} + p_t^{C} + q_t + c_t^{*} + \varphi n_t^{*} + n_{j,t}^{*} \\ &= -\tau_{i,t}^{X} - p_{i,t}^{H} + p_t^{C} + q_t + c_t - q_t + \varphi n_t^{*} + n_{j,t}^{*} = -\tau_{i,t}^{X} - p_{i,t}^{H} + p_t^{C} + c_t, \end{aligned}$$

where we have made use of the first-order conditions of the firms for labor demand and intermediate goods' demand, $x_{j,t}^{*} + p_{j,t}^{X,*} = n_{j,t}^{*} + w_t^{*}$, and of the fact that foreign variables are constant, in addition to the risk-sharing condition (A.7), the relation $p_{i,t}^{*} + x_{ji,t}^{*} = p_{j,t}^{X,*} + x_{j,t}^{*}$, the logarithmic transformation of (32), and the labor supply condition (A.5). Equivalently, under LCP,

$$\begin{aligned} x_{ji,t}^{H,*} &= p_{i,t}^{*} - \tau_{i,t}^{X} - p_{i,t}^{H,*} + x_{ji,t}^{*} = p_{i,t}^{*} - \tau_{i,t}^{X} - (p_{i,t}^{H} + e_t) + x_{ji,t}^{*} = p_{i,t}^{F} - \tau_{i,t}^{X} - p_{i,t}^{H} + x_{ji,t}^{*} \\ &= -\tau_{i,t}^{X} - p_{i,t}^{H} + p_t^{C} + q_t + c_t - q_t + \varphi n_t^{*} + n_{j,t}^{*} = -\tau_{i,t}^{X} - p_{i,t}^{H} + p_t^{C} + c_t \end{aligned}$$

so that the allocation under both invoicing schemes is identical. In addition, under PCP note that we have

$$\begin{aligned} x_{ji,t}^{F} &= p_{i,t}^{H} - \tau_{i,t}^{F} - p_{i,t}^{F} + x_{ji,t} = p_{i,t}^{H} - \tau_{i,t}^{F} - p_{i,t}^{F} + (p_{j,t}^{X} - p_{i,t}^{H}) + x_{j,t} \\ &= -\tau_{i,t}^{F} - p_{i,t}^{F} + p_t^{C} + (w_t - p_t^{C}) + n_{j,t} = -\tau_{i,t}^{F} - p_{i,t}^{F} + p_t^{C} + c_t + \varphi n_t + n_{j,t}. \end{aligned}$$

and under LCP,

$$\begin{aligned} x_{ji,t}^{F} &= p_{i,t}^{H} - \tau_{i,t}^{F} - p_{i,t}^{*} + x_{ji,t} = p_{i,t}^{H} - \tau_{i,t}^{F} - p_{i,t}^{*} + (p_{j,t}^{X} - p_{i,t}^{H}) + x_{j,t} \\ &= -\tau_{i,t}^{F} - p_{i,t}^{*} + p_t^{C} + (w_t - p_t^{C}) + n_{j,t} = -\tau_{i,t}^{F} - p_{i,t}^{*} + p_t^{C} + c_t + \varphi n_t + n_{j,t} \end{aligned}$$

Therefore, we have that:

$$\sum_{i=1}^{I} \sum_{j=1}^{I} \lambda_j \nu_{ji} (1 - \alpha_j) \zeta_{ji} \left(x_{ji,t}^{H,*} - x_{ji,t}^{F} \right) = \sum_{i=1}^{I} \sum_{j=1}^{I} \lambda_j \nu_{ji} (1 - \alpha_j) \zeta_{ji} \left(\tau_{i,t}^{F} + p_{i,t}^{F} - \tau_{i,t}^{X} - p_{i,t}^{H} - \varphi n_t - n_{j,t} \right) \tag{A.11}$$

$$
\begin{aligned}
&= \sum_{i=1}^{I}\sum_{j=1}^{I} \lambda_j \nu_{ji}(1-\alpha_j)\zeta_{ji}\left(\tau^F_{i,t} + e_t + p^*_{i,t} - \mathbb{1}e_t - \tau^X_{i,t} - p^H_{i,t} - \varphi n_t - n_{j,t}\right)\\
&= \sum_{i=1}^{I}\sum_{j=1}^{I} \lambda_j \nu_{ji}(1-\alpha_j)\zeta_{ji}\left(s_{i,t} - \tau^X_{i,t} - \varphi n_t - n_{j,t}\right)\\
&= \boldsymbol{\lambda}^\intercal \Omega_F(\boldsymbol{s}_t - \boldsymbol{\tau}^X_t) - \boldsymbol{\lambda}^\intercal \text{diag}\left(\Omega_F \mathbf{1}\right)\boldsymbol{n}_t - \omega_X \varphi\, y_t.
\end{aligned}
$$

In addition, we have made use of the fact that aggregate GDP equals the sales-weighted sum of sectoral value-added, and hence aggregate employment (using the labor market clearing condition $n_t = \sum_{i=1}^{I} \lambda_i \alpha_i n_{i,t}$), $y_t = \sum_{i=1}^{I} \lambda_i \alpha_i n_{i,t} = n_t$. Furthermore, $\omega_X = \sum_{i=1}^{I}\sum_{j=1}^{I} \lambda_j \nu_{ji}(1-\alpha_j)\zeta_{ji}$.

Next, plugging in (A.10) and (A.11) into (A.9), we obtain:

$$
(1+\varphi\omega_X)\, y_t = c_t + (\boldsymbol{\beta}_F^\intercal + \boldsymbol{\lambda}^\intercal \Omega_F)(\boldsymbol{s}_t - \boldsymbol{\tau}^X_t) - \boldsymbol{\lambda}^\intercal \text{diag}\left(\Omega_F \mathbf{1}\right)\boldsymbol{n}_t. \tag{A.12}
$$

Natural Equilibrium. In the natural allocation with flexible prices (denoted by superscript n) where prices equal marginal costs and nominal wages remain constant (Rubbo 2023), we have that $\mathbf{p}^{H,n}_t = \boldsymbol{\alpha} w^n_t + \Omega \mathbf{p}^{H,n}_t + \Omega_F \boldsymbol{s}^n_t$. Therefore, we can write:

$$
\mathbf{p}^{H,n}_t = (\boldsymbol{I} - \Omega)^{-1}\Omega_F \boldsymbol{s}^n_t = (\boldsymbol{I} - \Omega_H)^{-1}\Omega_F\left(\mathbf{p}^{F,n}_t + \boldsymbol{\tau}^F_t\right). \tag{A.13}
$$

That is, in response to an increase in import prices, domestic prices adjust through a direct effect via its exposure to foreign markets (measured by Ω_F) and an indirect effect via input-output linkages mediated by the domestic Leontief inverse $(\boldsymbol{I} - \Omega_H)^{-1}$.

Note also that making use of these results and of the relationship between consumption shares and Domar weights in steady state, $\boldsymbol{\lambda}^\intercal = \boldsymbol{\beta}^\intercal (\boldsymbol{I} - \Omega)^{-1}$, we have that the consumer price level in the flexible price equilibrium is given by:

$$
p^{C,n}_t = \boldsymbol{\beta}^\intercal \mathbf{p}^{H,n}_t + \boldsymbol{\beta}_F^\intercal \boldsymbol{s}^n_t = \left[\boldsymbol{\beta}_F^\intercal + \boldsymbol{\beta}^\intercal (\boldsymbol{I} - \Omega)^{-1}\Omega_F\right]\boldsymbol{s}^n_t = \left[\boldsymbol{\beta}_F^\intercal + \boldsymbol{\lambda}^\intercal (\boldsymbol{I} - \Omega)(\boldsymbol{I} - \Omega)^{-1}\Omega_F\right]\boldsymbol{s}^n_t = (\boldsymbol{\beta}_F^\intercal + \boldsymbol{\lambda}^\intercal \Omega_F)\boldsymbol{s}^n_t. \tag{A.14}
$$

Next, writing the expression for real GDP (A.12) in natural terms, and using (A.14) we obtain:

$$
(1+\varphi\omega_X)\, y^n_t = c^n_t + p^{C,n}_t - (\boldsymbol{\beta}_F^\intercal + \boldsymbol{\lambda}^\intercal \Omega_F)\boldsymbol{\tau}^X_t - \boldsymbol{\lambda}^\intercal \text{diag}\left(\Omega_F \mathbf{1}\right)\boldsymbol{n}^n_t.
$$

In addition, make use of the labor supply condition with flexible prices and constant nominal

wages, together with $y_t^n = n_t^n$ and $-\varphi y_t^n = c_t^n + p_t^{C,n}$ to obtain:

$$\left[1+\varphi(\omega_X+1)\right] y_t^n = -(\beta_F^\intercal + \boldsymbol{\lambda}^\intercal\Omega_F)\tau_t^X - \boldsymbol{\lambda}^\intercal \text{diag}\left(\Omega_F \mathbf{1}\right) \boldsymbol{n}_t^n \tag{A.15}$$

Phillips Curve in Gaps from Natural Equilibrium. First, we start by writing the labor supply condition (A.5), using the result that aggregate hours equal aggregate real GDP (or value added)

$$n_t = y_t, \tag{A.16}$$

which follows from the CRS production structure and $\sum_j \alpha_j \lambda_j = 1$; together with (A.12) to substitute out c_t:

$$w_t - p_t^C = \left[1+\varphi(\omega_X+1)\right] y_t - \left(\beta_F^\intercal + \boldsymbol{\lambda}^\intercal\Omega_F\right)\left(\boldsymbol{s}_t - \tau_t^X\right) + \boldsymbol{\lambda}^\intercal \text{diag}\left(\Omega_F \mathbf{1}\right) \boldsymbol{n}_t \tag{A.17}$$

Next, we add $p_t^{C,n}$ on both sides and use (A.14) to obtain:

$$w_t - \widetilde{p}_t^C = \left[1+\varphi(\omega_X+1)\right] y_t - \left(\beta_F^\intercal + \boldsymbol{\lambda}^\intercal\Omega_F\right)\widetilde{\boldsymbol{s}}_t + \left(\beta_F^\intercal + \boldsymbol{\lambda}^\intercal\Omega_F\right)\tau_t^X + \boldsymbol{\lambda}^\intercal \text{diag}\left(\Omega_F \mathbf{1}\right) \boldsymbol{n}_t \tag{A.18}$$

where tildes denote deviations from the flexible price equilibrium: $\widetilde{x}_t = x_t - x_t^n$ for a variable x. Finally, make use of (A.15) and add and subtract $\left[1+\varphi(\omega_X+1)\right] y_t^n$ on the right-hand side (using equation A.15) to obtain:

$$w_t - \widetilde{p}_t^C = \left[1+\varphi(\omega_X+1)\right] \widetilde{y}_t - \left(\beta_F^\intercal + \boldsymbol{\lambda}^\intercal\Omega_F\right)\widetilde{\boldsymbol{s}}_t + \boldsymbol{\lambda}^\intercal \text{diag}\left(\Omega_F \mathbf{1}\right) \widetilde{\boldsymbol{n}}_t. \tag{A.19}$$

We next work with the stacked sectoral Phillips curves (A.4). Subtracting $\boldsymbol{\Delta\alpha}\, p_t^C$ on both sides, and using (A.6) to write $p_t^C = \beta^\intercal \mathbf{p}_t^H + \beta_F^\intercal \boldsymbol{s}_t = \beta^\intercal \boldsymbol{\pi}_t^H + \beta^\intercal \mathbf{p}_{t-1}^H + \beta_F^\intercal \boldsymbol{s}_t$, we obtain:

$$(\boldsymbol{I} - \boldsymbol{\Delta\Omega} - \boldsymbol{\Delta\alpha\beta}^\intercal)\boldsymbol{\pi}_t^H = \boldsymbol{\Delta}\left[\boldsymbol{\alpha}(w_t - p_t^C) - (\boldsymbol{I} - \Omega - \boldsymbol{\alpha\beta}^\intercal)\mathbf{p}_{t-1}^H + (\Omega_F + \boldsymbol{\alpha}\beta_F^\intercal)\boldsymbol{s}_t\right] + \beta(\boldsymbol{I} - \boldsymbol{\Delta})\mathbb{E}_t\boldsymbol{\pi}_{t+1}^H \tag{A.20}$$

Adding and subtracting $p_t^{C,n}$ on the right-hand side, using (A.14), we obtain:

$$\begin{aligned}(\boldsymbol{I} - \boldsymbol{\Delta\Omega} - \boldsymbol{\Delta\alpha\beta}^\intercal)\boldsymbol{\pi}_t^H = \boldsymbol{\Delta}\Big[&\boldsymbol{\alpha}(w_t - \widetilde{p}_t^C) - (\boldsymbol{I} - \Omega - \boldsymbol{\alpha\beta}^\intercal)\mathbf{p}_{t-1}^H + (\Omega_F + \boldsymbol{\alpha}\beta_F^\intercal)\boldsymbol{s}_t \\ &- \boldsymbol{\alpha\beta}^\intercal \mathbf{p}_t^{H,n} - \boldsymbol{\alpha}\beta_F^\intercal \boldsymbol{s}_t^n\Big] + \beta(\boldsymbol{I} - \boldsymbol{\Delta})\mathbb{E}_t\boldsymbol{\pi}_{t+1}^H\end{aligned} \tag{A.21}$$

Next, adding and subtracting $(\boldsymbol{I} - \Omega)\mathbf{p}_t^{H,n}$ on the right-hand side and using the fact that

$(\boldsymbol{I}-\Omega)\mathbf{p}_t^{H,n} = \Omega_F \boldsymbol{s}_t^n$ to obtain:

(A.22)
$$(\boldsymbol{I}-\boldsymbol{\Delta}\Omega-\boldsymbol{\Delta}\boldsymbol{\alpha}\boldsymbol{\beta}^\intercal)\boldsymbol{\pi}_t^H = \boldsymbol{\Delta}\left[\boldsymbol{\alpha}(w_t-\widetilde{p}_t^C)-(\boldsymbol{I}-\Omega-\boldsymbol{\alpha}\boldsymbol{\beta}^\intercal)(\mathbf{p}_{t-1}^H-\mathbf{p}_t^{H,n})+(\Omega_F+\boldsymbol{\alpha}\boldsymbol{\beta}_F^\intercal)\widetilde{\boldsymbol{s}}_t\right] + \beta(\boldsymbol{I}-\boldsymbol{\Delta})\mathbb{E}_t\boldsymbol{\pi}_{t+1}^H$$

Finally, using (A.13) we obtain:

(A.23)
$$(\boldsymbol{I}-\boldsymbol{\Delta}\Omega-\boldsymbol{\Delta}\boldsymbol{\alpha}\boldsymbol{\beta}^\intercal)\boldsymbol{\pi}_t^H = \boldsymbol{\Delta}\left\{\boldsymbol{\alpha}(w_t-\widetilde{p}_t^C)-(\boldsymbol{I}-\Omega-\boldsymbol{\alpha}\boldsymbol{\beta}^\intercal)\left[\mathbf{p}_{t-1}^H-(\boldsymbol{I}-\Omega_H)^{-1}\Omega_F\left(\mathbf{p}_t^{F,n}+\boldsymbol{\tau}_t^F\right)\right] +(\Omega_F+\boldsymbol{\alpha}\boldsymbol{\beta}_F^\intercal)\widetilde{\boldsymbol{s}}_t\right\}+\beta(\boldsymbol{I}-\boldsymbol{\Delta})\mathbb{E}_t\boldsymbol{\pi}_{t+1}^H$$

Next, plugging in the expression for the real wage gap (A.19) into (A.23) to obtain:

(A.24)
$$\begin{aligned}(\boldsymbol{I}-\boldsymbol{\Delta}\Omega-\boldsymbol{\Delta}\boldsymbol{\alpha}\boldsymbol{\beta}^\intercal)\boldsymbol{\pi}_t^H =& \boldsymbol{\Delta}\boldsymbol{\alpha}\left[1+\varphi(\omega_X+1)\right]\widetilde{y}_t+\boldsymbol{\Delta}\boldsymbol{\alpha}\boldsymbol{\lambda}^\intercal\mathrm{diag}\left(\Omega_F\mathbf{1}\right)\widetilde{\boldsymbol{n}}_t+\boldsymbol{\Delta}(\boldsymbol{I}-\boldsymbol{\alpha}\boldsymbol{\lambda}^\intercal)\Omega_F\widetilde{\boldsymbol{s}}_t+\beta(\boldsymbol{I}-\boldsymbol{\Delta})\mathbb{E}_t\boldsymbol{\pi}_{t+1}^H\\ &-\boldsymbol{\Delta}(\boldsymbol{I}-\Omega-\boldsymbol{\alpha}\boldsymbol{\beta}^\intercal)\left[\mathbf{p}_{t-1}^H-(\boldsymbol{I}-\Omega_H)^{-1}\Omega_F\left(\mathbf{p}_t^{F,n}+\boldsymbol{\tau}_t^F\right)\right]\end{aligned}$$

Natural allocation and external resources. The natural allocation is obtained from the flexible-price private equilibrium. For the tariff-free welfare experiment, the flexible-price allocation is maintained as efficient; the tariff experiments below use it as a positive benchmark only. Domestic tariff receipts are internal transfers. In foreign currency, the consolidated external balance is

(A.25)
$$NX_t^* = \sum_i \frac{R_i}{1+\tau_{i,t}^X} - \sum_i (1+\tau_{i,t}^W)P_{i,t}^{F,*}M_{i,t}, \qquad M_{i,t} = C_{i,t}^F + \sum_j X_{ji,t}^F.$$

Here R_i is foreign expenditure before the export tariff and $P_{i,t}^{F,*}$ is the producer's foreign-currency price; the latter may depend on the exchange rate under LCP. The domestic tariff τ^M enters private prices through $\tau^F = \tau^M + \tau^W$, but is excluded from the external payment in (A.25). With given initial foreign wealth A_0, $A_0 + \mathbb{E}_0 \sum_t Q_{0t}^* NX_t^* = 0$; complete markets do not require $NX_t^* = 0$ in every period. Appendix B retains this restriction when deriving welfare.

We characterize the flexible-price allocation from the conditions that also hold under sticky prices: goods market clearing, the labor demand of firms, the labor supply condition and risk sharing. The natural allocation is therefore the flexible-price counterpart of the equilibrium conditions used in the main text, and the conditions in gaps inherit no residual tariff terms.

We start from sectoral employment. The market clearing condition for good i is

$$Y_{i,t}^n = C_{i,t}^{H,n} + C_{i,t}^{H,*,n} + \sum_j \left(X_{ji,t}^{H,n} + X_{ji,t}^{H,*,n} \right). \tag{A.26}$$

Substituting out the demands for good i of domestic and foreign households and firms, linearizing, and dividing by steady-state GDP, we have that:

$$\begin{aligned} \lambda_i y_{i,t}^n =& \beta_i \left(p_t^{C,n} + c_t^n - p_{i,t}^{H,n} \right) + \sum_j \Big[\lambda_j \nu_{ji} (1-\zeta_{ji})(1-\alpha_j)(p_t^{C,n} + c_t^n - p_{i,t}^{H,n} + \varphi n_t^n + n_{j,t}^n) \\ &+ \lambda_j \nu_{ji} \zeta_{ji} (1-\alpha_j)(p_t^{C,n} + c_t^n - p_{i,t}^{H,n}) \Big] - \left[\beta_{F,i} + \sum_j \lambda_j \nu_{ji} \zeta_{ji} (1-\alpha_j) \right] \tau_{i,t}^X, \end{aligned} \tag{A.27}$$

where the last term is the reduction in foreign demand for sector-i goods caused by the export tariff, as in the sticky-price market clearing condition (A.39) below. Next, using that in the flexible price equilibrium $p_t^{C,n} + c_t^n = -\varphi n_t^n = -\varphi y_t^n$, and collecting terms, the above expression simplifies to:

$$\begin{aligned} \lambda_i y_{i,t}^n =& - \left[\beta_i + \sum_j \lambda_j \nu_{ji} (1-\alpha_j) \right] p_{i,t}^{H,n} + \sum_j \lambda_j \nu_{ji} (1-\zeta_{ji})(1-\alpha_j) n_{j,t}^n \\ &- \left[\beta_i + \sum_j \lambda_j \nu_{ji} \zeta_{ji} (1-\alpha_j) \right] \varphi y_t^n - (\boldsymbol{\beta}_F + \boldsymbol{\Omega}_F^\intercal \boldsymbol{\lambda})_i \tau_{i,t}^X. \end{aligned} \tag{A.28}$$

In addition, note that at the steady state we have $\lambda_i = \beta_i + \sum_j \lambda_j \nu_{ji} (1-\alpha_j)$, leading to:

$$\lambda_i (p_{i,t}^{H,n} + y_{i,t}^n) = \sum_j \lambda_j \nu_{ji} (1-\zeta_{ji})(1-\alpha_j) n_{j,t}^n - \left[\beta_i + \sum_j \lambda_j \nu_{ji} \zeta_{ji} (1-\alpha_j) \right] \varphi y_t^n - (\boldsymbol{\beta}_F + \boldsymbol{\Omega}_F^\intercal \boldsymbol{\lambda})_i \tau_{i,t}^X. \tag{A.29}$$

Using the result that with flexible prices we have that $p_{i,t}^{H,n} + y_{i,t}^n = n_{i,t}^n$ from the labor demand condition of firms and stacking over i we have that $\boldsymbol{\Lambda} \boldsymbol{n}_t^n = \boldsymbol{\Omega}_H^\intercal \boldsymbol{\Lambda} \boldsymbol{n}_t^n - (\boldsymbol{\beta} + \boldsymbol{\Omega}_F^\intercal \boldsymbol{\lambda}) \varphi y_t^n - \text{diag}(\boldsymbol{\beta}_F + \boldsymbol{\Omega}_F^\intercal \boldsymbol{\lambda}) \boldsymbol{\tau}_t^X$, where $\boldsymbol{\Lambda} = \text{diag}\,(\boldsymbol{\lambda})$, and hence

$$\begin{aligned} \boldsymbol{n}_t^n &= -[(\boldsymbol{I} - \boldsymbol{\Omega}_H^\intercal)\boldsymbol{\Lambda}]^{-1} \left[(\boldsymbol{\beta} + \boldsymbol{\Omega}_F^\intercal \boldsymbol{\lambda}) \varphi y_t^n + \text{diag}(\boldsymbol{\beta}_F + \boldsymbol{\Omega}_F^\intercal \boldsymbol{\lambda}) \boldsymbol{\tau}_t^X \right] \\ &= -\mathbf{1} \varphi y_t^n - \mathcal{N}_x \boldsymbol{\tau}_t^X, \end{aligned} \tag{A.30}$$

where $\mathcal{N}_x$ is the export-demand matrix defined in the main text, equivalently $\mathcal{N}_x = [(\boldsymbol{I} - \boldsymbol{\Omega}_H^\top)\boldsymbol{\Lambda}]^{-1} \text{diag}(\boldsymbol{\beta}_F + \boldsymbol{\Omega}_F^\top \boldsymbol{\lambda})$, and the second equality uses $(\boldsymbol{I} - \boldsymbol{\Omega}_H^\intercal)\boldsymbol{\Lambda}\mathbf{1} = \boldsymbol{\lambda} - \boldsymbol{\Omega}_H^\intercal \boldsymbol{\lambda} = \boldsymbol{\beta} + \boldsymbol{\Omega}_F^\intercal \boldsymbol{\lambda}$,

which follows from $\boldsymbol{\lambda} = \boldsymbol{\beta} + \boldsymbol{\Omega}^\intercal \boldsymbol{\lambda}$.

Next, aggregate hours equal real GDP, $n_t = \sum_i \lambda_i \alpha_i n_{i,t} = y_t$, as in (A.12). In the flexible-price equilibrium,[14]

$$n_t^n = \boldsymbol{\alpha}^\intercal \boldsymbol{\Lambda} \boldsymbol{n}_t^n = y_t^n. \tag{A.32}$$

Premultiplying (A.30) by $\boldsymbol{\alpha}^\intercal \boldsymbol{\Lambda}$, and using $\boldsymbol{\alpha}^\intercal \boldsymbol{\Lambda} \mathbf{1} = \boldsymbol{\lambda}^\intercal \boldsymbol{\alpha} = 1$ and $\boldsymbol{\alpha}^\intercal \boldsymbol{\Lambda} [(\boldsymbol{I} - \boldsymbol{\Omega}_H^\intercal)\boldsymbol{\Lambda}]^{-1} = \left[(\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1} \boldsymbol{\alpha}\right]^\intercal$, we obtain $y_t^n = -\varphi y_t^n - \boldsymbol{\ell}_X^\intercal \boldsymbol{\tau}_t^X$, that is,

$$y_t^n = -\frac{1}{1+\varphi} \boldsymbol{\ell}_X^\intercal \boldsymbol{\tau}_t^X, \qquad \boldsymbol{\ell}_X \equiv \operatorname{diag}\left[(\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1} \boldsymbol{\alpha}\right] \left(\boldsymbol{\beta}_F + \boldsymbol{\Omega}_F^\intercal \boldsymbol{\lambda}\right), \tag{A.33}$$

which holds for both currency invoicing regimes. Each element of $(\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1} \boldsymbol{\alpha}$ is the domestic labor content of one unit of sectoral output, including the labor embodied in domestic intermediate inputs. Because $(\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1} \boldsymbol{\alpha} + (\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1} \boldsymbol{\Omega}_F \mathbf{1} = \mathbf{1}$ under constant returns to scale, the complement is the direct and indirect imported-input content, which does not generate domestic value added. One can verify that (A.33) and (A.30) also satisfy the flexible-price version of real GDP, (A.15).

In addition, using the risk-sharing condition $c_t^n = e_t^n - p_t^{C,n}$ together with $p_t^{C,n} + c_t^n = -\varphi y_t^n$, we have that

$$e_t^n = -\varphi y_t^n = \frac{\varphi}{1+\varphi} \boldsymbol{\ell}_X^\intercal \boldsymbol{\tau}_t^X \tag{A.34}$$

holds for both currency invoicing regimes, and sectoral employment can be written as $\boldsymbol{n}_t^n = \mathbf{1} e_t^n - \mathcal{N}_x \boldsymbol{\tau}_t^X$.

Under these conditions, we obtain that the response of foreign prices in the flexible price equilibrium is given by

$$\mathbf{p}_t^{F,n} = (\mathbf{1} - \mathbb{1}) e_t^n + \mathbf{p}_t^{*,n} = (\mathbf{1} - \mathbb{1}) e_t^n = \frac{\varphi}{1+\varphi} (\mathbf{1} - \mathbb{1}) \boldsymbol{\ell}_X^\intercal \boldsymbol{\tau}_t^X$$

and thus the response of domestic prices in the flexible price equilibrium is given by:

$$\mathbf{p}_t^{H,n} = (\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1} \boldsymbol{\Omega}_F \left[\boldsymbol{\tau}_t^F + (\mathbf{1} - \mathbb{1}) e_t^n\right] = (\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1} \boldsymbol{\Omega}_F \left[\boldsymbol{\tau}_t^F + \frac{\varphi}{1+\varphi} (\mathbf{1} - \mathbb{1}) \boldsymbol{\ell}_X^\intercal \boldsymbol{\tau}_t^X\right], \tag{A.35}$$

where we have used (A.13). Consequently, the response of the terms of trade under PCP is

[14] Combining (A.16) and (A.32), we can write

$$\widetilde{n}_t = \widetilde{y}_t. \tag{A.31}$$

given by:

$$\boldsymbol{s}_t^n = \boldsymbol{\tau}_t^F + \mathbf{p}_t^{F,n} - \mathbf{p}_t^{H,n} = \left[\boldsymbol{I} - (\boldsymbol{I} - \Omega_H)^{-1}\Omega_F\right]\left[\boldsymbol{\tau}_t^F + (\mathbf{1} - \mathbb{1})e_t^n\right]. \tag{A.36}$$

Finally, inserting all previous findings into the Phillips curve (A.24), we obtain

$$\begin{aligned}(\boldsymbol{I} - \boldsymbol{\Delta\Omega} - \boldsymbol{\Delta\alpha\beta}^\intercal)\boldsymbol{\pi}_t^H &= \boldsymbol{\Delta\alpha}\left[1 + \varphi(\omega_X + 1)\right]\widetilde{y}_t + \boldsymbol{\Delta\alpha\lambda}^\intercal \text{diag}\left(\Omega_F\mathbf{1}\right)\widetilde{\boldsymbol{n}}_t + \boldsymbol{\Delta}(\boldsymbol{I} - \boldsymbol{\alpha\lambda}^\intercal)\Omega_F\widetilde{\boldsymbol{s}}_t \\ &+ \beta(\boldsymbol{I} - \boldsymbol{\Delta})\mathbb{E}_t\boldsymbol{\pi}_{t+1}^H - \boldsymbol{\Delta}(\boldsymbol{I} - \boldsymbol{\Omega} - \boldsymbol{\alpha\beta}^\intercal)\left[\mathbf{p}_{t-1}^H - (\boldsymbol{I} - \Omega_H)^{-1}\Omega_F\left[\boldsymbol{\tau}_t^F + (\mathbf{1} - \mathbb{1})e_t^n\right]\right]\end{aligned} \tag{A.37}$$

Following the same transformations as in Rubbo (2023) to invert the term $(\boldsymbol{I} - \boldsymbol{\Delta\Omega} - \boldsymbol{\Delta\alpha\beta}^\intercal)$, we obtain

$$\boldsymbol{\pi}_{H,t} = \mathcal{B}\left(1 + \varphi + \varphi\omega_X\right)\widetilde{y}_t + \mathcal{B}\boldsymbol{\lambda}^\intercal \text{diag}\left(\Omega_F\mathbf{1}\right)\widetilde{\boldsymbol{n}}_t + \mathcal{V}(\boldsymbol{I} - \Omega)^{-1}\Omega_F\widetilde{\boldsymbol{s}}_t - \mathcal{V}\boldsymbol{\chi}_t + \beta(\boldsymbol{I} - \mathcal{V})\mathbb{E}_t\boldsymbol{\pi}_{H,t+1}. \tag{A.38}$$

where $\boldsymbol{\chi}_t = \mathbf{p}_{t-1}^H - \mathbf{p}_t^{H,n}$ is the bracket in (A.37), with $\mathbf{p}_t^{H,n}$ given by (A.35). Since $\mathbf{p}_{t-1}^H = \widetilde{\mathbf{p}}_{t-1}^H + \mathbf{p}_{t-1}^{H,n}$, this is equivalent to the gap form $\boldsymbol{\chi}_t = \widetilde{\mathbf{p}}_{t-1}^H - \Delta\mathbf{p}_t^{H,n}$ in (49). In (A.38), $\boldsymbol{\pi}_{H,t}$ is actual sectoral inflation.

Consider now the market clearing condition (A.27) in the rigid-price equilibrium, $\lambda_i y_{i,t} = \left[\beta_i + \sum_{j=1}^I \lambda_j\nu_{ji}(1-\alpha_j)\right]\left(p_t^C + c_t - p_{i,t}^H\right) - \left[\beta_{F,i} + \sum_{j=1}^I \lambda_j\nu_{ji}\zeta_{ji}(1-\alpha_j)\right]\tau_{i,t}^X + \sum_{j=1}^I \lambda_j\nu_{ji}(1-\zeta_{ji})(1-\alpha_j)(\varphi n_t + n_{j,t})$, where the export tariff reduces foreign demand for SOE sector i. Using $\lambda_i = \beta_i + \sum_{j=1}^I \lambda_j\nu_{ji}(1-\alpha_j)$ and $\beta_{F,i} + \sum_{j=1}^I \lambda_j\nu_{ji}\zeta_{ji}(1-\alpha_j) = (\boldsymbol{\beta}_F + \boldsymbol{\Omega}_F^\top\boldsymbol{\lambda})_i$, this becomes $\lambda_i y_{i,t} = \lambda_i\left(p_t^C + c_t - p_{i,t}^H\right) - (\boldsymbol{\beta}_F + \boldsymbol{\Omega}_F^\top\boldsymbol{\lambda})_i\tau_{i,t}^X + \sum_{j=1}^I \lambda_j\nu_{ji}(1-\zeta_{ji})(1-\alpha_j)(\varphi n_t + n_{j,t})$. Furthermore, we can write $p_t^C + c_t - p_{i,t}^H = e_t - p_{i,t}^H = p_{i,t}^F + \mathbb{1}_i e_t - p_{i,t}^H = s_{i,t} + \mathbb{1}_i e_t - \tau_{i,t}^F$, where we have used (A.7). Thus, we can write the previous expression as

$$\lambda_i y_{i,t} = \lambda_i\left(s_{i,t} + \mathbb{1}_i e_t - \tau_{i,t}^F\right) - (\boldsymbol{\beta}_F + \boldsymbol{\Omega}_F^\top\boldsymbol{\lambda})_i\tau_{i,t}^X + \sum_{j=1}^I \lambda_j\nu_{ji}(1-\zeta_{ji})(1-\alpha_j)(\varphi n_t + n_{j,t}) \tag{A.39}$$

Log-linearizing the production function (28)-(30),

$$\begin{aligned}y_{i,t} &= \alpha_i n_{i,t} + \sum_{j=1}^I (1-\alpha_i)\nu_{ij}(1-\zeta_{ij})x_{ij,t}^H + \sum_{j=1}^I (1-\alpha_i)\nu_{ij}\zeta_{ij}x_{ij,t}^F \\ &= \alpha_i n_{i,t} + \sum_{j=1}^I (1-\alpha_i)\nu_{ij}x_{ij,t}^H - \sum_{j=1}^I (1-\alpha_i)\nu_{ij}\zeta_{ij}s_{j,t}\end{aligned}$$

$$= \alpha_i n_{i,t} + \sum_{j=1}^{I}(1-\alpha_i)\nu_{ij}(s_{j,t} + \mathbb{1}_j e_t - \tau^F_{j,t} + \varphi n_t + n_{i,t}) - \sum_{j=1}^{I}(1-\alpha_i)\nu_{ij}\zeta_{ij}s_{j,t}$$

$$= n_{i,t} + (1-\alpha_i)\varphi n_t + (1-\alpha_i)\sum_{j=1}^{I}\nu_{ij}(1-\zeta_{ij})s_{j,t} + (1-\alpha_i)\sum_{j=1}^{I}\nu_{ij}\mathbb{1}_j e_t - (1-\alpha_i)\sum_{j=1}^{I}\nu_{ij}\tau^F_{j,t},$$

where the second equality made use of the international input demand curve $x^F_{ij,t} - x^H_{ij,t} = p^H_{j,t} - (p^F_{j,t} + \tau^F_{j,t}) = -s_{j,t}$, and the third equality made use of the domestic input demand curve $x^H_{ij,t} = x_{ij,t} + p_{ij,t} - p^H_{j,t} = p^X_{i,t} + x_{i,t} - p^H_{j,t} = w_t + n_{i,t} - p^H_{j,t} = p^C_t + c_t + \varphi n_t + n_{i,t} - p^H_{j,t} = p^F_{j,t} + \mathbb{1}_j e_t + \varphi n_t + n_{i,t} - p^H_{j,t} = \varphi n_t + n_{i,t} + s_{j,t} + \mathbb{1}_j e_t - \tau^F_{j,t}$.

Pre-multiplying this last expression by λ_i, and combining it with (A.39), we can write

$$\lambda_i n_{i,t} - \sum_{j=1}^{I}\lambda_j\nu_{ji}(1-\zeta_{ji})(1-\alpha_j)n_{j,t} + \lambda_i(1-\alpha_i)\varphi n_t - \sum_{j=1}^{I}\lambda_j\nu_{ji}(1-\zeta_{ji})(1-\alpha_j)\varphi n_t$$

$$= \lambda_i\left[s_{i,t} + \mathbb{1}_i e_t - (1-\alpha_i)\sum_{j=1}^{I}\nu_{ij}(1-\zeta_{ij})s_{j,t} - (1-\alpha_i)\sum_{j=1}^{I}\nu_{ij}\mathbb{1}_j e_t - \tau^F_{i,t} + (1-\alpha_i)\sum_{j=1}^{I}\nu_{ij}\tau^F_{j,t}\right] - (\boldsymbol{\beta}_F + \boldsymbol{\Omega}_F^\top\boldsymbol{\lambda})$$

which can be written in stacked form as

$$(\boldsymbol{I}-\boldsymbol{\Omega}_H^\intercal)\boldsymbol{\Lambda}\boldsymbol{n}_t = [\boldsymbol{\Lambda}\boldsymbol{\alpha}-(\boldsymbol{I}-\boldsymbol{\Omega}_H^\intercal)\boldsymbol{\lambda}]\varphi\, y_t + \boldsymbol{\Lambda}(\boldsymbol{I}-\boldsymbol{\Omega}_H)\boldsymbol{s}_t + \boldsymbol{\Lambda}(\boldsymbol{I}-\boldsymbol{\Omega})\mathbb{1}e_t - \boldsymbol{\Lambda}(\boldsymbol{I}-\boldsymbol{\Omega})\boldsymbol{\tau}^F_t - \text{diag}(\boldsymbol{\beta}_F + \boldsymbol{\Omega}_F^\top\boldsymbol{\lambda})\boldsymbol{\tau}^X_t.$$

Rearranging, we obtain (41), where we use the fact that, since $\boldsymbol{\Lambda}$ is diagonal and invertible, we have $\boldsymbol{\Lambda}^{-1}(\boldsymbol{I} - \boldsymbol{\Omega}_H^\top)^{-1}\boldsymbol{\Lambda} = \left(\boldsymbol{I} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_H^\top\boldsymbol{\Lambda}\right)^{-1}$. Rewriting this expression in terms of gaps from the flexible-price equilibrium,

$$\widetilde{\boldsymbol{n}}_t = (\boldsymbol{\mathcal{N}}_y - \boldsymbol{1})\varphi\widetilde{y}_t + \boldsymbol{\mathcal{N}}_s\widetilde{\boldsymbol{s}}_t + \boldsymbol{\mathcal{N}}_\tau\mathbb{1}\widetilde{e}_t. \tag{A.40}$$

Inserting this expression into (A.38), we obtain (48).

A.3. Additional Theoretical Results in the SOE

Slope of the Phillips curve. In contrast to the one-sector economy, the closed- and open-economy slopes of the Phillips curve (48) with respect to the output gap no longer coincide:

PROPOSITION A1. *The slope of the Phillips curve* (48) *with respect to the output gap is weakly steeper than its closed-economy counterpart. It is strictly steeper if at least one imported-input-using sector is connected, directly or indirectly through the domestic input-output network, to a sector with a positive labor share.*

PROOF. The Neumann series $(\mathbf{I} - \boldsymbol{\Omega}_H^\top)^{-1} = \sum_{n=0}^{\infty}(\boldsymbol{\Omega}_H^\top)^n$ converges, and each power $(\boldsymbol{\Omega}_H^\top)^n$ is entrywise non-negative because $\boldsymbol{\Omega}_H^\top \geq 0$ elementwise. Hence, $(\mathbf{I} - \boldsymbol{\Omega}_H^\top)^{-1} \geq 0$ entrywise. Premultiplying and postmultiplying by the diagonal matrices $\boldsymbol{\Lambda}^{-1}$ and $\boldsymbol{\Lambda}$, which have strictly positive diagonal elements, preserves entrywise non-negativity. It follows that $\boldsymbol{B} \equiv \boldsymbol{\Lambda}^{-1}(\mathbf{I} - \boldsymbol{\Omega}_H^\top)^{-1}\boldsymbol{\Lambda}$ is entrywise non-negative.

Consider now $\boldsymbol{\Theta} = \boldsymbol{\lambda}^\top \operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\,\boldsymbol{B}\,\boldsymbol{\alpha} = \sum_{j=1}^{I}\sum_{k=1}^{I}\lambda_j\left(\sum_{i=1}^{I}(\boldsymbol{\Omega}_F)_{ji}\right)B_{jk}\,\alpha_k$. Under weakly positive labor shares, each term $\lambda_j(\sum_i(\boldsymbol{\Omega}_F)_{ji})B_{jk}\alpha_k$ in the double sum is therefore non-negative, and we obtain $\Theta \geq 0$.

To show strict positivity, note first that by the definition of $\mathbf{B}$, $B_{jk} = \left[\boldsymbol{\Lambda}^{-1}(\mathbf{I} - \boldsymbol{\Omega}_H^\top)^{-1}\boldsymbol{\Lambda}\right]_{jk} = \frac{\lambda_k}{\lambda_j}\left[(\mathbf{I} - \boldsymbol{\Omega}_H^\top)^{-1}\right]_{jk}$. Under actual openness, there exists an index $j^\star$ such that $\sum_{i=1}^{I}(\boldsymbol{\Omega}_F)_{j^\star i} > 0$ (sector $j^\star$ uses imported intermediate inputs), $\lambda_{j^\star} > 0$, and, by the strict-connectivity condition in the proposition, there exists an index $k^\star$ with $\alpha_{k^\star} > 0$ such that the $(j^\star, k^\star)$ entry of $(\mathbf{I}-\boldsymbol{\Omega}_H^\top)^{-1}$ is strictly positive. The latter implies $B_{j^\star k^\star} > 0$.

Combining these conditions, there exist indices $(j^\star, k^\star)$ such that $\lambda_{j^\star} > 0$, $\sum_{i=1}^{I}(\boldsymbol{\Omega}_F)_{j^\star i} > 0$, $B_{j^\star k^\star} > 0$, $\alpha_{k^\star} > 0$. The corresponding term in the sum for Θ, $\lambda_{j^\star}\left(\sum_{i=1}^{I}(\boldsymbol{\Omega}_F)_{j^\star i}\right)B_{j^\star k^\star}\alpha_{k^\star}$ is therefore strictly positive, while all other terms are non-negative. Consequently, $\Theta > 0$ under this openness-and-connectivity condition.

□

Proposition A1 shows that international production networks steepen the Phillips curve relative to a closed economy. Importantly, this result does not arise because firms' pricing decisions are intrinsically more sensitive to demand. Rather, it reflects a general-equilibrium mechanism whereby producing a given increase in domestic output requires a larger increase in domestic labor input once imported intermediate goods and sectoral trade imbalances are taken into account.

To see this, consider an increase in the output gap holding the sectoral ToT gaps fixed. In an economy with international production networks, higher domestic production mechanically raises the demand for imported intermediate inputs. As shown in equation (42), this generates an output-consumption wedge that is proportional to $\boldsymbol{\lambda}^\top \operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\,\mathcal{N}_y$. Equivalently, along this margin the equilibrium relation implies that the consumption gap rises more than one-for-one with the output gap.

The presence of this output-consumption wedge has direct implications for labor supply. Through the labor-supply condition, a larger consumption gap for a given output gap raises the marginal rate of substitution between consumption and leisure. The real wage required to support the expansion in domestic production therefore increases by more than in the closed-economy benchmark. In this sense, international production networks amplify the labor-supply component of marginal costs associated with a given output gap.

Higher real wages feed directly into firms' marginal costs. As a result, for a given increase in the output gap, marginal costs rise by more in the presence of international production networks. This amplification is made explicit in equation (48), where the slope of the Phillips curve depends on the additional term $\varphi \boldsymbol{\lambda}^{\intercal} \operatorname{diag}(\boldsymbol{\Omega}_F \mathbf{1}) \mathcal{N}_y$, which reflects the network-induced amplification of the labor-supply channel: because imported inputs complement domestic production, higher output requires more domestic labor rather than allowing production to be shifted abroad.

Sensitivity to ToT gaps. The following result states that a deterioration in the ToT, such as the one caused by a price-wedge shock, tends to increase domestic prices through the two channels discussed above.

REMARK A1. *The elements in $\boldsymbol{\Psi}$ related to the first and second transmission channels, $\boldsymbol{\Delta}(\boldsymbol{I} - \boldsymbol{\Omega}\boldsymbol{\Delta})^{-1}\boldsymbol{\Omega}_F$ and $\mathcal{B}\boldsymbol{\beta}^{\intercal}\boldsymbol{\Delta}(\boldsymbol{I} - \boldsymbol{\Omega}\boldsymbol{\Delta})^{-1}\boldsymbol{\Omega}_F$, respectively, are entrywise non-negative. Moreover, any entry corresponding to a domestic sector that is directly or indirectly exposed to foreign sectors through $\boldsymbol{\Omega}_F$ is strictly positive. In particular, strict positivity holds entrywise under the additional assumption that the relevant network is sufficiently connected and that the associated IO shares are strictly positive.*

The following proposition also clarifies why the third term in $\boldsymbol{\Psi}$ is absent in a one-sector economy, even when intermediate goods are present:

PROPOSITION A2. *The matrix $\operatorname{diag}(\boldsymbol{\Omega}_F \mathbf{1}) \mathcal{N}_s - \boldsymbol{\Omega}_F$ is hollow (the elements on the diagonal are all zero).*

PROOF. Define $\boldsymbol{D} \equiv \operatorname{diag}(\boldsymbol{\Omega}_F \mathbf{1})$ and $\mathcal{N}_s \equiv \boldsymbol{D}^{-1}\boldsymbol{\Omega}_F$, so that $\boldsymbol{\Xi} \equiv \boldsymbol{D}\mathcal{N}_s - \boldsymbol{\Omega}_F$. We show that $\boldsymbol{\Xi}$ is hollow, i.e., $\Xi_{ii} = 0$ for all i.

Let i be arbitrary. Expanding the (i, i) entry of the product gives $\Xi_{ii} = \left[\boldsymbol{D}\mathcal{N}_s\right]_{ii} - (\Omega_F)_{ii} = \sum_{j=1}^{I} D_{ij}(\mathcal{N}_s)_{ji} - (\Omega_F)_{ii}$. Since $\boldsymbol{D}$ is diagonal, $D_{ij} = 0$ for $j \neq i$, and therefore $\Xi_{ii} = D_{ii}(\mathcal{N}_s)_{ii} - (\Omega_F)_{ii}$.

Next, note that the definition $\mathcal{N}_s = \boldsymbol{D}^{-1}\boldsymbol{\Omega}_F$ implies $\boldsymbol{D}\mathcal{N}_s = \boldsymbol{\Omega}_F$, or equivalently $\boldsymbol{D}^{-1}\boldsymbol{D}\mathcal{N}_s = \boldsymbol{D}^{-1}\boldsymbol{\Omega}_F$. Taking the (i, i) entry of $\mathcal{N}_s = \boldsymbol{D}^{-1}\boldsymbol{\Omega}_F$ yields $(\mathcal{N}_s)_{ii} = \sum_{j=1}^{I}(D^{-1})_{ij}(\Omega_F)_{ji}$. Again, $\boldsymbol{D}^{-1}$ is diagonal, so $(D^{-1})_{ij} = 0$ for $j \neq i$, hence $(\mathcal{N}_s)_{ii} = (D^{-1})_{ii}(\Omega_F)_{ii}$. Substituting into the expression for Ξ_{ii} gives $\Xi_{ii} = D_{ii}(D^{-1})_{ii}(\Omega_F)_{ii} - (\Omega_F)_{ii} = (\Omega_F)_{ii} - (\Omega_F)_{ii} = 0$. Since i was arbitrary, $\Xi_{ii} = 0$ for all i, and thus $\boldsymbol{D}\mathcal{N}_s - \boldsymbol{\Omega}_F$ is hollow.

□

The previous proposition illustrates that what matters are production linkages *across* sectors. That is, since the diagonal elements of $\operatorname{diag}(\boldsymbol{\Omega}_F \mathbf{1}) \mathcal{N}_s - \boldsymbol{\Omega}_F$ are zero, the term $\mathcal{B}\boldsymbol{\lambda}^{\intercal}(\operatorname{diag}(\boldsymbol{\Omega}_F \mathbf{1}) \mathcal{N}_s - \boldsymbol{\Omega}_F)$ would be absent not only in one-sector economies, but also in

multi-sector frameworks with heterogeneous sectors and no cross-sector relationships in the production network—where both Ω_F and Ω_H are diagonal matrices.

The intuition is as follows. Consider the one-sector economy with intermediate goods characterized by the Phillips curve (47). In that case, a 1% increase in the ToT translates one-to-one into higher demand for the domestic intermediate good, given the unitary elasticity of substitution. The associated increase in the demand for imports arising from higher sectoral employment exactly offsets this effect, given the Cobb–Douglas production function, thereby leaving net exports—and hence wages—unchanged. With cross-sector linkages, however, other domestic sectors also reallocate demand toward the now cheaper domestic sector, further increasing overall demand and generating trade imbalances that distort real wages and inflation.

Proof of Proposition 1. Define the mark-up gap as

$$\boldsymbol{\mu}_t = \widetilde{\boldsymbol{p}}_t^H - \widetilde{\mathrm{mc}}_t. \tag{A.41}$$

Rewriting the marginal cost curve (37) in gap terms, $\widetilde{\mathrm{mc}}_t = \boldsymbol{\alpha}\widetilde{w}_t + \Omega\widetilde{\boldsymbol{p}}_t^H + \Omega_F\widetilde{\boldsymbol{s}}_t$, and combining both equations,

$$\widetilde{\boldsymbol{p}}_t^H = \mathbf{1}\widetilde{w}_t + (\boldsymbol{I} - \Omega)^{-1}\boldsymbol{\mu}_t + (\boldsymbol{I} - \Omega)^{-1}\Omega_F\widetilde{\boldsymbol{s}}_t \tag{A.42}$$

where we used the identity

$$(\boldsymbol{I} - \Omega)^{-1}\boldsymbol{\alpha} = \mathbf{1}, \tag{A.43}$$

by CRS.

In the open economy, the CPI is given by (A.6), which in gap terms can be written as

$$\widetilde{p}_t^C = \boldsymbol{\beta}^\intercal \widetilde{\boldsymbol{p}}_t^H + \boldsymbol{\beta}_F^\intercal\widetilde{\boldsymbol{s}}_t. \tag{A.44}$$

Inserting (A.41)-(A.42) into (A.44), and using $\boldsymbol{\beta}^\intercal\mathbf{1} = 1$ and $\boldsymbol{\beta}^\intercal + \boldsymbol{\lambda}^\intercal\Omega = \boldsymbol{\lambda}^\intercal$, yields

$$\widetilde{w}_t - \widetilde{p}_t^C = -\boldsymbol{\lambda}^\intercal\boldsymbol{\mu}_t - (\boldsymbol{\beta}_F^\intercal + \boldsymbol{\lambda}^\intercal\Omega_F)\widetilde{\boldsymbol{s}}_t. \tag{A.45}$$

Combining the definition of ToT, (38), with the import-invoicing convention in (33)-(34), we can write the ToT in gap terms as $\widetilde{\boldsymbol{s}}_t = (\mathbf{1} - \mathbb{1})\widetilde{e}_t - \widetilde{\boldsymbol{p}}_t^H$. Inserting (A.42) in this expression,

$$\widetilde{\boldsymbol{s}}_t = (\boldsymbol{I} - \Omega_H)^{-1}\boldsymbol{\alpha}(\widetilde{e}_t - \widetilde{w}_t) - (\boldsymbol{I} - \Omega_H)^{-1}\boldsymbol{\mu}_t - (\boldsymbol{I} - \Omega_H)^{-1}(\boldsymbol{I} - \Omega)\mathbb{1}\widetilde{e}_t. \tag{A.46}$$

This step uses $\boldsymbol{I} + (\boldsymbol{I} - \boldsymbol{\Omega})^{-1}\boldsymbol{\Omega}_F = (\boldsymbol{I} - \boldsymbol{\Omega})^{-1}(\boldsymbol{I} - \boldsymbol{\Omega}_H)$ and (A.43).

Combining now the risk-sharing condition (A.7), rewritten in gap terms as $\widetilde{c}_t = \widetilde{e}_t - \widetilde{p}_t^C$, with the labor supply condition (39), which can be written in gap terms as

$$\widetilde{w}_t - \widetilde{p}_t^C = \widetilde{c}_t + \varphi \widetilde{y}_t, \tag{A.47}$$

we can write $\widetilde{e}_t - \widetilde{w}_t = -\varphi \widetilde{y}_t$. Inserting this last expression into (A.46),

$$\widetilde{\boldsymbol{s}}_t = -\varphi(\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1}\boldsymbol{\alpha}\widetilde{y}_t - (\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1}\boldsymbol{\mu}_t - (\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1}(\boldsymbol{I} - \boldsymbol{\Omega})\mathbb{1}\widetilde{e}_t. \tag{A.48}$$

Combining the risk-sharing condition (A.7) with the CPI definition (A.6), together with the definition of ToT (38), we can write, in gap terms,

$$\widetilde{c}_t = \boldsymbol{\beta}_H^{\mathsf{T}}\widetilde{\boldsymbol{s}}_t + \boldsymbol{\beta}^{\mathsf{T}}\mathbb{1}\widetilde{e}_t. \tag{A.49}$$

Combining (A.49) with (A.47) and (A.45), and using $\boldsymbol{\beta}^{\mathsf{T}} + \boldsymbol{\lambda}^{\mathsf{T}}\boldsymbol{\Omega}_F = \boldsymbol{\lambda}^{\mathsf{T}}(\boldsymbol{I} - \boldsymbol{\Omega}_H)$, gives

$$-\boldsymbol{\lambda}^{\mathsf{T}}\boldsymbol{\mu}_t = \varphi\widetilde{y}_t + \boldsymbol{\lambda}^{\mathsf{T}}(\boldsymbol{I} - \boldsymbol{\Omega}_H)\widetilde{\boldsymbol{s}}_t + \left[\boldsymbol{\lambda}^{\mathsf{T}}(\boldsymbol{I} - \boldsymbol{\Omega})\right]\mathbb{1}\widetilde{e}_t. \tag{A.50}$$

Introducing (A.49) into the gap version of (42), we obtain

$$\left(1 + \varphi\boldsymbol{\lambda}^{\mathsf{T}}\operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\mathcal{N}_y\right)\widetilde{y}_t = \left[\boldsymbol{\beta}^{\mathsf{T}} + \boldsymbol{\lambda}^{\mathsf{T}}\left(\boldsymbol{\Omega}_F - \operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\mathcal{N}_s\right)\right]\widetilde{\boldsymbol{s}}_t + \left[\boldsymbol{\beta}^{\mathsf{T}}\mathbb{1} + \boldsymbol{\lambda}^{\mathsf{T}}\operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\mathcal{N}_\tau\mathbb{1}\right]\widetilde{e}_t. \tag{A.51}$$

Inserting (A.48) into (A.51), and using $\mathcal{N}_y = \mathcal{N}_s(\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1}\boldsymbol{\alpha}$, gives

$$-\left[\boldsymbol{\beta}^{\mathsf{T}} + \boldsymbol{\lambda}^{\mathsf{T}}\left(\boldsymbol{\Omega}_F - \operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\mathcal{N}_s\right)\right](\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1}\boldsymbol{\mu}_t = (1+\varphi)\widetilde{y}_t - R_e\widetilde{e}_t, \tag{A.52}$$

where R_e is defined in (53). From the definition of $\mathcal{N}_s$,

$$\left[\boldsymbol{\beta}^{\mathsf{T}} + \boldsymbol{\lambda}^{\mathsf{T}}\left(\boldsymbol{\Omega}_F - \operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\mathcal{N}_s\right)\right](\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1} = \left[(\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1}\boldsymbol{\alpha}\right]^{\mathsf{T}}\boldsymbol{\Lambda}, \tag{A.53}$$

where the last equality uses the CRS identity $\mathbf{1} - (\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1}\boldsymbol{\Omega}_F\mathbf{1} = (\boldsymbol{I} - \boldsymbol{\Omega}_H)^{-1}\boldsymbol{\alpha}$. Combining (A.52) and (A.53),

$$-\sum_{i=1}^{I}\lambda_i d_i \mu_{i,t} = (1+\varphi)\widetilde{y}_t - R_e\widetilde{e}_t. \tag{A.54}$$

Finally, consider the sectoral Phillips curves (36). Reorganizing terms, multiplying the

Phillips curve of sector i by $\lambda_i d_i/\kappa_i$, and summing across sectors,

$$\sum_{i=1}^{I} \frac{\lambda_i}{\kappa_i} d_i \left(\pi^H_{i,t} - \beta \mathbb{E}_t \pi^H_{i,t+1} \right) = (1+\varphi)\widetilde{y}_t - R_e \widetilde{e}_t. \tag{A.55}$$

Using the definition of the open-economy DCI in (51), equation (A.55) yields (52). Since any constant-weight domestic inflation index whose Phillips curve is only a function of the aggregate output gap must aggregate the markup vector through the left-hand side of (A.54), the weights in (51) are pinned down up to normalization. Therefore, the strict Divine Coincidence holds if and only if the exchange-rate residual in (52) vanishes, that is, if and only if $R_e = 0$. □

Proposition 1 refines the comparison with the closed-economy benchmark. The closed-economy DCI of Rubbo (2023) is not the right aggregate in an open economy, because the output gap is not governed by sales-weighted markups alone. Once production uses imported intermediates, the relevant object is instead the domestic-content markup aggregate in (A.54). The weights in (51) are therefore the open-economy analogue of Rubbo's sales-and-rigidity adjustment: sectoral inflation is weighted by λ_i/κ_i, as in the closed economy, and by d_i, which measures how much domestic production content is embedded in sector i.

The proposition also clarifies why heterogeneous import invoicing remains a source of monetary-policy trade-offs. The open-economy DCI removes the sectoral markup residual associated with the output gap, but its Phillips curve in (52) contains the exchange-rate residual R_e. This residual collects the components of the exchange-rate gap that cannot be absorbed into the domestic-content markup aggregate. It depends on the import-invoicing vector $\mathbb{1}$, on imported-input exposure through Ω_F, and on the expenditure-switching and tariff components embedded in $\mathcal{N}_s$ and $\mathcal{N}_\tau$. Hence, heterogeneous currency pricing does not mechanically rule out a DCI index. Rather, it changes the condition under which that index delivers the strict Divine Coincidence.

When $R_e = 0$, closing the output gap closes the open-economy DCI index, up to expected future DCI inflation, exactly as in the closed-economy benchmark. When $R_e \neq 0$, closing $\widetilde{y}_t$ leaves a residual term proportional to $\widetilde{e}_t$. In that case the DCI remains informative because it is the unique domestic inflation aggregate aligned with the output-gap markup aggregate, but it is not a sufficient target for implementing the efficient allocation. The central bank must still confront relative-price movements generated by the exchange rate and by the currency denomination of imported inputs.

The One-Sector Phillips Curve. In the one-sector economy, we can write the Phillips curve as $\pi_{H,t} = \kappa(\mathrm{mc}_t - p_{H,t}) + \beta \mathbb{E}_t \pi_{H,t+1}$. Inserting the expression for nominal marginal costs (A.3)

in the one-sector economy, $\text{mc}_t = \alpha w_t + (1-\alpha)\, p_{H,t} + \omega_F s_t$—where we have made use of the CRS assumption $\omega_H = 1 - \alpha - \omega_F$—we can rewrite the Phillips curve as $\pi_{H,t} = \kappa[\alpha(w_t - p_{H,t}) + \omega_F s_t] + \beta \mathbb{E}_t \pi_{H,t+1}$. Inserting the one-sector expression for (A.6), $p_t^C = p_{H,t} + \beta_F s_t$, we can write $\pi_{H,t} = \kappa[\alpha(w_t - p_t^C) + (\alpha\beta_F + \omega_F) s_t] + \beta \mathbb{E}_t \pi_{H,t+1}$. Rewriting $p_t^C = \widetilde{p}_t^C + p_t^{C,n}$, and making use of (A.13) in the one-sector economy, $p_{H,t}^n = \omega_F/\alpha s_t^n$, and (A.14), which in the one-sector limit is given by $p_t^{C,n} = (\beta_F + \omega_F/\alpha) s_t^n$, we can write $\pi_{H,t} = \kappa[\alpha(w_t - \widetilde{p}_t^C) + (\alpha\beta_F + \omega_F)\widetilde{s}_t] + \beta \mathbb{E}_t \pi_{H,t+1}$.[15] Introducing (A.19), $w_t - \widetilde{p}_t^C = \left[1 + \varphi(\omega_X + 1)\right] \widetilde{y}_t - (\beta_F + \omega_F/\alpha)\widetilde{s}_t + \omega_F/\alpha \widetilde{n}_t$, where $\widetilde{y}_t = \widetilde{n}_t$ and having made use of the identity in footnote 15, we can write the Phillips curve as (47) (where we have used the identity $\omega_X = \boldsymbol{\lambda}^\intercal \boldsymbol{\Omega}_F \mathbf{1} = \omega_F/\alpha$).

Appendix B. Quadratic Welfare Loss Function in the SOE

In this appendix, we derive the quadratic welfare loss for foreign price-wedge shocks with $\boldsymbol{\tau}_t^M = \boldsymbol{\tau}_t^X = \mathbf{0}$, maintaining efficiency of the natural allocation. Import invoicing may differ across sectors. The price representation below applies when $\boldsymbol{\beta}^\top(\mathbf{1} - \mathbb{1}) > 0$, so that PCP imports have positive consumption weight; it includes uniform PCP and heterogeneous PCP/LCP invoicing. The all-LCP limit requires retaining the exchange-rate gap as an additional coordinate and is not covered by this reduced representation.

The starting point is the household period utility function (1) under the logarithmic consumption assumption (25). We want to expand welfare around the flexible-price allocation (C_t^n, N_t^n) which plays the role of the efficient benchmark.

Because the steady state is efficient thanks to the subsidy, define the log gaps relative to the flexible-price allocation as $\log C_t - \log C_t^n = (\log C_t - \log C) - (\log C_t^n - \log C) = \widehat{c}_t - \widehat{c}_t^n = \widetilde{c}_t$ and $\log N_t - \log N_t^n = (\log N_t - \log N) - (\log N_t^n - \log N) = \widehat{n}_t - \widehat{n}_t^n = \widetilde{n}_t$.

It is convenient to collect the small deviations around the flexible-price allocation into a vector $\widehat{\boldsymbol{g}}_t \equiv \begin{pmatrix} \widetilde{c}_t & \widetilde{n}_t \end{pmatrix}^\top$. Thus, $\widehat{\boldsymbol{g}}_t$ is just shorthand for the vector of gap variables with respect to the flexible-price benchmark. The notation ξ_t below is reserved for the exogenous state.

B.1. Second-order expansion to welfare

Because utility is separable in consumption and labor, there is no cross-derivative term. Therefore,

(B.1)

$$u(C_t, N_t) - u(C_t^n, N_t^n) = u_{C,t}^n (C_t - C_t^n) + u_{N,t}^n (N_t - N_t^n) + \frac{1}{2} u_{CC,t}^n (C_t - C_t^n)^2 + \frac{1}{2} u_{NN,t}^n (N_t - N_t^n)^2 + o(\|\widehat{\boldsymbol{g}}_t\|^2),$$

[15] In the one-sector limit, we can thus write $\boldsymbol{\lambda}^\intercal \boldsymbol{\Omega}_{\boldsymbol{F}}$ as ω_F/α.

where all derivatives are evaluated at the flexible-price allocation (C_t^n, N_t^n). The term $o(\|\widehat{\boldsymbol{g}}_t\|^2)$ collects all the remainder terms that are negligible relative to second-order terms.

Note that, for any time-t variable X_t, we can write $X_t = X_t^n e^{\widetilde{x}_t}$. Using the second-order approximation for the exponential, $e^{\widetilde{x}_t} = 1 + \widetilde{x}_t + \frac{1}{2}\widetilde{x}_t^2 + o(\|\widehat{\boldsymbol{g}}_t\|^2)$, we obtain $X_t - X_t^n = X_t^n\left(\widetilde{x}_t + \frac{1}{2}\widetilde{x}_t^2\right) + o(\|\widehat{\boldsymbol{g}}_t\|^2)$. Applying this to consumption and labor,

$$C_t - C_t^n = C_t^n\left(\widetilde{c}_t + \frac{1}{2}\widetilde{c}_t^2\right) + o(\|\widehat{\boldsymbol{g}}_t\|^2), \tag{B.2}$$

$$N_t - N_t^n = N_t^n\left(\widetilde{n}_t + \frac{1}{2}\widetilde{n}_t^2\right) + o(\|\widehat{\boldsymbol{g}}_t\|^2). \tag{B.3}$$

From (1) and (25), we get $u_{C,t} = C_t^{-1}$, $u_{CC,t} = -C_t^{-2}$, $u_{N,t} = -N_t^{\varphi}$, and $u_{NN,t} = -\varphi N_t^{\varphi-1}$. Evaluating these at the flexible-price allocation,

$$u_{CC,t}^n (C_t^n)^2 = -u_{C,t}^n C_t^n = -1, \tag{B.4}$$

$$u_{NN,t}^n (N_t^n)^2 = \varphi\, u_{N,t}^n N_t^n. \tag{B.5}$$

First, consider the consumption terms. Using (B.2),

$$\begin{aligned} u_{C,t}^n (C_t - C_t^n) + \frac{1}{2}u_{CC,t}^n (C_t - C_t^n)^2 &= u_{C,t}^n C_t^n\left(\widetilde{c}_t + \frac{1}{2}\widetilde{c}_t^2\right) + \frac{1}{2}u_{CC,t}^n (C_t^n)^2 \widetilde{c}_t^2 + o(\|\widehat{\boldsymbol{g}}_t\|^2) \\ &= u_{C,t}^n C_t^n \widetilde{c}_t + o(\|\widehat{\boldsymbol{g}}_t\|^2). \end{aligned} \tag{B.6}$$

where the first equality used the fact that, to second order, only the first-order piece of $C_t - C_t^n$ matters inside the square, and the last equality made use of (B.4).

Now consider the labor terms. Similarly, combining (B.3) and (B.5), we can write

$$u_{N,t}^n (N_t - N_t^n) + \frac{1}{2}u_{NN,t}^n (N_t - N_t^n)^2 = u_{N,t}^n N_t^n\left(\widetilde{n}_t + \frac{1+\varphi}{2}\widetilde{n}_t^2\right) + o(\|\widehat{\boldsymbol{g}}_t\|^2). \tag{B.7}$$

Inserting (B.6) and (B.7) into (B.1), we get

$$u_t - u_t^n = u_{C,t}^n C_t^n \widetilde{c}_t + u_{N,t}^n N_t^n\left(\widetilde{n}_t + \frac{1+\varphi}{2}\widetilde{n}_t^2\right) + o(\|\widehat{\boldsymbol{g}}_t\|^2). \tag{B.8}$$

Equation (B.8) is not yet the final welfare loss expression. It is the raw second-order expansion of utility around the flexible-price allocation, expressed in terms of the consumption gap and labor gap. The next step is to eliminate $\widetilde{c}_t$ and $\widetilde{n}_t$ in favor of the relevant distortions, the wedges used in the proposition.

B.2. Writing welfare in terms of the efficiency and labor wedges

In this step, it is useful to distinguish clearly between *log deviations from steady state* and *gaps relative to the flexible-price allocation*. For any endogenous variable $X_t(\xi_t)$, define the log deviation from steady state by

$$\widehat{x}_t(\xi_t) \equiv \log X_t(\xi_t) - \log X_{ss}, \tag{B.9}$$

and define the corresponding gap relative to the flexible-price allocation by

$$\widetilde{x}_t(\xi_t) \equiv \widehat{x}_t(\xi_t) - \widehat{x}^n_t(\xi_t). \tag{B.10}$$

It is now convenient to write the flexible-price labor share,

$$\Lambda^n_{N,t}(\xi_t) \equiv \frac{W^n_t N^n_t}{P^{C,n}_t C^n_t}(\xi_t). \tag{B.11}$$

with an steady-state value of $\Lambda_N \equiv \Lambda^n_{L,ss} = 1$ since we consider a zero-trade-balance in steady state.

Using the flexible-price intratemporal optimality condition,

$$-\frac{u^n_{N,t}}{u^n_{C,t}} = \frac{W^n_t}{P^{C,n}_t}, \tag{B.12}$$

we can also write the labor share as $\Lambda^n_{N,t}(\xi_t) = -\frac{u^n_{N,t}N^n_t}{u^n_{C,t}C^n_t}$. Because the flexible-price allocation is a smooth function of the exogenous state ξ_t, all of the flexible-price variables are perturbations of their steady-state values. Therefore,

$$\Lambda^n_{N,t}(\xi_t) = \Lambda_N + O(\|\widehat{\xi}_t\|). \tag{B.13}$$

We now follow Chari *et al.* (2007) and Qiu *et al.* (2026), and introduce two wedges, $A_{\text{agg},t}(\xi_t)$ and $\Gamma_{N,t}(\xi_t)$, through the system

$$C_t(\xi_t) = A_{\text{agg},t}(\xi_t)\, N_t(\xi_t)^{\Lambda^n_{N,t}(\xi_t)}, \tag{B.14}$$

$$-\frac{u_{N,t}(C_t, N_t)}{u_{C,t}(C_t, N_t)}(\xi_t) = \Gamma_{N,t}(\xi_t)\, \frac{\partial C_t}{\partial N_t}(\xi_t). \tag{B.15}$$

These two equations summarize the aggregate allocation: $A_{\text{agg},t}(\xi_t)$ measures shifts in the aggregate transformation from labor into consumption (the efficiency wedge), and $\Gamma_{N,t}(\xi_t)$

measures the wedge between the marginal rate of substitution and the marginal product of labor (the labor wedge).

LEMMA A1. *Up to a first-order approximation, the efficiency wedge in the sticky price equilibrium is the same as the efficiency wedge in the flexible-price equilibrium*

$$\widehat{A}_{agg,t}(\xi_t) - \widehat{A}^n_{agg,t}(\xi_t) = \widetilde{c}_t(\xi_t) - \Lambda_N \widetilde{n}_t(\xi_t) = 0. \tag{B.16}$$

The labor wedge, though, deviates from the flexible-price level, and the deviation is proportional to the output gap

$$\widehat{\Gamma}_{N,t}(\xi_t) - \widehat{\Gamma}^n_{N,t}(\xi_t) = \widehat{\Gamma}_{N,t}(\xi_t) = \frac{1+\varphi}{\Lambda_N}\widetilde{c}_t(\xi_t). \tag{B.17}$$

PROOF. Log-linearizing (B.14) around the steady state gives

$$\widehat{A}_{\text{agg},t}(\xi_t) = \widehat{c}_t(\xi_t) - \Lambda^n_{N,t}(\xi_t)\widehat{n}_t(\xi_t) = \widehat{c}_t(\xi_t) - \Lambda_N \widehat{n}_t(\xi_t) + o(\|\widehat{\xi}_t\|), \tag{B.18}$$

where the second equality made use of the approximation (B.13). The aggregate technology representation in (B.14)-(B.13) implies that (B.18) is, to first order, a function of productivity, import-price, and foreign-demand shocks, rather than of the sticky-price wedges themselves. Therefore the same first-order object must arise in both the sticky-price and the flexible-price equilibrium, because both equilibria are evaluated at the same realized exogenous state ξ_t. Subtracting the flexible-price counterpart of (B.18), we get $\widehat{A}_{\text{agg},t}(\xi_t) - \widehat{A}^n_{\text{agg},t}(\xi_t) = \left[\widehat{c}_t(\xi_t) - \Lambda_N \widehat{n}_t(\xi_t)\right] - \left[\widehat{c}^n_t(\xi_t) - \Lambda_N \widehat{n}^n_t(\xi_t)\right] + o(\|\widehat{\xi}_t\|) = \widetilde{c}_t(\xi_t) - \Lambda_N \widetilde{n}_t(\xi_t) + o(\|\widehat{\xi}_t\|) = o(\|\widehat{\xi}_t\|)$. Writing (42) in gap terms, with the direct export-demand terms netted against their flexible-price counterparts, and rearranging,

$$(1+\varphi \boldsymbol{\lambda}^\intercal \text{diag}\left(\boldsymbol{\Omega}_F \mathbf{1}\right) \mathcal{N}_y)\widetilde{y}_t = \widetilde{c}_t + \left[\boldsymbol{\beta}_F^\intercal + \boldsymbol{\lambda}^\intercal \left(\boldsymbol{\Omega}_F - \text{diag}\left(\boldsymbol{\Omega}_F \mathbf{1}\right) \mathcal{N}_s\right)\right] \widetilde{\boldsymbol{s}}_t - \boldsymbol{\lambda}^\intercal \text{diag}\left(\boldsymbol{\Omega}_F \mathbf{1}\right) \mathcal{N}_\tau \mathbb{1} \widetilde{e}_t.$$

and making use of (A.31), we can write

$$\begin{aligned}\widehat{A}_{\text{agg},t}(\xi_t) - \widehat{A}^n_{\text{agg},t}(\xi_t) = &\left[1+\varphi \boldsymbol{\lambda}^\intercal \text{diag}\left(\boldsymbol{\Omega}_F \mathbf{1}\right) \mathcal{N}_y - \Lambda_N\right] \widetilde{y}_t - \left[\boldsymbol{\beta}_F^\intercal + \boldsymbol{\lambda}^\intercal \left(\boldsymbol{\Omega}_F - \text{diag}\left(\boldsymbol{\Omega}_F \mathbf{1}\right) \mathcal{N}_s\right)\right] \widetilde{\boldsymbol{s}}_t \\ &+ \boldsymbol{\lambda}^\intercal \text{diag}\left(\boldsymbol{\Omega}_F \mathbf{1}\right) \mathcal{N}_\tau \mathbb{1} \widetilde{e}_t \\ = &\, o(\|\widehat{\xi}_t\|)\end{aligned}$$

Hence, up to a first-order approximation, (B.16) is satisfied. From (B.16), we obtain

$$\widetilde{n}_t(\xi_t) = \frac{1}{\Lambda_N}\widetilde{c}_t(\xi_t) + o(\|\widehat{\xi}_t\|). \tag{B.19}$$

Now turn to (B.15). Given the period utility (1) and (25), we have

$$-\frac{u_{N,t}}{u_{C,t}} = N_t^{\varphi} C_t. \tag{B.20}$$

Next, compute the marginal product of labor from the prototype aggregate technology (B.14). Differentiating with respect to N_t, we obtain $\frac{\partial C_t}{\partial N_t}(\xi_t) = A_{\text{agg},t}(\xi_t)\,\Lambda^n_{N,t}(\xi_t)\,N_t(\xi_t)^{\Lambda^n_{N,t}(\xi_t)-1}$. Using again $C_t = A_{\text{agg},t} N_t^{\Lambda^n_{N,t}}$, this can be rewritten as

$$\frac{\partial C_t}{\partial N_t}(\xi_t) = \Lambda^n_{N,t}(\xi_t)\,\frac{C_t(\xi_t)}{N_t(\xi_t)}. \tag{B.21}$$

Substituting (B.20) and (B.21) into (B.15), we get

$$N_t^{\varphi} C_t = \Gamma_{N,t}(\xi_t)\,\Lambda^n_{N,t}(\xi_t)\,\frac{C_t}{N_t} \implies \Gamma_{N,t}(\xi_t) = \frac{N_t^{1+\varphi}}{\Lambda^n_{N,t}(\xi_t)}, \tag{B.22}$$

which makes the labor wedge transparent under log utility: it increases with labor relative to the flexible-price labor share. In other words, $\Gamma_{N,t}$ measures how far the MRS is from the MPL.

Log-linearizing (B.22) around the steady state yields $\widehat{\Gamma}_{N,t}(\xi_t) = (1+\varphi)\widehat{n}_t(\xi_t) - \widehat{\Lambda}^n_{N,t}(\xi_t)$. Now subtract the flexible-price counterpart. Since $\Lambda^n_{N,t}(\xi_t)$ is defined from the flexible-price allocation at the realized state ξ_t, the term $\widehat{\Lambda}^n_{N,t}(\xi_t)$ is the same object in both the sticky-price and flexible-price comparisons, so it cancels out: $\widehat{\Gamma}_{N,t}(\xi_t) - \widehat{\Gamma}^n_{N,t}(\xi_t) = (1+\varphi)\widetilde{n}_t(\xi_t)$. Using (B.19), this becomes $\widehat{\Gamma}_{N,t}(\xi_t) - \widehat{\Gamma}^n_{N,t}(\xi_t) = \frac{1+\varphi}{\Lambda_N}\widetilde{c}_t(\xi_t) + o(\|\widehat{\xi}_t\|)$.

Finally, in the flexible-price equilibrium there is no intratemporal distortion, so the labor wedge equals one state by state: $\Gamma^n_{N,t}(\xi_t) = 1$ for all ξ_t. Hence $\widehat{\Gamma}^n_{N,t}(\xi_t) = 0$, and therefore (B.17) is satisfied. □

On the one hand, equation (B.16) says that nominal rigidities do not change aggregate efficiency at first order. They distort the allocation, but they do not shift the aggregate production frontier to first order. That is why the efficiency wedge is the same in sticky and flexible prices at first order. Equivalently, the combination (B.19) means that, to first order, the sticky-price equilibrium moves (C_t, N_t) along the aggregate production frontier rather than moving the frontier itself.

Importantly, (B.19) should not be interpreted as an additional equilibrium restriction equating the consumption gap and the output gap in this SOE. In the equilibrium block, these objects generally differ because output is affected not only by domestic absorption but also by sectoral terms of trade and imported intermediate inputs. The statement is instead an

envelope result for the welfare expansion around the constrained efficient flexible-price allocation: along feasible sticky-price deviations from a given realized shock vector, the linear component of $\widetilde{c}_t - \Lambda_L \widetilde{y}_t$ is already absorbed by the flexible-price benchmark and therefore does not generate an independent first-order welfare term. Its welfare consequences appear through the remaining second-order aggregate-gap, dispersion, and misallocation terms.

On the other hand, equation (B.17) implies that sticky prices generate a first-order labor wedge. So, unlike the efficiency wedge, the labor wedge is the object that summarizes the aggregate distortion introduced by nominal rigidities.

After substituting the flexible-price labor-supply optimality condition (B.12), the labor share (B.13), and using $u^n_{C,t} C^n_t = 1 + O(\|\widehat{\xi}_t\|)$, equation (B.8) becomes

$$u_t - u^n_t = \widetilde{c}_t - \Lambda_N \widetilde{n}_t - \frac{1+\varphi}{2} \Lambda_N \widetilde{n}_t^2 + o(\|\widehat{\xi}_t\|^2). \tag{B.23}$$

Using (B.19), we can write

$$-\frac{1+\varphi}{2} \Lambda_N \widetilde{n}_t^2 = -\frac{1+\varphi}{2} \Lambda_N \left(\frac{1}{\Lambda_N} \widetilde{c}_t \right)^2 + o(\|\widehat{\xi}_t\|^2) = -\frac{1}{2} \frac{1+\varphi}{\Lambda_N} \widetilde{c}_t^2 + o(\|\widehat{\xi}_t\|^2). \tag{B.24}$$

Now solve (B.17) for the output gap:

$$\widetilde{c}_t(\xi_t) = \frac{\Lambda_N}{1+\varphi} \widehat{\Gamma}_{N,t}(\xi_t) + o(\|\widehat{\xi}_t\|). \tag{B.25}$$

Substituting (B.25) into (B.24), we obtain

$$-\frac{1+\varphi}{2} \Lambda_N \widetilde{n}_t^2 = -\frac{1}{2} \frac{\Lambda_N}{1+\varphi} \widehat{\Gamma}_{N,t}(\xi_t)^2 + o(\|\widehat{\xi}_t\|^2). \tag{B.26}$$

Putting (B.16) and (B.26) into (B.23), we get

$$u_t(\xi_t) - u^n_t(\xi_t) = \underbrace{\widehat{A}_{\text{agg},t}(\xi_t) - \widehat{A}^n_{\text{agg},t}(\xi_t)}_{\text{efficiency wedge component}} - \frac{1}{2} \underbrace{\frac{\Lambda_N}{1+\varphi} \widehat{\Gamma}_{N,t}(\xi_t)^2}_{\text{labor wedge component}} + o(\|\widehat{\xi}_t\|^2). \tag{B.27}$$

Equation (B.27) is the bridge between the Taylor expansion in (B.8) and the final quadratic welfare loss. The two parts have different meanings: sticky prices do not move aggregate efficiency at first order, but they do create a first-order labor wedge, and that is why welfare loss can be written as a quadratic function of $\widehat{\Gamma}_{N,t}$.

The labor wedge in ToT gaps and markups. We are now interested in obtaining $\widehat{\Gamma}_{N,t}(\xi_t)^2$ from (B.27). Notice that, since the labor wedge is squared, writing it in first-order terms suffices.

Writing the labor wedge (B.17) in terms of the output gap, using (A.31), we can write

$$\widehat{\Gamma}_{N,t}(\xi_t) = (1+\varphi)\widetilde{y}_t + o(\|\widehat{\xi}_t\|). \tag{B.28}$$

We can rewrite (A.50) as

$$\widetilde{y}_t = -\frac{1}{\varphi}\boldsymbol{\lambda}^\intercal \boldsymbol{\mu}_t - \frac{1}{\varphi}\boldsymbol{\lambda}^\intercal(\boldsymbol{I}-\Omega_H)\widetilde{\boldsymbol{s}}_t = \boldsymbol{m}^\top \boldsymbol{z}_t \tag{B.29}$$

where $\boldsymbol{m} = \begin{bmatrix} -\frac{1}{\varphi}\boldsymbol{\lambda}^\intercal & -\frac{1}{\varphi}\boldsymbol{\lambda}^\intercal(\boldsymbol{I}-\Omega_H) \end{bmatrix}^\top$ and $\boldsymbol{z}_t = \begin{bmatrix} \boldsymbol{\mu}_t & \widetilde{\boldsymbol{s}}_t \end{bmatrix}^\top$. At this point, note that $\boldsymbol{m}^\top \boldsymbol{z}_t$ is a scalar. Hence its square can be written as

$$\left(\boldsymbol{m}^\top \boldsymbol{z}_t\right)^2 = \boldsymbol{z}_t^\top \boldsymbol{m}\boldsymbol{m}^\top \boldsymbol{z}_t. \tag{B.30}$$

Squaring (B.28) thus gives

$$\widehat{\Gamma}_{N,t}(\xi_t)^2 = (1+\varphi)^2\, \boldsymbol{z}_t^\top \boldsymbol{m}\boldsymbol{m}^\top \boldsymbol{z}_t + o(\|\widehat{\xi}_t\|^2). \tag{B.31}$$

Finally, multiply both sides by $\frac{\Lambda_N}{1+\varphi}$:

$$\frac{\Lambda_N}{1+\varphi}\widehat{\Gamma}_{N,t}(\xi_t)^2 = \Lambda_N(1+\varphi)\boldsymbol{z}_t^\top \boldsymbol{m}\boldsymbol{m}^\top \boldsymbol{z}_t + o(\|\widehat{\xi}_t\|^2). \tag{B.32}$$

Equation (B.32) states that the welfare-relevant labor-wedge term depends on markup wedges and ToT gaps through the single scalar projection $\boldsymbol{m}^\top \boldsymbol{z}_t$. The associated quadratic matrix is the rank-one positive semidefinite matrix $\boldsymbol{m}\boldsymbol{m}^\top$. Thus this component of welfare penalizes the particular combination of sectoral markups and ToT gaps that moves the aggregate labor wedge.

Substituting (B.32) into (B.27) yields

$$u_t(\xi_t) - u_t^n(\xi_t) = \widehat{A}_{\text{agg},t}(\xi_t) - \widehat{A}^n_{\text{agg},t}(\xi_t) - \frac{1}{2}\Lambda_N(1+\varphi)\boldsymbol{z}_t^\top \boldsymbol{m}\boldsymbol{m}^\top \boldsymbol{z}_t + o(\|\widehat{\xi}_t\|^2). \tag{B.33}$$

The efficiency wedge in ToT gaps and markups. We now turn to the efficiency-wedge component. To do so, it is useful to construct an equivalent economy in order to express that first term as a second-order quadratic form in the markup wedge and the sectoral ToT $\left(\widehat{\boldsymbol{\mu}}_t(\xi_t), \widetilde{\boldsymbol{s}}_t(\xi_t)\right)$.

We follow Qiu *et al.* (2026) and we construct an *equivalent economy* that satisfies all equilibrium conditions of the original economy except that the sectoral markups of sticky-price firms are now imposed directly so as to reproduce exactly those realized sectoral markup wedges. In particular, for each sector i, the markup of sticky-price firms in the equivalent

economy is chosen so that

$$\theta_i + (1-\theta_i)\mu_{if,t}^{1-\epsilon_i^p} = [\mu_{it}(\xi_t)]^{1-\epsilon_i^p}, \tag{B.34}$$

where $\widehat{\mu}_{it}(\xi_t)$ is the markup wedge of sector i in the sticky-price equilibrium.

In the original sticky-price equilibrium, the realized shocks ξ_t generate a vector of sectoral markup wedges $\left(\widehat{\boldsymbol{\mu}}_t(\xi_t), \widetilde{\boldsymbol{s}}_t(\xi_t)\right)$, and those wedges are the objects that summarize the sectoral distortions relevant for welfare. The equivalent economy simply takes those same realized markup wedges $\left(\widehat{\boldsymbol{\mu}}_t(\xi_t), \widetilde{\boldsymbol{s}}_t(\xi_t)\right)$ and treats them as the primitive distortion to be fed into the equilibrium system, holding the realized exogenous state fixed. With the state fixed, this auxiliary economy reproduces the same equilibrium allocations and prices as the sticky-price equilibrium under ξ_t. The subsequent step that sets the shock argument to zero is weaker: it establishes the second-order welfare equivalence in Lemma A2, not an exact replication of the shocked allocation.

The next lemma supplies the remaining step: it shows that the *total* welfare loss can be written as a quadratic function of the markup wedge vector. Combined with equation (B.32), which showed that the *labor-wedge component* is also a quadratic function of $\boldsymbol{z}_t(\widehat{\xi}_t)$. It follows that the remaining term,

$$\widehat{A}_{\text{agg},t}(\widehat{\xi}_t) - \widehat{A}^n_{\text{agg},t}(\widehat{\xi}_t), \tag{B.35}$$

must itself be a quadratic form in $\boldsymbol{z}_t(\widehat{\xi}_t)$.

LEMMA A2. *Let $\boldsymbol{z}_t = \left(\widehat{\boldsymbol{\mu}}_t(\xi_t), \widetilde{\boldsymbol{s}}_t(\xi_t)\right)$ denote the vector of realized sectoral markup wedges in the sticky-price equilibrium under shocks $\widehat{\xi}_t$. Up to second order, the welfare loss in the sticky-price equilibrium under any shock $\widehat{\xi}_t$ is equal to the welfare loss in the equivalent economy under the same sectoral markup wedges $\boldsymbol{z}_t$ and with the shock vector set to zero:*

$$u_t\left(\boldsymbol{z}_t, \widehat{\xi}_t\right) - u_t(0, \widehat{\xi}_t) = u_t(\boldsymbol{z}_t, 0) - u_t(0,0) + o(\|\widehat{\xi}_t\|^2) \tag{B.36}$$

$$= \boldsymbol{z}_t(\widehat{\xi}_t)^\top \mathbf{L}^u_{zz} \boldsymbol{z}_t(\widehat{\xi}_t) + o(\|\widehat{\xi}_t\|^2). \tag{B.37}$$

which is, therefore, a function of only sectoral markup wedges and ToT $\boldsymbol{z}_t$.

PROOF. Define

$$\mathcal{W}_t(\boldsymbol{z}_t, \widehat{\xi}_t) \equiv u_t(\boldsymbol{z}_t, \widehat{\xi}_t) - u_t(0, \widehat{\xi}_t). \tag{B.38}$$

This object measures the welfare loss in the equivalent economy from imposing the markup

wedge vector $\boldsymbol{z}_t$, holding fixed the exogenous state $\widehat{\xi}_t$. Therefore,

$$\mathcal{W}_t(0, \widehat{\xi}_t) = u_t(0, \widehat{\xi}_t) - u_t(0, \widehat{\xi}_t) = 0 \qquad \text{for every } \widehat{\xi}_t. \tag{B.39}$$

The flexible-price equilibrium implements the optimal allocation for the domestic social planner. Therefore, once the exogenous state $\widehat{\xi}_t$ is fixed, the allocation corresponding to zero markup wedges,

$$\boldsymbol{z}_t = (\mathbf{0}, \mathbf{0}), \tag{B.40}$$

is the welfare-maximizing allocation in the equivalent economy. Hence, for every realized $\widehat{\xi}_t$,

$$u_t(\boldsymbol{z}_t, \widehat{\xi}_t) \leq u_t(0, \widehat{\xi}_t), \tag{B.41}$$

or equivalently,

$$\mathcal{W}_t(\boldsymbol{z}_t, \widehat{\xi}_t) \leq 0 \qquad \text{for all } (\boldsymbol{z}_t, \widehat{\xi}_t), \tag{B.42}$$

with equality at $\boldsymbol{z}_t = (\mathbf{0}, \mathbf{0})$.

As a consequence, the gradient of $\mathcal{W}_t$ with respect to $\boldsymbol{z}_t$ must vanish at the point $\boldsymbol{z}_t = (\mathbf{0}, \mathbf{0})$, for every $\widehat{\xi}_t$:

$$\frac{\partial \mathcal{W}_t}{\partial \boldsymbol{z}_t}(\mathbf{0}, \widehat{\xi}_t) = \mathbf{0} \qquad \text{for every } \widehat{\xi}_t. \tag{B.43}$$

We now perform a second-order expansion of $\mathcal{W}_t(\boldsymbol{z}_t, \widehat{\xi}_t)$ around the point

$$(\boldsymbol{z}_t, \widehat{\xi}_t) = (\mathbf{0}, \mathbf{0}, \mathbf{0}). \tag{B.44}$$

Because $\boldsymbol{z}_t(\widehat{\xi}_t) = O(\|\widehat{\xi}_t\|)$, both arguments are small when the realized shock vector is small. Therefore a second-order expansion is appropriate:

$$\mathcal{W}_t(\boldsymbol{z}_t, \widehat{\xi}_t) = \mathcal{W}_t(\mathbf{0}, \mathbf{0}, \mathbf{0}) + \left(\frac{\partial \mathcal{W}_t}{\partial \boldsymbol{z}_t}\right)_0^\top \boldsymbol{z}_t + \left(\frac{\partial \mathcal{W}_t}{\partial \widehat{\xi}_t}\right)_0^\top \widehat{\xi}_t + \boldsymbol{z}_t^\top \mathbf{L}_{zz}^u \boldsymbol{z}_t + \widehat{\xi}_t^\top \mathbf{L}_{\xi z}^u \boldsymbol{z}_t + \widehat{\xi}_t^\top \mathbf{L}_{\xi\xi}^u \widehat{\xi}_t + o(\|\widehat{\xi}_t\|^2). \tag{B.45}$$

where $\mathbf{L}_{zz}^u$ is the matrix collecting the pure second derivatives with respect to $\boldsymbol{z}_t$, $\mathbf{L}_{\xi z}^u$ collects the mixed second derivatives in $(\widehat{\xi}_t, \boldsymbol{z}_t)$, and $\mathbf{L}_{\xi\xi}^u$ collects the pure second derivatives with respect to $\widehat{\xi}_t$, where the matrices absorb the usual 1/2 constants from the Taylor formula. The expansion is joint in $(\boldsymbol{z}_t, \widehat{\xi}_t)$. Since $\mathcal{W}_t$ is twice continuously differentiable in a neighborhood

of the origin, the remainder is uniform in the joint norm; along equilibrium paths with $\boldsymbol{z}_t = O(\|\widehat{\xi}_t\|)$, it can be written as $o(\|\widehat{\xi}_t\|^2)$.

From (B.39), we in particular have

$$\mathcal{W}_t(\mathbf{0}, \mathbf{0}, \mathbf{0}) = 0. \tag{B.46}$$

Also, because $\mathcal{W}_t(\mathbf{0}, \mathbf{0}, \widehat{\xi}_t) = 0$ for every $\widehat{\xi}_t$, the derivative with respect to $\widehat{\xi}_t$ at the origin must vanish:

$$\left(\frac{\partial \mathcal{W}_t}{\partial \widehat{\xi}_t}\right)_0 = \mathbf{0}. \tag{B.47}$$

And from (B.43), evaluated at $\widehat{\xi}_t = \mathbf{0}$,

$$\left(\frac{\partial \mathcal{W}_t}{\partial \boldsymbol{z}_t}\right)_0 = \mathbf{0}. \tag{B.48}$$

Substituting (B.46)–(B.48) into (B.45), we obtain

$$\mathcal{W}_t(\boldsymbol{z}_t, \widehat{\xi}_t) = \boldsymbol{z}_t^\top \mathbf{L}^u_{zz} \boldsymbol{z}_t + \widehat{\xi}_t^\top \mathbf{L}^u_{\xi z} \boldsymbol{z}_t + \widehat{\xi}_t^\top \mathbf{L}^u_{\xi\xi} \widehat{\xi}_t + o(\|\widehat{\xi}_t\|^2). \tag{B.49}$$

Now use the stronger property (B.43). Differentiate (B.49) with respect to $\boldsymbol{z}_t$:

$$\frac{\partial \mathcal{W}_t}{\partial \boldsymbol{z}_t}(\boldsymbol{z}_t, \widehat{\xi}_t) = \left(\mathbf{L}^u_{zz} + (\mathbf{L}^u_{zz})^\top\right) \boldsymbol{z}_t + (\mathbf{L}^u_{\xi z})^\top \widehat{\xi}_t + o(\|\widehat{\xi}_t\|). \tag{B.50}$$

Evaluating at $\boldsymbol{z}_t = (\mathbf{0}, \mathbf{0})$ gives

$$\frac{\partial \mathcal{W}_t}{\partial \boldsymbol{z}_t}(\mathbf{0}, \mathbf{0}, \widehat{\xi}_t) = (\mathbf{L}^u_{\xi z})^\top \widehat{\xi}_t + o(\|\widehat{\xi}_t\|). \tag{B.51}$$

But (B.43) says that the left-hand side is identically zero for every $\widehat{\xi}_t$. Therefore the coefficient on the first-order term $\widehat{\xi}_t$ must be zero:

$$\mathbf{L}^u_{\xi z} = \mathbf{0}. \tag{B.52}$$

Next use (B.39), which states that $\mathcal{W}_t(\mathbf{0}, \mathbf{0}, \widehat{\xi}_t) = 0$ for every $\widehat{\xi}_t$. Set $\boldsymbol{z}_t = (\mathbf{0}, \mathbf{0})$ in (B.49). Using (B.52), we obtain

$$\mathcal{W}_t(\mathbf{0}, \mathbf{0}, \widehat{\xi}_t) = \widehat{\xi}_t^\top \mathbf{L}^u_{\xi\xi} \widehat{\xi}_t + o(\|\widehat{\xi}_t\|^2). \tag{B.53}$$

But the left-hand side is exactly zero for every $\widehat{\xi}_t$, by (B.39). Therefore the quadratic form in

$\widehat{\xi}_t$ must vanish identically:

$$\mathbf{L}^u_{\xi\xi} = \mathbf{0}. \tag{B.54}$$

Substituting (B.52) and (B.54) into (B.49), we arrive at

$$\mathcal{W}_t(\boldsymbol{z}_t, \widehat{\xi}_t) = \boldsymbol{z}_t^\top \mathbf{L}^u_{zz} \boldsymbol{z}_t + o(\|\widehat{\xi}_t\|^2). \tag{B.55}$$

Recalling the definition (B.38), this means

$$u_t(\boldsymbol{z}_t, \widehat{\xi}_t) - u_t(0, \widehat{\xi}_t) = \boldsymbol{z}_t^\top \mathbf{L}^u_{zz} \boldsymbol{z}_t + o(\|\widehat{\xi}_t\|^2). \tag{B.56}$$

which is exactly the second equality in (B.37).

Now evaluate (B.55) at $\widehat{\xi}_t = \mathbf{0}$, holding fixed the realized wedge vector $\boldsymbol{z}_t = \boldsymbol{z}_t(\widehat{\xi}_t)$:

$$\mathcal{W}_t(\boldsymbol{z}_t, \mathbf{0}) = \boldsymbol{z}_t^\top \mathbf{L}^u_{zz} \boldsymbol{z}_t + o(\|\widehat{\xi}_t\|^2). \tag{B.57}$$

Using again the definition of $\mathcal{W}_t$, we have

$$u_t(\boldsymbol{z}_t, 0) - u_t(0, 0) = \boldsymbol{z}_t^\top \mathbf{L}^u_{zz} \boldsymbol{z}_t + o(\|\widehat{\xi}_t\|^2). \tag{B.58}$$

Comparing (B.56) and (B.58), we conclude that

$$u_t(\boldsymbol{z}_t, \widehat{\xi}_t) - u_t(0, \widehat{\xi}_t) = u_t(\boldsymbol{z}_t, 0) - u_t(0, 0) + o(\|\widehat{\xi}_t\|^2), \tag{B.59}$$

which is exactly the first equality in (B.36).

Finally, substituting $\boldsymbol{z}_t = \boldsymbol{z}_t(\widehat{\xi}_t)$ into (B.56) and (B.59), we obtain the full statement (B.36)-(B.37). □

The proof rests on two results. First, the flexible-price equilibrium is efficient for every realized exogenous state. Therefore, once the shocks are fixed, the point $\boldsymbol{z}_t = \mathbf{0}$ is a welfare maximum, which kills all first-order effects of markup wedges and also kills the mixed $(\widehat{\xi}_t, \boldsymbol{z}_t)$ term. Second, the welfare loss is defined relative to the flexible-price allocation at the *same* exogenous state. Therefore, when $\boldsymbol{z}_t = \mathbf{0}$, the welfare loss is exactly zero for every $\widehat{\xi}_t$. This kills all pure shock terms. So the only thing that can remain up to second order is the quadratic effect of markup wedges themselves.

B.3. The equivalent economy

From Lemma A2 onward, the appendix works in the *equivalent economy without realized shocks*, and treats the realized markup wedge vector as the primitive distortion. So we now write

$$\boldsymbol{z}_t \equiv \boldsymbol{z}_t(\widehat{\xi}_t), \tag{B.60}$$

and for any endogenous variable X_t,

$$\widehat{x}_t(\boldsymbol{z}_t) \equiv \widehat{x}_t(\boldsymbol{z}_t, 0) - \widehat{x}_t(0, 0). \tag{B.61}$$

Up to a first-order approximation,

$$\widehat{x}_t(\boldsymbol{z}_t, 0) - \widehat{x}_t(0, 0) = \widehat{x}_t(\boldsymbol{z}_t, \widehat{\xi}_t) - \widehat{x}_t(0, \widehat{\xi}_t) + o(\|\widehat{\xi}_t\|) = \widetilde{x}_t(\widehat{\xi}_t) + o(\|\widehat{\xi}_t\|). \tag{B.62}$$

So whenever only first-order accuracy is needed, the appendix replaces the gap notation by the no-shock equivalent-economy notation.

We also define the *percentage deviation from steady state* by

$$\widehat{\Delta X}_t(\boldsymbol{z}_t) \equiv \frac{X_t(\boldsymbol{z}_t) - X_{ss}}{X_{ss}}. \tag{B.63}$$

This differs from the log deviation

$$\widehat{x}_t(\boldsymbol{z}_t) = \log X_t(\boldsymbol{z}_t) - \log X_{ss}. \tag{B.64}$$

The two are related by

$$\widehat{x}_t(\boldsymbol{z}_t) = \log\Big(1 + \widehat{\Delta X}_t(\boldsymbol{z}_t)\Big) = \widehat{\Delta X}_t(\boldsymbol{z}_t) - \frac{1}{2}\widehat{\Delta X}_t(\boldsymbol{z}_t)^2 + o(\|\boldsymbol{z}_t\|^2). \tag{B.65}$$

The efficiency wedge. The efficiency-wedge component in the shockless equivalent economy is

$$\widehat{c}_t(\boldsymbol{z}_t) - \Lambda_N \widehat{n}_t(\boldsymbol{z}_t). \tag{B.66}$$

Using (B.65) for C_t and N_t,

$$\widehat{c}_t(\boldsymbol{z}_t) - \Lambda_N \widehat{n}_t(\boldsymbol{z}_t) = \widehat{\Delta C}_t(\boldsymbol{z}_t) - \Lambda_N \widehat{\Delta N}_t(\boldsymbol{z}_t) - \frac{1}{2}\widehat{\Delta C}_t(\boldsymbol{z}_t)^2 + \frac{1}{2}\Lambda_N \widehat{\Delta N}_t(\boldsymbol{z}_t)^2 + o(\|\boldsymbol{z}_t\|^2). \tag{B.67}$$

But from the first-order efficiency-wedge relation, in the shockless equivalent economy, $\widehat{c}_t(\boldsymbol{z}_t) - \Lambda_N \widehat{n}_t(\boldsymbol{z}_t) = o(\|\boldsymbol{z}_t\|)$, so, to first order, $\widehat{\Delta C}_t(\boldsymbol{z}_t) = \Lambda_N \widehat{\Delta N}_t(\boldsymbol{z}_t) + o(\|\boldsymbol{z}_t\|)$. Therefore, inside the quadratic term, we may replace

$$\widehat{\Delta C}_t(\boldsymbol{z}_t)^2 = \Lambda_N^2 \widehat{\Delta N}_t(\boldsymbol{z}_t)^2 + o(\|\boldsymbol{z}_t\|^2). \tag{B.68}$$

Substituting (B.68) into (B.67) yields $\widehat{c}_t(\boldsymbol{z}_t) - \Lambda_N \widehat{n}_t(\boldsymbol{z}_t) = \widehat{\Delta C}_t(\boldsymbol{z}_t) - \Lambda_N \widehat{\Delta N}_t(\boldsymbol{z}_t) + \frac{1}{2}\Lambda_N(1 - \Lambda_N)\widehat{\Delta N}_t(\boldsymbol{z}_t)^2 + o(\|\boldsymbol{z}_t\|^2)$. Since $\widehat{\Delta N}_t(\boldsymbol{z}_t)^2 = \widehat{n}_t(\boldsymbol{z}_t)^2 + o(\|\boldsymbol{z}_t\|^2)$, we can write this as

$$\widehat{c}_t(\boldsymbol{z}_t) - \Lambda_N \widehat{n}_t(\boldsymbol{z}_t) = \widehat{\Delta C}_t(\boldsymbol{z}_t) - \Lambda_N \widehat{\Delta N}_t(\boldsymbol{z}_t) + \frac{1}{2}\Lambda_N(1 - \Lambda_N)\widehat{n}_t(\boldsymbol{z}_t)^2 + o(\|\boldsymbol{z}_t\|^2). \tag{B.69}$$

Aggregate consumption. From the household side, domestic aggregate consumption is the Cobb–Douglas aggregate of sectoral consumptions, given by (27). Taking logs and deviations from steady state, $\widehat{c}_t = \sum_{i=1}^I \beta_i \widehat{c}_{it}$. Now use (B.65): $\widehat{\Delta C}_t - \frac{1}{2}\widehat{\Delta C}_t^2 = \sum_{i=1}^I \beta_i \left(\widehat{\Delta C}_{it} - \frac{1}{2}\widehat{\Delta C}_{it}^2\right) + o(\|\boldsymbol{z}_t\|^2)$. Rearranging,

$$\widehat{\Delta C}_t = \sum_{i=1}^I \beta_i \widehat{\Delta C}_{it} - \frac{1}{2}\left[\sum_{i=1}^I \beta_i \widehat{\Delta C}_{it}^2 - \widehat{\Delta C}_t^2\right] + o(\|\boldsymbol{z}_t\|^2). \tag{B.70}$$

Let $\bar{c}_t \equiv \sum_{i=1}^I \beta_i \widehat{\Delta C}_{it}$. Since $\widehat{\Delta C}_t = \bar{c}_t + o(\|\boldsymbol{z}_t\|)$, the bracketed term in (B.70) can be written as $\sum_{i=1}^I \beta_i \widehat{\Delta C}_{it}^2 - \widehat{\Delta C}_t^2 = \sum_{i=1}^I \beta_i \left(\widehat{\Delta C}_{it} - \widehat{\Delta C}_t\right)^2 + o(\|\boldsymbol{z}_t\|^2)$. Therefore,

$$\widehat{\Delta C}_t(\boldsymbol{z}_t) = \sum_{i=1}^I \beta_i \widehat{\Delta C}_{it}(\boldsymbol{z}_t) - \frac{1}{2}\sum_{i=1}^I \beta_i \left(\widehat{\Delta C}_{it}(\boldsymbol{z}_t) - \widehat{\Delta C}_t(\boldsymbol{z}_t)\right)^2 + o(\|\boldsymbol{z}_t\|^2). \tag{B.71}$$

Equation (B.71) says that aggregate consumption is, to second order, the weighted average of sectoral consumption deviations, minus a dispersion term. The dispersion term is the usual Jensen term that appears because the aggregate consumption bundle is Cobb–Douglas in $\{C_{it}\}$.

Domestic and foreign consumption. Within sector i, the household combines domestic and imported goods through a Cobb-Douglas aggregator given by (27). Taking logs and deviations from steady state, $\widehat{c}_{it} = (1 - \zeta_i)\widehat{c}_{it}^H + \zeta_i \widehat{c}_{it}^F$. Now use (B.65): $\widehat{\Delta C}_{it} - \frac{1}{2}\widehat{\Delta C}_{it}^2 = (1 - \zeta_i)\left(\widehat{\Delta C}_{it}^H - \frac{1}{2}(\widehat{\Delta C}_{it}^H)^2\right) + \zeta_i \left(\widehat{\Delta C}_{it}^F - \frac{1}{2}(\widehat{\Delta C}_{it}^F)^2\right) + o(\|\boldsymbol{z}_t\|^2)$. Rearranging,

$$\widehat{\Delta C}_{it} = (1 - \zeta_i)\widehat{\Delta C}_{it}^H + \zeta_i \widehat{\Delta C}_{it}^F - \frac{1}{2}\left[(1 - \zeta_i)(\widehat{\Delta C}_{it}^H)^2 + \zeta_i(\widehat{\Delta C}_{it}^F)^2 - \widehat{\Delta C}_{it}^2\right] + o(\|\boldsymbol{z}_t\|^2). \tag{B.72}$$

Using the two-point variance identity,

$$(1-\zeta_i)a^2+\zeta_i b^2-\big((1-\zeta_i)a+\zeta_i b\big)^2=\zeta_i(1-\zeta_i)(a-b)^2, \tag{B.73}$$

and replacing $\widehat{\Delta C}_{it}$ by its first-order average inside the square, we obtain

$$\widehat{\Delta C}_{it}(\boldsymbol{z}_t)=(1-\zeta_i)\widehat{\Delta C}^H_{it}(\boldsymbol{z}_t)+\zeta_i\widehat{\Delta C}^F_{it}(\boldsymbol{z}_t)-\frac{1}{2}\zeta_i(1-\zeta_i)\Big(\widehat{\Delta C}^H_{it}(\boldsymbol{z}_t)-\widehat{\Delta C}^F_{it}(\boldsymbol{z}_t)\Big)^2+o(\|\boldsymbol{z}_t\|^2). \tag{B.74}$$

Equation (B.74) says that sectoral consumption is the weighted average of domestic and imported consumption deviations, minus a cross-border misallocation term. The misallocation term is proportional to the square of the wedge between domestic and imported uses within the same sector.

Sectoral intermediate input index. From the firm side, domestic sectoral intermediate input index is the Cobb–Douglas aggregate of sectoral intermediate inputs, given by (28). Taking logs and deviations from steady state, $\widehat{x}_{it}=\sum_{j=1}^I \nu_{ij}\widehat{x}_{ij,t}$. Now use (B.65): $\widehat{\Delta X}_{it}-\frac{1}{2}\widehat{\Delta X}^2_{it}=\sum_{j=1}^I\nu_{ij}\left(\widehat{\Delta X}_{ij,t}-\frac{1}{2}\widehat{\Delta X}^2_{ij,t}\right)+o(\|\boldsymbol{z}_t\|^2)$. Rearranging,

$$\widehat{\Delta X}_{it}=\sum_{j=1}^I\nu_{ij}\widehat{\Delta X}_{ij,t}-\frac{1}{2}\left[\sum_{j=1}^I\nu_{ij}\widehat{\Delta X}^2_{ij,t}-\widehat{\Delta X}^2_{it}\right]+o(\|\boldsymbol{z}_t\|^2). \tag{B.75}$$

Let $\bar{x}_{it}\equiv\sum_{j=1}^I\nu_{ij}\widehat{\Delta X}_{ij,t}$. Since $\widehat{\Delta X}_{it}=\bar{x}_{it}+o(\|\boldsymbol{z}_t\|)$, the bracketed term in (B.75) can be written as $\sum_{j=1}^I\nu_{ij}\widehat{\Delta X}^2_{ij,t}-\widehat{\Delta X}^2_{it}=\sum_{j=1}^I\nu_{ij}\big(\widehat{\Delta X}_{ij,t}-\widehat{\Delta X}_{it}\big)^2+o(\|\boldsymbol{z}_t\|^2)$. Therefore,

$$\widehat{\Delta X}_{it}(\boldsymbol{z}_t)=\sum_{j=1}^I\nu_{ij}\widehat{\Delta X}_{ij,t}(\boldsymbol{z}_t)-\frac{1}{2}\sum_{j=1}^I\nu_{ij}\Big(\widehat{\Delta X}_{ij,t}(\boldsymbol{z}_t)-\widehat{\Delta X}_{it}(\boldsymbol{z}_t)\Big)^2+o(\|\boldsymbol{z}_t\|^2). \tag{B.76}$$

Equation (B.76) says that sectoral intermediate inputs are, to second order, the weighted average of sectoral intermediate inputs deviations, minus a dispersion term. The dispersion term is the usual Jensen term that appears because the intermediate input index bundle is Cobb–Douglas in $\{X_{ij,t}\}$.

Domestic and foreign intermediate inputs. Similarly, the composite intermediate input from sector j used by sector i is given by (28). Within sector i, the firm combines domestic and imported intermediate inputs through a Cobb-Douglas aggregator given by (28). Taking logs

and deviations from steady state, $\widehat{x}_{ij,t} = (1-\zeta_{ij})\widehat{x}^H_{ij,t} + \zeta_{ij}\widehat{x}^F_{ij,t}$. Now use (B.65): $\widehat{\Delta X}_{ij,t} - \frac{1}{2}\widehat{\Delta X}^2_{ij,t} = (1-\zeta_{ij})\left(\widehat{\Delta X}^H_{ij,t} - \frac{1}{2}(\widehat{\Delta X}^H_{ij,t})^2\right) + \zeta_{ij}\left(\widehat{\Delta X}^F_{ij,t} - \frac{1}{2}(\widehat{\Delta X}^F_{ij,t})^2\right) + o(\|\boldsymbol{z}_t\|^2)$. Rearranging,

(B.77)
$$\widehat{\Delta X}_{ij,t} = (1-\zeta_{ij})\widehat{\Delta X}^H_{ij,t} + \zeta_{ij}\widehat{\Delta X}^F_{ij,t} - \frac{1}{2}\left[(1-\zeta_{ij})(\widehat{\Delta X}^H_{ij,t})^2 + \zeta_{ij}(\widehat{\Delta X}^F_{ij,t})^2 - \widehat{\Delta X}^2_{ij,t}\right] + o(\|\boldsymbol{z}_t\|^2).$$

Using the two-point variance identity (B.73), and replacing $\widehat{\Delta X}_{ij,t}$ by its first-order average inside the square, we obtain

(B.78)
$$\widehat{\Delta X}_{ij,t}(\boldsymbol{z}_t) = (1-\zeta_{ij})\widehat{\Delta X}^H_{ij,t}(\boldsymbol{z}_t) + \zeta_{ij}\widehat{\Delta X}^F_{ij,t}(\boldsymbol{z}_t) - \frac{1}{2}\zeta_{ij}(1-\zeta_{ij})\left(\widehat{\Delta X}^H_{ij,t}(\boldsymbol{z}_t) - \widehat{\Delta X}^F_{ij,t}(\boldsymbol{z}_t)\right)^2 + o(\|\boldsymbol{z}_t\|^2).$$

Equation (B.78) says that intermediate input index is the weighted average of domestic and imported intermediate input deviations, minus a cross-border misallocation term. The misallocation term is proportional to the square of the wedge between domestic and imported uses within the same sector.

Aggregate labor. Aggregate labor is the sum of sectoral labor (19), which we can write as

(B.79)
$$N\widehat{\Delta N}_t = \sum_{i=1}^{I} N_i \widehat{\Delta N}_{it}.$$

Now use the steady-state identity $WN_i = \alpha_i P_i^H Y_i$, which follows from Cobb–Douglas cost minimization, together with $\lambda_i \equiv \frac{P_i^H Y_i}{P^C C}$ and $\Lambda_N \equiv \frac{WN}{P^C C}$. Hence,

(B.80)
$$\frac{N_i}{N} = \frac{\alpha_i \lambda_i}{\Lambda_N}.$$

Substituting (B.80) into (B.79) gives

(B.81)
$$\Lambda_N \widehat{\Delta N}_t(\boldsymbol{z}_t) = \sum_{i=1}^{I} \lambda_i \alpha_i \widehat{\Delta N}_{it}(\boldsymbol{z}_t).$$

Aggregate labor is a weighted average of sectoral labor, where the weights are the sectoral contributions to aggregate labor income.

Production. At the firm level, all firms in sector i share the same CRS Cobb–Douglas production function and face the same input prices. Therefore, cost minimization implies that all firms in sector i choose the same input ratios. As a result, if we denote $N_{it} \equiv \int_0^1 N_{if,t}\, df$ and $X_{it} \equiv \int_0^1 X_{if,t}\, df$, then aggregating firm-level production gives

$$\int_0^1 Y_{if,t}\, df = F_i(N_{it}, X_{it}). \tag{B.82}$$

Because sectoral output Y_{it} is the CES aggregate of differentiated firm outputs, it differs from the simple integral $\int_0^1 Y_{if,t}\, df$ whenever there is within-sector dispersion. Define

$$\iota_{it} \equiv \frac{Y_{it}}{\int_0^1 Y_{if,t}\, df}. \tag{B.83}$$

Then (B.82) becomes

$$Y_{it} = \iota_{it} F_i\big(L_{it}, \{X_{ij,t}\}_{j=1}^I\big), \tag{B.84}$$

Combining (28) and (30), (B.84) becomes

$$Y_{it} = \iota_{it} \left(N_{it}\right)^{\alpha_i} \prod_{j=1}^{I} \left(X_{ij,t}\right)^{(1-\alpha_i)\nu_{ij}}. \tag{B.85}$$

Taking logs,

$$\widehat{Y}_{it} = \widehat{\iota}_{it} + \alpha_i \widehat{n}_{it} + \sum_{j=1}^{I} (1-\alpha_i)\nu_{ij} \widehat{x}_{ij,t}. \tag{B.86}$$

Because at the efficient steady state there is no within-sector dispersion, $\iota_i = 1$. Around that point, price dispersion is a second-order object in relative-price gaps, so $\widehat{\iota}_{it} = O(\|\boldsymbol{z}_t\|^2)$.

Now replace the log deviations of Y_{it}, N_{it}, and X_{it} by percentage deviations: $\widehat{\Delta Y}_{it} - \frac{1}{2}\widehat{\Delta Y}_{it}^2 = \widehat{\iota}_{it} + \alpha_i \left(\widehat{\Delta N}_{it} - \frac{1}{2}\widehat{\Delta N}_{it}^2\right) + \sum_{j=1}^I (1-\alpha_i)\nu_{ij} \left(\widehat{\Delta X}_{ij,t} - \frac{1}{2}\widehat{\Delta X}_{ij,t}^2\right) + o(\|\boldsymbol{z}_t\|^2)$. Rearranging, $\widehat{\Delta Y}_{it} = \widehat{\iota}_{it} + \alpha_i \widehat{\Delta N}_{it} + \sum_{j=1}^I (1-\alpha_i)\nu_{ij} \widehat{\Delta X}_{ij,t} - \frac{1}{2}\left[\alpha_i \widehat{\Delta N}_{it}^2 + \sum_{j=1}^I (1-\alpha_i)\nu_{ij} \widehat{\Delta X}_{ij,t}^2 - \widehat{\Delta Y}_{it}^2\right] + o(\|\boldsymbol{z}_t\|^2)$. Using the weighted-variance identity, $\alpha_i \widehat{\Delta N}_{it}^2 + \sum_{j=1}^I (1-\alpha_i)\nu_{ij} \widehat{\Delta X}_{ij,t}^2 - \widehat{\Delta Y}_{it}^2 = \alpha_i \big(\widehat{\Delta N}_{it} - \widehat{\Delta Y}_{it}\big)^2 + \sum_{j=1}^I (1-\alpha_i)\nu_{ij} \big(\widehat{\Delta X}_{ij,t} - \widehat{\Delta Y}_{it}\big)^2 + o(\|\boldsymbol{z}_t\|^2)$. Hence,

$$\widehat{\Delta Y}_{it}(\boldsymbol{z}_t) = \widehat{\iota}_{it}(\boldsymbol{z}_t) + \alpha_i \widehat{\Delta N}_{it}(\boldsymbol{z}_t) + \sum_{j=1}^{I} (1-\alpha_i)\nu_{ij} \widehat{\Delta X}_{ij,t}(\boldsymbol{z}_t)$$

$$-\frac{1}{2}\left[\alpha_i\left(\widehat{\Delta N}_{it}(\boldsymbol{z}_t)-\widehat{\Delta Y}_{it}(\boldsymbol{z}_t)\right)^2+\sum_{j=1}^{I}(1-\alpha_i)\nu_{ij}\left(\widehat{\Delta X}_{ij,t}(\boldsymbol{z}_t)-\widehat{\Delta Y}_{it}(\boldsymbol{z}_t)\right)^2\right]+o(\|\boldsymbol{z}_t\|^2). \tag{B.87}$$

Equation (B.87) says that sectoral output is driven by within-sector dispersion $\widehat{\iota}_{it}$, plus the weighted average of sectoral labor and intermediate-input deviations, minus a misallocation term. The latter is zero only if every input used by sector i moves proportionally with Y_{it}.

Sectoral goods market clearing. Sector i's good market clearing condition is given by (18), which can be written as $Y_{it}=C_{it}^H+\sum_{j=1}^I X_{ji,t}^H+Y_{it}^X$, where $Y_{it}^X=C_{it}^{H,*}+\sum_{j=1}^I X_{ji,t}^{H,*}$ denotes sectoral exports. Subtracting its steady state counterpart, $Y_{it}-Y_i=C_{it}^H-C_i^H+\sum_{j=1}^I(X_{ji,t}^H-X_{ji}^H)+Y_{it}^X-Y_i^X$. Rewriting the expression using percentage deviations, we obtain $Y_i\widehat{\Delta Y_{it}}=C_i^H\widehat{\Delta C_{it}^H}+\sum_{j=1}^I X_{ji}^H\widehat{\Delta X_{ji,t}^H}+Y_i^X\widehat{\Delta Y_{it}^X}$. Multiplying both sides by $P_i^H/(P^Y Y)$, we can write

$$\lambda_i\widehat{\Delta Y_{it}}=\beta_i(1-\zeta_i)\widehat{\Delta C_{it}^H}+\sum_{j=1}^{I}\lambda_j(1-\alpha_j)\nu_i(1-\zeta_{ji})\widehat{\Delta X_{ji,t}^H}+\gamma_i^X\widehat{\Delta Y_{it}^X} \tag{B.88}$$

where $P_i^H C_i^H/(P^Y Y)=\beta_i(1-\zeta_i)$, $P_i^H X_{ji,t}^H/(P^Y Y)=\lambda_j(1-\alpha_j)\nu_i(1-\zeta_{ji})$, and $\gamma_i^X=P_i^H Y_i^X/(P^Y Y)$.

Sector i's output deviation is allocated across three uses: domestic final consumption, domestic intermediate use, and exports. The coefficients are exactly the steady-state shares of those three uses in aggregate GDP.

Real GDP and trade balance. Real GDP is given by (24), which can be written as

$$P^Y Y_t=P^C C_t+\sum_{i=1}^{I}P_i^H Y_{it}^X-\sum_{i=1}^{I}P_i^F Y_{it}^M, \tag{B.89}$$

where

$$Y_{it}^M=C_{it}^F+\sum_{j=1}^{I}X_{ji,t}^F \tag{B.90}$$

denotes sectoral imports. Substracting its steady-state counterpart, we can write $Y_{it}^M-Y_i^M=C_{it}^F-C_i^F+\sum_{j=1}^I(X_{ji,t}^F-X_{ji}^F)$. In percentage deviation from steady state, $Y_i^M\widehat{\Delta Y_{it}^M}=C_i^F\widehat{\Delta C_{it}^F}+$

$\sum_{j=1}^{I} X_{ji}^F \widehat{\Delta X_{ji,t}^F}$. Multiplying by $P_i^F/(P^Y Y)$,

$$\gamma_i^M \widehat{\Delta Y_{it}^M} = \beta_i \zeta_i \widehat{\Delta C_{it}^F} + \sum_{j=1}^{I} \lambda_j (1-\alpha_j) \nu_i \zeta_{ji} \widehat{\Delta X_{ji,t}^F} \tag{B.91}$$

where $\gamma_i^M = P_i^F Y_i^M/(P^Y Y)$, $P_i^F C_i^F/(P^Y Y) = \beta_i \zeta_i$, and $P_i^F X_{ji}^F/(P^Y Y) = \lambda_j(1-\alpha_j)\nu_i \zeta_{ji}$.

Similarly, subtracting the steady-state counterpart in (B.89), we can write $P^Y(Y_t - Y) = P^C(C_t - C) + \sum_{i=1}^{I} P_i^H (Y_{it}^X - Y_i^X) - \sum_{i=1}^{I} P_i^F (Y_{it}^M - Y_i^M)$, which can be rewritten in percentage deviations as $P^Y Y \widehat{\Delta Y_t} = P^C C \widehat{\Delta C_t} + \sum_{i=1}^{I} P_i^H Y_i^X \widehat{\Delta Y_{it}^X} - \sum_{i=1}^{I} P_i^F Y_i^M \widehat{\Delta Y_{it}^M}$. Dividing by nominal GDP, we can write

$$\widehat{\Delta Y_t} = \widehat{\Delta C_t} + \sum_{i=1}^{I} \gamma_i^X \widehat{\Delta Y_{it}^X} - \sum_{i=1}^{I} \gamma_i^M \widehat{\Delta Y_{it}^M} \tag{B.92}$$

where we used $P^C C = P^Y Y$. Inserting (B.91) in the rightmost term,

$$\widehat{\Delta Y_t} = \widehat{\Delta C_t} + \sum_{i=1}^{I} \gamma_i^X \widehat{\Delta Y_{it}^X} - \sum_{i=1}^{I} \beta_i \zeta_i \widehat{\Delta C_{it}^F} - \sum_{i=1}^{I} \sum_{j=1}^{I} \lambda_j (1-\alpha_j) \nu_i \zeta_{ji} \widehat{\Delta X_{ji,t}^F} \tag{B.93}$$

Equation (B.93) is the open-economy-specific feasibility condition. It says that imports are used for two purposes only: household consumption and imported intermediate inputs.

Combining all equations. Inserting (B.74) into (B.71),

$$\begin{aligned}\widehat{\Delta C_t} &= \sum_{i=1}^{I} \beta_i \left[(1-\zeta_i) \widehat{\Delta C_{it}^H} + \zeta_i \widehat{\Delta C_{it}^F} - \frac{1}{2} \zeta_i (1-\zeta_i) \left(\widehat{\Delta C_{it}^H} - \widehat{\Delta C_{it}^F} \right)^2 \right] - \frac{1}{2} \sum_{i=1}^{I} \beta_i \left(\widehat{\Delta C_{it}} - \widehat{\Delta C_t} \right)^2 + o(\|\boldsymbol{z}_t\|^2) \\ &= \sum_{i=1}^{I} \beta_i \left[(1-\zeta_i) \widehat{\Delta C_{it}^H} + \zeta_i \widehat{\Delta C_{it}^F} \right] - \frac{1}{2} \sum_{i=1}^{I} \beta_i \zeta_i (1-\zeta_i) \left(\widehat{\Delta C_{it}^H} - \widehat{\Delta C_{it}^F} \right)^2 - \frac{1}{2} \sum_{i=1}^{I} \beta_i \left(\widehat{\Delta C_{it}} - \widehat{\Delta C_t} \right)^2 + o(\|\boldsymbol{z}_t\|\end{aligned} \tag{B.94}$$

Rearranging the market clearing condition (B.88), and summing across sectors,

$$\sum_{i=1}^{I} \beta_i (1-\zeta_i) \widehat{\Delta C_{it}^H} = \sum_{i=1}^{I} \lambda_i \widehat{\Delta Y_{it}} - \sum_{i=1}^{I} \sum_{j=1}^{I} \lambda_j (1-\alpha_j) \nu_i (1-\zeta_{ji}) \widehat{\Delta X_{ji,t}^H} - \sum_{i=1}^{I} \gamma_i^X \widehat{\Delta Y_{it}^X} \tag{B.95}$$

Rearrange (B.78) as $(1-\zeta_{ij}) \widehat{\Delta X}_{ij,t}^H + \zeta_{ij} \widehat{\Delta X}_{ij,t}^F = \widehat{\Delta X}_{ij,t} + \frac{1}{2} \zeta_{ij} (1-\zeta_{ij}) \left(\widehat{\Delta X}_{ij,t}^H - \widehat{\Delta X}_{ij,t}^F \right)^2 + o(\|\boldsymbol{z}_t\|^2)$.

Multiplying by $\lambda_i(1-\alpha_i)\nu_{ij}$, and summing across sectors i and j, we can write the weighted sum of domestic and imported intermediate inputs as

$$\sum_{i=1}^{I}\sum_{j=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\left[(1-\zeta_{ij})\widehat{\Delta X}^{H}_{ij,t}+\zeta_{ij}\widehat{\Delta X}^{F}_{ij,t}\right]=\sum_{i=1}^{I}\sum_{j=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\widehat{\Delta X}_{ij,t}$$

$$+\frac{1}{2}\sum_{i=1}^{I}\sum_{j=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\zeta_{ij}(1-\zeta_{ij})\left(\widehat{\Delta X}^{H}_{ij,t}-\widehat{\Delta X}^{F}_{ij,t}\right)^2+o(\|\boldsymbol{z}_t\|^2) \tag{B.96}$$

Inserting (B.94) into (B.93)

$$\begin{aligned}\widehat{\Delta Y_t}&=\sum_{i=1}^{I}\beta_i\left[(1-\zeta_i)\widehat{\Delta C}^{H}_{it}+\zeta_i\widehat{\Delta C}^{F}_{it}\right]-\frac{1}{2}\sum_{i=1}^{I}\beta_i\zeta_i(1-\zeta_i)\left(\widehat{\Delta C}^{H}_{it}-\widehat{\Delta C}^{F}_{it}\right)^2-\frac{1}{2}\sum_{i=1}^{I}\beta_i\left(\widehat{\Delta C}_{it}-\widehat{\Delta C}_t\right)^2\\&+\sum_{i=1}^{I}\gamma^{X}_i\widehat{\Delta Y^{X}_{it}}-\sum_{i=1}^{I}\beta_i\zeta_i\widehat{\Delta C^{F}_{it}}-\sum_{i=1}^{I}\sum_{j=1}^{I}\lambda_j(1-\alpha_j)\nu_i\zeta_{ji}\widehat{\Delta X^{F}_{ji,t}}+o(\|\boldsymbol{z}_t\|^2)\\&=\sum_{i=1}^{I}\beta_i(1-\zeta_i)\widehat{\Delta C}^{H}_{it}+\sum_{i=1}^{I}\gamma^{X}_i\widehat{\Delta Y^{X}_{it}}-\sum_{i=1}^{I}\sum_{j=1}^{I}\lambda_j(1-\alpha_j)\nu_i\zeta_{ji}\widehat{\Delta X^{F}_{ji,t}}\\&-\frac{1}{2}\sum_{i=1}^{I}\beta_i\zeta_i(1-\zeta_i)\left(\widehat{\Delta C}^{H}_{it}-\widehat{\Delta C}^{F}_{it}\right)^2-\frac{1}{2}\sum_{i=1}^{I}\beta_i\left(\widehat{\Delta C}_{it}-\widehat{\Delta C}_t\right)^2+o(\|\boldsymbol{z}_t\|^2)\end{aligned} \tag{B.97}$$

Inserting (B.95), and then (B.96)

$$\begin{aligned}\widehat{\Delta Y_t}&=\sum_{i=1}^{I}\lambda_i\widehat{\Delta Y_{it}}-\sum_{i=1}^{I}\sum_{j=1}^{I}\lambda_j(1-\alpha_j)\nu_i\left[(1-\zeta_{ji})\widehat{\Delta X^{H}_{ji,t}}+\zeta_{ji}\widehat{\Delta X^{F}_{ji,t}}\right]\\&-\frac{1}{2}\sum_{i=1}^{I}\beta_i\zeta_i(1-\zeta_i)\left(\widehat{\Delta C}^{H}_{it}-\widehat{\Delta C}^{F}_{it}\right)^2-\frac{1}{2}\sum_{i=1}^{I}\beta_i\left(\widehat{\Delta C}_{it}-\widehat{\Delta C}_t\right)^2+o(\|\boldsymbol{z}_t\|^2)\\&=\sum_{i=1}^{I}\lambda_i\widehat{\Delta Y_{it}}-\sum_{i=1}^{I}\sum_{j=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\widehat{\Delta X}_{ij,t}-\frac{1}{2}\sum_{i=1}^{I}\sum_{j=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\zeta_{ij}(1-\zeta_{ij})\left(\widehat{\Delta X}^{H}_{ij,t}-\widehat{\Delta X}^{F}_{ij,t}\right)^2\\&-\frac{1}{2}\sum_{i=1}^{I}\beta_i\zeta_i(1-\zeta_i)\left(\widehat{\Delta C}^{H}_{it}-\widehat{\Delta C}^{F}_{it}\right)^2-\frac{1}{2}\sum_{i=1}^{I}\beta_i\left(\widehat{\Delta C}_{it}-\widehat{\Delta C}_t\right)^2+o(\|\boldsymbol{z}_t\|^2)\end{aligned} \tag{B.98}$$

Subtracting $\Lambda_N \widehat{\Delta N}_t$ from (B.98), and using (B.81)

$$\begin{aligned}\widehat{\Delta Y}_t - \Lambda_N \widehat{\Delta N}_t = &\sum_{i=1}^{I} \lambda_i \left[\widehat{\Delta Y}_{it} - \alpha_i \widehat{\Delta N}_{it} - \sum_{j=1}^{I} (1-\alpha_i) \nu_{ij} \widehat{\Delta X}_{ij,t} \right] \\ &- \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{I} \lambda_i (1-\alpha_i) \nu_{ij} \zeta_{ij} (1-\zeta_{ij}) \left(\widehat{\Delta X}^H_{ij,t} - \widehat{\Delta X}^F_{ij,t} \right)^2 \\ &- \frac{1}{2} \sum_{i=1}^{I} \beta_i \zeta_i (1-\zeta_i) \left(\widehat{\Delta C}^H_{it} - \widehat{\Delta C}^F_{it} \right)^2 - \frac{1}{2} \sum_{i=1}^{I} \beta_i \left(\widehat{\Delta C}_{it} - \widehat{\Delta C}_t \right)^2 + o(\|\boldsymbol{z}_t\|^2)\end{aligned} \tag{B.99}$$

Inserting the sectoral production function (B.87),

$$\begin{aligned}\widehat{\Delta Y}_t - \Lambda_N \widehat{\Delta N}_t = &\sum_{i=1}^{I} \lambda_i \widehat{\iota}_{it} - \frac{1}{2} \sum_{i=1}^{I} \lambda_i \left[\alpha_i \left(\widehat{\Delta N}_{it} - \widehat{\Delta Y}_{it} \right)^2 + \sum_{j=1}^{I} (1-\alpha_i) \nu_{ij} \left(\widehat{\Delta X}_{ij,t} - \widehat{\Delta Y}_{it} \right)^2 \right] \\ &- \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{I} \lambda_i (1-\alpha_i) \nu_{ij} \zeta_{ij} (1-\zeta_{ij}) \left(\widehat{\Delta X}^H_{ij,t} - \widehat{\Delta X}^F_{ij,t} \right)^2 \\ &- \frac{1}{2} \sum_{i=1}^{I} \beta_i \zeta_i (1-\zeta_i) \left(\widehat{\Delta C}^H_{it} - \widehat{\Delta C}^F_{it} \right)^2 - \frac{1}{2} \sum_{i=1}^{I} \beta_i \left(\widehat{\Delta C}_{it} - \widehat{\Delta C}_t \right)^2 + o(\|\boldsymbol{z}_t\|^2)\end{aligned} \tag{B.100}$$

Finally, inserting (B.76), we can write

$$\begin{aligned}\widehat{\Delta Y}_t - \Lambda_N \widehat{\Delta N}_t = &\sum_{i=1}^{I} \lambda_i \widehat{\iota}_{it} - \frac{1}{2} \sum_{i=1}^{I} \lambda_i \alpha_i \left(\widehat{\Delta N}_{it} - \widehat{\Delta Y}_{it} \right)^2 - \frac{1}{2} \sum_{i=1}^{I} \lambda_i \sum_{j=1}^{I} (1-\alpha_i) \nu_{ij} \left(\widehat{\Delta X}_{ij,t} - \widehat{\Delta Y}_{it} \right)^2 \\ &- \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{I} \lambda_i (1-\alpha_i) \nu_{ij} \zeta_{ij} (1-\zeta_{ij}) \left(\widehat{\Delta X}^H_{ij,t} - \widehat{\Delta X}^F_{ij,t} \right)^2 \\ &- \frac{1}{2} \sum_{i=1}^{I} \beta_i \zeta_i (1-\zeta_i) \left(\widehat{\Delta C}^H_{it} - \widehat{\Delta C}^F_{it} \right)^2 - \frac{1}{2} \sum_{i=1}^{I} \beta_i \left(\widehat{\Delta C}_{it} - \widehat{\Delta C}_t \right)^2 + o(\|\boldsymbol{z}_t\|^2)\end{aligned} \tag{B.101}$$

Now recall that the welfare component in (B.66) is $\widehat{\Delta C}_t - \Lambda_N \widehat{\Delta N}_t$. Hence, we can write

$$\begin{aligned}\widehat{\Delta Y}_t - \widehat{\Delta C}_t + \widehat{\Delta C}_t - \Lambda_N \widehat{\Delta N}_t &= o(\|\boldsymbol{z}_t\|) + \widehat{\Delta C}_t - \Lambda_N \widehat{\Delta N}_t \\ &= \sum_{i=1}^{I} \lambda_i \widehat{\iota}_{it} - \frac{1}{2} \sum_{i=1}^{I} \lambda_i \alpha_i \left(\widehat{\Delta N}_{it} - \widehat{\Delta Y}_{it} \right)^2\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\sum_{j=1}^{I}\sum_{i=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\Big(\widehat{\Delta X}_{ij,t}-\widehat{\Delta Y}_{it}\Big)^2 \\
&- \frac{1}{2}\sum_{i=1}^{I}\sum_{j=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\zeta_{ij}(1-\zeta_{ij})\Big(\widehat{\Delta X}^{H}_{ij,t}-\widehat{\Delta X}^{F}_{ij,t}\Big)^2 \\
&- \frac{1}{2}\sum_{i=1}^{I}\beta_i\zeta_i(1-\zeta_i)\Big(\widehat{\Delta C}^{H}_{it}-\widehat{\Delta C}^{F}_{it}\Big)^2 - \frac{1}{2}\sum_{i=1}^{I}\beta_i\Big(\widehat{\Delta C}_{it}-\widehat{\Delta C}_{t}\Big)^2 + o(\|\boldsymbol{z}_t\|^2)
\end{aligned}
\tag{B.102}
$$

B.4. Writing the second-order gaps in terms of prices

Labor supply and risk sharing give $\widetilde{w}_t = \widetilde{e}_t + \varphi\widetilde{y}_t$ under the labor–output identity maintained in this appendix. With heterogeneous invoicing, $\widetilde{\boldsymbol{p}}^H_t + \widetilde{\boldsymbol{s}}_t = (\mathbf{1}-\mathbb{1})\widetilde{e}_t$. Thus, when $\boldsymbol{\beta}^\top(\mathbf{1}-\mathbb{1}) > 0$, the exchange-rate gap equals $\boldsymbol{\beta}^\top(\widetilde{\boldsymbol{p}}^H_t + \widetilde{\boldsymbol{s}}_t)/[\boldsymbol{\beta}^\top(\mathbf{1}-\mathbb{1})]$. Substituting this expression and the goods-market restriction (46) into labor supply gives

$$
\widetilde{w}_t = \boldsymbol{w}_p^\top\widetilde{\boldsymbol{p}}^H_t + \boldsymbol{w}_s^\top\widetilde{\boldsymbol{s}}_t,
$$

where the two coefficient vectors $\boldsymbol{w}_p, \boldsymbol{w}_s \in \mathbb{R}^I$ are

$$
\begin{aligned}
\boldsymbol{w}_p &= \left[\frac{D_y + \varphi\big(1-\boldsymbol{\lambda}^\top\operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\mathcal{N}_\tau\mathbb{1}\big)}{D_y\,\boldsymbol{\beta}^\top(\mathbf{1}-\mathbb{1})} - \frac{\varphi}{D_y}\right]\boldsymbol{\beta}, \\
\boldsymbol{w}_s &= \frac{D_y + \varphi\big(1-\boldsymbol{\lambda}^\top\operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\mathcal{N}_\tau\mathbb{1}\big)}{D_y\,\boldsymbol{\beta}^\top(\mathbf{1}-\mathbb{1})}\boldsymbol{\beta} + \frac{\varphi}{D_y}\big(\boldsymbol{\Omega}_F - \operatorname{diag}(\boldsymbol{\Omega}_F\mathbf{1})\mathcal{N}_s\big)^\top\boldsymbol{\lambda}.
\end{aligned}
\tag{B.103}
$$

Here D_y is defined in the main text. Under uniform PCP, $\mathbb{1} = \mathbf{0}$ and $\boldsymbol{w}_p = \boldsymbol{\beta}$. This elimination uses the independent household and goods-market conditions; the CPI identity alone cannot determine the wage because $\boldsymbol{\lambda}^\top\boldsymbol{\alpha} = 1$ makes $\boldsymbol{I} - \boldsymbol{\alpha}\boldsymbol{\lambda}^\top$ singular.

We now express the allocation gaps directly in terms of prices. Let $\boldsymbol{\Omega}_{i\cdot}$ and $\boldsymbol{\Omega}_{F,i\cdot}$ denote row i of the existing input–output matrices, and let $\boldsymbol{e}_i$ be the i-th canonical basis vector. Define the following column vectors in $\mathbb{R}^I$:

$$
\begin{aligned}
\mathbf{a}^{(N)}_i &\equiv \boldsymbol{\Omega}_{i\cdot}^\top - (1-\alpha_i)\boldsymbol{w}_p, & \mathbf{b}^{(N)}_i &\equiv \boldsymbol{\Omega}_{F,i\cdot}^\top - (1-\alpha_i)\boldsymbol{w}_s, \\
\mathbf{a}^{(X)}_{ij} &\equiv \boldsymbol{\Omega}_{i\cdot}^\top - \boldsymbol{e}_j + \alpha_i\boldsymbol{w}_p, & \mathbf{b}^{(X)}_{ij} &\equiv \boldsymbol{\Omega}_{F,i\cdot}^\top - \zeta_{ij}\boldsymbol{e}_j + \alpha_i\boldsymbol{w}_s, \\
\mathbf{a}^{(C)}_i &\equiv \boldsymbol{e}_i - \boldsymbol{\beta}, & \mathbf{b}^{(C)}_i &\equiv \zeta_i\boldsymbol{e}_i - \boldsymbol{\beta}_F.
\end{aligned}
$$

The subscripts i and ij identify the allocation being represented, not a component of these vectors. In particular, $\boldsymbol{\beta} = (\beta_k)_{k=1}^I$ and $\boldsymbol{\beta}_F = (\beta_k\zeta_k)_{k=1}^I$ are column vectors: component k of

$\mathbf{a}_i^{(C)}$ is $\mathbf{1}_{\{k=i\}} - \beta_k$, and component k of $\mathbf{b}_i^{(C)}$ is $\zeta_i \mathbf{1}_{\{k=i\}} - \beta_k \zeta_k$. Their inner products with the price-gap vectors are scalars.

From firms' cost minimization, $W_t N_{it} = \alpha_i \mathrm{MC}_{it} Y_{it}$ to first order around zero dispersion. The marginal-cost equation (A.42) gives $\widetilde{mc}_{it} = \alpha_i \widetilde{w}_t + \boldsymbol{\Omega}_{i\cdot} \widetilde{\boldsymbol{p}}_t^H + \boldsymbol{\Omega}_{F,i\cdot} \widetilde{\boldsymbol{s}}_t$. Taking logs and gaps therefore yields

$$\widetilde{n}_{it} - \widetilde{y}_{it} = \widetilde{mc}_{it} - \widetilde{w}_t = (\mathbf{a}_i^{(N)})^\top \widetilde{\boldsymbol{p}}_t^H + (\mathbf{b}_i^{(N)})^\top \widetilde{\boldsymbol{s}}_t.$$

Similarly, the expenditure condition for the origin-composite input is $P_{ij,t}^X X_{ij,t} = (1 - \alpha_i)\nu_{ij} \mathrm{MC}_{it} Y_{it}$, with price gap $\widetilde{p}_{ij,t}^X = \widetilde{p}_{jt}^H + \zeta_{ij} \widetilde{s}_{jt}$. Hence,

$$\widetilde{x}_{ij,t} - \widetilde{y}_{it} = \widetilde{mc}_{it} - \widetilde{p}_{jt}^H - \zeta_{ij} \widetilde{s}_{jt} = (\mathbf{a}_{ij}^{(X)})^\top \widetilde{\boldsymbol{p}}_t^H + (\mathbf{b}_{ij}^{(X)})^\top \widetilde{\boldsymbol{s}}_t.$$

Finally, $\widetilde{p}_{it}^C = \widetilde{p}_{it}^H + \zeta_i \widetilde{s}_{it}$ and the aggregate CPI identity (A.44) imply

$$\widetilde{p}_{it}^C - \widetilde{p}_t^C = (\mathbf{a}_i^{(C)})^\top \widetilde{\boldsymbol{p}}_t^H + (\mathbf{b}_i^{(C)})^\top \widetilde{\boldsymbol{s}}_t.$$

The household intratemporal FOC, $P_{it}^C = \beta_i P_t^C C_t / C_{it}$, gives $\widetilde{c}_{it} - \widetilde{c}_t = -(\widetilde{p}_{it}^C - \widetilde{p}_t^C)$. Squaring these first-order expressions yields the corresponding second-order labor, input and consumption terms in (B.102), up to $o(\|\boldsymbol{z}_t\|^2)$.

From the domestic vs. foreign consumption FOC, we have $\frac{C_{it}^H}{C_{it}^F} = \frac{1-\zeta_i}{\zeta_i} \frac{(1+\tau_{it}^M)(1+\tau_{it}^W) P_{it}^F}{P_{it}^H}$. Taking logs and gaps gives $\widetilde{c}_{it}^H - \widetilde{c}_{it}^F = \widetilde{s}_{it}$. Therefore, we can write

$$\left(\widehat{\Delta C}_{it}^H - \widehat{\Delta C}_{it}^F\right)^2 = \widetilde{s}_{it}^2 + o(\|\boldsymbol{z}_t\|^2)$$

The same logic applies to firms' domestic vs. foreign input choice, $\frac{X_{ij,t}^H}{X_{ij,t}^F} = \frac{1-\zeta_{ij}}{\zeta_{ij}} \frac{(1+\tau_{jt}^M)(1+\tau_{jt}^W) P_{jt}^F}{P_{jt}^H}$. Taking logs and gaps gives $\widetilde{x}_{ij,t}^H - \widetilde{x}_{ij,t}^F = \widetilde{s}_{jt}$. Therefore, we can write

$$\left(\widehat{\Delta X}_{ij,t}^H - \widehat{\Delta X}_{ij,t}^F\right)^2 = \widetilde{s}_{jt}^2 + o(\|\boldsymbol{z}_t\|^2)$$

We can thus write (B.102) as

$$\begin{aligned}\widehat{\Delta C}_t - \Lambda_N \widehat{\Delta N}_t = &\sum_{i=1}^I \lambda_i \widehat{\iota}_{it} - \frac{1}{2} \sum_{i=1}^I \lambda_i \alpha_i \left[(\mathbf{a}_i^{(N)})^\top \widetilde{\boldsymbol{p}}_t^H + (\mathbf{b}_i^{(N)})^\top \widetilde{\boldsymbol{s}}_t\right]^2 \\ &- \frac{1}{2} \sum_{j=1}^I \sum_{i=1}^I \lambda_i (1 - \alpha_i) \nu_{ij} \left[(\mathbf{a}_{ij}^{(X)})^\top \widetilde{\boldsymbol{p}}_t^H + (\mathbf{b}_{ij}^{(X)})^\top \widetilde{\boldsymbol{s}}_t\right]^2\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\sum_{i=1}^{I}\sum_{j=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\zeta_{ij}(1-\zeta_{ij})\widetilde{s}_{jt}^{2} \\
&- \frac{1}{2}\sum_{i=1}^{I}\beta_i\zeta_i(1-\zeta_i)\widetilde{s}_{it}^{2} - \frac{1}{2}\sum_{i=1}^{I}\beta_i\left[(\mathbf{a}_i^{(C)})^{\top}\widetilde{\boldsymbol{p}}_t^{H} + (\mathbf{b}_i^{(C)})^{\top}\widetilde{\boldsymbol{s}}_t\right]^2 + o(\|\boldsymbol{z}_t\|^2)
\end{aligned}
$$

B.5. Within-sector price dispersion

The industry-level output is given by

$$
Y_{it} = \left[\int_0^1 Y_{if,t}^{(\epsilon_i^p-1)/\epsilon_i^p}\,df\right]^{\epsilon_i^p/(\epsilon_i^p-1)} \tag{B.104}
$$

An expenditure-minimization problem $\min_{Y_{if,t}}\int_0^1 P_{if,t}Y_{if,t}df$ subject to $\left[\int_0^1 Y_{if,t}^{(\epsilon_i^p-1)/\epsilon_i^p}\,df\right]^{\epsilon_i^p/(\epsilon_i^p-1)} \geq Y_{it}$ results in an implied sectoral price index

$$
P_{it}^{H} = \left[\int_0^1 (P_{if,t}^{H})^{\epsilon_i^p-1}df\right]^{1/(\epsilon_i^p-1)} \tag{B.105}
$$

and a relative demand equation

$$
Y_{if,t} = \left(\frac{P_{if,t}^{H}}{P_{it}^{H}}\right)^{-\epsilon_i^p} Y_{it} \tag{B.106}
$$

Define the firm-level relative-price gap by

$$
e_{if,t}^{H} \equiv \log P_{if,t}^{H} - \log P_{it}^{H}. \tag{B.107}
$$

Also define the cross-sectional mean and variance of relative prices by

$$
\bar{e}_{it}^{H} \equiv \int_0^1 e_{if,t}^{H}\,df, \tag{B.108}
$$

$$
\vartheta_{it}^{H} \equiv \int_0^1 (e_{if,t}^{H})^2\,df - (\bar{e}_{it}^{H})^2. \tag{B.109}
$$

Using (B.105), divide each firm price by the sectoral price index. Then

$$1 = \int_0^1 \left(\frac{P^H_{if,t}}{P^H_{it}} \right)^{1-\epsilon^p_i} df = \int_0^1 e^{(1-\epsilon^p_i) e^H_{if,t}} df. \tag{B.110}$$

Now define the within-sector efficiency term ι_{it} from (B.83)

$$\iota_{it}^{-1} = \int_0^1 \left(\frac{P^H_{if,t}}{P^H_{it}} \right)^{-\epsilon^p_i} df = \int_0^1 e^{-\epsilon^p_i e^H_{if,t}} df. \tag{B.111}$$

We now expand (B.110) and (B.111) to second order, using

$$e^{ax} = 1 + ax + \frac{a^2}{2} x^2 + o(x^2). \tag{B.112}$$

Applying (B.112) to (B.110), we obtain

$$\begin{aligned} 1 &= \int_0^1 \left[1 + (1-\epsilon^p_i) e^H_{if,t} + \frac{(1-\epsilon^p_i)^2}{2} (e^H_{if,t})^2 \right] df + o(\|\boldsymbol{z}_t\|^2) \\ &= 1 + (1-\epsilon^p_i) \bar{e}^H_{it} + \frac{(1-\epsilon^p_i)^2}{2} \int_0^1 (e^H_{if,t})^2 \, df + o(\|\boldsymbol{z}_t\|^2). \end{aligned} \tag{B.113}$$

Using (B.109), $\int_0^1 (e^H_{if,t})^2 \, df = \vartheta^H_{it} + (\bar{e}^H_{it})^2$, so (B.113) becomes

$$0 = (1-\epsilon^p_i) \bar{e}^H_{it} + \frac{(1-\epsilon^p_i)^2}{2} \left(\vartheta^H_{it} + (\bar{e}^H_{it})^2 \right) + o(\|\boldsymbol{z}_t\|^2). \tag{B.114}$$

Since $\bar{e}^H_{it} = O(\|\boldsymbol{z}_t\|^2)$, the term $(\bar{e}^H_{it})^2$ is fourth order and can be dropped. Therefore,

$$\bar{e}^H_{it} = \frac{\epsilon^p_i - 1}{2} \vartheta^H_{it} + o(\|\boldsymbol{z}_t\|^2). \tag{B.115}$$

Next expand (B.111). Using again (B.112),

$$\begin{aligned} \iota_{it}^{-1} &= \int_0^1 \left[1 - \epsilon^p_i e^H_{if,t} + \frac{(\epsilon^p_i)^2}{2} (e^H_{if,t})^2 \right] df + o(\|\boldsymbol{z}_t\|^2) = 1 - \epsilon^p_i \bar{e}^H_{it} + \frac{(\epsilon^p_i)^2}{2} \int_0^1 (e^H_{if,t})^2 \, df + o(\|\boldsymbol{z}_t\|^2) \\ &= 1 - \epsilon^p_i \bar{e}^H_{it} + \frac{(\epsilon^p_i)^2}{2} \left(\vartheta^H_{it} + (\bar{e}^H_{it})^2 \right) + o(\|\boldsymbol{z}_t\|^2). \end{aligned} \tag{B.116}$$

Substitute (B.115) into (B.116), and again drop $(\bar{e}_{it}^H)^2$ as higher order. This gives

$$\iota_{it}^{-1} = 1 - \epsilon_i^p \left(\frac{\epsilon_i^p - 1}{2} \vartheta_{it}^H \right) + \frac{(\epsilon_i^p)^2}{2} \vartheta_{it}^H + o(\|\boldsymbol{z}_t\|^2) = 1 + \frac{\epsilon_i^p}{2} \vartheta_{it}^H + o(\|\boldsymbol{z}_t\|^2). \tag{B.117}$$

Now take logs. Since $\vartheta_{it}^H = O(\|\boldsymbol{z}_t\|^2)$, we may use $\log(1+x) = x + o(x)$, so

$$-\log \iota_{it} = \frac{\epsilon_i^p}{2} \vartheta_{it}^H + o(\|\boldsymbol{z}_t\|^2), \tag{B.118}$$

Finally, using the convention $\widehat{\iota}_{it} \equiv -\log \iota_{it}$, we obtain

$$-\widehat{\iota}_{it} = \log \iota_{it} = -\frac{\epsilon_i^p}{2} \vartheta_{it}^H + o(\|\boldsymbol{z}_t\|^2). \tag{B.119}$$

Thus, up to second order, the within-sector efficiency loss is proportional to the cross-sectional variance of relative prices within the sector. The coefficient of proportionality is $\epsilon_i^p/2$: the larger is the elasticity of substitution across varieties, the larger is the output loss generated by a given amount of price dispersion.

Let us now relate ϑ_{it}^H to the Calvo rigidity. We have that

$$\vartheta_{it}^H = \int_0^1 (\log P_{if,t}^H - \log P_{it}^{H,n})^2 df - \left[\int_0^1 (\log P_{if,t}^H - \log P_{it}^{H,n}) df \right]^2 = \int_0^1 (\log P_{if,t}^H - \mu_{it}^p)^2 df$$

where $\mu_{it}^p = \int_0^1 \log P_{if,t}^H df$ is the cross-sectional mean of log prices. We can write

$$\begin{aligned} \int_0^1 (\log P_{if,t}^H - \log P_{it}^H)^2 df &= \int_0^1 \left[(\log P_{if,t}^H - \mu_{it}^p) - (\log P_{it}^H - \mu_{it}^p) \right]^2 df \\ &= \underbrace{\int_0^1 (\log P_{if,t}^H - \mu_{it}^p)^2 df}_{\vartheta_{it}^H} + 2(\mu_{it}^p - \log P_{it}^H) \underbrace{\int_0^1 (\log P_{if,t}^H - \mu_{it}^p) df}_{=0} + (\mu_{it}^p - \log P_{it}^H)^2 \\ &= \vartheta_{it}^H + (\mu_{it}^p - \log P_{it}^H)^2 \end{aligned} \tag{B.120}$$

Using

$$\begin{aligned} \log P_{it}^H - \log P_{it}^{H,n} &= \frac{1}{1-\epsilon_i^p} \log \left[\int_0^1 e^{e_{if,t}(1-\epsilon_i^p)} df \right] \\ &= \frac{1}{1-\epsilon_i^p} \log \left[1 + (1-\epsilon_i^p) \bar{e}_{it}^H + \frac{(1-\epsilon_i^p)^2}{2} \left(\vartheta_{it}^H + (\bar{e}_{it}^H)^2 \right) \right] + o(\|\boldsymbol{z}_t\|^2) \end{aligned}$$

$$= \frac{1}{1-\epsilon_i^p}\left[(1-\epsilon_i^p)\bar{e}_{it}^H + \frac{(1-\epsilon_i^p)^2}{2}\left(\vartheta_{it}^H + (\bar{e}_{it}^H)^2\right) - \frac{(1-\epsilon_i^p)^2}{2}(\bar{e}_{it}^H)^2\right] + o(\|\boldsymbol{z}_t\|^2)$$

$$= \bar{e}_{it}^H + \frac{1-\epsilon_i^p}{2}\vartheta_{it}^H + o(\|\boldsymbol{z}_t\|^2) \tag{B.121}$$

we can write

$$\log P_{it}^H = \log P_{it}^{H,n} + \bar{e}_{it}^H + \frac{1-\epsilon_i^p}{2}\vartheta_{it}^H + o(\|\boldsymbol{z}_t\|^2) = \mu_{it}^p + \frac{1-\epsilon_i^p}{2}\vartheta_{it}^H + o(\|\boldsymbol{z}_t\|^2) \tag{B.122}$$

where the last equality uses $\bar{e}_{it} = \mu_{it}^p - \log P_{it}^{H,n}$. Inserting (B.122) into (B.120)

$$\int_0^1 (\log P_{if,t}^H - \log P_{it}^H)^2 df = \vartheta_{it}^H + \left[\frac{1-\epsilon_i^p}{2}\vartheta_{it}^H\right]^2 + o(\|\boldsymbol{z}_t\|^2) = \vartheta_{it}^H$$

At date t, the unit mass of firms in sector i splits into two groups. Adjusting firms, with mass $1-\theta_i$, all choose the same optimal reset price $\overline{P}_{it}^H$, so $\log P_{if,t}^H = \log \overline{P}_{it}^H$. Non-adjusting firms keep their price from last period, $\log P_{if,t}^H = \log P_{if,t-1}$. Using this partition, we can write

$$\vartheta_{it}^H = (1-\theta_i)(\log \overline{P}_{it}^H - \log P_{it}^H)^2 + \theta_i \int_0^1 (\log P_{if,t-1}^H - \log P_{it}^H)^2 df \tag{B.123}$$

For non-adjusting firms, we can write

$$\log P_{if,t-1}^H - \log P_{it}^H = (\log P_{if,t-1}^H - \log P_{i,t-1}^H) - (\log P_{it}^H - \log P_{i,t-1}^H) = (\log P_{if,t-1}^H - \log P_{i,t-1}^H) - \pi_{it}^H$$

Therefore,

$$\theta_i \int_0^1 (\log P_{if,t-1}^H - \log P_{it}^H)^2 df = \theta_i \int_0^1 (\log P_{if,t-1}^H - \log P_{i,t-1}^H - \pi_{it}^H)^2 df$$

$$= \theta_i \left[\int_0^1 (\log P_{if,t-1}^H - \log P_{i,t-1}^H)^2 df - 2\pi_{it}^H \underbrace{\int_0^1 \left(\log P_{if,t-1}^H - \log P_{i,t-1}^H\right) df}_{M_{i,t-1}} + (\pi_{it}^H)^2\right] \tag{B.124}$$

From the CES price index approximation (B.121),

$$\log P_{i,t-1}^H = \int_0^1 \log P_{if,t-1}^H df + \frac{1-\epsilon_i^p}{2}\vartheta_{i,t-1} + o(\|\boldsymbol{z}_t\|^2)$$

and we can thus write

$$M_{i,t-1} = \int_0^1 \left(\log P^H_{if,t-1} - \log P^H_{i,t-1}\right) df = -\frac{1-\epsilon^p_i}{2}\vartheta_{i,t-1} + o(\|\boldsymbol{z}_t\|^2)$$

Since $\pi^H_{it} = O(\|\boldsymbol{z}_t\|)$ and $M_{i,t-1} = O(\|\boldsymbol{z}_t\|^2)$, the cross term in (B.124) is $2\pi^H_{it}M_{i,t-1} = o(\|\boldsymbol{z}_t\|^2)$. Thus, we can write (B.124) as

$$\theta_i \int_0^1 (\log P^H_{if,t-1} - \log P^H_{it})^2 df = \theta_i(\vartheta_{i,t-1} + \pi^2_{it})$$

Consider now (B.123). We need to obtain $\log \overline{P}^H_{it} - \log P^H_{it}$. From the CES price index under Calvo,

$$P^{1-\epsilon^p_i}_{it} = \theta_i (P^H_{i,t-1})^{1-\epsilon^p_i} + (1-\theta_i)(\overline{P}^H_{it})^{1-\epsilon^p_i}$$

Log-linearizing around a zero-inflation steady-state where $P_{it} = P^H_{i,t-1} = \overline{P}^H_{it}$,

$$\log P^H_{it} = \theta_i \log P^H_{i,t-1} + (1-\theta_i)\log \overline{P}^H_{it} + O(2) \implies \log P^H_{it} - \theta_i \log P^H_{i,t-1} = (1-\theta_i)\log \overline{P}^H_{it}$$

Adding and subtracting $\theta_i \log P^H_{it}$ on the LHS,

$$\begin{aligned}
\log P^H_{it} - \theta_i(\log P^H_{i,t-1} - \log P^H_{it}) - \theta_i \log P^H_{it} &= (1-\theta_i)\log \overline{P}^H_{it} \\
(1-\theta_i)\log P^H_{it} + \theta_i \pi^H_{it} &= (1-\theta_i)\log \overline{P}^H_{it} \\
\log \overline{P}^H_{it} - \log P^H_{it} &= \frac{\theta_i}{1-\theta_i}\pi^H_{it}
\end{aligned}$$

Hence, we can write

$$(1-\theta_i)(\log \overline{P}^H_{it} - \log P^H_{it})^2 = \frac{\theta^2_i}{1-\theta_i}(\pi^H_{it})^2 + o(\|\boldsymbol{z}_t\|^2)$$

and we can write (B.123) as

$$\vartheta^H_{it} = \frac{\theta^2_i}{1-\theta_i}(\pi^H_{it})^2 + \theta_i(\vartheta^H_{i,t-1} + (\pi^H_{it})^2) + o(\|\boldsymbol{z}_t\|^2) = \theta_i \vartheta^H_{i,t-1} + \frac{\theta_i}{1-\theta_i}(\pi^H_{it})^2 = \frac{\theta_i}{1-\theta_i}\sum_{s=0}^{t}\theta^{t-s}_i(\pi^H_{is})^2 + o(\|\boldsymbol{z}_t\|^2).$$

This is the law of motion for within-sector price dispersion. Multiplying the expression by

$\lambda_i \epsilon_i^p$ and summing over sectors i, and then computing the discounted infinite sum,

$$\sum_{t=0}^{\infty}\beta^t\sum_{i=1}^{I}\lambda_i\epsilon_i^p\vartheta_{it}^H = \sum_{t=0}^{\infty}\beta^t\sum_{i=1}^{I}\lambda_i\epsilon_i^p\frac{\theta_i}{1-\theta_i}\sum_{s=0}^{t}\theta_i^{t-s}\pi_{is}^2 + o(\|\boldsymbol{z}_t\|^2) = \sum_{i=1}^{I}\lambda_i\epsilon_i^p\frac{\theta_i}{1-\theta_i}\sum_{s=0}^{\infty}\pi_{is}^2\sum_{t=s}^{\infty}(\beta\theta_i)^{t-s}\beta^s + o(\|\boldsymbol{z}_t$$
$$= \sum_{i=1}^{I}\lambda_i\epsilon_i^p\frac{\theta_i}{1-\theta_i}\frac{1}{1-\beta\theta_i}\sum_{t=0}^{\infty}\beta^t(\pi_{it}^H)^2 + o(\|\boldsymbol{z}_t\|^2) = \sum_{i=1}^{I}\frac{\lambda_i\epsilon_i^p}{\kappa_i}\sum_{t=0}^{\infty}\beta^t(\pi_{it}^H)^2 + o(\|\boldsymbol{z}_t\|^2)$$

where we define the sectoral Phillips curve slope $\kappa_i \equiv \frac{(1-\theta_i)(1-\beta\theta_i)}{\theta_i}$.

Welfare loss function. The welfare loss function can thus be written as

$$\begin{aligned}\sum_{t=0}^{\infty}\beta^t(u_t(\xi_t)-u_t^n(\xi_t)) = -\frac{1}{2}\sum_{t=0}^{\infty}\beta^t\Bigg\{&(1+\varphi)\widetilde{y}_t^2 + \sum_{i=1}^{I}\frac{\lambda_i\epsilon_i^p}{\kappa_i}(\pi_{it}^H)^2 + \sum_{i=1}^{I}\lambda_i\alpha_i\left[(\mathbf{a}_i^{(N)})^\top\widetilde{\boldsymbol{p}}_t^H + (\mathbf{b}_i^{(N)})^\top\widetilde{\boldsymbol{s}}_t\right]^2\\ &+\sum_{j=1}^{I}\sum_{i=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\left[(\mathbf{a}_{ij}^{(X)})^\top\widetilde{\boldsymbol{p}}_t^H + (\mathbf{b}_{ij}^{(X)})^\top\widetilde{\boldsymbol{s}}_t\right]^2\\ &+\sum_{i=1}^{I}\sum_{j=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\zeta_{ij}(1-\zeta_{ij})\widetilde{s}_{jt}^2 + \sum_{i=1}^{I}\beta_i\zeta_i(1-\zeta_i)\widetilde{s}_{it}^2\\ &+\sum_{i=1}^{I}\beta_i\left[(\mathbf{a}_i^{(C)})^\top\widetilde{\boldsymbol{p}}_t^H + (\mathbf{b}_i^{(C)})^\top\widetilde{\boldsymbol{s}}_t\right]^2\Bigg\}\\ &+o(\|\widehat{\xi}_t\|^2).\end{aligned}$$

Let $\mathbf{x}_t \equiv \begin{pmatrix}\widetilde{\mathbf{p}}_t^H & \widetilde{\mathbf{s}}_t\end{pmatrix}^\top$. Each square term in the welfare loss is of the form

$$\left[\mathbf{a}^\top\widetilde{\mathbf{p}}_t^H + \mathbf{b}^\top\widetilde{\mathbf{s}}_t\right]^2 = \mathbf{x}_t^\top\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}^\top\mathbf{x}_t.$$

Hence the collection of all cross-sector and cross-border misallocation terms can be written as $(1/2)\mathbf{x}_t^\top\mathbf{H}\mathbf{x}_t$, where $\mathbf{H}$ is the sum of the corresponding rank-one positive semidefinite matrices.

Using these coefficient vectors, the labor–output misallocation block satisfies

$$\sum_{i=1}^{I}\lambda_i\alpha_i\left[(\mathbf{a}_i^{(N)})^\top\widetilde{\boldsymbol{p}}_t^H + (\mathbf{b}_i^{(N)})^\top\widetilde{\boldsymbol{s}}_t\right]^2 = \mathbf{x}_t^\top\left[\sum_{i=1}^{I}\lambda_i\alpha_i\begin{pmatrix}\mathbf{a}_i^{(N)}\\ \mathbf{b}_i^{(N)}\end{pmatrix}\begin{pmatrix}\mathbf{a}_i^{(N)}\\ \mathbf{b}_i^{(N)}\end{pmatrix}^\top\right]\mathbf{x}_t, \tag{B.125}$$

the input–output misallocation block satisfies

(B.126)
$$\sum_{i=1}^{I}\sum_{j=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\left[(\mathbf{a}_{ij}^{(X)})^\top\widetilde{\boldsymbol{p}}_t^H+(\mathbf{b}_{ij}^{(X)})^\top\widetilde{\boldsymbol{s}}_t\right]^2=\mathbf{x}_t^\top\left[\sum_{i=1}^{I}\sum_{j=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\begin{pmatrix}\mathbf{a}_{ij}^{(X)}\\ \mathbf{b}_{ij}^{(X)}\end{pmatrix}\begin{pmatrix}\mathbf{a}_{ij}^{(X)}\\ \mathbf{b}_{ij}^{(X)}\end{pmatrix}^\top\right]\mathbf{x}_t,$$

and the across-sector consumption block satisfies

$$\sum_{i=1}^{I}\beta_i\left[(\mathbf{a}_i^{(C)})^\top\widetilde{\boldsymbol{p}}_t^H+(\mathbf{b}_i^{(C)})^\top\widetilde{\boldsymbol{s}}_t\right]^2=\mathbf{x}_t^\top\left[\sum_{i=1}^{I}\beta_i\begin{pmatrix}\mathbf{a}_i^{(C)}\\ \mathbf{b}_i^{(C)}\end{pmatrix}\begin{pmatrix}\mathbf{a}_i^{(C)}\\ \mathbf{b}_i^{(C)}\end{pmatrix}^\top\right]\mathbf{x}_t. \tag{B.127}$$

The pure terms-of-trade terms can also be written as quadratic forms in $\mathbf{x}_t$:

(B.128)
$$\sum_{i=1}^{I}\sum_{j=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\zeta_{ij}(1-\zeta_{ij})\widetilde{s}_{j,t}^2=\mathbf{x}_t^\top\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \operatorname{diag}\left(\sum_{i=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\zeta_{ij}(1-\zeta_{ij})\right)_{j=1}^{I}\end{pmatrix}\mathbf{x}_t,$$

$$\sum_{i=1}^{I}\beta_i\zeta_i(1-\zeta_i)\widetilde{s}_{i,t}^2=\mathbf{x}_t^\top\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \operatorname{diag}\left(\beta_i\zeta_i(1-\zeta_i)\right)_{i=1}^{I}\end{pmatrix}\mathbf{x}_t. \tag{B.129}$$

Collecting (B.125)–(B.129), the matrix $\mathbf{H}$ is given by

$$\begin{aligned}\mathbf{H}=&\sum_{i=1}^{I}\lambda_i\alpha_i\begin{pmatrix}\mathbf{a}_i^{(N)}\\ \mathbf{b}_i^{(N)}\end{pmatrix}\begin{pmatrix}\mathbf{a}_i^{(N)}\\ \mathbf{b}_i^{(N)}\end{pmatrix}^\top+\sum_{i=1}^{I}\sum_{j=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\begin{pmatrix}\mathbf{a}_{ij}^{(X)}\\ \mathbf{b}_{ij}^{(X)}\end{pmatrix}\begin{pmatrix}\mathbf{a}_{ij}^{(X)}\\ \mathbf{b}_{ij}^{(X)}\end{pmatrix}^\top\\ &+\sum_{i=1}^{I}\beta_i\begin{pmatrix}\mathbf{a}_i^{(C)}\\ \mathbf{b}_i^{(C)}\end{pmatrix}\begin{pmatrix}\mathbf{a}_i^{(C)}\\ \mathbf{b}_i^{(C)}\end{pmatrix}^\top+\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \operatorname{diag}\left(\sum_{i=1}^{I}\lambda_i(1-\alpha_i)\nu_{ij}\zeta_{ij}(1-\zeta_{ij})\right)_{j=1}^{I}\end{pmatrix}\\ &+\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \operatorname{diag}\left(\beta_i\zeta_i(1-\zeta_i)\right)_{i=1}^{I}\end{pmatrix}.\end{aligned} \tag{B.130}$$

Conditional on the preceding welfare-reduction steps, this sum of squares has the compact form (54). The construction here establishes the price representation; it does not independently establish the efficiency of the natural allocation after shocks. Since each rank-one matrix in (B.130) is positive semidefinite, and the two diagonal blocks are also positive semidefinite, $\mathbf{H}$ is itself positive semidefinite.

Appendix C. Equilibrium Conditions under a Taylor Rule

This appendix derives the set of equilibrium conditions used to solve the model under a simple Taylor rule rather than under optimal monetary policy. Relative to the Ramsey problem in Section 4.2, which imposes the same GDP–consumption and risk-sharing restrictions, the monetary authority no longer chooses the allocation directly. Instead, the nominal interest rate is set according to a Taylor rule, while the output gap, consumption gap, CPI inflation, nominal exchange rate, and sectoral ToT gaps are jointly determined by the equilibrium conditions.

C.1. Sectoral Phillips Curves and Price Dynamics

The sectoral Phillips curves are the same as in equation (48). Sectoral domestic inflation is defined as (50). With $\boldsymbol{\tau}^F = \boldsymbol{\tau}^M + \boldsymbol{\tau}^W$, the inefficient-price wedge is given by

$$\boldsymbol{\chi}_t = \widetilde{\mathbf{p}}^H_{t-1} - \mathbf{A}_M \Delta \boldsymbol{\tau}^F_t - \mathbf{A}_X \Delta \boldsymbol{\tau}^X_t, \tag{C.1}$$

where

$$\mathbf{A}_M \equiv (\mathbf{I} - \boldsymbol{\Omega}_H)^{-1} \boldsymbol{\Omega}_F, \qquad \mathbf{A}_X \equiv \frac{\varphi}{1+\varphi} \mathbf{A}_M (\mathbf{1} - \mathbb{1})\, \boldsymbol{\ell}_X^{\mathsf{T}}, \tag{C.2}$$

and $\boldsymbol{\ell}_X$ is the value-added-weighted export exposure defined in (44).

C.2. Terms of Trade and CPI Inflation

The sectoral ToT gaps are defined as (56). The CPI level, in gap terms, is given by (A.44). Actual CPI inflation, which enters the Taylor rule and the Euler equation, is

$$\pi^C_t = \widetilde{p}^C_t - \widetilde{p}^C_{t-1} + p^{C,n}_t - p^{C,n}_{t-1}, \tag{C.3}$$

where $p^{C,n}_t$ is the natural CPI derived in (C.6) below. The change in the CPI gap alone is not CPI inflation: it omits the movement of the natural CPI induced by tariffs.

C.3. Output, Consumption, and Risk Sharing

The output gap, the consumption gap, the CPI gap, and the exchange-rate gap are linked by the risk-sharing condition, the CPI definition, and the GDP–consumption relation in (45), with the coefficients written explicitly in the main text.

C.4. Euler Equation and the Natural Real Interest Rate

The household Euler equation is

$$c_t = \mathbb{E}_t c_{t+1} - \left(i_t - \mathbb{E}_t \pi^C_{t+1}\right),$$

where i_t is the nominal interest rate expressed as a deviation from its steady-state value. Subtracting the flexible-price counterpart gives the gap Euler equation (63) in the main text, where the natural real interest rate is defined as

$$r^n_t \equiv \mathbb{E}_t c^n_{t+1} - c^n_t. \tag{C.4}$$

We now derive r^n_t. From risk sharing in the flexible-price allocation, $c^n_t = e^n_t - p^{C,n}_t$. The flexible-price exchange rate is pinned down by export tariffs. From (A.34):

$$e^n_t = \mathbf{a}_e^\intercal \boldsymbol{\tau}^X_t, \qquad \mathbf{a}_e^\intercal \equiv \frac{\varphi}{1+\varphi}\boldsymbol{\ell}_X^\intercal. \tag{C.5}$$

Using the natural ToT (A.36) and the natural domestic price (A.35), the natural CPI is

$$\begin{aligned} p^{C,n}_t = \boldsymbol{\beta}^\intercal \mathbf{p}^{H,n}_t + \boldsymbol{\beta}_F^\intercal \mathbf{s}^n_t &= \boldsymbol{\beta}^\intercal (\boldsymbol{I} - \Omega_H)^{-1}\Omega_F \left[\boldsymbol{\tau}^F_t + (\mathbf{1} - \mathbb{1})e^n_t\right] \\ &\quad + \boldsymbol{\beta}_F^\intercal \left[\boldsymbol{I} - (\boldsymbol{I} - \Omega_H)^{-1}\Omega_F\right]\left[\boldsymbol{\tau}^F_t + (\mathbf{1} - \mathbb{1})e^n_t\right]. \end{aligned} \tag{C.6}$$

Combining (C.5) and (C.6), natural consumption can be written as

$$c^n_t = \mathbf{a}^c_M \boldsymbol{\tau}^F_t + \mathbf{a}^c_X \boldsymbol{\tau}^X_t, \tag{C.7}$$

where

$$\mathbf{a}^c_M \equiv -\boldsymbol{\beta}_H^\intercal \mathbf{A}_M - \boldsymbol{\beta}_F^\intercal, \qquad \mathbf{a}^c_X \equiv \left[1 - \boldsymbol{\beta}_F^\intercal(\mathbf{1} - \mathbb{1})\right]\mathbf{a}_e^\intercal - \boldsymbol{\beta}_H^\intercal \mathbf{A}_X \tag{C.8}$$

Assume that wedge processes follow (35). Then

$$\mathbb{E}_t \boldsymbol{\tau}^F_{t+1} = \boldsymbol{\rho}_M \boldsymbol{\tau}^M_t + \boldsymbol{\rho}_W \boldsymbol{\tau}^W_t, \qquad \mathbb{E}_t \boldsymbol{\tau}^X_{t+1} = \boldsymbol{\rho}_X \boldsymbol{\tau}^X_t.$$

Using (C.4) and (C.7) gives the natural-rate expression (64) in the main text. That expression shows that, in the SOE, the natural real rate is driven by the expected growth of natural consumption, not by natural output alone. This distinction matters because, in an open economy, natural consumption and natural output generally differ due to movements in the terms of trade and the real exchange rate.

C.5. Taylor Rule

The monetary authority sets the nominal interest rate according to the Taylor rule (62) in the main text.

C.6. Equilibrium System

The Taylor-rule equilibrium is characterized by the following set of equations: the wedge processes (35); the definition of χ_t, (C.1); the sectoral Phillips curves (48); the sectoral domestic price dynamics, (50); the ToT definition (56); the CPI definition (C.3); the aggregate gap conditions (45); the Euler equation (63), with the natural rate (64); and the Taylor rule (62).

This system replaces the Ramsey optimality conditions in Section 4.2. In particular, the multipliers η_t, ν_t, ξ_t, and ϑ_t no longer appear. Instead, the nominal interest rate follows (62), while the nominal exchange rate is determined jointly by risk sharing, the Euler equation, the CPI, and the ToT definition.

Appendix D. Additional SOE Policy Derivations

D.1. Static Benchmark and Commitment Target Criterion

Under the tariff-free welfare benchmark of Proposition 2, the static problem isolates the contemporaneous trade-off and the dynamic problem yields a commitment target criterion.

A static detour. Following Rubbo (2023), it is useful to isolate the static trade-off faced by the planner. We therefore consider the one-period version of the problem, abstracting from price-level dynamics and inflation expectations. In this one-period problem the economy starts from $\widetilde{\mathbf{p}}^H_{t-1} = \mathbf{0}$, so $\chi_t = -\Delta p^{H,n}_t$ is exogenous and the intertemporal term in (59) is absent. The planner chooses $\{\widetilde{y}_t, \pi^H_t, \widetilde{\mathbf{p}}^H_t, \widetilde{\mathbf{s}}_t, \widetilde{e}_t\}$ to minimize the period loss $L^{\text{static}}_t = \frac{1}{2}(1+\varphi)\widetilde{y}^2_t + \frac{1}{2}(\pi^H_t)^\top \mathbf{W}_\pi \pi^H_t + \frac{1}{2}\begin{pmatrix}\widetilde{\mathbf{p}}^H_t \\ \widetilde{\mathbf{s}}_t\end{pmatrix}^\top \mathbf{H} \begin{pmatrix}\widetilde{\mathbf{p}}^H_t \\ \widetilde{\mathbf{s}}_t\end{pmatrix}$, subject to the static sectoral Phillips curves

$$\pi^H_t = \mathbf{b}_y \widetilde{y}_t + \Psi \widetilde{\mathbf{s}}_t + \Gamma \mathbb{1} \widetilde{e}_t - \mathcal{V} \chi_t, \tag{D.1}$$

to the structural ToT-definition constraint (56), and to the goods-market restriction (46). With the same multipliers as above, the first-order conditions are

$$\widetilde{y}_t = \frac{1}{1+\varphi}\left(\mathbf{b}_y^\top \boldsymbol{\eta}_t - D_y \vartheta_t\right), \tag{D.2}$$

$$\boldsymbol{\eta}_t = -\mathbf{W}_\pi \boldsymbol{\pi}_t^H, \tag{D.3}$$

$$\mathbf{H}_{pp}\widetilde{\mathbf{p}}_t^H + \mathbf{H}_{ps}\widetilde{\mathbf{s}}_t + \boldsymbol{\xi}_t + \boldsymbol{\beta}\,\vartheta_t = \mathbf{0}, \tag{D.4}$$

$$\mathbf{H}_{sp}\widetilde{\mathbf{p}}_t^H + \mathbf{H}_{ss}\widetilde{\mathbf{s}}_t + \boldsymbol{\xi}_t + \left\{\beta_F - \left[\beta_F + (\Omega_F - \operatorname{diag}(\Omega_F \mathbf{1})\mathcal{N}_s)^\top \boldsymbol{\lambda}\right]\right\}\vartheta_t = \boldsymbol{\Psi}^\top \boldsymbol{\eta}_t, \tag{D.5}$$

$$(\boldsymbol{\Gamma}\mathbb{1})^\top \boldsymbol{\eta}_t + (\mathbf{1} - \mathbb{1})^\top \boldsymbol{\xi}_t + \left(1 - \boldsymbol{\lambda}^\top \operatorname{diag}(\Omega_F \mathbf{1})\mathcal{N}_\tau \mathbb{1}\right)\vartheta_t = 0. \tag{D.6}$$

Combining (D.2) and (D.3) gives the static inflation-output trade-off

$$\mathbf{b}_y^\top \mathbf{W}_\pi \boldsymbol{\pi}_t^H = -(1+\varphi)\widetilde{y}_t - D_y \vartheta_t. \tag{D.7}$$

Equations (D.4) and (D.5) show that this trade-off is intertwined with the welfare costs of cross-sector misallocation in domestic prices, $\widetilde{\mathbf{p}}_t^H$, and international relative prices, $\widetilde{\mathbf{s}}_t$. Indeed, combining (D.4) and (D.5) yields

$$(\mathbf{H}_{sp} - \mathbf{H}_{pp})\widetilde{\mathbf{p}}_t^H + (\mathbf{H}_{ss} - \mathbf{H}_{ps})\widetilde{\mathbf{s}}_t - \left\{\boldsymbol{\beta} + [\Omega_F - \operatorname{diag}(\Omega_F \mathbf{1})\mathcal{N}_s]^\top \boldsymbol{\lambda}\right\}\vartheta_t = \boldsymbol{\Psi}^\top \boldsymbol{\eta}_t. \tag{D.8}$$

Finally, (D.6) characterizes the optimal use of the exchange rate. The term $(\boldsymbol{\Gamma}\mathbb{1})^\top \boldsymbol{\eta}_t$ captures the marginal effect of the exchange rate on inflationary pressures through the LCP component of the Phillips curve, $(\mathbf{1} - \mathbb{1})^\top \boldsymbol{\xi}_t$ captures its marginal effect on sectoral ToT gaps through PCP import prices, and $\left(1 - \boldsymbol{\lambda}^\top \operatorname{diag}(\Omega_F \mathbf{1})\mathcal{N}_\tau \mathbb{1}\right)\vartheta_t$ captures its effect on the goods-market restriction. Thus, in the static structural problem, the planner balances inflation stabilization, aggregate stabilization, and cross-sector misallocation subject to the pass-through structure linking $\widetilde{e}_t$, $\widetilde{\mathbf{s}}_t$, and $\widetilde{y}_t$.

An explicit target criterion under commitment. We now return to $\beta \in (0,1)$ and derive a target criterion by eliminating the multipliers associated with inflation and price-level dynamics. To keep the expression transparent, we impose the standard simplifying restriction that the expectational coefficient in (48) is the identity,[16] so that

$$\boldsymbol{\pi}_t^H = \mathbf{b}_y \widetilde{y}_t + \boldsymbol{\Psi}\widetilde{\boldsymbol{s}}_t + \boldsymbol{\Gamma}\mathbb{1}\widetilde{e}_t + \beta \mathbb{E}_t \boldsymbol{\pi}_{t+1}^H. \tag{D.9}$$

[16] Formally, $\mathcal{V} = \mathbf{0}$. The general case adds matrix-weighted lead–lag terms and $-\beta \mathcal{V}^\top \mathbb{E}_t \boldsymbol{\eta}_{t+1}$ to (59).

In the structural Ramsey problem, the ToT gap and the exchange-rate gap are linked by (56), and the output gap and the exchange-rate gap are linked by (46). Hence, relative to a reduced-form problem in which $\widetilde{\boldsymbol{s}}_t$ is treated as a free wedge and the exchange rate is taken as given, the structural problem features two additional shadow values, $\boldsymbol{\xi}_t$ and ϑ_t.

Using (58) to write

$$\boldsymbol{\nu}_t = \boldsymbol{W}_\pi \boldsymbol{\pi}_t^H + \boldsymbol{\eta}_t - \boldsymbol{\eta}_{t-1},$$

substituting into (59), and shifting (58) forward one period to eliminate $\mathbb{E}_t \boldsymbol{\nu}_{t+1}$, yields the following scalar target criterion after premultiplying by $\mathbf{b}_y^\top$ and using (57) to replace $\mathbf{b}_y^\top \boldsymbol{\eta}_t$ by $(1+\varphi)\widetilde{y}_t + D_y \vartheta_t$:

(D.10)

$$\begin{aligned}&\mathbf{b}_y^\top \boldsymbol{W}_\pi\, \boldsymbol{\pi}_t^H - \beta\, \mathbb{E}_t\left[\mathbf{b}_y^\top \boldsymbol{W}_\pi\, \boldsymbol{\pi}_{t+1}^H\right] + (1+\varphi)\Big((1+\beta)\widetilde{y}_t - \widetilde{y}_{t-1} - \beta\, \mathbb{E}_t \widetilde{y}_{t+1}\Big) \\ &\quad + D_y\Big((1+\beta)\vartheta_t - \vartheta_{t-1} - \beta\, \mathbb{E}_t \vartheta_{t+1}\Big) + \big(\mathbf{b}_y^\top \boldsymbol{\beta}\big)\vartheta_t + \mathbf{b}_y^\top\Big(\boldsymbol{H}_{pp}\widetilde{\mathbf{p}}_t^H + \boldsymbol{H}_{ps}\widetilde{\boldsymbol{s}}_t\Big) + \mathbf{b}_y^\top \boldsymbol{\xi}_t = 0.\end{aligned}$$

Equation (D.10) is a structural target criterion under commitment. The first term captures inflation stabilization. The second term captures the history dependence generated by commitment. The third term reflects that the output gap cannot be moved without moving the exchange rate. The fourth term captures the welfare cost of domestic price-level gaps and sectoral ToT gaps. The final term reflects that the planner cannot choose $\widetilde{\boldsymbol{s}}_t$ independently of $\widetilde{e}_t$ and $\widetilde{\mathbf{p}}_t^H$.

The shadow values $\boldsymbol{\xi}_t$ and ϑ_t are disciplined by the remaining optimality conditions,

(D.11)

$$\begin{aligned}\boldsymbol{\Psi}^\top \boldsymbol{\eta}_t &= \boldsymbol{H}_{sp}\widetilde{\mathbf{p}}_t^H + \boldsymbol{H}_{ss}\widetilde{\boldsymbol{s}}_t + \boldsymbol{\xi}_t \\ &\quad + \Big\{\boldsymbol{\beta}_F - \Big[\boldsymbol{\beta}_F + (\boldsymbol{\Omega}_F - \operatorname{diag}(\boldsymbol{\Omega}_F \mathbf{1})\mathcal{N}_s)^\top \boldsymbol{\lambda}\Big]\Big\}\vartheta_t,\end{aligned}$$

(D.12)

$$(\boldsymbol{\Gamma}\mathbb{1})^\top \boldsymbol{\eta}_t + (\mathbf{1} - \mathbb{1})^\top \boldsymbol{\xi}_t + \Big(1 - \boldsymbol{\lambda}^\top \operatorname{diag}(\boldsymbol{\Omega}_F \mathbf{1})\mathcal{N}_\tau \mathbb{1}\Big)\vartheta_t = 0.$$

Equation (D.11) states that the shadow value of relaxing the ToT-definition constraint is tied to the marginal welfare cost of sectoral relative-price misallocation, to the effect of ToT gaps on sectoral inflation, and to the effect of ToT gaps on the goods-market restriction. Equation (D.12) characterizes the optimal use of the nominal exchange rate. The planner balances the marginal effect of the exchange rate on inflationary pressures through LCP sectors, $(\boldsymbol{\Gamma}\mathbb{1})^\top \boldsymbol{\eta}_t$, its marginal effect on sectoral ToT gaps through PCP sectors, $(\mathbf{1} - \mathbb{1})^\top \boldsymbol{\xi}_t$, and its effect on the output gap through the goods market.

Equation (D.10) highlights four forces shaping optimal policy. First, the relevant inflation object is not a simple constant-weight index: the criterion involves the scalar $\mathbf{b}_y^\top \boldsymbol{W}_\pi \boldsymbol{\pi}_t^H$, which weights sectoral inflation by both welfare weights, $\boldsymbol{W}_\pi$, and the sector-specific sensitivity

of inflation to aggregate demand, summarized by $\mathbf{b}_y$. This is the natural open-economy analogue of Rubbo's sales-and-rigidity-adjusted aggregation.

Second, commitment introduces history dependence. The term $(1+\varphi)\Big((1+\beta)\widetilde{y}_t - \widetilde{y}_{t-1} - \beta\mathbb{E}_t\widetilde{y}_{t+1}\Big)$ captures the intertemporal trade-off that arises because current policy affects expected future inflation.

Third, the criterion features sectoral relative-price misallocation through $\widetilde{\boldsymbol{s}}_t$ and the structural shadow value $\boldsymbol{\xi}_t$. Because $\widetilde{\boldsymbol{s}}_t$ enters both inflation dynamics (D.9) and the welfare loss (54), while $\widetilde{e}_t$ affects the feasible configuration of ToT gaps through (56), the planner generally cannot replicate the closed-economy "divine coincidence" by targeting a constant-weight inflation index. Even if aggregate demand were adjusted to stabilize a particular inflation index, residual inefficient movements in international relative prices, together with the pass-through constraint linking exchange rates and ToT gaps, would continue to generate welfare-relevant trade-offs.

Fourth, the goods-market restriction (46) couples the output gap and the exchange-rate gap. Because the central bank has a single instrument, stabilizing the output gap has a cost in terms of exchange-rate-induced movements in ToT gaps and the CPI, summarized by ϑ_t and its own history-dependent term.

D.2. Optimal Monetary Policy under Discretion

Under discretion the central bank cannot make credible promises about future policy. As in Galí (2015, Ch. 5), each period it chooses the current allocation taking private-sector expectations as given. The only difference with the one-sector benchmark is that the economy has an endogenous state: last period's domestic price gaps enter the inefficient-price residual χ_t in (49) and, through the price-level gaps, the welfare loss. Expectations of future inflation are therefore not independent of today's choices: they depend on the price gaps that the central bank leaves to its successors. We focus on the Markov-perfect equilibrium (Oudiz and Sachs 1985; Söderlind 1999), in which future policy depends only on the state, so that

$$\mathbb{E}_t\boldsymbol{\pi}^H_{t+1} = \boldsymbol{\Phi}\,\widetilde{\mathbf{p}}^H_t + \boldsymbol{\Phi}_\tau\,\boldsymbol{\tau}_t, \tag{D.13}$$

where $\boldsymbol{\Phi}$ measures how expected inflation responds to the price gaps inherited by future policymakers, and $\boldsymbol{\tau}_t = \boldsymbol{\tau}^W_t$ under the tariff-free welfare benchmark. Both $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}_\tau$ are determined in equilibrium. In the one-sector benchmark the price level is not payoff relevant and $\boldsymbol{\Phi} = \mathbf{0}$.

Each period, the central bank minimizes the period loss L_t plus the discounted expected loss from $t+1$ onward. It does so subject to the current-period constraints (50), (56), (46), and

the Phillips curves (48), with (D.13) in place of expected inflation. With the same multipliers as under commitment, the conditions for the output gap, the ToT gaps, and the exchange-rate gap are again (57), (60), and (61). The conditions for inflation and domestic price gaps become[17]

$$\boldsymbol{\eta}_t = \boldsymbol{\nu}_t - \boldsymbol{W}_\pi \boldsymbol{\pi}_t^H, \tag{D.14}$$

$$\boldsymbol{\nu}_t = \beta \mathbb{E}_t \boldsymbol{\nu}_{t+1} - \beta \boldsymbol{\mathcal{V}}^\top \mathbb{E}_t \boldsymbol{\eta}_{t+1} + \beta \boldsymbol{\Phi}^\top (\boldsymbol{I} - \boldsymbol{\mathcal{V}})^\top \boldsymbol{\eta}_t - H_{pp} \widetilde{\mathbf{p}}_t^H - H_{ps} \widetilde{\boldsymbol{s}}_t - \boldsymbol{\xi}_t - \boldsymbol{\beta}\, \vartheta_t. \tag{D.15}$$

Relative to commitment, discretion changes two terms. First, the lagged multiplier $(\boldsymbol{I} - \boldsymbol{\mathcal{V}})^\top \boldsymbol{\eta}_{t-1}$ in (58) disappears. The central bank does not honor past promises, so policy is not history dependent, and it is subject to the usual stabilization bias (Clarida *et al.* 1999). Second, the term $\beta \boldsymbol{\Phi}^\top (\boldsymbol{I} - \boldsymbol{\mathcal{V}})^\top \boldsymbol{\eta}_t$ appears in (D.15). A discretionary central bank can still influence expectations, but only through the price gaps it leaves behind. By contrast, the network term $-\beta \boldsymbol{\mathcal{V}}^\top \mathbb{E}_t \boldsymbol{\eta}_{t+1}$ is common to both regimes. It reflects that lagged prices are a state variable in a multi-sector economy, not the ability to commit.

Target criterion under discretion. Combining (57) and (D.14) gives

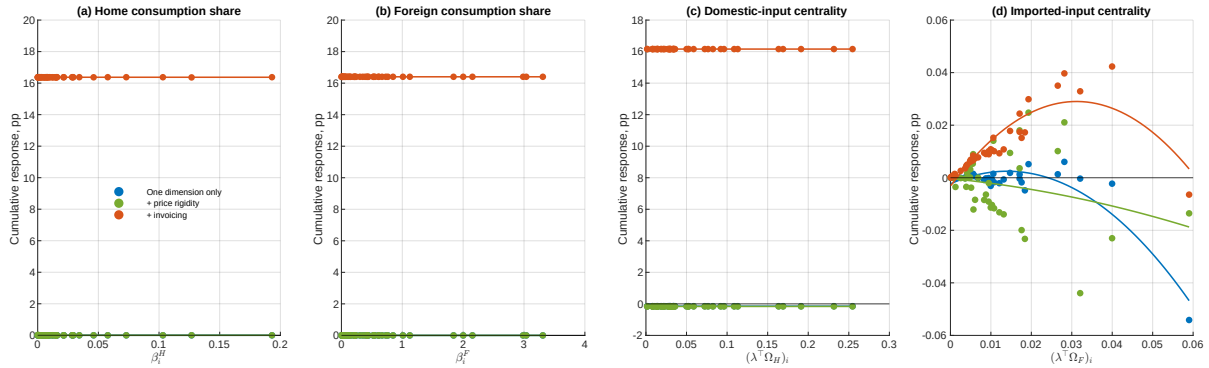
$$(1 + \varphi)\, \widetilde{y}_t = -\, \mathbf{b}_y^\top \boldsymbol{W}_\pi \boldsymbol{\pi}_t^H + \mathbf{b}_y^\top \boldsymbol{\nu}_t - D_y \vartheta_t. \tag{D.16}$$

In the one-sector closed economy, $\boldsymbol{\nu}_t = \mathbf{0}$ and $\vartheta_t = 0$, and (D.16) is the discretionary targeting rule of Galí (2015, Ch. 5): the central bank leans against inflation by contracting the output gap in proportion to it. In the networked open economy, two additional terms appear. The term $\mathbf{b}_y^\top \boldsymbol{\nu}_t$ is the shadow value of the cross-sector price and ToT misallocation that today's price gaps leave for the future, as implied by (D.15). The term $D_y \vartheta_t$ is the exchange-rate cost of moving the output gap. Under commitment, the output gap enters the target criterion (D.10) with its lag and lead; under discretion only its current value matters.

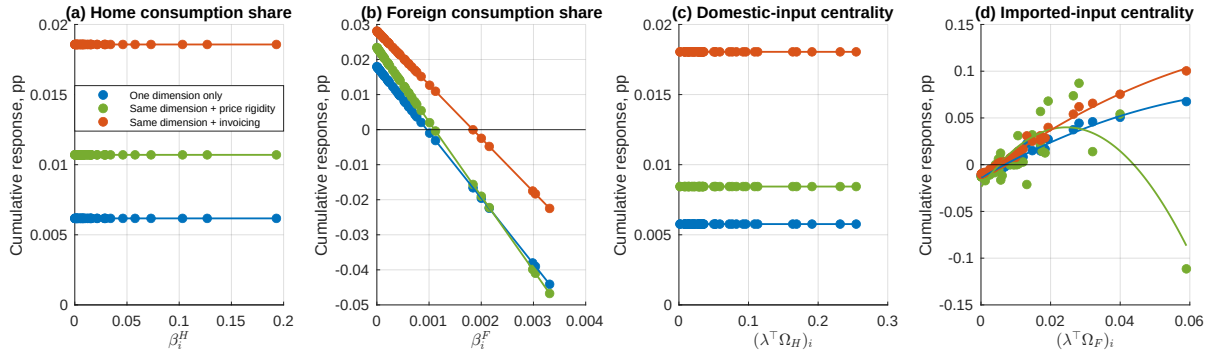
D.3. Sectoral Wedges under Discretion

We solve the linear-quadratic Markov-perfect problem with the welfare loss in Proposition 2, the tariff-free foreign-wedge processes, and the calibration of Section 4.4. Figures D.1–D.4 repeat the experiments of Figures 1–4, replacing commitment with discretion. The Taylor-rule paths and the sectoral counterfactual definitions are unchanged.

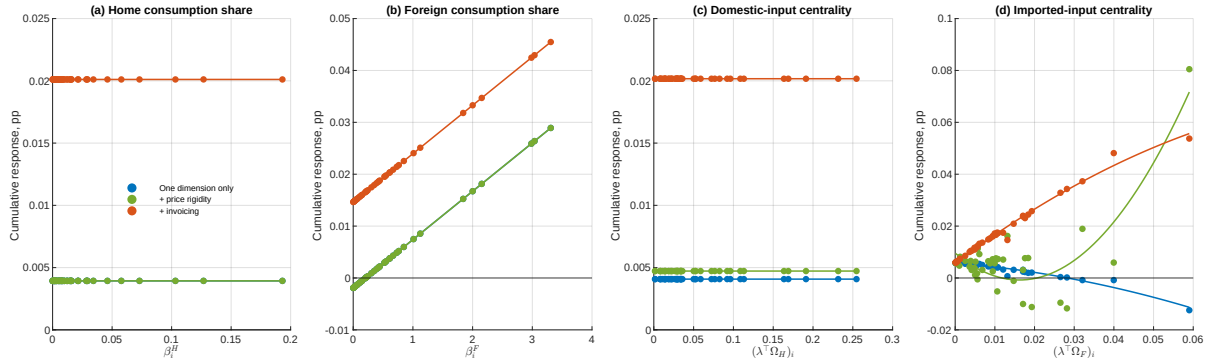
[17] The continuation loss depends on $\widetilde{\mathbf{p}}_t^H$ because it enters $\boldsymbol{\chi}_{t+1}$ and the price dynamics at $t+1$. Its marginal value, by the envelope condition, is $\boldsymbol{\mathcal{V}}^\top \mathbb{E}_t \boldsymbol{\eta}_{t+1} - \mathbb{E}_t \boldsymbol{\nu}_{t+1}$.



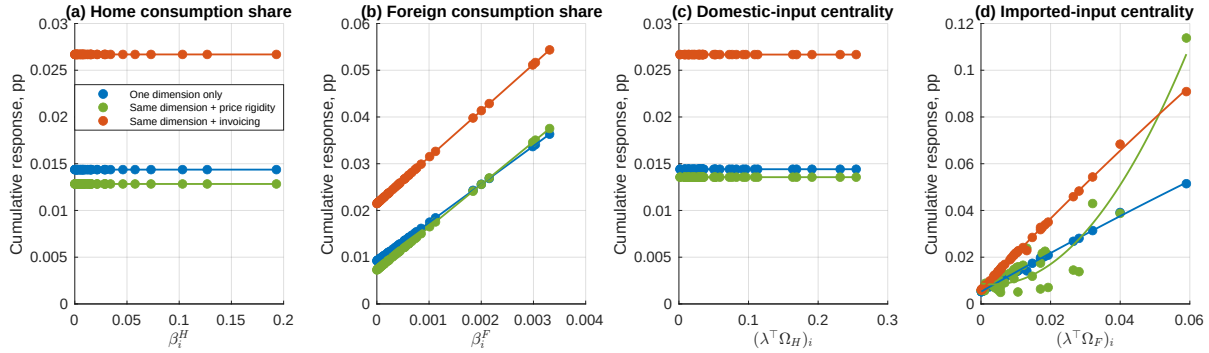
A. Output gap, discretion



B. Output gap, Taylor rule

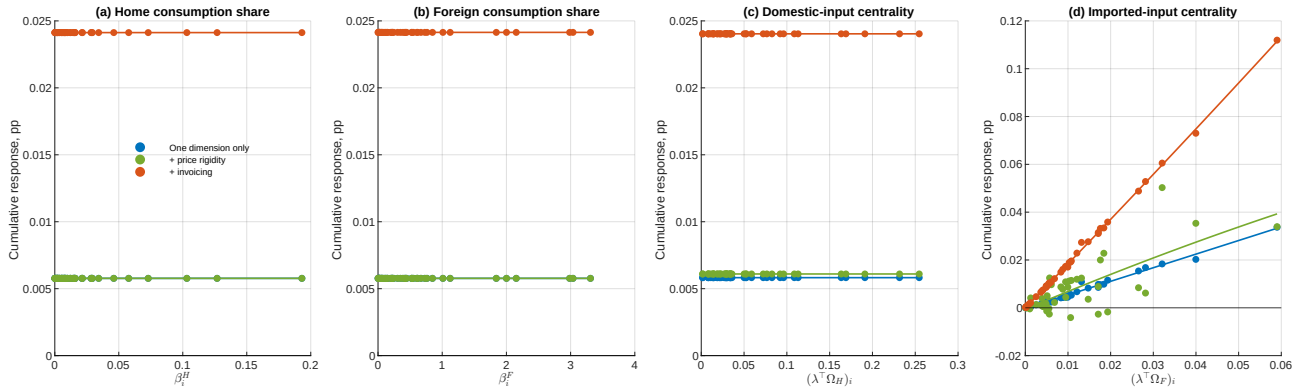


C. CPI inflation, discretion

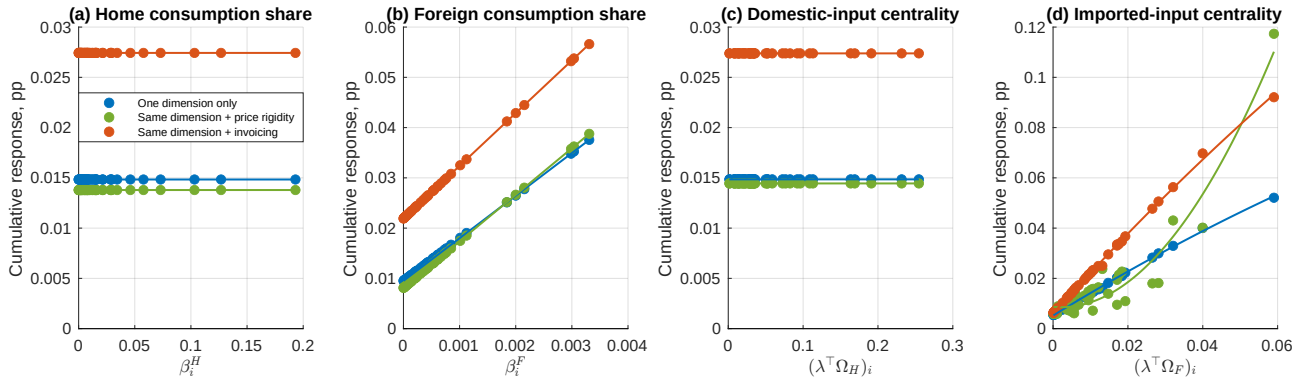


D. CPI inflation, Taylor rule

FIGURE D.1. Eight-quarter cumulative responses to 10 log-point sectoral foreign price wedges. The four horizontal-axis characteristics and three counterfactual economies in each panel match Figure 1.



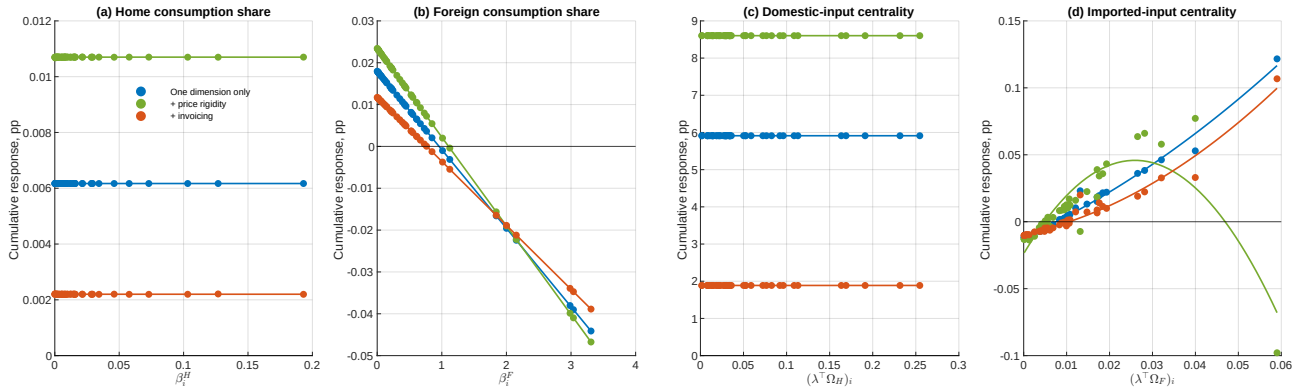
E. Policy rate, discretion



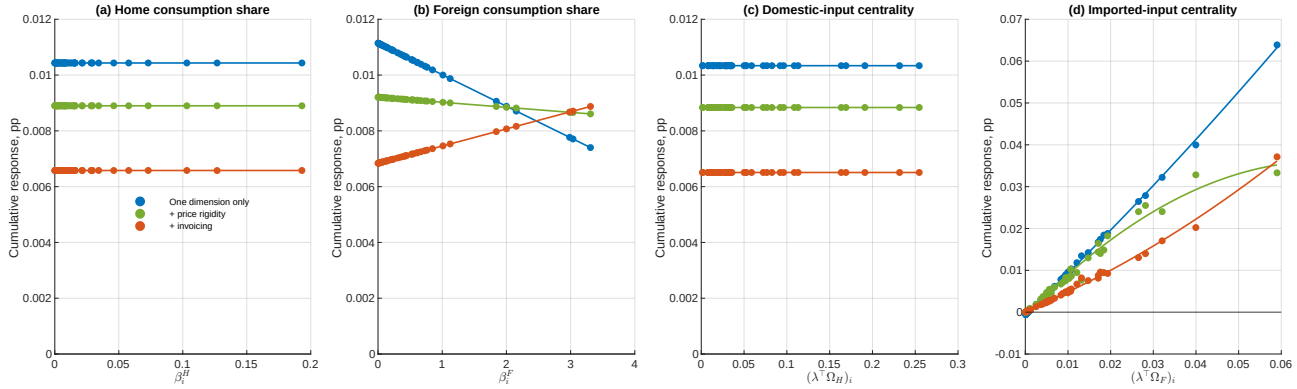
F. Policy rate, Taylor rule

FIGURE D.1. Sectoral foreign price-wedge responses under discretion and the Taylor rule, continued.

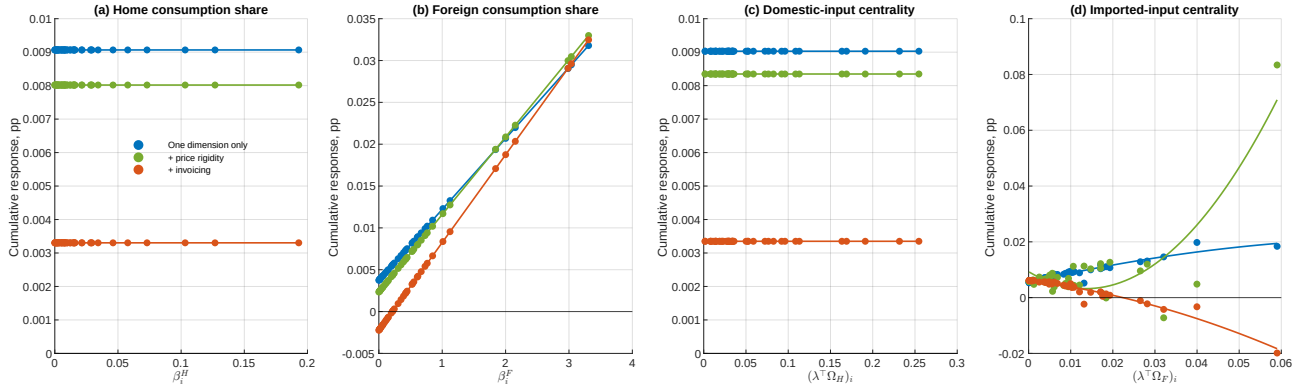
Imported-input centrality remains the clearest organizing characteristic: a wedge in an upstream input reaches many domestic price setters. Heterogeneous invoicing changes the dispersion around this relationship because exchange-rate movements alter the prices paid by PCP-like users but not LCP-like users. Discretion changes the policy response without changing this exposure mechanism.



A. Output gap



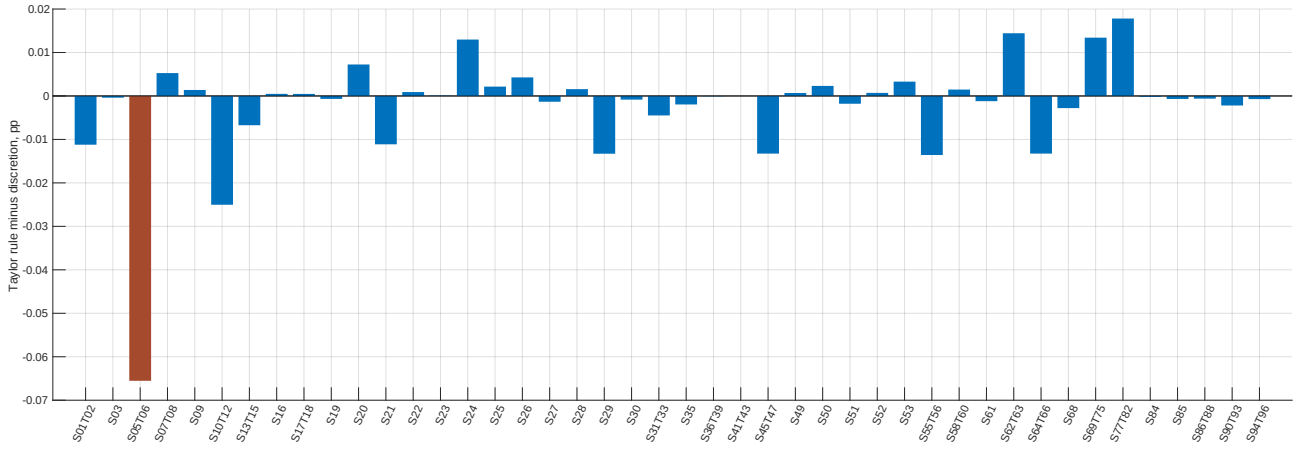
B. CPI inflation



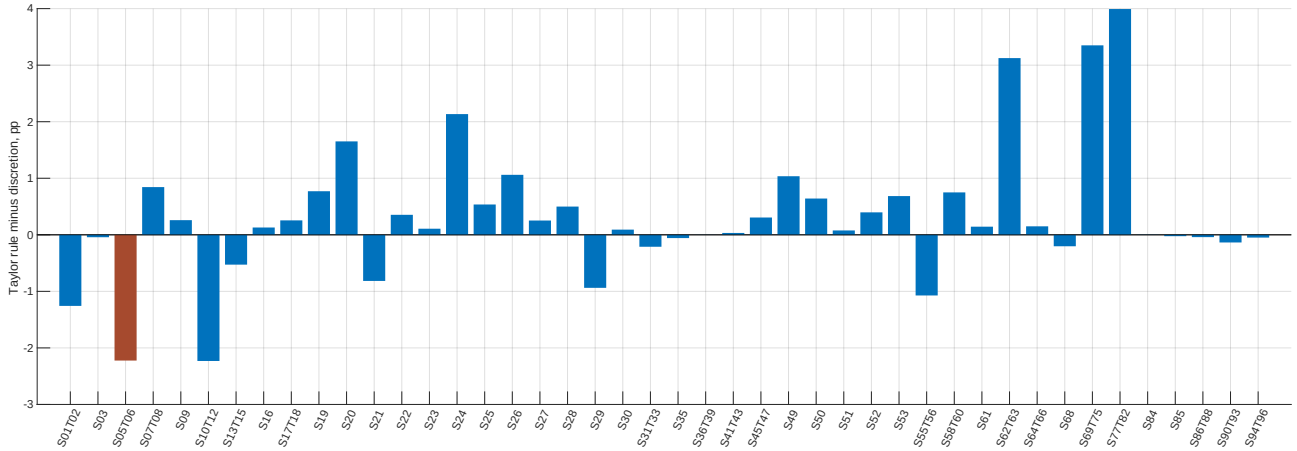
C. Policy rate

FIGURE D.2. Taylor-rule minus discretionary cumulative responses to sectoral foreign price-wedge shocks. The counterfactuals and eight-quarter horizon match Figure 2.

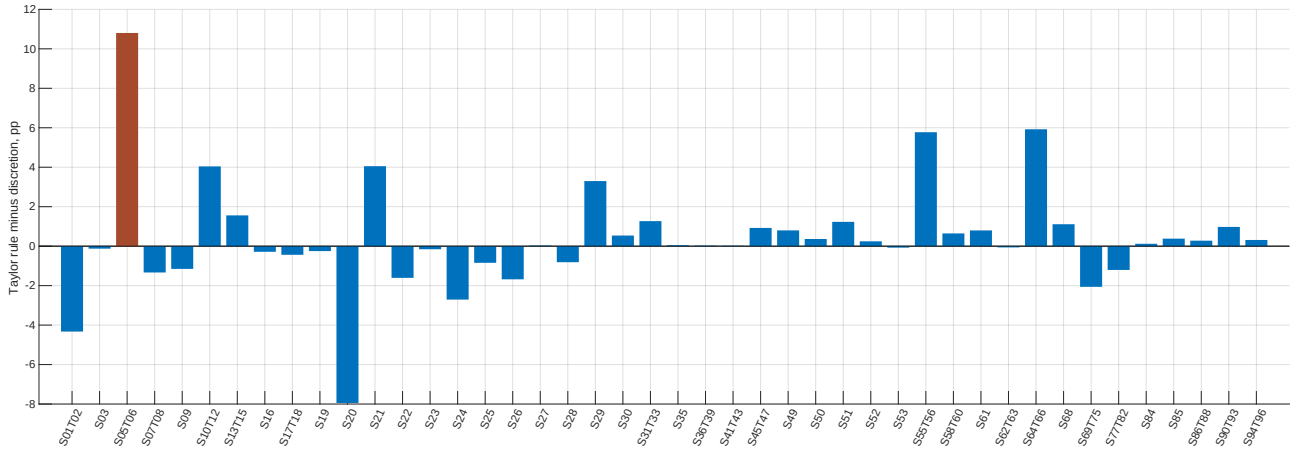
Figure D.2 isolates the policy difference. It need not be uniformly smaller than the commitment comparison: removing promises changes current stabilization and the future price-gap state in different directions. Figure D.3 retains sector identities and confirms that the policy difference is concentrated in a small number of shocks, with *Energy* prominent in the output-gap and policy-rate panels.



A. Output gap



B. CPI inflation



C. Policy rate

FIGURE D.3. Sectoral impact differences, Taylor rule minus discretion, after 10 log-point foreign price-wedge shocks. Bars refer to NACE Rev. 2 sectors; *Energy* (S05T06) is highlighted.

The aggregate *Energy* experiment makes the timing explicit. Impact headline inflation changes little across regimes because the first-round imported price increase is largely

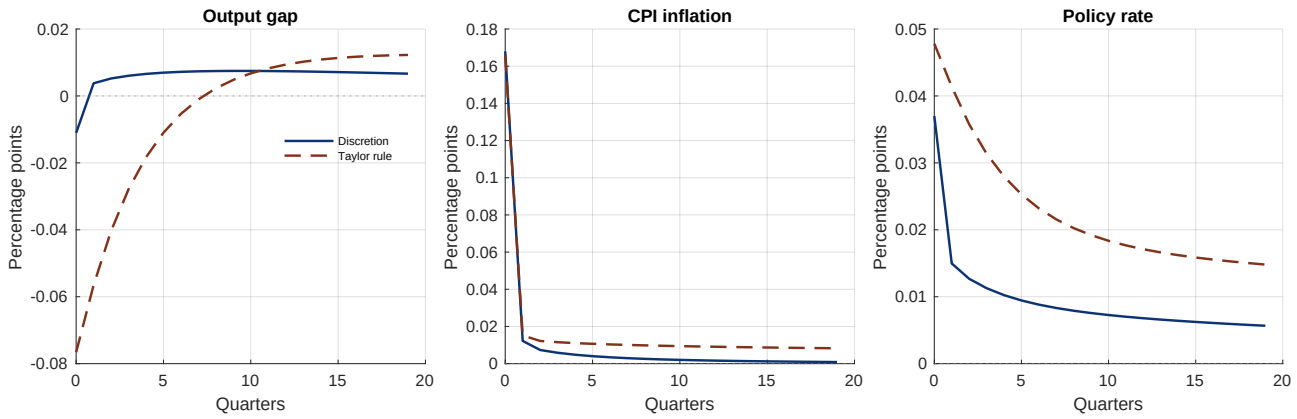


FIGURE D.4. Discretionary and Taylor-rule responses to a 10 log-point foreign price wedge in *Energy* in the fully heterogeneous SOE. Output gap, CPI inflation, and policy rate are in percentage points.

outside domestic monetary control. Discretion raises the nominal rate more and contracts the output gap more than commitment on impact, but less than the Taylor rule. The ranking of cumulative responses need not match the impact ranking because, without commitment, the inherited domestic price gaps alter subsequent policy incentives.

Appendix E. Quantitative Analysis

The analytical results isolate the mechanisms through which sectoral foreign shocks interact with production networks, currency invoicing, and monetary policy. This section takes the same mechanisms to a quantitative multi-country environment. The counterfactuals focus on two disturbances that have been central in recent policy debates: a homogeneous tariff increase by the United States on Chinese goods, and a common increase in the foreign-currency price of energy supplied by the rest of the world to importing economies. The first experiment maps directly into the tariff shocks studied in the small-open-economy analysis; the second captures the type of imported-energy disturbance that played an important role in the recent inflation surge in the Euro Area (Arce *et al.* 2024).

For this purpose, we extend the analytical framework presented in Section 3 on several dimensions. First, the economy contains several countries and a full bilateral input-output structure. Second, nominal wage rigidities operate at the country level. Third, the consumption and production nests distinguish energy from non-energy goods, allowing the substitution elasticities relevant for energy-price shocks to differ from those governing other goods. Fourth, the monetary block allows for country-specific monetary rules and for monetary unions.

The appendix is organized as follows. Sections E.1 and E.2 describe the model and calibra-

tion. Section E.3 then returns to the tariff-free SOE of the main text and compares welfare-based and aggregate-loss simple rules for foreign price-wedge shocks. Section E.4 gives the full multi-country results summarized in Section 4.6. Section E.5 assesses the robustness of the energy OSR to credible adjustments of the policy path.

E.1. Theoretical Model

The quantitative model enriches the analytical framework with features that are standard in quantitative open-economy monetary models and relevant for the tariff and energy experiments. This subsection highlights the main differences relative to the environment of Section 3; Appendix F contains the full characterization of the first-order and equilibrium conditions of the quantitative model.

E.1.1. Assumptions

First, as specified in the general environment in Section 2, we consider $K \geq 2$ countries, potentially different in their preferences and technologies. In particular, we no longer restrict utility (1) to be logarithmic and introduce external habit formation of the keeping-up-with-the-Joneses form (Abel 1990): $U(C_{k,t}, \bar{C}_{k,t-1}) = (C_{k,t} - h_k \bar{C}_{k,t-1})^{1-\sigma}/(1-\sigma)$, where households take lagged aggregate consumption $\bar{C}_{k,t-1}$ as given.

We consider a more general CES production function (6), no longer imposing a unitary elasticity of substitution:

$$Y_{kif,t} = \left[\widetilde{\alpha}_{ki}^{\frac{1}{\psi}} N_{kif,t}^{\frac{\psi-1}{\psi}} + \widetilde{\vartheta}_{ki}^{\frac{1}{\psi}} X_{kif,t}^{\frac{\psi-1}{\psi}} \right]^{\frac{\psi}{\psi-1}\psi_{ki}}, \tag{E.1}$$

where $\psi_{ki} \in (0, 1]$ modulates the degree of returns to scale in production. Similarly, we assume a CES structure for the consumption (2) and intermediate (9) good aggregators. Given the quantitative focus on energy shocks, we introduce an additional layer to distinguish between energy and non-energy goods,[18]

$$C_{k,t} = \left[\widetilde{\beta}_k^{\frac{1}{\gamma}} C_{kE,t}^{\frac{\gamma-1}{\gamma}} + \left(1 - \widetilde{\beta}_k\right)^{\frac{1}{\gamma}} C_{kM,t}^{\frac{\gamma-1}{\gamma}} \right]^{\frac{\gamma}{\gamma-1}}, \tag{E.2}$$

$$X_{ki,t} = \left[\widetilde{\beta}_{ki}^{\frac{1}{\phi}} X_{kiE,t}^{\frac{\phi-1}{\phi}} + (1 - \widetilde{\beta}_{ki})^{\frac{1}{\phi}} X_{kiM,t}^{\frac{\phi-1}{\phi}} \right]^{\frac{\phi}{\phi-1}}, \tag{E.3}$$

[18] This specification allows us to introduce a specific elasticity of substitution of the energy consumption that does not necessarily need to be equal to the elasticity of substitution between the rest of goods and services.

where $C_{kE,t}$ and $C_{kM,t}$ denote the consumption of energy and non-energy goods, respectively, and $X_{kiE,t}$ and $X_{kiM,t}$ sector i intermediate goods' demand for energy and non-energy goods, respectively. These are given by:

$$\text{(E.4)} \qquad C_{kE,t} = \left[\sum_{i\in I_E} \widetilde{\nu}_{ki}^{\frac{1}{\eta}} C_{ki,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}, \qquad C_{kM,t} = \left[\sum_{i\in I_M} \widetilde{\upsilon}_{ki}^{\frac{1}{\iota}} C_{ki,t}^{\frac{\iota-1}{\iota}}\right]^{\frac{\iota}{\iota-1}},$$

$$\text{(E.5)} \qquad X_{kiE,t} = \left[\sum_{j\in I_E} \widetilde{\nu}_{kij}^{\frac{1}{\chi}} X_{kij,t}^{\frac{\chi-1}{\chi}}\right]^{\frac{\chi}{\chi-1}}, \qquad X_{kiM,t} = \left[\sum_{j\in I_M} \widetilde{\upsilon}_{kij}^{\frac{1}{\xi}} X_{kij,t}^{\frac{\xi-1}{\xi}}\right]^{\frac{\xi}{\xi-1}},$$

where I_E and I_M denote the sets of sectors producing energy and non-energy goods, respectively. The aggregation over sectoral goods produced in different countries is given by CES aggregators:

$$\text{(E.6)} \qquad C_{ki,t} = \left[\sum_{l=1}^{K} \widetilde{\zeta}_{kli}^{\frac{1}{\delta}} C_{kli,t}^{\frac{\delta-1}{\delta}}\right]^{\frac{\delta}{\delta-1}} \quad \text{and} \quad X_{kij,t} = \left[\sum_{l=1}^{K} \widetilde{\zeta}_{klij}^{\frac{1}{\mu}} X_{klij,t}^{\frac{\mu-1}{\mu}}\right]^{\frac{\mu}{\mu-1}}.$$

Second, we consider the case of incomplete financial markets, where only one international bond is traded, denominated in country K's currency. This means that an analogous risk-sharing condition of the type (31) no longer holds.

Third, regarding the price-setting structure, we dispense with the sectoral production subsidies $\tau_{k,i}^{p}$, allowing for positive profits at steady state. In addition, we consider the DCP paradigm, on top of PCP and LCP (Devereux and Engel 2003; Gopinath *et al.* 2020).

Fourth, we allow non-reset prices to be partially indexed to lagged inflation. Specifically, prices that cannot be reoptimized inherit a weighted combination of lagged inflation in the same sector and lagged producer-country CPI inflation. This specification follows the medium-scale monetary models of Christiano *et al.* (2005) and Smets and Wouters (2007), and introduces intrinsic inflation persistence without altering the Calvo timing assumption.

Fifth, we also allow for nominal wage rigidities (Erceg *et al.* 2000), heterogeneous across countries but homogeneous across sectors, and assume that non-reset wages are partially indexed to lagged CPI inflation.

Finally, we assume that the (log-)tariff, introduced in (5), follows an AR(1) process:

$$\text{(E.7)} \qquad \tau_{kli,t} = \rho_{kli}^{\tau} \tau_{kli,t-1} + \varepsilon_{kli,t}^{\tau}$$

where $\varepsilon_{kli,t}^{\tau} \sim \mathcal{N}\left(0, (\sigma_{kli}^{\tau})^2\right)$. By contrast, we assume that the (log-)price wedge introduced in

(5) follows an AR(2) process:

$$\tau^{w}_{kl\,i,t} = \rho^{\tau,w}_{1,kl\,i}\tau^{w}_{kl\,i,t-1} + \rho^{\tau,w}_{2,kl\,i}\tau^{w}_{kl\,i,t-2} + \varepsilon^{\tau,w}_{kl\,i,t} \tag{E.8}$$

where $\varepsilon^{\tau,w}_{kl\,i,t} \sim \mathcal{N}\left(0, (\sigma^{\tau,w}_{kl\,i})^2\right)$.

E.1.2. Monetary Authority

There is a monetary authority in each country $k \in K$. We distinguish between countries that conduct an independent monetary policy and countries that are members of a currency union.

Non-members of Currency Unions. Each central bank follows a Taylor rule:

$$i_{k,t} = \rho_{kr} i_{k,t-1} + (1-\rho_{kr})\left(\phi_{k\pi}\pi_{k\phi,t} + \phi_{k\,y}\widehat{y}_{k,t} + \phi_{k,\Delta\pi}\Delta\pi_{k\phi,t} + \phi_{k,\Delta\,y}\Delta\widehat{y}_{k,t}\right) + \varepsilon^{r}_{k,t} \tag{E.9}$$

where the coefficients are allowed to vary by country; ρ_{kr} denotes the degree of interest rate smoothing in the monetary instrument, coefficients $\{\phi_{k\pi}, \phi_{k\,y}\}$ modulate the elasticity of the policy rate with respect to changes in a given inflation index $\pi_{k\phi,t}$, and output $\widehat{y}_{k,t}$, measured as the log-deviation from its steady-state value. The Taylor rule features a monetary policy shock, which follows $\varepsilon^{r}_{k,t} \sim \mathcal{N}\left(0, (\sigma_{kr})^2\right)$.

Furthermore, we allow the monetary authority to choose the particular inflation measure (CPI, PCE, PPI, GDP deflator, etc.) that they aim to stabilize, $\pi_{k\phi,t} = \mathbf{\Phi}^{\intercal}\boldsymbol{\pi}_{k,t} = \sum_{l=1}^{K}\sum_{i=1}^{I}\phi_{kl\,i}\pi_{kl\,i,t}$, where $\sum_{l=1}^{K}\sum_{i=1}^{I}\phi_{kl\,i} = 1$, and $\pi_{kki,t} = \pi_{ki,t}$. For example, when $\phi_{kl\,i}$ is equal to the consumption share of sector i in the country k, then the central bank targets headline inflation.

Members of Currency Unions. Suppose that a subset $K^{MU} \subset K$ of countries belongs to a monetary union. Without loss of generality, we assume that the central bank of a country $k^{MU} \in K^{MU}$ sets the nominal interest rate to stabilize the *union-wide* price inflation index and output deviations, π^{MU}_{t} and $\widehat{y}^{MU}_{t}$,[19]

$$i_{k^{MU},t} = \rho^{\mathrm{MU}r}_{k^{\mathrm{MU}}} i_{k^{MU},t-1} + \left(1-\rho^{\mathrm{MU}r}_{k^{\mathrm{MU}}}\right)\left(\phi_{\mathrm{MU}\pi}\pi^{\mathrm{MU}}_{t} + \phi_{\mathrm{MU}\,y}\widehat{y}^{\mathrm{MU}}_{t} + \phi_{\mathrm{MU},\Delta\pi}\Delta\pi^{\mathrm{MU}}_{t} + \phi_{\mathrm{MU},\Delta\,y}\Delta\widehat{y}^{\mathrm{MU}}_{t}\right) + \varepsilon^{r}_{\mathrm{MU},t} \tag{E.10}$$

where π^{MU}_{t} and $\widehat{y}^{MU}_{t}$ are defined as the GDP-weighted sum of member states' price inflation and output deviations, $\pi^{\mathrm{MU}}_{t} = \sum_{k=1}^{K}\phi^{\mathrm{MU}}_{k}\pi_{k\phi,t}$ and $\widehat{y}^{\mathrm{MU}}_{t} = \sum_{k=1}^{K}\phi^{\mathrm{MU}}_{k}\widehat{y}_{k,t}$, where

[19] The specific location of the union-wide central bank is innocuous as long as it targets union-wide variables.

$\phi_k^{\text{MU}} = y_k / \sum_{l=1}^{K^{\text{MU}}} y_l$ is the measure of the (steady-state) relative size of country k in the monetary union in terms of nominal GDP.

The central banks in the rest of countries $l \neq k^{MU}$ that belong to the monetary union adopt a peg vis-a-vis the country k^{MU} that sets the monetary stance:

$$\mathcal{E}_{k,k^{MU},t} = \mathcal{E}_{k,k^{MU}} \quad \forall k \in K^{MU} \tag{E.11}$$

where $\mathcal{E}_{k,k^{MU}}$ is the bilateral nominal exchange rate in steady state.

E.2. Calibration

The quantitative model is solved at quarterly frequency for four regions: the Euro Area, China, the rest of the world, and the United States. The United States is the currency of denomination of the internationally traded bond.

Multi-sectoral Dimension. Each region contains 44 sectors. The sectoral classification is chosen to preserve the input-output structure of the data while separating the energy block into mining and crude petroleum, refined petroleum products, and electricity and gas. The bilateral expenditure shares, input-output coefficients, and labor shares are constructed from the international input-output data (ICIO, OECD). The calibration therefore preserves both the domestic production network and the bilateral sourcing patterns that determine the exposure of each country-sector pair to tariffs and foreign-price shocks.

Table E.1 summarizes the main externally calibrated parameters.

Households. Preferences are standard at quarterly frequency. The discount factor is $\beta = 0.99$, the intertemporal elasticity is one, the Frisch elasticity is one, and external habit formation is set to $h = 0.70$. The consumption aggregator has a low elasticity between energy and non-energy goods, $\gamma_C = 0.10$, while substitution within the energy and non-energy blocks is higher, $\eta_C = \iota_C = 0.90$. The trade elasticity in final demand is set to one in the baseline.

Firms. In production, the elasticity between labor and intermediate inputs is $\psi = 0.50$. Energy and non-energy intermediates are also difficult to substitute, with $\gamma_X = 0.10$, and within-block substitution in intermediate demand is set to $\eta_X = \iota_X = 0.20$. These values make short-run production relatively exposed to energy and imported-input cost shocks, as in quantitative multi-sector models with low input substitutability (Atalay 2017; Böhringer and Rivers 2021).

Parameter	Value	Description
Households		
β	0.99	Quarterly discount factor
σ	1	Inverse intertemporal elasticity of substitution
φ	1	Inverse Frisch elasticity of labor supply
h	0.7	External habit in consumption
γ_C	0.1	Elasticity between energy and non-energy consumption
η_C	0.9	Elasticity within energy consumption block
ι_C	0.9	Elasticities within non-energy consumption block
δ_C	1	Armington elasticity in final demand
Firms		
ψ	0.5	Elasticity between labor and intermediate inputs
γ_X	0.1	Elasticity between energy and non-energy intermediates
η_X	0.2	Elasticity within energy intermediate block
ι_X	0.2	Elasticity within non-energy intermediate block
δ_X	1	Armington elasticity in intermediate demand
Nominal Rigidities		
θ_i	sectoral	Sectoral Calvo probabilities from micro price data
θ^w	0.75	Calvo probability for wages
χ^w	0.5	Wage indexation to lagged CPI inflation
ι^p	0.2	Price indexation to own sectoral lagged inflation
χ^p	0.1	Price indexation to producer-country CPI inflation
Monetary Policy		
ρ_r	0.7	Interest-rate smoothing in the Taylor rule
ϕ_π	1.50	Response to CPI inflation (Galí 2015)
ϕ_y	0	Response to (log-)output
$\phi_{\Delta\pi}$	0	Response to CPI inflation growth
$\phi_{\Delta y}$	0.125	Response to output growth
Foreign Shocks		
ρ^τ	0.96	Persistence of tariff shocks
$\rho_1^{\tau,w}$	1.17	Persistence of foreign-price wedges
$\rho_2^{\tau,w}$	-0.20	Persistence of foreign-price wedges

Notes: The table reports the baseline quarterly calibration used in the quantitative exercises. Data-based objects, including expenditure shares, input-output shares, labor shares, and sectoral weights, are allowed to vary by country and sector.

TABLE E.1. Main Parameters in the Quantitative Calibration

Staggered Price- and Wage-Setting. Nominal rigidities are disciplined by micro evidence. Sectoral price stickiness is based on the distribution of price changes in Gautier *et al.* (2024), with energy sectors assigned near-flexible prices following the evidence summarized by Dhyne *et al.* (2006). The baseline assigns the same sectoral Calvo vector to the four regions, which keeps cross-country differences in the quantitative exercises tied to expenditure shares, production networks, trade linkages, monetary regimes, and invoicing rather than to country-specific price durations. Wage setting follows Erceg *et al.* (2000), with a quarterly Calvo probability of 0.75 and partial indexation to lagged CPI inflation. Non-reset prices inherit a weighted average of lagged inflation in the same sector and lagged producer-country CPI inflation, as in medium-scale monetary models with intrinsic inflation persistence (Christiano *et al.* 2005; Smets and Wouters 2007).

Currency Invoicing. The invoicing calibration combines producer-currency pricing, local-currency pricing, and dominant-currency pricing. The assignment of sectors to invoicing regimes is based on the empirical evidence that trade prices are neither exclusively producer-currency nor local-currency denominated, and that the U.S. dollar has a disproportionate role in the pricing of international transactions (Boz *et al.* 2020; Gopinath *et al.* 2020); Table G.3 in Appendix G.3 reports the sector-by-sector assignment. This feature is relevant for both counterfactuals. In the tariff experiment, it governs how the exchange rate maps into the domestic-currency price of Chinese goods purchased by the United States and by other regions through the production network. In the energy experiment, it determines how a common foreign-currency energy-price disturbance is transmitted into local marginal costs and headline CPI inflation.

Monetary Policy. The monetary calibration follows the rule in (E.9) and (E.10). The baseline rule responds to CPI inflation and output growth, with $\phi_\pi = 1.5$ and $\phi_{\Delta y} = 0.125$, and features interest-rate smoothing $\rho_r = 0.70$. The quantitative optimal-simple-rule exercises use the same instrument but choose the coefficients to minimize a quadratic objective in union output growth, CPI inflation, and interest-rate changes. Section E.3 assesses the corresponding aggregate criterion in the tractable SOE before it is used here.

Foreign Shocks. The stochastic processes match the two counterfactuals. Tariff shocks follow (E.7) with persistence 0.96. We use an AR(1) as a parsimonious representation of the persistence of trade-policy changes. The active tariff innovations are bilateral between the United States and China and apply to the set of sectors classified as tariff-relevant in the quantitative data. The results normalize the shock as a homogeneous increase in U.S. tariffs on Chinese goods. Foreign-price wedges follow (E.8). In the energy-price experiment, the

Objective	ρ_i	ϕ_π	ϕ_y	$\phi_{\Delta\pi}$	$\phi_{\Delta y}$
Welfare loss	0.986	1.000	2.414	15.000	10.554
Aggregate quadratic loss	0.996	10.471	0.289	0.000	9.771

Notes: The search imposes $0.01 \le \rho_i \le 0.9999$, $1 \le \phi_\pi \le 15$, $0 \le \phi_y \le 5$, $0 \le \phi_{\Delta\pi} \le 15$, and $0 \le \phi_{\Delta y} \le 15$. Some coefficients reach these bounds. It uses three starting points followed by local refinement and determinate first-order solutions.

TABLE E.2. Simple rules for imported-energy wedges in the SOE

active disturbance is a single common innovation that raises the rest-of-world energy-price wedge faced by each importing economy identically. In contrast to the tariff process, its AR(2) coefficients, (1.17, −0.20), are estimated from quarterly log Brent crude oil prices, as in Aguilar *et al.* (2026). Unlike a tariff, this wedge accrues to the energy exporter, so it operates as an international transfer—the imported-energy bill—rather than as domestic revenue. This specification isolates the imported-energy channel while keeping the global production network, currency invoicing, and monetary-policy block active.

E.3. From Commitment Policy to Simple Rules

The tariff-free SOE of Sections 3 and 4 permits a direct comparison between the commitment allocation and implementable interest-rate rules after foreign price-wedge shocks. The multi-country model considered next uses a different, operational objective. Here we assess how much that objective changes policy in the tractable SOE.

We retain the instrument rule (62). One rule minimizes the welfare loss (54); another minimizes the aggregate quadratic criterion

$$L_t^Q = (\pi_t^C)^2 + 0.2\,\widetilde{y}_t^2 + 1.4\,(\Delta i_t)^2. \tag{E.12}$$

The second objective is a policy criterion, not a second-order approximation to household utility. Both searches minimize the conditional discounted loss $(1-\beta)\mathbb{E}_0 \sum_{t\ge 0} \beta^t L_t$, with $\beta = 0.99$, zero prehistory, and innovations in the three imported-energy sectors. A discounted Lyapunov equation evaluates the infinite sum. Table E.2 reports the resulting coefficients; the local search does not certify global optimality.

Relative to commitment policy, the welfare-based simple rule raises the welfare loss by 0.45% in this shock environment; the rule chosen under L_t^Q raises it by 10.37%. Figure E.1 shows the response to a wedge in *Energy*. Impact CPI inflation is similar across policies, but the aggregate-loss rule contracts output more than the welfare-based rule. The commitment interest rate is recovered from the Euler equation; the simple rules set it directly.

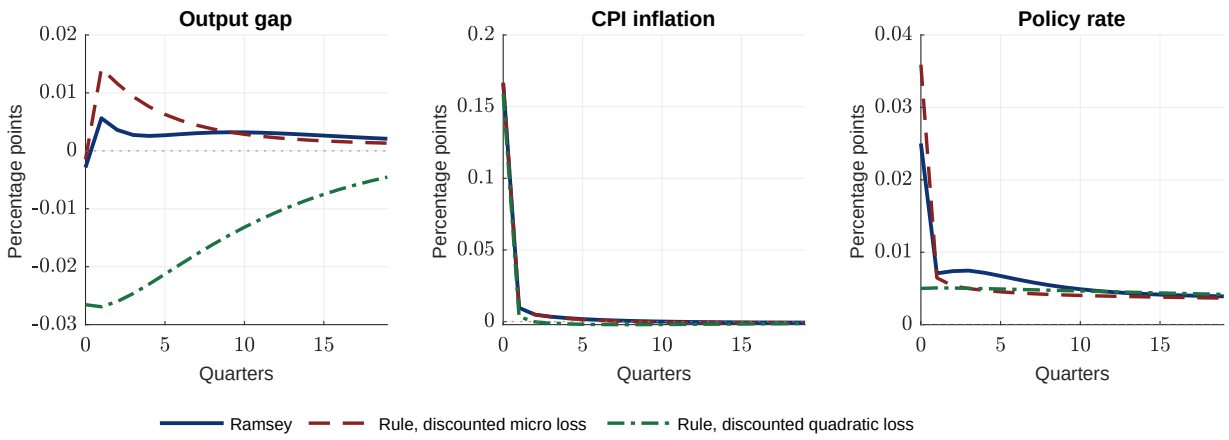


FIGURE E.1. Commitment and optimal-simple-rule responses to a 10 log-point foreign price wedge in *Energy*. Rates and inflation are in percentage points.

The quantitative exercise replaces the SOE output gap in L_t^Q with union output growth. Its results therefore compare stabilization rules under an observable aggregate objective, rather than consumption-equivalent welfare.

E.4. Optimal Simple Rules under Tariff and Energy Shocks

This subsection takes the calibrated model to the question that motivates the paper: how should the monetary authority of a networked open economy respond to tariff and imported-energy shocks, and how far does a standard inertial Taylor rule fall short of that benchmark? We focus on the Euro Area (EA) central bank and compare the baseline Taylor rule of Section E.2 with an optimal simple rule (OSR) of the same functional form. The OSR chooses the three coefficients $\{\rho_r, \phi_\pi, \phi_{\Delta y}\}$ in (E.10) to minimize a quadratic loss in union output growth, CPI inflation, and interest-rate changes. Section E.3 shows what this operational criterion captures, and where it departs from Ramsey policy, in the tractable SOE. The exercise here asks whether the network mechanisms isolated in the main text remain quantitatively important in a multisectoral multi-country calibration.

We study three experiments, each scaled to a 10 log-point shock so that magnitudes are comparable with the SOE exercises: a homogeneous U.S. tariff on Chinese goods, the reciprocal Chinese tariff on U.S. goods, and a common rest-of-world energy-price shock to the importing economies. In the two tariff experiments the EA is a third party, exposed only through the global production network, world relative prices, and currency invoicing; in the energy experiment it is directly exposed through its imported-energy bill. The OSR coefficients are re-optimized separately for the tariff environment and for the energy environment, reflecting that the optimal rule depends on the shocks the economy is stabilizing.

Each rule is optimized over the unconditional stabilization loss induced by the relevant family of shocks, not over a single impulse. In the tariff environment the active shocks are the bilateral U.S.–China sectoral tariff innovations, in both directions and across all tariffed sectors,[20] drawn independently; these are tariff wedges, with revenue rebated domestically, and follow an AR(1) with persistence 0.96. In the energy environment the disturbance is a single common rest-of-world energy-price innovation that raises the imported energy input of all three importing economies (the Euro Area, the United States, and China) identically; it is a price wedge—an international transfer toward the energy exporter rather than domestic revenue—and follows the estimated AR(2). The impulse responses reported below normalize, for transparency, a homogeneous 10 log-point realization of each family.

This split carries over the distinction identified in the analytical economy, while changing the incidence of the tariff experiment. The SOE is directly exposed to its own import tariff; the EA is a bystander to bilateral U.S.–China tariffs. In both settings, network centrality and invoicing determine which foreign price movements reach domestic producers, but bystander exposure also scales their aggregate importance. The energy shock instead hits the EA directly and combines a large first-round headline component from a flexible-price imported input with a downstream second round inside sticky-price sectors. The quantitative exercise adds the global input–output structure, wage rigidity, indexation, energy-specific substitution, and a monetary union.

Optimal coefficients. Table E.3 reports the baseline and optimized coefficients. Two features stand out. First, both optimal rules are far more inertial than the baseline, with smoothing coefficients between 0.92 and 0.96. Highly inertial rules approximate the commitment solution: they concentrate the stabilization burden in expectations of sustained future responses rather than in large realized movements of the policy rate (Rotemberg and Woodford 1997; Woodford 2003). Second, the composition of the optimal response flips across environments. Against tariffs, the rule is in effect inertial strict inflation targeting: $\phi_\pi \approx 14.9$ with $\phi_{\Delta y}/\phi_\pi \approx 0.09$. Against the energy shock, the rule shifts toward a speed-limit policy in which the output-growth response is of the same order as the inflation response: $\phi_{\Delta y}/\phi_\pi \approx 0.92$. The tariff rule reflects an approximate quantitative property of the calibrated bystander exposure: the domestic trade-off created by the foreign tariff is small, so stabilizing CPI inflation is nearly costless for the EA. It is not a general property of tariff shocks in a directly exposed economy. The energy rule is the instrument-rule expression of the core-versus-headline result of Bodenstein *et al.* (2008): when the inflationary impulse originates in a flexible-price imported

[20] A sector is classified as tariffed if the observed 2025 U.S.–China schedule places a nonzero effective tariff on it in either direction, which selects 22 of the 44 sectors—broadly agriculture, mining, and manufacturing—and leaves utilities, construction, and services untariffed.

	ρ_r	ϕ_π	$\phi_{\Delta y}$
Baseline Taylor rule	0.70	1.50	0.125
OSR — tariff environment	0.92	14.87	1.34
OSR — energy environment	0.96	6.84	6.30

Notes: Coefficients of the union Taylor rule (E.10). The OSR minimizes a quadratic loss in union output growth, CPI inflation, and interest-rate changes, over the indicated shock environment. The remaining coefficients ϕ_y and $\phi_{\Delta\pi}$ are set to zero.

TABLE E.3. Baseline and Optimal Simple Rule Coefficients (EA)

input, the optimized rule redirects its response toward the component of the allocation that monetary policy can actually stabilize.

EA responses. Figure 5 reports the EA responses of real GDP growth, CPI inflation, and the policy rate under the baseline Taylor rule and under the OSR, with one row per experiment. A first glance at the scales conveys the central quantitative message: for the EA, the spillover from a 10% bilateral tariff war between the United States and China is several times to an order of magnitude smaller than the effect of an energy shock of the same size. The model therefore assigns the EA central bank a watching brief in the trade war but an active and difficult stabilization problem in the energy crisis—a ranking consistent with the euro-area experience of 2018–19 relative to 2022 and 2026.

U.S. tariff on China. The tariff reaches the EA through four channels that the model separates explicitly. First, an imported-input cost channel: although the EA does not levy the tariff, the shock changes equilibrium world prices, exchange rates, and dollar-invoiced import prices for Chinese intermediates used directly and indirectly by EA producers. Second, trade diversion: Chinese exportables displaced from the U.S. market lower the relative price of EA imports from China, a disinflationary force that builds gradually. Third, a demand spillover from the slowdown in the two largest economies. Fourth, the exchange-rate adjustment that reconciles these flows under dominant-currency pricing, which mutes the conventional expenditure-switching response (Gopinath *et al.* 2020; Egorov and Mukhin 2023). On net, EA CPI inflation rises by about 0.05 percentage points on impact—the cost-push channel dominates initially—and then dips below baseline as trade diversion takes over, while GDP growth contracts and recovers. The two rules resolve this small trade-off differently. The OSR, with its near-strict inflation coefficient, raises the policy rate on impact by roughly fifty percent more than the baseline rule; it accepts a deeper impact contraction in GDP growth (about -0.03 versus -0.01 percentage points) in exchange for an inflation impact roughly one-third smaller and a milder subsequent disinflation. Consistent with the SOE results of

Section 4.5, the size of these responses is organized by the imported-input centrality of the tariffed flows: the tariff matters for the EA not because Chinese goods loom large in final consumption, but because the affected goods are central inputs in the production network.

Chinese tariff on the United States. The reciprocal experiment delivers qualitatively similar EA dynamics at one-third to one-quarter of the scale. The asymmetry is informative about the transmission mechanism. China's tariff on U.S. goods reaches the EA mainly through lower U.S. activity and the reallocation of U.S. exportables, whereas the U.S. tariff on China moves a set of Chinese intermediates that are more central in the EA's imported-input network and disproportionately dollar-invoiced. Two wedges of identical statutory size therefore generate very different spillovers depending on where they sit in the global input–output structure—the open-economy, third-country analogue of the result that network position, not sector size, determines the aggregate footprint of sectoral shocks, and the quantitative confirmation of the location dependence derived in Section 3.3. For policy, the implication is that a bystander central bank cannot read its exposure to a trade war off bilateral trade shares; the relevant statistic is the network centrality of the tariffed flows in domestic production.

Rest-of-world energy shock. The energy experiment is where the monetary trade-off is steep, and where the choice of rule matters most. The impact response of headline inflation (about 0.17 percentage points) is virtually identical across the two rules: with nearly flexible energy prices, the first-round effect is determined by the shock and the exchange rate, not by policy. This is the quantitative expression of the first-round/second-round distinction of Bodenstein *et al.* (2008). The rules differ in how they treat the second round. The baseline Taylor rule responds mechanically to headline inflation, more than doubles the OSR's impact increase in the policy rate, and keeps the rate elevated for years; yet it ends up with *more* persistent inflation and a GDP-growth contraction nearly twice as deep on impact. The OSR instead raises the rate by less than half as much on impact and uses its output-growth coefficient as a speed-limit term: it avoids reinforcing the initial contraction, while promising a persistent response if activity rebounds too quickly. This dynamic response still delivers lower inflation from the second quarter onward. The standard rule is thus dominated across the components of the loss function. Two model mechanisms produce this result. First, the persistent component of inflation is generated downstream, by sticky-price sectors processing the energy cost increase through the input–output structure and by indexed wages; because this component is domestically produced, reducing the volatility of output growth dampens the medium-run pass-through without trying to undo the imported first round. Second, the OSR's strong inertia shifts the stabilization burden to expectations: agents anticipate a sustained response, which restrains price setting today without requiring large realized rate increases. The prescription echoes the energy-shock literature (Auclert *et al.* 2023; Chan *et al.* 2024), but the channel here is distinct: it is the production network, not household heterogeneity, that makes over-

tightening counterproductive, because the medium-run inflation the central bank is fighting is domestically produced cost pass-through, not the imported first round.

Magnitude of the gains. To quantify the value of optimization we evaluate the central bank's loss—the same quadratic objective in union output growth, CPI inflation, and the change in the policy rate that the OSR minimizes—under the baseline Taylor rule and under the OSR, in a long stochastic simulation with independent innovations across active bilateral sectoral tariff wedges. The same random draws are used under both policy rules. In the tariff environment the result is striking: the optimized rule lowers the weighted loss by only 0.01%. This near-equivalence does not reflect a passive optimizer but a flat frontier. The OSR reshuffles the composition of volatility substantially—it roughly halves the variance of CPI inflation, to 0.47 of its baseline value, while raising the variance of output growth more than fivefold and that of the policy-rate change by a factor of 2.3—yet, weighted by the loss function, these movements very nearly cancel. For the EA bystander the stabilization trade-off against tariff spillovers is thus second order in a precise sense: the loss is essentially invariant to where on the frontier the rule sits, so the inertial Taylor rule of normal times is already close to optimal. This is the loss-based counterpart of the small impact responses in Figure 5.

The energy experiment is the opposite case. There the baseline rule over-tightens and the OSR reduces all three components of the loss, so the gains from optimization—negligible for tariffs—are first order: the optimized rule lowers the weighted loss by 24.7% relative to the baseline rule, and it does so by reducing output-growth, inflation, and policy-rate volatility rather than trading one against another—the variances of output growth and of the rate change fall by roughly one half and four fifths respectively, with a smaller reduction in inflation variance. The asymmetry of the gains carries a broader message for the design of simple rules in networked economies: a single rule calibrated to normal times can be nearly optimal for one class of imported shocks and substantially suboptimal for another, because the two classes load on different coefficients of the rule. This is the instrument-rule counterpart of the result that the welfare-relevant inflation index is shock- and network-dependent (Rubbo 2023; Qiu *et al.* 2026).

Non-EA economies. Appendix G.5 reports the responses of the United States and China to the same three shocks under the EA baseline and OSR (Figure G.5). Two observations stand out. First, the U.S. tariff on China is stagflationary for the United States—CPI inflation rises by about 0.13 percentage points on impact and remains elevated while GDP growth turns negative after a brief expansion—and contractionary for China, whose impact inflation spike gives way to deflation as external demand collapses. Both patterns line up with the predictions of Auclert *et al.* (2025) and Kalemli-Özcan *et al.* (2025) and with the pass-through evidence

in Cavallo *et al.* (2021), providing an external check on the quantitative block. Second, the Taylor-rule and OSR lines for the large economies nearly coincide: because the EA is small relative to the United States and China, its choice of monetary rule has a negligible effect on their dynamics. The EA optimal-simple-rule problem studied here is therefore, from the perspective of the large economies, close to a partial-equilibrium stabilization problem, which is what makes the bystander question well posed.

Sectoral anatomy of the trade-war spillover. The homogeneous experiments make the comparison with the analytical SOE transparent but leave the cross-sector richness of the model unexploited. We therefore shock the bilateral U.S.–China tariff one sector at a time, at the common 10% size, and report sector by sector the difference between the cumulative eight-quarter EA response under the baseline Taylor rule and under the OSR (Figure E.2), the quantitative analogue of the scatter analysis of Section 4.5. Two patterns emerge. First, the rule-induced wedge is uniformly small: across all tariffed sectors the gap in the cumulative EA output response is at most 0.05 percentage points and the gap in cumulative CPI inflation never exceeds 0.002 percentage points. This confirms sector by sector the flat-frontier result of the loss accounting—no single tariffed flow is central enough in the EA's bystander exposure to make the choice of rule consequential. Second, the wedge is organized by network position rather than statutory symmetry: the U.S. tariff on China generates a systematically larger and oppositely signed wedge than the reciprocal Chinese tariff, and within each direction the differences concentrate in a handful of manufacturing sectors with high imported-input centrality and dollar invoicing. The bystander transmission is thus ranked by the same network statistics—centrality, invoicing, and price rigidity—that organize the directly exposed economy, even though its aggregate footprint is an order of magnitude smaller.

The observed 2025 tariff schedule. Replacing the homogeneous 10% vector with the observed 2025 U.S.–China tariff schedule turns the model into an accounting device for the EA macroeconomic footprint of actual trade policy, in the spirit of Kalemli-Özcan *et al.* (2025). We build a quarterly, sector-level series of effective bilateral tariffs from product-level data in the IMF Tariff Tracker, concorded to the NACE Rev. 2 sectors of the calibration with active-day and import-value weights and first-differenced in log gross wedges, $\log(1+\tau)$, to recover the tariff-wedge innovations; Appendix G.4 details the construction. Figure E.3 reports the simulated EA paths from 2025Q2 onward. The signature of the bystander problem survives the move to data. EA headline inflation rises by about 0.12 percentage points on impact under the baseline rule and then turns mildly negative as trade diversion takes hold, while output growth dips by a few hundredths of a percentage point and the policy rate barely moves. The OSR trims the inflation peak by roughly a third, to about 0.08 percentage points, at the cost of

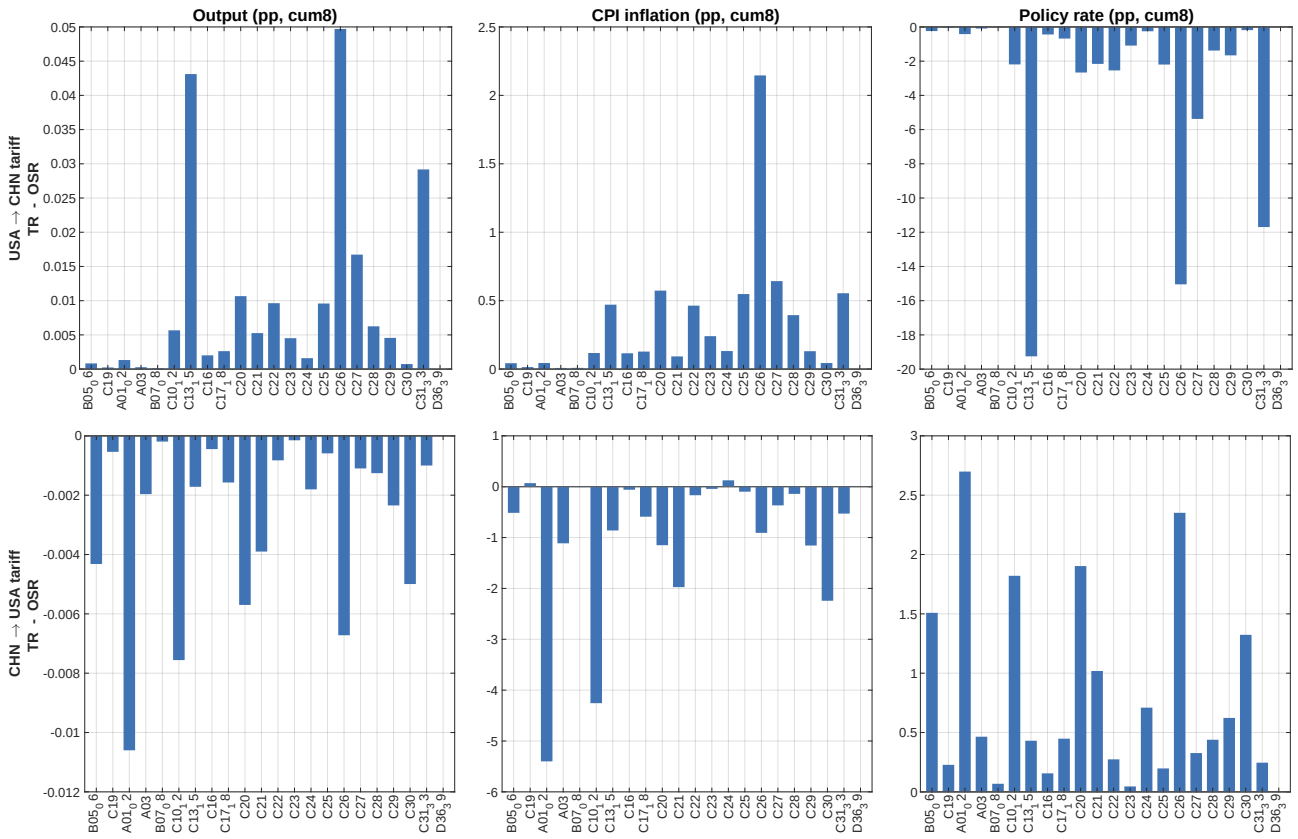


FIGURE E.2. Sectoral decomposition of the trade-war spillover to the Euro Area. Each bar is the difference between the cumulative eight-quarter EA response under the baseline Taylor rule and under the optimal simple rule, when the bilateral U.S.–China tariff is applied to one sector at a time at a 10% size. Top row: U.S. tariff on China; bottom row: Chinese tariff on the United States. Columns: real output, CPI inflation, and the policy rate, in percentage points. Sectors are labelled by NACE code along the horizontal axis.

a deeper but still small impact contraction in output growth—the same trade-off identified in the homogeneous experiment. These magnitudes are an order of magnitude smaller than those borne by the directly involved economies: in the same simulation U.S. CPI inflation peaks near 0.44 and Chinese CPI inflation near 0.37 percentage points, with Chinese output contracting by a comparable amount (Appendix G.5, Figure G.6), the stagflation-for-the-imposer, contraction-for-the-target pattern found by Auclert *et al.* (2025) and Kalemli-Özcan *et al.* (2025). The model thus attributes to the 2025 trade war only a modest, front-loaded, and quickly reversing inflationary impulse for the EA.

Isolating the dollar-pricing channel. Of the four transmission channels identified above, the muting of expenditure switching by dominant-currency pricing is the one tied to the invoicing structure of the data. We isolate its contribution with a counterfactual that shuts down dominant-currency pricing: holding the calibration and the U.S. tariff on China fixed, we

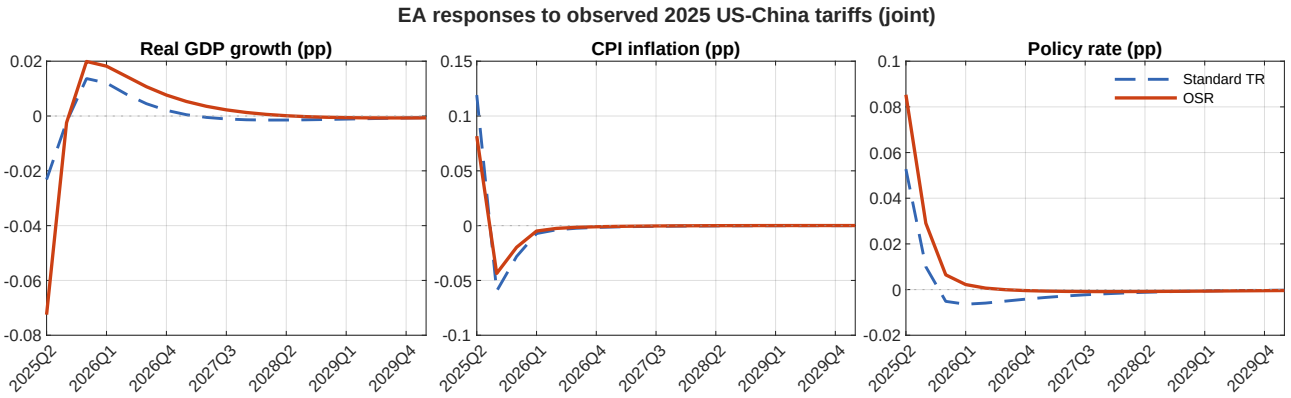


FIGURE E.3. Euro Area responses to the observed 2025 U.S.–China tariff schedule under the baseline Taylor rule (dashed) and the optimal simple rule (solid). The observed sectoral tariff levels are converted into wedge innovations by first-differencing the log gross tariff; the simulation runs from 2025Q2. Columns: real GDP growth, CPI inflation, and the policy rate, in percentage points.

replace the empirical mix of producer-, local-, and dominant-currency invoicing with uniform producer-currency pricing and re-solve for the EA responses under both rules (Figure E.4). The result is stark. Under the baseline invoicing, the tariff raises EA CPI inflation by about 0.05 percentage points on impact; under pure producer-currency pricing the same tariff raises it by 0.001 percentage points—a spillover two orders of magnitude smaller—and the EA output response changes sign, turning marginally expansionary. The inflationary footprint of the U.S.–China tariff on the EA is therefore almost entirely a dollar-pricing phenomenon: because the affected trade flows are invoiced in dollars, the tariff and the dollar movements that accompany it pass into EA import prices irrespective of the euro's bilateral exchange rate, severing the expenditure-switching insulation that producer-currency pricing would otherwise provide (Gopinath *et al.* 2020; Egorov and Mukhin 2023). The same logic explains why the choice of rule becomes immaterial in the counterfactual: with the cost-push channel nearly switched off, there is essentially no inflation–output trade-off left for the central bank to resolve, and the Taylor-rule and OSR responses collapse onto each other. Dominant-currency pricing is the feature of the global economy that turns a bystander into an exposed party: the EA's insulation in a trade war is contingent on the invoicing structure of its imports, not merely on its distance from the tariff.

Core versus headline targeting. The loss accounting establishes that re-optimizing the rule is worth roughly a quarter of the baseline loss in the energy environment, but not where that gain comes from. The energy-shock dynamics point to a specific diagnosis—the baseline rule over-tightens because it responds mechanically to a first-round headline spike that monetary policy cannot control—and the classic remedy for that failure is to target core rather than

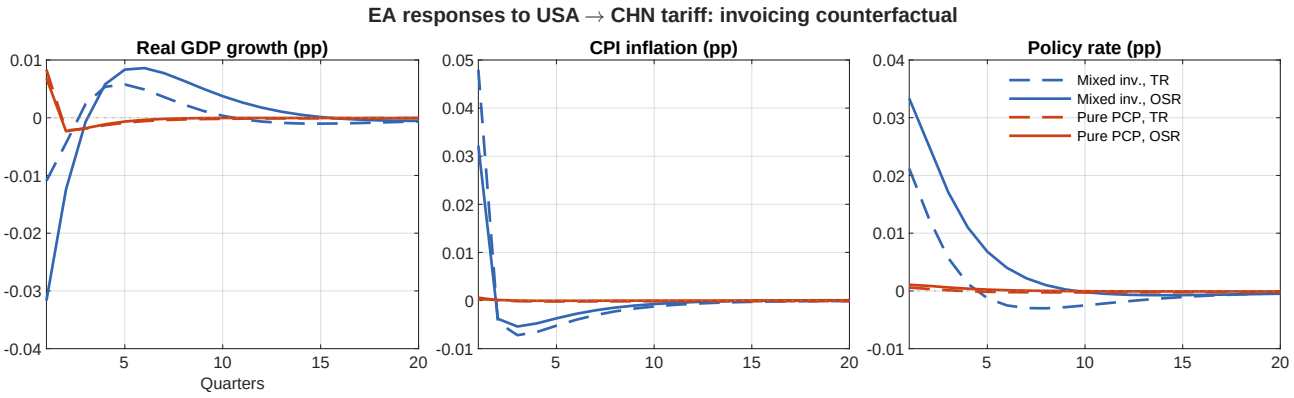


FIGURE E.4. Isolating the dollar-pricing channel in the Euro Area spillover of the U.S. tariff on China. Each panel reports the EA response under the baseline mixed invoicing of the calibration and under a pure producer-currency-pricing counterfactual, for both the baseline Taylor rule and the optimal simple rule. Columns: real GDP growth, CPI inflation, and the policy rate, in percentage points. The tariff is scaled to a 10 log-point increase. Under producer-currency pricing the spillover to the EA nearly vanishes and the two rules coincide.

headline inflation (Bodenstein *et al.* 2008). We test it directly, feeding the baseline rule core (energy-excluding) CPI in place of headline CPI while holding its coefficients fixed, and compare the resulting EA responses with the headline Taylor rule and the OSR (Figure E.5). The change in target alone closes most of the gap. The headline rule raises the policy rate by about 0.07 percentage points on impact and holds it elevated for years, chasing a first-round headline impulse of 0.17 percentage points; the core rule, responding instead to the much smaller core impulse of about 0.04 percentage points, raises the rate by only 0.02 percentage points, roughly halves the impact contraction in output growth, and—because the first round is policy-invariant—delivers essentially the same headline inflation path. In loss terms the core rule attains a 22.8% reduction relative to the headline rule, against the 24.7% delivered by the fully optimized OSR: looking through the first round recovers about ninety percent of the gains from optimizing the simple rule. The residual is the network's contribution. The OSR improves on core targeting not by treating the first round differently—neither rule can affect it—but by leaning on output growth to damp the second-round inflation that sticky-price sectors generate as they pass the energy cost through the input–output structure and indexed wages. The core-versus-headline prescription of Bodenstein *et al.* (2008) therefore survives, and indeed accounts for the bulk of the gain from optimizing the simple rule, in the multisectoral open economy; what the production network adds is a second-round, domestically produced component of energy inflation that a fixed-coefficient core rule leaves on the table and that the speed-limit form of the OSR is built to reach.

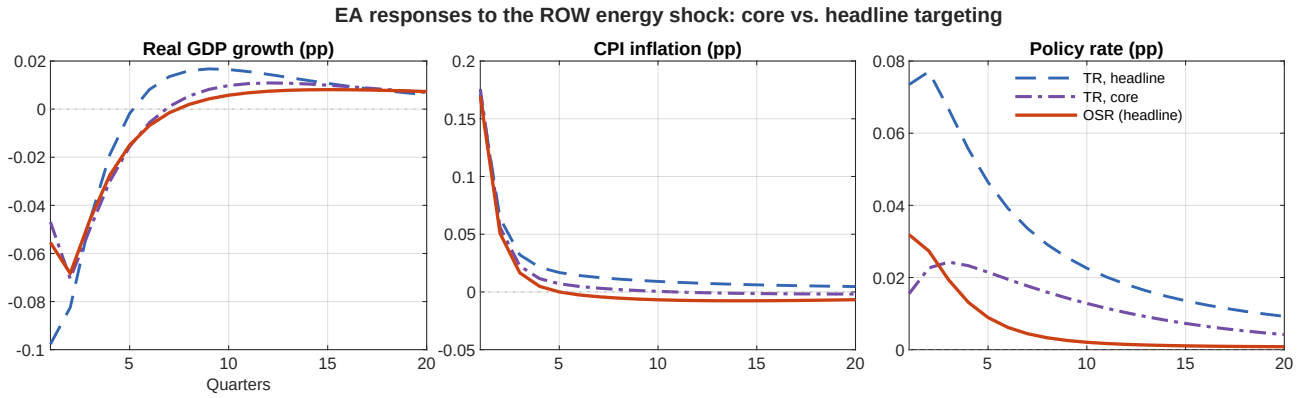


FIGURE E.5. Euro Area responses to the rest-of-world energy shock under three rules: the baseline Taylor rule targeting headline CPI (dashed), the same rule targeting core CPI (dash-dotted), and the energy-environment optimal simple rule (solid). Columns: real GDP growth, CPI inflation, and the policy rate, in percentage points. The shock is scaled to a 10 log-point increase. Switching the baseline rule from headline to core inflation recovers most of the loss reduction delivered by the fully optimized rule.

E.5. Policy-Path Flexibility: Optimal Policy Perturbations

The OSR restricts the monetary response to a three-coefficient feedback rule. We assess its robustness by allowing additional announced adjustments to the policy path, using the optimal-policy-perturbation approach of Barnichon and Mesters (2023). The exercise retains the calibrated multi-country model and the common 10 log-point rest-of-world energy innovation of Section E.4. Forecasts and monetary transmission responses are generated by this same model. The benchmark therefore examines the OSR response within an expanded space of policy interventions.

Criterion and policy announcements. Let $t = 0$ be the arrival of the energy shock, with zero prehistory. We evaluate the conditional discounted criterion

$$J_\beta = \sum_{j=0}^{\infty} \beta^j \left[\pi^2_{C,MU,j} + 0.2(\Delta y_{MU,j})^2 + 1.4(\Delta i_{1,j})^2 \right], \qquad \beta = 0.99. \tag{E.13}$$

Inflation and GDP growth are quarter-on-quarter deviations from their steady-state values, with zero targets; the nominal rate is in quarterly units. None of these variables is annualized. The first policy-rate change is measured relative to the pre-shock rate, so $\Delta i_{1,0} = i_{1,0}$.

The EA rule is augmented by an additive monetary wedge:

$$i_{1,t} = \rho_r i_{1,t-1} + (1-\rho_r)\left[\phi_\pi \pi_{C,MU,t} + \phi_{\Delta y}\Delta y_{MU,t}\right] + m_t,$$

$$m_t = \varepsilon_t^{MP,0} + \sum_{h=1}^{H} \varepsilon_{t-h}^{MP,h}. \tag{E.14}$$

All perturbations are observed at date zero. A realization $\varepsilon_0^{MP,h} = x$ announces a wedge of x at date h, which is delivered with full credibility. Its effects on expectations, activity, inflation, and the endogenous component of the interest rate can begin immediately. The announcement fixes the future additive wedge in (E.14); the total rate also responds through the feedback rule. The wedge has unit loading outside the smoothing factor $(1-\rho_r)$. The central case includes the contemporaneous direction and one news direction for each horizon from one to eight quarters; sensitivity exercises set $H = 4, 12, 16$.

Construction and interpretation. Stack the baseline OSR paths of inflation, GDP growth, and policy-rate changes in $\boldsymbol{q}^0$. Let $\mathcal{R}_H$ collect their responses to the $H+1$ date-zero perturbations. At first order, the adjusted path is $\boldsymbol{q}^0 + \mathcal{R}_H\boldsymbol{\delta}$. With period weights $W = \mathrm{diag}(1, 0.2, 1.4)$ and discounted path weights $\mathcal{W}_\beta = \mathrm{diag}(W, \beta W, \beta^2 W, \ldots)$, the minimizing perturbation is

$$\boldsymbol{\delta}^* = -\left(\mathcal{R}_H^\top \mathcal{W}_\beta \mathcal{R}_H\right)^{-1} \mathcal{R}_H^\top \mathcal{W}_\beta \boldsymbol{q}^0. \tag{E.15}$$

The resulting path minimizes (E.13) within these announcement directions. The loss includes the response after the last announcement has been delivered. Numerically, extending the evaluation horizon from 400 to 800 quarters changes minimized loss by less than 4.3×10^{-7} of baseline loss across the reported cases. A direct equilibrium simulation of the energy shock and the chosen date-zero announcements reproduces the superposed OPP path.

The OSR coefficients are those used for the Section E.4 energy experiment. They were optimized for the unconditional-variance criterion; OPP is optimized for (E.13). All paths below are evaluated with this common conditional discounted criterion. The gap can therefore reflect both policy-path flexibility and the difference between the original optimization criteria. Isolating the contribution of the simple-rule restriction alone would require reoptimizing its coefficients under (E.13). The criterion remains an operational stabilization objective rather than a household-welfare measure.

Robustness of the energy OSR. Table E.4 normalizes the discounted loss under the baseline Taylor rule to 100. The existing OSR lowers it to 74.77; the eight-quarter OPP lowers it further to 68.34. Thus the additional reduction is 8.59% of the OSR loss, or 6.42 points of the initial

Policy	H	Loss (Taylor = 100)	Gain over OSR (%)
Taylor rule	–	100.00	–
OSR	–	74.77	–
OSR + OPP	0	74.21	0.75
OSR + OPP	4	68.39	8.53
OSR + OPP	8	68.34	8.59
OSR + OPP	12	68.23	8.75
OSR + OPP	16	67.97	9.10

Notes: Conditional discounted losses after the common 10 log-point energy innovation, with $\beta = 0.99$ and period weights $(1, 0.2, 1.4)$. The Taylor-rule loss is normalized to 100. Each H includes one contemporaneous direction and all news horizons $1, \ldots, H$; $H = 0$ permits only the contemporaneous perturbation. The final column measures the reduction relative to the existing OSR, evaluated under the same discounted criterion.

TABLE E.4. Energy OSR robustness to announced policy perturbations

Loss component	OSR	OSR + OPP	Change (%)
Headline inflation	66.62	57.17	-14.18
GDP growth	4.80	10.90	+126.97
Policy-rate changes	3.35	0.28	-91.76
Total	74.77	68.34	-8.59

Notes: Discounted weighted contributions to (E.13), on the same scale as Table E.4. Components sum to total loss before rounding. Changes are measured relative to the corresponding OSR component; positive values indicate a higher loss.

TABLE E.5. Composition of the eight-quarter OPP gain

Taylor-rule loss. The OSR captures about 80% of the total loss reduction attained by this perturbed policy path relative to Taylor. The additional improvement is 8.53% with four quarters of news and 9.10% with sixteen, showing that extending the announcement horizon beyond the central eight-quarter specification adds little to the measured gain. The policy spaces are nested, so minimized loss is non-increasing in H by construction; the informative result is the small size of the additional improvement. A contemporaneous perturbation alone delivers only 0.75%, so the announcement directions account for most of the additional gain.

The additional improvement involves a stabilization trade-off. Figure E.6 shows a smaller initial nominal-rate increase and lower impact headline inflation under OSR plus OPP, alongside a deeper initial contraction in GDP growth. Table E.5 shows that lower inflation and policy-rate-change losses outweigh a higher output-growth loss under the stated weights. The robustness result concerns the aggregate criterion and the attainable announced policy paths; its loss decomposition identifies the additional real-activity cost.

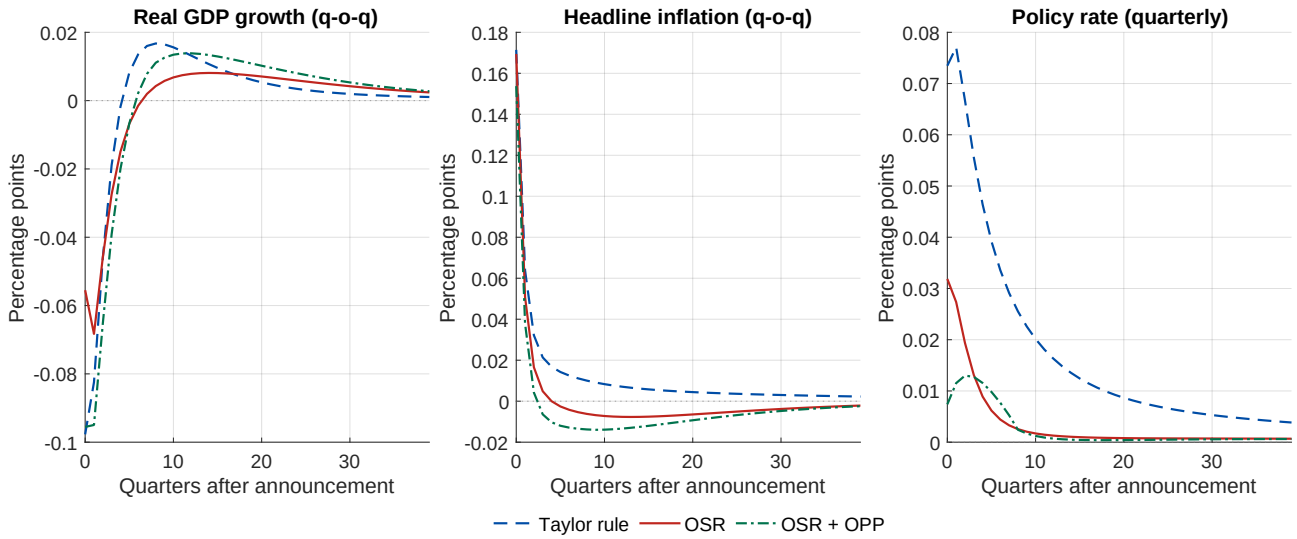


FIGURE E.6. Policy-path robustness of the energy OSR. Responses to the common 10 log-point rest-of-world energy shock under the baseline Taylor rule, the existing energy OSR, and OSR plus the eight-quarter OPP. All news are announced when the energy shock arrives. GDP growth and headline inflation are quarter-on-quarter, without annualization; the policy rate is quarterly. All panels report deviations in percentage points.

Appendix F. Model Derivation and Log-linearization

In this section, we derive model equilibrium conditions, outlining the final set of log-linearized equations. We further enlarge the theoretical model presented in the main text—which only contains exogenous perturbations to the foreign price wedges and the monetary policy shock—to accommodate the standard shocks in the literature: an internal demand shock, sectoral TFP, sectoral price cost-push shocks, and wage cost-push shocks.

F.1. Households

Each household is made up of a continuum of members, each specialized in a different labor service, indexed by $g \in [0,1]$. Income is pooled within each household, acting as a risk-sharing mechanism. The per-period utility function (1) is modified to accommodate nominal wage rigidities, external habit formation, and a discount factor disturbance. Namely,

$$U_t = \left(\frac{\left(C_{k,t} - h_k \bar{C}_{k,t-1} \right)^{1-\sigma}}{1-\sigma} - \int_0^1 \frac{\mathcal{N}_{gk,t}^{1+\varphi}}{1+\varphi} \, dg \right) Z_{k,t}, \tag{F.1}$$

where $\mathcal{N}_{gk,t}$ denotes the labor supply of labor service g, $Z_{k,t}$ is an exogenous preference shifter, σ denotes the inverse of the intertemporal elasticity of substitution, $h_k \in [0,1)$ controls the

degree of keeping up with the Joneses, and $\bar{C}_{k,t-1}$ denotes lagged aggregate consumption, taken as given by the household.[21]

Consumption Demand Curves. The allocation optimal allocation between energy and non-energy goods is the result of a cost minimization programme min $P^{C,E}_{k,t}C^E_{k,t} + P^{C,M}_{k,t}C^M_{k,t}$ subject to (E.3). Similarly, the optimal allocation between energy (non-energy) consumption is the result of a cost minimization programme min $\sum_{i\in I_E} P^C_{ki,t}C_{ki,t}$ (min $\sum_{i\in I_M} P^C_{ki,t}C_{ki,t}$) subject to (E.4). Finally, the optimal allocation between the different consumption goods is the result of a cost minimization programme min $\sum_{l=1}^{K}(1+\tau_{kl\,i,t})(1+\tau^w_{kl\,i,t})P^k_{kl\,i,t}C_{kl\,i,t}$ subject to (E.6). The implied demand curves are given by

$$P^{C,E}_{k,t} = P^C_{k,t}\left(\frac{\widetilde{\varrho}^C_k C_{k,t}}{C^E_{k,t}}\right)^{\frac{1}{\gamma_C}} \quad \text{and} \quad P^{C,M}_{k,t} = P^C_{k,t}\left(\frac{(1-\widetilde{\varrho}^C_k)C_{k,t}}{C^M_{k,t}}\right)^{\frac{1}{\gamma_C}} \tag{F.2}$$

$$P^C_{ki,t} = P^{C,E}_{k,t}\left(\frac{\widetilde{\nu}^C_{ki}C^E_{k,t}}{C_{ki,t}}\right)^{\frac{1}{\eta_C}} \quad \forall \quad i\in I_E \quad \text{and} \quad P^C_{ki,t} = P^{C,M}_{k,t}\left(\frac{\widetilde{\upsilon}^C_{ki}C^M_{k,t}}{C_{ki,t}}\right)^{\frac{1}{\iota_C}} \quad \forall \quad i\in I_M \tag{F.3}$$

$$(1+\tau_{kl\,i,t})(1+\tau^w_{kl\,i,t})P^k_{kl\,i,t} = P^C_{ki,t}\left(\frac{\widetilde{\zeta}^C_{kl\,i}C_{ki,t}}{C_{kl\,i,t}}\right)^{\frac{1}{\delta_C}} \quad \forall \quad l\in K. \tag{F.4}$$

The log-linearized versions of the consumption demand curves (F.2)-(F.4) are given by

$$\widehat{p}^{C,E}_{k,t} = \frac{1}{\gamma_C}(\widehat{c}_{k,t} - \widehat{c}^E_{k,t}) \quad \text{and} \quad \widehat{p}^{C,M}_{k,t} = \frac{1}{\gamma_C}(\widehat{c}_{k,t} - \widehat{c}^M_{k,t}) \tag{F.5}$$

$$\widehat{p}^C_{ki,t} - \widehat{p}^{C,E}_{k,t} = \frac{1}{\eta_C}(\widehat{c}^E_{k,t} - \widehat{c}_{ki,t}) \quad \text{and} \quad \widehat{p}^C_{ki,t} - \widehat{p}^{C,M}_{k,t} = \frac{1}{\iota_C}(\widehat{c}^M_{k,t} - \widehat{c}_{ki,t}) \tag{F.6}$$

$$\tau_{kl\,i,t} + \tau^w_{kl\,i,t} + \widehat{p}^k_{kl\,i,t} - \widehat{p}^C_{ki,t} = \frac{1}{\delta_C}\left(\widehat{c}_{ki,t} - \widehat{c}_{kl\,i,t}\right) \tag{F.7}$$

where $\widehat{p}^{C,E}_{k,t} = p^{C,E}_{k,t} - p^C_{k,t}$, $\widehat{p}^{C,M}_{k,t} = P^{C,M}_{k,t} - p^C_{k,t}$, $\widehat{p}^C_{ki,t} = P^C_{ki,t} - p^C_{k,t}$, and $\widehat{p}^k_{kl\,i,t} = p^k_{kl\,i,t} - p^C_{k,t}$ are well-defined as a ratio of prices.[22]

[21] Each household takes as given labor income since wages are set by labor unions and employment is decided by firms. Thus, the only decisions made by the household are the optimal allocation of consumption expenditures among different good varieties across different countries, and the optimal intertemporal allocation of consumption. In equilibrium, $\bar{C}_{k,t-1} = C_{k,t-1}$.

[22] The individual price levels are not well-defined in steady state, but their ratio is.

Currency Pricing Paradigm. Under PCP, using the log-linearized version of (12), we obtain

(F.8)
$$\widehat{p}^{k}_{kl\,i,t} = p^{k}_{kl\,i,t} - p^{C}_{k,t} = e^{l}_{k,t} + p_{kl\,i,t} - p^{C}_{k,t} = e^{l}_{k,t} + (p_{l\,i,t} - p^{C}_{l,t}) + (p^{C}_{l,t} - p^{C}_{k,t}) = \widehat{p}_{l\,i,t} + q^{l}_{k,t},$$

(F.9)
$$\widehat{p}^{k}_{l\,ki,t} = p^{k}_{l\,ki,t} - p^{C}_{k,t} = p_{l\,ki,t} - p^{C}_{k,t} = p_{ki,t} - p^{C}_{k,t} = \widehat{p}_{ki,t},$$

where we have used that, under PCP, $p_{kl\,i,t} = p_{l\,i,t}$ since the law of one price (LOP) is satisfied; the transformation $\widehat{p}_{l\,i,t} = p_{l\,i,t} - p^{C}_{l,t}$; and the definition of the bilateral real exchange rate between country k and country m: $\mathcal{Q}_{kl,t} = P_{l\,C,t}\mathcal{E}_{kl,t}/P^{C}_{k,t}$, which log-linearized takes the form

(F.10)
$$q^{m}_{k,t} = e^{m}_{k,t} + p^{C}_{m,t} - p^{C}_{k,t}.$$

Under LCP, using the log-linearized version of (14), we can write

(F.11)
$$\widehat{p}^{k}_{kl\,i,t} = p^{k}_{kl\,i,t} - p^{C}_{k,t} = \widehat{p}_{kl\,i,t},$$

(F.12)
$$\widehat{p}^{k}_{l\,ki,t} = p^{k}_{l\,ki,t} - p^{C}_{k,t} = e^{l}_{k,t} + p_{l\,ki,t} - p^{C}_{k,t} = (p_{l\,ki,t} - p^{C}_{l,t}) + e^{l}_{k,t} + p^{C}_{l,t} - p^{C}_{k,t} = q^{l}_{k,t} + \widehat{p}_{l\,ki,t},$$

where we have made use of (F.10), and of the definition $\widehat{p}_{kl\,i,t} = p_{kl\,i,t} - p^{C}_{k,t}$.

Under DCP, using the log-linearized version of (16), we obtain

(F.13)
$$\widehat{p}^{k}_{kl\,i,t} = p^{k}_{kl\,i,t} - p^{C}_{k,t} = e^{\mathrm{US}}_{k,t} + p_{l\,i,t} - p^{C}_{k,t} = e^{\mathrm{US}}_{k,t} + (p_{l\,i,t} - p^{C}_{\mathrm{US},t}) + p^{C}_{\mathrm{US},t} - p^{C}_{k,t} = q^{\mathrm{US}}_{k,t} + \widehat{p}_{l\,i,t},$$

(F.14)
$$\widehat{p}^{k}_{l\,ki,t} = p^{k}_{l\,ki,t} - p^{C}_{k,t} = e^{\mathrm{US}}_{k,t} + p_{ki,t} - p^{C}_{k,t} = e^{\mathrm{US}}_{k,t} + (p_{ki,t} - p^{C}_{\mathrm{US},t}) + p^{C}_{\mathrm{US},t} - p^{C}_{k,t} = q^{\mathrm{US}}_{k,t} + \widehat{p}_{ki,t}$$

where we have used that, under DCP, $p_{kl\,i,t} = p_{l\,i,t}$ although it should be noted that the LOP is not satisfied, the transformation $\widehat{p}_{l\,i,t} = p_{l\,i,t} - p^{C}_{\mathrm{US},t}$, and the definition of the bilateral real exchange rate (F.10).

To summarize pricing, using (F.8), (F.11), (F.13), (F.9), (F.12), and (F.14), we have that

(F.15)
$$\widehat{p}^{k}_{kl\,i,t} = \begin{cases} q^{l}_{k,t} + \widehat{p}_{l\,i,t}, & \text{if } i \in I_{\mathrm{PCP}}, \\ \widehat{p}_{kl\,i,t}, & \text{if } i \in I_{\mathrm{LCP}}, \\ q^{\mathrm{US}}_{k,t} + \widehat{p}_{l\,i,t}, & \text{if } i \in I_{\mathrm{DCP}}. \end{cases} \qquad \widehat{p}^{k}_{l\,ki,t} = \begin{cases} \widehat{p}_{ki,t}, & \text{if } i \in I_{\mathrm{PCP}}, \\ q^{l}_{k,t} + \widehat{p}_{l\,ki,t}, & \text{if } i \in I_{\mathrm{LCP}}, \\ q^{\mathrm{US}}_{k,t} + \widehat{p}_{ki,t}, & \text{if } i \in I_{\mathrm{DCP}}. \end{cases}$$

Consumption Baskets. The log-linearized consumption aggregator (E.3) is given by

$$\widehat{c}_{k,t} = \varrho_k^C \widehat{c}_{k,t}^E + (1 - \varrho_k^C)\widehat{c}_{k,t}^M \tag{F.16}$$

where $\varrho_k^C = \frac{P_k^{C,E} C_k^E}{P_k^C C_k} = (\widetilde{\varrho}_k^C)^{\frac{1}{\gamma_C}} \left(\frac{C_k^E}{C_k}\right)^{\frac{\gamma_C - 1}{\gamma_C}}$ and $1 - \varrho_k^C = \frac{P_k^{C,M} C_k^M}{P_k^C C_k} = (1 - \widetilde{\varrho}_k^C)^{\frac{1}{\gamma_C}} \left(\frac{C_k^M}{C_k}\right)^{\frac{\gamma_C - 1}{\gamma_C}}$ can be verified using the steady-state consumption aggregator (E.3) and the demand curves (F.2).

The log-linearized versions of the energy and non-energy consumption aggregators (E.4) are given by

$$\widehat{c}_{k,t}^E = \sum_{i \in I_E} \nu_{ki} \widehat{c}_{ki,t} \qquad \text{and} \qquad \widehat{c}_{k,t}^M = \sum_{i \in I_M} \upsilon_{ki} \widehat{c}_{ki,t} \tag{F.17}$$

where $\nu_{ki} = \frac{P_{ki}^C C_{ki}}{P_k^{C,E} C_k^E} = (\widetilde{\nu}_{ki}^C)^{\frac{1}{\eta_C}} \left(\frac{C_{ki}}{C_k^E}\right)^{\frac{\eta_C - 1}{\eta_C}}$ and $\upsilon_{ki} = \frac{P_{ki}^C C_{ki}}{P_k^{C,M} C_k^M} = (\widetilde{\upsilon}_{ki}^C)^{\frac{1}{\iota_C}} \left(\frac{C_{ki}}{C_k^M}\right)^{\frac{\iota_C - 1}{\iota_C}}$ can be verified using the steady-state energy and non-energy consumption aggregators (E.4) and the demand curves (F.3).

The log-linearized version of the final layer of the consumption aggregator, (E.6), is given by

$$\widehat{c}_{ki,t} = \sum_{l=1}^{K} \zeta_{kl\,i} \widehat{c}_{kl\,i,t} \tag{F.18}$$

where $\zeta_{kl\,i} = \frac{P_{kl\,i} C_{kl\,i}}{P_{ki}^C C_{ki}} = (\widetilde{\zeta}_{kl\,i}^C)^{\frac{1}{\delta_C}} \left(\frac{C_{kl\,i}}{C_{ki}}\right)^{\frac{\delta_C - 1}{\delta_C}}$ can be verified using the steady-state international consumption aggregator (E.6) and the consumption demand curves (F.4).

Price Indices. The different price indices can be derived by combining the consumption demand curves previously derived with the different consumption aggregators. The consumption price index, the energy and non-energy price index, and the consumption import price index are given by $P_{k,t}^C = \left[\widetilde{\varrho}_k^C (P_{k,t}^{C,E})^{1-\gamma_C} + (1 - \widetilde{\varrho}_k^C)(P_{k,t}^{C,M})^{1-\gamma_C}\right]^{\frac{1}{1-\gamma_C}}$, $P_{k,t}^{C,E} = \left[\sum_{i \in I_E} \widetilde{\nu}_{ki}^C (P_{ki,t}^C)^{1-\eta_C}\right]^{\frac{1}{1-\eta_C}}$, $P_{k,t}^{C,M} = \left[\sum_{i \in I_M} \widetilde{\upsilon}_{ki}^C (P_{ki,t}^C)^{1-\iota_C}\right]^{\frac{1}{1-\iota_C}}$, and $P_{ki,t}^C = \left\{\sum_{l=1}^K \widetilde{\zeta}_{kl\,i}^C [(1 + \tau_{kl\,i,t}) P_{kl\,i,t}^k]^{1-\delta_C}\right\}^{\frac{1}{1-\delta_C}}$. Their log-linearized counterparts are given by

$$0 = \varrho_k^C \widehat{p}_{k,t}^{C,E} + (1 - \varrho_k^C)\widehat{p}_{k,t}^{C,M} \tag{F.19}$$

$$\widehat{p}_{k,t}^{C,E} = \sum_{i \in I_E} \nu_{ki} \widehat{p}_{ki,t}^C \quad \text{and} \quad \widehat{p}_{k,t}^{C,M} = \sum_{i \in I_M} \upsilon_{ki} \widehat{p}_{ki,t}^C \tag{F.20}$$

(F.21)
$$\widehat{p}^{C}_{ki,t} = \sum_{l=1}^{K} \zeta_{kl\,i}(\widehat{p}^{k}_{kl\,i,t} + \tau_{kl\,i,t} + \tau^{w}_{kl\,i,t})$$

Intertemporal Household Problem. International financial markets are incomplete, $\mathcal{H} = 1$ in (4), with households in each country only having access to two risk-free bonds. More precisely, the household in country k has access to a domestic bond, and an internationally traded bond, $B^{K}_{k,t}$, issued, without loss of generality, by country K and denominated in country K's currency. The agent maximizes the present discounted value of per-period utility flows (F.1), with discount factor β, subject to her budget constraint,

(F.22)
$$\begin{aligned} &P^{C}_{k,t}C_{k,t} + B_{k,t} + B^{K}_{k,t}\left[1 - \Gamma(\text{NFA}^{K}_{k,t})\right]^{-1}\mathcal{E}^{K}_{k,t} + \Xi_{k,t} \leq \\ &B_{k,t-1}(1+i_{k,t-1}) + B^{K}_{k,t-1}\mathcal{E}^{K}_{k,t}(1+i_{K,t-1}) + \int_0^1 W_{gk,t}\mathcal{N}_{gk,t}dg + \Pi_{k,t} - T_{k,t} \end{aligned}$$

where $\int_0^1 W_{gk,t}\mathcal{N}_{gk,t}dg$ is the nominal labor income received by the representative household, $\text{NFA}^{K}_{k,t} = B^{K}_{k,t}\mathcal{E}^{K}_{k,t}$ is the net foreign asset position of households in the country k, and where $\Gamma(x) = \gamma_*\left(\exp\left\{x/\mathcal{Y}_{k,t}\right\} - 1\right)$ is an external financial intermediary premium that depends on the economy-wide net holdings of internationally traded foreign bonds as a ratio to the national nominal GDP $\mathcal{Y}_{k,t}$, with $\gamma_* > 0$.[23] The incurred intermediation premium is rebated to households in a lump-sum manner through the fiscal instrument $\Xi_{k,t}$. Finally, $T_{k,t}$ denotes government transfers, also rebated to households in lump sum.

The above program delivers two sets of different Euler conditions,

(F.23)
$$\left(C_{k,t} - h_k C_{k,t-1}\right)^{-\sigma} = \mathbb{E}_t\beta\left(C_{k,t+1} - h_k C_{k,t}\right)^{-\sigma}\frac{1+i_{k,t}}{1+\pi^{C}_{k,t+1}}\frac{Z_{k,t+1}}{Z_{k,t}}$$

(F.24)
$$\left(C_{k,t} - h_k C_{k,t-1}\right)^{-\sigma} = \mathbb{E}_t\beta\left(C_{k,t+1} - h_k C_{k,t}\right)^{-\sigma}\frac{1+i_{K,t}}{1+\pi^{C}_{k,t+1}}\left[1 - \Gamma(\text{NFA}^{K}_{k,t})\right]\frac{\mathcal{E}^{K}_{k,t+1}}{\mathcal{E}^{K}_{k,t}}\frac{Z_{k,t+1}}{Z_{k,t}} \quad \forall k \neq K$$

where we assume that the (log-)demand shock follows an AR(1) process:

(F.25)
$$z_{k,t} = \rho^{z}_{k} z_{k,t-1} + \varepsilon^{z}_{k,t}$$

[23] The role of this intermediation premium is to stabilize the net foreign asset position in response to transitory shocks, a common practice in open economies with incomplete financial markets (Schmitt-Grohe and Uribe 2003). Furthermore, this specification guarantees that, in the non-stochastic steady state, households have no incentive to hold foreign bonds and the economy's net foreign asset position is zero.

where $z_{k,t} := \log Z_{k,t}$, and $\varepsilon^z_{k,t} \sim \mathcal{N}\left(0, \sigma^2_{kz}\right)$.

The log-linearized version of the household's first-order conditions (F.23)-(F.24) are given by

$$\widehat{c}_{k,t} = \frac{h_k}{1+h_k}\widehat{c}_{k,t-1} + \frac{1}{1+h_k}\mathbb{E}_t\widehat{c}_{k,t+1} - \frac{1-h_k}{\sigma(1+h_k)}(i_{k,t} - \mathbb{E}_t\pi^C_{k,t+1}) + \frac{1-h_k}{\sigma(1+h_k)}(1-\rho^z_k)z_{k,t} \tag{F.26}$$

$$\widehat{c}_{k,t} = \frac{h_k}{1+h_k}\widehat{c}_{k,t-1} + \frac{1}{1+h_k}\mathbb{E}_t\widehat{c}_{k,t+1} - \frac{1-h_k}{\sigma(1+h_k)}(i_{K,t} - \mathbb{E}_t\pi^C_{k,t+1}) + \frac{1-h_k}{\sigma(1+h_k)}(1-\rho^z_k)z_{k,t}$$

$$- \frac{1-h_k}{\sigma(1+h_k)}\mathbb{E}_t\Delta e^K_{k,t+1} + \frac{1-h_k}{\sigma(1+h_k)}\gamma_*\text{nfa}^K_{k,t} \quad \forall k \neq K \tag{F.27}$$

where we define the different log-linear NFA positions as $\text{nfa}^K_{k,t} = B^K_{k,t}\mathcal{E}^K_{k,t}/\mathcal{Y}_k$ and $\text{nfa}^{\text{MU}}_{k,t} = B^{\text{MU}}_{k,t}\mathcal{E}_{k\text{MU},t}/\mathcal{Y}_k$ since $B^K_{k,t} = 0$ and $B^{\text{MU}}_{k,t} = 0$ in the steady state, and

$$\Delta e^l_{k,t} = e^l_{k,t} - e^l_{k,t-1}. \tag{F.28}$$

Combining the log-linearized first-order conditions for the holdings of domestic and internationally traded bonds (F.26)-(F.27), yields a risk-adjusted Uncovered Interest Parity (UIP) condition

$$i_{k,t} - i_{K,t} = \mathbb{E}_t\Delta e^K_{k,t+1} - \gamma_*\text{nfa}^K_{k,t}. \tag{F.29}$$

F.2. Firms

We augment the production function (E.1) to include a sectoral TFP shock $A_{ki,t}$,

$$Y_{kif,t} = A_{ki,t}\left[\widetilde{\alpha}^{\frac{1}{\psi}}_{ki}N^{\frac{\psi-1}{\psi}}_{kif,t} + \widetilde{\vartheta}^{\frac{1}{\psi}}_{ki}X^{\frac{\psi-1}{\psi}}_{kif,t}\right]^{\frac{\psi}{\psi-1}\psi_{ki}}, \tag{F.30}$$

where the (log-)TFP shock follows an AR(1) process:

$$a_{ki,t} = \rho^a_{ki}a_{ki,t-1} + \varepsilon^a_{ki,t}, \tag{F.31}$$

where $a_{ki,t} := \log A_{ki,t}$, and $\varepsilon^a_{ki,t} \sim \mathcal{N}\left(0, \sigma^2_{kia}\right)$.

Intermediate Input Demand Curves. The optimal allocation between labor and intermediate inputs is the result of a cost minimization programme $\min W_{k,t}N_{kif,t} + P^X_{ki,t}X_{kif,t}$ subject to (F.30). The optimal allocation between energy and non-energy intermediate goods is the result of a cost minimization programme $\min P^{X,E}_{ki,t}X^E_{ki,t} + P^{X,M}_{ki,t}X^M_{ki,t}$ subject to (E.3). Similarly,

the optimal allocation between energy (non-energy) intermediate inputs is the result of a cost minization programme min $\sum_{j\in I_E} P^X_{kij,t}X_{kij,t}$ (min $\sum_{j\in I_M} P^X_{kij,t}X_{kij,t}$) subject to (E.5). Finally, the optimal allocation across source countries for each intermediate input is the result of a cost minimization programme min $\sum_{l=1}^{K}(1+\tau_{klj,t})(1+\tau^w_{klj,t})P^k_{klj,t}X_{klij,t}$ subject to (E.6). The implied intermediate input demand curves are given by

$$W_{k,t} = \text{MC}_{kif,t}A_{ki,t}^{\frac{\psi-1}{\psi\psi_{ki}}}\psi_{ki}Y_{kif,t}^{\frac{(1-\psi_{ki})(1-\psi)}{\psi\psi_{ki}}}\left(\frac{\widetilde{\alpha}_{ki}Y_{kif,t}}{N_{kif,t}}\right)^{\frac{1}{\psi}} \tag{F.32}$$

$$P^X_{ki,t} = \text{MC}_{ki,t}A_{ki,t}^{\frac{\psi-1}{\psi\psi_{ki}}}\psi_{ki}Y_{kif,t}^{\frac{(1-\psi_{ki})(1-\psi)}{\psi\psi_{ki}}}\left(\frac{\widetilde{\vartheta}_{ki}Y_{kif,t}}{X_{kif,t}}\right)^{\frac{1}{\psi}} \tag{F.33}$$

$$P^{X,E}_{ki,t} = P^X_{ki,t}\left(\frac{\widetilde{\varrho}^X_{ki}X_{ki,t}}{X^E_{ki,t}}\right)^{\frac{1}{\gamma_X}} \quad \text{and} \quad P^{X,M}_{ki,t} = P^X_{ki,t}\left(\frac{(1-\widetilde{\varrho}^X_{ki})X_{ki,t}}{X^M_{ki,t}}\right)^{\frac{1}{\gamma_X}} \tag{F.34}$$

(F.35)

$$P^X_{kij,t} = P^{X,E}_{ki,t}\left(\frac{\widetilde{\nu}^X_{kij}X^E_{ki,t}}{X_{kij,t}}\right)^{\frac{1}{\eta_X}} \quad \forall \quad j\in I_E \quad \text{and} \quad P^X_{kij,t} = P^{X,M}_{ki,t}\left(\frac{\widetilde{\upsilon}^X_{kij}X^M_{ki,t}}{X_{kij,t}}\right)^{\frac{1}{\iota_X}} \quad \forall \quad j\in I_M$$

$$(1+\tau_{klj,t})(1+\tau^w_{klj,t})P^k_{klj,t} = P^X_{kij,t}\left(\frac{\zeta^X_{klij}X_{kij,t}}{X_{klij,t}}\right)^{\frac{1}{\delta_X}} \quad \forall \quad l\in K \tag{F.36}$$

The log-linearized versions of the labor and intermediate inputs demand curves (F.32)-(F.36) are given by

$$\widehat{w}_{k,t} - \widehat{\text{mc}}_{ki,t} = \frac{\psi-1}{\psi\psi_{ki}}a_{ki,t} + \frac{(1-\psi_{ki})(1-\psi)}{\psi\psi_{ki}}\widehat{y}_{kif,t} + \frac{1}{\psi}\left(\widehat{y}_{kif,t} - \widehat{n}_{kif,t}\right) \tag{F.37}$$

$$\widehat{p}^X_{ki,t} - \widehat{\text{mc}}_{ki,t} = \frac{\psi-1}{\psi\psi_{ki}}a_{ki,t} + \frac{(1-\psi_{ki})(1-\psi)}{\psi\psi_{ki}}\widehat{y}_{kif,t} + \frac{1}{\psi}\left(\widehat{y}_{kif,t} - \widehat{x}_{kif,t}\right) \tag{F.38}$$

(F.39)

$$\widehat{p}^{X,E}_{ki,t} - \widehat{p}^X_{ki,t} = \frac{1}{\gamma_X}\left(\widehat{x}_{ki,t} - \widehat{x}^E_{ki,t}\right) \quad \text{and} \quad \widehat{p}^{X,M}_{ki,t} - \widehat{p}^X_{ki,t} = \frac{1}{\gamma_X}\left(\widehat{x}_{ki,t} - \widehat{x}^M_{ki,t}\right)$$

(F.40)

$$\widehat{p}^X_{kij,t} - \widehat{p}^{X,E}_{ki,t} = \frac{1}{\eta_X}\left(\widehat{x}^E_{ki,t} - \widehat{x}_{kij,t}\right) \quad \forall \quad j\in I_E \quad \text{and} \quad \widehat{p}^X_{kij,t} - \widehat{p}^{X,M}_{ki,t} = \frac{1}{\iota_X}\left(\widehat{x}^M_{ki,t} - \widehat{x}_{kij,t}\right) \quad \forall \quad j\in I_M$$

$$\tau_{klj,t} + \tau^w_{klj,t} + \widehat{p}^k_{klj,t} - \widehat{p}^X_{kij,t} = \frac{1}{\delta_X}\left(\widehat{x}_{kij,t} - \widehat{x}_{klij,t}\right) \quad \forall \quad l\in K \tag{F.41}$$

where $\widehat{w}_{k,t} = w_{k,t} - p^C_{k,t}$, $\widehat{\text{mc}}_{ki,t} = \text{mc}_{ki,t} - p^C_{k,t}$, $\widehat{p}^X_{ki,t} = p^X_{ki,t} - p^C_{k,t}$, $\widehat{p}^{X,E}_{ki,t} = p^{X,E}_{ki,t} - p^C_{k,t}$,

$\widehat{p}^{X,M}_{ki,t} = p^{X,M}_{ki,t} - p^{C}_{k,t}$, and $\widehat{p}^{X}_{kij,t} = p^{X}_{kij,t} - p^{C}_{k,t}$.

Intermediate Inputs Baskets. The log-linearized intermediary input aggregator (E.3) is given by

$$\widehat{x}_{ki,t} = \varrho^{X}_{ki}\widehat{x}^{E}_{ki,t} + (1-\varrho^{X}_{ki})\widehat{x}^{M}_{ki,t} \tag{F.42}$$

where $\varrho^{X}_{ki} = \frac{P^{X,E}_{ki}X^{E}_{ki}}{P^{X}_{ki}X_{ki}} = (\widetilde{\varrho}^{X}_{ki})^{\frac{1}{\gamma_X}}\left(\frac{X^{E}_{ki}}{X_{ki}}\right)^{\frac{\gamma_X-1}{\gamma_X}}$ and $1-\varrho^{X}_{ki} = \frac{P^{X,M}_{ki}X^{M}_{ki}}{P^{M}_{ki}X_{ki}} = (1-\widetilde{\varrho}^{X}_{ki})^{\frac{1}{\gamma_X}}\left(\frac{X^{M}_{ki}}{X_{ki}}\right)^{\frac{\gamma_X-1}{\gamma_X}}$ can be verified using the steady-state intermediate input aggregator (E.3) and the input demand curves (F.34).

The log-linearized versions of the energy and non-energy intermediate input aggregators (E.5) are given by

$$\widehat{x}^{E}_{ki,t} = \sum_{j\in I_E}\nu^{X}_{kij}\widehat{x}_{kij,t} \qquad \text{and} \qquad \widehat{x}^{M}_{ki,t} = \sum_{j\in I_M}\upsilon^{X}_{kij}\widehat{x}_{kij,t} \tag{F.43}$$

where $\nu^{X}_{kij} = \frac{P^{X}_{kij}X_{kij}}{P^{X,E}_{ki}X^{E}_{ki}} = (\widetilde{\nu}^{X}_{kij})^{\frac{1}{\eta_X}}\left(\frac{X_{kij}}{X^{E}_{ki}}\right)^{\frac{\eta_X-1}{\eta_X}}$ and $\upsilon^{X}_{kij} = \frac{P^{X}_{kij}X_{kij}}{P^{X,M}_{ki}X^{M}_{ki}} = (\widetilde{\upsilon}^{X}_{kij})^{\frac{1}{\iota_X}}\left(\frac{X_{kij}}{X^{M}_{ki}}\right)^{\frac{\iota_X-1}{\iota_X}}$ can be verified using the steady-state energy and non-energy intermediate input aggregators (E.5) and the demand curves (F.35).

The log-linearized version of the final layer of the intermediate input aggregator, (E.6), is given by

$$\widehat{x}_{kij,t} = \sum_{l=1}^{K}\zeta^{X}_{kl\,ij}\widehat{x}_{kl\,ij,t} \tag{F.44}$$

where $\zeta^{X}_{kl\,ij} = \frac{P_{kl\,j}X_{kl\,ij}}{P^{X}_{kij}X_{kij}} = (\widetilde{\zeta}^{X}_{kl\,ij})^{\frac{1}{\delta_X}}\left(\frac{X_{kl\,ij}}{X_{kij}}\right)^{\frac{\delta_X-1}{\delta_X}}$ can be verified using the steady-state international intermediate input aggregators (E.6) and the demand curve (F.36).

Price Indices. The different price indices can be derived by combining the intermediate input demand curves previously derived with the other intermediate input aggregators. The marginal cost of production, the intermediate input price index, the energy and non-energy input price index, and the input import price index are given by $\text{MC}_{ki,t} = A^{-1/\psi_{ki}}_{ki,t}\psi^{-1}_{ki}Y^{(1-\psi_{ki})/\psi_{ki}}_{ki,t}\left[\widetilde{\alpha}_{ki}W^{1-\psi}_{k,t} + \widetilde{\vartheta}_{ki}(P^{X}_{ki,t})^{1-\psi}\right]^{\frac{1}{1-\psi}}$, $P^{X}_{ki,t} = \left[\widetilde{\varrho}^{X}_{ki}(P^{X,E}_{ki,t})^{1-\gamma_X} + (1-\widetilde{\varrho}^{X}_{ki})(P^{X,M}_{ki,t})^{1-\gamma_X}\right]^{\frac{1}{1-\gamma_X}}$, $P^{X,E}_{ki,t} = \left[\sum_{j\in I_E}\widetilde{\nu}^{X}_{kij}(P^{X}_{kij,t})^{1-\eta_X}\right]^{\frac{1}{1-\eta_X}}$, $P^{X,M}_{ki,t} =$

$\left[\sum_{j\in I_M}\widetilde{\upsilon}^X_{kij}(P^X_{kij,t})^{1-\iota_X}\right]^{\frac{1}{1-\eta_X}}$, and $P^X_{kij,t} = \left\{\sum_{l=1}^K \widetilde{\zeta}^X_{kl\,ij}[(1+\tau_{kl\,j,t})P^k_{kl\,j,t}]^{1-\delta_X}\right\}^{\frac{1}{1-\delta_X}}$. Their log-linearized counterparts are given by

$$\widehat{\text{mc}}_{ki,t} = \frac{1-\psi_{ki}}{\psi_{ki}}\widehat{y}_{ki,t} - \frac{1}{\psi_{ki}}a_{ki,t} + \frac{\mathcal{M}_{ki}}{\psi_{ki}}\frac{W_kN_{ki}}{P_{ki}Y_{ki}}\widehat{w}_{k,t} + \frac{\mathcal{M}_{ki}}{\psi_{ki}}\frac{P^X_{ki}X_{ki}}{P_{ki}Y_{ki}}\widehat{p}^X_{ki,t} \tag{F.45}$$

$$\widehat{p}^X_{ki,t} = \varrho^X_{ki}\widehat{p}^{X,E}_{ki,t} + (1-\varrho^X_{ki})\widehat{p}^{X,M}_{ki,t} \tag{F.46}$$

$$\widehat{p}^{X,E}_{ki,t} = \sum_{j\in I_E}\nu^X_{kij}\widehat{p}^X_{kij,t} \quad \text{and} \quad \widehat{p}^{X,M}_{ki,t} = \sum_{j\in I_M}\upsilon^X_{kij}\widehat{p}^X_{kij,t} \tag{F.47}$$

$$\widehat{p}^X_{kij,t} = \sum_{l=1}^K \zeta^X_{kl\,ij}(\widehat{p}^k_{kl\,j,t} + \tau_{kl\,j,t} + \tau^w_{kl\,j,t}) \tag{F.48}$$

where $\frac{\mathcal{M}_{ki}}{\psi_{ki}}\frac{W_kN_{ki}}{P_{ki}Y_{ki}} = \left(\frac{W_kY_{ki}^{\frac{1-\psi_{ki}}{\psi_{ki}}}}{\text{MC}_{ki}\psi_{ki}}\right)^{1-\psi}\widetilde{\alpha}_{ki}$ and $\frac{\mathcal{M}_{ki}}{\psi_{ki}}\frac{P^X_{ki}X_{ki}}{P_{ki}Y_{ki}} = \left(\frac{P^X_{ki}Y_{ki}^{\frac{1-\psi_{ki}}{\psi_{ki}}}}{\text{MC}_{ki}\psi_{ki}}\right)^{1-\psi}\widetilde{\vartheta}_{ki}$ can be derived using (F.32)-(F.33) in steady-state.

Production Structure. The log-linearized version of the production function (F.30) is given by

$$\widehat{y}_{kif,t} = a_{ki,t} + \mathcal{M}_{ki}\alpha_{ki}\widehat{n}_{kif,t} + \mathcal{M}_{ki}\vartheta_{ki}\widehat{x}_{kif,t} \tag{F.49}$$

where the identities $\mathcal{M}_{ki}\alpha_{ki} = \mathcal{M}_{ki}\frac{W_kN_{ki}}{P_{ki}Y_{ki}} = \frac{N_{ki}}{Y_{ki}}\psi_{ki}Y_{ki}^{\frac{(1-\psi_{ki})(1-\psi)}{\psi\psi_{ki}}}\left(\frac{\widetilde{\alpha}_{ki}Y_{ki}}{N_{ki}}\right)^{\frac{1}{\psi}}$ and $\mathcal{M}_{ki}\vartheta_{ki} = \mathcal{M}_{ki}\frac{P^X_{ki}X_{ki}}{P_{ki}Y_{ki}} = \frac{X_{ki}}{Y_{ki}}\psi_{ki}Y_{ki}^{\frac{(1-\psi_{ki})(1-\psi)}{\psi\psi_{ki}}}\left(\frac{\widetilde{\vartheta}_{ki}Y_{ki}}{X_{ki}}\right)^{\frac{1}{\psi}}$ can be verified using the first-order conditions from the firms' problem (F.32) and (F.33) in steady-state and the standard monopolistic competition pricing condition in steady-state, $P_{ki} = \mathcal{M}_{ki}\text{MC}_{ki}$. Using the previous identities, together with the production function (F.30) in steady-state, one can verify that

$$\mathcal{M}_{ki}\frac{W_kN_{ki}}{P_{ki}Y_{ki}} + \mathcal{M}_{ki}\frac{P^X_{ki}X_{ki}}{P_{ki}Y_{ki}} = \frac{W_kN_{ki}}{\text{MC}_{ki}Y_{ki}} + \frac{P^X_{ki}X_{ki}}{\text{MC}_{ki}Y_{ki}} = \psi_{ki}. \tag{F.50}$$

F.3. Price–Setting

Relative to the price-setting block in the main text, the quantitative model allows non-reset prices to be partially indexed to lagged inflation. This section derives the corresponding optimal reset-price condition and Phillips curves.

We extend our framework to allow for a time-varying elasticity of substitution between

different good varieties, in order to micro-found price cost-push shocks. We extend (3) to

$$Y^m_{kl\,i,t} = \left(\int_0^1 (Y^m_{kl\,if,t})^{(\epsilon^p_{ki,t}-1)/\epsilon^p_{ki,t}} df \right)^{\epsilon^p_{ki,t}/(\epsilon^p_{ki,t}-1)}, \tag{F.51}$$

where $Y^m_{kl\,i,t}$ denotes the demand for domestic sector i from country k invoiced in currency m in country l. The implied sectoral price index is

$$P^m_{kl\,i,t} = \left(\int_0^1 (P^m_{kl\,if,t})^{1-\epsilon^p_{ki,t}} df \right)^{\frac{1}{1-\epsilon^p_{ki,t}}}. \tag{F.52}$$

Producers of each differentiated variety face the demand function

$$Y^m_{kl\,i,t+s|t} = \left(\overline{P}^m_{l\,ki,t} \mathcal{I}^m_{l\,ki,t,s} / P^m_{kl\,i,t+s} \right)^{-\epsilon^p_{ki,t}} Y^m_{kl\,i,t+s}, \tag{F.53}$$

where $\mathcal{I}^m_{l\,ki,t,0} = 1$ and, for $s \geq 1$,

$$\mathcal{I}^m_{l\,ki,t,s} = \prod_{r=1}^{s} \left(\Pi^m_{l\,ki,t+r-1} \right)^{\iota^p_{ki}} \left(\Pi^C_{k,t+r-1} \right)^{\chi^p_{ki}}. \tag{F.54}$$

Firms set prices à la Calvo, and non-reset prices are indexed according to $\mathcal{I}^m_{l\,ki,t,s}$. The aggregate price dynamics are therefore described by

$$(\Pi^m_{l\,ki,t})^{1-\epsilon^p_{ki,t}} = \theta^p_{ki} \left[\left(\Pi^m_{l\,ki,t-1} \right)^{\iota^p_{ki}} \left(\Pi^C_{k,t-1} \right)^{\chi^p_{ki}} \right]^{1-\epsilon^p_{ki,t}} + (1-\theta^p_{ki}) \left(\overline{P}^m_{l\,ki,t} / P^m_{l\,ki,t-1} \right)^{1-\epsilon^p_{ki,t}}, \tag{F.55}$$

which can be written in log-linearized terms as

$$\pi^m_{l\,ki,t} = \theta^p_{ki} \left(\iota^p_{ki} \pi^m_{l\,ki,t-1} + \chi^p_{ki} \pi^C_{k,t-1} \right) + (1-\theta^p_{ki})(\overline{p}^m_{l\,ki,t} - p^m_{l\,ki,t-1}). \tag{F.56}$$

Denote by $P^m_{l\,ki,t}$ the price of a good from sector i, originating in country k, sold in country l, and invoiced in the currency of country m. The per-period nominal profits of the domestic firm producing good from sector i are then given by

$$\Pi_{ki,t} = \sum_{l=1}^{K} \sum_{m=1}^{K} \mathcal{E}^m_{k,t} P^m_{l\,ki,t} Y^m_{l\,ki,t} - \mathcal{C}_{ki,t}(Y_{ki,t}) \tag{F.57}$$

where $Y_{ki,t} = \sum_{l=1}^{K} \sum_{m=1}^{K} Y^m_{l\,ki,t}$ denotes the total demand across destination markets l and

invoicing currencies k. $\mathcal{C}_{ki,t}(Y_{ki,t})$ denotes total costs of country k firms in their home currency.

Consider the pricing problem of a firm from country k selling in country l and invoicing in currency m, and denote $\overline{P}^m_{l\,ki,t}$ its optimally-chosen reset price. Written market by market, the problem is

$$\max_{\overline{P}^m_{l\,ki,t}} \sum_{s=0}^{\infty} (\theta^p_{ki})^s \mathbb{E}_t \left\{ \frac{\Lambda_{t,t+s}}{P_{ki,t+s}} \left[\mathcal{E}^m_{k,t+s} \overline{P}^m_{l\,ki,t} \mathcal{I}^m_{l\,ki,t,s} Y^m_{l\,ki,t+s|t} - \mathcal{C}_{ki,t+s} \left(Y^m_{l\,ki,t+s|t} \right) \right] \right\} \tag{F.58}$$

subject to the sequence of demand constraints (F.53), with $P_{ki,t} = P^k_{kki,t}$. The optimality condition associated with the problem takes the form

$$\sum_{s=0}^{\infty} (\theta^p_{ki})^s \mathbb{E}_t \left\{ \frac{\Lambda_{t,t+s} Y^m_{l\,ki,t+s|t}}{P_{ki,t+s}} [\mathcal{E}^m_{k,t+s} \overline{P}^m_{l\,ki,t} \mathcal{I}^m_{l\,ki,t,s} - \mathcal{M}_{ki,t+s} \mathrm{MC}_{ki,t+s|t}] \right\} = 0. \tag{F.59}$$

Let $i^{p,m}_{l\,ki,t,s} \equiv \sum_{r=1}^{s} \left(\iota^p_{ki} \pi^m_{l\,ki,t+r-1} + \chi^p_{ki} \pi^C_{k,t+r-1} \right)$ denote the log cumulative indexation adjustment. Thus, CPI indexation is always to producer-country CPI inflation. A first-order Taylor expansion of (F.59) around the zero inflation steady state yields

$$\overline{p}^m_{l\,ki,t} = (1 - \beta\theta^p_{ki}) \sum_{s=0}^{\infty} (\beta\theta^p_{ki})^s \mathbb{E}_t \left(\mathrm{mc}_{ki,t+s|t} - e^m_{k,t+s} + \mu_{ki,t+s} - i^{p,m}_{l\,ki,t,s} \right). \tag{F.60}$$

The log marginal cost for an individual firm that last set its price in period t is given by (F.37),

$$\mathrm{mc}_{ki,t+s|t} = w_{k,t+s} - \frac{\psi - 1}{\psi\psi_{ki}} a_{ki,t+s} - \log \psi_{ki} - \frac{(1 - \psi_{ki})(1 - \psi)}{\psi\psi_{ki}} y_{ki,t+s|t} - \frac{1}{\psi} \left[\log \widetilde{\alpha}_{ki} + y_{ki,t+s|t} - n_{ki,t+s|t} \right],$$

where $n_{ki,t+s|t}$ denotes the log employment in period $t+s$ for a firm that last reset its price in period t, and where we have made use of (F.37). Letting $\mathrm{mc}_{ki,t} = \int_0^1 \mathrm{mc}_{kif,t}\, df = (1 - \theta^p_{ki}) \sum_{s=0}^{\infty} (\theta^p_{ki})^s \mathrm{mc}_{ki,t|t-s}$ represent the log average marginal cost, it follows that

$$\mathrm{mc}_{ki,t} = w_{k,t} - \frac{\psi - 1}{\psi\psi_{ki}} a_{ki,t} - \log \psi_{ki} - \frac{(1 - \psi_{ki})(1 - \psi)}{\psi\psi_{ki}} y_{ki,t} - \frac{1}{\psi} \left[\log \widetilde{\alpha}_{ki} + y_{ki,t} - n_{ki,t} \right].$$

Thus, the following relation holds between firm-specific and economy-wide marginal costs

$$\mathrm{mc}_{ki,t+s|t} = \mathrm{mc}_{ki,t+s} - \frac{(1 - \psi_{ki})(1 - \psi)}{\psi\psi_{ki}} \left(y_{ki,t+s|t} - y_{ki,t+s} \right) - \frac{1}{\psi} \left[\left(y_{ki,t+s|t} - y_{ki,t+s} \right) - \left(n_{ki,t+s|t} - n_{ki,t+s} \right) \right].$$

Notice that, making use of both marginal cost expressions (F.32)-(F.33), the identity $x_{ki,t+s|t} -$

$x_{ki,t+s} = n_{ki,t+s|t} - n_{ki,t+s}$ is satisfied. Hence, we can write

$$y_{ki,t+s|t} - y_{ki,t+s} = \left[\mathcal{M}_{ki}\frac{W_k N_{ki}}{P_{ki}Y_{ki}} + \mathcal{M}_{ki}\frac{P_{ki}^X X_{ki}}{P_{ki}Y_{ki}}\right](n_{ki,t+s|t} - n_{ki,t+s}) = \psi_{ki}(n_{ki,t+s|t} - n_{ki,t+s}),$$

where we have used the identity (F.50), and where we have used the linearized production function (F.49). We can finally write the relation between marginal costs as

$$\text{mc}_{ki,t+s|t} = \text{mc}_{ki,t+s} + \frac{1-\psi_{ki}}{\psi_{ki}}\left(y_{ki,t+s|t} - y_{ki,t+s}\right) = \text{mc}_{ki,t+s} - \frac{(1-\psi_{ki})\epsilon_{ki}^p}{\psi_{ki}}\left(\overline{p}_{l\,ki,t}^m + i_{l\,ki,t,s}^{p,m} - p_{l\,ki,t+s}^m\right).$$

Introducing this last expression into the log-linearized firms' FOC (F.60), we can write

(F.61)

$$\overline{p}_{l\,ki,t}^m = (1-\beta\theta_{ki}^p)\sum_{s=0}^{\infty}(\beta\theta_{ki}^p)^s\mathbb{E}_t\left[\text{mc}_{ki,t+s} - \frac{(1-\psi_{ki})\epsilon_{ki}^p}{\psi_{ki}}\left(\overline{p}_{l\,ki,t}^m + i_{l\,ki,t,s}^{p,m} - p_{l\,ki,t+s}^m\right) - e_{k,t+s}^m + \mu_{ki,t+s} - i_{l\,ki,t,s}^{p,m}\right]$$

(F.62)

$$= (1-\beta\theta_{ki}^p)\sum_{s=0}^{\infty}(\beta\theta_{ki}^p)^s\mathbb{E}_t\left[\Theta_{ki}\text{mc}_{ki,t+s} + (1-\Theta_{ki})\,p_{l\,ki,t+s}^m - \Theta_{ki}e_{k,t+s}^m + \Theta_{ki}\mu_{ki,t+s} - i_{l\,ki,t,s}^{p,m}\right]$$

Rewriting the (linearized) inflation dynamics (F.56) as

$$\overline{p}_{l\,ki,t}^m - p_{l\,ki,t}^m = \frac{\theta_{ki}^p}{1-\theta_{ki}^p}\left[\pi_{l\,ki,t}^m - \iota_{ki}^p\pi_{l\,ki,t-1}^m - \chi_{ki}^p\pi_{k,t-1}^C\right]. \tag{F.63}$$

Subtracting $p_{l\,ki,t}^m$ on both sides of (F.60), and inserting (F.63) on the LHS,

$$\begin{aligned}\frac{\theta_{ki}^p}{1-\theta_{ki}^p}\left[\pi_{l\,ki,t}^m - \iota_{ki}^p\pi_{l\,ki,t-1}^m - \chi_{ki}^p\pi_{k,t-1}^C\right] &= (1-\beta\theta_{ki}^p)\sum_{s=0}^{\infty}(\beta\theta_{ki}^p)^s\mathbb{E}_t\left[\Theta_{ki}(\text{mc}_{ki,t+s} - p_{l\,ki,t+s}^m - e_{k,t+s}^m) + p_{l\,ki,t+s}^m\right.\\ &= \Theta_{ki}(1-\beta\theta_{ki}^p)\sum_{s=0}^{\infty}(\beta\theta_{ki}^p)^s\mathbb{E}_t\left(\text{mc}_{ki,t+s} - p_{l\,ki,t+s}^m - e_{k,t+s}^m + \mu_{ki,t+s}\right)\end{aligned}$$

(F.64)

$$+ (1-\beta\theta_{ki}^p)\sum_{s=1}^{\infty}(\beta\theta_{ki}^p)^s\mathbb{E}_t\left[\pi_{l\,ki,t+s}^m - \iota_{ki}^p\pi_{l\,ki,t+s-1}^m - \chi_{ki}^p\pi_{k,t+s-1}^C\right].$$

Taking conditional expectations at time $t+1$, multiplying by $\beta\theta_{ki}^p$, and rearranging terms

using (F.56), we can write

$$\pi^m_{l\,ki,t} = \frac{1}{1+\beta\iota^p_{ki}}\left\{\kappa_{ki}\left(\mathrm{mc}_{ki,t} - p^m_{l\,ki,t} - e^m_{k,t}\right) + \beta\mathbb{E}_t\pi^m_{l\,ki,t+1} + \iota^p_{ki}\pi^m_{l\,ki,t-1}\right.$$
$$\left. +\chi^p_{ki}\pi^C_{k,t-1} - \beta\chi^p_{ki}\pi^C_{k,t} + u^p_{ki,t}\right\}, \tag{F.65}$$

where $\kappa_{ki} = \Theta_{ki}(1-\theta^p_{ki})(1-\beta\theta^p_{ki})/\theta^p_{ki}$.

We assume that the sectoral price cost-push shocks $u^p_{ki,t} = \kappa_{ki}\widehat{\mu}_{ki,t} = \kappa_{ki}(\mu_{ki,t} - \mu_{ki})$, micro-founded through a time-varying elasticity of substitution $\epsilon^p_{ki,t}$ in (3), follow independent AR(1) processes:

$$u^p_{ki,t} = \rho^p_{ki}u^p_{ki,t-1} + \varepsilon^p_{ki,t}, \tag{F.66}$$

where the innovation is $\varepsilon^p_{ki,t} \sim \mathcal{N}\left(0, \sigma^2_{ki\,p}\right)$.

Producer Currency Pricing. Under PCP ($m = k$), the price is set in the producer currency and $\pi^k_{l\,ki,t} = \pi_{ki,t}$, so we can write (F.65) as

$$\begin{aligned}
\pi_{ki,t} &= \frac{1}{1+\beta\iota^p_{ki}}\left\{\kappa_{ki}\left(\mathrm{mc}_{ki,t} - p_{ki,t}\right) + \beta\mathbb{E}_t\pi_{ki,t+1} + \iota^p_{ki}\pi_{ki,t-1}\right.\\
&\quad \left. +\chi^p_{ki}\pi^C_{k,t-1} - \beta\chi^p_{ki}\pi^C_{k,t} + u^p_{ki,t}\right\}\\
&= \frac{1}{1+\beta\iota^p_{ki}}\left\{\kappa_{ki}\left[(\mathrm{mc}_{ki,t} - p^C_{k,t}) - (p_{ki,t} - p^C_{k,t})\right] + \beta\mathbb{E}_t\pi_{ki,t+1} + \iota^p_{ki}\pi_{ki,t-1}\right.\\
&\quad \left. +\chi^p_{ki}\pi^C_{k,t-1} - \beta\chi^p_{ki}\pi^C_{k,t} + u^p_{ki,t}\right\}\\
&= \frac{1}{1+\beta\iota^p_{ki}}\left\{\kappa_{ki}\left(\widehat{\mathrm{mc}}_{ki,t} - \widehat{p}_{ki,t}\right) + \beta\mathbb{E}_t\pi_{ki,t+1} + \iota^p_{ki}\pi_{ki,t-1}\right.\\
&\quad \left. +\chi^p_{ki}\pi^C_{k,t-1} - \beta\chi^p_{ki}\pi^C_{k,t} + u^p_{ki,t}\right\},
\end{aligned} \tag{F.67}$$

where the sectoral-inflation rates and sectoral-level real prices ($\widehat{p}_{ki,t}$) are related through the identity

$$\pi_{ki,t} = p_{ki,t} - p_{ki,t-1} = \widehat{p}_{ki,t} - \widehat{p}_{ki,t-1} + \pi^C_{k,t}, \tag{F.68}$$

where $\widehat{p}_{ki,t} = p_{ki,t} - p^C_{k,t}$.

Local Currency Pricing. Under LCP ($m = l$), we have $\pi^{l}_{l\,ki,t} = \pi_{l\,ki,t}$, and we can write (F.65) as

$$\begin{aligned}
\pi_{l\,ki,t} &= \frac{1}{1+\beta\iota^{p}_{ki}}\left\{\kappa_{ki}\left(\mathrm{mc}_{ki,t} - p_{l\,ki,t} - e^{l}_{k,t}\right) + \beta\mathbb{E}_t\pi_{l\,ki,t+1} + \iota^{p}_{ki}\pi_{l\,ki,t-1}\right. \\
&\quad \left. +\chi^{p}_{ki}\pi^{C}_{k,t-1} - \beta\chi^{p}_{ki}\pi^{C}_{k,t} + u^{p}_{ki,t}\right\} \\
&= \frac{1}{1+\beta\iota^{p}_{ki}}\left\{\kappa_{ki}\left[(\mathrm{mc}_{ki,t} - p^{C}_{k,t}) - (p_{l\,ki,t} - p^{C}_{k,t}) - e^{l}_{k,t}\right] + \beta\mathbb{E}_t\pi_{l\,ki,t+1} + \iota^{p}_{ki}\pi_{l\,ki,t-1}\right. \\
&\quad \left. +\chi^{p}_{ki}\pi^{C}_{k,t-1} - \beta\chi^{p}_{ki}\pi^{C}_{k,t} + u^{p}_{ki,t}\right\} \\
&= \frac{1}{1+\beta\iota^{p}_{ki}}\left\{\kappa_{ki}\left[(\mathrm{mc}_{ki,t} - p^{C}_{k,t}) - (p_{l\,ki,t} - p^{C}_{l,t}) + p^{C}_{k,t} - p^{C}_{l,t} - e^{l}_{k,t}\right]\right. \\
&\quad \left. +\beta\mathbb{E}_t\pi_{l\,ki,t+1} + \iota^{p}_{ki}\pi_{l\,ki,t-1} + \chi^{p}_{ki}\pi^{C}_{k,t-1} - \beta\chi^{p}_{ki}\pi^{C}_{k,t} + u^{p}_{ki,t}\right\} \\
&= \frac{1}{1+\beta\iota^{p}_{ki}}\left\{\kappa_{ki}\left(\widehat{\mathrm{mc}}_{ki,t} - \widehat{p}_{l\,ki,t} - q^{l}_{k,t}\right) + \beta\mathbb{E}_t\pi_{l\,ki,t+1} + \iota^{p}_{ki}\pi_{l\,ki,t-1}\right. \\
&\quad \left. +\chi^{p}_{ki}\pi^{C}_{k,t-1} - \beta\chi^{p}_{ki}\pi^{C}_{k,t} + u^{p}_{ki,t}\right\}, \qquad \text{(F.69)}
\end{aligned}$$

which is the export Phillips curve, where we have used (F.10), and where

$$\pi_{l\,ki,t} = p_{l\,ki,t} - p_{l\,ki,t-1} = \widehat{p}_{l\,ki,t} - \widehat{p}_{l\,ki,t-1} + \pi^{C}_{l,t} \qquad \text{(F.70)}$$

denotes price inflation in sector i exported to country l from country k, where $\widehat{p}_{l\,ki,t} = p_{l\,ki,t} - p^{C}_{l,t}$.

Dominant Currency Pricing. Under DCP ($m = \mathrm{US}$), we have $\pi^{\mathrm{US}}_{l\,ki,t} = \pi_{ki,t}$, and we can write (F.65) as

$$\begin{aligned}
\pi_{ki,t} &= \frac{1}{1+\beta\iota^{p}_{ki}}\left\{\kappa_{ki}\left(\mathrm{mc}_{ki,t} - p_{ki,t} - e^{\mathrm{US}}_{k,t}\right) + \beta\mathbb{E}_t\pi_{ki,t+1} + \iota^{p}_{ki}\pi_{ki,t-1}\right. \\
&\quad \left. +\chi^{p}_{ki}\pi^{C}_{k,t-1} - \beta\chi^{p}_{ki}\pi^{C}_{k,t} + u^{p}_{ki,t}\right\} \\
&= \frac{1}{1+\beta\iota^{p}_{ki}}\left\{\kappa_{ki}\left[(\mathrm{mc}_{ki,t} - p^{C}_{k,t}) - (p_{ki,t} - p^{C}_{\mathrm{US},t}) + p^{C}_{k,t} - p^{C}_{\mathrm{US},t} - e^{\mathrm{US}}_{k,t}\right]\right. \\
&\quad \left. +\beta\mathbb{E}_t\pi_{ki,t+1} + \iota^{p}_{ki}\pi_{ki,t-1} + \chi^{p}_{ki}\pi^{C}_{k,t-1} - \beta\chi^{p}_{ki}\pi^{C}_{k,t} + u^{p}_{ki,t}\right\} \\
&= \frac{1}{1+\beta\iota^{p}_{ki}}\left\{\kappa_{ki}\left(\widehat{\mathrm{mc}}_{ki,t} - \widehat{p}_{ki,t} - q^{\mathrm{US}}_{k,t}\right) + \beta\mathbb{E}_t\pi_{ki,t+1} + \iota^{p}_{ki}\pi_{ki,t-1}\right. \\
&\quad \left. +\chi^{p}_{ki}\pi^{C}_{k,t-1} - \beta\chi^{p}_{ki}\pi^{C}_{k,t} + u^{p}_{ki,t}\right\}. \qquad \text{(F.71)}
\end{aligned}$$

where the sectoral-inflation rates and sectoral-level real prices ($\widehat{p}_{ki,t}$) are related through the identity

$$\pi_{ki,t} = p_{ki,t} - p_{ki,t-1} = \widehat{p}_{ki,t} - \widehat{p}_{ki,t-1} + \pi^{C}_{\text{US},t}, \tag{F.72}$$

where $\widehat{p}_{ki,t} = p_{ki,t} - p^{C}_{\text{US},t}$. For domestic sales, $k = l$, we have that

$$\pi_{kki,t} = \pi_{ki,t} \quad \text{and} \quad \widehat{p}_{kki,t} = \widehat{p}_{ki,t}, \tag{F.73}$$

and both the Phillips curve and the sectoral inflation rates are given by (F.67) and (F.68), respectively.

Marginal costs and aggregate inflation. In this open input-output (IO) economy, the price Phillips curves depends on the international supply network through the real marginal costs faced by firm i in country k, $\widehat{\text{mc}}_{ki,t}$. Combining the log-linearized intermediate input prices indices (F.45)-(F.48), we obtain the marginal cost equation,

$$\widehat{\text{mc}}_{ki,t} = \frac{1-\psi_{ki}}{\psi_{ki}}\widehat{y}_{ki,t} - \frac{1}{\psi_{ki}}a_{ki,t} + \mathcal{M}_{ki}\alpha_{ki}\widehat{w}_{k,t} + \sum_{l=1}^{K}\sum_{j=1}^{I}\mathcal{M}_{ki}\omega_{kl\,ij}\left(\widehat{p}^{k}_{kl\,j,t} + \tau_{kl\,j,t} + \tau^{w}_{kl\,j,t}\right) \tag{F.74}$$

where, in the absence of a production subsidy, $\alpha_{ki} = \frac{W_k N_{ki}}{P_{ki}Y_{ki}} = \frac{W_k N_{ki}}{\mathcal{M}_{ki}\text{MC}_{ki}Y_{ki}}$ denotes the (steady-state) labor income share of total sales of firm i, $\omega_{kl\,ij} = \frac{P_{kl\,j}X_{kl\,ij}}{P_{ki}Y_{ki}} = \frac{P_{kl\,j}X_{kl\,ij}}{\mathcal{M}_{ki}\text{MC}_{ki}Y_{ki}}$ denotes the (steady-state) IO expenditure share of total sales of firm i, and $\mathcal{M}_{ki} = \epsilon^{p}_{ki}/(\epsilon^{p}_{ki} - 1)$ denotes the steady-state markup charged by firm i.

Writing (F.19)-(F.21) in first-differences, we can obtain consumer price inflation,

$$\pi^{C}_{k,t} = \sum_{i=1}^{I}\sum_{l=1}^{K}\beta_{kl\,i}\pi^{k}_{kl\,i,t} \tag{F.75}$$

where $\beta_{kl\,i} = \frac{P_{kl\,i}C_{kl\,i}}{P^{C}_{k}C_{k}} = \zeta_{kl\,i}\left[\nu_{ki}\varrho^{C}_{k}\mathbb{1}_{\{i\in I_E\}} + \upsilon_{ki}(1-\varrho^{C}_{k})\left(1-\mathbb{1}_{\{i\in I_E\}}\right)\right]$, and

$$\pi^{k}_{kl\,i,t} = p^{k}_{kl\,i,t} - p^{k}_{kl\,i,t-1} = \widehat{p}^{k}_{kl\,i,t} - \widehat{p}^{k}_{kl\,i,t-1} + \pi^{C}_{k,t}. \tag{F.76}$$

F.4. Wage–Setting

Following Erceg *et al.* (2000), wage stickiness is introduced in a way analogous to price stickiness. Labor unions specialized in any given labor type can reset their nominal wage only

with probability $1-\theta_k^w$ each period, independently of the time elapsed since they last adjusted their wage. Unions that cannot reset their wage partially index it to lagged CPI inflation, with indexation parameter $\chi_k^w \in [0,1]$. We assume that firms employ a continuum of differentiated labor services. In particular, $N_{kif,t}$ is an index of labor input used by firm f, and defined by

$$N_{kif,t} = \left(\int_0^1 N_{kigf,t}^{\frac{\epsilon_k^w - 1}{\epsilon_k^w}} dg \right)^{\frac{\epsilon_k^w}{\epsilon_k^w - 1}}, \tag{F.77}$$

where $N_{kigf,t}$ denotes the quantity of type-g labor employed by firm f in period t. Note that $\epsilon_{k,t}^w$ represents the elasticity of substitution among labor varieties. Note also the assumption of a continuum of labor types, indexed by $g \in [0,1]$.

Let $W_{gk,t}$ denote the nominal wage for type-g labor prevailing in period t. Nominal wages are set by workers of each type (or a union representing them) and taken as given by firms. Given the wages effective at any point in time for the different types of labor services, cost minimization yields a corresponding set of demand schedules for each firm f and labor type g, given the firm's total employment $N_{kf,t}$,

$$N_{kigf,t} = \left(\frac{W_{gk,t}}{W_{k,t}} \right)^{-\epsilon_{k,t}^w} N_{kif,t}, \tag{F.78}$$

where

$$W_{k,t} \equiv \left(\int_0^1 W_{gk,t}^{1-\epsilon_{k,t}^w} dg \right)^{\frac{1}{1-\epsilon_{k,t}^w}} \tag{F.79}$$

is an aggregate wage index. Combining the previous conditions, one can obtain a convenient aggregation result, $\int_0^1 W_{gk,t} N_{kigf,t} dg = W_{k,t} N_{kif,t}$. That is, the wage bill of any given firm can be expressed as the product of the wage index and the firm's employment index.

Consider a union resetting its members' wage in period t, and let $W_{k,t}^*$ denote the newly set wage. If the wage remains in place until period $t+s$, it is indexed according to $W_{k,t+s|t} = W_{k,t}^* \prod_{r=1}^{s} (\Pi_{k,t+r-1}^C)^{\chi_k^w}$. The union chooses $W_{k,t}^*$ in a way consistent with utility maximization of its members' households, taking as given the decisions of other unions as well as the paths of aggregate consumption and prices. Specifically, the union seeks to maximize

$$\max_{W_{k,t}^*} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta \theta_k^w)^s \left(\frac{\Lambda_{k,t+s}}{\Lambda_{k,t}} \frac{W_{k,t+s|t} N_{k,t+s|t}}{P_{t+s}^c} - \frac{N_{k,t+s|t}^{1+\varphi}}{1+\varphi} \right)$$

subject to the sequence of labor demand schedules

$$N_{k,t+s|t} = \left(\frac{W_{k,t+s|t}}{W_{k,t+s}}\right)^{-\epsilon^w_{k,t+s}} \int_0^1 N_{gk,t+s} dg,$$

where $N_{k,t+s|t}$ denotes the level of employment in period $t+s$ among workers that last reset their wage in period t, and $\Lambda_{k,t} = Z_{k,t}\left(C_{k,t} - h_k C_{k,t-1}\right)^{-\sigma}$ is the household's marginal utility of nominal income. The first-order condition is given by

$$\sum_{s=0}^{\infty} (\beta\theta^w_k)^s \mathbb{E}_t \left[N_{k,t+s|t} \Lambda_{k,t+s} \left(\frac{W_{k,t+s|t}}{P^c_{t+s}} - \mathcal{M}^w_{k,t+s} \text{MRS}_{k,t+s|t} \right) \right] = 0,$$

where $\mathcal{M}^w_{k,t+s} = \frac{\epsilon^w_{k,t+s}}{\epsilon^w_{k,t+s}-1}$, and $\text{MRS}_{k,t+s|t} = \left(C_{k,t+s} - h_k C_{k,t+s-1}\right)^{\sigma} N^{\varphi}_{k,t+s|t}$ denotes the marginal rate of substitution between household consumption and employment in period $t+s$ relevant to the workers resetting their wage in period t. Log-linearizing the above expression around a zero inflation steady-state, and denoting the cumulative indexation adjustment by $i^w_{k,t,s} \equiv \chi^w_k \sum_{r=1}^{s} \pi^C_{k,t+r-1}$, yields the wage setting rule

$$w^*_{k,t} = (1 - \beta\theta^w_k) \sum_{s=0}^{\infty} (\beta\theta^w_k)^s \mathbb{E}_t \left(\text{mrs}_{t+s|t} + \mu^w_{k,t} + p^C_{k,t+s} - i^w_{k,t,s} \right) \tag{F.80}$$

where $\mu^w_{k,t} = \log \mathcal{M}^w_{k,t}$ and $\text{mrs}_{t+s|t} = \frac{\sigma}{1-h_k}(c_{k,t+s} - h_k c_{k,t+s-1}) + \varphi n_{k,t+s|t}$.

Letting $\text{mrs}_{t+s} = \frac{\sigma}{1-h_k}(c_{k,t+s} - h_k c_{k,t+s-1}) + \varphi n_{k,t+s}$ define the economy's average marginal rate of substitution, where $n_{k,t+s} = \log \int_0^1 \int_0^1 N_{kgf}\, df\, dg$ denotes the log aggregate employment. Up to a first-order approximation,

$$\text{mrs}_{t+s|t} = \text{mrs}_{t+s} + \varphi(n_{k,t+s|t} - n_{k,t+s}) = \text{mrs}_{t+s} - \epsilon^w_k \varphi (w^*_{k,t} + i^w_{k,t,s} - w_{k,t+s}).$$

Hence, we can write (F.80) as

$$w^*_{k,t} = \frac{1 - \beta\theta^w_k}{1 + \epsilon^w_k \varphi} \sum_{s=0}^{\infty} (\beta\theta^w_k)^s \mathbb{E}_t \left[(1 + \epsilon^w_k \varphi) w_{k,t+s} - \left(w_{k,t+s} - p^C_{k,t+s} - \text{mrs}_{k,t+s} - \mu^w_{k,t+s} \right) - (1 + \epsilon^w_k \varphi) i^w_{k,t,s} \right] \tag{F.81}$$

where $\mu^w_{k,t+s}$ denotes the deviations of the economy's log average wage markup.

Given the assumed wage setting structure with CPI indexation, the evolution of the aggre-

gate wage index is given by

$$W_{k,t} = \left(\theta_k^w \left[W_{k,t-1}(\Pi_{k,t-1}^C)^{\chi_k^w}\right]^{1-\epsilon_k^w} + (1-\theta_k^w)(W_{k,t}^*)^{1-\epsilon_k^w}\right)^{\frac{1}{1-\epsilon_k^w}}.$$

Log-linearized,

$$w_{k,t} = \theta_k^w(w_{k,t-1} + \chi_k^w \pi_{k,t-1}^C) + (1-\theta_k^w)w_{k,t}^*.$$

Combining the last expression with (F.81), and letting $\pi_{k,t}^w = w_{k,t} - w_{k,t-1}$, we obtain the wage inflation equation:

$$\begin{aligned}\pi_{k,t}^w = \kappa_k^w \left(\frac{\sigma}{1-h_k}\left(\widehat{c}_{k,t} - h_k \widehat{c}_{k,t-1}\right) + \varphi \widehat{n}_{k,t} - \widehat{w}_{k,t}\right) + \beta \mathbb{E}_t \pi_{k,t+1}^w \\ + \chi_k^w \pi_{k,t-1}^C - \beta \chi_k^w \pi_{k,t}^C + u_{k,t}^w,\end{aligned} \tag{F.82}$$

where

$$\pi_{k,t}^w = w_{k,t} - w_{k,t-1} = \widehat{w}_{k,t} - \widehat{w}_{k,t-1} + \pi_{k,t}^C, \qquad \kappa_k^w = \frac{(1-\theta_k^w)(1-\beta\theta_k^w)}{\theta_k^w(1+\epsilon_k^w\varphi)} \tag{F.83}$$

The first expression defines wage inflation and the second defines the wage Phillips curve slope. Both aggregate consumption $\widehat{c}_{k,t}$ and employment $\widehat{n}_{k,t}$ appear in log-deviations from their steady-state values, and $u_{k,t}^w = \kappa_k^w \widehat{\mu}_{k,t}^w = \kappa_k^w(\widehat{\mu}_{k,t}^w - \widehat{\mu}_k^w)$.

We assume that the wage cost-push shock, micro-founded through a time-varying elasticity of substitution in the labor demand aggregator, follows an AR(1) process:

$$u_{k,t}^w = \rho_k^w u_{k,t-1}^w + \varepsilon_{k,t}^w \tag{F.84}$$

where the innovation is $\varepsilon_{k,t}^w \sim \mathcal{N}\left(0, \sigma_{kw}^2\right)$.

F.5. Monetary Authority

The log-linearized bilateral nominal exchange rate (E.11) is given by $e_{k,t}^{k^{\text{MU}}} = e_{k,}^{k^{\text{MU}}}, \forall k \in K^{MU}$. In stationary terms, taking first differences, this can be written as

$$\Delta e_{k,t}^{k^{\text{MU}}} = 0 \quad \forall k \in K^{MU} \tag{F.85}$$

Log-linearizing the expression for the real exchange rate (F.10) and first-differencing, we

obtain

$$\Delta q^{l}_{k,t} = \Delta e^{l}_{k,t} + \pi_{l,t} - \pi_{k,t}. \tag{F.86}$$

The nominal exchange rates are cross-related by

$$e^{l}_{k,t} = e^{m}_{k,t} - e^{m}_{l,t}, \tag{F.87}$$

and symmetry between nominal exchange rates is satisfied,

$$e^{l}_{k,t} = -e^{k}_{l,t}. \tag{F.88}$$

Log-linearizing and first-differencing the symmetry of nominal exchange rates condition $\mathcal{E}^{l}_{k,t} = (\mathcal{E}^{k}_{l,t})^{-1}$ yields

$$\Delta e^{l}_{k,t} = -\Delta e^{k}_{l,t}. \tag{F.89}$$

F.6. Market Clearing, GDP, and Trade Balance

Market Clearing. We first consider the goods market clearing condition (18). Pre-multiplying by $\frac{P_{ki,t}}{P_{k,t}C_{k,t}} = \frac{P_{ki,t}}{E_{k,t}}$, and making use of (F.17),

$$\begin{aligned}
\frac{P_{ki,t}Y_{ki,t}}{E_{k,t}} &= \sum_{l=1}^{K} \frac{P_{ki,t}C_{l\,ki,t}}{E_{k,t}} + \sum_{l=1}^{K}\sum_{j=1}^{I} \frac{P_{ki,t}X_{l\,kji,t}}{E_{k,t}} \\
&= \sum_{l=1}^{K} \frac{P_{ki,t}}{P_{l\,ki,t}} \frac{P_{l\,ki,t}C_{l\,ki,t}}{E_{k,t}} + \sum_{l=1}^{K}\sum_{j=1}^{I} \frac{P_{ki,t}}{P_{l\,ki,t}} \frac{P_{lj,t}Y_{lj,t}}{E_{k,t}} \frac{P_{l\,ki,t}X_{l\,kji,t}}{P_{lj,t}Y_{lj,t}} \\
&= \sum_{l=1}^{K} \frac{P_{ki,t}}{P_{l\,ki,t}} \frac{E_{l,t}}{E_{k,t}} \frac{P_{l\,ki,t}C_{l\,ki,t}}{E_{l,t}} + \sum_{l=1}^{K}\sum_{j=1}^{I} \frac{P_{ki,t}}{P_{l\,ki,t}} \frac{E_{l,t}}{E_{k,t}} \frac{P_{lj,t}Y_{lj,t}}{E_{l,t}} \frac{P_{l\,ki,t}X_{l\,kji,t}}{P_{lj,t}Y_{lj,t}} \quad \forall i \in I
\end{aligned} \tag{F.90}$$

which we can write in steady-state as

$$\lambda_{ki} = \sum_{l=1}^{K} \mathcal{Y}_{l\,k}\beta_{l\,ki} + \sum_{l=1}^{K}\sum_{j=1}^{I} \mathcal{Y}_{l\,k}\lambda_{lj}\omega_{l\,kji} \quad \forall i \in I$$

where the Domar weight for sector i in country k is $\lambda_{ki} = \frac{P_{ki}Y_{ki}}{\mathcal{Y}_k}$, the cross-country scale factor is defined as $\mathcal{Y}_{l\,k} = \frac{P_{l\,C}C_l}{P^{C}_{k}C_k}$ and equals the nominal GDP ratio under the zero-net-

export steady-state normalization, and the IO share is given by $\omega_{lkji} = \frac{P_{lki}X_{lkji}}{P_{lj}Y_{lj}} = \frac{P_{ki}X_{lkji}}{P_{lj}Y_{lj}} = \zeta_{lkji}\vartheta_{lj}\left[\nu_{lji}\beta_{lj}\mathbb{1}_{\{i\in I_E\}} + \upsilon_{lji}(1-\beta_{lj})\left(1-\mathbb{1}_{\{i\in I_E\}}\right)\right]$, where we have made use of the law of one price in steady-state, $P_{klj} = P_{lj}$. Notice that β_{lki} and ω_{lkji} can be extracted directly from the data.

Hence, we can write the log-linearized version of the goods market clearing condition (18), $Y_{ki}\widehat{y}_{ki,t} = \sum_{l=1}^{K}\left(C_{lki}\widehat{c}_{lki,t} + \sum_{j=1}^{I}X_{lkji}\widehat{x}_{lkji}\right)$. Pre-multiplying the expression by P_{ki}/E_k, we can write

$$\lambda_{ki}\widehat{y}_{ki,t} = \sum_{l=1}^{K}\mathcal{Y}_{lk}\left(\beta_{lki}\widehat{c}_{lki,t} + \sum_{j=1}^{I}\lambda_{lj}\omega_{lkji}\widehat{x}_{lkji}\right) \tag{F.91}$$

where we have pre-multiplied the first expression by $\frac{P_{ki}}{E_k}$.

The log-linearized version of the labor market clearing condition (19) is given by

$$\widehat{n}_{k,t} = \sum_{i=1}^{I}\delta_{ki}\widehat{n}_{ki,t} \tag{F.92}$$

where $\delta_{ki} = \frac{N_{ki}}{N_k} = \frac{W_kN_{ki}}{P_{ki}Y_{ki}}\frac{P_{ki}Y_{ki}}{P_k^CC_k}\frac{P_k^CC_k}{W_kN_k} = \frac{\alpha_{ki}\lambda_{ki}}{1-\sum_{j=1}^{I}\left(1-\frac{\psi_{kj}}{\mathcal{M}_{kj}}\right)\lambda_{kj}}$ can be derived using (F.97).

The NFA from the "global" country K (20) can be log-linearized to

$$\frac{1}{\beta}\sum_{k=1}^{K-1}\text{nfa}_{k,t-1}^{K} - \sum_{k=1}^{K-1}\text{nfa}_{k,t}^{K} = \Upsilon_K\left(\widehat{\text{exp}}_{K,t} - \widehat{\text{imp}}_{K,t} + \widehat{p}_{K,t}^{\text{EXP}} - \widehat{p}_{K,t}^{\text{IMP}}\right), \tag{F.93}$$

the NFA from country $k \neq K$ $k \notin$ MU, (20) can be log-linearized to

$$\text{nfa}_{k,t}^{K} - \frac{1}{\beta}\text{nfa}_{k,t-1}^{K} = \Upsilon_k\left(\widehat{\text{exp}}_{k,t} - \widehat{\text{imp}}_{k,t} + \widehat{p}_{k,t}^{\text{EXP}} - \widehat{p}_{k,t}^{\text{IMP}}\right) \tag{F.94}$$

where $\widehat{p}_{k,t}^{\text{IMP}} = p_{k,t}^{\text{IMP}} - p_{k,t}^{C}$ and $\widehat{p}_{k,t}^{\text{EXP}} = p_{k,t}^{\text{EXP}} - p_{k,t}^{C}$. The linearized export and import price deflators are given by:

(F.95)

$$\widehat{p}_{k,t}^{\text{IMP}} = \sum_{l\neq k}\sum_{i=1}^{I}\frac{P_{kli}C_{kli} + \sum_{j=1}^{I}P_{klji}X_{klji}}{P_k^{\text{IMP}}\text{IMP}_k}\left(\tau_{kli,t}^{w} + \widehat{p}_{kli,t}^{k}\right) = \sum_{l\neq k}\sum_{i=1}^{I}\Upsilon_k^{-1}\left(\beta_{kli} + \sum_{j=1}^{I}\lambda_{kj}\omega_{klji}\right)\left(\tau_{kli,t}^{w} + \widehat{p}_{kli,t}^{k}\right)$$

(F.96)

$$\widehat{p}^{\text{EXP}}_{k,t} = \sum_{l\neq k}\sum_{i=1}^{I} \frac{P_{ki}C_{l\,ki} + \sum_{j=1}^{I} P_{ki}X_{l\,kji}}{P^{\text{EXP}}_{k}\text{EXP}_{k}} \left(\tau^{w}_{l\,ki,t} + \widehat{p}^{k}_{l\,ki,t}\right) = \sum_{l\neq k}\sum_{i=1}^{I} \frac{\mathcal{Y}_{l\,k}}{\Upsilon_{k}} \left(\beta_{l\,ki} + \sum_{j=1}^{I} \lambda_{l\,j}\omega_{l\,kji}\right) \left(\tau^{w}_{l\,ki,t} + \widehat{p}^{k}_{l\,ki,t}\right)$$

Gross Domestic Product and Net Exports. Let us now move to nominal GDP (23). In steady state, assuming zero net exports, $P^{\text{EXP}}_{k}\text{EXP}_{k} - P^{\text{IMP}}_{k}\text{IMP}_{k} = 0$, we can write $\mathcal{Y}_{k} = P^{C}_{k}C_{k}$. Using the household's budget constraint (F.22) in steady state, we can write

$$\mathcal{Y}_{k} = P^{C}_{k}C_{k} = W_{k}N_{k} + \Pi_{k} = W_{k}N_{k} + \sum_{i=1}^{I}\left(1 - \frac{\psi_{ki}}{\mathcal{M}_{ki}}\right)P_{ki}Y_{ki} \tag{F.97}$$

where the last equality makes use of (F.50).

Log-linearizing the real GDP (24) definition,

$$\widehat{y}_{k,t} = \frac{P^{C}_{k}C_{k}}{\mathcal{Y}_{k}}\widehat{c}_{k,t} + \frac{P^{\text{EXP}}_{k}\text{EXP}_{k}}{\mathcal{Y}_{k}}\widehat{\text{exp}}_{k,t} - \frac{P^{\text{IMP}}_{k}\text{IMP}_{k}}{\mathcal{Y}_{k}}\widehat{\text{imp}}_{k,t} = \widehat{c}_{k,t} + \Upsilon_{k}\left(\widehat{\text{exp}}_{k,t} - \widehat{\text{imp}}_{k,t}\right) \tag{F.98}$$

where second equality uses that nominal consumption expenditures will be equal nominal GDP in steady state, and $\Upsilon_{k} = \frac{P^{\text{EXP}}_{k}\text{EXP}_{k}}{\mathcal{Y}_{k}} = \frac{P^{\text{IMP}}_{k}\text{IMP}_{k}}{\mathcal{Y}_{k}}$ is the export (or import) share of nominal GDP.

The nominal exports expression (21) can be log-linearized to:

$$\begin{aligned}\widehat{\text{exp}}_{k,t} &= \sum_{l\neq k}\sum_{i\in I}\left(\frac{P_{ki}C_{l\,ki}}{P^{\text{EXP}}_{k}\text{EXP}_{k}}\widehat{c}_{l\,ki,t} + \sum_{j=1}^{I}\frac{P_{ki}X_{l\,ki}}{P^{\text{EXP}}_{k}\text{EXP}_{k}}\widehat{x}_{l\,kji,t}\right)\\ &= \sum_{l\neq k}\sum_{i\in I}\frac{\mathcal{Y}_{l\,k}}{\Upsilon_{k}}\left(\beta_{l\,ki}\widehat{c}_{l\,ki,t} + \sum_{j=1}^{I}\lambda_{l\,j}\omega_{l\,kji}\widehat{x}_{l\,kji,t}\right)\end{aligned} \tag{F.99}$$

where the export share of nominal GDP is given by

$$\Upsilon_{k} = \frac{P^{\text{EXP}}_{k}\text{EXP}_{k}}{\mathcal{Y}_{k}} = \sum_{l\neq k}\sum_{i=1}^{I}\mathcal{Y}_{l\,k}\left[\beta_{l\,ki} + \sum_{j=1}^{I}\lambda_{l\,j}\omega_{l\,kji}\right] = \left(\sum_{i=1}^{I}\lambda_{ki}\right) - \sum_{i=1}^{I}\left(\beta_{kki} + \sum_{j=1}^{I}\lambda_{kj}\omega_{kkji}\right) \tag{F.100}$$

$$= \frac{P^{\text{IMP}}_{k}\text{IMP}_{k}}{\mathcal{Y}_{k}} = \sum_{l\neq k}\sum_{i=1}^{I}\left[\beta_{kl\,i} + \sum_{j=1}^{I}\lambda_{kj}\omega_{kl\,ji}\right] \tag{F.101}$$

Similarly, the nominal imports expression (22) can be log-linearized to

$$\begin{aligned}\widehat{\text{imp}}_{k,t} &= \sum_{l\neq k}\sum_{i\in I}\left(\frac{P_{kl\,i}C_{kl\,i}}{P_k^{\text{IMP}}\text{IMP}_k}\widehat{c}_{kl\,i,t} + \sum_{j=1}^{I}\frac{P_{kl\,i}X_{kl\,ji}}{P_k^{\text{IMP}}\text{IMP}_k}\widehat{x}_{kl\,ji,t}\right)\\ &= \sum_{l\neq k}\sum_{i\in I}\Upsilon_k^{-1}\left(\beta_{kl\,i}\widehat{c}_{kl\,i,t} + \sum_{j=1}^{I}\lambda_{kj}\omega_{kl\,ji}\widehat{x}_{kl\,ji,t}\right)\end{aligned} \tag{F.102}$$

Now we can combine the linearized expression for real gdp, and the expressions for real imports and exports:

$$\begin{aligned}\widehat{y}_{k,t} &= \widehat{c}_{k,t} + \sum_{l\neq k}\sum_{i\in I}\left(\frac{P_{ki}C_{l\,ki}}{y_k}\widehat{c}_{l\,ki,t} + \sum_{j=1}^{I}\frac{P_{ki}X_{l\,kji}}{y_k}\widehat{x}_{l\,kji,t} - \frac{P_{kl\,i}C_{kl\,i}}{y_k}\widehat{c}_{kl\,i,t} - \sum_{j=1}^{I}\frac{P_{kl\,i}X_{kl\,ji}}{y_k}\widehat{x}_{kl\,ji,t}\right)\\ &= \widehat{c}_{k,t} + \sum_{l\neq k}\sum_{i\in I}\left(y_{l\,k}\beta_{l\,ki}\widehat{c}_{l\,ki,t} + \sum_{j=1}^{I}y_{l\,k}\lambda_{l\,j}\omega_{l\,kji}\widehat{x}_{l\,kji,t} - \beta_{kl\,i}\widehat{c}_{kl\,i,t} - \sum_{j=1}^{I}\lambda_{kj}\omega_{kl\,ji}\widehat{x}_{kl\,ji,t}\right)\end{aligned}$$

Appendix G. Additional Figures and Tables

G.1. SOE Calibration

This subsection documents the sectoral objects used in the fully heterogeneous SOE calibration of Section 4.4. Table G.2 reports the import-invoicing classification sector by sector. The binary classification records whether import prices are foreign-currency-sensitive, denoted PCP-like, or local-currency-priced in the short run, denoted LCP-like. The classification is based on the evidence on global invoicing patterns and dominant-currency pricing in Boz *et al.* (2020) and Gopinath *et al.* (2020). The PCP-like group includes sectors whose empirical pricing is either producer-currency pricing or dominant/vehicle-currency pricing. This mapping is the relevant one for the binary SOE model because both cases generate domestic-currency import-price movements with external exchange rates. The calibration assigns 19 sectors to the PCP-like group and 25 sectors to the LCP-like group.

Table G.1 summarizes the aggregate parameter values.

Parameter	Value	Target or source
Households		
β	0.99	R = 4.5% p.a.
φ	1	Chetty *et al.* (2011)
$\boldsymbol{\beta}_H, \boldsymbol{\beta}_F$	Sector-specific	ICIO tables (OECD)
Firms		
I	44	ICIO tables (OECD)
$\boldsymbol{\Omega}_H, \boldsymbol{\Omega}_F$	Sector-specific	ICIO tables (OECD)
$\boldsymbol{\alpha}$	Sector-specific	ICIO tables (OECD)
ϵ_i^p	11	Ten percent markup (Galí 2015)
θ_i^p	Sector-specific	Dhyne *et al.* (2005)
Import invoicing	Sector-specific	Boz *et al.* (2020); Gopinath *et al.* (2020)
Monetary Policy		
ρ_i	0.8	Standard Taylor rule
ϕ_π	1.5	Galí (2015)
ϕ_y	0.125	Galí (2015)
$\phi_{\Delta\pi}$	0	Galí (2015)
$\phi_{\Delta y}$	0	Galí (2015)
Foreign Shocks		
ρ_i^M	0.99	Persistent import tariff shocks
ρ_i^W	0.99	Persistent foreign price-wedge shocks
ρ_i^X	0.99	Persistent export tariff shocks
σ_i^M	1	Normalization
σ_i^W	1	Normalization
σ_i^X	1	Normalization

TABLE G.1. Calibration of the SOE numerical exercise

TABLE G.2. Sectoral import-invoicing classification

Sector	EArW code	Import-invoicing classification
1	S01T02	PCP-like / foreign-currency-sensitive
2	S03	PCP-like / foreign-currency-sensitive
3	S05T06	PCP-like / foreign-currency-sensitive
4	S07T08	PCP-like / foreign-currency-sensitive
5	S09	PCP-like / foreign-currency-sensitive

Continued on next page

TABLE G.2. Sectoral import-invoicing classification (continued)

Sector	EArW code	Import-invoicing classification
6	S10T12	LCP-like / local-currency-priced
7	S13T15	PCP-like / foreign-currency-sensitive
8	S16	PCP-like / foreign-currency-sensitive
9	S17T18	LCP-like / local-currency-priced
10	S19	PCP-like / foreign-currency-sensitive
11	S20	PCP-like / foreign-currency-sensitive
12	S21	LCP-like / local-currency-priced
13	S22	PCP-like / foreign-currency-sensitive
14	S23	PCP-like / foreign-currency-sensitive
15	S24	PCP-like / foreign-currency-sensitive
16	S25	PCP-like / foreign-currency-sensitive
17	S26	PCP-like / foreign-currency-sensitive
18	S27	PCP-like / foreign-currency-sensitive
19	S28	PCP-like / foreign-currency-sensitive
20	S29	LCP-like / local-currency-priced
21	S30	LCP-like / local-currency-priced
22	S31T33	LCP-like / local-currency-priced
23	S35	LCP-like / local-currency-priced
24	S36T39	LCP-like / local-currency-priced
25	S41T43	LCP-like / local-currency-priced
26	S45T47	LCP-like / local-currency-priced
27	S49	LCP-like / local-currency-priced
28	S50	PCP-like / foreign-currency-sensitive
29	S51	PCP-like / foreign-currency-sensitive
30	S52	PCP-like / foreign-currency-sensitive
31	S53	LCP-like / local-currency-priced
32	S55T56	LCP-like / local-currency-priced
33	S58T60	LCP-like / local-currency-priced
34	S61	LCP-like / local-currency-priced

Continued on next page

TABLE G.2. Sectoral import-invoicing classification (continued)

Sector	EArW code	Import-invoicing classification
35	S62T63	LCP-like / local-currency-priced
36	S64T66	LCP-like / local-currency-priced
37	S68	LCP-like / local-currency-priced
38	S69T75	LCP-like / local-currency-priced
39	S77T82	LCP-like / local-currency-priced
40	S84	LCP-like / local-currency-priced
41	S85	LCP-like / local-currency-priced
42	S86T88	LCP-like / local-currency-priced
43	S90T93	LCP-like / local-currency-priced
44	S94T96	LCP-like / local-currency-priced

Figures G.1 and G.2 summarize the remaining heterogeneous calibration objects. The first heatmap reports sectoral final-demand weights, labor shares, Calvo parameters, the invoicing indicator, domestic-input exposure, imported-input exposure, and imported final-consumption exposure. The domestic-input and imported-input columns are row sums of the domestic and foreign input-output matrices, respectively; the imported-final-consumption column is the sectoral imported share in household final demand. The second figure reports the full domestic and imported input-output matrices. These objects are the sources of cross-sector variation in the numerical exercises.

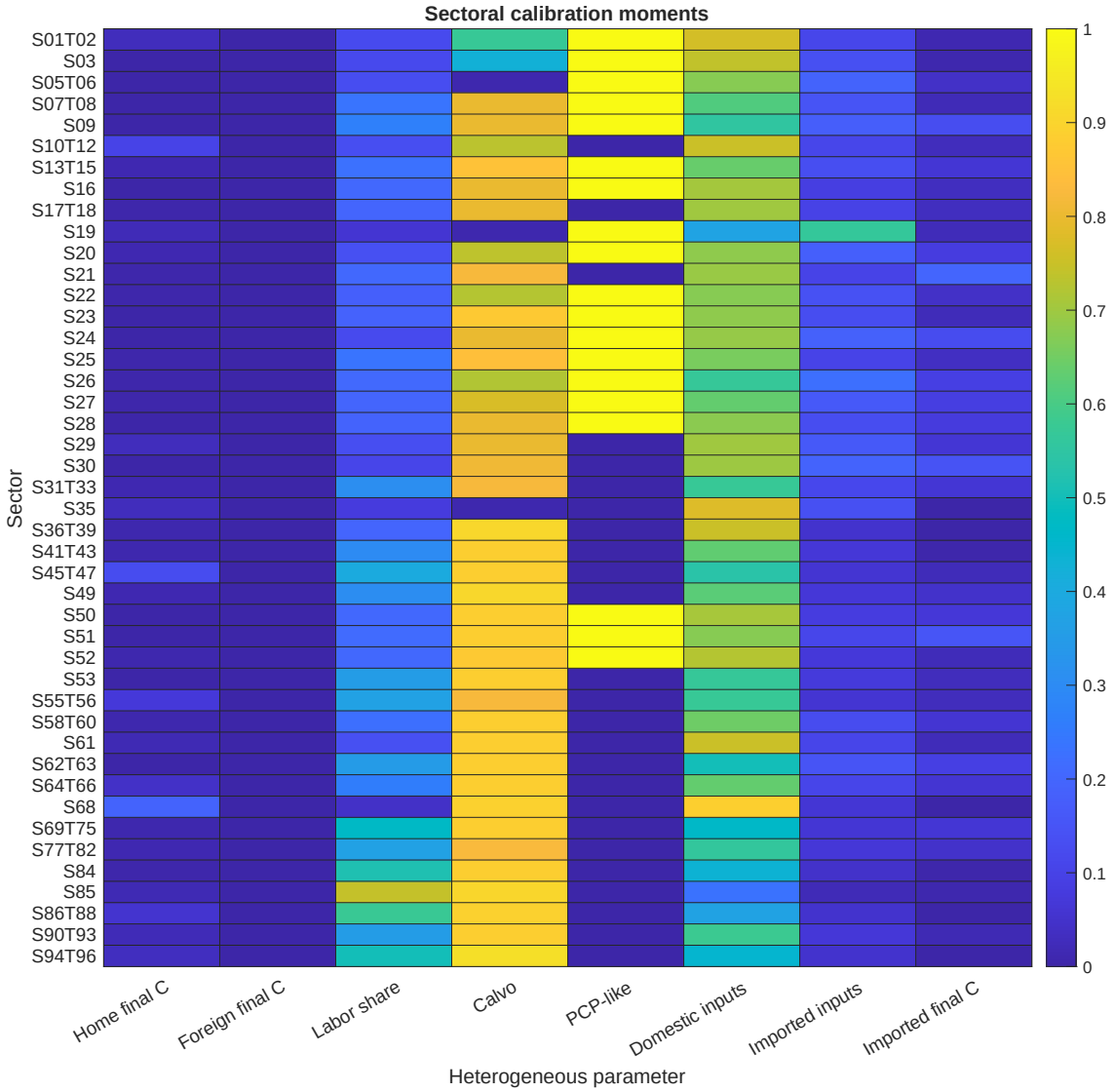


FIGURE G.1. Sectoral heterogeneity in the SOE calibration. The entries are shares or indicators; the PCP-like column equals one for foreign-currency-sensitive import pricing. Domestic inputs and imported inputs are row sums of the domestic and foreign input-output matrices. Imported final C is the imported share of sectoral final consumption.

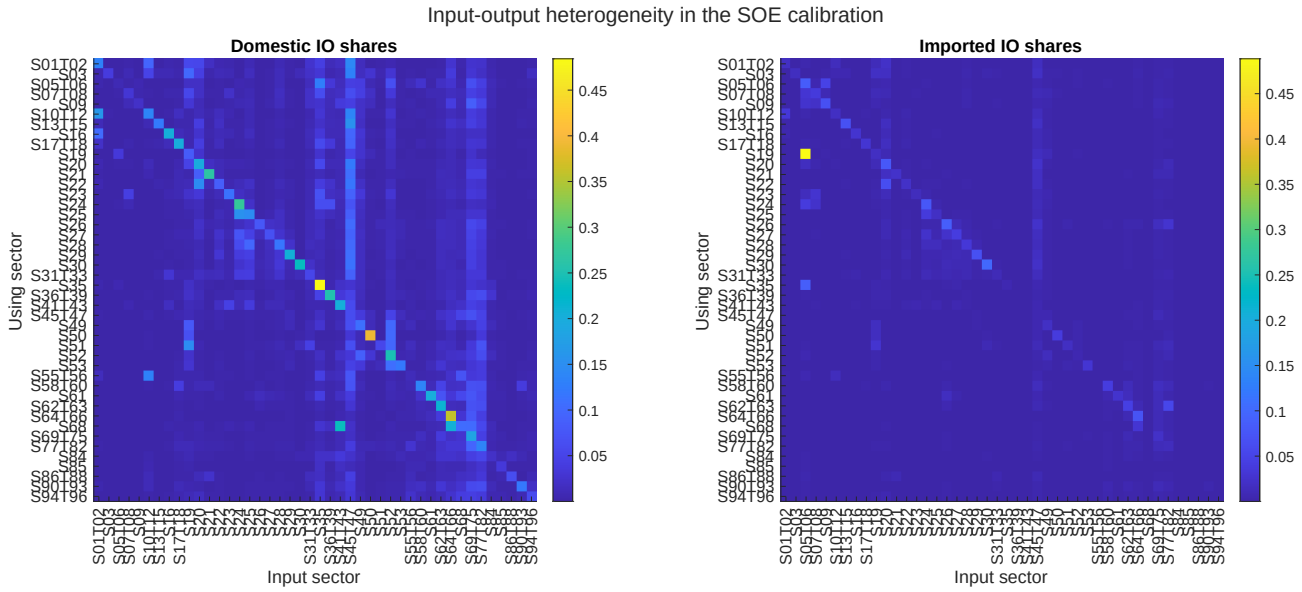
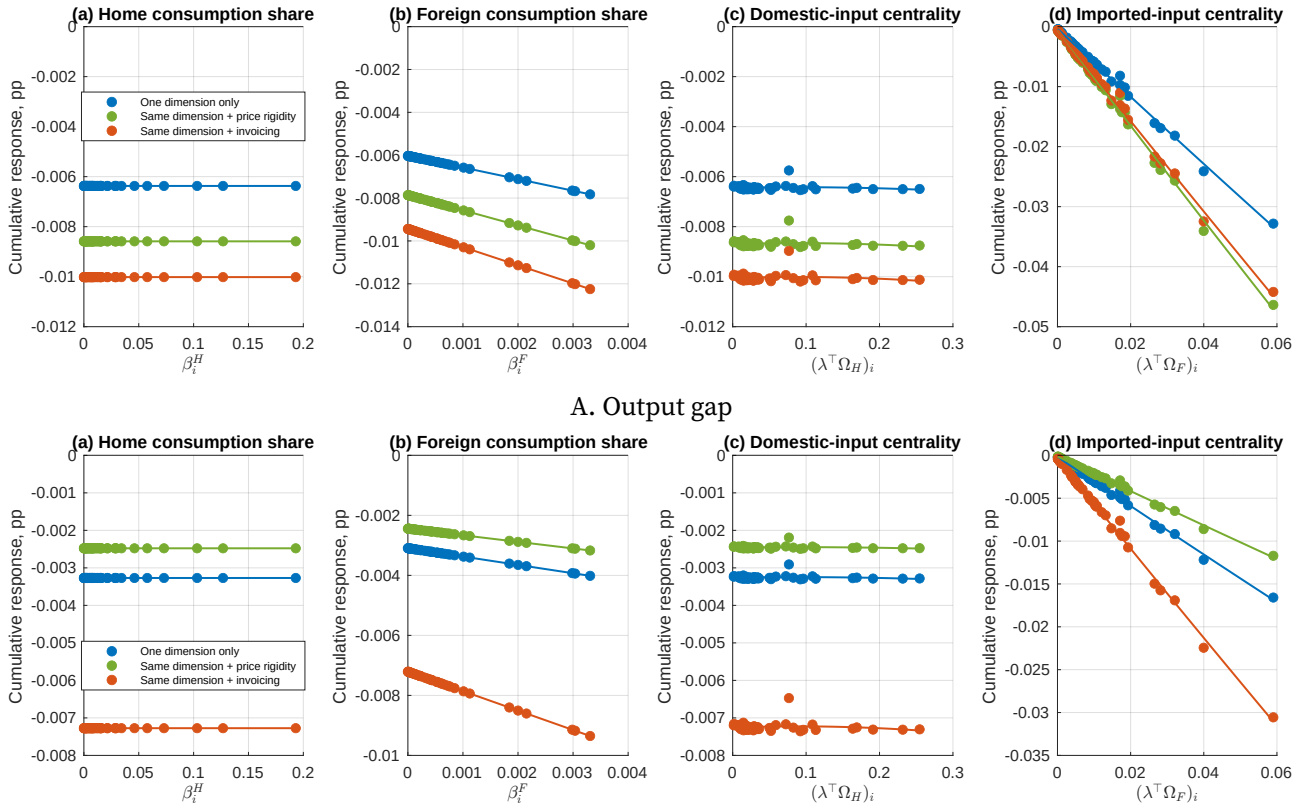


FIGURE G.2. Domestic and imported input-output shares in the SOE calibration. Rows index using sectors and columns index input sectors.



A. Output gap

B. CPI inflation

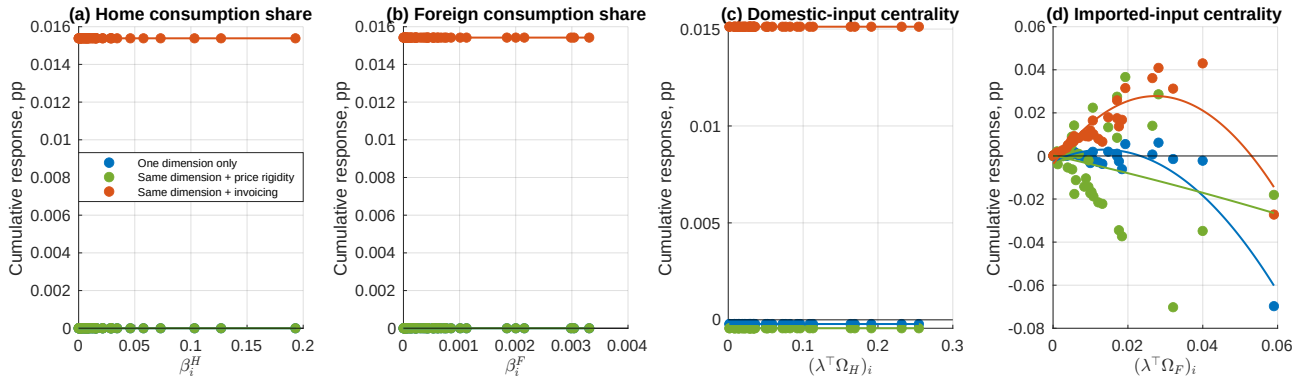
FIGURE G.3. Taylor-rule cumulative responses after sectoral export-tariff shocks.

G.2. Sectoral Transmission under Export Tariffs and Foreign Wedges

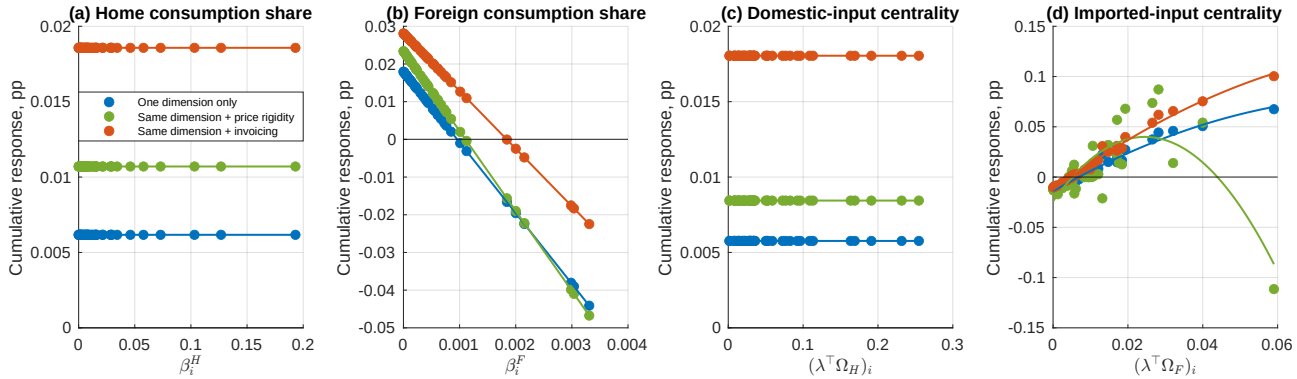
Figure G.3 extends the Taylor-rule transmission exercise of Figure 1 to sectoral export tariffs. These plots concern the positive equilibrium only. Figure G.4 reports the corresponding foreign price-wedge responses under commitment and the Taylor rule. The wedge has the same first-order buyer-price loading as an import tariff, while the welfare comparison uses the tariff-free benchmark of Proposition 2.

G.3. Currency Invoicing in the Quantitative Model

The quantitative model distinguishes three invoicing regimes: producer-currency pricing (PCP), local-currency pricing (LCP), and dominant-currency pricing (DCP). Table G.3 reports the sector-by-sector assignment used in the baseline mixed-invoicing calibration, the quantitative counterpart of the binary SOE classification in Table G.2. Sector indices refer to the model ordering, in which the three energy sectors—mining and extraction of energy products, coke and refined petroleum, and electricity and gas—are placed first; the NACE Rev. 2 code identifies each sector unambiguously. The classification assigns 7 sectors to PCP, 20 sectors to LCP, and 17 sectors to DCP. Traded manufacturing inputs and commodities are predominantly



A. Output gap under commitment



B. Output gap under the Taylor rule

FIGURE G.4. Cumulative responses after sectoral foreign price-wedge shocks.

dollar-invoiced, consistent with the evidence in Boz *et al.* (2020) and Gopinath *et al.* (2020), while consumer-facing services are local-currency-priced; the PCP group consists of sectors with little cross-border trade, for which the invoicing convention is largely immaterial.

TABLE G.3. Sectoral invoicing classification in the quantitative model

Sector	NACE Rev. 2	Description	Invoicing
1	B05–06	Mining and extraction of energy products	DCP
2	C19	Coke and refined petroleum products	DCP
3	D35	Electricity, gas, steam and air conditioning	PCP
4	A01–02	Agriculture and forestry	DCP
5	A03	Fishing and aquaculture	DCP
6	B07–08	Mining of metal ores and other quarrying	LCP
7	B09	Mining support services	DCP
8	C10–12	Food, beverages and tobacco	DCP

Continued on next page

TABLE G.3. Sectoral invoicing classification in the quantitative model (continued)

Sector	NACE Rev. 2	Description	Invoicing
9	C13–15	Textiles, apparel and leather	LCP
10	C16	Wood and cork products	DCP
11	C17–18	Paper and printing	DCP
12	C20	Chemicals and chemical products	LCP
13	C21	Pharmaceuticals	DCP
14	C22	Rubber and plastic products	DCP
15	C23	Other non-metallic mineral products	DCP
16	C24	Basic metals	DCP
17	C25	Fabricated metal products	DCP
18	C26	Computer, electronic and optical products	DCP
19	C27	Electrical equipment	DCP
20	C28	Machinery and equipment	LCP
21	C29	Motor vehicles	LCP
22	C30	Other transport equipment	LCP
23	C31–33	Furniture and other manufacturing	LCP
24	E36–39	Water supply and waste management	PCP
25	F41–43	Construction	PCP
26	G45–47	Wholesale and retail trade	LCP
27	H49	Land transport	LCP
28	H50	Water transport	LCP
29	H51	Air transport	DCP
30	H52	Warehousing and transport support	DCP
31	H53	Postal and courier activities	LCP
32	I55–56	Accommodation and food services	LCP
33	J58–60	Publishing and media	LCP
34	J61	Telecommunications	LCP
35	J62–63	IT and information services	LCP
36	K64–66	Financial and insurance activities	LCP
37	L68	Real estate activities	PCP

Continued on next page

TABLE G.3. Sectoral invoicing classification in the quantitative model (continued)

Sector	NACE Rev. 2	Description	Invoicing
38	M69–75	Professional, scientific and technical activities	LCP
39	N77–82	Administrative and support services	LCP
40	O84	Public administration and defence	PCP
41	P85	Education	PCP
42	Q86–88	Human health and social work	PCP
43	R90–93	Arts, entertainment and recreation	LCP
44	S94–96	Other service activities	LCP

Notes: Sector indices follow the model ordering (energy sectors first). Source: baseline mixed-invoicing calibration of the quantitative model.

G.4. Construction of the Observed 2025 Tariff Series

The observed-tariff exercise of Section E.4 uses a quarterly, sector-level measure of effective U.S.–China tariffs that we build from product-level data in the IMF Tariff Tracker, our primary source for the timing and level of the bilateral measures. The raw data record, for each product at the HS2022 level, the statutory tariff rate and the date on which it becomes applicable.

We first convert the sequence of recorded changes into continuous validity spells. Each tariff is held in force from its application date until the day before the next recorded change for the same product, and the final spell is extended to the end of the sample. Intersecting these spells with the calendar yields, for every product h, in-force rate s, and quarter q, the number of days $d_{h,s,q}$ for which that rate is active within the quarter. The quarterly product-level tariff is the active-day–weighted average of the rates in force,

$$\tau_{h,q} = \frac{\sum_s \tau_{h,s}\, d_{h,s,q}}{\sum_s d_{h,s,q}},$$

so that a change occurring within a quarter contributes in proportion to the number of days it is effective.

We then map the quarterly HS2022 series into the NACE Rev. 2 sectors of the calibration in three steps. HS2022 products are first concorded to HS2017 using the UNSD HS2022–HS2017 correspondence tables; HS2017 products are then assigned to OECD TiVA/NACE sectors using the OECD HS–ISIC/NACE key; and tariffs are finally aggregated to NACE Rev. 2 sectors as import-weighted averages of the underlying product-level rates, so that products carrying

larger trade flows receive proportionally greater weight. Letting m_h denote the import value of product h and $\mathcal{H}(k)$ the set of products assigned to sector k, the sectoral tariff is

$$\tau_{k,q} = \frac{\sum_{h \in \mathcal{H}(k)} m_h \tau_{h,q}}{\sum_{h \in \mathcal{H}(k)} m_h}.$$

Service and other non-goods sectors that carry no direct tariff exposure are assigned a zero rate, yielding a balanced quarterly tariff dataset across all 44 sectors for 2025Q1–2025Q4. The model innovations are the first differences of the log gross tariff wedge, $\Delta \log(1 + \tau_{k,q})$, so zero-rate sectors are well defined and receive zero innovations unless their assigned tariff changes.

G.5. Non-Euro-Area Responses in the Quantitative Model

This subsection reports the United States and China responses that accompany the Euro Area results in Section E.4. Figure G.5 plots real GDP growth, CPI inflation, and the policy rate of the two large economies to the same three shocks—the U.S. tariff on China, the Chinese tariff on the U.S., and the rest-of-world energy-price shock—under the Euro Area baseline Taylor rule and under the Euro Area optimal simple rule. For each economy the two lines almost coincide: because the Euro Area is small relative to the United States and China, its choice of monetary rule has a negligible effect on their dynamics. The Euro Area optimal-simple-rule problem of Section E.4 is therefore, from the perspective of the large economies, close to a partial-equilibrium stabilization problem.

Figure G.6 reports the analogous United States and China responses to the observed 2025 U.S.–China tariff schedule simulated in Section E.4. Consistent with the homogeneous experiments, the two large economies absorb the bulk of the disturbance—U.S. CPI inflation peaks near 0.44 and Chinese CPI inflation near 0.37 percentage points—while the Euro Area lines (Figure E.3) remain an order of magnitude smaller, and the Euro Area rule again leaves the large-economy dynamics essentially unchanged.

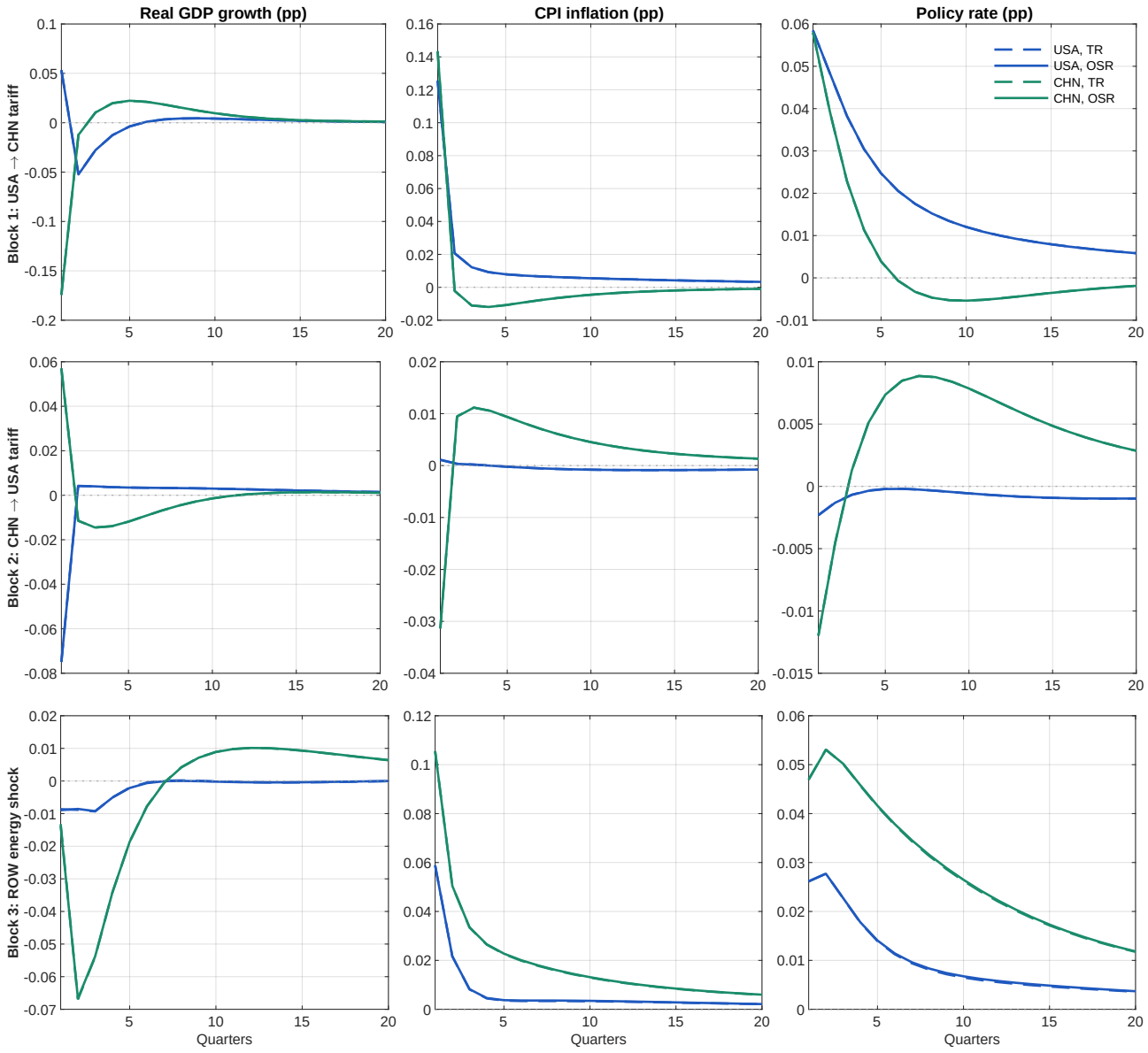


FIGURE G.5. United States and China responses to tariff and energy shocks under the Euro Area baseline Taylor rule and optimal simple rule. Top row: U.S. tariff on China; middle row: Chinese tariff on the United States; bottom row: rest-of-world energy-price shock. Columns: real GDP growth, CPI inflation, and the policy rate, in percentage points. Colour denotes the country (USA, China) and line style denotes the Euro Area rule (Taylor rule, OSR). Each shock is scaled to a 10 log-point increase.

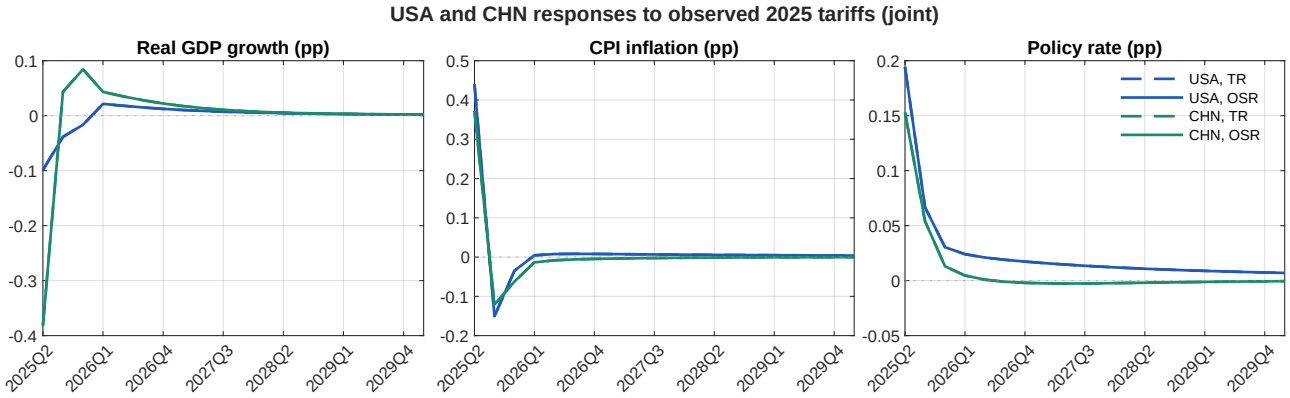


FIGURE G.6. United States and China responses to the observed 2025 U.S.–China tariff schedule under the Euro Area baseline Taylor rule and optimal simple rule. Columns: real GDP growth, CPI inflation, and the policy rate, in percentage points. Colour denotes the country (USA, China) and line style denotes the Euro Area rule. The observed sectoral tariff levels are converted into wedge innovations by first-differencing the log gross tariff; the simulation runs from 2025Q2.