Trade Credit Chains and Monetary Policy*

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Abstract

We develop a general equilibrium monetary model of supply chain finance where firms are intermediaries in a production chain. Firms buy inputs upstream and sell output downstream where payments are financed internally with money or externally with trade credit. Limited commitment generates two jointly determined debt limits that reinforce one another through a vertical credit channel. When credit is scarce, credit limits reinforce each other and can generate multiple steady states while the availability of money prevents a collapse of credit at one link from spreading to the other. Inflation simultaneously worsens the defaulter's outside option and reduces liquidity elsewhere in the chain, giving monetary policy potentially opposing effects on trade credit. The Friedman rule is optimal but does not attain the first best since bargaining creates a downstream holdup wedge that distorts upstream production.

Keywords: money, trade credit, supply chains, limited commitment, monetary policy
JEL codes: E40, E51, E52, G32

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1 Introduction

Trade credit is a central feature of firm finance. Firms in intermediate positions buy inputs on account from their suppliers and sell on account to business customers. The same balance sheet therefore carries both payables and receivables. Trade credit is the largest source of short-term external funds for U.S. small businesses (Petersen and Rajan 1997) and about 60 percent of them use it (SSBF 2003). In the aggregate, trade payables in the United States and Europe have held near 20 percent of GDP for a quarter century (Boissay, Patel, and Shin 2020) while receivables are roughly one fifth of firm assets (Hill, Kelly, and Highfield 2010). Importantly, trade credit use also responds systematically to monetary policy: when liquidity and bank finance become more costly, firms substitute toward trade credit (Meltzer 1960; Nilsen 2002). Distress then travels along chains as firms pass on the credit losses they suffer (Jacobson and von Schedvin 2015; Costello 2020).

These observations raise important questions for monetary theory and policy. How do credit conditions at one link of a chain determine credit conditions at another? How does monetary policy affect trade credit when payments along the chain are made in sequence? And what is optimal policy when linkages of this kind are present?

To answer these and related questions, we develop a general equilibrium monetary model of supply chain finance where firms are intermediaries in a production supply chain. The environment is based on the New Monetarist model of Lagos and Wright (2005) and Rocheteau and Wright (2005) where limited commitment makes credit self-enforcing (Kehoe and Levine 1993) and limited record-keeping makes money socially valuable (Lotz and Zhang 2016). Firms in the model obtain intermediate inputs from suppliers in an upstream market and sell output to downstream customers in a downstream market.[1] Trade can be settled with money or trade credit where the amount of credit available depends on the borrower's incentive to repay. Money holdings are the firm's internal finance, as in Rocheteau, Wright, and Zhang (2018), so retained earnings finance inputs only by being carried as cash and bear the same opportunity cost as any liquid balance. As in Kehoe and Levine (1993), a borrower who defaults is excluded from future credit in that market and can self-finance with money alone. This generates endogenous debt limits in both markets that depend on one another. Because exclusion upstream costs the firm its ability to run a downstream business, the downstream credit environment enters the upstream limit. Upstream credit in turn determines input stocks and hence the value to downstream buyers of credit access. The two limits therefore form a fixed-point system.

[1] We call the downstream buyer a consumer since it values the final good, though a closer empirical counterpart is a business customer buying on account such as a distributor or a retailer.

Our first result is that conditions at the customer end of the chain support credit at the supplier end. If a firm sells more downstream, or its customers borrow more easily, its downstream business is worth more and default upstream becomes more costly. Notice that the discipline is reputational. The supplier never takes the receivable as security, and the firm repays because it does not want to lose the credit that keeps its downstream operation running.

Importantly, the two debt limits reinforce each other. Downstream credit raises the value of the firm's business, which supports upstream credit. Upstream credit in turn lets the firm buy more inputs and sell more goods which raises what its customers lose by defaulting. The feedback between the limits generates a vertical multiplier. A small improvement in downstream liquidity has a larger total effect and a small deterioration is amplified the same way.

The feedback can also generate multiple steady states. As in any model where debt limits rest on beliefs about future limits, there is a pessimistic outcome where nobody expects to be repaid and so nobody lends, and this is as true with one market as with two. The chain adds something different: steady states in which credit is active at both links. With one market there is exactly one of these. In the chain there can be several when the feedback is strong, and credit can then fall from one to another with no change in fundamentals, each limit falling because the other does. None of this uses money. The vertical channel, the multiplier, and the multiplicity survive in a chain that runs on credit alone (Appendix D.5). Money matters for the size of the limits and for what policy does to them.

Monetary policy has opposing effects on trade credit. Higher inflation taxes real balances and drains liquidity from the chain. It also makes the money-only fallback after default less attractive, which strengthens repayment incentives and supports credit. We call the first force the cross-market effect and the second the own-market effect. The own-market effect is familiar from the one-market model of Lotz and Zhang (2016), where the surplus a credit relationship protects is priced by the borrower's own liquidity, so inflation relaxes debt limits by making default worse. The cross-market effect is new. In the chain, the surplus one relationship protects is priced by liquidity that a counterparty holds in another market, and by draining that liquidity inflation can tighten debt limits.

Which force dominates depends on fundamentals and on whether money and credit coexist. If all sellers accept credit, self-enforcing credit crowds out money and the own-market effect dominates. If some sellers accept only cash, borrowers use money for those meetings and both forces are active. Just above the inflation rate at which credit becomes sustainable, credit limits are small and the cross-market effect is weak, so moderate inflation lowers real balances, relaxes debt limits, and shifts payments from money toward trade credit. When

inflation is sufficiently high, this expansion reverses and further inflation contracts credit. We call the intermediate region over which inflation expands credit the credit-expansion band. A natural empirical implication is that the substitution between money and trade credit documented by Meltzer (1960) and Nilsen (2002) should be most pronounced at moderate inflation rates.

On welfare, no monetary policy attains the first best, including the Friedman rule. With bargaining, the firm values inputs at its share of the downstream surplus, so a friction that lies entirely downstream ends up distorting upstream production. This vertical holdup wedge is present at every nominal rate, and it is specific to the chain. With one market and proportional bargaining, the Friedman rule attains the first best (Aruoba, Rocheteau, and Waller 2007; Lotz and Zhang 2016). Here the wedge sits on the input purchase and not on the liquidity margin. It is the search-theoretic counterpart of the vertical externality in Spengler (1950), carried by bargaining instead of pricing. Only one margin is distorted, since the input payment is already sunk by the time the firm bargains downstream.

Nor can the wedge be bargained away. Efficient input purchases require the customer's downstream bargaining share to be zero, while money is valued only if customers receive some of the surplus their liquidity generates. The same share governs both margins, so no assignment of bargaining power implements the first best in a monetary equilibrium. The conclusion does depend on the contracting environment. If the supplier were paid only when the firm later sells downstream, the wedge would disappear. Section 6.2 explains why this contract removes the distortion and why it cannot be written under our record-keeping assumptions.

Related Literature

We build on monetary search models in which money and credit coexist because credit needs record keeping and is bounded by limited commitment (Kocherlakota 1998; Berentsen, Camera, and Waller 2007; Sanches and Williamson 2010; Carapella and Williamson 2015; Lotz and Zhang 2016; Gu, Mattesini, and Wright 2016; Araujo and Hu 2018). What is new is unsecured credit and money inside a production chain. In one-market models inflation relaxes self-enforcing limits by worsening the defaulter's money-only fallback (Aiyagari and Williamson 2000; Gomis-Porqueras and Sanches 2013; Antinolfi, Azariadis, and Bullard 2016; Lotz and Zhang 2016). There the surplus a credit relationship protects is priced by the borrower's own liquidity, which drops out of the flow cost of default, so the comparative static is monotone. In the chain it is priced partly by liquidity in another relationship, and the sign becomes conditional. Self-fulfilling credit dynamics in a single market come from

beliefs about future limits (Gu, Mattesini, Monnet, and Wright 2013; Bethune, Rocheteau, and Rupert 2015; Bethune, Hu, and Rocheteau 2018), while multiplicity here comes from two limits at adjacent links. Firms hold money in Rocheteau, Wright, and Zhang (2018) as they do here, but with one decentralized market its return is a primitive surplus rather than an option on whether the firm's customers can pay.

Firms in our model are middlemen, which connects the paper to the search literature on intermediation that begins with Rubinstein and Wolinsky (1987) and is surveyed in Gong, Qiao, and Wright (2026). Middlemen there have an advantage over direct trade in meeting rates, information, bargaining power, entry costs, or inventories (Biglaiser 1993; Li 1998; Shevchenko 2004; Masters 2008; Watanabe 2010; Wright and Wong 2014; Nosal, Wong, and Wright 2015; Gong and Wright 2024; Gong, Qiao, and Wright 2025). In Gong, Qiao, Wong, and Wright (2025) the advantage is credit itself, and endogenous debt limits deliver multiple equilibria. Their economy has no money, their middleman intermediates in exchange rather than production, and their two limits constrain the same debtor and move in proportion, which collapses the system to one equation. Ours constrain different debtors at different links, and the fixed point is two dimensional.

Bocola and Bornstein (2026) study reputation-enforced trade credit between firms in production and quantify a credit multiplier with Italian data. Their chain has one reputational link, firms hold no cash, and the multiplier comes from suppliers discounting receivables with banks. Ours comes from enforcement at two links, without banks, and a defaulter keeps trading with money. They name chains in which firms both receive and extend trade credit as a direction for future work, and those chains are the object of this paper. In Gottardi and Zhang (2026) firms pledge downstream receivables as collateral for input IOUs, in a non-monetary competitive economy with default on path. As here, the enforcement constraint at one link is priced by the firm's position at the other. There the link is a pledged claim, while here the firm forfeits its downstream relationship, since IOUs are bilateral. Our setup fits open-account trade credit and theirs fits factored or pledged receivables, and Section 7.2 sketches how to nest them. Hu, Watanabe, and Zhang (2025) study supplier finance in a search environment, also without money.

Paying production costs before sales revenue arrives is standard in working-capital models (Blinder 1987; Fuerst 1992; Jermann and Quadrini 2012), and interlocking obligations that propagate shocks go back to Kiyotaki and Moore (1997) and Kim and Shin (2012, 2024). Bigio and La'O (2020), Altinoglu (2021), and Reischer (2024) put working-capital constraints inside production networks, with financing wedges imposed rather than derived from limited commitment. Boissay, Garcia-Appendini, and Ongena (2026) find that a monetary tightening reaches firms through their constrained suppliers and customers. With

nominal rigidities, a production network amplifies monetary shocks because prices adjust slowly (Pasten, Schoenle, and Weber 2020; Ghassibe 2021; La'O and Tahbaz-Salehi 2022; Rubbo 2023). Our prices are flexible and policy works through the payments system. Finally, corporate finance explains why suppliers lend (Biais and Gollier 1997; Petersen and Rajan 1997; Burkart and Ellingsen 2004; Cuñat 2007) and how trade credit interacts with bank finance (Nilsen 2002; Love, Preve, and Sarria-Allende 2007; Garcia-Appendini and Montoriol-Garriga 2013; Adelino, Ferreira, Giannetti, and Pires 2023). We abstract from banks to isolate bilateral payment arrangements and how they interact across links.

The paper is organized as follows. Section 2 presents the environment and Section 3 characterizes equilibrium taking debt limits as given. Section 4 derives the limits, studies the feedback between them, and works out how monetary policy affects them. Section 5 solves a benchmark economy, Section 6 studies welfare, Section 7 sketches extensions, and Section 8 concludes.

2 Environment

Time is discrete and continues forever. There are three types of infinitely lived agents, consumers, firms, and suppliers, each of unit measure. Each period splits into three stages. Trade opens in an upstream decentralized market, which we abbreviate UM, moves to a downstream decentralized market, or DM, and closes in a frictionless centralized market, or CM, where everyone meets, settles debts, and chooses what to carry into the next period.

Figure 1 shows the market structure. The same firm appears in all three markets within a period. It buys upstream, sells downstream, and settles both positions in the centralized market that closes the period.

Goods and production. There are three goods. A perfectly divisible numeraire good is produced and consumed in the CM. An intermediate good is traded in the UM, and a final good is traded in the DM.

Suppliers use labor $h_U \geq 0$ to produce the intermediate good with the technology

$$q_U = f(h_U), \tag{1}$$

where f satisfies $f' > 0$, $f'' \leq 0$, and $f(0) = 0$. The intermediate good is sold to firms in the upstream market and nobody values it directly.

Firms turn intermediate inputs and their own labor into the final good. A firm that has

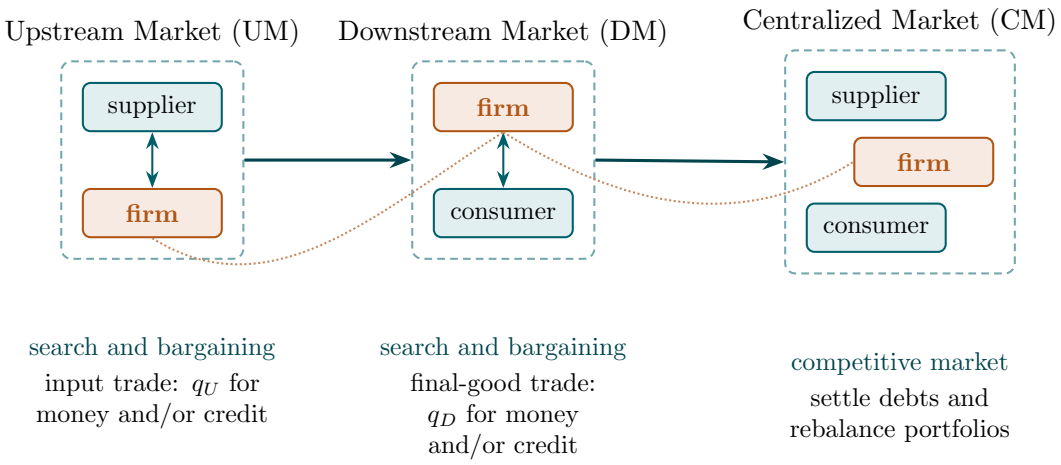


Figure 1: Market structure within a period

purchased q_U units of the input supplies labor $h_D \geq 0$ to produce

$$q_D = G(q_U, h_D), \tag{2}$$

where G is increasing in each argument and jointly concave, with $G(0, h_D) = G(q_U, 0) = 0$. The final good is sold to consumers in the downstream market. Neither good is storable across periods.

Preferences. Consumers value the final good but cannot produce it, while suppliers can make the intermediate good but do not value it. In the CM everyone can consume the numeraire good $x \in \mathbb{R}_+$ and produce it one-for-one with CM labor $y \in \mathbb{R}_+$. All agents discount across periods at rate $\beta \in (0, 1)$ and do not discount between subperiods.

Instantaneous utility for a consumer is

$$U^c(q_D, x, y) = u(q_D) + x - y, \tag{3}$$

for a firm it is

$$U^f(q_D, h_D, x, y) = -h_D + x - y, \tag{4}$$

and for a supplier it is

$$U^s(h_U, x, y) = -h_U + x - y, \tag{5}$$

where the disutility of labor enters directly for both firms and suppliers. The function $u(\cdot)$ is the consumer's utility from the final good, with $u' > 0$, $u'' < 0$, $u(0) = 0$, and $u'(0) = \infty$. As in Lagos and Wright (2005), utility is linear in the numeraire good and in CM labor.

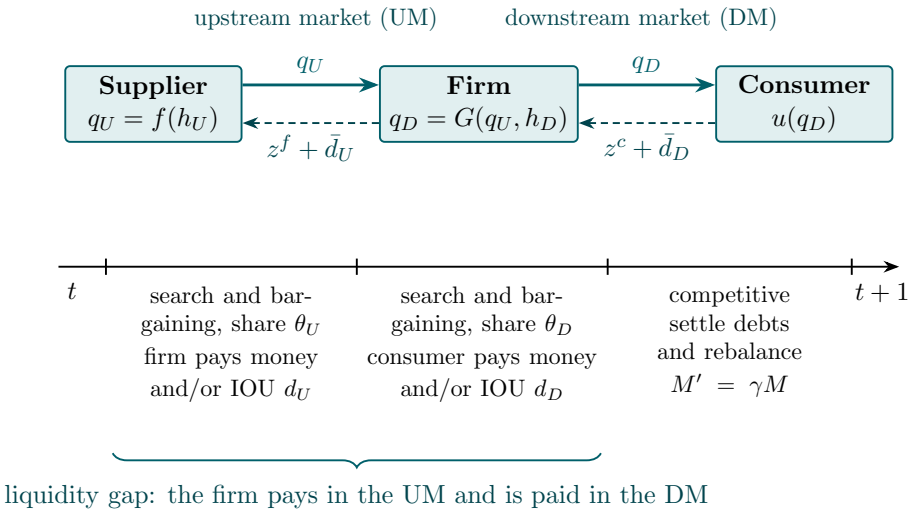


Figure 2: Agents, goods, payments, and timing within a period

Agents can then always produce enough numeraire to settle their debts, and default is never a matter of being unable to pay. Linearity also makes what an agent carries out of the CM independent of what it carried in. We never have to track a distribution of money holdings.

Timing. Figure 2 summarizes the environment. Goods move from suppliers to firms to consumers, and payments move the other way. Matching in each decentralized market is bilateral, and the buyer can pay with money, with an IOU settled in the CM, or with a combination of the two, provided the seller is able to accept IOUs. Which sellers can do so is a matter of record keeping, described below.

The figure also shows the key friction. The UM opens before the DM, and a firm must pay for its inputs before it collects any revenue from consumers. This within-period liquidity gap ties together money demand, trade credit, and production. It is our model of working capital. In the CM debts are settled in numeraire, portfolios are rebalanced, and the money stock grows at rate γ.

Upstream and Downstream Markets

The decentralized part of the period has two rounds of bilateral trade, the UM followed by the DM. In the UM each firm meets a supplier and buys q_U units of the input, produced with (1), and we set the matching probability to one, since the friction that matters is downstream.[2] In the DM a firm meets a consumer with probability $\sigma_D \in (0, 1]$ and sells q_D

[2]Reinstating $\sigma_U < 1$ scales the expected decentralized surpluses on both sides of the chain by σ_U. This lowers both demonetization thresholds of Section 4.6 by the same factor and leaves the sustainability

units of the final good, made with (2) out of its input stock. A firm that meets no consumer is stuck: its inputs perish, as the intermediate good is neither consumed nor traded in the CM. The two quantities are bargained separately, q_D downstream and q_U upstream. Still, the firm bargaining upstream already has its downstream revenue in mind, and that is how conditions at the two ends of the chain come to depend on each other.

Terms of trade come from proportional bargaining, with consumer share $\theta_D \in (0,1)$ downstream and firm share $\theta_U \in (0,1)$ upstream. Only the holdup wedge of Proposition 6 and the demonetization thresholds of Section 4.6 depend on bargaining. Everything else survives price taking (Section 6.1).

Centralized Market

In the CM agents trade the numeraire competitively, settle debts, and choose what to hold going into the next period.

The only storable asset is fiat money. It is divisible and perfectly recognizable, with price ϕ_t in numeraire. The money stock grows at the constant gross rate $\gamma = M_{t+1}/M_t \geq \beta$, with new money injected through lump-sum transfers to consumers and firms at the beginning of the CM. The nominal interest rate is $1 + i = \gamma/\beta$, the return on an illiquid nominal bond, so i is the opportunity cost of holding money.

Agents are anonymous in the decentralized markets, so credit requires the lender to have a record-keeping technology, the primitive that substitutes for money in Kocherlakota (1998). In the main analysis access to it is a fixed characteristic of sellers. A fraction $\Lambda_D \in [0,1]$ of firms operates the technology at no cost and can accept consumer IOUs $d_D \leq \bar{d}_D$, while the rest face a prohibitive record-keeping cost and sell only against money. Likewise a fraction $\Lambda_U \in [0,1]$ of suppliers accepts firm IOUs $d_U \leq \bar{d}_U$. We treat (Λ_U, Λ_D) as exogenous here and endogenize it in Section 7.1. The two corners bracket the analysis: at $\Lambda_U = \Lambda_D = 0$ the chain runs entirely on money, and at $\Lambda_U = \Lambda_D = 1$ every seller keeps records. One restriction remains. IOUs are bilateral claims that cannot be transferred or pledged, so a firm cannot hand its downstream receivable to a supplier.

Borrowers cannot commit. Nobody can force them to repay, so repayment has to be in their own interest. A default is recorded by lenders in the market where it happens and leads to permanent exclusion from credit in that market, while the defaulter can still trade with money.[3] This punishment bounds how much a borrower can credibly promise.

thresholds unchanged.

[3]Exclusion is from credit at the link where the default occurred, not from a single relationship and not from the economy. A firm that defaults on one supplier is cut off from all record-keeping suppliers, since the record is kept by the technology at that link, but it goes on buying inputs with money and selling

Social optimum. It is convenient to work with cost functions rather than production functions. Let $c(q_U) \equiv f^{-1}(q_U)$ be the supplier labor needed to make q_U, and let $g(q_D \mid q_U) \equiv h_D$ solve $G(q_U, h_D) = q_D$, the firm labor needed to make q_D out of an input stock q_U. The timing of the two markets matters for the efficient allocation (q_U^*, q_D^*). Supplier labor goes in upstream, before the downstream match is realized, while firm labor goes in only inside a downstream match. The downstream matching probability therefore enters the upstream margin and not the downstream one. The planner maximizes

$$\sigma_D\big[u\big(G(q_U, h_D)\big) - h_D\big] - h_U \tag{6}$$

subject to $q_U = f(h_U)$, with first-order conditions

$$u'(q_D^*)\, G_2(q_U^*, h_D^*) = 1, \tag{7}$$

$$\sigma_D\, u'(q_D^*)\, G_1(q_U^*, h_D^*)\, f'(h_U^*) = 1, \tag{8}$$

where $G_2 \equiv \partial G(q_U, h_D)/\partial h_D > 0$ and $G_1 \equiv \partial G(q_U, h_D)/\partial q_U > 0$.

Condition (7) equates the marginal benefit of downstream labor to its marginal cost conditional on a match. In (8), $G_1 f'$ is the marginal product of upstream labor transmitted through the chain, discounted by the probability that the input ever finds a downstream buyer.

3 Equilibrium with Exogenous Debt Limits

We solve the model backward within a period: first the CM value functions, then the two bargaining problems, then money demand. The downstream problem comes first, because the firm values inputs by what they will earn downstream. Debt limits are taken as given here. Section 4 derives them from incentive constraints.

Let real balances be $z \equiv \phi m$. In a stationary equilibrium $\phi_t/\phi_{t+1} = \gamma$, so acquiring $\hat{z}$ real balances for next period costs $\gamma\hat{z}$ units of numeraire today. Accepting downstream credit changes a firm's upstream payoff, and we index firms by $n \in \{0, 1\}$, with $n = 1$ meaning the firm accepts consumer IOUs. The two population shares are Λ_D and $1 - \Lambda_D$.

downstream on its usual terms. Appendix D.4 compares this punishment with the economy-wide exclusion of Kehoe and Levine (1993). It also shows that exclusion need not be permanent: if a defaulter's record is erased with probability δ each period, every result holds with r replaced by $(r + \delta)/(1 - \delta)$.

3.1 Centralized Market Value Functions

Debts are settled in the CM, where agents produce the numeraire at constant marginal disutility. Repayment is therefore always feasible. The only question is whether it is in the borrower's interest.

Consumers. Let $W^c(z, -d_D)$ be the CM value of a consumer in good credit standing who enters with z real balances and owes d_D in numeraire to a firm. He solves

$$W^c(z, -d_D) = \max_{x,y,\hat{z}\geq 0} \; x - y + \beta V^c(\hat{z}) \quad \text{s.t.} \quad x + \gamma\hat{z} = y + z - d_D + T^c, \tag{9}$$

where V^c is his value at the start of next period's decentralized stage, $\hat{z}$ is next period's real balances, and T^c is the lump-sum transfer. Substituting the budget constraint into the objective,

$$W^c(z, -d_D) = z - d_D + T^c + \max_{\hat{z}\geq 0} \left\{-\gamma\hat{z} + \beta V^c(\hat{z})\right\}. \tag{10}$$

The value W^c is linear in net wealth $z - d_D$ with unit slope, and the choice of $\hat{z}$ is independent of (z, d_D), so every consumer in good standing leaves the CM with the same real balances. Both properties come from linearity of utility in the numeraire. The value function of a consumer with a recorded default, $\widehat{W}^c(z)$, is (10) with $d_D = 0$ and continuation value $\widehat{V}^c$.

Firms. A firm enters the CM holding money from its downstream sale, owing d_U to a supplier, and holding a receivable d_D from a consumer if it extended credit in the DM. For a firm of type n,

$$W_n^f(z, -d_U, d_D) = z - d_U + d_D + T^f + \max_{\hat{z}\geq 0} \left\{-\gamma\hat{z} + \beta V_n^f(\hat{z})\right\}. \tag{11}$$

Again W_n^f is linear in net wealth $z - d_U + d_D$, the portfolio choice does not depend on history, and $\widehat{W}_n^f(z, d_D)$ is the analogue of (11) with $d_U = 0$. Notice what default on a trade payable does and does not do. It excludes the firm from upstream credit. The firm keeps operating, still pays suppliers with money, and keeps both its downstream record-keeping technology and its ability to lend. This is the punishment of Lotz and Zhang (2016), extended to the chain. A harsher punishment would raise the cost of default and relax the upstream limit further.

Suppliers. A supplier enters the CM with money z received upstream and a receivable d_U:

$$W^s(z, d_U) = z + d_U + \max_{\hat{z}\geq 0} \{-\gamma \hat{z} + \beta V^s(\hat{z})\}. \tag{12}$$

Suppliers are sellers in their only decentralized meeting. Money buys them no liquidity services, yet holding it costs $\gamma - \beta > 0$ per unit whenever $i > 0$, and $\hat{z} = 0$ solves their portfolio problem. This confirms $m^s = 0$ in any equilibrium away from the Friedman rule and gives $W^s(z, d_U) = z + d_U + \beta V^s(0)$.

3.2 Terms of Trade

The firm bargains upstream over an input whose value is determined downstream. We therefore work out the downstream problem first and feed its solution into the upstream one.

3.2.1 Downstream Market (DM)

Consider a meeting between a consumer and a type-n firm holding q_U units of the input. The consumer's payment capacity is

$$\omega_n^c = z^c + n\bar{d}_D,$$

where z^c is his real balances. A type-1 firm can be paid with money and credit, a type-0 firm with money alone. Linearity of W^c and W_n^f means that only the total payment matters, not its split between money and IOUs. The consumer's surplus from trading (q_D, p_D) is $u(q_D) - p_D$ and the firm's is $p_D - g(q_D \mid q_U)$, where $g(q_D \mid q_U)$ is the labor the firm supplies to make q_D out of q_U, and p_D corresponds to the payment for the exchanged good.

The firm's upstream payment does not appear in this problem. It is owed in the CM whether or not this meeting happens, so it is sunk here and nets out of both surpluses through the linearity of the CM value functions. Section 6.2 shows that this feature, and not the timing of settlement, is what creates the holdup wedge of Section 6. From $G(q_U, h_D) = q_D$, the conditional cost function satisfies

$$g'_D \equiv \frac{\partial g(q_D \mid q_U)}{\partial q_D} = \frac{1}{G_2} > 0, \qquad g'_U \equiv \frac{\partial g(q_D \mid q_U)}{\partial q_U} = -\frac{G_1}{G_2} < 0,$$

so more output requires more labor while a bigger input stock economizes on labor for given output.

Proportional bargaining with consumer share θ_D solves

$$\begin{aligned} \max_{q_D, p_D} \quad & \underbrace{u(q_D) - p_D}_{\text{consumer's surplus}} \\ \text{s.t.} \quad & \underbrace{p_D - g(q_D \mid q_U)}_{\text{firm's surplus}} = (1-\theta_D)\ \underbrace{\big[u(q_D) - g(q_D \mid q_U)\big]}_{\text{total surplus } S_D}, \\ & p_D \le \omega_n^c. \end{aligned} \tag{13}$$

The rule pins the payment down as a function of the quantity,

$$p_D = (1-\theta_D)\, u(q_D) + \theta_D\, g(q_D \mid q_U), \tag{14}$$

and the quantity is then the largest q_D the consumer can pay for. Write $q_D^*(q_U)$, defined by $u'(q_D^*) = g_D'(q_D^* \mid q_U)$, for the quantity that maximizes match surplus given the input stock. The notation is consistent with the planner's, since $q_D^* = q_D^*(q_U^*)$. Then

$$q_D = \begin{cases} q_D^*(q_U) & \text{if } p_D(q_D^*(q_U)) \le \omega_n^c \\ q_D(\omega_n^c; q_U) & \text{otherwise,} \end{cases} \tag{15}$$

where $q_D(\omega_n^c; q_U)$ solves $(1-\theta_D)u(q_D) + \theta_D g(q_D \mid q_U) = \omega_n^c$. When the payment constraint binds the consumer hands over all his wealth and the traded quantity rises with ω_n^c:

$$\frac{\partial q_D}{\partial \omega_n^c} = \frac{1}{(1-\theta_D)u'(q_D) + \theta_D g_D'(q_D \mid q_U)} > 0. \tag{16}$$

Total match surplus is $S_D(q_D; q_U) \equiv u(q_D) - g(q_D \mid q_U)$, split θ_D to the consumer and $1-\theta_D$ to the firm.

3.2.2 The Value of Inputs

The two markets are linked through the firm's valuation of the input. Define

$$\Omega_n(q_U) \equiv \sigma_D(1-\theta_D)\, S_D\big(q_D; q_U\big), \tag{17}$$

the expected downstream surplus going to a type-n firm that starts the DM with q_U, where q_D comes from (15). With no inputs there is nothing to sell, and $\Omega_n(0) = 0$. The discount σ_D reflects that inputs perish: with probability $1-\sigma_D$ the firm meets nobody and its stock is worthless. Downstream conditions enter Ω_n through the consumer's payment capacity

$\omega_n^c = z^c + n\bar{d}_D$ and the bargaining share θ_D. This is how they reach the upstream market.

One regularity condition is needed. It plays the role that concavity of the bilateral surplus plays in one-market models, and it keeps the portfolio problems and the debt-limit equations well behaved.

Assumption 1. For each n, the upstream surplus $S_U^n(q_U) \equiv \Omega_n(q_U) - c(q_U)$ defined in (24) below is strictly concave on the payment-constrained regime, regime (i) of Lemma 1, and Ω_n' does not jump upward at the regime boundary. Appendix B verifies the assumption for isoelastic u, Cobb–Douglas G, and linear f.

Assumption 1 says there are diminishing returns to liquidity along the chain, which makes money demand well behaved and gives each debt-limit equation a single positive solution. It has bite only on the payment-constrained regime. On the unconstrained regime $\Omega_n(q_U)/[\sigma_D(1-\theta_D)] = \max_h\{u(G(q_U,h))-h\}$, which is concave because G is jointly concave.

Lemma 1. *The marginal value of inputs is strictly positive, $\Omega_n'(q_U) > 0$, in every trading regime:*

(i) Payment-constrained ($q_D = q_D(\omega_n^c; q_U)$):

$$\Omega_n'(q_U) = \sigma_D(1-\theta_D)\left[\big(u'(q_D) - g_D'\big)\frac{\partial q_D}{\partial q_U} - g_U'\right], \quad \frac{\partial q_D}{\partial q_U} = \frac{-\theta_D\, g_U'}{(1-\theta_D)u'(q_D) + \theta_D g_D'} > 0.$$

(ii) Unconstrained ($q_D = q_D^*(q_U)$): by the envelope theorem,

$$\Omega_n'(q_U) = -\sigma_D(1-\theta_D)\, g_U'\big(q_D^*(q_U) \mid q_U\big) > 0.$$

If the consumer's payment constraint is slack, an extra input just lowers the firm's downstream labor cost. If it binds, there is a second effect. A bigger input stock lowers the bargained payment for any given quantity, so a consumer with fixed wealth can buy more output. The marginal value of inputs now moves with downstream liquidity. This is where the vertical channel first appears.

Substituting $\partial q_D/\partial q_U$ into regime (i) and collecting terms gives a representation we use throughout. Write $\tilde{D} \equiv (1-\theta_D)\, u'(q_D) + \theta_D\, g_D'(q_D \mid q_U)$ for the bargained price of a marginal unit of output, which is the denominator in (16). Then

$$\Omega_n'(q_U) = -\sigma_D(1-\theta_D)\, g_U'\, \frac{u'(q_D)}{\tilde{D}} > 0. \tag{18}$$

The marginal value of inputs is a product of two terms. The input-saving term g_U' is there in both regimes. The payment-relaxation term $u'(q_D)/\tilde{D} \geq 1$ is there only when the consumer's

payment constraint binds, and it equals $1 + \theta_D L_D$ once the downstream liquidity premium (29) is defined below. Whether Ω'_n rises or falls with downstream liquidity is a contest between the two: a richer consumer buys more, which shrinks the premium and, when the input and labor are complements, raises the input saving.

Notice that (18) raises q_U at a fixed payment capacity ω_n^c. Changing downstream liquidity instead holds q_U fixed and moves q_D at the rate (16), so we can index that experiment by q_D. The curvature restriction we impose next governs it.

The next restriction settles this contest. We state it in terms of elasticities of the primitives u and G, evaluated where the payment constraint binds:

$$\eta(q_D) \equiv -\frac{q_D\, u''}{u'} > 0, \qquad \zeta(q_D \mid q_U) \equiv \frac{q_D\, g''_{DD}}{g'_D} \geq 0, \qquad \varepsilon(q_D \mid q_U) \equiv \frac{q_D\, g''_{UD}}{g'_U}. \tag{19}$$

Here η is the consumer's relative risk aversion, ζ measures how fast the firm's marginal labor requirement rises with output, and ε measures how fast the labor saved by an extra input rises with output. Joint concavity of G makes g jointly convex, so $\zeta \geq 0$, but it does not sign the cross derivative g''_{UD}, which can take either sign. Since $-g'_U = G_1/G_2$, the elasticity ε has the sign of $\partial(G_1/G_2)/\partial h_D$. It is positive, with $g''_{UD} < 0$, if and only if the marginal rate of substitution between the input and labor rises with labor, so that an extra input saves more labor the more the firm produces. This is the case when the input and labor are complements, $G_{12} > 0$, and for any homothetic G with strictly convex isoquants, including the Cobb–Douglas benchmark. A negative ε requires $G_{12} < 0$ and a nonhomothetic technology, in which more labor lowers the marginal product of the input relative to that of labor.

Assumption 2 (Chain curvature)**.** On the payment-constrained regime, regime (i) of Lemma 1,

$$\theta_D\big[\eta(q_D) + \zeta(q_D \mid q_U)\big] \leq \varepsilon(q_D \mid q_U). \tag{20}$$

Assumption 2 asks the input-saving term in (18) to grow with output at least as fast as the payment-relaxation term dies away. The sum $\eta + \zeta$ measures how fast the premium falls as the consumer buys more, and ε measures how fast the input saving grows. The restriction is weakest as $\theta_D \to 0$ and tightest at $\theta_D = 1$, where it reads $\eta + \zeta \leq \varepsilon$. In the benchmark of Appendix B it is a bound on risk aversion that always exceeds one. It restricts a cross partial of Ω_n and is independent of Assumption 1. Lemma 3 below shows that it delivers complementarity in both directions along the chain.

3.2.3 Upstream Market (UM)

Consider a UM meeting between a supplier and a type-n firm. The firm's payment capacity is $w = z_n^f + \bar{d}_U^n$ if the supplier accepts credit and $w = z_n^f$ if not, where z_n^f is its real balances and $\bar{d}_U^n$ its upstream credit limit. Using linearity of the CM value functions, the supplier's surplus from trading (q_U, p_U) is $p_U - c(q_U)$, with $c' = 1/f' > 0$ and $c'' \geq 0$, while the firm's surplus is $\Omega_n(q_U) - p_U$, the expected value of selling downstream net of what it pays for the input. The payment p_U is unconditional and settled in the CM whether or not the firm meets a consumer, so it enters the upstream surplus additively and drops out of the downstream problem. Proportional bargaining with firm share θ_U solves

$$\begin{aligned} \max_{q_U, p_U} \quad & \underbrace{\Omega_n(q_U) - p_U}_{\text{firm's surplus}} \\ \text{s.t.} \quad & \underbrace{p_U - c(q_U)}_{\text{supplier's surplus}} = (1-\theta_U)\, \underbrace{\big[\Omega_n(q_U) - c(q_U)\big]}_{\text{total surplus } S_U^n}, \\ & p_U \leq w, \end{aligned} \tag{21}$$

which gives

$$p_U = (1-\theta_U)\,\Omega_n(q_U) + \theta_U\, c(q_U), \tag{22}$$

and the quantity

$$q_U = \begin{cases} \hat{q}_U^n & \text{if } p_U(\hat{q}_U^n) \leq w \\ q_U^n(w) & \text{otherwise,} \end{cases} \tag{23}$$

where $\hat{q}_U^n$ solves $\Omega_n'(q_U) = c'(q_U)$ and $q_U^n(w)$ solves $(1-\theta_U)\Omega_n(q_U) + \theta_U c(q_U) = w$. Total upstream match surplus is

$$S_U^n(q_U) \equiv \Omega_n(q_U) - c(q_U), \tag{24}$$

split θ_U to the firm and $1-\theta_U$ to the supplier. Equations (17)–(24) make the vertical structure explicit. The upstream surplus contains the downstream surplus, so a single upstream trade carries two bargaining shares and two layers of liquidity constraints. The firm's share θ_U applies to a surplus from which the consumer's share θ_D has already been netted out, and the firm's payment capacity w sits on top of the consumer's ω_n. As Section 6 makes precise, only the downstream share ends up distorting quantities.

3.3 Value Functions in the UM and DM

We can now write the decentralized value functions using the bargaining solutions. Let $q_U^{n,a}$ be the input bought by a type-n firm, where $a = 1$ means the supplier it meets accepts credit, which happens with probability Λ_U, and $a = 0$ means it does not. So $q_U^{n,1} = q_U(z_n^f + \bar{d}_U^n)$ and $q_U^{n,0} = q_U(z_n^f)$ from (23).

Consumers. A consumer meets a firm with probability σ_D, and the firm is of type $n = 1$ with probability Λ_D. Hence

$$V^c(z^c) = z^c + W^c(0,0) + \sigma_D\theta_D\Big\{\Lambda_D\,\mathbb{E}_a\Big[S_D\big(q_D(z^c + \bar{d}_D; q_U^{1,a}); q_U^{1,a}\big)\Big] + (1-\Lambda_D)\,\mathbb{E}_a\Big[S_D\big(q_D(z^c; q_U^{0,a}); q_U^{0,a}\big)\Big]\Big\}, \quad (25)$$

where $\mathbb{E}_a$ averages over the supplier's record keeping with weights $(\Lambda_U, 1-\Lambda_U)$. The first two terms use linearity: whatever the consumer pays in the DM nets out one-for-one against his CM wealth, leaving only the bargaining surplus $\theta_D S_D$. The value function of a defaulter, $\widehat{V}^c(z)$, is identical except that ω_n is replaced by z in every meeting, because a defaulter still meets credit-accepting firms but can only pay them with money.

Firms. A type-n firm holding z^f meets a supplier, and the meeting permits credit with probability Λ_U:

$$V_n^f(z^f) = z^f + W_n^f(0,0,0) + \theta_U\Big[\Lambda_U\,S_U^n\big(q_U(z^f + \bar{d}_U^n)\big) + (1-\Lambda_U)\,S_U^n\big(q_U(z^f)\big)\Big]. \quad (26)$$

There is no separate downstream term in (26). The firm's expected downstream rents, $\Omega_n(q_U)$, are already inside S_U^n through (24). This reflects the within-period liquidity gap. The firm's payoff from the whole chain is determined in the UM, financed by z^f and $\bar{d}_U^n$ alone, before any downstream revenue arrives. The value function of a firm excluded from upstream credit, $\widehat{V}_n^f(z^f)$, replaces the bracketed term with $S_U^n(q_U(z^f))$ in all meetings.

Suppliers. Suppliers hold no money, so

$$V_a^s(0) = W^s(0,0) + (1-\theta_U)\,\mathbb{E}_n\Big[S_U^n\big(q_U^{n,a}\big)\Big] \quad (27)$$

for a supplier of type a, where $\mathbb{E}_n$ averages over the type of firm met with weights $(\Lambda_D, 1-\Lambda_D)$.

3.4 Portfolio Choice

Substituting the decentralized value functions into the CM problems gives the following portfolio problems, up to constants and a factor β.

Consumers.

$$\max_{z^c \geq 0} \; -iz^c + \sigma_D \theta_D \Big\{ \Lambda_D \, \mathbb{E}_a \Big[S_D\big(q_D(z^c + \bar{d}_D; q_U^{1,a}); q_U^{1,a}\big) \Big] + (1 - \Lambda_D) \, \mathbb{E}_a \Big[S_D\big(q_D(z^c; q_U^{0,a}); q_U^{0,a}\big) \Big] \Big\}. \tag{28}$$

The consumer trades off the carrying cost of money against the liquidity services it provides across the meetings he might face. Using (16), the marginal surplus per unit of wealth in a constrained meeting defines the downstream liquidity premium

$$L_D(\omega; q_U) \equiv \left. \frac{u'(q_D) - g_D'(q_D \mid q_U)}{(1 - \theta_D) \, u'(q_D) + \theta_D \, g_D'(q_D \mid q_U)} \right|_{q_D = q_D(\omega; q_U)}, \tag{29}$$

with $L_D = 0$ in meetings where the payment constraint is slack. The premium is the surplus a marginal unit of wealth buys inside the match, that is, the wedge between marginal utility and marginal cost, deflated by the bargained price of a marginal unit. It is large when the consumer is a long way from the surplus-maximizing quantity and vanishes as liquidity accumulates. The first-order condition for $z^c > 0$ is

$$i = \sigma_D \theta_D \Big\{ \Lambda_D \, \mathbb{E}_a \Big[L_D\big(z^c + \bar{d}_D; \, q_U^{1,a}\big) \Big] + (1 - \Lambda_D) \, \mathbb{E}_a \Big[L_D\big(z^c; \, q_U^{0,a}\big) \Big] \Big\}. \tag{30}$$

This is the money demand condition of Lotz and Zhang (2016) with one new feature. The liquidity premium depends on the firm's input stock $q_U^{n,a}$. Consumer money demand therefore depends on upstream conditions: anything that tightens upstream finance lowers q_U, shifts the downstream cost function, and changes the marginal value of consumer money.

Firms. For each type n,

$$\max_{z^f \geq 0} \; -iz^f + \theta_U \Big[\Lambda_U \, S_U^n\big(q_U(z^f + \bar{d}_U^n)\big) + (1 - \Lambda_U) \, S_U^n\big(q_U(z^f)\big) \Big]. \tag{31}$$

Define the upstream liquidity premium

$$L_U^n(w) \equiv \left. \frac{\Omega_n'(q_U) - c'(q_U)}{(1 - \theta_U) \, \Omega_n'(q_U) + \theta_U \, c'(q_U)} \right|_{q_U = q_U(w)}, \tag{32}$$

with $L_U^n = 0$ when the firm is unconstrained. This is the upstream counterpart of (29), except that the numerator contains Ω_n' in place of a marginal utility. The firm's return on money is therefore an option value, priced by the liquidity of the customers it expects to meet. The first-order condition for $z_n^f > 0$ is

$$i = \theta_U \Big[\Lambda_U \, L_U^n\big(z_n^f + \bar{d}_U^n\big) + (1 - \Lambda_U) \, L_U^n\big(z_n^f\big) \Big]. \tag{33}$$

Equation (33) shows how the nominal rate reaches the firm through the chain. The firm's marginal value of money is governed by Ω_n', which by Lemma 1 depends on downstream liquidity $\omega_n = z^c + n\bar{d}_D$ and on (σ_D, θ_D). A higher i hits the firm twice. It raises the cost of carrying z^f. It also reduces consumer balances through (30), which lowers Ω_n' in payment-constrained meetings and with it the firm's return to holding money. Think of the firm's money as working capital whose return depends on whether its customers can pay. A tax on consumer balances is then also a tax on the firm's. Both margins contract upstream trade, and neither is present with a single decentralized market.

The next lemma records the properties we use below. Money demand is well behaved, falls with the nominal rate, and is crowded out by credit, although total liquidity rises with credit access.

Lemma 2. *Under Assumption 1, in the portfolio problems* (28) *and* (31)*:*

(i) the objectives are concave in real balances and an optimal choice exists,

(ii) optimal real balances are nonincreasing in i, and strictly decreasing whenever interior,

(iii) money and credit are substitutes while total liquidity rises with credit access: z^c is nonincreasing in $\bar{d}_D$ and $z^c + \bar{d}_D$ is nondecreasing in $\bar{d}_D$, and likewise z_n^f is nonincreasing in $\bar{d}_U^n$ and $z_n^f + \bar{d}_U^n$ is nondecreasing in $\bar{d}_U^n$.

Money is held only for the liquidity it provides, and demand for it falls with its carrying cost. Credit provides the same services at no carrying cost, so it displaces money at the margin while raising total purchasing power.

A higher nominal rate therefore works through real balances and through the cost of default. Through real balances it lowers consumer money, the marginal value of inputs, and the firm's return to money, and so the value of credit access, since each credit relationship protects a surplus priced by liquidity in the other market. Through the cost of default it makes the defaulter's money-only fallback worse. Section 4.5 sorts out the two.

4 Equilibrium with Endogenous Debt Limits

We now endogenize the credit limits. In each market the borrower repays only if repayment leaves him at least as well off as default followed by exclusion from credit there. The CM value functions are linear, so current wealth drops out of that comparison and the incentive constraint becomes a flow equation in the limit alone.

4.1 Downstream Debt Limit

A consumer repays d_D in the CM if and only if

$$-d_D + W^c(z,0) \geq \widehat{W}^c(z) \quad \Longleftrightarrow \quad d_D \leq \bar{d}_D \equiv W^c(0,0) - \widehat{W}^c(0), \tag{34}$$

where the equivalence uses linearity of both value functions. There are no wealth effects, so the debt limit is independent of the consumer's money holdings. To get a flow equation, substitute (10) and its defaulter analogue into (34), use (25) and $\widehat{V}^c$, and note that $W^c(0,0) - \widehat{W}^c(0) = \bar{d}_D$ reappears inside the continuation values. Collecting terms,

$$r\bar{d}_D = \Psi_D(\bar{d}_D) \equiv \max_{z\geq 0}\left\{-iz + \Sigma_D(z;\bar{d}_D)\right\} - \max_{z\geq 0}\left\{-iz + \Sigma_D(z;0)\right\}, \tag{35}$$

where

$$\Sigma_D(z;d) \equiv \sigma_D\theta_D\Big\{\Lambda_D\,\mathbb{E}_a\Big[S_D\big(q_D(z+d;\, q_U^{1,a});q_U^{1,a}\big)\Big] + (1-\Lambda_D)\,\mathbb{E}_a\Big[S_D\big(q_D(z;\, q_U^{0,a});q_U^{0,a}\big)\Big]\Big\}$$

is the consumer's expected downstream surplus and d is a generic level of credit access, so (35) evaluates Σ_D at $d = \bar{d}_D$ and at $d = 0$. The left side of (35) is the flow return from the loan and the right side is the flow cost of default, the difference between the maximized value of a consumer with credit access and one restricted to money. Writing a self-enforcing limit as a flow condition is standard (Kehoe and Levine 1993; Alvarez and Jermann 2000; Gu, Mattesini, Monnet, and Wright 2013), and the structure is that of Lemma 1 in Lotz and Zhang (2016). What is new is that Ψ_D depends on firms' input stocks $q_U^{n,a}$, and hence on the upstream block. Exclusion cuts a consumer off from sellers who take credit, and how much that hurts depends on how much those sellers have to sell. Proposition 1 holds input stocks fixed, and Section 4.3 takes up this dependence.

Proposition 1 (Determination of the downstream debt limit)**.** *Suppose a consumer who has defaulted holds money and Assumption 1 holds. Then, holding firms' input stocks fixed:*

(i) $\Psi_D(0) = 0$ *and* $\Psi_D'(0) = s_D \equiv \sigma_D\theta_D\Lambda_D\,\mathbb{E}_a\big[L_D(z_0;q_U^{1,a})\big]$*, where* z_0 *is the real balances*

of a consumer who has defaulted. The slope satisfies $s_D \leq i$, with $s_D = i\Lambda_D$ when $\Lambda_D = 1$ or when type-1 and type-0 firms hold the same input stocks, as in the upstream block that goes with $\bar{d}_D = 0$. Moreover Ψ_D is nondecreasing, concave, and bounded, and it is constant for all $\bar{d}_D$ above a satiation level $\hat{d}_D$ at which credit alone finances the bargained quantity in every credit meeting.

(ii) $\bar{d}_D = 0$ *solves* (35) *for all parameter values, so a no-credit equilibrium is self-fulfilling. If $s_D \leq r$, it is the only solution. If $s_D > r$, there is in addition a unique $\bar{d}_D^* > 0$ solving* (35)*.*

(iii) At the positive solution, $\bar{d}_D^*$ is strictly increasing in i whenever $\bar{d}_D^* < \hat{d}_D$, so that the payment constraint still binds in some credit meeting, and the defaulter then holds strictly more money than a consumer in good standing.

The flow cost of default starts at zero with slope s_D. The first unit of credit saves the carrying cost of a marginal unit of money, but only in credit meetings, so the slope is $i\Lambda_D$ when all firms hold the same input stocks. As the limit rises, credit crowds out money and the marginal value of further credit falls, until credit alone finances the desired trade. A positive limit exists when the flow cost starts out above the ray $r\bar{d}_D$, and concavity makes it unique. A higher nominal rate makes the defaulter's fallback worse, which shifts the flow cost up and relaxes the limit.

4.2 Upstream Debt Limit and the Vertical Credit Channel

A type-n firm repays d_U if and only if

$$d_U \leq \bar{d}_U^n \equiv W_n^f(0,0,0) - \widehat{W}_n^f(0,0). \tag{36}$$

The same steps as above, now using (11), (26), and $\widehat{V}_n^f$, deliver

$$r\bar{d}_U^n = \Psi_U^n(\bar{d}_U^n) \equiv \max_{z\geq 0}\left\{-iz + \Sigma_U^n(z;\bar{d}_U^n)\right\} - \max_{z\geq 0}\left\{-iz + \Sigma_U^n(z;0)\right\}, \tag{37}$$

where

$$\Sigma_U^n(z;d) \equiv \theta_U\Big[\Lambda_U\, S_U^n\big(q_U(z+d)\big) + (1-\Lambda_U)\, S_U^n\big(q_U(z)\big)\Big]$$

is the firm's expected upstream surplus with credit access d. A firm that defaults keeps its downstream business but has to finance every input purchase with money, so the cost of default is the compression of the chain that follows.

The new element is inside Ψ_U^n. Through $S_U^n = \Omega_n - c$ and Lemma 1, the flow cost of default to the firm depends on the downstream credit environment $(\bar{d}_D, z^c)$ and on (σ_D, θ_D). Hold the firm's constraint structure fixed and raise $\bar{d}_D$. This raises Ω_1 and steepens it in payment-constrained meetings, which raises the surplus the firm loses by being shut out of upstream credit, and therefore raises $\bar{d}_U^1$. Differentiating (37),

$$\frac{\partial \bar{d}_U^n}{\partial \bar{d}_D} = \frac{\partial \Psi_U^n / \partial \bar{d}_D}{r - \partial \Psi_U^n / \partial \bar{d}_U^n}, \tag{38}$$

which is positive at any stable solution, where $r > \partial \Psi_U^n / \partial \bar{d}_U^n$, provided downstream credit raises the marginal value of upstream credit access. Since the upstream liquidity premium (32) is strictly increasing in Ω_n', we need downstream liquidity to raise the marginal value of inputs. Assumption 2 delivers this, along with its mirror image in the other market.

Lemma 3 (Complementarity along the chain)**.** *Suppose Assumption 2 holds. Then, on the payment-constrained regime:*

(i) Ω_n' is nondecreasing in the consumer's payment capacity ω_n, holding the firm's input stock q_U fixed. Hence $L_U^1(w)$ is nondecreasing in $\bar{d}_D$ for every w, and $L_U^1(w) \geq L_U^0(w)$ for all w whenever $\bar{d}_D > 0$.

(ii) L_D, evaluated along the binding payment constraint at fixed consumer wealth, is nondecreasing in the firm's input stock q_U.

Part (i) says liquid consumers make inputs more valuable to firms. Part (ii) says well-stocked firms make liquidity more valuable to consumers, because a bigger input stock lowers the bargained payment. Neither is automatic: more downstream liquidity strengthens the input-saving term in (18) and weakens the payment-relaxation term, and Assumption 2 requires the first to dominate. Where it fails, both directions reverse together and retail credit tightens supplier credit. Signing the response of the upstream limit when only some suppliers keep records takes one more restriction, Assumption E.1 in Appendix E, because the firm reoptimizes its money holdings as the downstream limit moves.

Proposition 2 (Upstream debt limits and the vertical credit channel)**.** *Fix the downstream environment $(\bar{d}_D, z^c)$ and suppose a firm that has defaulted holds money and Assumption 1 holds. Then:*

(i) For each n, Ψ_U^n has the properties of Proposition 1(i) with $\Psi_U^{n\prime}(0) = i\Lambda_U$. The origin always solves (37), and a unique positive limit $\bar{d}_U^{n*} > 0$ exists if and only if $i\Lambda_U > r$, in which case it is strictly increasing in i whenever $\bar{d}_U^{n*}$ lies below the firm's satiation level.

(ii) Suppose in addition Assumption 2 holds and $\Lambda_U = 1$. Then $\bar{d}_U^{1*} \geq \bar{d}_U^{0*}$: a firm that accepts downstream credit sustains a weakly higher upstream credit limit. Moreover, at any stable positive solution, $\partial \bar{d}_U^{1*}/\partial \bar{d}_D \geq 0$ as in (38), so conditions at the retail end of the chain support credit at the supplier end.

(iii) For $\Lambda_U \in (0, 1]$, the conclusions of part (ii) hold under Assumptions 2 and E.1.

Proposition 2 is the vertical credit channel. The firm's promise to its supplier is disciplined by how much its downstream business shrinks when it must finance inputs with money alone, so anything that raises the value of that business raises the upstream limit. The proof shows where the channel operates. While the firm still holds money, a marginal unit of upstream credit is worth only the carrying cost i it saves, so customers matter for supplier credit only once the credit line exceeds the firm's money demand. Nothing here uses money or bargaining. The channel needs only that punishment be exclusion at the link where the default happens and that one agent span both links, and similar interdependence appears in Gottardi and Zhang (2026) and Gong, Qiao, Wong, and Wright (2025). Money and the pricing protocol determine its size and sign.

Propositions 1 and 2 imply a simple threshold, because self-enforcing trade credit needs the defaulter's money-only fallback to be unattractive enough.

Corollary 1 (A threshold for self-enforcing credit)**.** *(i) In any stationary equilibrium in which money is valued, $\bar{d}_U^n > 0$ requires $i > r/\Lambda_U$, and $\bar{d}_D > 0$ requires $s_D > r$ and hence $i > r$. Positive downstream limits branch off zero at $i = r/\Lambda_D$, and at $\Lambda_D = 1$ this threshold is exact. As $i \to 0$ trade credit collapses at both stages and all exchange along the chain is monetary.*

(ii) If $\Lambda_k = 1$, any positive limit at link k exceeds the borrower's total-liquidity target, so a borrower in good standing at that link holds no money.

The corollary concerns only the part of trade credit sustained by the threat of exclusion. Supplier credit is also supported by collateral, legal enforcement, and relationship rents, which can sustain it at low nominal rates. Part (ii) says that universal record keeping at a link leaves no composition of payments to move there, which is why interior record keeping matters for the policy results below.

4.3 The Fixed-Point System of Debt Limits

Figure 3 shows the credit block when $\Lambda_D = \Lambda_U = 1$, a benchmark we explain in Section 4.4. There is then a single firm type, and every supplier accepts credit. The curve UU

gives the upstream limit supported by a given downstream limit, $\bar{d}_U = \Phi_U(\bar{d}_D)$, and DD gives the downstream limit supported by a given upstream limit, $\bar{d}_D = \Phi_D(\bar{d}_U)$. Under the complementarity conditions both curves slope up. More downstream credit raises what firms lose from upstream exclusion, and more upstream credit raises input stocks and with them what consumers lose from downstream exclusion. Each curve picks the largest self-enforcing limit given the other, and steady states with credit active at both links are where they cross. The other solutions, in which beliefs shut credit down at one link or both, sit on the axes: at the origin and at the two intercepts.

The downstream limit also moves input stocks directly, because firms bargain upstream with their downstream revenue in view, so the downstream reaction has to be consistent with the input stocks it induces. Write $\phi(\bar{d}_U, y)$ for the positive solution of (35) at the input stocks that go with an upstream limit $\bar{d}_U$ and a downstream limit y. The downstream limit supported by $\bar{d}_U$ solves $y = \phi(\bar{d}_U, y)$, and we let $\Phi_D(\bar{d}_U)$ be the largest solution. When Φ_D is differentiable,

$$\Phi_D'(\bar{d}_U) = \frac{\phi_1}{1-\rho}, \qquad \rho \equiv \frac{\partial \phi}{\partial y}, \tag{39}$$

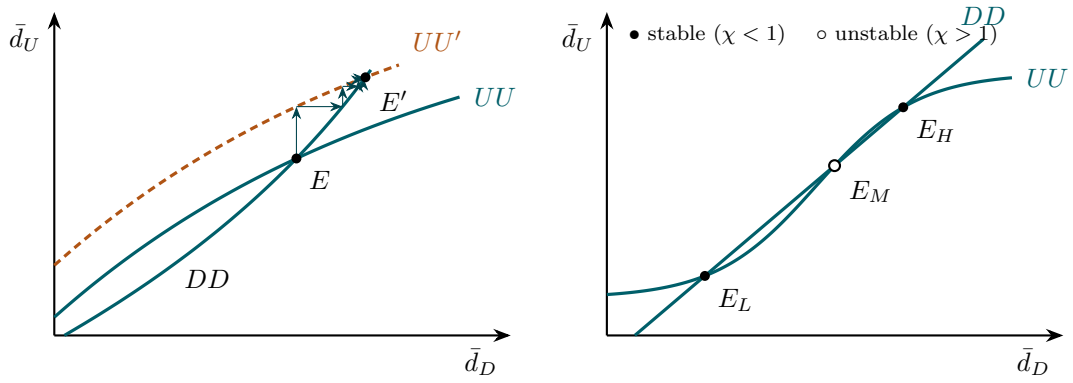
where $\phi_1 \equiv \partial\phi/\partial\bar{d}_U$ is the effect through the upstream limit and ρ is the effect of the downstream limit on its own reaction through input stocks, with $\rho \leq 1$ at the largest solution. The vertical feedback χ is the slope of the composite map $\bar{d}_D \mapsto \Phi_D(\Phi_U(\bar{d}_D))$.

Panel (a) of Figure 3 illustrates the vertical multiplier. A higher nominal rate shifts UU up. The first-round increase in the upstream limit raises input purchases and downstream surplus, which relaxes the downstream limit and with it the upstream limit again, and the total change exceeds the initial shift by the factor $1/(1-\chi)$. A steady state is stable when $\chi < 1$, so that this staircase converges to it. Panel (b) shows multiple steady states when the feedback is strong.

Corollary 2 (Payment constraints and the vertical feedback)**.** *Consider a steady state with positive limits.*

(i) If consumers in good standing are unconstrained in credit meetings at every input stock up to the one firms buy in good standing, the upstream limits do not respond to $\bar{d}_D$. If firms in good standing are unconstrained in upstream credit meetings, the downstream limit does not respond to the upstream limits. In either case $\chi = 0$, and the limits at the two links are determined separately.

(ii) Suppose a defaulter at link k holds money. Along a branch of positive limits at link k, as the nominal rate approaches the sustainability threshold of Corollary 1, borrowers in good standing at link k become payment constrained in credit meetings.



(a) unique positive steady state and the vertical multiplier

(b) multiple steady states

Figure 3: The fixed-point system of debt limits ($\Lambda_D = \Lambda_U = 1$)

(iii) *Let $\Lambda_U \in (0, 1)$, and suppose the upstream premium is twice continuously differentiable on the constrained range. Along a branch of positive upstream limits that leaves zero at $i\Lambda_U = r$, the response of a firm's upstream limit to its customers' payment capacity vanishes with the limit: $\partial \bar{d}_U^n / \partial \omega_n = O(\bar{d}_U^n)$.*

If retail credit already pays for everything consumers want to buy, more of it does not change what firms sell, and the value of their business stays put. The same holds upstream. Part (ii) says both constraints bind near the sustainability thresholds, where credit lines are small, and part (iii) says the feedback is weak there too. A small credit line mostly replaces money the firm would otherwise carry, so what its customers can pay changes its value only at second order. The feedback is cut where credit lines are large and vanishes where they are small. It can even be negative. At $\Lambda_U < 1$ a firm with more upstream credit carries less money for its cash-only suppliers and buys less from them, and if those meetings carry enough weight the downstream reaction falls with the upstream limit.

4.4 Stationary Equilibrium: Existence and Multiplicity

Given (Λ_U, Λ_D), a stationary monetary equilibrium is a list

$$\left(z^c, \{z_n^f\}_n, \{q_U^{n,a}\}_{n,a}, \{q_D^{n,a}\}_{n,a}, \bar{d}_D, \{\bar{d}_U^n\}_n \right)$$

in which some agent in good standing holds money, $z^c > 0$ or $z_n^f > 0$ for some n, such that quantities and payments solve the bargaining problems (14)–(15) and (22)–(23), real balances solve the portfolio conditions (30) and (33), and the debt limits solve (35) and (37).

Parts (ii) and (iii) of the next proposition set $\Lambda_D = \Lambda_U = 1$, so there is a single pair

$(\bar{d}_D, \bar{d}_U)$ and Figure 3 can be drawn in the plane. This corner is a benchmark for the credit block and not a monetary steady state, since nobody in good standing carries money once both limits are positive (Corollary 1(ii)). We therefore solve the flow equations taking as given that a defaulter can self-finance at cost i. At interior Λ_D the vertical channel is type specific. Retail credit crowds out consumer money, and cash-only meetings are the ones type-0 firms are in, so a higher $\bar{d}_D$ lowers $\bar{d}_U^0$ while it raises $\bar{d}_U^1$. Corollary E.1 in Appendix E shows that the credit block stays monotone once the order on $\bar{d}_U^0$ is reversed, and Section 5 measures the gap between the two types.

Proposition 3 (Existence and multiplicity)**.** *Suppose Assumption 1 holds.*

(i) Let $\bar{\imath} \equiv \min\{\sigma_D\theta_D/(1-\theta_D),\, \theta_U/(1-\theta_U)\}$. For all $i \in (0, \bar{\imath})$ a stationary monetary equilibrium with $\bar{d}_D = \bar{d}_U^n = 0$ exists: a pure-money supply chain in which $z^c > 0$ and $z_n^f > 0$.

(ii) Let $\Lambda_D = \Lambda_U = 1$ and $r < i < \bar{\imath}$, and suppose Assumption 2 holds. Given $\bar{d}_D$, the upstream flow equation has exactly one positive solution, $\Phi_U(\bar{d}_D)$. Given $\bar{d}_U$, let $\Phi_D(\bar{d}_U)$ be the largest solution of $y = \phi(\bar{d}_U, y)$, the downstream reaction of Section 4.3, and let $T(\bar{d}_D, \bar{d}_U) \equiv \big(\Phi_D(\bar{d}_U), \Phi_U(\bar{d}_D)\big)$. Both reactions are strictly positive and nondecreasing. The fixed points of T form a nonempty complete lattice, all of them are strictly positive, and the greatest, $(\bar{d}_D^+, \bar{d}_U^+)$, is the greatest solution of the system (35)–(37). If $\rho < 1$ wherever it is defined, $\Phi_D(\bar{d}_U)$ is the only positive downstream limit consistent with $\bar{d}_U$, and the solutions of the system are the fixed points of T together with the pessimistic solutions $(0,0)$, $(0, \Phi_U(0))$, and $(\Phi_D(0), 0)$, in which beliefs shut credit down at one link or both. Otherwise there can also be solutions in which the downstream limit is a smaller solution of $y = \phi(\bar{d}_U, y)$.

(iii) Let $\chi(\bar{d}_D) \equiv \Phi_U'(\bar{d}_D)\, \Phi_D'\big(\Phi_U(\bar{d}_D)\big)$ denote the slope of the composite reaction map, with Φ_D' given by (39). If $\chi > 1$ at some fixed point of T, then T has at least two further fixed points, one below it and one above it, and generically the fixed points alternate between stable ($\chi < 1$) and unstable ($\chi > 1$), as in Figure 3(b). Conversely, if $\rho < 1$ wherever it is defined and $\chi < 1$ at every fixed point of T, the fixed point is unique.

Part (i) shows that a chain financed entirely with money exists at low nominal rates. Part (ii) sorts the steady states into two kinds. The pessimistic ones come from the one-market model: if nobody expects to be repaid at a link, nobody lends there. The fixed points of T are where the chain shows up, and they have a worst and a best element. Part (iii) locates the source of multiplicity. With one credit market the positive solution is unique

(Proposition 1(ii)), so any second steady state with credit at both links must come from the feedback between the links, and by Corollary 2 that feedback needs payment constraints that bind at both links. A little legal enforcement moves the pessimistic solutions from zero to the enforced amount without removing them, as wherever debt limits rest on beliefs about future limits (Gu, Mattesini, Monnet, and Wright 2013).

Proposition 4 (Money contains credit collapses)**.** *Let $\Lambda_D = \Lambda_U = 1$, suppose Assumption 1 holds and $r < i < \bar{\imath}$, and compare the monetary chain with the nonmonetary chain of Appendix D.5, where a defaulter is excluded from trade at the link where it defaulted.*

(i) With money, a collapse of credit at either link leaves credit at the other link positive: $\Phi_U(0) > 0$ and $\Phi_D(0) > 0$.

(ii) Without money, a collapse of credit at either link shuts down the chain: $\Phi_U(0) = \Phi_D(0) = 0$, and the origin is a fixed point of T.

(iii) Holding conditions at the other link fixed, the flow cost of default at each link is weakly lower with money than without it, so each debt limit is weakly lower.

A firm whose customers lose their credit still sells to them for cash, so it still has a downstream business to lose by defaulting on its supplier, and consumers who face firms without supplier credit still find goods to buy. Without money neither is true, and a collapse at either link brings down the other. The money-only fallback that keeps a collapse local is also the defaulter's outside option, and it lowers every limit. In the benchmark of Section 5 the positive steady state is unique at every rate and parametrization we compute, so the steady states the chain produces at every parameter value are the pessimistic ones.

4.5 Monetary Policy and the Vertical Multiplier

Write the interior solutions of the two flow equations as reaction functions, $\bar{d}_U = B_U(\bar{d}_D, i)$ and $\bar{d}_D = B_D(\bar{d}_U, i)$, so $B_{U,i}$ and $B_{D,i}$ are the direct effects of the nominal rate holding the other limit fixed and $\chi \equiv B'_U B'_D$. Each flow cost depends on i directly, through the borrower's own money demand, because inflation makes the defaulter's fallback worse. We call this the own-market effect. It also depends on i through the other side of the trade. The upstream flow cost depends on downstream liquidity $\omega_D \equiv z^c + \bar{d}_D$, and the downstream flow cost on firms' input stocks, and at fixed limits both move with i through real balances. We call this the cross-market effect. It runs through a two-sided money demand system, and Lemma E.1 in Appendix E shows that when the product of the cross-to-own sensitivities of

the two money demands, ξ, is below one, the chain amplifies the inflation tax on each side by at least $1/(1-\xi)$. A steady state exhibits own-market dominance if $B_{U,i} \geq 0$ and $B_{D,i} \geq 0$.

Proposition 5 (The nominal rate and credit along the chain)**.** *Let $\Lambda_D, \Lambda_U \in (0,1]$, not both equal to one, suppose Assumptions 1, 2, and E.1 hold, and consider a regular interior steady state of the credit block with $\chi < 1$. Regularity also requires that real balances on both sides be nonincreasing in i once the two-sided portfolio system* (30) *and* (33) *adjusts, for which $\xi < 1$ is sufficient by Lemma E.1.*

(i) Decomposition. Holding the other limit fixed, the response of each flow cost to the nominal rate decomposes into an own-market and a cross-market incentive effect,

$$\frac{\partial \Psi_D}{\partial i} = \underbrace{z^c(0) - z^c(\bar{d}_D)}_{own \geq 0} + \underbrace{\frac{\partial \Psi_D}{\partial q_U} \cdot \frac{dq_U}{di}}_{cross}, \qquad \frac{\partial \Psi_U}{\partial i} = \underbrace{z^f(0) - z^f(\bar{d}_U)}_{own \geq 0} + \underbrace{\frac{\partial \Psi_U}{\partial \omega_D} \frac{d\omega_D}{di}}_{cross \leq 0}, \tag{40}$$

where $z(d)$ denotes optimal real balances given credit access d and q_U is the vector of firms' input stocks. The own-market terms are nonnegative because inflation falls more heavily on the defaulter, who self-finances with money. The upstream cross-market term is nonpositive. The downstream one is nonpositive at $\Lambda_D = 1$ and ambiguous at interior Λ_D, because the input stocks of cash-only firms enter Ψ_D with the opposite sign.

(ii) The vertical multiplier. The equilibrium responses are

$$\frac{d\bar{d}_U}{di} = \frac{B_{U,i} + B_U' B_{D,i}}{1-\chi}, \qquad \frac{d\bar{d}_D}{di} = \frac{B_{D,i} + B_D' B_{U,i}}{1-\chi}. \tag{41}$$

Each limit responds through its own direct effect plus the pass-through of the other's, scaled by the vertical multiplier $1/(1-\chi)$, whatever the signs of the direct effects.

Proposition 5 takes the steady state as given, and Section 5 computes steady states with partial record keeping at both links. Under own-market dominance real balances fall with i while both limits rise, so the share of payments financed by credit rises with inflation. If dominance fails, higher inflation contracts the limits and the same multiplier scales the contraction. In Lotz and Zhang (2016) the cross-market term is absent, the sign is unambiguous, and $\chi = 0$.

4.6 Coexistence and Own-Market Dominance

Where does own-market dominance hold? The own-market term is the money a defaulter would have to add to replace the lost credit line, so it cannot exceed the borrower's money demand. The cross-market term needs money to be held on the other side of the trade. Define the demonetization thresholds

$$\bar{\imath}_D^z \equiv \frac{\sigma_D \theta_D}{1-\theta_D}, \qquad \bar{\imath}_U^z \equiv \frac{\theta_U}{1-\theta_U}, \tag{42}$$

the nominal rates above which consumers and firms stop holding money, since the premia are bounded by $1/(1-\theta_D)$ and $1/(1-\theta_U)$. They are also the arguments of the bound $\bar{\imath}$ in Proposition 3(i).

Near the sustainability threshold the limit is small, and a defaulter must replace almost the whole credit line with money. One can check, by the expansion behind Corollary 2(iii), that the own-market term is then of first order in the limit and the cross-market term of second order, so higher inflation relaxes credit and shifts payments toward it. As the rate rises, borrowers economize on balances, and the base of the own-market effect shrinks while the credit-financed region grows. Once the borrower's own side of the chain stops holding money, the own-market effect is gone, and further inflation contracts credit wherever the other side is still payment constrained. Each market therefore has a switch point i_k^s between its sustainability threshold and its demonetization threshold. By the argument behind Corollary 2(i), the cross-market effect on a firm's upstream limit requires its customers to be payment constrained. A firm that sells for cash always faces such customers, while a firm that sells on credit faces them only until retail credit covers their purchases, so the switch point comes earlier for firms whose customers pay with money. Section 5 locates it.

5 The Benchmark Economy

Several results of Section 4 at interior record keeping rest on restrictions that shut channels down (Appendix E), and the multiplier and the reversal of the effect of policy both need payment constraints that bind. We now solve the model with partial record keeping at both links, where money and trade credit coexist on both sides of the chain and every channel operates at once. The computation measures the vertical channel, the strength of the feedback where both payment constraints bind, and the location of the switch point of Section 4.6.

Table 1: Benchmark statistics across parametrizations

	Feedback χ where both constraints bind	Supplier credit ratio $\bar{d}_U^1/\bar{d}_U^0$ at $i = 10\%$	Switch point i^s for firms selling for cash (%)	Cash side demonetizes (%)
Baseline	-0.01 to 0.37	3.1	16	32
$\Lambda_U = 0.6$	-0.03 to 0.00	3.5	16	40
$\Lambda_U = 0.9$	0.00 to 0.35	2.2	21	36
$\theta_D = 0.7$	0.00 to 0.24	1.9	42	75
$\theta_U = 0.3$	0.00 to 0.37	2.4	19	32
$\sigma_D = 0.5$	-0.09 to 0.08	8.7	9	20
$\alpha = 0.25$	-0.01 to 0.03	2.7	17	32
$\eta = 0.8$	0.00 to 0.13	1.9	29	>31

Each row changes one parameter from the baseline ($\Lambda_D = 0.6$, $\Lambda_U = 0.8$, $\theta_D = \theta_U = 0.5$, $\sigma_D = 1$, $\alpha = 0.4$, $\eta = 0.5$, $r = 0.04$). The feedback column reports the range of χ on a grid of rates spaced 0.15 to 0.3 percentage points within the window where both payment constraints bind, and $\chi = 0$ outside it. The demonetization rate extrapolates consumer balances to zero. For $\eta = 0.8$ the solver does not converge above 31 percent, where consumer balances are still positive. We also computed $\Lambda_U \in \{0.7, 0.75, 0.85\}$ and variations in each parameter around $\Lambda_U = 0.6$, and χ stays between -0.33 and 0.39 in all of them.

5.1 Parametrization

We use the benchmark of Appendix B, with $u(q) = q^{1-\eta}/(1-\eta)$, $G(q_U, h_D) = q_U^{\alpha} h_D^{1-\alpha}$, and $f(h_U) = h_U$. A period is a year and $r = 0.04$. We set $\eta = 0.5$, which satisfies Assumption 2 by Lemma B.1, $\alpha = 0.4$, symmetric bargaining shares $\theta_D = \theta_U = 1/2$, and $\sigma_D = 1$. Suppliers keep records more often than retailers, $\Lambda_U = 0.8$ and $\Lambda_D = 0.6$, so self-enforcing supplier credit requires a nominal rate above 5 percent and self-enforcing retail credit a rate above 6.7 percent. These choices illustrate the mechanism and are not a calibration. Table 1 reports the main statistics when one parameter at a time is changed, and Appendix B.2 describes the solution method.

5.2 The Vertical Channel Across Firms

Figure 4(a) plots the three debt limits against the nominal rate, on a log scale. Retail credit becomes self-enforcing at 6.7 percent and grows quickly with the rate. Supplier credit is positive at both types of firm, but the two types diverge as soon as retail credit appears. At a nominal rate of 10 percent, firms that sell on account have a supplier credit limit 3.1 times that of firms that sell for cash. The ratio is 4.9 at 15 percent and 8.4 at 20 percent. The two types differ only in whether they keep records on their customers, so the gap is the vertical channel of Proposition 2 at work. A firm that sells on account faces customers who can pay more, so its downstream business is worth more and it has more to lose by defaulting on its supplier. A firm that sells for cash faces customers whose money retail credit has crowded

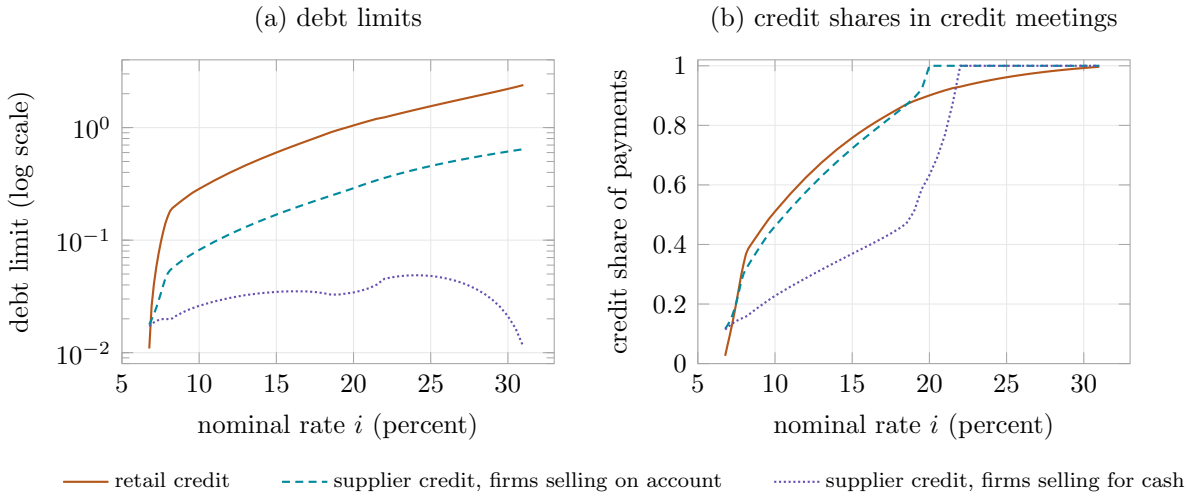


Figure 4: Debt limits and credit shares in the benchmark

out, and its supplier credit grows slowly.

Panel (b) shows the composition of payments in credit meetings. The share financed by credit rises with the nominal rate at both links. Retail credit finances 34 percent of payments in credit meetings at 8 percent, half at 10 percent, and 90 percent at 20 percent. Supplier credit finances all input purchases in credit meetings once the rate reaches 20 percent for firms that sell on account and 22 percent for firms that sell for cash. Above those rates firms in good standing carry no money at all. With 80 percent of suppliers keeping records, money is useful to them only in the remaining fifth of upstream meetings, and the premium there is bounded by $\theta_U(1-\Lambda_U)/(1-\theta_U) = 0.2$.

5.3 The Strength of the Feedback

Figure 5(a) plots χ. Both payment constraints bind only between 6.7 and 8.2 percent. Within that window χ is close to zero just above the threshold, as Corollary 2(iii) predicts, rises to a peak of 0.37 at a rate of 7.8 percent, and falls back as retail credit approaches what consumers want to buy. At the peak the vertical multiplier is 1.6 and retail credit finances 30 percent of payments in credit meetings. Above 8.2 percent retail credit covers what consumers want to buy from firms that sell on account, the upstream limits stop responding to it, and $\chi = 0$ by Corollary 2(i). At the rate of peak feedback the composite reaction map $\bar{d}_D \mapsto \Phi_D\big(\Phi_U^0(\bar{d}_D), \Phi_U^1(\bar{d}_D)\big)$ crosses the diagonal once, with slope below 0.4 everywhere on the range we compute, so the positive steady state is unique.

Table 1 shows the same pattern across parametrizations. In every variation we compute, χ stays between -0.33 and 0.39, and it is zero once retail credit covers what consumers

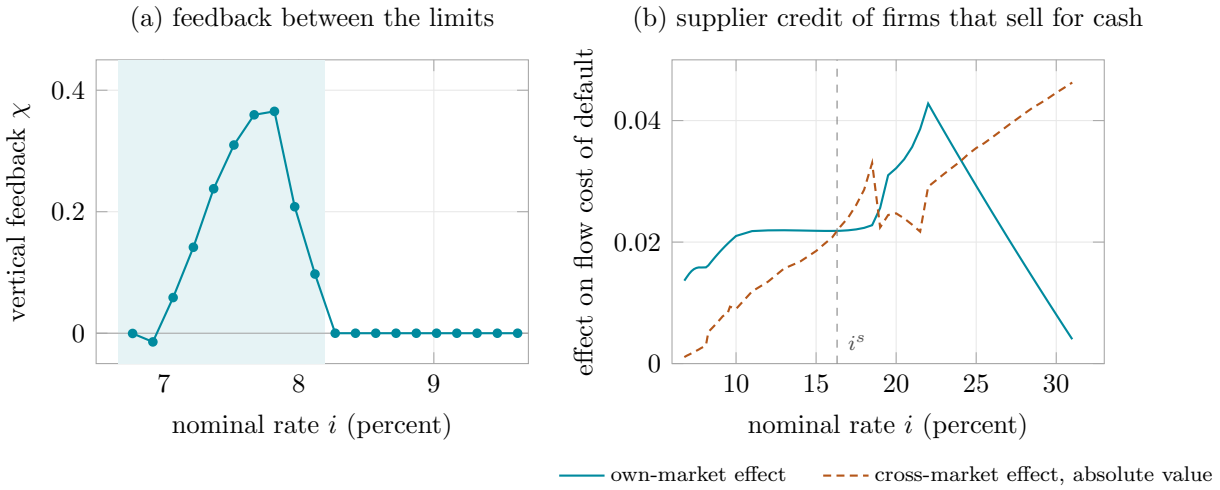


Figure 5: Vertical feedback and the effects of the nominal rate in the benchmark. Shading in panel (a) marks the rates at which both payment constraints bind.

want to buy at firms that sell on account, which happens within a few percentage points of the sustainability threshold. The sign depends mostly on upstream record keeping. With $\Lambda_U = 0.6$, a firm with more supplier credit carries less money for its cash-only suppliers and buys less from them, and χ is negative or zero throughout the window. With more suppliers keeping records that channel weakens, and from $\Lambda_U = 0.75$ up χ is positive over most of the window. No parametrization comes close to $\chi = 1$, so the multiplicity of Proposition 3(iii) does not arise in the benchmark. Where the chain matters quantitatively here is in the cross section of Figure 4, through the gap between firms that sell on account and firms that sell for cash.

5.4 Monetary Policy Along the Chain

Retail credit and the supplier credit of firms that sell on account rise with the nominal rate at every rate we compute. For both, the own-market effect in (40) dominates throughout. For retail credit the cross-market term is positive. At fixed limits a higher rate leads firms to carry less money, so firms that sell for cash, and the cash purchases of all firms, shrink. A consumer who defaults has to shop with cash, partly at those firms, so a defaulter's money buys less and default becomes more costly. This is the loop through cash-only firms described in Appendix E. For firms that sell on account the cross-market term vanishes once retail credit covers their customers' purchases, as Corollary 2(i) implies.

Firms that sell for cash are different, because their customers pay with money and are payment constrained at every rate. Figure 5(b) decomposes the effect of the nominal rate on

their flow cost of default. The own-market effect is roughly flat between 11 and 18 percent, while the cross-market effect grows steadily as inflation drains their customers' balances. The two cross at $i^s = 16.3$ percent, and the supplier credit of these firms stops growing there. Above i^s the path depends on the firms' own money holdings. Between 19 and 22 percent firms in good standing run down the last of their money, so a defaulter would have to replace the entire credit line with money. The own-market effect jumps, and the limit recovers until 24 percent. Beyond 24 percent the cross-market effect dominates again, and the limit falls steeply as the cash side of the chain demonetizes near 32 percent. The kinks in Figure 5(b) mark the rates at which each type of firm stops carrying money. Meanwhile the supplier credit of firms that sell on account keeps rising, so high inflation widens the gap between the two types, from a factor of 3.1 at 10 percent to 29 at 30 percent.

Consumers stop carrying money near 32 percent, far below $\bar{\imath}_D^z = 1$. Once retail credit covers purchases at firms that sell on account, consumers carry money only for cash meetings, which occur with probability $1-\Lambda_D$, so the expected premium on consumer money is bounded by $\sigma_D\theta_D(1-\Lambda_D)/(1-\theta_D) = 0.4$. The money demands of consumers and of firms that sell for cash also support each other through the portfolio feedback of Lemma E.1, and the monetary solution of the portfolio system disappears before that bound is reached. We do not report results above this rate.

6 Optimal Monetary Policy

Given that policy has opposing effects on credit, what should it do? Money transfers and debt repayments net out across agents, and record keeping is costless for the sellers who operate it, so steady-state welfare is expected chain surplus,

$$\mathcal{W}(i) \equiv \sum_n \pi_n \, \mathbb{E}_a\big[\sigma_D S_D(q_D^{n,a}; q_U^{n,a}) - c(q_U^{n,a})\big], \tag{43}$$

evaluated at stationary equilibrium quantities, where $\pi_1 = \Lambda_D$ and $\pi_0 = 1 - \Lambda_D$ are the population shares of the two firm types and $\mathbb{E}_a$ averages over the supplier's record keeping as in (25). Write $\mathcal{V}(q) \equiv \sigma_D S_D(q_D^*(q); q) - c(q)$ for chain surplus when the downstream quantity is unconstrained, so that $\mathcal{V}$ is the planner's objective (6), and let $\hat{q}_U$ denote the input trade at the Friedman rule, which solves $\Omega' = c'$ with the downstream constraints slack.

Proposition 6 (Optimal monetary policy)**.** *Suppose Assumption 1 holds and the planner's objective is strictly concave, as it is in the benchmark of Appendix B.*

(i) As $i \to 0$, every payment constraint becomes slack, all trade is monetary by Corollary 1,

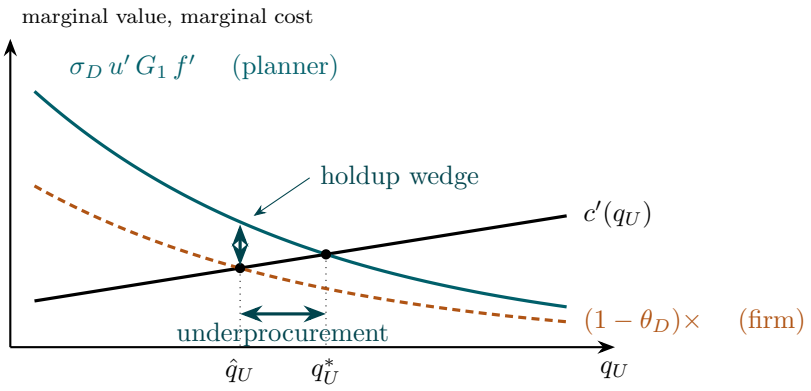


Figure 6: The vertical holdup wedge

the allocation in every realized chain converges to $\big(\hat{q}_U, q_D^*(\hat{q}_U)\big)$*, and welfare converges to* $\mathcal{W}(0) = \mathcal{V}(\hat{q}_U)$*.*

(ii) Within the class of unconditional input contracts, no monetary policy attains the first best. For every $i \geq 0$*, either a downstream constraint binds in a positive measure of meetings, so* $u' > g_D'$ *there and* (7) *fails, or no downstream constraint binds, in which case* Ω' *carries the factor* $1-\theta_D$ *and input trade satisfies* $q_U \leq \hat{q}_U < q_U^*$*.*

(iii) Under Assumption 2, input trade never exceeds the Friedman rule benchmark, $q_U^{n,a} \leq \hat{q}_U$ *in every upstream meeting, and* $\mathcal{W}(i) \leq \mathcal{W}(0)$ *for all* $i \geq 0$*, with strict inequality whenever a payment constraint binds in a positive measure of meetings. The Friedman rule is therefore optimal despite extinguishing self-enforcing trade credit.*

The Friedman rule closes every liquidity wedge at no cost. Total liquidity is at satiation on both sides, and the credit that positive rates make sustainable only substitutes for money whose carrying cost the policy itself created. The holdup wedge survives. At the Friedman rule the downstream trade maximizes match surplus, but the surplus is still split, and only the firm's slice, $(1-\theta_D)S_D$, reaches the upstream meeting. Figure 6 illustrates. The planner equates the expected marginal value of inputs, $\sigma_D u' G_1 f'$, to marginal cost, while the firm equates the same schedule scaled by $1-\theta_D$. Positive rates move the economy further from q_U^* through the payment constraints, and by part (iii) never toward it. Part (iii) uses Assumption 2. Constrained customers make inputs more valuable at the margin, through the factor $1+\theta_D L_D$ in (18), but the input saving is evaluated at a smaller downstream quantity, and the assumption makes this second force dominate.

The wedge cannot be bargained away. Efficient input purchases require $\theta_D \to 0$, but consumers hold money only to capture downstream surplus, and as $\theta_D \to 0$ the return on consumer money vanishes. The same parameter gives money its value and creates the

wedge. The wedge is a holdup on an investment made before the meeting in which its return is divided (Grout 1984). In monetary economies the familiar case is the buyer's money (Lagos and Wright 2005; Berentsen, Rocheteau, and Shi 2007), and closer to ours is the holdup on capital in Aruoba, Waller, and Wright (2011). What the chain adds is where the parties sit. The firm buys its input from a third party in an efficient bargain, and the consumer whose share is missing from its valuation is not in that meeting, so the remedy lies in the form of the input contract rather than in the division of surplus or the cost of liquidity.

6.1 The Pricing Protocol

The welfare result requires that the firm bargain downstream with market power and that its upstream obligation be unconditional. If either fails, the wedge closes. Under price taking, consumers equate u' to the final-good price, firms equate that price to g'_D, the firm values inputs as the planner does, and the Friedman rule attains the first best, as in Rocheteau and Wright (2005). For trade credit, though, market power is the relevant case, since terms of sale reflect bargaining positions (Klapper, Laeven, and Rajan 2012; Murfin and Njoroge 2015). The model then predicts a larger upstream distortion where downstream buyers have more power.

Price taking removes bargaining but leaves exclusion intact, as in Kehoe and Levine (1993). The flow equations (35) and (37) need only that the defaulter's continuation value fall short of the borrower's and that the gap depend on the other link. The flow cost of default would still depend on firms' input stocks, now through the price of the final good, so a competitive version delivers the vertical channel, the multiplier, and the decomposition of the effect of the nominal rate. Bargaining does matter for the demonetization thresholds, which bound the premia by $1/(1-\theta_k)$. Under price taking money is valued at every finite i, so the failure of own-market dominance at high rates becomes parametric rather than guaranteed.

6.2 The Form of the Input Contract

The upstream payment p_U in (21) is unconditional: the firm owes it whether or not it meets a consumer. By the time the firm bargains downstream the payment is sunk and does not enter the downstream surplus, so the consumer takes the share θ_D of all of it. This is where the wedge comes from. Under consignment the supplier is paid only if the firm sells downstream, the trading protocol Rubinstein and Wolinsky (1987) contrast with outright purchase. The payment is then a cost of completing the downstream sale, it enters the downstream surplus,

and firm and supplier jointly recover part of what the consumer would otherwise extract. Appendix C shows that the firm's upstream surplus becomes proportional to the planner's objective $\mathcal{V}(q_U)$, so consignment attains the first best for every bargaining split, technology, and matching probability.

Consignment is not available here. It conditions on whether the firm sold downstream, but record keeping is link specific, so a supplier observes the firm's repayment to it and not its trades with consumers. Even with verification the consignment payment is unsecured, so it must satisfy $p_U \leq \bar{d}_U$ in the selling state, and by Corollary 1 that bound is zero near the Friedman rule, where the contract would be needed most. The model therefore predicts a smaller vertical wedge where suppliers observe downstream sales, as under vendor-managed inventory.

7 Extensions

We have kept the model simple: record keeping is a fixed characteristic of sellers, IOUs cannot be transferred, the chain has two links, and the only instrument is money growth.

7.1 Endogenous Record Keeping

Endogenizing record keeping follows Lotz and Zhang (2016). Suppose each seller draws a per-period record-keeping cost from a nondegenerate distribution F_k. A supplier with cost κ_U invests if $\kappa_U \leq \bar{\kappa}_U \equiv (1-\theta_U)\,\mathbb{E}_n\big[S_U^n(q_U^{n,1}) - S_U^n(q_U^{n,0})\big]$, so $\Lambda_U = F_U(\bar{\kappa}_U)$, and similarly for firms. If F_k has positive density on $(0,\infty)$, then $\Lambda_k \in (0,1)$ whenever credit at link k relaxes a binding constraint, so the coexistence of money and credit becomes an equilibrium outcome and the results of Sections 3 to 5 apply at those shares. The chain adds a new return to downstream record keeping. A firm that accepts consumer credit values inputs more and supports a higher upstream limit (Proposition 2), so part of what it earns by lending downstream is collected upstream. The shares then move with i, and where own-market dominance holds a higher rate both relaxes existing limits and makes record keeping more valuable.

7.2 Transferable and Pledgeable Receivables

Suppose a firm can pledge a fraction $\lambda \in [0,1]$ of its downstream receivable against its upstream payable, so upstream payment capacity becomes $z_n^f + \bar{d}_U^n + \lambda\,\mathbb{E}[d_D]$. The receivable then supports supplier credit through the reputational channel of Proposition 2 and through

a collateral channel. The two interact, because seizure substitutes for exclusion and makes the reputational punishment worth less at the margin. At $\lambda = 1$ the model approaches Gottardi and Zhang (2026). Pledgeability enlarges upstream payment capacity and leaves the payment unconditional, so unlike consignment it leaves the holdup wedge intact.

7.3 Longer Chains

With K links the reaction functions form a system, and the counterpart of χ is the spectral radius of the Jacobian of the composite map, much as with input-output matrices in production networks (Bigio and La'O 2020; Altinoglu 2021; La'O and Tahbaz-Salehi 2022). Propagation is a product of local elasticities. A firm in the middle of a long chain is both a borrower and a lender, and we conjecture that the middle carries the largest multipliers. With several suppliers per firm, exclusion by one supplier is a weaker punishment, so debt limits should fall with supplier multiplicity.

7.4 Other Policy Instruments

Interest on money paid to everyone lowers the carrying cost for defaulters and borrowers in good standing alike, which is the same as lowering i. Direct support for trade credit, of the kind the 2020 reinsurance backstops provided, is not a liquidity instrument. In the model it selects equilibria, moving the chain away from a pessimistic steady state without changing any primitive. This matters only when the nominal rate stays above the Friedman rule for reasons outside the model, and the case is for a contingent guarantee that is never exercised on path. Gottardi and Zhang (2026) study interventions of this kind in a nonmonetary chain. A policy that extends record keeping, such as a payment instrument that leaves a verifiable record, raises Λ_k and lowers the sustainability threshold, and with universal record keeping at both links it takes money out of use (Corollary 1(ii)).

8 Conclusion

We study trade credit in a monetary general equilibrium model of a supply chain. A firm's repayment upstream is disciplined by the value of its downstream business, so the debt limits at the two links are determined jointly, and firms whose customers buy on account sustain more supplier credit than firms whose customers pay cash. The feedback between the limits can amplify a change at either link and, when strong, support several steady states. It needs payment constraints that bind at both links, though, and it is modest in a solved benchmark.

Money keeps a collapse of credit at one link from spreading to the other, at the cost of tighter limits. Inflation relaxes credit where its effect on the defaulter's fallback dominates its drain on liquidity at the other link. That holds near the rates at which trade credit becomes self-enforcing and fails first for firms whose customers pay cash. The Friedman rule closes every liquidity wedge, yet no monetary policy attains the first best when input contracts do not condition on the downstream sale.

A quantitative version would need a source of persistently binding payment constraints, such as shocks to the size of desired trades, disciplined with data on payment terms and trade credit use (Barrot 2016; Amberg, Jacobson, von Schedvin, and Townsend 2021).

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Appendix A: Proofs

Throughout, $z(d)$ denotes an optimal real-balance choice in the relevant portfolio problem when credit access is d, and $\omega(d) \equiv z(d) + d$ denotes total liquidity.

Proof of Lemma 1

Regime (i). With $q_D = q_D(\omega_n^c; q_U)$, the traded quantity solves the payment constraint at equality, $(1-\theta_D)u(q_D) + \theta_D g(q_D \mid q_U) = \omega_n^c$. Differentiating with respect to q_U at fixed ω_n^c,

$$\big[(1-\theta_D)u'(q_D) + \theta_D g'_D\big]\frac{\partial q_D}{\partial q_U} + \theta_D g'_U = 0 \quad \Longrightarrow \quad \frac{\partial q_D}{\partial q_U} = \frac{-\theta_D g'_U}{(1-\theta_D)u'(q_D) + \theta_D g'_D} > 0,$$

because $g'_U < 0$. Then $\Omega_n(q_U) = \sigma_D(1-\theta_D)\big[u(q_D) - g(q_D \mid q_U)\big]$ gives

$$\Omega'_n(q_U) = \sigma_D(1-\theta_D)\Big[\big(u'(q_D) - g'_D\big)\frac{\partial q_D}{\partial q_U} - g'_U\Big] > 0,$$

where $u'(q_D) > g'_D$ because the payment constraint binds only when $q_D < q_D^*(q_U)$.
Regime (ii). With $q_D = q_D^*(q_U)$ interior, q_D maximizes $S_D(q_D; q_U)$ over q_D, so the envelope theorem gives $\Omega'_n(q_U) = \sigma_D(1-\theta_D)\,\partial S_D/\partial q_U = -\sigma_D(1-\theta_D)\,g'_U > 0$. □

Proof of Lemma 2

We first record a simple property of the two liquidity premia: on the constrained range both are decreasing in wealth. Downstream, write $s \equiv u'(q_D) - g'_D$ and note $(1-\theta_D)u' + \theta_D g'_D = g'_D + (1-\theta_D)s$, so

$$L_D = \frac{s}{g'_D + (1-\theta_D)s} = \frac{1}{g'_D/s + (1-\theta_D)}. \tag{A.1}$$

As ω rises, $q_D(\omega; q_U)$ rises by (16), while s falls because $u'' < 0$ and g'_D is nondecreasing in q_D, so g'_D/s rises and L_D falls, reaching zero when the constraint stops binding. The same argument works upstream, replacing $u - g$ with $\Omega_n - c$.

(i) Differentiating $S_D(q_D(\omega; q_U); q_U)$ with respect to wealth using (16) gives the downstream premium $L_D(\omega; q_U)$, and similarly $dS_U^n(q_U(w))/dw = L_U^n(w)$. The premia are nonnegative and nonincreasing, so the surplus terms in (28) and (31) are concave in real balances. Together with the linear carrying cost this gives concavity and existence.

(ii) The objective has decreasing differences in (z, i) because the cross-partial is -1. By Topkis's theorem (Topkis 1978), optimal real balances are nonincreasing in i. If the optimum is interior the first-order condition holds, and the marginal benefit of money is strictly decreasing in z on the constrained range, so the response is strict.

(iii) The cross-partial of $\Sigma_D(z; d)$ in (z, d) is $\sigma_D\theta_D\Lambda_D\,\mathbb{E}_a[\partial L_D(z+d)/\partial\omega] \le 0$. Hence Σ_D has nonincreasing differences and $z(d)$ is nonincreasing by Topkis. To show total liquidity rises, suppose $d_2 > d_1$ but $\omega(d_2) < \omega(d_1)$, with interior optima. Then $z(d_2) < z(d_1)$ and the

first-order conditions imply

$$\begin{aligned} i &= \sigma_D \theta_D \Big\{ \Lambda_D \mathbb{E}[L_D(\omega(d_2))] + (1-\Lambda_D)\mathbb{E}[L_D(z(d_2))] \Big\} \\ &> \sigma_D \theta_D \Big\{ \Lambda_D \mathbb{E}[L_D(\omega(d_1))] + (1-\Lambda_D)\mathbb{E}[L_D(z(d_1))] \Big\} = i, \end{aligned}$$

a contradiction, because L_D is decreasing. Boundary cases follow by the same monotonicity argument, and the upstream proof is identical. □

Proof of Proposition 1

(i) $\Psi_D(0) = 0$ is immediate from (35). Let $V(d) \equiv \max_{z \geq 0}\{-iz + \Sigma_D(z;d)\}$, so $\Psi_D(d) = V(d) - V(0)$. By the envelope theorem of Milgrom and Segal (2002), V is absolutely continuous with

$$\Psi'_D(d) = V'(d) = \frac{\partial \Sigma_D}{\partial d}\Big|_{z=z(d)} = \sigma_D \theta_D \Lambda_D \, \mathbb{E}_a \Big[L_D\big(\omega(d); q_U^{1,a}\big) \Big] \geq 0, \tag{A.2}$$

so Ψ_D is nondecreasing. By Lemma 2(iii), $\omega(d)$ is nondecreasing in d, and each L_D is nonincreasing in ω, so Ψ'_D is nonincreasing and Ψ_D is concave. The premia are bounded above by $1/(1-\theta_D)$ and vanish for ω above the satiation level $\hat{d}_D \equiv \max_a p_D\big(q_D^*(q_U^{1,a}); q_U^{1,a}\big)$, so Ψ_D is bounded and constant for $d \geq \hat{d}_D$. For the slope at the origin, (A.2) at $d = 0$ gives $\Psi'_D(0) = \sigma_D \theta_D \Lambda_D \, \mathbb{E}_a[L_D(z_0; q_U^{1,a})] = s_D$ with $z_0 = z(0)$. Because $z_0 > 0$, the defaulter's first-order condition holds with equality, $i = \sigma_D \theta_D \big\{ \Lambda_D \, \mathbb{E}_a[L_D(z_0; q_U^{1,a})] + (1-\Lambda_D) \, \mathbb{E}_a[L_D(z_0; q_U^{0,a})] \big\}$. The second term is nonnegative, so $s_D \leq i$. If $\Lambda_D = 1$ it is absent and $s_D = i$, and if $q_U^{1,a} = q_U^{0,a}$ for each a the two expectations coincide and $s_D = i\Lambda_D$. The two firm types hold the same input stocks in the upstream block that goes with $\bar{d}_D = 0$, because they then face the same consumer liquidity, solve the same portfolio problem (31), and face the same upstream limits.

(ii) $\bar{d}_D = 0$ solves (35) because both sides vanish. Define $h(d) \equiv \Psi_D(d) - rd$, which is concave with $h(0) = 0$. If $s_D \leq r$, concavity gives $\Psi_D(d) \leq \Psi'_D(0)\, d = s_D \, d \leq rd$ for all d, with strict inequality for $d > 0$ because L_D is strictly decreasing on the constrained range, so 0 is the unique solution. If $s_D > r$, then $h'(0) > 0$, so $h > 0$ in a right neighborhood of 0. Since Ψ_D is bounded, $h(d) \to -\infty$, so h has a positive root by the intermediate value theorem, and a concave function with $h(0) = 0$ and $h'(0) > 0$ has exactly one positive root, $\bar{d}_D^*$.

(iii) Applying the envelope theorem in i to both maximized values, $\partial \Psi_D / \partial i = z(0) - z(\bar{d}_D) \geq 0$ by Lemma 2(iii). Under the stated hypothesis the inequality is strict. If $z(0) = z(\bar{d}_D) \equiv z^*$ with both interior, the defaulter's first-order condition reads $i = \sigma_D \theta_D \, \mathbb{E}[L_D(z^*)]$ while the consumer's reads $i = \sigma_D \theta_D \big\{ \Lambda_D \, \mathbb{E}[L_D(z^* + \bar{d}_D)] + (1-\Lambda_D) \, \mathbb{E}[L_D(z^*)] \big\}$, which forces $\mathbb{E}[L_D(z^* + \bar{d}_D)] = \mathbb{E}[L_D(z^*)]$ and contradicts strict monotonicity of the premium on the constrained range when $\bar{d}_D^* < \hat{d}_D$. The corner $z(\bar{d}_D) = 0 < z(0)$ is strict directly. Implicit differentiation of $r\bar{d}_D^* = \Psi_D(\bar{d}_D^*)$ gives $d\bar{d}_D^*/di = (\partial \Psi_D / \partial i)/(r - \Psi'_D(\bar{d}_D^*))$, and at the unique positive root of a concave h we have $h'(\bar{d}_D^*) < 0$, that is $\Psi'_D(\bar{d}_D^*) < r$, so the derivative is strictly positive. □

Proof of Lemma 3

Both parts reduce to the same inequality. Throughout, derivatives are taken on regime (i) of Lemma 1, where the payment constraint binds and $u' > g'_D$, so $\tilde{D} = (1-\theta_D)u' + \theta_D g'_D > g'_D > 0$. Write $s \equiv u' - g'_D > 0$ and recall the elasticities (19).

Part (i). By (16), q_D is strictly increasing in ω at fixed q_U, so it is enough that the right side of (18) be nondecreasing in q_D. The experiment is a change in ω at a fixed input stock, so ω is what adjusts here and $\partial q_D/\partial q_U$ from Lemma 1(i), which holds ω fixed instead, plays no role. Taking logarithms of (18) and differentiating with respect to $\ln q_D$ at fixed q_U,

$$\frac{d\ln\Omega'_n}{d\ln q_D} = \varepsilon - \eta - \frac{q_D\big[(1-\theta_D)u'' + \theta_D g''_{DD}\big]}{\tilde{D}} = \varepsilon - \eta + \frac{(1-\theta_D)\,\eta\,u' - \theta_D\,\zeta\,g'_D}{\tilde{D}}, \tag{A.3}$$

using $d\ln(-g'_U)/d\ln q_D = \varepsilon$ and $d\ln u'/d\ln q_D = -\eta$. Multiplying by $\tilde{D} > 0$ and cancelling the terms in $(1-\theta_D)\eta u'$,

$$\tilde{D}\,\frac{d\ln\Omega'_n}{d\ln q_D} = \varepsilon\,\tilde{D} - \theta_D\big(\eta+\zeta\big)g'_D. \tag{A.4}$$

Since $\tilde{D} > g'_D > 0$ and (20) requires $\varepsilon > 0$, the right side of (A.4) is bounded below by $g'_D\big[\varepsilon - \theta_D(\eta+\zeta)\big]$, which is nonnegative under (20). Hence Ω'_n is nondecreasing in q_D and therefore in ω_n. Taking ω from z^c to $z^c + \bar{d}_D$ gives $\Omega'_1 \geq \Omega'_0$ pointwise. The statements about L^1_U follow because the upstream premium (32) is strictly increasing in Ω'_n, its derivative in Ω'_n being c' divided by the square of the denominator, and $\omega_1 = z^c + \bar{d}_D > \omega_0 = z^c$. Holding w fixed, a higher ω_n also raises the payment (22) for any quantity, so the firm buys a smaller input stock, and the premium is decreasing in the input stock on the constrained range (proof of Lemma 2), so this effect moves $L^n_U(w)$ in the same direction.

Part (ii). From (29), $1/L_D = \tilde{D}/s = (1-\theta_D) + Q$ with $Q \equiv g'_D/s$, so L_D is nondecreasing in q_U if and only if Q is nonincreasing in q_U. Fix the consumer's wealth ω and move along the binding payment constraint, so that $\partial q_D/\partial q_U = \theta_D(-g'_U)/\tilde{D} > 0$ by Lemma 1(i). Let

$$e \equiv \frac{q_U}{q_D}\frac{\partial q_D}{\partial q_U} > 0, \qquad \psi \equiv \frac{q_U\,g''_{DU}}{g'_D} < 0.$$

The sign of ψ follows from (20), which requires $\varepsilon > 0$ and hence $g''_{DU} = g''_{UD} < 0$. Then $d\ln g'_D/d\ln q_U = \psi + \zeta e$, and

$$q_U\frac{ds}{dq_U} = -e\,\eta\,u' - e\,\zeta\,g'_D - \psi\,g'_D,$$

so that

$$s\,\frac{d\ln Q}{d\ln q_U} = \psi\,(s+g'_D) + \zeta\,e\,(s+g'_D) + e\,\eta\,u' = u'\big[\psi + e(\eta+\zeta)\big],$$

using $s + g'_D = u'$. Hence Q is nonincreasing if and only if $e(\eta+\zeta) \leq -\psi$. Substituting $e = (q_U/q_D)\,\theta_D(-g'_U)/\tilde{D}$ and, from $\varepsilon = q_D\,g''_{UD}/g'_U$, the identity $-\psi = q_U\,\varepsilon\,(-g'_U)/(q_D\,g'_D)$,

and cancelling the common factor $(q_U/q_D)(-g'_U) > 0$, this becomes

$$\theta_D\big(\eta + \zeta\big)\, g'_D \le \varepsilon\, \tilde{D},$$

which is (A.4) again and holds under (20) by the argument in part (i). □

Proof of Proposition 2

(i) The arguments of Proposition 1 apply verbatim to (37), with the firm's first-order condition (33) delivering $\Psi^{n\prime}_U(0) = i\Lambda_U$, Lemma 2 delivering concavity and the monotonicity of total liquidity, and the envelope theorem in i delivering $\partial\Psi^n_U/\partial i = z^f(0) - z^f(\bar{d}^n_U) \ge 0$.

(ii) Let $\Lambda_U = 1$. When the firm still holds money at credit access d, its first-order condition reads $i = \theta_U L^n_U(\omega)$, which pins total liquidity at the constant ω^{n*} solving this equation, so $z_n(d) = \omega^{n*} - d$ for $d \le \omega^{n*}$ and $z_n(d) = 0$ thereafter. From (A.2) adapted to the firm,

$$\Psi^{n\prime}_U(d) = \begin{cases} \theta_U\, L^n_U(\omega^{n*}) = i & d \le \omega^{n*}, \\ \theta_U\, L^n_U(d) & d > \omega^{n*}. \end{cases} \tag{A.5}$$

By Lemma 3(i), $L^1_U(w) \ge L^0_U(w)$ for all w, so $\omega^{1*} \ge \omega^{0*}$ because the premia are decreasing, and (A.5) gives $\Psi^{1\prime}_U(d) \ge \Psi^{0\prime}_U(d)$ pointwise. On $[0, \omega^{0*}]$ both slopes equal i; on $(\omega^{0*}, \omega^{1*}]$ the type-1 slope is i while the type-0 slope is $\theta_U L^0_U(d) < i$; and beyond ω^{1*} the ranking of premia applies directly. Integrating from 0, $\Psi^1_U \ge \Psi^0_U$ pointwise. Each $\bar{d}^{n*}_U$ is the unique positive root of the concave function $\Psi^n_U(d) - rd$, and $\Psi^1_U(\bar{d}^{0*}_U) \ge \Psi^0_U(\bar{d}^{0*}_U) = r\bar{d}^{0*}_U$, so the root of the type-1 equation lies weakly to the right and $\bar{d}^{1*}_U \ge \bar{d}^{0*}_U$.

For the cross-partial, fix $d > \omega^{1*}(\bar{d}_D)$ and integrate (A.5):

$$\Psi^1_U(d; \bar{d}_D) = i\,\omega^{1*}(\bar{d}_D) + \int_{\omega^{1*}(\bar{d}_D)}^{d} \theta_U\, L^1_U(w; \bar{d}_D)\, dw. \tag{A.6}$$

Differentiating in $\bar{d}_D$, the boundary terms cancel because $\theta_U L^1_U(\omega^{1*}) = i$ by construction, leaving

$$\frac{\partial \Psi^1_U}{\partial \bar{d}_D} = \int_{\omega^{1*}}^{d} \theta_U\, \frac{\partial L^1_U(w; \bar{d}_D)}{\partial \bar{d}_D}\, dw \ge 0 \tag{A.7}$$

by Lemma 3(i), and $\partial\Psi^1_U/\partial\bar{d}_D = 0$ for $d \le \omega^{1*}$. Implicit differentiation of $r\bar{d}^1_U = \Psi^1_U(\bar{d}^1_U; \bar{d}_D)$ then yields (38), which is nonnegative at any stable solution because $r > \partial\Psi^1_U/\partial\bar{d}^1_U$ there.

(iii) Let $\Lambda \equiv \Lambda_U \in (0,1)$, the case $\Lambda = 1$ being part (ii), and write $L(w; b)$ for the upstream premium of a credit-accepting firm whose downstream credit limit is b, so $L^1_U = L(\cdot\,; \bar{d}_D)$ and $L^0_U = L(\cdot\,; 0)$. The two firm types face the same premium evaluated at different b, because they differ only through the consumer liquidity $\omega = z^c + b$ entering Ω'. Fix d and suppose the firm's money holdings are interior, so its first-order condition reads $i = \theta_U\big[\Lambda L(z + d; b) + (1 - \Lambda)L(z; b)\big]$. By the envelope theorem, $\Psi'_U(d; b) = \theta_U \Lambda\, L(\omega(d); b)$ with $\omega(d) = z(d) + d$. Differentiating the first-order condition in b, solving for $\partial z/\partial b$, and

substituting,

$$\frac{\partial \Psi'_U(d;b)}{\partial b} = \theta_U \, \frac{\Lambda(1-\Lambda)\big[L_b(\omega)\, L_w(z) - L_w(\omega)\, L_b(z)\big]}{\Lambda\, L_w(\omega) + (1-\Lambda)\, L_w(z)}, \tag{A.8}$$

where subscripts denote partial derivatives. The denominator is negative because $L_w < 0$ on the constrained range, so the expression is nonnegative if and only if $L_b(\omega)/|L_w(\omega)| \geq L_b(z)/|L_w(z)|$, which is Assumption E.1 because $\omega(d) \geq z(d)$. Where the firm holds no money, $\Psi'_U(d;b) = \theta_U \Lambda\, L(d;b)$ and $\partial \Psi'_U/\partial b = \theta_U \Lambda\, L_b \geq 0$ directly by Lemma 3(i). Hence $\Psi'_U(\cdot\,;b)$ is nondecreasing in b pointwise, and integrating from $\Psi_U(0;b) = 0$, so is $\Psi_U(\cdot\,;b)$. Both conclusions follow. Taking b from 0 to $\bar{d}_D$ gives $\Psi^1_U \geq \Psi^0_U$ pointwise, so the largest positive roots rank as $\bar{d}^{1*}_U \geq \bar{d}^{0*}_U$. And the largest root of the concave equation $\Psi_U(d;b) = rd$ is nondecreasing in b, which is (38) with a nonnegative numerator. At a stable solution $r > \partial \Psi^1_U/\partial \bar{d}^1_U$, so $\partial \bar{d}^{1*}_U/\partial \bar{d}_D \geq 0$. At $\Lambda = 1$ the factor $1-\Lambda$ in (A.8) recovers the flat money region in (A.5). □

Proof of Corollary 1

By concavity and the slopes at the origin established in Propositions 1(i) and 2(i), $\Psi^n_U(d) \leq i\Lambda_U d$ and $\Psi_D(d) \leq s_D d$ for all $d \geq 0$. A positive solution of $rd = \Psi_k(d)$ therefore requires $i\Lambda_U > r$ upstream and $s_D > r$ downstream, and $s_D \leq i$ by Proposition 1(i). At $\bar{d}_D = 0$ the two firm types hold the same input stocks, so $s_D = i\Lambda_D$ there. Along any sequence of equilibria with $\bar{d}_D \to 0$ the input stocks of the two types converge and $s_D \to i\Lambda_D$, so positive downstream solutions near zero require $i\Lambda_D$ near r, and a branch of them can leave zero only at $i\Lambda_D = r$. At $\Lambda_D = 1$, $s_D = i$ everywhere. As $i \to 0$ both necessary conditions fail, so the only solutions are $\bar{d}_D = \bar{d}^n_U = 0$.

(ii) Let $\Lambda_D = 1$. While a consumer holds money, (30) pins his total liquidity at the level ω^*_D that equates the expected premium to i, so $z(d) = \omega^*_D - d$ on $[0, \omega^*_D]$ and $\Psi_D(d) = id$ there, as in (A.5). A positive root of $\Psi_D(d) = rd$ requires $i > r$, and on $(0, \omega^*_D]$ we have $\Psi_D(d) - rd = (i-r)d > 0$, so any positive root exceeds ω^*_D and $z^c(\bar{d}_D) = 0$. The corner is strict, because the marginal value of money to a consumer in good standing at $z = 0$ is $\sigma_D \theta_D\, \mathbb{E}_a[L_D(\bar{d}_D)] < \sigma_D \theta_D\, \mathbb{E}_a[L_D(\omega^*_D)] = i$. The argument for a firm at $\Lambda_U = 1$ is the same, with (33) and (A.5). □

Proof of Proposition 3

(i) Set $\bar{d}_D = \bar{d}^n_U = 0$, so both firm types solve identical problems and the equilibrium objects are (z^c, z^f) solving (30) and (33). As wealth vanishes, traded quantities vanish and, because $u'(0) = \infty$, the premia approach their upper bounds: $L_D \to 1/(1-\theta_D)$ as $\omega \to 0$ for any $q_U > 0$, and $L^n_U(w) \to 1/(1-\theta_U)$ as $w \to 0$, using Lemma 1 to pass the Inada property of u through Ω'_n. Fix $i < \bar{\imath}$, with $\bar{\imath}$ as in the statement. By continuity of the premia in own wealth and in the other side's stock, there exists $\delta > 0$ such that whenever the other side holds at least δ, the marginal return to the first δ units of own balances exceeds i, so best responses map $[\delta, Z] \times [\delta, Z]$ into itself, where Z bounds the satiation levels. Best responses are single-valued by strict concavity (Lemma 2(i)) and continuous by the maximum theorem, so Brouwer's theorem delivers a fixed point $(z^c, z^f) \in [\delta, Z]^2$, which is a monetary steady state

with inactive credit. Two-sided liquidity problems of this form are treated in Rocheteau, Wright, and Zhang (2018).

(ii) Let $K \geq \max\{\bar{\Psi}_D, \bar{\Psi}_U\}/r$, where bars denote the bounds from Propositions 1(i) and 2(i), so all solutions of the flow equations lie in $[0, K]$. Upstream, the slope of Ψ_U at the origin is $i\Lambda_U = i > r$ whatever $\bar{d}_D$ is, by Proposition 2(i), so given $\bar{d}_D$ the upstream flow equation has exactly two solutions, zero and $\Phi_U(\bar{d}_D) > 0$. Downstream, for $y \geq 0$ take the input stocks that go with upstream limit $\bar{d}_U$ when consumers bring payment capacity y to firms. At those stocks the slope of Ψ_D at the origin is $i > r$ by Proposition 1(i), so $\phi(\bar{d}_U, y)$ is the unique positive root of a concave equation, and it is continuous in y because input stocks are continuous in consumers' payment capacity. At a positive solution a of (35), consumers in good standing hold no money (Corollary 1(ii)), so their payment capacity is a itself, and a solves (35) given $\bar{d}_U$ if and only if $a = \phi(\bar{d}_U, a)$. The map $y \mapsto \phi(\bar{d}_U, y)$ is continuous from $[0, K]$ into $(0, K]$, so it has fixed points, all of them positive, and a largest one, $\Phi_D(\bar{d}_U)$. A solution (a, b) of the system therefore has $b \in \{0, \Phi_U(a)\}$ and either $a = 0$ or $a = \phi(b, a) \leq \Phi_D(b)$.

Both reactions are nondecreasing. An increase in $\bar{d}_D$ shifts Ψ_U up pointwise by (A.7) under Assumption 2, and the positive root of the concave equation $\Psi_U(d) = rd$ is nondecreasing in upward shifts of Ψ_U. An increase in $\bar{d}_U$ raises the firm's total liquidity, which at $\Lambda_U = 1$ is $\max\{\bar{d}_U, \omega_U^*\}$ (proof of Proposition 2(ii)), and with it the input stock the firm buys at any y. A larger input stock shifts Ψ_D up pointwise, by the envelope expression for $\partial\Psi_D/\partial q_U^1$ in the proof of Corollary E.1, which uses Lemma 3(ii). So $\phi(\cdot\,, y)$ is nondecreasing for each y. The largest fixed point inherits this. If $\phi(b', \cdot) \geq \phi(b, \cdot)$ pointwise, then $\phi(b', \Phi_D(b)) \geq \Phi_D(b)$ and $\phi(b', K) \leq K$, so by the intermediate value theorem $y \mapsto \phi(b', y)$ has a fixed point in $[\Phi_D(b), K]$, and $\Phi_D(b') \geq \Phi_D(b)$. So T is monotone on the complete lattice $[0, K]^2$, and by Tarski (1955) its fixed points form a nonempty complete lattice. Every fixed point x satisfies $x = T(x) \geq T(0, 0) = (\Phi_D(0), \Phi_U(0)) \gg 0$, so all of them are strictly positive. Any solution (a, b) of the system satisfies $(a, b) \leq T(a, b)$ by the previous paragraph, and by Tarski's theorem the greatest fixed point is the supremum of all such points, so it is the greatest solution.

If $\rho < 1$ wherever it is defined, then $y \mapsto \phi(b, y) - y$ is strictly decreasing, its unique zero is $\Phi_D(b)$, and the solutions of (35) given b are 0 and $\Phi_D(b)$. A solution (a, b) then has $a \in \{0, \Phi_D(b)\}$ and $b \in \{0, \Phi_U(a)\}$. If $a = 0$ then $b \in \{0, \Phi_U(0)\}$, and if $b = 0$ then $a \in \{0, \Phi_D(0)\}$, which gives the pessimistic solutions listed in part (ii). Otherwise $(a, b) = T(a, b)$.

(iii) The fixed points of T correspond to the fixed points of the composite $h(x) \equiv \Phi_D(\Phi_U(x))$ on $[0, K]$, which is nondecreasing, with $h(0) > 0$ and $h(K) \leq K$. Differentiating $y = \phi(\bar{d}_U, y)$ at $y = \Phi_D(\bar{d}_U)$ gives (39), and $h'(x) = \chi(x)$ where the derivatives exist. At the largest fixed point, $\phi(\bar{d}_U, y) - y$ is nonpositive to the right because $\phi(\bar{d}_U, K) \leq K$, so $\rho \leq 1$ there, with equality only at a tangency. Let x_0 be a fixed point with $\chi(x_0) > 1$. Then $h(x_1) < x_1$ for some $x_1 < x_0$ and $h(x_2) > x_2$ for some $x_2 > x_0$. Because h is nondecreasing, it maps $[0, x_1]$ into $[h(0), h(x_1)] \subset [0, x_1]$ and $[x_2, K]$ into $[h(x_2), h(K)] \subset [x_2, K]$, so by Tarski (1955) it has a fixed point in each interval, and the one in $[0, x_1]$ is at least $h(0) > 0$. Each gives a fixed point $(x, \Phi_U(x))$ of T, strictly positive by part (ii), and because Φ_U is nondecreasing the two lie below and above the fixed point at x_0. Generically h crosses

the diagonal from above at the flanking fixed points, so $\chi < 1$ there, giving the alternating pattern of Figure 3(b). For the converse, suppose $\rho < 1$ wherever it is defined. Then $\Phi_D(\bar{d}_U)$ is the unique zero of $y \mapsto \phi(\bar{d}_U, y) - y$, which is strictly decreasing in y and continuous in $(\bar{d}_U, y)$, so Φ_D is continuous and so is $g(x) \equiv h(x) - x$, with $g(0) > 0$ and $g(K) \leq 0$. Suppose $\chi < 1$ at every zero of g. Then g is strictly decreasing at each zero, so the zeros are isolated, hence finite in number, and g changes sign from positive to negative at each of them. Two consecutive zeros would require g to be negative just to the right of the first and positive just to the left of the second, and so to have another zero in between. Hence g has exactly one zero. □

Proof of Proposition 4

(i) With $\bar{d}_D = 0$, consumers solve the money-only portfolio problem, and $i < \bar{\imath} \leq \sigma_D \theta_D/(1 - \theta_D)$ gives them positive real balances, as in the proof of Proposition 3(i). Firms' customers then have positive payment capacity, so $\Omega' > 0$ by Lemma 1, and by Proposition 2(i) the upstream flow equation has a unique positive root because $i\Lambda_U = i > r$. Hence $\Phi_U(0) > 0$. With $\bar{d}_U = 0$, a firm finances its inputs with money, which it holds because $i < \theta_U/(1 - \theta_U)$. At any downstream limit y, consumers in good standing hold money up to their total-liquidity target when y is below it, so their payment capacity is bounded away from zero and firms buy positive input stocks. At those stocks the slope of Ψ_D at the origin is $i > r$ by Proposition 1(i), so $\phi(0, y) > 0$ for every y. Zero is then not a fixed point of $y \mapsto \phi(0, y)$, and $\Phi_D(0) > 0$ by the argument in the proof of Proposition 3(ii).

(ii) Without money a consumer pays only with credit and a firm buys inputs only with credit. If $\bar{d}_D = 0$, no consumer can pay, no downstream trade takes place, and $\Omega \equiv 0$. The upstream surplus is then $-c(q_U) \leq 0$, so no upstream trade takes place whatever the upstream limit, the firm's flow cost of default is identically zero, and the only solution of the upstream flow equation is zero. If $\bar{d}_U = 0$, no firm can pay its supplier, so $q_U = 0$, $G(0, h_D) = 0$, and firms have nothing to sell. The consumer's flow cost of default is identically zero, and so is the downstream limit. Hence $\Phi_U(0) = \Phi_D(0) = 0$ and $T(0, 0) = (0, 0)$.

(iii) Fix the other link and let $\Sigma(z; d)$ be the borrower's expected surplus at link k with money z and credit access d, as in (35) or (37) at $\Lambda_k = 1$. With money the flow cost of default is $\Psi^m(d) = V(d) - V(0)$, where $V(d) \equiv \max_{z \geq 0}\{-iz + \Sigma(z; d)\}$. Without money a borrower in good standing pays with credit alone and a defaulter does not trade at link k, so $\Psi^n(d) = \Sigma(0; d) - \Sigma(0; 0) = \Sigma(0; d)$. Let $G(d) \equiv V(d) - \Sigma(0; d) \geq 0$ be the value of the money option given credit access d. Then $\Psi^m(d) - \Psi^n(d) = G(d) - G(0)$, and by the envelope theorem $G'(d) = \Sigma_d\big(z(d); d\big) - \Sigma_d(0; d) \leq 0$, because the cross-partial of Σ in (z, d) is nonpositive (proof of Lemma 2(iii)). Hence $\Psi^m \leq \Psi^n$ pointwise. Both functions are concave and vanish at zero, and the largest root of $\Psi(d) = rd$ is nondecreasing in upward shifts of Ψ, so the limit with money is weakly below the limit without it. □

Proof of Corollary E.1

At $\Lambda_U = 1$ every upstream meeting accepts credit, so a type-n firm buys $q_U^n = q_U(\omega_n^f)$, given its customers' payment capacity, with total liquidity $\omega_n^f \equiv z_n^f + \bar{d}_U^n$, which is nondecreasing in

$\bar{d}_U^n$ by Lemma 2(iii) when the firm's money responds only to its own limit. Write $\hat{S}(\omega, q) \equiv S_D\big(q_D(\omega; q); q\big)$ for match surplus as a function of the consumer's payment capacity and the firm's input stock. Its derivative in ω is the premium L_D, so by Lemma 3(ii) the cross-partial $\partial^2 \hat{S}/\partial\omega\,\partial q = \partial L_D/\partial q$ is nonnegative, and $\partial\hat{S}/\partial q$ is nondecreasing in ω. The envelope theorem applied to both maximized values in (35) gives

$$\frac{\partial \Psi_D}{\partial q_U^1} = \sigma_D \theta_D \Lambda_D \Big[\frac{\partial \hat{S}}{\partial q}\big(z(\bar{d}_D) + \bar{d}_D,\, q_U^1\big) - \frac{\partial \hat{S}}{\partial q}\big(z(0),\, q_U^1\big)\Big],$$
$$\frac{\partial \Psi_D}{\partial q_U^0} = \sigma_D \theta_D (1 - \Lambda_D) \Big[\frac{\partial \hat{S}}{\partial q}\big(z(\bar{d}_D),\, q_U^0\big) - \frac{\partial \hat{S}}{\partial q}\big(z(0),\, q_U^0\big)\Big].$$

By Lemma 2(iii), $z(\bar{d}_D) + \bar{d}_D \geq z(0) \geq z(\bar{d}_D)$, so the first derivative is nonnegative and the second nonpositive: a consumer in good standing has more to spend than a defaulter at credit-accepting firms and less at cash-only firms, and a larger input stock is worth more to whoever has more to spend. The largest root of the concave equation $\Psi_D(d) = rd$ is nondecreasing in upward shifts of Ψ_D. This holds at the input stocks that go with any downstream limit y, and the largest solution of $y = \phi(\bar{d}_U^0, \bar{d}_U^1, y)$ inherits it by the argument in the proof of Proposition 3(ii), so Φ_D is nondecreasing in $\bar{d}_U^1$ and nonincreasing in $\bar{d}_U^0$.

Upstream, the flow cost of a type-n firm depends on $\bar{d}_D$ only through its customers' liquidity, $\omega_1 = z^c + \bar{d}_D$ and $\omega_0 = z^c$, which by Lemma 2(iii) are nondecreasing and nonincreasing in $\bar{d}_D$ respectively. At $\Lambda_U = 1$ the argument of (A.7), with customers' liquidity as the shift parameter, shows that Ψ_U^n shifts up pointwise with ω_n, as in the proof of Proposition 5(i). Hence Φ_U^1 is nondecreasing and Φ_U^0 nonincreasing in $\bar{d}_D$.

Reversing the $\bar{d}_U^0$ coordinate therefore makes all three reactions nondecreasing on $[0, K]^3$, and by Tarski (1955) the fixed points form a nonempty complete lattice. At a fixed point, $\bar{d}_U^n = \Phi_U^n(\bar{d}_D) > 0$ because $i\Lambda_U = i > r$, which follows from $i\Lambda_D > r$. If $\bar{d}_D$ were zero, the two firm types would hold the same limits, and at $y = 0$ they would face the same customers and hold the same input stocks, so the slope of Ψ_D at the origin would be $i\Lambda_D > r$ by Proposition 1(i). Then $\phi(\bar{d}_U^0, \bar{d}_U^1, 0) > 0$, so zero does not solve $y = \phi(\bar{d}_U^0, \bar{d}_U^1, y)$ and $\Phi_D > 0$, a contradiction. Finally, the fixed points correspond to the fixed points of $h(x) \equiv \Phi_D\big(\Phi_U^0(x), \Phi_U^1(x)\big)$, where $h(0) > 0$ by the same argument and $h(K) \leq K$, and the proof of Proposition 3(iii) applies verbatim. □

Proof of Lemma E.1

Write the two first-order conditions as $i = R_D(z^c, z^f)$ and $i = R_U(z^f, z^c)$, where R_D is the right side of (30), with the input stocks $q_U^{n,a}$ evaluated at the firm's liquidity, and R_U is the right side of (33), with Ω_n' evaluated at the consumer's liquidity. On the constrained range the own-partials are negative, $R_{D,c} < 0$ and $R_{U,f} < 0$, by the strict monotonicity of the premia established in the proof of Lemma 2. The cross-partials are nonnegative: $R_{U,c} \geq 0$ by Lemma 3(i) and $R_{D,f} \geq 0$ by Lemma 3(ii), using $q_U'(w) = \big[(1-\theta_U)\Omega_n' + \theta_U c'\big]^{-1} > 0$ from (23). Total differentiation gives

$$R_{D,c}\, dz^c + R_{D,f}\, dz^f = di, \qquad R_{U,c}\, dz^c + R_{U,f}\, dz^f = di,$$

with determinant $\Delta = R_{D,c}R_{U,f} - R_{D,f}R_{U,c} = R_{D,c}R_{U,f}(1-\xi)$. Since $R_{D,c}R_{U,f} > 0$, $\xi < 1$ implies $\Delta > 0$ and the system is regular. Cramer's rule gives

$$\frac{dz^c}{di} = \frac{R_{U,f} - R_{D,f}}{\Delta} < 0, \qquad \frac{dz^f}{di} = \frac{R_{D,c} - R_{U,c}}{\Delta} < 0,$$

because each numerator is negative and $\Delta > 0$, and at fixed debt limits $d\omega = dz$. Finally,

$$\left|\frac{dz^c}{di}\right| = \frac{|R_{U,f}| + R_{D,f}}{|R_{D,c}|\,|R_{U,f}|\,(1-\xi)} \geq \frac{1}{|R_{D,c}|} \cdot \frac{1}{1-\xi},$$

and $1/|R_{D,c}|$ is the response of z^c to i with z^f held fixed. The firm's side is symmetric. □

Proof of Proposition 5

(i) Each maximized value in (35) and (37) depends on i directly, through the carrying cost of the borrower's own balances, and indirectly, through the other side of the chain, because $\omega_D = z^c + \bar{d}_D$ enters Ψ_U through Ω' and firms' input stocks enter Ψ_D. Differentiating at fixed other limit, the direct part is the envelope term of Milgrom and Segal (2002), $z(0) - z(d) \geq 0$ by Lemma 2(iii), and the indirect part is the product in (40). For its first factor, the argument of (A.7) applies with ω_D as the shift parameter, because $\Psi_U(d;\omega_D) = i\,\omega^*(\omega_D) + \int_{\omega^*(\omega_D)}^{d} \theta_U\, L_U(w;\omega_D)\,dw$ for $d > \omega^*$, the boundary terms cancel, and Lemma 3(i) gives $\partial\Psi_U/\partial\omega_D \geq 0$. For $\Lambda_U \in (0,1)$ the same conclusion follows from (A.8) with ω_D as the shift parameter, as in the proof of Proposition 2(iii), with the numerator signed by Assumption E.1 where the firm holds money and by Lemma 3(i) directly where it does not. For the second factor, at fixed limits the indirect arguments move with i only through real balances, $d\omega_D/di = dz^c/di$ and $d\omega_F/di = dz^f/di$. Lemma 2(ii) gives the partial effects, and regularity of the portfolio system (30) and (33) requires that the two-sided feedback preserve their sign, so both are nonpositive, with Lemma E.1 giving $\xi < 1$ as a sufficient condition. For the downstream term, the proof of Corollary E.1 gives $\partial\Psi_D/\partial q_U^{1,a} \geq 0$ and $\partial\Psi_D/\partial q_U^{0,a} \leq 0$, using Lemma 3(ii). At $\Lambda_D = 1$ there are no cash-only firms and consumers in good standing hold no money, so at fixed limits input stocks move with i only through firms' balances and rise with them, and the downstream term has the sign of dz^f/di, which is nonpositive. This establishes (40). Under own-market dominance, $\partial\Psi_k/\partial i \geq 0$. At a stable interior root $B_{U,i} = (\partial\Psi_U/\partial i)/(r - \Psi'_U) \geq 0$, and differentiating $y = \phi(\bar{d}_U, y)$ gives $B_{D,i} = (\partial\Psi_D/\partial i)/\big[(r-\Psi'_D)(1-\rho)\big] \geq 0$, where $1 - \rho > 0$ at the largest solution except at a tangency.

(ii) Totally differentiate the system $\bar{d}_U = B_U(\bar{d}_D, i)$, $\bar{d}_D = B_D(\bar{d}_U, i)$:

$$d\bar{d}_U = B'_U\, d\bar{d}_D + B_{U,i}\, di, \qquad d\bar{d}_D = B'_D\, d\bar{d}_U + B_{D,i}\, di.$$

Substituting the second equation into the first and collecting terms,

$$(1 - B'_U B'_D)\, d\bar{d}_U = \big(B_{U,i} + B'_U B_{D,i}\big) di,$$

which is (41) because $\chi = B'_U B'_D < 1$ at a regular stable steady state, and the expression

for $d\bar{d}_D/di$ is symmetric. No step restricts the signs of $B_{U,i}$ and $B_{D,i}$. $\square$

Proof of Corollary 2

(i) By Lemma 1(ii), when the downstream payment constraint is slack, $\Omega_n(q) = \sigma_D(1-\theta_D)S_D\big(q_D^*(q);q\big)$ and $\Omega_n'(q)$ do not depend on the consumer's payment capacity. Suppose consumers in good standing are unconstrained in credit meetings at every input stock up to the one firms buy in good standing. Every stock that enters the firm's problem, with or without upstream credit, is in this range, so a small change in $\bar{d}_D$ leaves Ω_1 and Ω_1' unchanged at each of them, and so it leaves S_U^1, the premia L_U^1, and Ψ_U^1 unchanged. It also leaves consumer money unchanged, because the first-order condition (30) then involves only cash meetings, whose premia do not depend on $\bar{d}_D$. So $\omega_0 = z^c$ does not move, and neither does Ψ_U^0. Each upstream limit is the root of a flow equation that does not move with $\bar{d}_D$, so $\Phi_U^{n\prime} = 0$. Suppose instead that firms in good standing are unconstrained in upstream credit meetings. Those meetings trade the unconstrained quantity $\hat{q}_U^n$, which does not depend on the firm's wealth, and at $\Lambda_U < 1$ the first-order condition (33) involves only cash meetings, so the firm's money does not depend on its upstream limit either. Input stocks then do not move with the upstream limits, and neither does $\phi(\cdot\,, y)$ at any y, so $\phi_1 = 0$ and $\Phi_D' = 0$ by (39). In either case the composite reaction map has slope zero.

(ii) If $\Lambda_k < 1$, the limit at link k converges to zero as the rate approaches the threshold, upstream because $\Psi_U'(0) = i\Lambda_U$ with Ψ_U' strictly decreasing near the origin on the constrained range, and downstream because positive limits branch off zero there (Corollary 1). The borrower's payment capacity in credit meetings then converges to its money holdings, and the premia in credit and cash meetings converge to each other, downstream because the two firm types' input stocks converge (proof of Corollary 1). The first-order condition (30) or (33) sets their weighted average equal to $i/(\sigma_D\theta_D)$ or i/θ_U, so in the limit the premium in credit meetings is positive and the constraint binds. If $\Lambda_k = 1$, the threshold is $i = r$, any positive limit exceeds the borrower's total-liquidity target ω_k^* (Corollary 1(ii)), and the expected premium at ω_k^* equals $i/(\sigma_D\theta_D)$ or i/θ_U, which is positive because the defaulter holds money. Beyond ω_k^* the slope of Ψ_k is the scaled premium, which equals i at ω_k^* and falls strictly above it, while $\Psi_k(d) - rd = (i-r)\omega_k^*$ at ω_k^*. So for any $\epsilon > 0$, $\Psi_k(d) - rd < 0$ at $d = \omega_k^* + \epsilon$ once i is close enough to r, and the positive root converges to ω_k^*. By continuity of the premium, the constraint binds for i close enough to r.

(iii) Fix a type n, write ω for its customers' payment capacity, and write $S(w;\omega)$ and $L(w;\omega)$ for the firm's upstream surplus and premium, so $S_w = L$. With money holdings interior, the envelope theorem applied to both maximized values in (37) gives

$$\frac{\partial \Psi_U}{\partial \omega} = \theta_U\Big[\Lambda_U\, S_\omega\big(z(d)+d\big) + (1-\Lambda_U)\, S_\omega\big(z(d)\big)\Big] - \theta_U\, S_\omega\big(z(0)\big).$$

Differentiating the first-order condition $\theta_U\big[\Lambda_U L(z+d) + (1-\Lambda_U)L(z)\big] = i$ in d gives $z'(d) = -\Lambda_U L_w(z+d)/\big[\Lambda_U L_w(z+d) + (1-\Lambda_U)L_w(z)\big]$, so $z'(0) = -\Lambda_U$, $z(d) = z(0) - \Lambda_U d + O(d^2)$, and $z(d) + d = z(0) + (1-\Lambda_U)d + O(d^2)$. Expanding S_ω around $z(0)$, the first-order terms are $\theta_U S_{\omega w}\big[\Lambda_U(1-\Lambda_U) - (1-\Lambda_U)\Lambda_U\big]d = 0$, so $\partial\Psi_U/\partial\omega = O(d^2)$. By the envelope theorem $\Psi_U'(d) = \theta_U\Lambda_U L\big(z(d)+d\big)$, so $\Psi_U''(0) = \theta_U\Lambda_U(1-\Lambda_U)L_w\big(z(0)\big) < 0$ on the constrained

range. At the positive root $r = \Psi_U(\bar{d}_U)/\bar{d}_U = \Psi'_U(0) + \Psi''_U(0)\bar{d}_U/2 + O(\bar{d}_U^2)$, while $\Psi'_U(\bar{d}_U) = \Psi'_U(0) + \Psi''_U(0)\bar{d}_U + O(\bar{d}_U^2)$, so $r - \Psi'_U(\bar{d}_U) = -\Psi''_U(0)\bar{d}_U/2 + O(\bar{d}_U^2)$. Implicit differentiation of $r\bar{d}_U = \Psi_U(\bar{d}_U;\omega)$ then gives $\partial\bar{d}_U/\partial\omega = \big(\partial\Psi_U/\partial\omega\big)/\big(r - \Psi'_U(\bar{d}_U)\big) = O(\bar{d}_U)$. □

Proof of Proposition 6

(i) As $i \to 0$, the first-order conditions (30) and (33) force the expected premia to zero. The premia are nonnegative and vanish only where the relevant payment constraint is slack, so every constraint is slack in the limit, and Corollary 1 gives $\bar{d}_D = \bar{d}_U^n = 0$ once $i < r$. With slack constraints the downstream quantity is $q_D^*(q_U)$ in every meeting, $L_D = 0$, and by Lemma 1 the firm's surplus is $\Omega(q_U) = \sigma_D(1-\theta_D)\, S_D\big(q_D^*(q_U); q_U\big)$ on the unconstrained regime, so the bargained input purchase is $\hat{q}_U$ solving $\Omega' = c'$. Substituting into (43) gives $\mathcal{W} \to \mathcal{V}(\hat{q}_U)$.

(ii) The first best requires $u'(q_D) = g'_D$ in every realized downstream match and $q_U = q_U^*$ in every upstream match. If a payment constraint binds in a positive measure of downstream meetings, $u' > g'_D$ there. If none binds, then $L_D = 0$ in every meeting, Ω is as in (i), and bargained input trade satisfies $q_U \le \hat{q}_U$ by (23). On the unconstrained regime, $\mathcal{V}'(q) = \Omega'(q)/(1-\theta_D) - c'(q) = \big[S'_U(q) + \theta_D\, c'(q)\big]/(1-\theta_D)$, so $\mathcal{V}'(\hat{q}_U) = \theta_D\, c'(\hat{q}_U)/(1-\theta_D) > 0$. Since $\mathcal{V}$ is strictly concave with $\mathcal{V}'(q_U^*) = 0$ by (7) and (8), it follows that $\hat{q}_U < q_U^*$.

(iii) We first show that input trade never exceeds $\hat{q}_U$. Fix a type-n firm and an input stock q. On the payment-constrained regime, $\Omega'_n(q)$ is nondecreasing in its customers' payment capacity by Lemma 3(i). As that capacity rises to the level at which the constraint stops binding, $L_D \to 0$ and $q_D \to q_D^*(q)$, so (18) converges to $-\sigma_D(1-\theta_D)\, g'_U\big(q_D^*(q) \mid q\big)$, the slope at the Friedman rule. Hence $\Omega'_n(q) \le \Omega'(q)$ for every q and every payment capacity, where Ω is the Friedman rule valuation of (i). The Friedman rule surplus satisfies $S_U = (1-\theta_D)\mathcal{V} - \theta_D c$, which is strictly concave, so $S'_U(q) < 0$ for $q > \hat{q}_U$, and therefore $\Omega'_n(q) - c'(q) < 0$ there. The unconstrained input purchase $\hat{q}_U^n$ is then at most $\hat{q}_U$, and a firm that is payment constrained upstream buys less than $\hat{q}_U^n$ by (23), so $q_U^{n,a} \le \hat{q}_U$ in every upstream meeting.

Now fix a stationary equilibrium at $i \ge 0$ and a realized chain with quantities (q_U, q_D). Since $u' > g'_D$ on $[0, q_D^*(q_U))$, $S_D(\cdot\,; q_U)$ is strictly increasing there, and $q_D \le q_D^*(q_U)$ gives $\sigma_D S_D(q_D; q_U) - c(q_U) \le \mathcal{V}(q_U)$. Writing $S_U \equiv \Omega - c$ for the upstream surplus of (i), $\mathcal{V}'(q) = \big[S'_U(q) + \theta_D\, c'(q)\big]/(1-\theta_D)$, and strict concavity of S_U together with $S'_U(\hat{q}_U) = 0$ gives $S'_U \ge 0$ on $(0, \hat{q}_U]$, so $\mathcal{V}' > 0$ there and $\mathcal{V}(q_U) \le \mathcal{V}(\hat{q}_U)$ because $q_U \le \hat{q}_U$. Averaging across meeting types with the weights in (43) yields $\mathcal{W}(i) \le \mathcal{V}(\hat{q}_U) = \mathcal{W}(0)$. If a payment constraint binds in a positive measure of meetings, then $q_D < q_D^*(q_U)$ strictly there and the first inequality is strict. □

Appendix B: A Parametric Benchmark

This appendix verifies the maintained conditions in a benchmark parametrization with isoelastic utility $u(q) = q^{1-\eta}/(1-\eta)$ where $\eta \in (0,1)$, linear upstream technology $f(h_U) = h_U$, so $c(q_U) = q_U$, and Cobb–Douglas downstream technology $G(q_U, h_D) = q_U^{\alpha} h_D^{1-\alpha}$ with $\alpha \in (0,1)$. Let $\mu \equiv \alpha/(1-\alpha) > 0$. Inverting G gives

$$g(q_D \mid q_U) = q_D^{1+\mu}\, q_U^{-\mu}, \qquad g'_D = (1+\mu)\, q_D^{\mu}\, q_U^{-\mu}, \qquad g'_U = -\mu\, q_D^{1+\mu}\, q_U^{-(1+\mu)}, \tag{B.1}$$

and the match-efficient quantity satisfies

$$q_D^*(q_U) = \Big(\frac{q_U^{\mu}}{1+\mu}\Big)^{1/(\eta+\mu)}. \tag{B.2}$$

Lemmas B.1 and B.2 verify Assumptions 2 and 1 analytically, and we check Assumption E.1 numerically at the end.

Lemma B.1 (Assumption 2)**.** *Under the parametrization, $\zeta = \mu$ and $\varepsilon = 1+\mu$ everywhere, so* (20) *holds if and only if*

$$\eta \;\le\; 1 + \frac{(1+\mu)(1-\theta_D)}{\theta_D}, \tag{B.3}$$

a bound that always exceeds one. Moreover the sharp inequality (A.4) *holds pointwise on regime (i) for every $\eta \le 1$, whatever the remaining parameters.*

Proof. From (B.1), $g''_{DD} = \mu(1+\mu) q_D^{\mu-1} q_U^{-\mu}$ and $g''_{UD} = -\mu(1+\mu) q_D^{\mu} q_U^{-(1+\mu)}$, so $\zeta = q_D g''_{DD}/g'_D = \mu$ and $\varepsilon = q_D g''_{UD}/g'_U = 1+\mu$. Substituting into (20) gives $\theta_D(\eta+\mu) \le 1+\mu$, which rearranges to (B.3), and the right side exceeds one because $(1+\mu)(1-\theta_D)/\theta_D > 0$. At $\alpha = \theta_D = 1/2$, so $\mu = 1$, the bound is $\eta \le 3$. For the sharp statement, substituting $\zeta = \mu$ and $\varepsilon = 1+\mu$ into (A.4) gives

$$\varepsilon \tilde{D} - \theta_D(\eta+\zeta) g'_D = (1+\mu)(1-\theta_D)\, u' - \theta_D\, g'_D\, (\eta - 1), \tag{B.4}$$

whose right side is positive whenever $\eta \le 1$. The restriction $\eta \in (0,1)$ in the benchmark serves only the normalization $u(0) = 0$, while the curvature restriction itself requires only (B.3). □

Lemma B.2 (Assumption 1)**.** *Under the parametrization, $S_U^n = \Omega_n - c$ is strictly concave on each trading regime, and Ω'_n does not jump upward at the regime boundary, for all parameter values. Hence Assumption 1 holds.*

Proof. Since c is linear, it is enough that Ω'_n be decreasing in q_U at fixed ω. *Regime (ii)*: substituting (B.2) into S_D, the u and g terms are proportional to the same power of q_U, so $\Omega_n(q_U) = \Gamma q_U^{\nu}$ with $\Gamma > 0$ and $\nu = \mu(1-\eta)/(\eta+\mu) \in (0,1)$, which is strictly concave. *Regime (i)*: write $\Omega'_n = \sigma_D(1-\theta_D) M$ with, by (18), $\ln M = \ln\mu + (1+\mu-\eta)\ln q_D - (1+\mu)\ln q_U - \ln \tilde{D}$. Along the binding payment constraint at fixed ω, $d\ln q_D / d\ln q_U = e$, the elasticity defined in

the proof of Lemma 3(ii), which here equals $\left[\theta_D\mu/(1+\mu)\right] g'_D/\tilde{D} \in (0,1)$, and differentiating,

$$\frac{d\ln M}{d\ln q_U} = e\,(1+\mu-\eta) - (1+\mu) + \frac{\eta(1-\theta_D)\,u'\,e + \mu\,\theta_D\,g'_D\,(1-e)}{\tilde{D}}. \tag{B.5}$$

The right side is linear in e, and at $e=1$ it equals $-\eta\theta_D g'_D/\tilde{D} < 0$ while at $e=0$ it equals $\mu\theta_D g'_D/\tilde{D} - (1+\mu) < 0$, so it is negative for every $e \in [0,1]$ and Ω'_n is strictly decreasing within regime (i). *Boundary*: at the boundary between regimes (i) and (ii) the payment constraint just binds, so $s \to 0$ and $L_D \to 0$, and (18) converges to the regime-(ii) slope $-\sigma_D(1-\theta_D)\,g'_U$, so Ω'_n is continuous. □

Assumption E.1 has no useful closed form, because it involves the upstream premium evaluated at the bargained input purchase, so we verify it numerically. It has content only where the downstream meeting is payment constrained, since otherwise $\partial L_U/\partial \bar{d}_D = 0$ identically by Lemma 1(ii). The check reduces to one dimension. Firm wealth enters the premium only through the bargained input stock, which is increasing in w, and with linear c the ratio in Assumption E.1 at input stock q_U equals

$$\mathcal{R}(q_U;\omega) = (1-\theta_U)\,\Omega_\omega + \left[(1-\theta_U)\Omega' + \theta_U\right]\frac{\Omega'_\omega}{-\Omega''},$$

where ω is the consumer's payment capacity, the subscript ω denotes a partial derivative in it, and $\Omega_\omega = \sigma_D(1-\theta_D)L_D$. Assumption E.1 holds if and only if $\mathcal{R}$ is nondecreasing in q_U on the range where both payment constraints bind. We evaluate $\mathcal{R}$ at 160 input stocks spanning that range, on a grid of $(\alpha,\eta,\theta_D,\theta_U)$ in $\{0.3,0.4,0.5,0.6,0.7\}^4$ with $\sigma_D = 1$ and ω equal to the sum of consumer money holdings and downstream limits in $\{0.05,0.2,0.5,1.0\}$, and on 4,000 random draws with $\alpha \in [0.15,0.85]$, $\eta,\theta_D,\theta_U \in [0.05,0.95]$, $\sigma_D \in [0.1,1]$, and ω log-uniform on $[0.01,3]$. Both payment constraints bind somewhere in 2,110 grid cells and 1,826 random cells, and $\mathcal{R}$ is nondecreasing in every one of them. For replication, note that finite-difference steps must be small enough not to cross a regime boundary, and that the range where both constraints bind can be very narrow, so the grid of input stocks should be scaled to its width.

The same routine confirms the two analytical lemmas outside the ranges they are stated for, as well as the property of the marginal value of inputs used in the proof of Proposition 3(i). As $q_U \to 0$ at fixed consumer liquidity the downstream meeting is eventually unconstrained, $\Omega_n(q_U) = \Gamma q_U^\nu$ with $\nu = \mu(1-\eta)/(\eta+\mu) \in (0,1)$ by Lemma B.2, and hence $\Omega'_n(q_U) = \Gamma\nu\, q_U^{\nu-1} \to \infty$. This is how the Inada condition on u passes through the chain to the firm's marginal valuation, and it delivers $L_U^n(w) \to 1/(1-\theta_U)$ as $w \to 0$.

B.2 Solution Method

The numerical results in Section 5 solve the full model at interior record keeping, with both firm types, both supplier types, and the two-sided money demand system, without the restrictions used in Corollary E.1. Every decentralized object has a closed form along the binding payment constraint once the constraint is parametrized by the traded quantity. Downstream, for an input stock q_U we trace q_D from zero to $q_D^*(q_U)$ and record the payment

(14), the match surplus, and the premium (29). Upstream, for a consumer payment capacity ω the downstream constraint binds when q_U exceeds the level at which $p_D(q_D^*(q_U)) = \omega$. Above that level the binding constraint can be inverted for q_U as a function of q_D,

$$q_U = \left[\frac{\theta_D \, q_D^{1+\mu}}{\omega - (1-\theta_D)\, u(q_D)}\right]^{1/\mu},$$

and below it $q_D = q_D^*(q_U)$. Tracing either branch gives Ω_n, Ω_n' from (18), the payment (22), the upstream surplus, and the premium (32) in closed form. We interpolate the surpluses with cubic Hermite splines, using the fact that the derivative of each surplus with respect to the buyer's wealth is the corresponding premium, so the interpolants are consistent with the first-order conditions.

Given the three limits $(\bar{d}_D, \bar{d}_U^0, \bar{d}_U^1)$, we solve the portfolio system (30)–(33) as a scalar equation in consumer real balances, solving each firm type's first-order condition inside it. This system always has a nonmonetary solution, in which consumers hold no money and cash-only firms are idle, and we select its largest root. Given portfolios, each limit is the largest root of its flow equation, with the defaulter's portfolio solved separately. We iterate on the three limits to a tolerance of 10^{-9} or tighter, starting from the solution at the neighboring rate. The reaction functions of Section 4.3 are computed the same way, holding one block fixed, and χ is the slope of the composite map $\bar{d}_D \mapsto \Phi_D\big(\Phi_U^0(\bar{d}_D), \Phi_U^1(\bar{d}_D)\big)$, obtained by central differences. The decomposition (40) is computed at fixed limits, with the own-market term from the envelope expression and the total from a central difference in i that lets portfolios adjust. As a check, the multiplier formula (41), built from the computed reaction slopes, reproduces the equilibrium response of $\bar{d}_D$ to i computed by finite differences to within half a percent at the rates we check (7.2, 7.8, and 8.1 percent).

Appendix C: Consignment Contracts

This appendix records the benchmark referred to in Section 6.2. Modify the upstream protocol so that the firm's obligation p_U is settled in the CM if and only if the firm meets a consumer in the DM. For this exercise, suppose the supplier can verify the downstream match and that all payment constraints are slack, while everything else is as in Section 2. Write

$$S_D^*(q_U) \equiv u\big(q_D^*(q_U)\big) - g\big(q_D^*(q_U) \mid q_U\big)$$

for the gross downstream match surplus at the match-efficient quantity, recall $\mathcal{V}(q_U) = \sigma_D S_D^*(q_U) - c(q_U)$ from Section 6, and let

$$A \equiv 1 - (1-\theta_U)\,\theta_D \in (0,1).$$

Downstream. The firm owes p_U when it meets the consumer, so the surpluses are $u(q_D) - p_D$ for the consumer and $p_D - g(q_D \mid q_U) - p_U$ for the firm, and the match surplus is $u(q_D) - g(q_D \mid q_U) - p_U$. Since p_U does not vary with q_D, the quantity is unchanged at $q_D = q_D^*(q_U)$, defined by $u' = g_D'$. Proportional bargaining with consumer share θ_D delivers the firm $(1-\theta_D)\big[S_D^*(q_U) - p_U\big]$.

Upstream. The firm's expected surplus is $\sigma_D(1-\theta_D)\big[S_D^*(q_U) - p_U\big]$, and the supplier's is $\sigma_D p_U - c(q_U)$, because the supplier bears the production cost for certain and is paid only when a downstream match occurs. Total upstream surplus is

$$S_U^{\kappa}(q_U, p_U) = \sigma_D(1-\theta_D)\, S_D^*(q_U) + \sigma_D \theta_D\, p_U - c(q_U). \tag{C.1}$$

Unlike (24), this depends on the payment. Raising p_U shrinks the downstream surplus and hence the consumer's claim on it, and the difference accrues to the upstream pair.

Proposition C.1 (Consignment attains the first best)**.** *Under the consignment protocol with slack payment constraints, proportional bargaining upstream with firm share θ_U gives*

$$\sigma_D p_U = \frac{\theta_U\, c(q_U) + (1-\theta_U)(1-\theta_D)\,\sigma_D S_D^*(q_U)}{A}, \tag{C.2}$$

and the firm's upstream surplus is

$$\frac{\theta_U(1-\theta_D)}{A}\,\mathcal{V}(q_U). \tag{C.3}$$

The bargained input purchase is therefore $q_U = q_U^*$ and the downstream quantity is $q_D = q_D^*(q_U^*)$, so the allocation is first best for every $(\theta_U, \theta_D) \in (0,1)^2$, every $\sigma_D \in (0,1]$, and every (u, f, G) satisfying the assumptions of Section 2.

Proof. Proportional bargaining requires $\sigma_D p_U - c = (1-\theta_U) S_U^{\kappa}$. Substituting (C.1) and collecting terms in $\sigma_D p_U$,

$$\sigma_D p_U\big[1 - (1-\theta_U)\theta_D\big] = \theta_U\, c(q_U) + (1-\theta_U)(1-\theta_D)\,\sigma_D S_D^*(q_U),$$

which is (C.2). The firm's surplus is $\sigma_D(1-\theta_D)S_D^* - (1-\theta_D)\,\sigma_D p_U$. Substituting (C.2),

$$(1-\theta_D)\,\frac{A\,\sigma_D S_D^* - \theta_U c - (1-\theta_U)(1-\theta_D)\,\sigma_D S_D^*}{A} = (1-\theta_D)\,\frac{\theta_U\big[\sigma_D S_D^* - c\big]}{A},$$

where the second equality uses $A-(1-\theta_U)(1-\theta_D) = 1-(1-\theta_U)\big[\theta_D+1-\theta_D\big] = \theta_U$. This is (C.3). The prefactor is a positive constant, so the bargained q_U maximizes $\mathcal{V}$, which is the planner's problem (6). Hence $q_U = q_U^*$, and the downstream quantity is match efficient at that input stock by the paragraph above. The supplier receives $(1-\theta_U)S_U^\kappa = (1-\theta_U)(1-\theta_D)\mathcal{V}(q_U)/A$, so both parties earn positive fractions of $\mathcal{V}$ and the contract is individually rational whenever there are gains from trade. □

Notice that the result does not depend on the bargaining shares. The prefactor in (C.3) varies with (θ_U, θ_D) but the maximand does not, so there is no Hosios-type condition to satisfy. It does not depend on the technology or on the matching probability either, because $\mathcal{V}$ appears intact. Nor does it come from deferred settlement. Under the unconditional contract of Section 3 the payment is also settled in the CM, yet the firm's surplus is $\sigma_D(1-\theta_D)S_D^* - c$ and the wedge is present. The two protocols differ in whether the obligation is still live when the firm bargains downstream.

The consignment payment is itself unsecured credit, so it is subject to the limited commitment of Section 4. The obligation falls due only in the state where the firm sells, while the punishment for default does not depend on that state, so incentive compatibility requires $p_U \le \bar{d}_U$ there, with $\bar{d}_U$ the limit of Proposition 2. By Corollary 1, $\bar{d}_U = 0$ whenever $i\Lambda_U < r$, so the contract is unenforceable near the Friedman rule. And since a contingent obligation pledges $\sigma_D \bar{d}_U$ in expectation against $\bar{d}_U$ for an unconditional one, a given limit supports less expected payment under consignment. Characterizing equilibrium under consignment therefore requires solving the debt-limit system jointly with (C.2), which we do not pursue. The exercise locates the welfare result of Proposition 6 in the contract space and does not propose consignment as an equilibrium arrangement.

Appendix D: The Role of the Assumptions

Section 6 discusses the pricing protocol and the input contract. This appendix covers record keeping, matching, bargaining, timing and punishment, and the role of money. Table 2 records which results use which assumptions.

Table 2: What the main results use

Result	Assumptions	Valued money	Bargaining	UM before DM
Vertical credit channel (Prop. 2)	2; E.1 if $\Lambda_U < 1$	no	no	no
Multiplicity (Prop. 3)	1, 2	no	no	no
Containment (Prop. 4)	1	yes	no	no
Own and cross effects (Prop. 5)	1, 2, E.1	yes	no	upstream own effect
Switch point (Section 4.6)	1, 2, $\Lambda_k < 1$	yes	reversal only	upstream threshold
Welfare (Prop. 6)	1, 2	yes	yes	no

D.1 Record Keeping

The share Λ_k scales the origin slope of the flow cost of default, and with it the sustainability threshold of Corollary 1. At $\Lambda_U = \Lambda_D = 0$ no credit is sustainable and the chain runs on money. At $\Lambda_U = \Lambda_D = 1$ any positive limit displaces the borrower's money (Corollary 1(ii)), so once credit is active at both links money has no value. We use this corner as a benchmark for the credit block, and as an equilibrium it is the nonmonetary chain of Appendix D.5. In earlier work partial record keeping is what generates coexistence (Sanches and Williamson 2010; Lotz and Zhang 2016; Gu, Mattesini, and Wright 2016). Here coexistence is a precondition, because the composition of payments is a margin only when neither instrument has displaced the other.

The mixed corners show what the two-sided structure adds. With $\Lambda_D = 0$ and $\Lambda_U > 0$, upstream credit survives whenever $i\Lambda_U > r$, disciplined by a downstream business financed with customers' money, but the downstream reaction is identically zero, so $\chi = 0$ and there is at most one positive limit. Multiplicity and the vertical multiplier therefore require credit at both links, and self-fulfilling collapses should be found where firms both take and extend trade credit. At interior Λ_D, a retail credit expansion raises the supplier credit of firms that extend credit and lowers that of firms that do not, whose customers arrive with smaller balances, so it widens the gap between them.

D.2 Matching Frictions

No result requires $\sigma_D < 1$. The sustainability threshold does not depend on it, because the same prefactor scales the borrower's money demand and the flow cost of default. The matching friction moves the downstream demonetization threshold $\sigma_D\theta_D/(1-\theta_D)$, and it is the model's demand risk: a firm finances its inputs before it knows whether it will sell, which is why the return on working capital is an option value.

D.3 Bargaining Powers

The consumer's share θ_D sizes the holdup wedge, caps the downstream premium, and gives money its value, and these roles cannot be separated (Section 6). The firm's share θ_U caps the upstream premium and gives firm money its value but creates no wedge, because it divides a surplus that is already determined. The proofs use the fact that the liquidity premia fall with wealth, which proportional bargaining delivers. Under generalized Nash bargaining the premia can move nonmonotonically (Aruoba, Rocheteau, and Waller 2007), the portfolio problems lose concavity, and the flow equations can have several positive solutions. Kalai's (1977) proportional solution replaces independence of irrelevant alternatives with monotonicity, which in a monetary economy says that a buyer who arrives with more wealth never ends up with a smaller share of a larger surplus, the property the proofs use, and Hu and Rocheteau (2020) give it strategic foundations. The wedge itself does not rely on this protocol. It arises between the two markets under any protocol that is efficient match by match and gives the consumer a positive share of the surplus.

D.4 Timing and Punishment

The sequencing of the two markets is our model of working capital, and the results that turn on firm money demand require it: the upstream own-market effect, the upstream demonetization threshold, and the liquidity multiplier of Lemma E.1. The vertical channel and the credit multiplier operate through continuation values and survive any ordering. Exclusion is from credit at the link where the default occurred. Relationship-specific exclusion sustains nothing here, because counterparties are drawn anew each period. With persistent relationships, as in Bocola and Bornstein (2026), the limit at each link would be bounded by the value of one relationship, and the cross-link dependence would survive. Chain-wide exclusion, the punishment of Kehoe and Levine (1993), raises the cost of default and strengthens the cross-link dependence, so our limits are lower bounds within the class of exclusion punishments. Exclusion need not be permanent either. If a defaulter's record is erased with probability δ each CM, the gap between good standing and default satisfies $\bar{d} = (1-\delta)\beta\big[\bar{d} + \Psi(\bar{d})\big]$, so every result holds with r replaced by $r_\delta \equiv (r+\delta)/(1-\delta)$ and Ψ unchanged.

D.5 The Role of Money

Money is essential for the usual reasons: sellers who cannot keep records accept nothing else (Kocherlakota 1998; Wallace 2001). The chain mechanism, though, does not need valued money. Consider the economy at $\Lambda_U = \Lambda_D = 1$ in which money is not available, so a defaulter is excluded from trade at the link where it defaulted and payment capacity is $\bar{d}_D$ downstream and $\bar{d}_U$ upstream. The limits still solve flow conditions of the form (35) and (37), Lemma 3 still applies, both reaction curves slope up, and there can be several steady states with credit at both links. Versions of these results appear in nonmonetary settings (Gottardi and Zhang 2026; Gong, Qiao, Wong, and Wright 2025). Other results do require money. With a monetary fallback the flow cost of default is smaller and the limits tighter (Proposition 4(iii)), self-enforcing credit needs $i > r/\Lambda_k$, and the comparative statics in i, the cross-market effect, and the question of whether the Friedman rule attains the first best

have no nonmonetary counterpart. And without money a link with no credit has no trade, so a collapse at either link shuts down the chain (Proposition 4(ii)).

Appendix E: General Results at Interior Record Keeping

This appendix collects results at interior record keeping that rest on restrictions on equilibrium objects or on how money demand responds. Section 5 solves the benchmark without these restrictions. Proofs are in Appendix A.

E.1 Uniform Complementarity

Assumption E.1 (Uniform complementarity)**.** Write the upstream premium of a credit-accepting firm as $L_U(w;\bar{d}_D)$, so $L_U^1(w) = L_U(w;\bar{d}_D)$ and $L_U^0(w) = L_U(w;0)$. On the constrained range, the ratio $\big(\partial L_U/\partial \bar{d}_D\big)\big/\big|\partial L_U/\partial w\big|$ is nondecreasing in w.

Assumption E.1 requires downstream credit to lift the whole schedule of upstream liquidity premia, not just its cash-poor end, so that a firm reoptimizing its money holdings cannot undo the lift. Unlike Assumption 2, it restricts an equilibrium object, and we do not have a primitive condition that implies it. It is used only when $\Lambda_U < 1$, where it enters Proposition 2(iii) and Proposition 5. Appendix B verifies it numerically in the benchmark.

E.2 Interior Downstream Record Keeping

At interior Λ_D the vertical channel is type specific. Retail credit crowds out consumer money (Lemma 2), and cash-only meetings are the ones type-0 firms are in, so by Lemma 3(i) a higher $\bar{d}_D$ lowers Ω_0' and hence $\bar{d}_U^0$, while it raises $\bar{d}_U^1$. Now follow the loop around. Retail credit crowds out cash, cash-only firms lose upstream credit and buy fewer inputs, and a defaulter's cash then buys less, which supports retail credit. The last step works because a consumer who has defaulted carries more money than one in good standing (Proposition 1(iii)), and cash-only firms are where he spends it. The loop through cash-only firms therefore reinforces the loop through credit-accepting firms, and Corollary E.1 shows that the credit block stays monotone once the order on $\bar{d}_U^0$ is reversed. Other channels of ambiguous sign work through portfolios. Consumer money demand rises with input stocks (Lemma 3(ii)), so type-1 credit can lift the cash-only firms' limit and work against the downstream limit. And at $\Lambda_U < 1$ firms carry less money for cash-only suppliers as their upstream limit rises, which lowers input stocks in those meetings and can reverse the sign of the downstream reaction (Section 4.3). Proposition 3(ii)–(iii) and Corollary E.1 therefore set $\Lambda_U = 1$.

Corollary E.1 (Interior downstream record keeping)**.** *Let $\Lambda_D \in (0,1)$, $\Lambda_U = 1$, and $i\Lambda_D > r$, and suppose money is valued and Assumption 2 holds. Let each side's money demand respond to its own credit limit but not to the other side's liquidity, and let Φ_D, Φ_U^0, and Φ_U^1 select the largest solutions of the flow equations, with the downstream solution consistent with the input stocks it induces as in Section 4.3. Then Φ_D is nondecreasing in $\bar{d}_U^1$ and nonincreasing in $\bar{d}_U^0$, Φ_U^1 is nondecreasing in $\bar{d}_D$, and Φ_U^0 is nonincreasing in $\bar{d}_D$. The credit block is therefore monotone in the order that reverses $\bar{d}_U^0$, its fixed points form a nonempty complete lattice in that order, and all of them are strictly positive. Part (iii) of Proposition 3 holds with χ the slope of $\bar{d}_D \mapsto \Phi_D\big(\Phi_U^0(\bar{d}_D), \Phi_U^1(\bar{d}_D)\big)$.*

The reversed order has a plain meaning. In the best configuration of credit, retail credit and the upstream credit of firms that sell on account are at their highest, while firms that sell for cash have the least upstream credit of any steady state. This is the dispersion prediction of Appendix D.1 in equilibrium form. The corollary's restrictions shut the channels of ambiguous sign described above, which run through money held for cash-only suppliers and through the portfolio feedback of Lemma E.1 below.

E.3 The Liquidity Multiplier

Conditions (30) and (33) form a two-sided money demand system. The consumer's premium depends on the firm's input stock, and the firm's premium on the consumer's liquidity. The portfolio problem thus has the same shape as the credit block. With a single decentralized market, real balances solve one Euler equation against a primitive surplus, as they do for the firm in Rocheteau, Wright, and Zhang (2018). Here, instead, the firm's return on money is an option on whether its customers can pay. The two conditions have to be solved jointly, and a feedback parameter governs their comparative statics much as χ does for credit.

Lemma E.1 (Regularity and the liquidity multiplier)**.** *Fix the debt limits, suppose Assumption 2 holds, and consider a solution of* (30) *and* (33) *with both real balances interior. Write the system as* $i = R_D(z^c, z^f)$ *and* $i = R_U(z^f, z^c)$*, where* R_D *and* R_U *are the right sides of* (30) *and* (33) *and subscripts denote partial derivatives, and define the portfolio feedback*

$$\xi \equiv \frac{R_{D,f}\, R_{U,c}}{R_{D,c}\, R_{U,f}} \geq 0, \tag{E.1}$$

the product of the cross-to-own sensitivity ratios of the two money demands. If $\xi < 1$ *at the solution, then the system is regular, real balances and total liquidity on both sides are strictly decreasing in* i*, and each side's response to the nominal rate exceeds its one-sided counterpart, computed holding the other side fixed, by at least the factor* $1/(1-\xi)$*.*

Lemma E.1 gives a sufficient condition for the regularity requirement in Proposition 5, stated in the same objects that discipline the credit block. It also turns up a second multiplier. A higher i lowers consumer liquidity, which lowers the firm's return to money, which lowers input stocks, and so on. The drain on each side exceeds its direct cause by at least $1/(1-\xi)$. The chain amplifies the tax through ξ, and where the limits reinforce each other it amplifies the credit response through χ as well.