On the Optimal Design of GDP-Linked Bonds with Trend Growth*

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Abstract

We study the pricing and welfare consequences of GDP-indexed bonds in an environment featuring permanent and transitory income shocks. Using a sovereign default model with lenders who fear model misspecification, we find that commonly-used structures, including the value-recovery instruments often proposed in restructurings, generate welfare losses for the government relative to delivering the restructured value through plain bonds. Indexed bonds can still be welfare improving when issued in normal times and when designed optimally. The optimal design of indexed bonds balances matching liabilities to debt-payment capacity, which incentivizes capturing the trend, with avoiding lock-in effects. Our findings suggest potential for untapped gains from indexation when the instruments are optimally designed.

JEL Classification E43, E44, F34, G12, H63, O16

Keywords Sovereign debt, default risk, state-contingent debt instruments, robust control, risk premia

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Introduction

Governments around the world borrow by mostly issuing bonds stipulating fixed payments. In constrast, state-contingent debt instruments (SCDIs), which make payments dependent on the state of the economy, would allow to modulate debt-servicing requirements to payment capacity. As a consequence, their use could improve international risk sharing, reduce the procyclicality of fiscal policy, and avoid costly sovereign debt crises. In practice, sovereign GDP-linked bond usage remains scarce, and such instruments have been issued almost exclusively during debt restructurings. In such contexts, they have traded at deep discounts (Costa et al., 2008; Igan et al., 2022) and have hence found limited success. Moreover, SCDIs issued in restructurings are typically viewed as a way to reconcile divergent projections for growth and debt-carrying capacity. This suggests that uncertainty about future growth could be a key factor in the design of SCDIs. Uncertainty about future growth is also a salient feature of emerging-markets economies, but one that most sovereign-debt models, based on stationary environments, are ill-equipped to capture.

In this paper, we study the pricing, welfare implications, and design of state-contingent sovereign debt in an economy subject to both transitory (cyclical) and permanent (trend) shocks to output. We build on the quantitative sovereign default framework with long-term debt and international lenders who fear model misspecification (Hansen and Sargent, 2001; Pouzo and Presno, 2016; Roch and Roldán, 2023). We show how to represent the types of SCDIs observed in practice in restructurings, how the model rationalizes the discounts at which these bonds have traded, and then move on to the optimal design of SCDIs in our environment with trend and cycle shocks.

A central question when designing GDP-linked bonds in our non-stationary environment is whether to index to transitory or permanent components. On the one hand, transitory shocks are ones for which the value of smoothing is highest, as positive trend growth shocks also predict future income, creating incentives to borrow rather than save. On the other hand, permanent shocks are more powerful in predicting future debt-carrying capacity. In other words, both types of shocks create demand for insurance at different horizons.

A related design question concerns how to design GDP-linked securities when it is not possible to index on the trend and cycle components separately. This includes for example a commitment to a particular statistical filter to extract the two components. To make progress on this question, we consider bonds indexed to cumulative growth rates of GDP and ask what the optimal cumulation horizon is. In this case, a similar tradeoff arises: when the cumulation horizon is too long, the bond becomes not indexed as shocks average out, when it is too short, the bond payment could become too volatile, reflecting unimportant short-term fluctuations; finally, inter-

mediate cumulation horizons might create lock-in effects by which good growth shocks increase the debt service but, upon reversion, it takes too long for the increased debt service to come back down.

Our preliminary results indicate that (*i*) the model is able to generate realistic pricing of SCDIs in restructurings, (*ii*) through the lens of the model, welfare for the government is higher when it does not use SCDIs in restructurings and rather delivers the entire face value of restructured claims through plain bonds; (*iii*) it is preferrable to index to the cyclical component rather than the trend growth, although optimal designs for each yield positive welfare gains (in each case following a linear index rather than thresholds, as Roch and Roldán, 2023, find in the context of a stationary environment); and finally (*iv*) optimal cumulative growth bonds use a very short cumulation horizon. The short cumulation horizon corresponds to hardwiring an estimate of the cyclical component in the bond payment structure.

Discussion of the Literature TBW

Layout The remainder of the paper is organized as follows. Section 2 describes the model, discusses its calibration, and characterizes some key pricing equations. Section 3 describes the validation of the quantitative model in an application to value-recovery instruments. Section 4 focuses on the optimal design of state-contingent debt instruments with different designs or constraints. Finally, Section 5 concludes. Several Appendices provide additional details about normalization, robustness, decompositions of spreads, model extensions, and an improvement upon standard methods for smoothing the debt-issuance choice during the numerical implementation.

2. Model

Endowment There is a single tradable good. The economy receives a stochastic endowment stream of this good Y_t in period t. While the true stochastic process is unknown, all agents in the economy share a *baseline* or approximating model according to which

$$\begin{aligned} Y_t &= \exp(z_t)\Gamma_t \\ \log \Gamma_t &= \log \Gamma_{t-1} + g_t \\ g_t &= \rho_g g_{t-1} + (1-\rho_g)\mu_g + \sigma_g \epsilon_t^g \\ z_t &= \rho_z z_{t-1} + \sigma_z \epsilon_t^z \end{aligned} \tag{1}$$

where the ϵ's are *iid* standard Normal random variables.

Estimation Equation (1) can be estimated by maximum likelihood using state-space methods given data on real output Y_t. By taking first differences and calling $\tilde{g}_t = g_t - \mu_g$, we obtain

$$\log Y_t - \log Y_{t-1} - \mu_g = z_t - z_{t-1} + \tilde{g}_t = F\, S_t \tag{Obs}$$

where $S_t = [z_t, z_{t-1}, \tilde{g}_t]$, $F = [1, -1, 1]$, and

$$\underbrace{\begin{pmatrix} z_t \\ z_{t-1} \\ \tilde{g}_t \end{pmatrix}}_{S_t} = \underbrace{\begin{bmatrix} \rho_z & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \rho_g \end{bmatrix}}_{A} \underbrace{\begin{pmatrix} z_{t-1} \\ z_{t-2} \\ \tilde{g}_{t-1} \end{pmatrix}}_{S_{t-1}} + \underbrace{\begin{bmatrix} \sigma_z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_g \end{bmatrix}}_{B} \epsilon_t \tag{Evol}$$

In this form, given parameters $(\rho_z, \rho_g, \mu_g, \sigma_z, \sigma_g)$, applying the Kalman filter to the system (Obs)-(Evol) yields the likelihood function. We estimate this state-space model on annual data for Argentina's real GDP between 1900 and 2001. Table 1 summarizes the estimation.

Table 1: Estimated income process

	Parameter	Value
Autocorr. cyclical	ρ_z	0.9369
Std. dev. cyclical	σ_z	0.0222
Autocorr. trend	ρ_g	0.7418
Std. dev. trend	σ_g	0.01807
Mean trend growth	μ_g	0.0067

Note: Parameters reported for a quarterly frequency

Normalization Let lowercase values denote detrended variables, normalized at time t by the lagged trend Γ_{t-1}. For example, $y_t = Y_t/\Gamma_{t-1} = \exp(z_t + g_t)$. Homotheticity of CRRA preferences implies that $u(C_t) = \Gamma_{t-1}^{1-\gamma} u(c_t)$, implying that a $\Gamma_{t-1}^{1-\gamma}$ term factors out of the government's value functions and making its effective discount factor $\tilde{\beta}(g) = \beta e^{(1-\gamma)g}$ state-dependent. The detrended problem is stationary in the state variables (b, z, g), where $b_t = B_t/\Gamma_{t-1}$ is debt normalized by lagged trend. In later sections, we consider debt-indexation schemes which may rely on an extra state variable, for which we reserve a placeholder symbol ξ.

Let $s = (z, g, \xi)$ denote the exogenous part of the state vector, including ξ in its relevant form (or not at all) depending on what is needed to describe the indexation of debt.

Preferences The government makes choices sequentially to maximize the representative household's welfare. At the beginning of a period, assuming it is not already excluded from international capital markets, the government decides whether to repay the debt or default after receiving

preference shocks (ϵ_R, ϵ_D)[1] and obtains the value

$$\mathscr{V}(b,s) = \mathbb{E}\left[\max\left\{v_R(b,s) + \epsilon_R, v_D(b,s) + \epsilon_D\right\}\right] \tag{2}$$

where v_R and v_D respectively denote the value attained after repayment and after default, described below. As is well-known (Chatterjee et al., 2018; Dvorkin et al., 2021), the following closed-forms for the value function and the ex-post probability of default obtain if the ϵ's follow a Type 1 Extreme Value distribution.

$$\begin{aligned}\mathscr{V}(b,s) &= \chi \log\left(\exp\left(\frac{1}{\chi}v_R(b,s)\right) + \exp\left(\frac{1}{\chi}v_D(b,s)\right)\right)\\ \mathscr{P}(b,s) &= \frac{\exp\left(\frac{1}{\chi}v_D(b,s)\right)}{\exp\left(\frac{1}{\chi}v_R(b,s)\right) + \exp\left(\frac{1}{\chi}v_D(b,s)\right)}\end{aligned} \tag{3}$$

In repayment, the government issues new debt at a price q. Debt is long-term (Leland, 1998; Hatchondo and Martinez, 2009; Arellano and Ramanarayanan, 2012) and stipulates a sequence of (possibly state-contingent) coupon payments: a bond pays $\kappa(s)$ in the following period and then becomes a perfect substitute for $(1-\delta)$ units of freshly-issued bonds. When debt is non-contingent, we set $\kappa(s) = \bar{\kappa} = r^\star + \delta$, where $r^\star$ is the risk-free rate.[2] We use the convention that the government chooses its debt level normalized to units of the following period, b', which results in a current value of $e^g b'$ (of which $(1-\delta)b$ corresponds to rollover of existing debt). The value of repayment summarizes this discussion:

$$\begin{aligned} v_R(b,s) &= \max_{b'} u(c) + \tilde{\beta}(g)\mathbb{E}\left[v(b',s') \mid s\right]\\ \text{subject to} \quad & c + \kappa(s)b = y(s) + q_R(b',s)(e^g b' - (1-\delta)b)\end{aligned} \tag{4}$$

For numerical stability, we consider (slight) randomness in the choice of b' similar to the preference shocks described above, with a scale parameter χ_b.[3] Our formulation makes the density of issuing an amount b' proportional to the exponential of the (scaled) objective function, or

$$\phi_b(b' \mid b,s) \propto \exp\left(\frac{1}{\chi_b}\left(u(c(b',b,s)) + \tilde{\beta}(g)\mathbb{E}[v(b',s') \mid s]\right)\right).$$

from which it is clear that ϕ_b converges to issuing the maximizer in (4) with probability one as $\chi_b \longrightarrow 0$.

[1] In non-stationary form, one would need to write $V = \mathbb{E}\left[\max\{V_R + \Gamma^{1-\gamma}\epsilon_R, V_D + \Gamma^{1-\gamma}\epsilon_D\}\right]$ to keep the variance/scale parameter of the ϵ's constant in the stationary version.

[2] This setting of the coupon rate makes the price of a bond expected to repay with probability 1 equal to 1. In other words, it makes the face value of b bonds equal b. When we introduce indexed bonds later, we will keep versions of this anchor $\bar{\kappa}$.

[3] Appendix A.4 describes the mathematical formulation in detail.

In default, the government is excluded from financial markets and suffers a direct output cost $h(s)$. The cost is proportional to output but is assumed to depend on the cyclical component only.[4] The government then reenters markets with constant hazard ψ and receives a haircut $\hbar$ on the face value of existing defaulted claims. The value of default is

$$v_D(b,s) = u(y(s)(1-h(s))) + \tilde{\beta}(g)\mathbb{E}\left[\psi\mathscr{V}((1-\hbar)be^{-g},s') + (1-\psi)v_D(be^{-g},s')\,\middle|\,s\right] \tag{5}$$

Following Aguiar and Amador (2014), we specify an increasing and convex function for the default costs by setting $h(s) = \max\left\{0, -d_1 + \frac{d_1}{d_0}y(s)\right\}$, written so that defaults do not entail costs when $y < d_0$ and the slope of the cost function at d_0 is d_1.

Lenders The small open economy borrows from risk-neutral lenders who fear model misspecification. Lenders are competitive and discount the future (or access funds) at a discount rate $\beta_L = \frac{1}{1+r^\star}$. Given a possibly random sequence of consumption $c_L(b,s)$ in state (b,s), they attain a value

$$v_L(b,s) = \mathbb{E}\left[c_L(b,s) + \frac{\beta_L}{-\theta}\log\left(\mathbb{E}\left[\exp\left(-\theta e^g v_L(b',s')\right) \mid s\right]\right)\right] \tag{6}$$

The outer expectation refers to the government's choice of default and issuance in the current period. Consumption of lenders is the sum of their endowment (normalized to zero) plus any payments from or to the government[5]

$$c_L(b',b,s) = \begin{cases} \kappa(s)b - q_R(b',s)(e^g b' - (1-\delta)b) & \text{in repayment} \\ 0 & \text{in default} \end{cases}$$

The following Euler equations ensure that lenders hold the bonds in repayment and default, respectively

$$\begin{aligned} q_R(x,s) &= \beta_L\mathbb{E}\left[\begin{array}{c}(1-\mathbb{1}_{\mathscr{D}'})M_R(x',s')\big(\kappa(s') + (1-\delta)q_R(x',s')\big) \\ +\mathbb{1}_{\mathscr{D}'}M_D(xe^{-g},s')q_D(xe^{-g},s')\end{array}\,\middle|\,s\right] \\ q_D(x,s) &= \beta_L\mathbb{E}\left[\begin{array}{c}\psi(1-\mathbb{1}_{\mathscr{D}'})M_R(x',s')(1-\hbar)\left(\kappa(s') + (1-\delta)(q_R(x',s'))\right) \\ +\psi\mathbb{1}_{\mathscr{D}'}M_D((1-\hbar)xe^{-g},s')(1-\hbar)q_D((1-\hbar)xe^{-g},s') \\ +(1-\psi)M_D(xe^{-g},s')q_D(xe^{-g},s')\end{array}\,\middle|\,s\right] \end{aligned} \tag{7}$$

where x refers to the debt level observed at the time of pricing the bonds (b' in repayment but b in default), x' refers to the following-period expected issuance (as perceived by creditors), and finally the stochastic discount factor M is given by

$$M_i(x,s') = \frac{\exp(-\theta e^g v_L(x,s',D=i))}{\mathbb{E}\left[\exp(-\theta e^g v_L(x,s',D')) \mid s\right]} \tag{8}$$

[4] This assumption is implicitly made by all studies which filter the data and consider stationary models.

[5] In default, even though the debt stock has a positive price, no payments are taking place

When debt prices satisfy (7), bondholders break even in expectations (after applying the risk adjustment required by their desire for robustness): the price of a performing bond reflects coupon payments $\kappa(s')$ and the resale value $q_R(x', s')$ of the remaining $(1 - \delta)$ units, in case of repayment, and the resale value $q_D(xe^{-g}, s')$ in case of default. In repayment, the government issues new debt x', which is (distributed with) $\phi_b(xe^{-g}, s')$ in the RHS of the equation for q_R, or $\phi_b((1 - \hbar)xe^{-g}, s')$ when returning from default in the RHS of the equation for q_D. When the government remains in default, we write explicitly the (mechanical) evolution of amount of debt x, which reflects the drift e^{-g} from the normalization and the haircut $(1 - \hbar)$ when an opportunity to restructure appears and the government re-defaults.

2.1 Debt prices and spreads

Following Roch and Roldán (2023), we decompose the price of debt into components related the creditors' preference for robustness at the core of our model. We write the state-contingent payouts $R_R(b', s') = \kappa(s') + (1 - \delta)q_R(b'', s')$ and $R_D(b', s') = q_D(b'e^{-g}, s')$, as well as the mixtures of repayments $R = \mathbb{1}_{\mathscr{D}} R_D + (1 - \mathbb{1}_{\mathscr{D}})R_R$ and discount factors $M = \mathbb{1}_{\mathscr{D}} M_D + (1 - \mathbb{1}_{\mathscr{D}})M_R$, to obtain

$$\begin{aligned} q_R(b', s) &= \beta_L \mathbb{E}\left[\left((1 - \mathbb{1}_{\mathscr{D}'})M_R R_R + \mathbb{1}_{\mathscr{D}'} M_D R_D\right) \mid s\right] \\ &= \beta_L \underbrace{\mathbb{E}[M]}_{=1} \mathbb{E}[R] + \beta_L \operatorname{cov}(M, R) \end{aligned} \tag{9}$$

where the first term corresponds to how a lender unaffected by ambiguity would price the bond understanding that it can be resold at the equilibrium price $q_R(b'', s')$. The second term summarizes the distortions introduced in the current period by robustness.[6] We propagate this decomposition over time in this dynamic case with long-term debt and recovery, and write the equilibrium bond price as

$$q_R(b', s) = q_{RE}(b', s) - \left(\Delta^{cont} + \Delta^{def} + \Delta^{\times}\right), \tag{10}$$

where q_{RE} is the price that an individual investor, unaffected by ambiguity, would bid when intending to hold the bond to maturity. The rational-expectations price q_{RE} satisfies a recursion analog to the debt-pricing equation (7) but without any adjustment for robustness at any point in time. It correctly incorporates the government's equilibrium strategy for default and repayment, which does depend on the prices the government faces (which are affected by the robustness of the investor base). The three Δ terms are discounts for the ambiguity in stipulated coupon payments $\kappa(s)$, in the default decision, and a cross-product term.

[6] Such distortions relate to the ambiguity in the evolution of the state s and, consequently, in the future decision of repayment and the future stipulated payments $\kappa(s')$. Roch and Roldán (2023) shows in a two-period example how to split that covariance term into risk premia for the ambiguity of default and that of stipulated payments.

We show in Appendix A.3 how to compute the three Δ terms in detail. First, we define the one-period covariance terms π^{cont}, π^{def}, and $\pi^{\times}$

$$\begin{aligned}
\pi^{cont} &= -\beta_L \mathbb{E}\left[1 - \mathbb{1}_{\mathscr{D}'} \mid s\right] \cdot \operatorname{cov}(M, R_R \mid s) \\
\pi^{def} &= -\beta_L \mathbb{E}\left[R_R - R_D \mid s\right] \cdot \operatorname{cov}(M, (1 - \mathbb{1}_{\mathscr{D}'}) \mid s) \\
\pi^{\times} &= -\beta_L \operatorname{cov}\big(M, (1-\mu)R_D + ((1 - \mathbb{1}_{\mathscr{D}'}) - \mu)(L - \mathbb{E}\left[L \mid s\right]) \mid s\big)
\end{aligned} \tag{11}$$

where $\mu = \mathbb{E}\left[1 - \mathbb{1}_{\mathscr{D}'} \mid s\right]$ denotes the perceived probability of repayment and $L = R_R - R_D$ the payout of a performing bond relative to default. The term π^{cont} measures the covariance between the discount factor and the value of the bond conditional on repayment.[7] The term π^{def} depends on the covariance between the stochastic discount factor and the government's repayment strategy, weighed by the difference in returns between default and repayment. Finally, the residual term $\pi^{\times}$ depends on the covariance between the discount factor and products of conditional payouts and the default probability.

Finally, accumulating these one-period covariance terms into an appropriate present-value notion yields, starting from repayment (a similar formula applies starting from default)

$$\begin{aligned}
\Delta^i(b', s) &= \pi^i(b', s) + \underbrace{\beta_L \mathbb{E}\left[(1 - \mathbb{1}_{\mathscr{D}'})(1-\delta)\Delta^i_R(b'', s') + \mathbb{1}_{\mathscr{D}'}\Delta^i_D(b'e^{-g}, s')\right]}_{= \mathscr{T}(\Delta^i)_R} \\
&= \sum_{j=0}^{\infty} \mathscr{T}^j \left(\pi^i\right)_R \qquad \text{for } i \in \{cont, def, \times\}
\end{aligned} \tag{12}$$

A decomposition of the spreads can likewise be obtained from the decomposition of the price q_R. For any state-contingent bond traded at price q_t at time t, we compute its yield as the discount rate that rationalizes the observed price assuming no default, by solving

$$q_t = \mathbb{E}\left[\sum_{j=1}^{\infty} \frac{(1-\delta)^{j-1}\kappa(s_{t+j})}{(1 + r_y)^j} \,\middle|\, \mathscr{F}_t\right] \tag{13}$$

for r_y.[8] We then obtain the spread by subtracting the risk-free rate $r^\star$ from r_y. We consequently define the rational-expectations spread as $spr_{RE} = r_y(q_{RE}) - r^\star$, the premium from the ambiguity of contingency as $spr^{cont}_\theta = r_y(q_{RE} - \Delta^{cont}) - r_y(q_{RE})$, the premium from the ambiguity of default as $spr^{def}_\theta = r_y(q_{RE} - \Delta^{cont} - \Delta^{def}) - r_y(q_{RE} - \Delta^{cont})$, and the cross-product term as $spr^{\times}_\theta = r_y(q_{RE} - \Delta^{cont} - \Delta^{def} - \Delta^{\times}) - r_y(q_{RE} - \Delta^{cont} - \Delta^{def})$. By construction, $spr = spr_{RE} + spr^{cont}_\theta + spr^{def}_\theta + spr^{\times}_\theta$.

[7] Notice that the volatility of R_R is itself a mix of the state-contingency of the coupon $\kappa(s')$ and the volatility of the resale price, as emphasized also by Pouzo and Presno (2016).

[8] When debt is noncontingent, the yield r_y can be solved in closed form as $r_y = \frac{\kappa}{q} - \delta$.

2.2 Calibration

We parametrize our model at a quarterly frequency following the literature on sovereign debt and default. We rely on external calibration as much as possible. Parameters related to the income process (1) are set according to our estimation described above; and parameters related to risk-free rates, intertemporal substitution, debt maturity, and duration of default spells are set to standard values in this literature following the calibration to Argentina of Chatterjee and Eyigungor (2012).

Table 2 describes our parametrization.

Table 2: Baseline parameter values

	Parameter	Value
Sovereign's discount factor	β	0.9627
Sovereign's risk aversion	γ	2
Preference shock (default) scale	χ	0.0005
Issuance entropy cost	χ_b	0.0005
Lender's robustness	θ	3.31
Risk-free interest rate	r	0.01
Duration of debt	δ	0.05
Income cycle autocorrelation coefficient	ρ_z	0.9369
Standard deviation of z_t	σ_z	0.0222
Income trend autocorrelation coefficient	ρ_g	0.7418
Standard deviation of g_t	σ_g	0.01807
Reentry probability	ψ	0.0385
Default cost: intercept	d_0	0.8543
Default cost: slope	d_1	0.2129

We are left with four parameters (the sovereign's discount rate β, the lenders' robustness θ, and the two parameters describing default costs d_0 and d_1). We choose them to match four data moments: the average external debt-to-GDP ratio, the average and standard deviation of spreads, and the default frequency. Following the literature, we contrast simulated data from the model to observations for Argentina in 1993Q1:2001Q4. In the model, we extract 35-period samples that end just before a default and condition on no default ocurring within those 35 periods or two years prior.[9]

The model with non-stationary income and both trend and cycle shocks is able to generate the features of the data that are typically targeted in the sovereign debt literature, which are

[9]Because we include a positive recovery rate, that second conditional is less relevant than in other studies.

Table 3: Calibration targets

	Data	Benchmark	$\theta = 0$	Rational Expectations
Average spread (bps)	815	839	1,091	688
Debt-to-GDP ratio (%)	17.4	17.4	21.1	17.5
Std. deviation of spreads (bps)	443	391	802	547
Frequency of default (%)	3	3.02	14.5	8.55

Note: Data moments correspond to Argentina 1993Q1:2001Q4. Model moments are computed on pre-default samples, except the default frequency which is calculated on the ergodic distribution. Spread measures are annualized and debt is reported as share of yearly GDP. The column marked 'Benchmark' refers to the full parametrization described in Table 2. The column marked '$\theta = 0$' shuts down robustness starting from that parametrization. The column marked 'Rational Expectations' is a recalibration exercise imposing $\theta = 0$ and dropping the frequency of default as a target moment.

based on stationary models compared to filtered data. The unconditional frequency of default required to match the spreads in the data, under rational expectations for international creditors, is very large, even larger than that found by stationary models (Chatterjee and Eyigungor, 2012). However, like in stationary models (Pouzo and Presno, 2016), incorporating robustness is enough to match the data along with a much more reasonable default frequency.

3. Value-recovery instruments

We begin by considering value-recovery instruments (VRIs). These are a type of state-contingent bond typically used in the context of debt restructuring. These bonds are often designed to share upside risk with creditors in case macroeconomic outcomes turn out stronger than expected at the time of issuance, hence bridging differences of views between creditors and debtors. Because of the popularity of this type of bond design in debt restructurings, we compare model-implied prices with data on VRIs to validate our model and calibration.

We consider the following design: VRIs are issued only at the time of restructuring, and are indexed using a threshold bond to a "baseline" level of income, which is defined as the expected path of trend income from the time of the restructuring. To make the design concrete, let baseline income (in logs) be $\log Y^b_t = 0$ at the time of restructuring, growing by the expected trend growth rate μ_g thereafter so that $\log Y^b_{t+1} = \log Y^b_t + \mu_g$. Thus, in normalized log terms $\zeta_{t+1} = \log \frac{Y^b_{t+1}}{\Gamma_t} = \zeta_t + \mu_g - g_t$. The VRI pays a coupon $\bar{\kappa}$ whenever actual income Y_{t+s} exceeds baseline income Y^b_{t+s},

or equivalently,

$$\kappa_W(z,g,\zeta) = \bar{\kappa}\max\{0, y - y^b\} = \bar{\kappa}\max\{0, e^{z+g} - e^{\zeta}\}$$

We assume that the VRIs are only issued at the time of restructuring because the calculation of baseline income depends on the time of issuance.[10] At the time of restructuring, the government receives a haircut $\hbar$ that applies to the face value of defaulted obligations; we assume that this is delivered by issuing plain bonds and VRIs so that a share α of the required face value is delivered through VRIs.

To compute how many bonds of each type the government needs to issue, notice that the face value $fv(w;z,g,\zeta)$ of an amount w in VRIs at state (z,g,ζ) is $wm(z,g,\zeta)$, where m is the face value of one unit, and is given recursively[11] by

$$m(z,g,\zeta) = \bar{\kappa}\max\{0, e^{z+g} - e^{\zeta}\} + \frac{1-\delta}{1+r^\star}\mathbb{E}\left[m(z',g',\zeta') \mid z,g\right]$$

Appendix B describes how the introduction of VRIs affects the government's problems and pricing equations for all bonds in repayment and default. However, a few features are worth discussing in detail. When the government reenters markets at the end of a default spell, we only need to know the total value of defaulted claims, regardless of their composition. Thus, when the government defaults, we compute the total face value $x = b + fv(w;z,g,\zeta)$ and carry that state variable into the default event.[12] Upon reentry, the split between plain-vanilla bonds and VRIs is determined implicitly by the following equations for b_R and w_R:

$$b_R = (1-\alpha)(1-\hbar)x; \quad \text{fv}(w_R;z,g,0) = \alpha(1-\hbar)x$$

The choice between default and repayment with VRIs when b plain-vanilla bonds and w VRIs are outstanding in state $s = (z,g,\zeta)$ is therefore

$$v(b,w,s) = \max\left\{v_R(b,w,s) + \epsilon_R, v_D(x(b,w,s),z,g) + \epsilon_D\right\} \tag{14}$$

where $x(b,w,s) = b + fv(w;s)$ is the total face value of defaulted claims discussed above. Notice that the value of default does not depend on ζ. Similarly, the value of repayment now carries the VRIs as a state variable, although only plain-vanilla bonds are issued:

$$\begin{aligned} v_R(b,w,s) = \max_{b'}\ & u(c) + \tilde{\beta}(g)\mathbb{E}\left[v(b',w',s') \mid s\right] \\ \text{subject to}\quad & c + \kappa b + \kappa_W(s)w = y(s) + q_R(b',w,s)(e^g b' - (1-\delta)b) \\ & w' = (1-\delta)e^{-g}w \end{aligned} \tag{15}$$

[10] Issuing these bonds in normal times would therefore blow up the state space by making bonds issued at different times only imperfect substitutes.

[11] Note that the Bellman equation for m is a contraction since $(1-\delta)/(1+r^\star) < 1$.

[12] The face value of a unit of plain-vanilla bonds is 1 thanks to our construction with coupon rate $\bar{\kappa} = r^\star + \delta$.

Table 4 summarizes the effect of VRIs on the economy, for $\alpha = 0.5$, when half of the restructured face value is given as VRIs.

Table 4: Value-recovery instruments

	Benchmark	**VRI**
Avg. spread (bps)	740	763
o/w Spread RE	194	201
Std. dev. spreads (bps)	358	400
Debt-to-GDP (%)	17	13.2
Avg. spread on warrants (bps)	-	2,534
o/w Spread RE	-	130
Std. dev. spreads on warrants (bps)	-	798
Default freq. (%)	3.01	2.59
Welfare gains (%)	-	-0.158

Note: Statistics based on ergodic simulations of each model, conditional on repayment. Welfare gains are given as consumption equivalent increases, averaging over the ergodic density of the benchmark model, also conditional on repayment.

On an average period of the equilibrium with only plain-vanilla bonds, the government values announcing that future restructured values would be partly delivered through VRIs (in other words, a change of α from 0 to 0.5) as a permanent *fall* in consumption by about 0.16%. This fall in welfare is supported by low prices for VRIs of the order of 2000bps, the vast majority of which comes from amplification through the lenders' misspecification concerns. The rational-expectations component of the VRI spreads is quite mild, below 150bps, reflecting the low probability of default in those states in which the VRI is yielding positive coupon payments. The bad prices at which the VRIs trade imply that, during restructurings, creditors discount the face value delivered through these instruments. Consequently, in normal times they act as if the loss given default was greater. This worsens the spreads for the government on its plain bonds too, increasing borrowing costs, and reducing debt capacity (the debt ratio reported in Table 4 refers to plain-vanilla bonds only). When VRIs are present, the government sustains a lower debt ratio, and higher spreads, despite a lower default frequency.

4. Designing state-contingent bonds

We now turn to the problem of designing state-contingent bonds in our model environment with trend and cycle shocks. We begin by introducing simple designs in which the coupon rate of a bond equals

$$\kappa(x) = \bar{\kappa} \cdot (1 + \alpha x) \mathbb{1}_{x \geq \tau} \tag{16}$$

for x an indexing variable, either the cyclical or the trend component, z or g.

Figure 1 shows the welfare gains from these designs, relative to a plain-vanilla bond which promises constant coupons $\bar{\kappa}$. Welfare gains for indexing on the cycle z are shown on the left panel, while gains for indexing on the trend g are shown on the right panel. In either case, gains can only be achieved for sufficiently low values of τ, meaning that while a threshold for payments can be introduced, it must be sufficiently low.

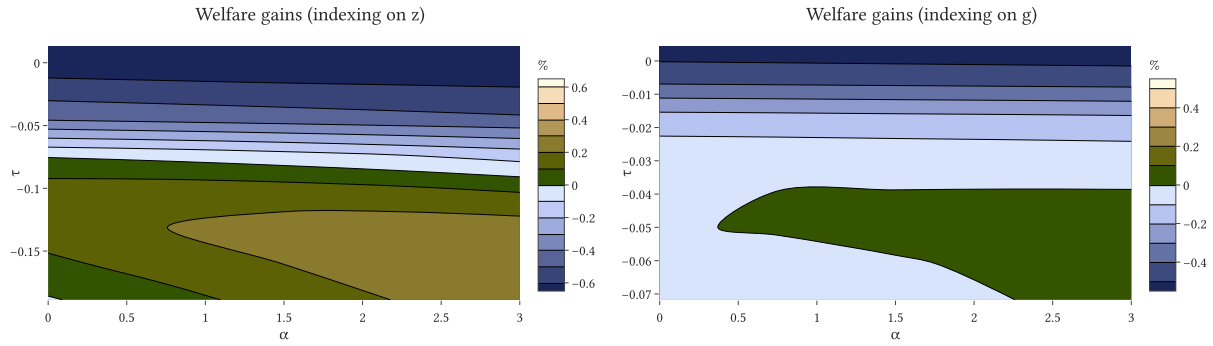


Figure 1: Welfare gains from indexed bonds

Indexing on the cycle component z yields higher welfare gains than using the trend component g. While it is possible that indexing on the trend component just requires more aggresive parameters, its maximum welfare gain is about how of the maximum welfare gain attained from indexing on the cycle component. This suggests that smoothing transitory shocks is more important than allocating higher payments when repayment capacity is high. In equilibrium, prices of *regular*, but long-term, plain-vanilla bonds also improve strongly upon positive cycle shocks, so perhaps implicit indexation to the cycle is already a feature of the equilibrium without indexed bonds.

4.1 Designs based on cumulative growth

To represent bonds indexed to the level of income, we define a *cumulative growth rate*, for a gain parameter $1-\rho$, where $\rho \in [0,1)$, as

$$\xi_0 = 1 \quad \text{and} \quad \xi_{t+1} = \rho\xi_t + (1-\rho)\log\left(\frac{Y_{t+1}}{Y_t}\right) \tag{17}$$

Notice that if $1-\rho = 1/t$ (in particular, if ρ is not constant), then $\xi_t = \frac{1}{t}\log(Y_t/Y_0)$, the average growth rate so far. Setting $\rho = 0$ makes ξ_t equal the current growth rate, while $\rho \longrightarrow 1$ on an infinite history makes $\xi_t \longrightarrow \mu_g$, the unconditional average growth rate.

To cast the law of motion of ξ, notice that $\frac{Y_{t+1}}{Y_t} = \frac{\exp(z_{t+1})\Gamma_{t+1}}{\exp(z_t)\Gamma_t} = \exp(z_{t+1}-z_t+g_{t+1})$, and therefore $\xi_{t+1} = \rho\xi_t + (1-\rho)(z_{t+1} - z_t + g_{t+1})$.

4.2 Designs based on growth surprises

We now consider bonds indexed to the surprise component of the income process, $\log(y_t) - \mathbb{E}_{t-1}[\log(y_t)]$. For this, we define

$$\xi_t = \mathbb{E}_{t-1}[\log(y_t)] = \rho_z z_{t-1} + \rho_g g_{t-1} + (1-\rho_g)\mu_g$$

and set the indexing variable $x_t = z_t + g_t - \xi_t$.

5. Concluding remarks

GDP-indexed bonds in sovereign debt have so far been used mainly through value-recovery instruments (VRIs) issued in debt restructurings and typically tying payouts to performance relative to a baseline level or path of GDP (though the specific contractual details vary across episodes). In addition, macroeconomic fluctuations in emerging-market economies are shaped by a combination of both permanent trend and transitory cyclical shocks. Putting these two facts together calls for analyzing sovereign debt and bond indexation in an environment where growth, and uncertainty about it, explicitly matters.

Studying this environment through a quantitative sovereign default model with trend and cycle shocks and lenders who fear model misspecification, we find that the model generates steep discounts for the VRIs, in line with the data. Such discounts make the government prefer to deliver the required value in the restructuring through plain-vanilla bonds rather than VRI, rationalizing the lack of popularity of VRIs in practice.

At the same time, indexation can be welfare-improving when bonds feature continuous, linear schedules rather than high thresholds. Our preliminary results indicate that indexing to the cyclical component outperforms indexing to trend growth, suggesting that debt-capacity considerations are first-order when desgining SCDIs. Among contracts restricted to observable GDP growth, a very short cumulation horizon tends to performe best, by approximating cyclical fluctuations and avoiding lock-in effects.

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A. Details about normalization and robustness

A.1 Normalization

To cast the model in stationary form we normalize by Γ_{t-1}. Let lowercase denote normalized values. For example, $y_t = \exp(z_t + g_t)$. For the value functions, notice that

$$u(C) = \frac{C^{1-\gamma}}{1-\gamma} = \Gamma^{1-\gamma}\frac{(C/\Gamma)^{1-\gamma}}{1-\gamma} = \Gamma^{1-\gamma}u(c)$$

and thus in a typical Bellman equation we can guess and verify forms like $\Gamma_0^{\gamma-1}V(B,z,g,\Gamma_0) = v(B/\Gamma_0,z,g) = v(b,z,g)$, where Γ_0 is the previous period trend so that $\Gamma = \Gamma_0 e^g$:

$$\begin{aligned}
V(B,z,g,\Gamma_0) &= \max_{B'} u(C) + \beta\mathbb{E}\left[V(B',z',g',\Gamma)\right] \\
V(B,z,g,\Gamma_0) &= \max_{B'} \Gamma_0^{1-\gamma}u(c) + \beta\mathbb{E}\left[V(B',z',g',\Gamma)\right] \\
\Gamma_0^{\gamma-1}V(B,z,g,\Gamma_0) &= \max_{B'} u(c) + \beta\mathbb{E}\left[(\Gamma_0/\Gamma)^{\gamma-1}\Gamma^{\gamma-1}V(B',z',g',\Gamma)\right] \\
v(b,z,g) &= \max_{b'} u(c) + \beta\mathbb{E}\left[e^{(1-\gamma)g}v(b',z',g')\right]
\end{aligned}$$

while for the budget constraint we have

$$\begin{aligned}
C + \kappa B &= Y + q(B' - (1-\delta)B) \\
c + \kappa b &= y + q(B'/\Gamma_0 - (1-\delta)b) \\
c + \kappa b &= y + q(b'(\Gamma/\Gamma_0) - (1-\delta)b) \\
c + \kappa b &= y + q(b'e^g - (1-\delta)b)
\end{aligned}$$

where of course the second line uses the definition of $b' = B'/\Gamma$.

A.2 Debt prices

Consider one lender (with robustness parameter θ/Γ_0) with current holdings of debt $a = A/\Gamma_0$ at aggregate state s. To cast the problem in stationary form, once more guess-and-verify a form $V = \Gamma_0 v$

$$\begin{aligned}
V(A,s) &= \max_{C,A'} C + \frac{\beta_L}{-\theta/\Gamma_0}\log\left(\mathbb{E}\left[\exp\left(-(\theta/\Gamma_0)V(A',s')\right) \mid s\right]\right) \\
\Gamma_0 v(a,s) &= \max_{c,x_a} \Gamma_0 c + \frac{\beta_L}{-\theta/\Gamma_0}\log\left(\mathbb{E}\left[\exp\left(-(\theta/\Gamma_0)\Gamma v(e^{-g}x_a,s')\right) \mid s\right]\right) \\
v(a,s) &= \max_{c,a'} c + \frac{\beta_L}{-\theta}\log\left(\mathbb{E}\left[\exp\left(-\theta e^g v(e^{-g}a',s')\right) \mid s\right]\right)
\end{aligned}$$

Since $1/\theta$ is the marginal cost of relative entropy for the evil agent, it seems sensible that it would scale with the size of the economy (see Cagetti et al., 2002, for a similar discussion).

To obtain the debt price equation (7), start from the lender's value function

$$\begin{aligned} v(a,s) &= \max_{c,a'} c + \frac{\beta_L}{-\theta} \log\left(\mathbb{E}\left[\exp\left(-\theta e^{g} v(e^{-g}a', s')\right) \mid s\right]\right) \\ \text{subject to} \quad & qa' + c = R(s)a + w \end{aligned}$$

The envelope condition is $v'(a,s) = R(s)$ under risk-neutrality. The first-order conditions are

$$\begin{aligned} \lambda &= 1 \\ \lambda q &= \frac{\beta_L}{-\theta} \frac{1}{\mathbb{E}\left[\exp\left(-\theta e^{g} v(e^{-g}a', s')\right)\right]} \mathbb{E}\left[-\theta e^{g} \exp\left(-\theta e^{g} v(e^{-g}a', s')\right) e^{-g} v'(e^{-g}a', s')\right] \end{aligned}$$

or, in the stochastic discount factor notation of equation (8),

$$q = \beta_L \mathbb{E}\left[\frac{\exp\left(-\theta e^{g} v(e^{-g}a', s')\right)}{\mathbb{E}\left[\exp\left(-\theta e^{g} v(e^{-g}a', s')\right)\right]} R(s')\right] = \beta_L \mathbb{E}\left[M(a', s') R(s')\right]$$

A.3 Decomposition of debt prices

We show how to obtain the decomposition of debt prices (10-12), which is straightforward conceptually but does require keeping track of the timing and haircuts, decay rates, and discount factors that price the debt. These factors are different in repayment and default, which complicates the writing of simple present-value formulas.

We start by separating out coupon payments from continuation prices. Let $\bar{q} = (q_R, q_D)$ be the equilibrium price vector in repayment and default, respectively. Let $X_R(\bar{q})$ and $X_D(\bar{q})$ be the payoffs associated with prices $\bar{q}$:

$$\begin{aligned} X_R(\bar{q}, s) &= (1 - \mathbb{1}_{\mathscr{D}})\left(\kappa(s) + (1-\delta) q_R(b', s)\right) + \mathbb{1}_{\mathscr{D}} q_D(be^{-g}, s) \\ X_D(\bar{q}, s) &= \psi(1-\hbar)\Big((1 - \mathbb{1}_{\mathscr{D}})\left(\kappa(s) + (1-\delta) q_R(b', s)\right) + \mathbb{1}_{\mathscr{D}} q_D((1-\hbar)be^{-g}, s)\Big) + (1-\psi) q_D(be^{-g}, s) \end{aligned} \tag{18}$$

where b' corresponds to the anticipated issuance strategy of the government at the relevant state in each case. Notice that both $X_R(\bar{q}, s)$ and $X_D(\bar{q}, s)$ are affine in continuation prices $\bar{q}$ in the sense that

$$X(\bar{q}, s) = C(s) + \mathscr{L}(\bar{q}, s) \tag{19}$$

where $C = (C_R, C_D)$ and $C_R(s) = (1 - \mathbb{1}_{\mathscr{D}})\kappa(s)$ and $C_D(s) = \psi(1 - \mathbb{1}_{\mathscr{D}})(1-\hbar)\kappa(s)$, to collect the terms associated with coupon payments, and the operator $\mathscr{L}$, which collects the terms related to continuation prices, is linear in $\bar{q}$.

Using the payoff $X(\bar{q}, s)$ and the likelihood ratio M from (8), we define the following price-updating operators

$$\mathscr{H}(\bar{q})(s) = \beta_L \mathbb{E}\left[X(\bar{q}, s') \mid s\right] = \beta_L \mathbb{E}\left[C(s') + \mathscr{L}(\bar{q}, s') \mid s\right]$$
$$\mathscr{H}_\theta(\bar{q})(s) = \beta_L \mathbb{E}\left[M(s')X(\bar{q}, s') \mid s\right] = \beta_L \mathbb{E}\left[M(s')\left(C(s') + \mathscr{L}(\bar{q}, s')\right) \mid s\right]$$

and notice that (*i*) the equilibrium price $\bar{q}$ is a fixed point of the distorted operator $\mathscr{H}_\theta$, (*ii*) the rational-expectations price $\bar{q}_{RE}$ is a fixed point of the undistorted operator $\mathscr{H}$, and (*iii*) both $\mathscr{H}$ and $\mathscr{H}_\theta$ are affine in $\bar{q}$. Let $\mathscr{T}(\bar{q}) = \beta_L \mathbb{E}\left[\mathscr{L}(\bar{q}, s') \mid s\right]$ be the linear part of $\mathscr{H}$, so that $\mathscr{H}(\bar{q}) = \mathscr{H}(0) + \mathscr{T}(\bar{q})$.

Let the one-period discount $\pi(\bar{q})$ at prices $\bar{q}$ be given by the difference between the distorted and undistorted operators

$$\pi(\bar{q}) = \mathscr{H}(\bar{q}) - \mathscr{H}_\theta(\bar{q}) = \beta_L\big(\mathbb{E}\left[X(\bar{q})\right] - \mathbb{E}\left[MX(\bar{q})\right]\big) = -\beta_L \mathrm{cov}(M, X(\bar{q})) \tag{20}$$

Collecting everything, since $\bar{q}$ is a fixed point of $\mathscr{H}_\theta$ and $\bar{q}_{RE}$ is a fixed point of $\mathscr{H}$, we have that

$$\underbrace{\bar{q}_{RE} - \bar{q}}_{=\Delta} = \mathscr{H}(\bar{q}_{RE}) - \mathscr{H}_\theta(\bar{q}) = \big(\mathscr{H}(\bar{q}_{RE}) - \mathscr{H}(\bar{q})\big) + \big(\mathscr{H}(\bar{q}) - \mathscr{H}_\theta(\bar{q})\big) = \underbrace{\mathscr{T}(\bar{q}_{RE} - \bar{q})}_{=\mathscr{T}(\Delta)} + \pi(\bar{q})$$

Finally, iterating this forward, we obtain the total ambiguity premium as a present-value sum of one-period discounts

$$\Delta = \bar{q}_{RE} - \bar{q} = \sum_{j=0}^{\infty} \mathscr{T}^j \pi(\bar{q})$$

Now, to break the total discount Δ into the three components shown in (10), we first break up each $\pi(\bar{q})$ term and then aggregate up. For this, we repeatedly use bilinearity of the covariance operator. Given random variables M, Y, and Z and an event $\mathscr{A}$, letting $X = \mathbb{1}_{\mathscr{A}} Y + (1 - \mathbb{1}_{\mathscr{A}})Z$, $\mu = \mathbb{E}\left[\mathbb{1}_{\mathscr{A}}\right]$, and $L = Y - Z$, we have that

$$\begin{aligned}\mathrm{cov}(M, X) &= \mathrm{cov}(M, \mathbb{1}_{\mathscr{A}} Y) + \mathrm{cov}(M, (1 - \mathbb{1}_{\mathscr{A}})Z) \\ &= \mu \mathrm{cov}(M, Y) + \mathbb{E}\left[L\right] \mathrm{cov}(M, \mathbb{1}_{\mathscr{A}}) + \mathrm{cov}\big(M, (1 - \mu)Z + (\mathbb{1}_{\mathscr{A}} - \mu)(L - \mathbb{E}\left[L\right])\big)\end{aligned}$$

Applying this formula to the covariance term in (20) yields the desired decomposition when setting the following labels: for X_R, set the event $\mathscr{A}$ to repayment in the following period, $1 - \mathbb{1}_{\mathscr{D}'}$, $Y = R_R$, and $Z = R_D$. For X_D, set $\mathscr{A}$ equal to reentry followed by repayment, Y equal to the residual value of the restructured bond, $(1 - \hbar)\big(\kappa(s') + (1 - \delta)q_R(b'', s')\big)$, and Z equal to the remaining payoff.

Because of the linearity of $\mathscr{T}$, decomposing π allows to decompose Δ as

$$\Delta^{cont} = \sum_{j=0}^{\infty} \mathscr{T}^j \pi^{cont}, \qquad \Delta^{def} = \sum_{j=0}^{\infty} \mathscr{T}^j \pi^{def}, \qquad \Delta^{\times} = \sum_{j=0}^{\infty} \mathscr{T}^j \pi^{\times}$$

A.4 A formulation for smooth issuance choices

We smooth the default versus repayment decision by including preference shocks. These are assumed to be distributed as *iid* Gumbel random variables, so that their difference is logistic with parameter χ, giving a probability of default of

$$\mathscr{P}(b,s) = \frac{\exp\left(\frac{1}{\chi}v_D(b,s)\right)}{\exp\left(\frac{1}{\chi}v_D(b,s)\right) + \exp\left(\frac{1}{\chi}v_R(b,s)\right)} \tag{21}$$

along with a value function

$$\begin{aligned}\mathscr{V}(b,s) &= \mathbb{E}\left[\max\left\{v_D(b,s)+\epsilon_D, v_R(b,s)+\epsilon_R\right\}\right] \\ &= \chi \log\left(\exp\left(\frac{1}{\chi}v_D(b,s)\right) + \exp\left(\frac{1}{\chi}v_R(b,s)\right)\right)\end{aligned} \tag{22}$$

For the choice of issuance b' in repayment, we seek a formulation which retains (*i*) the logistic shape for the distribuion of choices and (*ii*) the exponential tilting closed-form representation of the value function. Many studies in sovereign debt simply adapt the Gumbel-shocks formulation by fixing a finite choice set $\mathscr{B}$ for b' and including one shock per point. As a result, the choice set becomes a primitive of the problem, and the solution depends on features like the number and position of points in $\mathscr{B}$. We describe here an alternative which retains properties (*i*) and (*ii*) but approximates a fixed continuous choice and thus remains robust to the particular discretization chosen.

We are looking for a problem whose solution is a density: choosing a certain level b' must have a density proportional to the exponential of the objective function associated with choosing b' (perhaps scaled by a parameter playing the role of the scale parameter χ above). As it turns out, this is quite simple to reverse engineer, by posing the following problem

$$v_R(b,s) = \max_{\phi} \int f(b';b,s)d\phi(b') - \chi_b D_{KL}(\phi||\mu) \tag{23}$$

where $f(b';b,s) = u(c(b';b,s)) + \beta\mathbb{E}\left[\mathscr{V}(b',s') \mid s\right]$ is the objective function and $D_{KL}(\phi||\mu)$ is the Kullback-Leibler divergence between the chosen density ϕ and the reference measure μ (we take this to be a uniform). The interpretation of (23) is that the agent chooses a distribution for the issuance b' to maximize the expected value of issuing under this distribution, but is penalized for tighter control, at cost χ_b per unit of relative entropy compared to a uniform distribution. For example, there could be a delegation problem from the decision maker to the agent in charge of the bond auction; whereby precisely communicating the target for b' entails some effort.

The solution to (23) is the following pair for the value function and policy (density) function

$$\begin{aligned} v_R(b,s) &= \chi \log\left(\mathbb{E}\left[\exp\left(\frac{1}{\chi_b} f(b';b,s)\right)\right]\right) \\ \phi(b';b,s) &= \frac{\exp\left(\frac{1}{\chi_b} f(b';b,s)\right)}{\mathbb{E}\left[\exp\left(\frac{1}{\chi_b} f(b';b,s)\right)\right]} \end{aligned} \tag{24}$$

where the expectations are with respect to a uniform density on the choice set. These are readily computed on an arbitrary grid in the numerical implementation (and actually correspond to subtracting $\chi \log N_b$ from the solution to the typical discretized $\mathscr{B}$ + Gumbel shocks formulation).

The Kullback-Leibler divergence penalty leads to a log-mean-exp representation of the value function and a choice density proportional to the exponential of the objective function, scaled by the inverse of the marginal cost of relative entry. This formulation is intimately related to the way in which robustness to model misspecification is introduced in the robust control literature, including this paper. The same KL divergence, with marginal cost $1/\theta$, leads to worst-case models whose densities are proportional to the exponential of the value function, scaled by $-\theta$ (the minus sign coming from minimization in the robust-control problem).

B. Value-recovery instruments

With VRIs, a fraction α of claims is delivered as VRIs in the event of restructuring. VRIs are only issued in default and each unit pays a coupon $\kappa_W(s)$ in state s reflecting the difference between actual and baseline income:

$$\kappa_W(z,g,\zeta) = \bar{\kappa} \max\{0, e^{z+g} - e^{\zeta}\}$$

where $\bar{\kappa}$ is also the base coupon rate of plain-vanilla bonds.

The discussion in Section 3 is summarized by the following Bellman equations for the government's problem in the presence of VRIs. The repayment versus default choice now features the total face value x

$$v(b,w,s) = \max\left\{v_R(b,w,s) + \epsilon_R, v_D(x(b,w,s),z,g) + \epsilon_D\right\} \tag{25}$$

where $x(b,w,s) = b + fv(w;s)$ is the face value of defaulted claims. Note that the function $x(\cdot)$ links the state in repayment, (b,w,s), with the state in default, (x,z,g). We continue to use $s = (z,g,\zeta)$ to designate the exogenous part of the state vector when it does not lead to confusion. As before the probability of default is

$$\mathscr{P}(b,w,s) = \frac{\exp\{v_D(x(b,w,s),z,g)/\chi\}}{\exp\{v_D(x(b,w,s),z,g)/\chi\} + \exp\{v_R(b,w,s)/\chi\}}$$

In repayment, the government chooses the issuance of plain-vanilla bonds b', but no new VRIs are issued and $(1-\delta)$ units survive after the coupon payment. In normalized terms, $w' = (1-\delta)e^{-g}w$. The value of repayment is then

$$\begin{aligned} v_R(b,w,s) &= \max_{b'} u(c) + \tilde{\beta}(g)\mathbb{E}\left[v(b',w',s') \mid s\right] \\ \text{subject to} \quad & c + \kappa b + \kappa_W(s)w = y(s) + q_R(b',w,s)(e^g b' - (1-\delta)b) \\ & w' = (1-\delta)e^{-g}w \end{aligned} \tag{26}$$

In default, no debt-service payments take place but there are important subtleties in timing. The government starts a period of exclusion with a defaulted face value x and income $y(s)$. It first receives a reentry shock which restores market access, and arrives with probability ψ. Upon reentry, the haircut $\hbar$ is applied to the value x, which is then split into plain-vanilla bonds and VRIs; and the baseline income is reset to $\zeta = 0$. At this point, the government can choose to default again, prolonging the exclusion period. The value of entering a period in exclusion is

$$v_D(x,z,g) = u(y(z,g)(1-h(y(z,g)))) + \tilde{\beta}(g)\mathbb{E}\left[\begin{array}{c}(1-\psi)v_D(e^{-g'}x,z',g') \\ +\psi v(b_R,w_R,z',g',0)\end{array} \mid z,g\right] \tag{27}$$

where $b_R = e^{-g'}(1-\alpha)(1-\hbar)x$ and $\text{fv}(w_R;z',g',0) = e^{-g'}\alpha(1-\hbar)x$. Similarly to the $x_D(\cdot)$ function introduced above, the recovered values b_R and w_R (and the reset $\zeta = 0$) link the state in default (x,z,g) with the state in repayment (b,w,z,g,ζ).

To evaluate the welfare effects of a VRI issuance, we compare the value function of the government in the equilibrium with only noncontingent debt with that in the equilibrium with noncontingent debt and VRIs. We consider three possible criteria: (*i*) the VRI is first introduced in a restructuring, therefore the comparison in value functions is taken as an average over the ergodic distribution of the equilibrium without VRIs, conditional on a restructuring event (and also study the underlying map of (z,g) values for prices and welfare gains), (*ii*) the VRI is introduced at an arbitrary date, therefore the comparison is taken as an average over the ergodic distribution of the equilibrium without VRIs and comparing with the value of holding no VRIs, $x = 0$, (*iii*) the VRI is introduced at the start of the economy, so the comparison is taken at $b = x = 0$ and averaging over the ergodic distribution of income (z,g).

B.1 Bond prices

Lenders price plain-vanilla bonds and VRIs separately. Let q_R and q_W denote, respectively, the repayment prices of one unit of plain-vanilla debt and one unit of VRI, and let q_D denote the price of one unit of defaulted face value. As in (7), prices are formed under the lenders' worst-case measure through the stochastic discount factors M_R and M_D defined in (8). Continuation

states follow the laws of motion $w' = (1-\delta)e^{-g}w$ and $\zeta' = \zeta + \mu_g - g$, and the face value of defaulted claims next period is $x' = x_D(b', w', z', g', \zeta') = b' + w'\, m(z', g', \zeta')$.

The price of plain-vanilla debt satisfies

$$q_R(b', w, s) = \beta_L \mathbb{E}\left[\begin{array}{c}(1 - \mathbb{1}_{\mathscr{D}'})M_R\,(\kappa + (1-\delta)q_R(b', w', s')) \\ +\mathbb{1}_{\mathscr{D}'} M_D\, q_D(x', z', g')\end{array} \mid z, g\right] \tag{28}$$

while the price of a VRI unit satisfies

$$q_W(b', w, s) = \beta_L \mathbb{E}\left[\begin{array}{c}(1 - \mathbb{1}_{\mathscr{D}'})M_R\,(\kappa_W(s') + (1-\delta)q_W(b', w', s')) \\ +\mathbb{1}_{\mathscr{D}'} M_D\, m(s')\, q_D(x', z', g')\end{array} \mid z, g\right] \tag{29}$$

The two prices share the same default event but differ in their cash flows: a plain-vanilla bond pays the fixed coupon κ and recovers q_D per unit, whereas a VRI pays the state-contingent coupon $\kappa_W(z', g', \zeta')$ and, upon default, recovers $m(z', g', \zeta')$ units of defaulted face value, each priced at q_D.

Finally, the price of one unit of defaulted face value is determined by the expected prices of the instruments which will be received by creditors upon restructuring. While the default continues, creditors continue the hold the defaulted claim and nothing happens except the evolution of the state (x, z, g). When an opportunity to restructure arises, the defaulted claims are replaced with plain-vanilla bonds and VRIs as described above. The resulting break-even price is

$$q_D(x, z, g) = \beta_L \mathbb{E}\left[\begin{array}{c}(1-\psi)M_D\, q_D(e^{-g'}x, z', g') \\ +\psi\,\dfrac{b_R}{e^{-g'}x}(1 - \mathbb{1}_{\mathscr{D}'_R})M_R\,(\kappa + (1-\delta)q_R(b_R, w_R, z', g', 0)) \\ +\psi\,\dfrac{w_R}{e^{-g'}x}(1 - \mathbb{1}_{\mathscr{D}'_R})M_R\,(\kappa_W(z', g', 0) + (1-\delta)q_W(b_R, w_R, z', g', 0)) \\ +\psi\,\mathbb{1}_{\mathscr{D}'_R}\left(\dfrac{b_R}{e^{-g'}x}M_D\, q_D(x_R, z', g') + \dfrac{w_R}{e^{-g'}x}M_D\, m(z', g', 0)\, q_D(x_R, z', g')\right)\end{array} \mid z, g\right] \tag{30}$$

where the ratios simplify to $b_R/(e^{-g'}x) = (1-\alpha)(1-\hbar)$ and $w_R/(e^{-g'}x) = \alpha(1-\hbar)/m(z', g', 0)$.