Incorporating non-unitary income elasticities, choke prices and choke incomes into applied general-equilibrium models

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Abstract

Traditional applied general-equilibrium (AGE) models have always faced trade-offs between analytical and computational tractability and counter-empirical restrictions. One is the assumption of homothetic preferences implying unitary income elasticities of demand. Similarly, there is no "choke" income level, below which a certain good is not purchased and there is no choke price above which a good is not purchased, implying no changes in the extensive margin of trade. Here I analyze three different formulations of preferences which incorporate these features. The first two, Stone-Geary Modified (SGM) and direct explicit additivity (here labeled CRIE - constant relative income elasticity) have been rarely used due to inherent limitations. I avoid these constraints by introducing virtual substitute goods into a standard general-equilibrium framework. While other authors have explored these properties in alternative ways, my approaches have advantages for high-dimension simulation models in that they retain homothetic CES structures and functional forms so that they can slot right into existing modeling formats. A non-linear programming estimation of CRIE using US time-series data gives the ratio of the income elasticities of demand for services to goods as 1.510, and the elasticity of substitution between goods and services as 0.487. Parameterizing a general-equilibrium model to estimated parameters, counter-factuals show substantial deviations in predicted consumption changes between a homothetic model and the CRIE model when per-capita income rises or falls.

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Comments and additional references most appreciated. Two of the three numerical models in the paper require a solver for non-linear complementarity problems to compute boundary solutions and regime shifts. I use the MCP (mixed complementarity problem) solver PATH, accessed from GAMS, but also available through other software including R and Julia, maybe Matlab. The CRIE estimation requires the NLP (non-linear programming) solver in GAMS but also available elsewhere. See https://pages.cs.wisc.edu/~ferris/path.html

JEL codes: F10, F17, C63, C68

Key words: income elasticities, complementarity, non-linear programming, choke incomes, choke prices, applied general equilibrium, non-homothetic preferences, CES functions.

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1. Introduction

High-dimension applied general-equilibrium modeling (AGE) is now 50+ years old, with theoretical and algorithmic roots going back further. A good history of early techniques is Shoven and Whalley (1984). It is fair to say that these early efforts had to impose strict limits as to the types of microeconomic features that they could incorporate. A second major development at about the same time was the formulation of an alternative approach based on complementarity developed by Mathiesen (1985), who extended the mathematical programming (optimization) theorems of Karush (1939) and Kuhn and Tucker (1951) to economic equilibrium problems. This alternative was then computationally operationalized by Rutherford (1985, 1995) in his software mps/ge (mathematical programming system for general equilibrium).

Many limitations remain of course, some due to data restrictions with others due to a lack of insights as to how to convert empirically-relevant features into algebra and code. The purpose of this paper is to try to make progress on two constraining restrictions that are clearly counter-empirical. The first of these, in no particular order, is the continuing reliance on homothetic preferences: all income elasticities of demand are unity. Simple functional forms, CES in particular, allow for solutions to unit cost and expenditure functions depending only on prices, with the application of Shepard's lemma to these costs functions giving unit demands for goods and factors. These features immensely simplify high-dimension general-equilibrium models by neutralizing a critical endogeneity between trade, prices and per-capita income.

Data do not support this convenient assumption. Empirical estimates of income elasticities showing significant deviations from unity are presented in Caron, Fally and Markusen (2014, 2020), with further estimation and counter-factual simulations showing important quantitative effects on trade volume and factor prices from incorporating non-homotheticity. Table 1 is cross-section estimation from Caron, Fally and Markusen (2014) which lists estimated income elasticities for the GTAP data base, sorted in order from smallest to largest. The sample statistics (mean and standard deviations) on the right show a rather substantial departures from unity. Sectors, with an acknowledged bit of arbitrariness, are classified into primary, manufacturing and services. A simple OLS regression of income elasticities on 0-1 dummies for the three aggregate sectors is shown on the right: primary sectors have the lowest income elasticities, manufacturing in the middle, and services having the highest. Below those is the regression for just goods (primary plus manufacturing) and services.

Additional motivation is provided in Figure 1, where the top panel shows the shares of consumer expenditure on goods and services since 1950 and the growth of GDP per capita in constant dollars. While this fits nicely with Table 1, it could be that Figure 1 is explained by relative price changes between goods and services. The lower panel of Figure 1 shows the actual relative price change of services to goods, rising from (normalized) 1.0 to 3.4. Combined with literature estimates of the elasticity of substitution σ between goods and services of 0.5, this secular price change could contribute in part to the top-panel's data. But if we assume that preferences are homothetic CES and the elasticity of substitution is $\sigma = 0.5$, the relative price of services would have to rise by a factor of 11.4 in order to explain the share ratio change (also

Table 1: GTAP Sectors and Income Elasticities:
Caron, Fally, Markusen QJE 2014

GTAP sectors	Income elasticities	primary	manufac	services
Cereal grains nec	0.110	x		
Paddy rice	0.254	x		
Processed rice	0.352	x		
Sugar cane, sugar beet	0.433	x		
Animal products nec	0.444	x		
Bovine cattle, sheep and goats, horses	0.458	x		
Vegetable oils and fats	0.545	x		
Sugar	0.588	x		
Forestry	0.623	x		
Vegetables, fruit, nuts	0.640	x		
Petroleum, coal products	0.664	x	x	
Beverages and tobacco products	0.667		x	
Textiles	0.707		x	
Food products nec	0.777		x	
Dairy products	0.826		x	
Electricity	0.848		x	
Mineral products nec	0.874		x	
Chemical, rubber, plastic products	0.880		x	
Construction	0.880			x
Wheat	0.883	x		
Fishing	0.886	x		
Oil seeds	0.889	x		
Crops nec	0.893	x		
Air transport	0.929			x
Water transport	0.932			x
Machinery and equipment	0.938		x	
Wood products	0.970		x	
Transport equipment nec	0.981		x	
Leather products	0.981		x	
Transport nec	0.990			x
Metal products	0.992		x	
Bovine meat products	1.023		x	
Public Administration and services	1.033			x
Motor vehicles and parts	1.034		x	
Water	1.039			x
Paper products, publishing	1.044		x	
Meat products nec	1.052		x	
Wearing apparel	1.057		x	
Recreational and other services	1.075			x
Electronic equipment	1.094		x	
Manufactures nec	1.095		x	
Trade	1.106			x
Raw milk	1.118	x		
Communication	1.152			x
Business services nec	1.324			x
Financial services nec	1.331			x
Plant-based fibers	1.339	x		
Insurance	1.392			x
Wool, silk-worm cocoons	1.420	x		
Gas manufacture, distribution	2.221		x	x

Unweighted mean and standard deviation

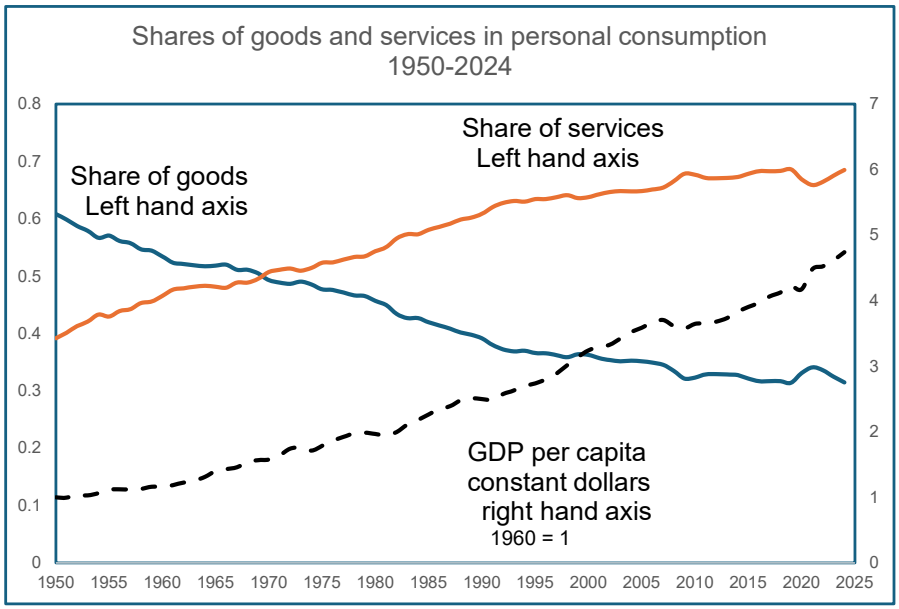
Mean	0.916
Standard deviation	0.342
Mean plus one SD	1.258
Mean minus one SD	0.573

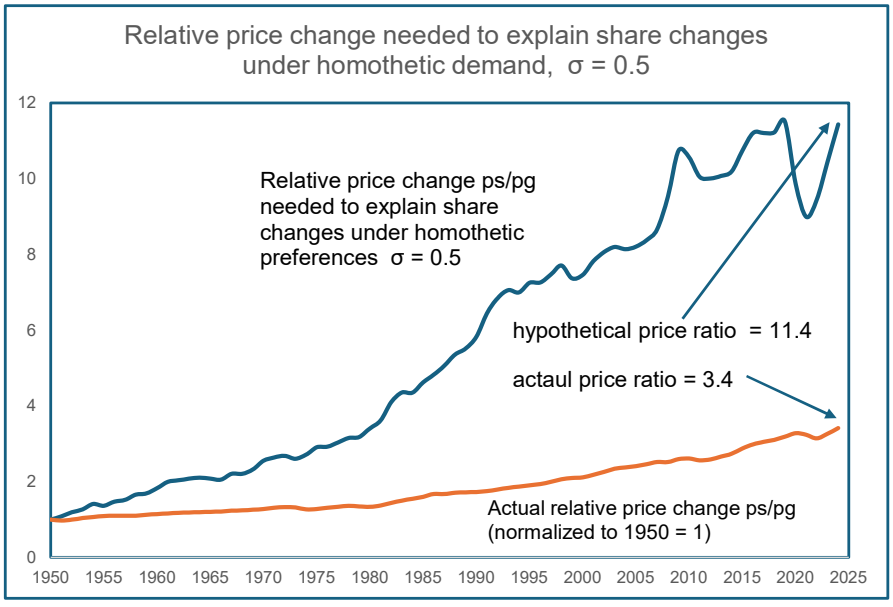
Regression of income elasticity on sector 0-1 dummies

	coefficient	standard error
Primary	0.646	0.066
Manufac	0.903	0.061
Services	1.115	0.078
Adjusted R2		0.899

	coefficient	standard error
Goods	0.828	0.043
Services	1.121	0.074
Adjusted R2		0.907

Figure 1: US personal consumption expenditure:
shares of goods and services in total, 1950-2024
σ = elasticity of substitution between goods and services





The actual relative price ps/pg tracks the relative shares G/S (top panel, not shown) closely.

=> With homothetic preferences, relative price and share changes are only consistent if
σ approximately 0: fixed coeifficient preferences

https://www.bea.gov/itable/national-gdp-and-personal-income

National data
National Income and Product Accounts
Table 2.3.5. Personal Consumption Expenditures by Major Type of Product
Publication Category Interactive Data NIPA Tables
Table 1.6.4 Price Indecies for Gross Domestic Purchases

about a factor of 3.4). To put it another way, with homothetic CES, the actual change in the share ratio and the actual change in the relative price (both about 3.4) are only consistent if the elasticity of substitution is approximately $\sigma = 0$: fixed coefficient preferences.

In addition, the homotheticity assumption does not allow for "choke" income levels below which certain goods or services are not purchased. This is clear counter-empirical since the world trade matrix is full of zeros (export from country i to country j in good k). The general tradition in AGE modeling is to hold initially-zero cells fixed at zero when doing counter-factuals. Thus there can be no changes in trade at the extensive margin in simulations. Without in any way demeaning their important contributions, this was embodied in the Armington (1969) assumption and in the popular Eaton-Kortum (2002) approach. These formulations also imply that any good that has a positive entry in the trade matrix can never go to zero. In the terminology of this paper, there is no "choke" income level for produced and traded goods.

The flip side of the absence of choke income levels is the absence, by assumption, of choke prices above which a good is not purchased. With CES preferences with any value for the elasticity of substitution σ, the demand price for a good goes to infinity as the quantity goes to zero. So any good produced anywhere in the world will always be imported be every single country no matter how high the tariff or transport cost. Again, the exceptions are trade cells initially zero and constrained to remain zero in running scenarios.

Figure 2 provides a motivating example of why income elasticities are important for general-equilibrium counter-factuals. Suppose that points A and B are observations on the per-capita income of two households/countries at the same set of prices for goods X_1 and X_2. There are two very different ways of calibrating preferences. One, which is the normal way in general-equilibrium modeling, is to assume that the countries have different, but homothetic preferences. If the countries' per-capita incomes grow in the same proportion, the Engel's curve for the sum of their purchases moves out along the steeper line. On the other hand, if we assume that the countries have identical but non-homothetic preferences, then equal growth in their incomes moves aggregate consumption out along the flatter (not necessarily linear) income curve. Beginning with the same data, the two options generate very different counter-factual results.

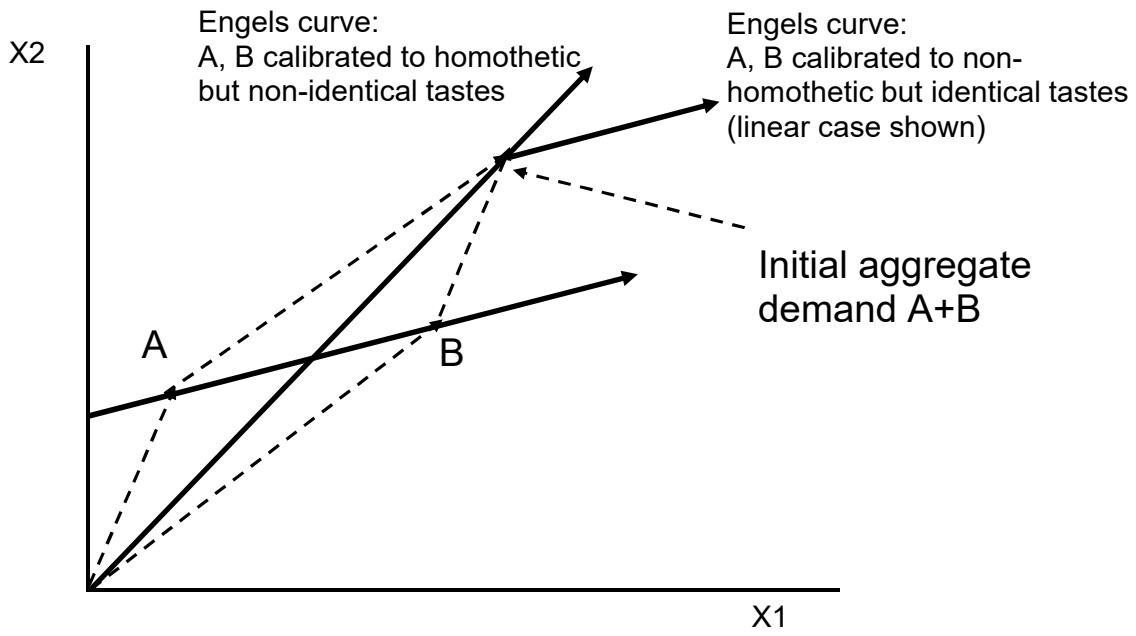
My first alternative proposes and analyzes a simple framework that allows for (a) non-unitary income elasticities of demand, (b) choke income levels below which a good is not purchased, and (c) choke prices above which a good is not purchased. These then allow for rich possibilities in counter-factuals including changes in the extensive marginal of trade (trade from i to j in good k switching on or off), reversals in the direction of trade, and an important role for per capita income in explaining trade flows.

My formulation is a straightforward modification of traditional Stone-Geary preferences (a great review and extensions to the AIDS formulation is in Deaton and Muellbauer (1980)). Whereas the Stone-Geary version has traditionally used negative parameters subtracted from consumption purchases and interpreted as minimum consumption requirements (e.g., Markusen, 1986), my version uses positive endowment parameters which could be thought of as (unobserved) fixed endowment goods/leisure time which are perfect substitutes for the

Figure 2: Why this is important for counter-factuals

Example: endowments X1E >0, X2E = 0
X1 has income elasticity of demand > 1, X2 income elasticity < 1

Calibrate (per capita) observations A and B
Homothetic but non-identical tastes
Non-homothetic but identical tastes



Alternative calibrations will yield very different counter-factual results when incomes rise or fall

(capital accumulation, technical change, lower trade costs)

consumption goods.[1] This implies that goods with large endowment parameters relative to their utility shares have income elasticities great than one, and have maximum choke prices and minimum choke income. This specification has been previously used by Markusen (2013), Simonovska (2015) and Jung, Simonovska and Weinberger (2019). The latter two papers are focused on price elasticities whereas this current paper is more interested in income elasticities.

My second alternative is to modify CRIE (constant relative income elasticities) preferences, an idea that has a long but small history: Hanock, (1975), Chao, Kim and Manne (1982). They have been exploited in their most basic form (direct explicit additivity in Hanock's terminology) in Fieler (2011) and Caron, Fally and Markusen (2014, 2020) in trade models. They do not allow for choke income and prices but offer advantages when there are large differences in per-capita income.. There is some commonality to my SGM version, in that the CRIE version uses endowed sector-specific fixed factors which, combined with purchased goods, map into consumption goods. As in the case of SGM, the resulting cost functions are all homothetic CES. My version also has the advantage of allowing for the independence of income and price elasticities which has not been possible in earlier analyses.

The third alternative is to assume traditional homothetic CES Armington preferences (domestic and foreign goods in an industry are imperfect substitutes), but adds inefficient latent domestic activities that can produce perfect substitutes for foreign varieties if prices are high enough. This introduces choke prices, although not choke incomes, on foreign imports, endogenizing zeros in the world trade matrix. Figure 3 gives a quick look at the three cases which may be useful early on: details to follow in subsequent sections.

I hope my formulations are valuable to researchers working with AGE models, either structural or calibrated. They retain homothetic CES preferences with all their advantages in large simulation models. These include simple solutions to cost and expenditure functions and the easy derivation of closed-form goods and factors demand equations via Shepard's lemma. Another advantage is that all functions are globally regular, making it an attractive choice for simulating large changes in trade costs, technologies, natural disasters etc. The first and third alternative do require that the models are formulated as non-linear complementarity problems, endogenizing corner/boundary solutions, but the software for that is well developed.

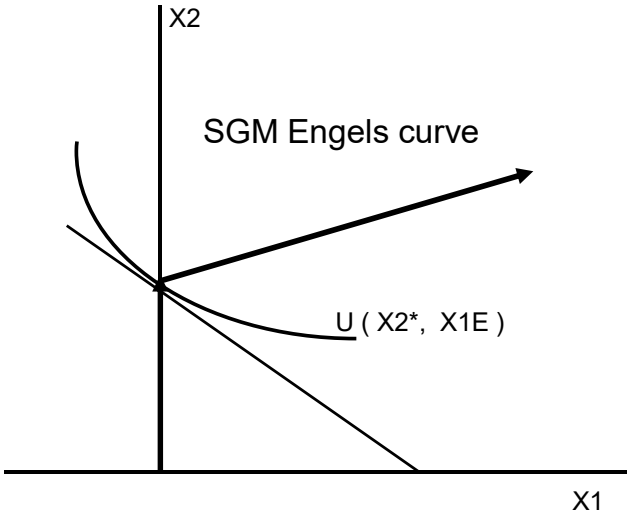
The paper includes an empirical exercise using the US time-series data shown in Figure 1 for CRIE preferences. The non-linear estimating equation is solved as an NLP (non-linear programming) problem for the CRIE parameters minimizing the sum of squared errors. Results are that the ratio of the income elasticity of demand for services to that for goods is 1.510. The estimated elasticity of substitution between goods and services is 0.487, which is consistent with literature estimates. I calibrate the CRIE general-equilibrium model to the parameter estimates and perform counter-factuals in which per-capita income rises or falls. Results show significant differences in predicted consumption changes when preferences are homothetic versus the CRIE estimated preferences.

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[1]No one has to watch 3.5 hour American football games. "Free" endowment substitutes like leisure include going for a walk, hike, or cycle, birdwatching, playing with your kids, etc. Other applications involving "virtual" fixed-endowment goods/factors include modeling a large open economy and specific factors of production in order to add strict concavity to some function.

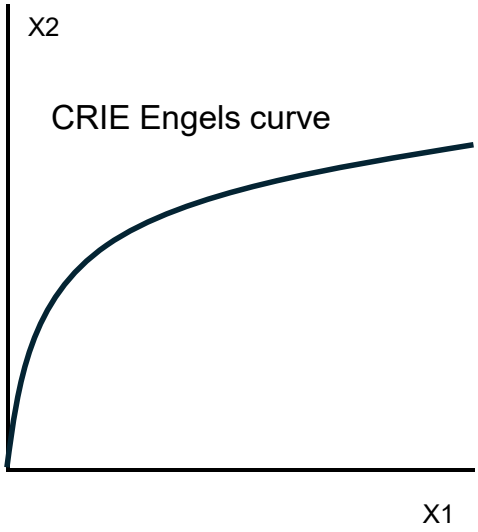
Figure 3: SMG, CRIE and homothetic Armington

X1 income elasticity > 1, X2 income elasticity < 1



Advantage: allows non-unitary income elasticities, choke incomes and choke prices

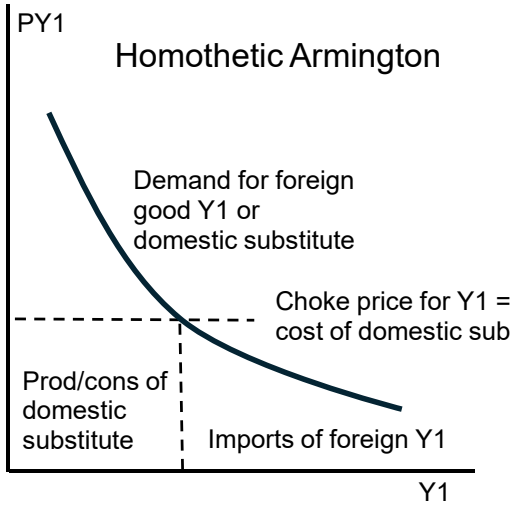
Disadvantage: income elasticities all converge to one as income grows large



Advantage: ratio of income elasticities constant over entire range of incomes

Allows for independence of price and income elasticities

Disadvantage: does not by itself allow for choke prices and incomes



Advantage: allows for choke incomes and choke prices while retaining all the simplicity of homothetic CES

Disadvantage: does not by itself allow for non-unitary income elasticities, but can be combined with one of the other options

There have been many contributions focusing on non-homothetic preferences over the decades, and a few focusing on choke prices or incomes. But I am not aware of any that treat these simultaneously in the same model framework (references welcome!). Linder (1961) is an important early work that focuses on the role of per capita income in explaining trade in non-primary products, although there is no formal model in his work. Some of the earlier work that did formalize the role of per capita income in general equilibrium were Markusen (1986), Bergstrand (1990) and Hunter (1991). After an apparent gap, interest reappeared in the new millennium with contributions by Matsuyama (2000), Fieler (2011), Markusen (2013) and Caron, Fally and Markusen (2014, 2020). For large-dimension simulation models, CRIE's usefulness has been limited by the fact that there are no close-form solutions for cost or demand functions, but I have found a work-around that retains all the features of homothetic CES.

Some other valuable contributions in this same time period include Simonovska (2015), Fajgelbaum and Khandelwal (2016), and Bertoletti, Federico and Simonovska (2018), Jung, Simonovska and Weinberger (2019) and Matsuyama (2019). These papers generally focus on different issues not closely related to this current paper (e.g., industrial-organization and income-distribution questions) and/or also use alternative preferences (e.g., indirect implicit additivity) which do not lend themselves easily to multi-sector general-equilibrium models. A number of references to papers finding an elasticity of substitution about 0.5 across broad sectors are found in Heblich, Redding, and Zylberberg (2025).

The three papers by Simonovska and co-authors just mentioned do feature choke prices. Wales and Woodland (1985) is a significant early contribution which is unfortunately unknown to trade economists. Wales and Woodland uses quadratic preferences which are also not easy to accommodate in large AGE models. Much more recently, Eaton, Kortum and Sotelo (2013), Gandri, Lu and Shi (2023) and Anderson and Zhang (2025) focus on the econometrics of endognizing zeros. Like Wales and Woodland, these combine theory and estimation, but are perhaps not particularly easy to slot into large simulations models. Fajgelbaum and Khandelwal and Anderson and Zhang use an AIDS formulation derived from Stone-Geary.

2. Stone-Geary-modified preferences, demand, income and price elasticities

First, let's consider the basics in just a partial-equilibrium framework: prices and money income held constant. For each purchased good, there is an unobserved perfect-substitute endowment good (some of which may be zero). This is the notation for the simple case used to derive demand functions and income elasticities of demand. [2]

[2]Earlier versions of the paper, available from the author, also derived price elasticities of demand. But these are not very informative and tangential to the objectives of the paper and so have been dropped from this version for both SGM and CRIE.

Non-negative variables

X_{ip}	amount of good i purchased from the market	(*p* for purchased)
X_{ic}	amount consumed: purchase plus endowment	(*c* for consumed)
p_{ic}	implicit consumer price of good i - equal to p_i if $X_{ip} > 0$, less if no i purchased	
e	CES price index	
MT	total "income" including the value of the endowments	
W	welfare (utility)	

Parameters

X_{ie}	perfect substitute endowment good $X_{ie} \geq 0$, can't be traded (*e* for endowment)
p_i	market price of purchased good i - parameter in partial equilibrium model
β	substitution parameter
α_i	share parameters on the goods
σ	elasticity of substitution among goods
M	money income that can be spent in the market - parameter in the PE model

The modified Stone-Geary (SGM) welfare (utility) function is a simple case, but the quantities of each good are the sum of the purchased amount plus the perfect substitute endowment good.

$$W = \left[\sum_i \alpha_i (X_{ip} + X_{ie})^{\beta}\right]^{1/\beta} \quad 0 < \alpha_i < 1, \quad -\infty < \beta < 1, \quad \sum_i \alpha_i = 1, \quad X_{ie} \geq 0 \qquad (1)$$

Definitional equations for X_i consumed (sum of purchases and endowments), elasticity of substitution σ, and income including the value of endowment.

$$X_{ic} = X_{ip} + X_{ie} \qquad \sigma = \frac{1}{1-\beta} \qquad MT = M + \sum_i p_{ic} X_{ie} \qquad (2)$$

CES price index giving the cost of one unit of welfare.

$$e = \left[\sum_i \alpha_i^{\sigma} p_{ic}^{(1-\sigma)}\right]^{\frac{1}{1-\sigma}} \qquad (3)$$

Demand function for X_{ic}.

$$X_{ic} = X_{ip} + X_{ie} = (\alpha_i / p_{ic})^{\sigma} e^{\sigma - 1} MT \qquad X_{ip} > 0 \iff (\alpha_i / p_{ic})^{\sigma} e^{\sigma - 1} MT > X_{ie} \qquad (4)$$

Figure 3 top panel shows this basic two-good case where $X_{1e} > 0$ and $X_{2e} = 0$. X_{1p} is constrained to be non-negative, so the Engel's curve for observed purchases runs up the X_2 axis to the choke point, and then is linear as show holding prices constant. At some set of income and prices M^*, p_1^*, p_2^* the solution is $X_{1p} = 0$, $X_{2p|} = M^*/p_2^*$ and $X_{1c} = X_{1e}$.

The SGM formulation automatically ensures that no endowment is sold. Multiple through by p_{ic} and sum over i.

$$\sum_i p_{ic}X_{ic} \;=\; \sum_i (\alpha_i^{\sigma} p_{ic}^{1-\sigma})\, e^{\sigma-1} MT \;=\; e^{1-\sigma} e^{\sigma-1} MT \;=\; MT \tag{5}$$

$$\sum_i p_{ic}X_{ic} = \sum_i (p_{ic}X_{ip} + p_{ic}X_{ie}) \;=\; M + \sum_i p_{ic}X_{ie} \;\Rightarrow\; \sum_i p_{ic}X_{ip} \;=\; M \tag{6}$$

Thus market purchases equal money income so adding up is always (globally) satisfied. Note that it will be the case that $p_{ic} = p_i \;\; if \;\; X_{ip} > 0$ (perfect substitutes X_{ip} and X_{ie} will have the same price if there are market purchases $X_{ip} > 0$). The implicit price p_{ic} will be less than the market price p_i if no market is purchased: $X_{ip} = 0$.

Equation (4) can be used to solve for the "choke" income level, the value of M below which there is no X purchased at given prices, and also for the choke price above which no X is purchased for the income level and other goods' prices. For good X_1, these values are implicitly

$$(\alpha_1/p_{1c})^{\sigma} e^{\sigma-1}(M + \sum_i p_i X_{ie}) \;\le\; X_{1e} \qquad \Rightarrow \qquad X_{1p} \;=\; 0 \tag{7}$$

The Cobb-Douglas case with $\sigma = 1$ makes this easier to interpret

$$X_{1p} \;=\; 0 \quad \Leftrightarrow \quad \alpha_1(M + \sum_i p_i X_{ie}) \;\le\; p_1 X_{1e} \tag{8}$$

with the choke income level M^* give by

$$M_1^* \;=\; Max\left[\frac{1}{\alpha_1}(p_1 X_{1e} - \alpha_1 \sum_i p_i X_{ie}),\; 0\right] \qquad M_1^* > 0 \;\Leftrightarrow\; \frac{p_1 X_{1e}}{\sum_i p_i X_{ie}} > \alpha_1 \tag{9}$$

M^* is strictly positive for good X_1 iff its value share of the endowment is great than its utility weight, otherwise there is no choke income level. If $X_{1e} = 0$, there is no choke income level.

From (8) (Cobb-Douglas), the choke price for X_1 given M and prices of other goods is

$$p_1^* \;=\; Min\left[\frac{\alpha_1(M + \sum_{j \neq i} p_j X_{je})}{(1-\alpha_1)X_{1e}},\; +\infty\right] \qquad p_1^* < +\infty \;\Leftrightarrow\; X_{1e} > 0 \tag{10}$$

A particular simple case for capturing the intuition occurs when good 1 is the only one with a positive endowment.

$$p_1^* \;=\; \frac{\alpha_1 M}{(1-\alpha_1)X_{1e}} \qquad \text{special case with Cobb-Douglas, only } X_{1e} > 0. \tag{11}$$

The top panel of Figure 3 shows this basic case where $X_{1e} > 0$ and $X_{2e} = 0$ (nothing in the picture relies on Cobb-Douglas).

Now turn to income elasticities of demand. The expenditure on all X_i consumption (purchased plus endowment) as a share of all income (MT = money income plus endowment income) is given by

$$\frac{p_{ic}X_{ic}}{MT} = \frac{\alpha_i^{\sigma} p_{ic}^{1-\sigma}}{\sum_i \alpha_i^{\sigma} p_{ic}^{1-\sigma}} = s_i \qquad p_{ic}X_{ic} = p_{ic}(X_{ip} + X_{ie}) = s_i(M + \sum_i p_{ic}X_{ie}) \tag{12}$$

There is one qualifying limitation here: the share defined in this way is the share of total consumption including the endowment quantity $X_{ic} = X_{ip} + X_{ie}$ in total income including the endowment income. That is not the quantity of X and the money income M observed in the data. The amount of purchased, X_{ip}, is given by

$$X_{ip} = \frac{s_i M}{p_{ic}} + \frac{s_i \sum_i p_{ic}X_{ie}}{p_{ic}} - X_{ie} \tag{13}$$

Let $X_{ih} = s_i M/p_{ic}$ denote the purchase of X_i under homothetic demand with no endowments. Note also from (13) that X_{ih} is the asymptotic amount of X_{ip} purchased when M grows arbitrarily large. Alternatively, s_i is the marginal share of added income spent on X_{ip}. Equation (14) then implies that the purchase of a good is less than what it would be under standard homothetic preferences (X_{ih}) when the following holds:

$$X_{ih} = \frac{s_i M}{p_{ic}} > X_{ip} \qquad <=> \qquad \frac{p_{ic}X_{ie}}{\sum_i p_{ic}X_{ie}} > s_i \tag{14}$$

The amount purchased X_{ip} is less than the amount under homothetic preferences (all $X_{ie} = 0$) if the share of X_{ie} in endowment value is greater than the marginal share spent on X_{ip}.

The income elasticity of demand for X_{ip} with respect to income M is found from (13).

$$\frac{M}{X_{ip}}\frac{dX_{ip}}{dM} = \frac{s_i M}{s_i M + s_i \sum_i p_{ic}X_{ie} - p_{ic}X_{ie}} \tag{15}$$

$$\frac{M}{X_{ip}}\frac{dX_{ip}}{dM} > 1 \qquad <=> \qquad \frac{p_{ic}X_{ie}}{\sum_i p_{ic}X_{ie}} > s_i \tag{16}$$

Note that the right-hand inequalities in (14) and (16) are the same. The income elasticity of demand for X_{ip} is great than one if the share of X_{ie} in endowment value is greater than the marginal share spent on X_{ip}. For example, with two goods and a positive endowment for only X_1, the income elasticity will be greater than one for X_1 and less than one for X_2. But (15) also

implies that all the income elasticities converge to one as (money) income becomes large. This property is specific to the Stone-Geary formulation, but for all non-homothetic preferences the income elasticities of demand for high elasticity goods must fall toward one as income grows large. This is required by the budget constraint, and the algebra will be presented later.

3. Small open-economy general-equilibrium model

I begin with a very simple, straight-forward case to illustrate the determination of choke prices and income. There are two goods, X_1 and X_2 with prices p_{x1} and p_{x2}, and two factors of production L and K in inelastic supply, with (endogenous), prices p_l and p_k. One consumer with fixed non-traded endowments X_{1e} and/or X_{2e} with X_{1e} and X_{2e} being perfect substitutes for the produced and trade goods respectively.

The country faces fixed world relative prices for traded goods X_1 and X_2 (no Armington product differentiation). If a good is not purchased (not produced or imported), then its consumer price p_{xic} will be determined by its endowment substitute only and will be less than the world price: the cost inequalities for domestic production or imported X_i are strict. Several other variables and notation are as follows. Over bars indicate fixed values.

$c_{x1}(p_l,p_k)$ $c_{x2}(p_l,p_k)$ $c_w(p_{x1c},p_{x2c})$	unit cost functions for X_1, X_2, and W
	partials give unit demands for X_1, X_2, and W by Shepard's lemma
T_{12} T_{21}	activities which export X_1 for X_2, X_2 for X_1 respectively
	the ratio of inputs to output is the world price ratio
tot	parameter which give world price ratio p_{x1}/p_{x2} in the T activities
CONS	consumer income including value of endowments

The model given below consists of 19 weak inequalities with 19 matched non-negative variables. These are divided into three blocks. The first are pricing inequalities derived from optimization conditions, the direction being marginal cost greater-than-or-equal to price, with the complementary variable being the output of that activity. Note that welfare is just treated as a produced good with price p_w, the consumer price index (cost of buying one unit of utility).

Price inequalities (e.g., marginal cost ≥ price)		quantities as complementary variables	
$c_{x1}(p_l,p_k) \geq p_{x1}$	$\perp$	X_1	(17)
$c_{x2}(p_l,p_k) \geq p_{x2}$	$\perp$	X_2	(18)
$p_{x1} \geq p_{x1c}$	$\perp$	X_{1p}	(19)
$p_{x2} \geq p_{x2c}$	$\perp$	X_{2p}	(20)
$p_{x1e} \geq p_{x1c}$	$\perp$	X_{1e}	(21)
$p_{x2e} \geq p_{x2c}$	$\perp$	X_{2e}	(22)
$p_{x2}tot \geq p_{x1}$	$\perp$	T_{21}	(23)

$$p_{x1} \geq p_{x2}tot \quad \perp \quad T_{12} \tag{24}$$

$$c_w(p_{x1c}, p_{x2c}) \geq p_w \quad \perp \quad W \tag{25}$$

The second block are market-clearing inequalities, with their direction being supply greater-than-or-equal to demand. The complementary variables are always prices. It is important for the modeler to emphasize that these good and factor demands on the right-hand side must be consistent with and derived from the cost functions above via Shepard's lemma.

Market clearing (e.g., supply ≥ demand) prices as complementary variables

$$X_1 \geq X_{1p} - T_{12} + T_{21}/tot \quad \perp \quad p_{x1} \tag{26}$$

$$X_2 \geq X_{2p} + T_{21} - T_{12}tot \quad \perp \quad p_{x2} \tag{27}$$

$$X_{1p} + X_{1e} \geq \partial c_w/\partial p_{x1c} W \quad \perp \quad p_{x1c} \tag{28}$$

$$X_{2p} + X_{2e} \geq \partial c_w/\partial p_{x2c} W \quad \perp \quad p_{x2c} \tag{29}$$

$$\bar{X}_{1e} \geq X_{1e} \quad \perp \quad p_{x1e} \tag{30}$$

$$\bar{X}_{2e} \geq X_{2e} \quad \perp \quad p_{x1e} \tag{31}$$

$$\bar{L} \geq \partial c_{x1}/\partial p_l X_1 + \partial c_{x2}/\partial p_l X_2 \quad \perp \quad p_l \tag{32}$$

$$\bar{K} \geq \partial c_{x1}/\partial p_k X_1 + \partial c_{x2}/\partial p_k X_2 \quad \perp \quad p_k \tag{33}$$

$$W \geq CONS/p_w \quad \perp \quad p_w \tag{34}$$

The final block is income-balance inequalities. Here there is just a single one for the representative consumer. This is effectively just a definitional equation, since it could be substituted into the market-clearing equation for welfare just above.

Income balance income is the complementary variable

$$CONS \geq p_l\bar{L} + p_k\bar{K} + p_{x1e}\bar{X}_{1e} + p_{x2e}\bar{X}_{2e} \quad \perp \quad CONS \tag{35}$$

4. A calibrated example with simulations

The following matrix represents a micro-consistent set of initial values for this model. Columns are inputs and output for the sectors of the model including the "production" of welfare and consumer income. Rows are markets with outputs (supplies) positive and inputs (demands) negative. Micro-consistency means that all row and column sums should be zero. There are two features to note. The first is that I have calibrated the model to zero trade initially (no Armington differentiation, so this is possible). Second, X_1 has a large amount of its perfect substitute endowment good, $p_{x1e} = 150$, while $p_{x2e} = 1$. This spread is much greater than the

difference in consumption shares, so good X_1 will have an income elasticity of demand greater than one and vice versa for good X_2.

	Production Sectors						Trade		Welfare	Consumer
	X1	X2	X1P	X1E	X2P	X2E	TX21	TX12	W	CONS
Markets										
PX1	100		-100				-0	0		
PX2		100			-100		0	-0		
PL	-25	-75								100
PK	-75	-25								100
PW									351	-351
PX1C			100	150					-250	
PX1E				-150						150
PX2C					100	1			-101	
PX2E						-1				1

These values then allow for the numerical calibration of the cost functions for the two goods and welfare. For simplicity and clarity, I'll just impose Cobb-Douglas functional forms on the goods and welfare. The matrix of values then gives us the numerical versions for producing X_1, X_2 and welfare W.

$$c_{x1}(p_l,p_k) = p_l^{0.25}p_k^{0.75} \qquad c_{x2}(p_l,p_k) = p_l^{0.75}p_k^{0.25} \tag{36}$$

These give unit factor demand equations in (35) and (36) by Shepard's lemma as follows.

$$\frac{\partial c_{x1}(p_l,p_k)}{\partial p_l} = 0.25p_l^{0.25-1}p_k^{0.75} \qquad \frac{\partial c_{x1}(p_l,p_k)}{\partial p_k} = 0.75p_l^{0.25}p_k^{0.75-1} \tag{37}$$

$$\frac{\partial c_{x2}(p_l,p_k)}{\partial p_l} = 0.75p_l^{0.75-1}p_k^{0.25} \qquad \frac{\partial c_{x2}(p_l,p_k)}{\partial p_k} = 0.25p_l^{0.75}p_k^{0.25-1} \tag{38}$$

The cost (expenditure) function for producing one unit of utility is given by

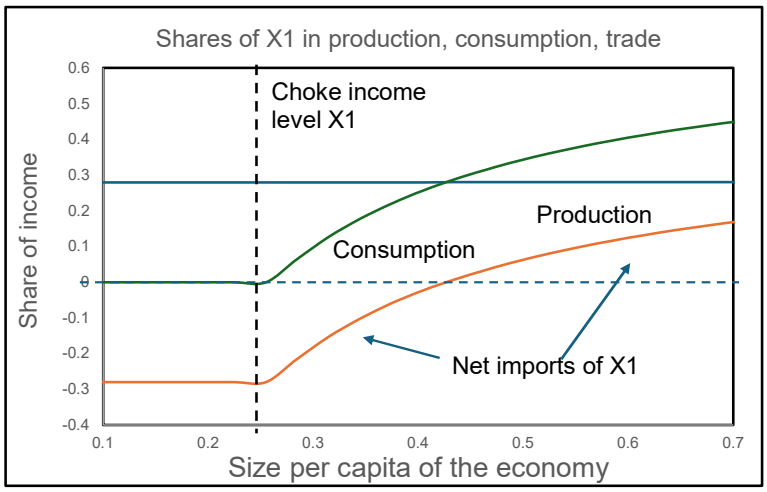
$$c_w(p_{x1c},p_{x2c}) = p_{x1c}^{\alpha_1}p_{x2c}^{\alpha_2} \qquad \alpha_1 = 250/351 \qquad \alpha_2 = 101/351 \tag{39}$$

Shepard's lemma give the unit commodity demand functions for producing a unit of W in (43) as

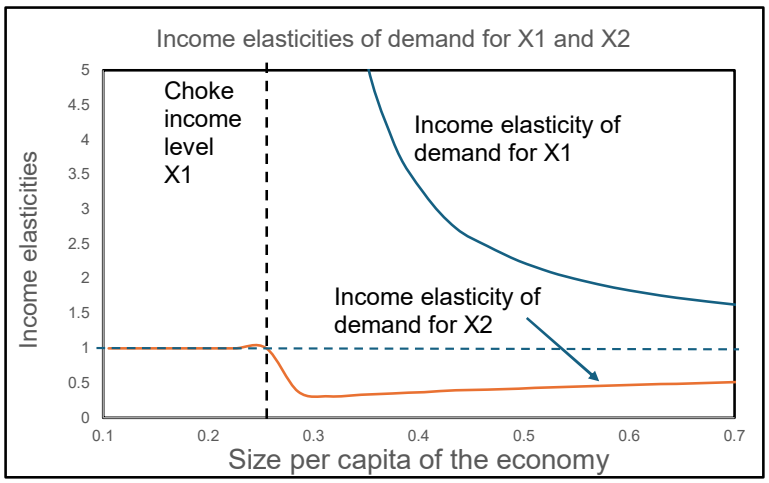
$$\frac{\partial c_w(p_{x1c},p_{x2c})}{\partial p_{x1c}} = \alpha_1 p_{x1c}^{\alpha_1-1}p_{x2c}^{\alpha_2} \qquad \frac{\partial c_w(p_{x1c},p_{x2c})}{\partial p_{x2c}} = \alpha_2 p_{x1c}^{\alpha_1}p_{x2c}^{\alpha_2-1} \tag{40}$$

Figure 4 presents results for this model. Panel A and B loop over the size of the economy, increasing the endowments of labor and capital in the same proportion while holding the levels of the endowment substitutes constant. Panel A focuses on purchased level of good X_{1p} with the high income-elasticity of demand (high endowment substitute), expressing results in shares of income (observed M, labor and capital income). The share of X_1 production is

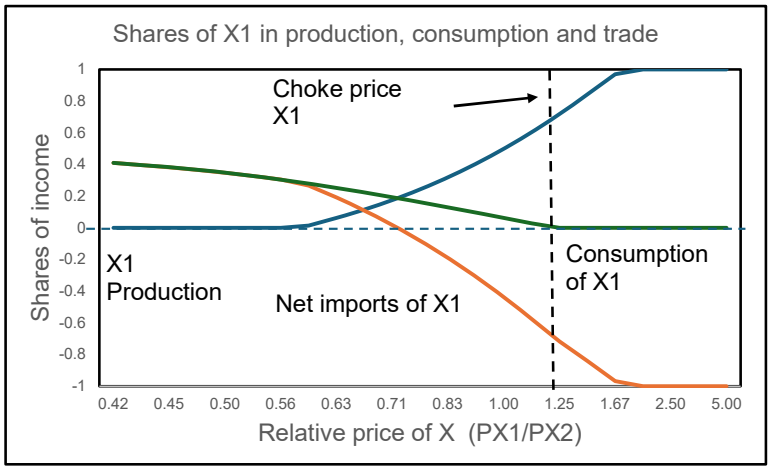
Figure 4: Small open economy - fixed world prices
SGM preferences - loop over size per capita, relative price of X1



Panel A: increase per-capita income holding world prices constant shares of production, consumption and trade



Panel B: increase per-capita income holding world prices constant income elasticities of demand



Panel C: increase relative price of X1 factor income fixed

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constant due to the small economy assumption. The vertical dashed line is the choke income level for X_{1p} consumption, so in additional to consumption being zero, all production of X_1 is exported. As per-capita income rises, consumption of X_{1p} gradually increases once choke income is passed and trade gradually shifts from exporting all X_1. The curves pass through an autarky point and then the direction of trade switches to importing X_1.

Panel B of Figure 4 graphs the income elasticities of demand for X_{1p} and X_{2p}. When income is below the choke level for X_{1p} the income elasticity for X_{2p} is equal to one. When the choke level is first passed, the income elasticity of demand for X_{1p} is (locally) infinity (at choke income the denominator of (15) is zero) This elasticity falls steeply, and the two income elasticities gradually converge toward one as discussed and shown above.

Panel C of Figure 4 graphs an increase in the relative (world) price of X_1 relative to X_2. At the low price at the left edge, the economy is specialized in the production of X_2. All consumption of X_1 is from imports (which of course equal consumption). As the relative price of X_1 continues to increase that good is produced and eventually the direction of trade reverses. Those results are completely "normal". What is different is that we eventually reach a choke price level in which the consumption of X_1 goes to zero. All production of X_1 is exported.

5. CRIE preferences, demand, income and price elasticities

Now we turn to a similar model in many respects, but now preferences are changed to what we have labeled CRIE, constant relative income elasticity. In simple terms, a CRIE function is like a CES with different exponents on different goods: $W = \sum_i X_i^{\gamma_i} \quad 0 < \gamma_i \leq 1$.

As noted in the introduction, these preferences have been around for a very long time. CRIE has the advantage that income elasticities do not all converge to one even with large differences in per-capita income. But it has been the view that they are awkward to use in general-equilibrium models since there are no closed form solutions for cost/expenditure and demand functions. Second, this simple version used in Fieler (2011) and Caron-Fally-Markusen (2014, 2020) does not allow for elasticities of substitution across sectors to be less than one. Third, income elasticities and price elasticities are bound together. These difficulties were noted in Hanock's (1975) original investigation, but all three will be overcome here.

I present a very simple version of the model in order to develop the intuition and clearly show its principal properties. The model is a two-final-good, two (conventional, observed) primary factors. It is a small, open economy facing fixed world prices, which can be adjusted as counter-factuals. Each produced and traded good is a perfect substitute in trade for same good from the (not-modeled) rest of world.

I model CRIE preferences in a way that retains constant-returns, homothetic CES functions so that these preferences slot right into AGE models. There are parameters which we will refer to as sector-specific factors that create the non-homotheticity. They are not observed in the data. The observed goods are combined with the specific factors/goods in fixed supply to produce "utility goods". It may be useful to think of these fixed endowments as leisure, which must be combined with the physical goods to produce welfare. Results are qualitatively similar

if there is a single endowment good (leisure) which has to be divided between sectors, but the algebra is slightly more involved. CRIE preferences do not allow for choke income and prices. However, the latter can be done by adding the SGM endowments to this formulation.

Notation for quantity variables:

X_1	production of good 1 - observed in data
X_2	production of good 2 - observed in data
X_{1p}	domestic purchases of good 1 - observed in data
X_{2p}	domestic purchases of good 2 - observed in data
TX_{12}	trade activity exports X_1 in exchange for X_2
TX_{21}	trade activity exports X_2 in exchange for X_1
L_1 K_1	labor and capital used to produce X_1
L_2 K_2	labor and capital used to produce X_2
L, K	total fixed labor and capital - scaled up/down to change the size of the economy
R_1	specific factor used by the Y_1 sector - fixed supply - not observed in data
R_2	specific factor used by the Y_2 sector - fixed supply - not observed in data
Y_1	utility good produced X_1 and R_1 - Cobb-Douglas - not observed in data
Y_2	utility good produced X_2 and R_2 - Cobb-Douglas - not observed in data
W	welfare produced from Y_1 and Y_2 - CES

The following are the key properties of the production, consumption, and trade with the rest of world for the two-sector model.

$$X_1 = L_1^{\beta_{1l}} K_1^{\beta_{1k}} \qquad X_2 = L_2^{\beta_{2l}} K_2^{\beta_{2k}} \qquad L = L_1 + L_2 \quad K = K_1 + K_2 \tag{41}$$

$$X_{1p} = X_1 - TX_{12} + TX_{21} \qquad X_{2p} = X_2 - TX_{21} + TX_{12} \tag{42}$$

$$Y_1 = X_{1p}^{\alpha_{x1}} R_1^{\alpha_{r1}} \qquad Y_2 = X_{2p}^{\alpha_{x2}} R_2^{\alpha_{r2}} \qquad \alpha_{xi} + \alpha_{ri} = 1 \tag{43}$$

$$W = (Y_1^{\rho} + Y_2^{\rho})^{1/\rho} \qquad -\infty \le \rho \le 1 \qquad \sigma = \frac{1}{1-\rho} \qquad \rho = \frac{\sigma - 1}{\sigma} \tag{44}$$

All production activities have constant returns to scale (CRS), which greatly facilitates computation in large models, especially using Rutherford's mps/ge.

The relationship between α_{x1} and α_{x2} determines which good has the higher income elasticity. Letting $R_1 = R_2 = 1$ for simplicity, W can be written as

$$W = (X_{1p}^{\gamma_1} + X_{2p}^{\gamma_2})^{1/\rho} \qquad \gamma_1 = \rho\alpha_{x1} \qquad \gamma_2 = \rho\alpha_{x2} \tag{45}$$

If $\rho > 0$ or equivalently $\sigma > 1$, then the good with the *higher* α will have the higher income elasticity of demand. However, if $\rho < 0$ or equivalently $\sigma < 1$, then the good with the *lower* α will have the higher income elasticity; e.g., X_1 has the higher elasticity with a lower α because its exponent is less negative: $\rho\alpha_{x2} < \rho\alpha_{x1} < 0$.

Notation for price and income $(tot = p_{x1}/p_{x2})$:

p_{x1}	price of production good X_1
p_{x2}	price of production good X_2
p_{x1p}	price of the observed consumption good X_1
p_{x2p}	price of the observed consumption good X_2
p_{r1}	price of the endowment good R_1
p_{r2}	price of the endowment good R_2
p_{y1}	price of consumption good Y_1
p_{y2}	price of consumption good Y_2
p_w	consumer price index (unit expenditure function)
p_l	price of labor
p_k	price of capital
CONS	representative consumer income

CRIE, as its name suggests, has the nice feature that the ratio of income elasticities between two goods is constant, which could aid greatly in calibration. Let p_{x1p} and p_{x2p} denote goods price, held constant here. Maximize utility subject to a budget constraint with income M and Lagrangean multiplier λ.

$$\max W = (X_{1p}^{\gamma_1} + X_{2p}^{\gamma_2})^{1/\rho} \quad s.t. \quad M = p_{x1p}X_{1p} + p_{x2p}X_{2p} \tag{46}$$

$$\delta\gamma_1 X_{1p}^{\gamma_1 - 1} - \lambda p_{x1p} = 0 \quad \delta\gamma_2 X_{2p}^{\gamma_2 - 1} - \lambda p_{x2p} = 0 \quad \delta = (1/\rho)(X_{1p}^{\gamma_1} + X_{2p}^{\gamma_2})^{1/\rho - 1} \tag{47}$$

An increase in M is proxied here by a change (fall) in λ. Hold prices constant. Forming the ratio, λ and δ cancel out, giving

$$\frac{dX_{1p}/d\lambda}{dX_{2p}/d\lambda} = \frac{p_{x1p}}{p_{x2p}} \frac{\gamma_2(1-\gamma_2)X_{2p}^{\gamma_2 - 2}}{\gamma_1(1-\gamma_1)X_{1p}^{\gamma_1 - 2}} \qquad \frac{dX_{1p}/X_{1p}}{dX_{2p}/X_{2p}} = \frac{p_{x1p}}{p_{x2p}} \frac{\gamma_2(1-\gamma_2)X_{2p}^{\gamma_2 - 1}}{\gamma_1(1-\gamma_1)X_{1p}^{\gamma_1 - 1}} \tag{48}$$

$$\frac{p_{x1p}}{p_{x2p}} \frac{\gamma_2 X_{2p}^{\gamma_2 - 1}}{\gamma_1 X_{1p}^{\gamma_1 - 1}} = 1 \text{ from the first-order conditions (47)}$$

$$\frac{dX_{1p}/X_{1p}}{dX_{2p}/X_{2p}} = \frac{1-\gamma_2}{1-\gamma_1} \tag{49}$$

$$\frac{dX_{1p}/X_{1p}}{dX_{2p}/X_{2p}} > 1 \qquad if\ \gamma_i > 0 \quad and \quad \gamma_1 > \gamma_2 \quad or \quad \gamma_i < 0 \quad and\ \gamma_1 < \gamma_2 \quad \text{alternatively}$$

$$\frac{dX_{1p}/X_{1p}}{dX_{2p}/X_{2p}} > 1 \qquad if\ \rho > 0 \quad and \quad a_{x1} > a_{x2} \quad or \quad \rho < 0 \quad and \quad a_{x1} < a_{x2} \tag{50}$$

This relationship is verified in the simulations, both for $\sigma > 1$ ($0 < \rho < 1$) and $\sigma < 1$ ($\rho < 0$).

Next, consider the effect on demands of an increase in income M holding the prices of the two goods constant.

$$M = p_{x1p}X_{1p} + p_{x2p}X_{2p} \qquad 1 = p_{x1p}\frac{dX_{1p}}{dM} + p_{x2p}\frac{dX_{2p}}{dM} \tag{51}$$

$$1 = \left[\frac{p_{x1p}X_{1p}}{M}\right]\frac{dX_{1p}/X_{1p}}{dM/M} + \left[\frac{p_{x2p}X_{2p}}{M}\right]\frac{dX_{2p}/X_{2p}}{dM/M} \tag{52}$$

Let s_{xi} denote the expenditure share on good i and let η_i denote the income elasticity of demand for good i. Equation (52) reduces to

$$\sum_i s_{xi}\eta_i = 1 \qquad i = 1, 2 \tag{53}$$

Note that, although the ratio of income elasticities is constant as noted above, there levels cannot be constant as *M* increases (unless all income elasticities are one - homothetic preferences). As the expenditure share of the good with the high η increases with income, the η's cannot remain constant or (53) is violated. Both income elasticities must fall to maintaining their ratio as income grows. Note that this coming exclusively from the budget constraint: it has nothing to do with the particular specification of preferences. Equation (53) is verified in the simulations.

The middle diagram in Figure 3 gives a (hand drawn) picture of the Engels curve for the CRIE case. As noted, these preferences by themselves do not allow for choke income and prices, but they have significant advantages in that the income elasticities do not converge to one, particularly relevant for goods with low income elasticities. It is hard to swallow these goods having unitary income elasticities at high levels of income.

The next section gives the inequalities and complementary variables for the small-economy general-equilibrium model in the same format as the SGM model in (17)-(35). I have tried to construct the model in a manner that stay close to the SGM formulation and, as I'll comment later, the two can be combined with both endowment goods and specific fixed factors.

6. CRIE small open-economy general-equilibrium model

Price inequalities — quantities as complementary variables

$$c_{x1}(p_l, p_k) \geq p_{x1} \quad \perp \quad X_1 \tag{54}$$

$$c_{x2}(p_l, p_k) \geq p_{x2} \quad \perp \quad X_2 \tag{55}$$

$$p_{x1} \geq p_{x1p} \quad \perp \quad X_{1p} \tag{66}$$

$$p_{x2} \geq p_{x2p} \quad \perp \quad X_{2p} \tag{57}$$

$$p_{x2}tot \geq p_{x1} \quad \perp \quad T_{21} \tag{58}$$

$$p_{x1} \geq p_{x2}tot \quad \perp \quad T_{12} \tag{59}$$

$$c_{y1}(p_{x1p}, p_{r1}) \geq p_{y1} \quad \perp \quad Y_1 \tag{60}$$

$$c_{y2}(p_{x2p}, p_{r2}) \geq p_{y2} \quad \perp \quad Y_1 \tag{61}$$

$$c_w(p_{y1}, p_{y2}) \geq p_w \quad \perp \quad W_1 \tag{62}$$

Market clearing — prices as complementary variables

$$X_1 \geq X_{1p} - T_{12} + T_{12}/tot \quad \perp \quad p_{x1} \tag{63}$$

$$X_2 \geq X_{2p} + T_{21} - T_{12}tot \quad \perp \quad p_{x2} \tag{64}$$

$$X_{1p} \geq \partial c_{y1}/\partial p_{x1p} Y_1 \quad \perp \quad p_{x1p} \tag{65}$$

$$X_{2p} \geq \partial c_{y2}/\partial p_{x2p} Y_2 \quad \perp \quad p_{x2p} \tag{66}$$

$$L \geq \partial c_{x1}/\partial p_l X_1 + \partial c_{x2}/\partial p_l X_2 \quad \perp \quad p_l \tag{67}$$

$$K \geq \partial c_{x1}/\partial p_k X_1 + \partial c_{x2}/\partial p_k X_2 \quad \perp \quad p_k \tag{68}$$

$$R_1 \geq \partial c_{y1}/\partial p_{r1} Y_1 \quad \perp \quad p_{r1} \tag{69}$$

$$R_2 \geq \partial c_{y2}/\partial p_{r2} Y_2 \quad \perp \quad p_{r2} \tag{70}$$

$$Y_1 \geq \partial c_w/\partial p_{y1} W \quad \perp \quad p_{y1} \tag{71}$$

$$Y_2 \geq \partial c_w/\partial p_{y2} W \quad \perp \quad p_{y2} \tag{72}$$

$$W \geq CONS/p_w \quad \perp \quad p_w \tag{73}$$

Income balance — consumer income as complementary variable

$$CONS \geq p_l L + p_k K + p_{r1} R_1 + p_{r2} R_{r2} \quad \perp \quad CONS \tag{74}$$

The initial calibration of the model is as follows, with a micro-consistent set of numbers

requiring row and column sums equal to zero. As in the SGM case, I calibrated it to zero trade initially. The share of the specific factor in Y_1 is higher than in Y_2 so that, with an elasticity of substitution less than one, Y_1 is the sector with the higher income elasticity of demand (52, 53).

	Production Sectors						Trade		Welfare	Consumer
	X1	X1P	X2	X2P	Y1	Y2	TX21	TX12	W	CONS
Markets										
PX1	100	-100					-0	0		
PX1P		100			-100					
PX2			100	-100			0	-0		
PX2P				100		-100				
PR1					-90					90
PR2						-10				10
PY1					190				-190	
PY2						110			-110	
PW									300	-300
PL	-25		-75							100
PK	-75		-25							100

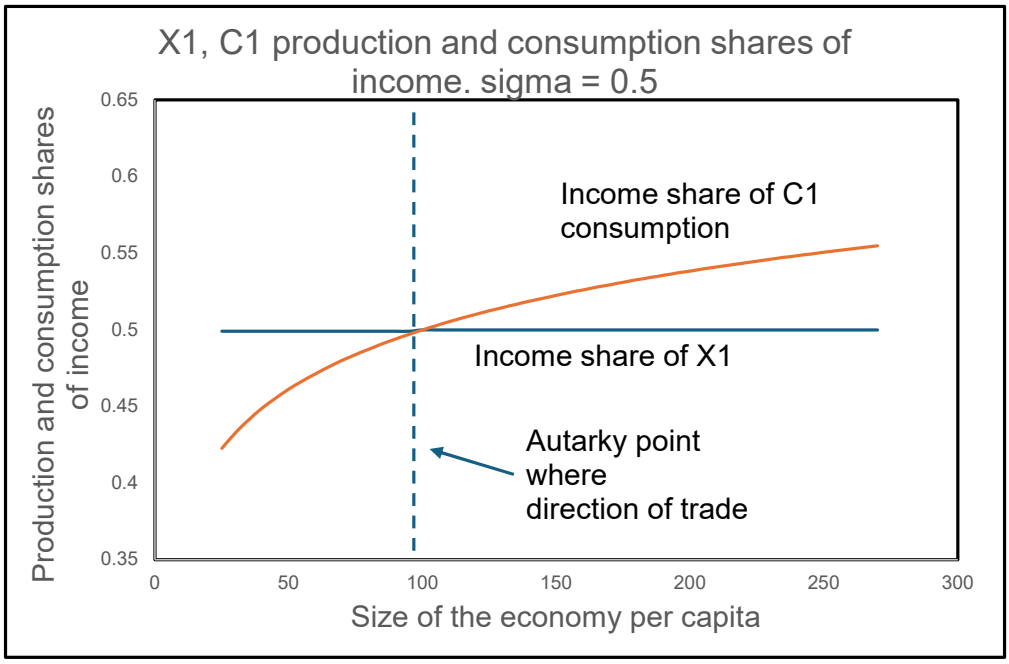
Figure 5 presents a simulation of a case with the elasticity of substitution σ between Y_1 and Y_2 in (47) equal to 0.5. Panels A and B correspond to the results shown in panels A and B of Figure 4, looping over the endowment size of the economy. Endowments of the specific factors are held fixed as L and K increase, similar to holding the endowment goods constant in SGM.

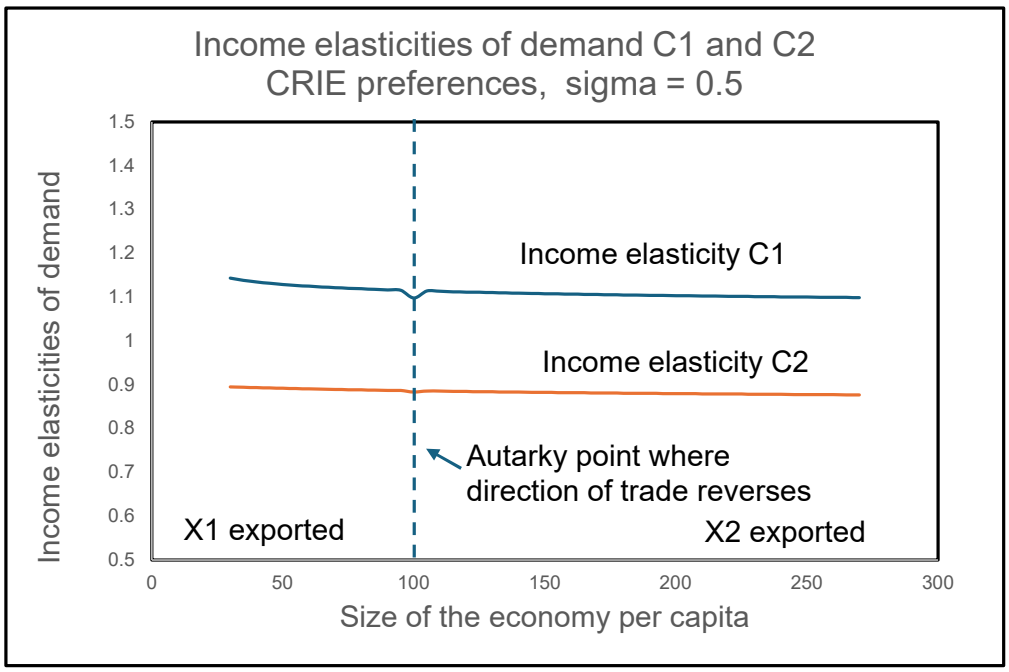
Panel A of Figure 5 shows how the income shares of X_1 consumption and production change with per-capita size. The production share of X_1 is constant by virtue of the small-country assumption. The dashed line is the autarky size and also the calibration point as just noted. At a small size, X_1 is exported while at large size it is imported. The vertical distance between the production and consumption lines is the share of imports or exports. Panel B of Figure 5 give the income elasticities of demand for the same per-capita income experiment. As noted a couple of times, a feature of CRIE is that these income elasticity differences persist over a much larger range of income and the good with the low income elasticity of demand never rises (and of course doesn't approach one) contrary to SGM.

It is certainly not a surprise that a high-income country specializes in consuming the income elastic good. I should note that the result here that this maps directly into the direction and balance of trade is not general however. Fieler (2011) and especially Caron-Fally-Markusen (2014, 2020) show that in models with much richer production structures, high income countries are relatively specialized in the production of skilled-labor-intensive goods that are also the high income-elasticity goods, so those countries may export the high income elasticity goods.

My CRIE formulation allows for elasticities of substitution between sectors to be less than one, which I believe is supported by empirical evidence (literature references in Heblich, Redding and Zylberberg (2025)). In earlier versions of the paper I also computed results for an elasticity of substitution of 2.5. Results don't qualitatively differ much from those I just presented for σ = 2.5, so I don't show them here. The code and figures for the case with σ = 2.5 are available from myself.

Figure 5: Small open economy - fixed world prices
CRIE preferences - loop over size (income) per capita





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c:\jim\RESEARCH\SGM and CRIE\SG CRIE gms xlsx\crie-open.xlsx

7. Estimation using US time-series data

The data used to construct Figure 1 can be use for a simple estimation of the income elasticities and the elasticity of substitution between goods and services using the CRIE formulation of preferences. Drawing on different sources, the knowns for 1950-2024 are

Expenditures on goods and services in total consumer expenditures
  from this, the ratio gives expenditure shares (SH_{1t} / SH_{2t}) services = 1, goods = 2
Prices of goods and services
  divide expenditures by prices => real quantity indices of goods and services
Population data
  divide real quantities by population => real quantities per capita (X_1, X_2)

The consumer optimization problem is give as follows, with sector/subscript 1 being services and sector/subscript 2 being goods.

$$W = \left[\beta X_1^{\alpha_1 \rho} + (1-\beta) X_2^{\alpha_2 \rho}\right]^{1/\rho} + \lambda(M - p_1 X_1 - p_2 X_2) \tag{75}$$

The first-order conditions can form a ratio which eliminates λ and M.

$$\frac{p_{1t}}{p_{2t}} = \frac{\beta}{1-\beta}\frac{\alpha_1}{\alpha_2}\left(\frac{X_{1t}^{\alpha_1}}{X_{2t}^{\alpha_2}}\right)^{\rho-1} \quad \text{or in shares} \quad \frac{p_1 X_1}{p_2 X_2} = \frac{\beta}{(1-\beta)}\frac{\alpha_1}{\alpha_2}\left(\frac{X_1^{\alpha_1}}{X_2^{\alpha_2}}\right)^{\rho} \tag{76}$$

Prices, quantities and therefore market shares are observed. The data are micro-consistent (satisfying adding up) so the budget constraint is redundant. Parameters to be estimated are β, α_1, α_2, and ρ. I estimate (76) by solving a non-linear programming problem (NLP) minimizing the sum of squared errors between the ratio of observed market shares on the left and the non-linear quantity ratio of observed X_i on the right side of (76). I use the second version as the share ratio is easier to interpret than the price ratio (magnitudes of the terms in the NLP below are different in the two versions, but the estimated parameters are virtually identical).

However, the software taught me what now seems apparent: there is a continuum of solutions among the three variables β, α_1, and α_2. The model solves, but different starting values yield completely different solutions (solving for four variables from three equations yields infinitely many solutions). Fixing β allows for a (unique) solution for the α_i, or fixing the ratio of the α_i allows for a solution for beta (including identical α_i - homothetic preferences). Since the goal is to estimate the differences in the α_i along with the elasticity of substitution σ = 1/(1-ρ), I fixed β. Constraining $\alpha_i \leq 1$ for calibrating counter-factuals, the NLP is (t for time/year):

$$Min_{\alpha_1,\alpha_2,\rho} \sum_t \left[\frac{SH_{1t}}{SH_{2t}} - C\frac{\alpha_1}{\alpha_2}\left(\frac{X_{1t}^{\alpha_1}}{X_{2t}^{\alpha_2}}\right)^{\rho}\right]^2 \quad \text{s.t.} \quad \alpha_1, \alpha_2 \leq 1 \qquad C = \frac{\beta}{(1-\beta)} \quad SH_i = \frac{p_i X_i}{M} \tag{77}$$

The chosen value of *C* is the average of the ratio of expenditure shares on services to goods over the sample period, which has a value of 1.445. This is admittedly rather arbitrary. The key results, which are the ratio of the income elasticities η_1/η_2 and the elasticity of substitution σ, do change with *C* but are not very sensitive to the value of *C*. I believe that *C* is just a weighting parameter perhaps analogous to weighted least squares with a higher value giving more weight to services (sector 1). The GAMS code for the NLP estimation is given in an appendix.

Table 2 presents the results for the NLP estimation in (77) for the 75 years 1950-2024. The (time invariant) values for α_s, α_g and ρ are shown (α non-negativity constraints not binding). From (48) we recover γ_s and γ_g, and then from these (52) gives us the ratio of income elasticities of demand: 1.510. Using this (time invariant) ratio along with the Engels adding-up condition for each year in (53) gives the level values of η_s and η_g for each year. Table 2 then gives the simple average of these level values over the 75 years. Table 2 then gives the estimated value of $\sigma = 0.487$ from the estimated value of $\rho = -1.051$ ($\sigma = 1/(1-\rho)$). This estimate of the elasticity of substitution is consistent with a literature review in Heblich, Redding, and Zylberberg (2025) and cross-section results in Caron, Fally and Markusen (2014) shown in Table 1 above.

Below the results, I give the implied utility function using the relationship between the estimated α and ρ values. The sector weights 0.59 and 0.49 (rounded) come from the fixed value of $C = 1.445$ solving for the beta weights $C = \beta/(1-\beta)$. As noted earlier, with $\rho < 0$ ($\sigma < 1$), the less negative coefficient on services implies the higher income elasticity of demand.

Returning to the lower part of Table 2 shortly, Figure 6 plots the residuals from the NLP estimation in the top panel: the share ratio in (76)-(77) minus the "fitted" right hand term using the estimated coefficients α_s, α_g, and ρ. These show increasing residuals as the share ratio increases over time, but the striking thing is the covid disruption. This could be called a demand or preference shock, resulting in households switching away from services and into goods: the share ratio falls significantly 2020-2022 but the right-hand (fitted) terms falls a lot more.

I do not want to throw out observations since there may be other situations with preference shocks (note the 1990s). What I have done in the lower panel of Figure 6 is to truncate the sample to 1950-2019 which still leaves 70 observations. Then I re-run the NLP to get new parameter values and these are shown in the lower panel of Table 2. Not surprisingly, the ratio and spread of the income elasticities increases but the effect is not dramatic. Also intuitive is that the R^2 increases when these covid outliers are eliminated. There is very little change in the estimated elasticity of substitution. Are the residuals large? For each panel of Figure 6, I note both the mean of the absolute value of the deviations and the mean expenditure share ratio. The ratio of these for the two panels are 0.076 and 0.075 respectively.[3]

Using the estimated parameters and the average expenditure share ratio in the data for the full sample, I recalibrate the small-economy general-equilibrium model to these values. The calibration and the code is shown in an appendix at the end of the paper: except for re-calibrated data, the model is identical to that in the previous section. I run the new version and compare its

[3]I do not compute standard errors of the estimated parameters, in large part because it is not clear how to do this in the NLP formulation (Caron, Fally, and Markusen (2014) use bootstrapping).

Table 2: Estimation of income and substitution elasticities goods and services, 1950-2024 real personal consumption expenditures per capita

Parameter estimates - solutions from NLP optimization $R^2 = 0.852$

alpha α_s	0.225	exponent on services
alpha α_g	0.826	exponent on goods
rho ρ	-1.051	CES parameter for elasticity of substitution

Calculated from the estimated parameters

eta η_s / η_g	1.510	ratio of income elasticities services/goods
average η_s	1.170	average income elasticity - services*
average η_g	0.775	average income elasticity - goods**
sigma σ	0.487	elasticity of substitution

* standard deviation = 0.0416

** standard deviation = 0.0276

$$W = [\, 0.59\, X_s^{-0.237} + 0.41\, X_g^{-0.868} \,]^{-0.951}$$

-0.237 = -1.051*0.225
-0.868 = -1.051*0.826
-0.951 = 1/(-1.051)

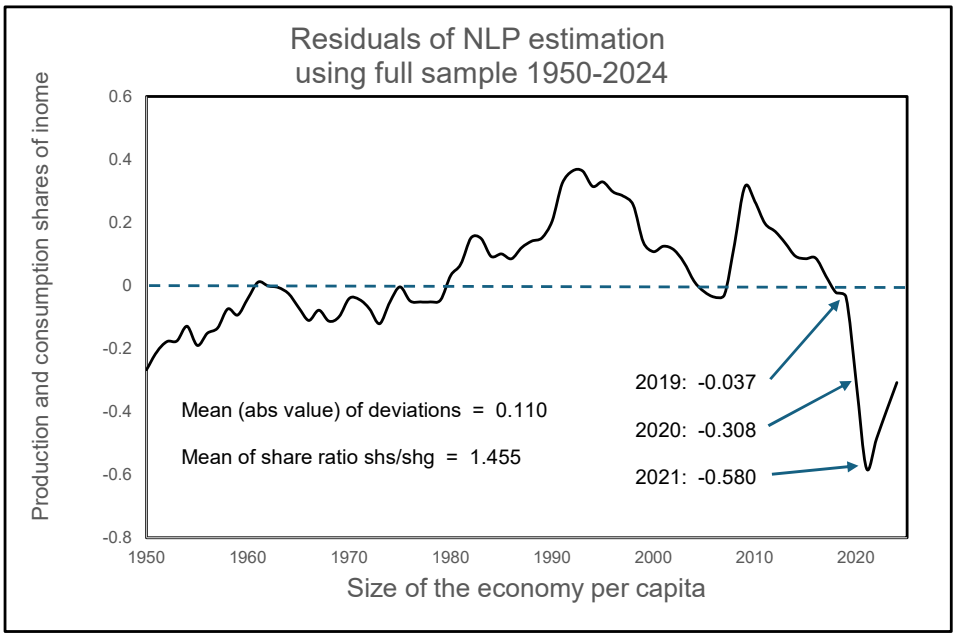
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Reduce sample years to 1950-2019 to eliminate COVID demand/preference shock

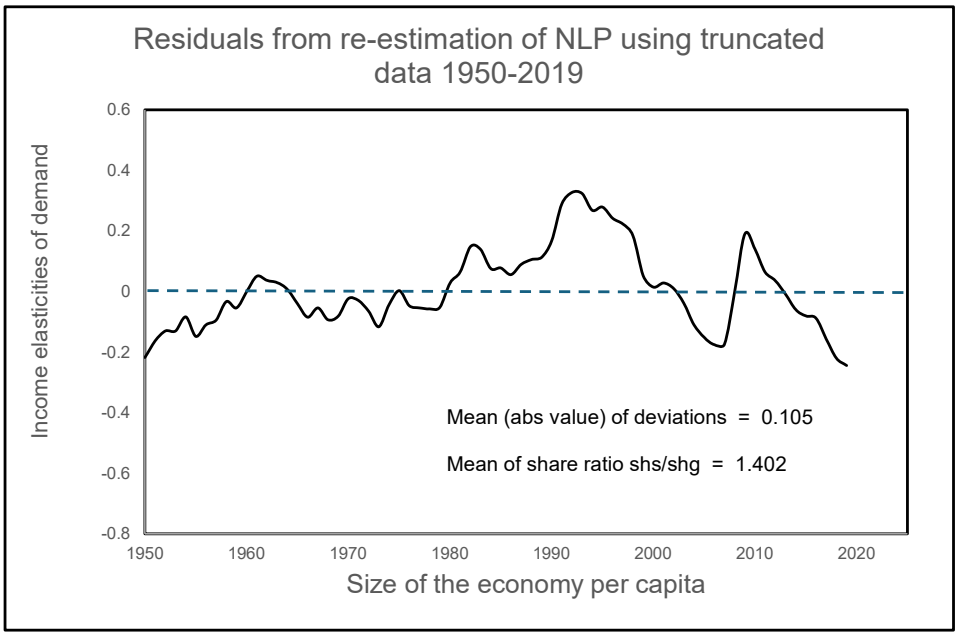
Re-run estimation with 70 year sample $R^2 = 0.925$

eta η_s / η_g	1.604	ratio of income elasticities services/goods
average η_s	1.197	average income elasticity - services
average η_g	0.746	average income elasticity - goods
sigma σ	0.481	elasticity of substitution

Figure 6: Residuals (Y - Yhat) NLP optimization illustrating COVID demand shock
Main results from Table 2 in upper Figure
Results with truncated sample from Table 2 in lower Figure



0.0756014



0.074893

c:\jim\RESEARCH\SGM and CRIE\SG CRIE gms xlsx\Calculate 2.gms (upper) Calculate 2b.gms (lower)
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Table 3: Run model calibrated to estimated parameters

Counter-factuals illustrate difference from homothetic

Estimation results		Small open economy model calibrated using (rounded) estimated α_s α_g ρ^* and share ratio rounded to 1.40 Benchmark solution / replication	
eta η_s / η_g	1.510	eta η_s / η_g	1.527
average η_s	1.170	η_s	1.170
average η_g	0.775	η_g	0.766
SH_s / SH_g**	1.414	SH_s / SH_g	1.400

Counter-factuals on model solution (benchmark ratio normalized = 1)	reduce size to 0.75 of initial	increase size to 1.50 of initial
	SH_s / SH_g	SH_s / SH_g
homothetic	1.000	1.000
calibrated CRIE	0.892	1.172

*these elasticities are arc elasticities, so not quite equal to the true values

** mid-point of the sample, 1986 (not a constant in the data)

benchmark results to the estimated parameters and data share mean. Results are shown in Table 3, where I re-enter the estimated values from above on the left-hand side and the model values on the right. I don't quite match the data, but I think it is "close".

I then perform the same counter-factual experiment with the new calibration, changing the per-capita income from the benchmark value to 0.75 and then 1.5 times the original value. Results are shown in the lower part of Table 3 where I normalize the initial expenditure share ratio to one. For the homothetic case, the expenditure share ratio remains 1.000 (the small economy assumption keeps relative prices constant). For the calibrate model counter-factuals, the ratio in the smaller economy falls to 0.892, while the ratio for the larger economy is 1.172.

Are these differences between the homothetic case and the CRIE model large? In many counter-factual experiments in international trade models, change in per-capita incomes are much smaller, although in models with disaggregated household types, these levels of real income changes are not especially large. Second and over several decades, per-capita income changes for many countries are far larger. Even for high-income US, real per-capita income increase by 60 percent between 1980 and 1999.

8. Added results 1: Independence of price and income elasticities in CRIE

As noted at the beginning of Section 5 on CRIE preferences, the third "nail in the coffin" of the simplest form of CRIE preferences,

$$W = \sum_i X_i^{\gamma_i} \quad 0 < \gamma_i \leq 1, \text{ labeled "direct explicit additivity" by Hanock (1975),} \qquad (78)$$

is that this functional form binds together price and income elasticities. An analogous problem exists in the commonly-used intertemporally additive utility function in macroeconomics: the intertemporal elasticity of substitution and the degree or relative risk aversion are tied together. This is particularly troublesome in that literature estimates referred to above of the elasticity of substitution between broad sectors in less than one (around 0.5) and the second nail in the coffin is that (88) does not permit this.

We have already seen that my formulation avoids this second problem and this section constitutes a brief note that it also avoids the third. I illustrate this by simulating a very simple two-good, partial-equilibrium model holding money income constant. Recall that, letting $R_1 = R_2 = 1$ for simplicity, W can be written as

$$W = (X_{1p}^{\rho\alpha_{x1}} + X_{2p}^{\rho\alpha_{x2}})^{1/\rho} \text{ allowing } \rho \text{ to take on positive or negative values} \qquad (79)$$

The relative α_{xi} values primarily determine the relative income elasticities while the ρ value primarily determines the prices elasticities. Like the income elasticities, the substitution and price elasticities are not constant, but they can be calibrated independently at a benchmark. The procedure for separating price and income elasticities is as follows.

(a) Pick a value of σ (e.g., from a literature estimate) and set $\rho = \sigma/(\sigma - 1)$ in (79).

(b) Using literature estimates of the relative income elasticities η and expenditure shares, use the procedure outlined in the appendix to determine the α_{xi} values.

(c) Equation (53) then determines the initial level values of the η (they are not constant).

Figure 7 presents simple simulations of a partial-equilibrium case holding money income constant to illustrated this (code available from the author includes parameter values). The top two panels are for σ = 2.5 (ρ = 0.6) and σ = 0.5 (ρ = -1), giving own and cross-price elasticities for and increase in p_1. Other than the difference in the levels of the elasticities, the important difference in the two cases is that with σ > 1, the share of expenditure on X_1 is falling in own price as we expect, while with σ < 1, the expenditure share on X_1 rises with own price.

The bottom panel of Figure 7 notes that the results for the ratio of the income elasticities in these two simulations are identical. In short, these results are illustrating that the second and third problems with the simple direct implicit additivity version are "solved" in my version: substitution elasticities less than one are permitted and price and income elasticities can be set independently.

9. Added Results 2: A short comment on homothetic Armington with choke prices

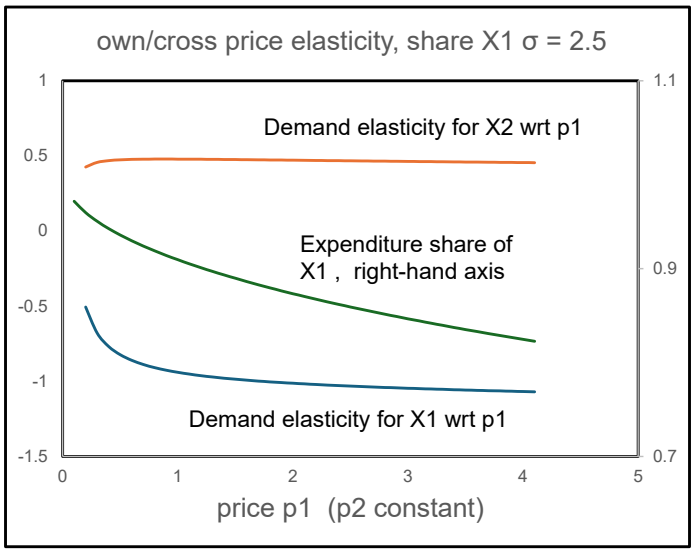
There are many other cases and situations that may be of interest to general-equilibrium modelers that can be formulated and simulated using more or less the same methodologies used above. For example, a modeler might want to retain homothetic CES preferences but introduce choke prices in order to endogenize initial zeros in the trade matrix and/or allow initially-positive values to go to zero at higher trade costs.

As always, the problem with homothetic CES is that the demand price for a good goes off to infinity as the quantity consumed of that good goes to zero. Thus there can be no endogenous zero values in the trade matrix: anything zero initially is dropped from the model and thus remains zero, and any initially positive value cannot go to zero. But suppose instead of assuming the existence of fixed endowment substitutes as we have done above, there are domestic technologies that can produce perfect substitutes for foreign (Armington differentiated) goods, which are inefficient at benchmark prices such that those technologies are inactive initially. I don't find this implausible when thinking about a broad range of goods such as clothing, footwear, kitchen goods and implements, etc. High trade costs can induce foreign firms to produce inside the domestic country, although it is beyond the scope of this paper to get into endogenizing multinational firms.

At this point, perhaps the algebra of my approaches is well understood, so let me just illustrated a simple model the makes the point. The computer code is available from myself. There are two industries 1 and 2. The domestic and foreign varieties are X_1 and Y_1 in industry 1 and X_2 and Y_2 in industry 2. In the usual homothetic CES Armington case, the domestic economy will always import some Y_1 and Y_2 no matter how high the price. But suppose that in the benchmark, there are latent technologies which can produce perfect substitutes for Y_1 and Y_2 , but are inefficient at benchmark prices.

Figure 7: Independence of price and income elasticities: Partial equilibrium example

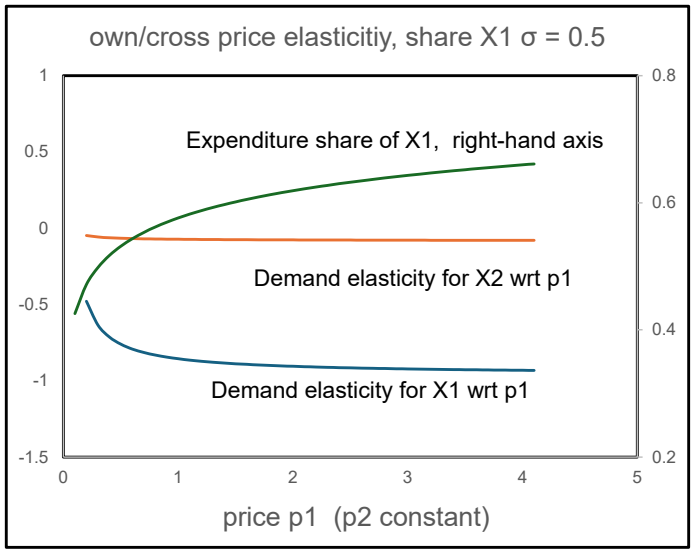
Holding ratio of income elasticities constant, contrast σ = 2.5 with σ = 0.5



$\sigma = 2.5 \quad (\rho = 0.6)$

$\alpha_1 = 0.7333 \quad \alpha_2 = 0.2667$

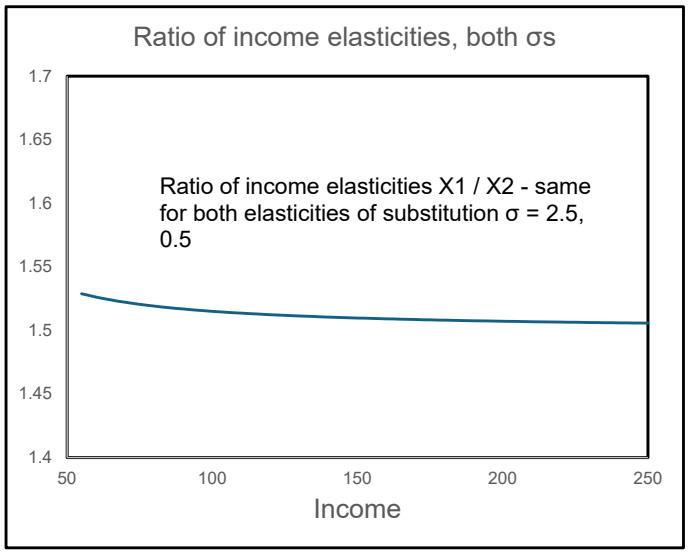
$\eta_1 / \eta_2 = 1.50$



$\sigma = 0.5 \quad (\rho = -1.0)$

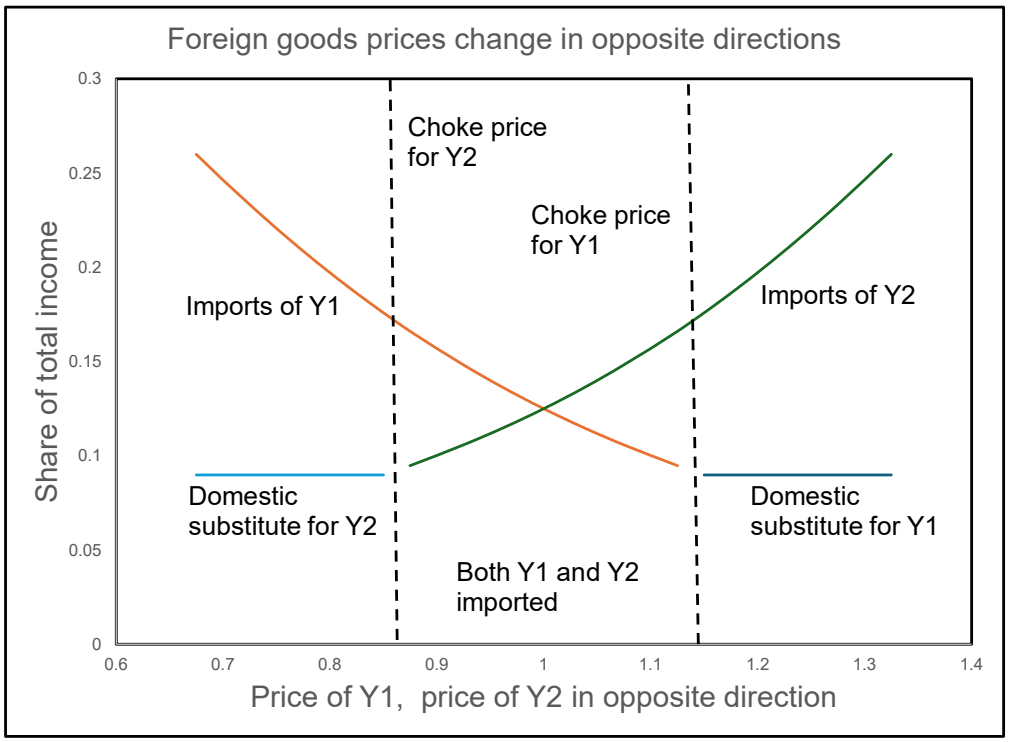
$\alpha_1 = 0.200 \quad \alpha_2 = 0.800$

$\eta_1 / \eta_2 = 1.50$



Note: income elasticity ratio is 0.5, but the calculation is "arc elasticities", so the ratio is a little larger for small incomes but trending down. Very small intervals or very large income does get the ratio converging to 0.5

Figure 8: Choke prices for foreign goods in a traditional Armington model



Foreign goods can be replaced by inefficiently proudced (in the benchmark) domestic goods
which are perfect substitutes for the imported goods
this example: in the benchmark, domestic substitutes 15 percent more costly

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\Armington-choke.gms

Figure 8 presents a simulation of a small economy calibrated to a symmetric initial benchmark with the rest of world with foreign prices equal to one for Y_1 and Y_2. The upper nest of utility is Cobb-Douglas while there is an elasticity of substitution equal to 4 between the domestic and foreign varieties in a sector. At initial world prices of one, both Y_1 and Y_2 are imported. But when either price gets 15 percent higher, production of the domestic substitute kicks in. Figure 8 shows the resulting trade pattern and the world choke prices for the two foreign goods in a symmetric case with the prices of the two goods moving symmetrically in opposite directions on the horizontal axis. In the mid-section, the model behaves exactly like the classic cge homothetic Armington model: preferences and all production technologies are homothetic of degree one. But the (latent) technologies impose choke prices for foreign goods.

As discussed above and will be discussed a little more in the following section, calibration will be a challenge as will be setting or solving for the degree of unprofitability of initially inactive trade links, or the upper cutoff for initially actively traded goods.

10. Strengths, Weaknesses and Challenges

As noted early on, a number of other authors have considered adding non-homothetic preferences to general-equilibrium trade theory and empirics. Many or most of these are significantly more technically complex than this present paper and proposal. Some have analytical solutions and some do not. Having an analytical solution is a great strength, but the restrictive assumptions needed have a cost for modelers seeking to formulate empirically-relevant models calibrated to large, multi-sector, multi-country, multi-factor and multi-household-type data sets. Other formulations without analytical solutions are equally awkward for high-dimension models. The CRIE preferences used in Caron, Fally and Markusen (2014, 2020) do not yield analytical solutions.

I am arguing that the simplicity of my formulation is a strength and an attractive feature for AGE modelers. In particular, it retains all the valuable features of constant returns CES preferences which generate solutions for cost and expenditure functions depending only on prices. This in turn generates straightforward and simple derivations of factor and goods-demand equations via Shepard's lemma. These features then allow a model to be specified in a complementarity format where pricing (optimization) inequalities have quantities as complementary variables and market clearing inequalities have prices as complementary variables. The final step is that these characteristics allow computation of solutions with choke income and prices levels.

However, I am the first to acknowledge that there are two significant problems to implementation on general-equilibrium data sets. The first of these is calibration as discussed in the appendix to follow: we have to solve for the level of the unobserved endowment goods in the SGM version or unobserved factor endowments in the CRIE version in order to get the income elasticities. Imagine that we have a set of estimates of income elasticities of demand such as those from Caron, Fallly and Markusen (2014, 2020) shown earlier, or the more aggregate level income elasticities derived here. The appendix to the paper and accompanying Table 4 illustrates a calibration procedure for CRIE that is essentially what I have did in calibrating the CRIE model to estimated parameters with results shown in the lower part of Table 2. If we

know relative income elasticities, expenditure shares, and an the elasticity of substitution from our own or earlier work, then we can solve for the parameters such and the alphas and rho that are need to calibrate a general-equilibrium model for counter-factual simulations.

The second difficulty lies in assessing the choke prices and incomes for goods not traded or not produced initially as just discussed above. How unprofitable are these trade links for latent goods at the benchmark? This is another difficult problem but one which I think AGE modelers can crack. I should point out that we have always had this problem. The standard procedure is to simply hold latent (zero) trade links in the benchmark constant at zero. In effect, there is an unbounded unobserved and implicit trade barrier on these links. This comes at a huge cost in that there cannot be an expansion of trade at the extensive margin following a liberalization. Conversely, an initially-active link cannot go to zero given standard CES preferences. The other approach is the Eaton-Kortum formulation in which (as I understand it) there are no zeros initially, just extremely small but positive trade flows on all links. So again, there is no change in trade at the extensive margin. When trade costs fall just a little, there will be immediate positive or negative (but still non-zero) responses on all links.

The CRIE formulation with its unobserved factor not only has theoretical advantages (the non-convergence of income elasticities to one), but is probably a better candidate for calibration to existing data sets. Given a set of income elasticities as in Caron, Fally and Markusen shown earlier and those derived here, these allow for relatively easy parameterization of expenditure and demand functions. Further, the CRIE preferences permit the independent calibration of price and income elasticities: in the notation of the paper: the ρ parameter can be combined with the differences in the α_i across sectors to achieve independence in the two elasticities. The drawback of CRIE, given the goals of the paper, is that the basic formulation does not allow for choke income and prices. SGM and CRIE can be combined (just add the endowment goods to CRIE), but this then revisits the calibration issues.

11. Addendum notes

Available from the author directly: .gms and excel files for all models, a couple which have both mcp and mps/ge formats. Also code for the partial equilibrium examples of price versus income elasticities, and code focusing on price elasticities in addition to income elasticities.

I have attached the GAMS program for the SGM model (no Armington differentiation). This is in MPC format (mixed complementarity problem), in which all weak inequalities are written out, and the model statement gives the complementary variable associated with each inequality.

Then the code for the basic CRIE model calibrated to the estimated parameters follows in mcp format. Again, no Armington differentiation to keep things simple. As noted at the end of the concluding section, more version are available from myself. The equivalent code in Rutherford's mps/ge format is available from myself.

The final program is for the calculation of the income and substitution elasticities in the

empirical example. As noted above, this formulates the estimation problem as an NLP, bi-passing the tradition econometrics' approach. I am unsure how common this might be, but I have been told it is used in non-parametric statistics. Treating estimation as a non-linear programming problem has considerable advantages that might be of interest. First, almost any estimating objective function can be used: there is no need to restrict it to linear or log linear. Second, inequalities bounds can be used on variables in complementarity format, such as restricting certain coefficients t non-negative values. Third, other side constraints can be imposed on the optimization problem, such as restricting a technology to constant returns to scale (restriction on the sum of input coefficients).

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Appendix: A Note on Calibration Challenges

Potential difficulties for calibrating either SGM or CRIE preferences to large multi-sector models seem formidable. A thorough analysis of these challenges is beyond the scope of this paper. However, this section will present a brief outline of how this might be done for CRIE preferences using available data and literature estimates for some parameters. Calibrating the endowment parameters in the SGM is a straightforward task of solving the system of equations in (15) where the left-hand side income elasticities are known from literature estimates and the endowment parameters are the unknowns. The shares s_i can be calibrated from expenditure data. However, for the SGM case, the important task is determining the degree of unprofitability of initially-zero trade links, something left for future work.

The CRIE example I will use here assumes that there are five sectors. After discovering a number of "roadblocks", I found I need to break out one sector and treat it as a "numeraire" sector in order to remove degeneracy of the solution. In math programming terms, degeneracy is

multiple equilibria and are all the same in real terms but causes computational difficulties and indeterminacy. We are all familiar with this insofar as we know that fixing one price in general-equilibrium models is needed to avoid indeterminacy of the price level. Let j denote a set indexing the five sectors. Set i(j) is a sub-set, consisting of sectors 2-5 with sector one considered the numeraire.

Sets j sectors {1...5}, i(j) equal subset of sectors {2...5}, sector 1 "numeraire"

The following are the parameters of the calibration model. While made up here, these are knows from data or literature estimates: a vector of income elasticities, the expenditure share across sectors, the elasticity of substitution among the five final goods, plus one parameter that is simply picked by the modeler in order to avoid an indeterminacy. This parameter is the total share of the fixed factors *R* in benchmark income.

Parameters

η_{i1}	literature estimate of income elasticity for sector i, divided by the literature estimate for sector 1
s_i, s_1	expenditure share of sector i
σ	elasticity of substitution across sectors, $\rho = (\sigma - 1)/\sigma$
s_r	share of fixed factor *R* in total income (picked by modeler)

Variables to be solved for are as follows, using the same notation as in the CRIE section.

Variables to be solved for

η_i, η_1	income elasticities consistent with Engels aggregation
γ_i, γ_1	given by $\gamma_i = \rho\alpha_{xi}$
α_{xi}, α_{x1}	share of good X_i in producing Y_i

The first difficulty is that the literature estimates of income elasticites as in Caron, Fally and Markusen (2014) will not satisfy the Engels aggregation condition (from the budget constraint) in (56) for most/all countries given their different expenditure shares. So the task is to modify the literature income elasticities for a country to (a) retain the relative values of those elasticities from the estimated ones, and (b) set the levels of those elasticities to satisfy Engels aggregation given each country's vector of expenditure shares. This is done by solving five equations in the five modified income elasticities using the Engels condition to solve for the income elasticity of the sector 1.

$$\eta_i/\eta_1 = \eta_{i1} \quad \perp \quad \eta_i \tag{80}$$

$$\sum_i s_i\eta_i + s_1\eta_1 = 1 \quad \perp \quad \eta_1 \tag{81}$$

Note that these five equations can be solved first and separately from the other unknowns. From the ratio of now-known income elasticities, the values of γ for industries 2-5 can now be found.

$$\eta_i / \eta_1 \;=\; (1-\gamma_1)/(1-\gamma_i) \qquad \perp \qquad \gamma_i \tag{82}$$

But an indeterminacy remains, and needs to be pinned down by specifying some relationship between the γ_i and ρ in the utility function (45) (with CES, all the $\gamma_i = \rho$). I have chosen to pin down the levels of the γ_i by setting the expenditure-weighted sum of the shares of the specific factor in each industries (equal to $\alpha_{ri} = (1-\alpha_{xi})$) equal to the aggregate shares of the specific factors in total income denoted s_r. In the Table that follows, I arbitrarily use $s_r = 1/3$ which is its value in the simulations in the previous section.

$$\sum_i (1-\alpha_{xi})s_i + (1-\alpha_{x1})s_1 \;=\; s_r \qquad \perp \qquad \gamma_1 \tag{83}$$

Note that the value of s_r does not affect the modified income elasticities of demand. The final step need for the calibration is to set the α_{xj} equal to the formula in (48).

$$\alpha_{xj} \;=\; \gamma_j(\sigma-1)/\sigma \qquad \perp \qquad \alpha_j \tag{84}$$

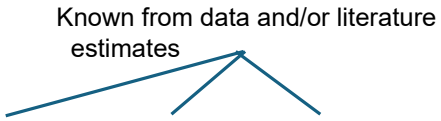
Equations (82), (83) and (84) are solved as a simultaneous system (in the code, I solve all equations (80)-(84) together as a single solve).

Table 4 gives some simulation results to show how this procedure works. In all sets of numbers, the three right-hand columns are made-up parameter values, but all could come from data and literature estimates in real data. The top two sets of results use a value $\sigma = 2.5$. The first solutions and the second for $\sigma = 2.5$ use different expenditure shares across sectors. The first use equal expenditure shares across the five sectors while the second has expenditures concentrated on he lower income-elasticity goods. We could imagine that the first set represents a high-income country and the second represents a low-income country.

The results in Table 4 for $\sigma = 2.5 > 1$ are consistent with the theory developed above. The η values that are consistent with the expenditures shares to satisfy Engels aggregation preserve their relative values η_{i1} (RELETA in the Table). The η values are somewhat higher for the "poorer" country consistent with our theory results. The values of γ and α are increasing in their sector's income elasticity η. The α are the key values for calibration. The inputs into *Y* production are composed of the observed *X* values and the (unobserved) *R* values, the latter given values according to the calculated α values and then added to the country's endowment.

The lower two sets of numbers in Table 4 give results for $\sigma = 0.5$, which is a generally-accepted value for the elasticity of substitution across broad sectors which is what we find in this paper and in literature references presented in Heblich, Redding and Zylberberg (2025). Now the γ values are negative as noted in (53) above. With $\sigma = 0.5$, $\rho = (\sigma - 1)/\sigma = -1$, implying $\alpha = -\gamma$ for this particular value of σ. The γ values are increasing with the income elasticity (becoming less negative) and the α are decreasing with increases in η as discussed in (53).

Table 4: Example of calculating/calibrating ALPHA input shares from known data for CRIE preferences



SIGMA = 2.5

	ETA	GAMMA	ALPHA	RELETA	EXPSH	SIGMA
Sector 1	0.833	0.290	0.484	1.00	0.20	2.50
Sector 2	0.917	0.355	0.591	1.10	0.20	2.50
Sector 3	1.000	0.409	0.681	1.20	0.20	2.50
Sector 4	1.083	0.454	0.757	1.30	0.20	2.50
Sector 5	1.167	0.493	0.822	1.40	0.20	2.50

alternative expenditure shares

	ETA	GAMMA	ALPHA	RELETA	EXPSH	SIGMA
Sector 1	0.866	0.315	0.525	1.00	0.20	2.50
Sector 2	0.952	0.377	0.629	1.10	0.40	2.50
Sector 3	1.039	0.429	0.715	1.20	0.15	2.50
Sector 4	1.126	0.473	0.788	1.30	0.15	2.50
Sector 5	1.212	0.511	0.851	1.40	0.10	2.50

EXPSH	Observed data on expenditure shares across sectors
RELETA	Relative income elasticities (sector 1 numerarie) from literature estimates
SIGMA	Elasticity of substitution across sectors from literature estimates
ETA	Income elasticities calculated from EXPSH and RELATA satisfying Engles aggregation
GAMMA	Parameters calculated from ETAs, EXPSH, and simultaneous with ALPHAs
ALPHA	Shares of X in producing Y, calculated from SIGMAs and simulteneously from GAMMAs

SIGMA = 0.5

	ETA	GAMMA	ALPHA	RELETA	EXPSH	SIGMA
Sector 1	0.833	-0.972	0.972	1.00	0.20	0.50
Sector 2	0.917	-0.793	0.793	1.10	0.20	0.50
Sector 3	1.000	-0.644	0.644	1.20	0.20	0.50
Sector 4	1.083	-0.517	0.517	1.30	0.20	0.50
Sector 5	1.167	-0.409	0.409	1.40	0.20	0.50

alternative expenditure shares

	ETA	GAMMA	ALPHA	RELETA	EXPSH	SIGMA
Sector 1	0.866	-0.904	0.904	1.00	0.20	0.50
Sector 2	0.952	-0.731	0.731	1.10	0.40	0.50
Sector 3	1.039	-0.587	0.587	1.20	0.15	0.50
Sector 4	1.126	-0.465	0.465	1.30	0.15	0.50
Sector 5	1.212	-0.360	0.360	1.40	0.10	0.50

c:\\jim\research\SGM and CRIE\SG CRIE gms xlsx\CRIE calibration 2.gms
c:\\jim\research\SGM and CRIE\SG CRIE gms xlsx\CRIE CALIB2.xlsx

Concluding this section, the point is that adding non-homothetic CRIE preferences to general-equilibrium is possible, and it will certain improve the accuracy of counter-factual experiments in which incomes are growing. SGM preferences add in the important features of choke prices and incomes. However, the important and challenging question of how to calibrate the unprofitability of initially inactive trade links leads me to leave that to a future project.

```
$Title: SGM-simulation in mcp format

PARAMETERS
   SCALEX1  scales up or down the endowment X1E /1/
   SCALEX2  scales up or down the endowment X2E /1/
   ENDOW    scales up or down the endowment of L and K/1/
   TOT      world price PX1 over PX2 /1/;

NONNEGATIVE VARIABLES

    X1        Output of sector X1
    X2        Output of sector X2
    X1P       Output of X1P (produces PX1C from PX1)
    X1E       Output of X1E (produces PX1C from endowment PX1E)
    X2P       Output of X2P (produces PX2C from PX2)
    X2E       Output of X2E (produces PX2C from endowment PX2E)
    W         Produces welfare from PW1 and PX2
    TX21      Export X2 in eX1change for X1
    TX12      Export X1 in eX1change for X2

    PX1       Price of commoditX2 X1
    PX2       Price of commoditX2 X2  (used as numeraire here)
    PL        Price of primarX2 factor L
    PK        Price of primarX2 factor K
    PX1C      Price of the X1 and X1E perfect substitutes  X1C = X1 + X1E
    PX2C      Price of the X2 and X2E perfect substitutes  X2C = X2 + X2E
    PW        Price of welfare
    PX1E      Price of the endowment good - perfect substitute for X1 cannot be traded
    PX2E      Price of the endowment good - perfect substitute for X2 cannot be traded

    CONS      Consumer;

EQUATIONS

    COSTX1     Marginal cost - price X1
    COSTX2     Marginal cost - price X2
    COSTX1P    Cost of producing X1 consumption from X1 production
    COSTX1E    Cost of producing X1 consumption from X1E endowment
    COSTX2P    Cost of producing X2 consumption from X2 production
    COSTX2E    Cost of producing X2 consumption from X2E endowment
    COSTW      Cost of producing a unit of welfare
    COSTTX21   Cost of exporting X2 in exchange for X1
    COSTTX12   Cost of exporting X1 in exchange for X2

    MKTPX1     Supply-demand for X1
    MKTPX2     Supply-demand for X2
    MKTPL      Supply-demand for labor
    MKTPK      Supply-demand for capital
    MKTPX1C    Market for X1 consumption of good 1
    MKTPX2C    Market for X2 consumption of good 2
    MKTPW      Supply demand for welfare
    MKTPX1E    Supply-demand for endowment good 1
    MKTPX2E    Supply demand for endowment good 2

    INC        Consumer income balance;

COSTX1..   PL**0.25*PK**(0.75) =G= PX1;
COSTX2..   PL**0.75*PK**(0.25) =G= PX2;
COSTX1P..  PX1 =G= PX1C;
COSTX1E..  PX1E =G= PX1C;
COSTX2P..  PX2 =G= PX2C;
COSTX2E..  PX2E =G= PX2C;

COSTW..    PX1C**(250/351)*PX2C**(101/351) =G= PW;

COSTTX21.. PX2*TOT =G= PX1;
COSTTX12.. PX1*1.001 =G= PX2*TOT;
```

```
MKTPX1..    X1 =G= X1P + TX12 - TX21/TOT;
MKTPX2..    X2 =G= X2P + TX21 - TX12*TOT/1.001;

MKTPL..     100*ENDOW*PL =G= 0.25*(PL**0.25*PK**(0.75))*100*X1 +
                             0.75*(PL**0.75*PK**(0.25))*100*X2;
MKTPK..     100*ENDOW*PK =G= 0.75*(PL**0.25*PK**(0.75))*100*X1 +
                             0.25*(PL**0.75*PK**(0.25))*100*X2;

MKTPX1C..   100*(X1P + X1E)*PX1C =G= (250/351)*(PX1C**(250/351)*PX2C**(101/351))*W*351;
MKTPX2C..   100*(X2P + X2E)*PX2C =G= (101/351)*(PX1C**(250/351)*PX2C**(101/351))*W*351;

MKTPW..     W*351*(PX1C**(250/351)*PX2C**(101/351)) =G= CONS*351;

MKTPX1E..   1.5*SCALEX1 =G= X1E;
MKTPX2E..   (1/100)*SCALEX2 =G= X2E;

INC..       CONS*351 =G= PL*100*ENDOW + PK*100*ENDOW + PX1E*150*SCALEX1 + PX2E*1*SCALEX2;

MODEL SG1 /COSTX1.X1, COSTX2.X2, COSTX1P.X1P, COSTX1E.X1E, COSTX2P.X2P, COSTX2E.X2E,
           COSTW.W, COSTTX21.TX21, COSTTX12.TX12,
           MKTPX1.PX1, MKTPX2.PX2, MKTPL.PL, MKTPK.PK, MKTPX1C.PX1C, MKTPX2C.PX2C,
           MKTPW.PW, MKTPX1E.PX1E, MKTPX2E.PX2E, INC.CONS/;

PL.L = 1; PK.L = 1; PX1C.L = 1; PX2C.L = 1; PX1.L = 1; PX2.L = 1; PX1E.L = 1; PX2E.L = 1;
PW.L = 1; X1.L = 1; X1P.L = 1; X1E.L = 1.5; X2.L = 1; X2P.L = 1; X2E.L = 1/100; W.L = 1;
PX2.FX = 1; CONS.L = 1;

OPTION ITERLIM = 0;
SOLVE SG1 USING MCP;

OPTION ITERLIM = 1000;
SOLVE SG1 USING MCP;


SETS I /I1*I25/;

PARAMETERS RESULTS1(I, *) econom X2 size, RESULTS2(I, *) terms of trade,
           X10, X20, M, M0, PX10;

X10 = 0; X20 = 0; M0 = PL.L*ENDOW +PK.L*ENDOW;
SCALEX1 = 1; SCALEX2 = 1; TOT = 1/1.25;

SOLVE SG1 USING MCP;

LOOP(I,

ENDOW = 0.03*ORD(I) + 0.045;

SOLVE SG1 USING MCP;

M = PL.L*ENDOW +PK.L*ENDOW;

RESULTS1(I, "ENDOW") = ENDOW;
RESULTS1(I, "X1") = MAX(EPS, PX1.L*X1.L/M);
RESULTS1(I, "TX21") = PX1.L*TX21.L/TOT/M;
RESULTS1(I, "TX12") = PX1.L*TX12.L*1.001/M;
RESULTS1(I, "SHX1") = PX1.L*X1P.L/M;
RESULTS1(I, "INCELASX1")$(X10 GT 0) = ((X1P.L-X10)/X10)/((M-M0)/M0);
RESULTS1(I, "INCELASX2")$(X20 GT 0) = ((X2P.L-X20)/X20)/((M-M0)/M0);

X10 = X1P.L; X20 = X2P.L; M0 = PL.L*ENDOW +PK.L*ENDOW;

);
DISPLAY RESULTS1;
```

```
$TITLE: CRIE SIMULATION USING PARAMETER VALUES FROM THE NLP ESTIMATION    SIGMA = 0.487
*       C:\jim\RESEARCH\SGM and CRIE\SG CRIE gms xlsx\CRIE-open-m-05-C3-text.gms
*       USES ALPHAS, ALPHAG, AND SIGMA (RHO) estimates, S/G EXPENDITURE RATIO
*       USED IN TABLE 3 SHOWING HOW NON-HOMOTHETICITY CHANGES COUNTER-FACTUAL RESULTS

$ONTEXT
How to do CRIE+SG in mps/ge
   CRIE converted from standard CES by fixed factors R1 and R2

Two produced goods, X1 and X2, observed in the data
then think of two CRS utility goods Y1 and Y2 produced respectively
with X1 and a fixed factor R1, X2 and a fixed factor R2 so Y1 and Y2 have CRS

    W  =  (Y1**rho + Y2**rho)**(1/rho)     rho = (sigma - 1)/sigma

    Y1  = ((X1)**alpha1x)*(R1**alpha1r)      alpha1x + alpha1r = 1
    Y2  = ((X2)**alpha2x)*(R2**alpha2r)      alpha2x + alpha2r = 1
        X1 = (L1**beta1l)*(K1**beta1k)       beta1l + beta1k = 1
        X2 = (L2**beta2l)*(K2**beta2k)       beta2l + beta2k = 1
        LBAR  =  L1 + L2    (LBAR = 100*SIZE)
        KBAR  =  K1 + K2    (KBAR = 100*SIZE)
        R1 and R2 specific factors in fixed supply

Income elasticities are calculated as the response of produced (and observed) goods
X1 and X2 to increases in the labor and capital supply (observed income)

      The benchmark accounting matrix:
                                                                          Parameters taken from NLP
                 Production Sectors                      Consumers        estimation, used in calibration
 Markets    X1      X1P      X2      X2P      Y1      Y2      W      CONS      alpha1x = 140/621.6 = 0.225
  PX1      140     -140                                                      alpha1r = 481.6/621.6 = 0.775
  PXIC              140                    -140                             alpha2x = 100/121    = 0.826
  PX2                      100     -100                                     alpha2r = 21/121 = 0.174
  PX2C                             100            -100                     expenditure ratio S/G = 1.400
  PR1                                     -481.6                  481.6    sigma = 0.487 (rho = -1.051)
  PR2                                             -21              21
  PY1                                      621.6        -621.6
  PY2                                              121   -121
  PW                                                      742.6   -742.6
  PL       -35             -75                                    110
  PK      -105             -25                                    130
$OFFTEXT

Parameters
 SIGMA   elasticity of substitution between Y1 and Y2 /0.487/,
 SIZE    multiplier on endowment of L and K primary factors /1/,
 TOT     world relative price p1 over p2 /1/;

NONNEGATIVE VARIABLES

   X1    production of X1 from L K
   X2    production of X2 from L K
   X1P   consumption of X1
   X2P   consumption of X2
   Y1    produces consumption good Y1 from X1 and R1
   Y2    produces consumption good Y2 from X2 and R2
   TX21  Export X2 in exchange for X1
   TX12  Export X1 in exchange for X2
   W     Produces welfare from PW1 and PX2

   PX1   price of X1
   PX2   price of X2
   PX1P  consumper price of the X1
   PX2P  consumer price of X2
   PR1   price of endowment of R1
   PR2   price of endowment of R2
   PY1   price of consumption good Y1
   PY2   price of consumption good Y2
```

```
   PW     consumer price index
   PL     price of labor
   PK     price of capital

   CONS   !representative consumer's income

EQUATIONS

   COSTX1      Marginal cost - price X1
   COSTX2      Marginal cost - price X2
   COSTX1P     Cost of producing X1 consumption from X1 production
   COSTX2P     Cost of producing X2 consumption from X2 production
   COSTY1      Cost of producing X1 consumption from X1E endowment
   COSTY2      Cost of producing X2 consumption from X2E endowment
   COSTTX21    Cost of exporting X2 in exchange for X1
   COSTTX12    Cost of exporting X1 in exchange for X2
   COSTW       Cost of producing a unit of welfare

   MKTPX1      Supply-demand for X1
   MKTPX2      Supply-demand for X2
   MKTPX1P     Supply-demand for X1
   MKTPX2P     Supply-demand for X2
   MKTPR1
   MKTPR2
   MKTY1
   MKTY2
   MKTPW       Supply demand for welfare
   MKTPL       Supply-demand for labor
   MKTPK       Supply-demand for capital

   INC         Consumer income balance;

COSTX1..   140*PL**0.25*PK**0.75 =G= 140*PX1;
COSTX2..   100*PL**0.75*PK**0.25 =G= 100*PX2;
COSTX1P..  140*PX1 =G= 140*PX1P;
COSTX2P..  100*PX2 =G= 100*PX2P;

COSTY1..   621.6*PR1**(481.6/621.6)*PX1P**(140/621.6) =G= 621.6*PY1;
COSTY2..   121*PR2**(21/121)*PX2P**(100/121) =G= 121*PY2;

COSTTX21.. 100*PX2*TOT =G= 100*PX1;
COSTTX12.. 140.00001*PX1 =G= 140*PX2*TOT;

COSTW..    742.6*((621.6/742.6)*PY1**(1-SIGMA) + (121/742.6)*PY2**(1-SIGMA))**(1/(1-SIGMA)) =G= 742.6*PW;

MKTPX1..   140*X1 =G= 140*X1P + 140*TX12*1.00001 - 100*TX21/TOT;
MKTPX2..   100*X2 =G= 100*X2P + 100*TX21 - 140*TX12*TOT;

MKTPX1P..  PX1P*140*X1P =G= (140/621.6)*PR1**(481.6/621.6)*PX1P**(140/621.6)*621.6*Y1;
MKTPX2P..  PX2P*100*X2P =G= (100/121)*PR2**(21/121)*PX2P**(100/121)*121*Y2;

MKTPR1..   PR1*481.6 =G= (481.6/621.6)*PR1**(481.6/621.6)*PX1P**(140/621.6)*621.6*Y1;
MKTPR2..   PR2*21 =G= (21/121)*PR2**(21/121)*PX2P**(100/121)*121*Y2;

MKTY1..    621.6*Y1 =G= (621.6/742.6)*PY1**(-SIGMA)*((621.6/742.6)*PY1**(1-SIGMA)
                      + (121/742.6)*PY2**(1-SIGMA))**(SIGMA/(1-SIGMA))*742.6*W;
MKTY2..    121*Y2   =G= (121/742.6)*PY2**(-SIGMA)*((621.6/742.6)*PY1**(1-SIGMA)
                      + (121/742.6)*PY2**(1-SIGMA))**(SIGMA/(1-SIGMA))*742.6*W;

MKTPW..   742.6*W*((621.6/742.6)*PY1**(1-SIGMA) + (121/742.6)*PY2**(1-SIGMA))**(1/(1-SIGMA)) =G= CONS;

MKTPL..    110*SIZE*PL =G= 0.25*(PL**0.25*PK**(0.75))*140*X1 +
                           0.75*(PL**0.75*PK**(0.25))*100*X2;
MKTPK..    130*SIZE*PK =G= 0.75*(PL**0.25*PK**(0.75))*140*X1 +
                           0.25*(PL**0.75*PK**(0.25))*100*X2;

INC..      CONS =G= PL*110*SIZE + PK*130*SIZE + PR1*481.6 + PR2*21;
```

```
MODEL CRIEMCP /COSTX1.X1, COSTX2.X2, COSTX1P.X1P, COSTX2P.X2P, COSTY1.Y1, COSTY2.Y2,
           COSTTX21.TX21, COSTTX12.TX12, COSTW.W,
           MKTPX1.PX1, MKTPX2.PX2,  MKTPX1P.PX1P, MKTPX2P.PX2P, MKTPR1.PR1, MKTPR2.PR2,
           MKTY1.PY1, MKTY2.PY2, MKTPW.PW, MKTPL.PL, MKTPK.PK,INC.CONS/;


PL.L = 1; PK.L = 1; PR1.L = 1; PR2.L = 1; PX1P.L = 1; PX2P.L = 1; PX1.L = 1; PX2.L = 1;
PW.L = 1; X1.L = 1; PY1.L = 1; PY2.L = 1; PW.L = 1;
X1.L = 1; X1P.L = 1;  X2.L = 1; X2P.L = 1; Y1.L = 1; Y2.L = 1; W.L = 1;
PX2.FX = 1; CONS.L = 742.6;

OPTION ITERLIM = 0;
SOLVE CRIEMCP USING MCP;

OPTION ITERLIM = 1000;
SOLVE CRIEMCP USING MCP;

SETS I /I1*I51/;
PARAMETERS
 RESULTS1(I, *) L = SIZE  SIGMA = 0.487
 RESULTS2(I, *) INC and SHARES
 SIZE0, X10, X20, INCOME, INC0
 INCELAS1, INCELAS2, CONS0, Y10, Y20;

SIZE0 = 0.15; X10 = .1; X20 = .1; INC0 = 1;
INCELAS1 = 1; INCELAS2 = 1; CONS0 = .1;

TOT = 1.;
SIGMA = 0.487;

LOOP (I,

SIZE =  0.25 + 0.05*ORD(I);

SOLVE CRIEMCP USING MCP;

INCOME = PL.L*110*SIZE + PK.L*130*SIZE;

INCELAS1 = (((140*X1P.L - 140*X10)/(140*X10))/((INCOME-INC0)/INC0))$(X10 GT 0);
INCELAS2 = (((100*X2P.L - 100*X20)/(100*X20))/((INCOME-INC0)/INC0))$(X20 GT 0);

RESULTS1(I, "L") = SIZE*100;
RESULTS1(I, "X1") = X1.L;
RESULTS1(I, "X2") = X2.L;
RESULTS1(I, "T12") = TX12.L;
RESULTS1(I, "T21") = TX21.L;
RESULTS1(I, "C1") = PX1.L*X1P.L*140;
RESULTS1(I, "C2") = PX2.L*X2P.L*100;
RESULTS1(I, "C1/C2") = (PX1.L*X1P.L*140/(PX2.L*X2P.L*100))$(X2.L GT 0);
RESULTS1(I, "X1/X2") =  140*X1.L/(100*X2.L);
RESULTS1(I, "INELAS1") = INCELAS1$(ORD(I) GT 1);
RESULTS1(I, "INELAS2") = INCELAS2$(ORD(I) GT 1);
RESULTS1(I, "RATIOINELAS") = (INCELAS1/INCELAS2)$(ORD(I) GT 1);

RESULTS2(I, "INCOME") = INCOME;
RESULTS2(I, "X1") = MAX(EPS, PX1.L*X1.L*140/INCOME);
RESULTS2(I, "TX21") = PX1.L*TX21.L*100*TOT/INCOME;
RESULTS2(I, "TX12") = -PX1.L*TX12.L*140*TOT*1.*1.0001/INCOME;
RESULTS2(I, "SHX1") = PX1.L*X1P.L*140/INCOME;

SIZE0 = SIZE;  X10 = X1P.L; X20 = X2P.L;
Y10 = Y1.L; Y20 = Y2.L; CONS0 = CONS.L;
*INC0 =  PL.L*100*SIZE + PK.L*100*SIZE + PR1.L*25 + PR2.L*75;
INC0 = PL.L*110*SIZE + PK.L*130*SIZE;
);

DISPLAY RESULTS1, RESULTS2;
```

```
$exit

Execute_Unload 'CRIE-open-counter.gdx' RESULTS1
execute 'gdxxrw.exe CRIE-open-counter.gdx par=RESULTS1 rng=SHEET1!A3:M55'

Execute_Unload 'CRIE-open-counter.gdx' RESULTS2
execute 'gdxxrw.exe CRIE-open-counter.gdx par=RESULTS2 rng=SHEET1!A56:F109'
```

```
$TITLE: Calculate 2.gms James Markusen, University of Colorado, Boulder
* uses the BEA and other data to create an excel file calculate.xlsx
* this excel file must be used with this GAMS file to read in data from Sheet2
* it is load into the program, which solves for alpha1, alpha2 and rho
* using non-linear programming to minimize the sum of squared erros

SETS I /1*75/
     J /1*3/;

PARAMETERS
  MATRIX(I,J);

execute 'gdxxrw calculate.xlsx par=MATRIX rng=Sheet2!A1:D76'
execute_load 'calculate.gdx' MATRIX;

DISPLAY MATRIX;

SETS K /1*2/;
PARAMETERS
  GOODS(I), SERVICES(I), SHRATIO(I), ETA12, SHARES(I,K), ETARAT(I), SUMALPHA, SIGMA;

GOODS(I) = MATRIX(I,"1");
SERVICES(I) = MATRIX(I,"2");
SHRATIO(I) = MATRIX(I,"3");
DISPLAY GOODS, SERVICES, SHRATIO;

* minimize the sum of square errors solving for the ALPHA parameters
* alphas constrainted to lie between 0 and 1 - not binding in the solution

VARIABLES
  OBJ      objective function minimize the sum of squared errors
  RHO      elasticity parameter sigma = 1 over (1-rho);

NONNEGATIVE VARIABLES
  C   average ratio of expenditure shares services over goods,
  ALPHA1, ALPHA2;

EQUATIONS
  MINSSE    sum of squared errors
  CONST1, CONST2  ALPHA LE 1;


MINSSE..  OBJ =E= SUM(I, POWER(SHRATIO(I) - (C*ALPHA1/ALPHA2)*(SERVICES(I)**(RHO*ALPHA1))
                    /(GOODS(I)**(RHO*ALPHA2)), 2));

CONST1..   1 =G= ALPHA1;
CONST2..   1 =G= ALPHA2;

MODEL
  calculate /MINSSE, CONST1, CONST2/;

* fix C as mean of share ratio from calculate.xlsx Sheet1
ALPHA1.L = 0.25; ALPHA2.L = 0.75; RHO.L = -1; C.L = 1;

C.FX = SUM(I, SHRATIO(I))/75;

OPTION DECIMALS = 6;
SOLVE calculate USING NLP MINIMIZING OBJ;

* N.B.  this is the formula when rho is negative (sigma < 1)
* if sigma > 1, change +alpha to -alpha

ETA12 = (1 - RHO.L*ALPHA2.L)/(1 - RHO.L*ALPHA1.L);
SIGMA = 1/(1-RHO.L);
SUMALPHA = (ALPHA1.L + ALPHA2.L) + RHO.L;

DISPLAY ALPHA1.L, ALPHA2.L, ETA12, SUMALPHA, RHO.L, SIGMA;
```

```
execute 'gdxxrw calculate.xlsx par=SHARES rng=Sheet3!A1:C76'
execute_load 'calculate.gdx' SHARES;

DISPLAY SHARES;

* calculateulate the sigma levels using the budget adding up condition

NONNEGATIVE VARIABLES
  ETA1(I), ETA2(I);

EQUATIONS
  RATIO(I)
  ADDING(I);

RATIO(I)..  ETA1(I)=E= ETA12*ETA2(I);
ADDING(I).. SHARES(I,"2")*ETA1(I)+ SHARES(I,"1")*ETA2(I)=E=1;

MODEL LEVELS /RATIO.ETA1, ADDING.ETA2/;

ETA1.L(I) = 0.5; ETA2.L(I) = 0.5;

SOLVE LEVELS USING MCP;
ETARAT(I) = ETA1.L(I)/ETA2.L(I);
DISPLAY ETA1.L, ETA2.L, ETARAT;

PARAMETERS
 RESULTS(I,*), AVETA1, AVETA2;

RESULTS(I, "ETA1") = ETA1.L(I);
RESULTS(I, "ETA2") = ETA2.L(I);
RESULTS(I, "RATIO") = ETARAT(I);
AVETA1 = SUM(I, ETA1.L(I))/75;
AVETA2 = SUM(I, ETA2.L(I))/75;
DISPLAY RESULTS, AVETA1, AVETA2;

$ontext
execute_unload 'calculate.gdx' RESULTS
execute 'gdxxrw.exe calculate.gdx par=RESULTS rng=SHEET7!A1:D76'
$offtext

PARAMETERS
 YHAT(I), DEV(I),  AVDEV, YBAR, SST, SSR, RSQR, RESULTSDEV(I,*);

YHAT(I) = (C.L*ALPHA1.L/ALPHA2.L)*(SERVICES(I)**(RHO.L*ALPHA1.L))/(GOODS(I)**(RHO.L*ALPHA2.L));
DEV(I) = SHRATIO(I) - YHAT(I);
DEV(I) = (SHRATIO(I) - YHAT(I))/YHAT(I);
AVDEV = SUM(I, DEV(I))/75;
YBAR = SUM(I, SHRATIO(I))/75;
SST = SUM(I, POWER(SHRATIO(I) - YBAR, 2));
SSR = SUM(I, POWER(SHRATIO(I) - YHAT(I), 2));
RSQR = 1 - OBJ.L/SST;
*RSQR = 1 - SSR/SST;
DISPLAY YBAR, DEV, AVDEV, YHAT, OBJ.L, SSR, SST, RSQR;

$exit
RESULTSDEV(I,"DEV") = DEV(I);
DISPLAY RESULTSDEV;
execute_unload 'calculate.gdx' RESULTSDEV
execute 'gdxxrw.exe calculate.gdx par=RESULTSDEV rng=SHEET10!A1:D76'
*execute 'gdxxrw.exe calculate.gdx par=RESULTSDEV rng=SHEET10!G1:I76'
```

Data used in Figure 1 and in NLP estimation

US data 1950 - 2024

Derived from raw data: these columns give the three variables for the NLP estimation

	Raw Data Expenditures on goods and services BEA tables on consumption shares		Raw Data Prices of goods and services BEA table on prices		Raw Data Population normalized 1950 = 1	Real consuption quantities per capita Expenditure / Price / Pop		Expenditure share ratio
	Goods	Services	Goods	Services	Population	Goods	Services	Serrvices/Goods
1950	116.80	75.20	24.061	7.888	1.000	4.854	9.533	0.644
1951	124.70	83.60	26.003	8.276	1.016	4.722	9.946	0.670
1952	128.80	90.50	26.253	8.579	1.031	4.757	10.228	0.703
1953	134.80	97.90	26.133	8.909	1.048	4.920	10.482	0.726
1954	135.80	103.80	26.033	9.133	1.067	4.889	10.652	0.764
1955	147.40	110.90	25.880	9.288	1.087	5.241	10.988	0.752
1956	152.20	119.00	26.320	9.508	1.106	5.226	11.312	0.782
1957	159.60	126.70	27.143	9.786	1.126	5.222	11.499	0.794
1958	161.60	134.00	27.761	10.028	1.147	5.073	11.645	0.829
1959	172.60	144.50	27.903	10.297	1.169	5.292	12.006	0.837
1960	177.00	154.20	28.122	10.571	1.218	5.166	11.972	0.871
1961	178.80	162.70	28.227	10.759	1.239	5.113	12.207	0.910
1962	189.00	173.60	28.373	10.963	1.258	5.295	12.588	0.919
1963	198.20	183.90	28.609	11.138	1.276	5.428	12.937	0.928
1964	212.30	198.40	28.920	11.342	1.294	5.673	13.517	0.935
1965	229.70	213.30	29.224	11.553	1.310	5.998	14.090	0.929
1966	249.60	230.30	29.868	11.886	1.326	6.304	14.617	0.923
1967	259.00	247.70	30.394	12.280	1.340	6.359	15.052	0.956
1968	284.60	272.20	31.454	12.815	1.354	6.685	15.693	0.956
1969	304.70	299.00	32.673	13.476	1.367	6.823	16.233	0.981
1970	318.80	327.90	33.919	14.223	1.383	6.797	16.671	1.029
1971	342.10	357.80	34.938	14.999	1.400	6.992	17.034	1.046
1972	373.80	394.30	35.833	15.634	1.416	7.370	17.817	1.055
1973	416.60	432.90	37.970	16.389	1.429	7.677	18.483	1.039
1974	451.50	478.60	42.709	17.778	1.442	7.330	18.666	1.060
1975	491.30	539.20	46.159	19.302	1.457	7.308	19.179	1.097
1976	546.30	601.40	47.966	20.641	1.470	7.746	19.815	1.101
1977	600.40	673.60	50.526	22.203	1.485	8.001	20.426	1.122
1978	663.60	758.70	53.626	23.910	1.501	8.244	21.139	1.143
1979	737.90	847.50	58.698	25.915	1.518	8.283	21.547	1.149
1980	799.80	950.90	65.271	28.610	1.532	7.996	21.689	1.189
1981	869.40	1064.60	70.120	31.541	1.548	8.012	21.811	1.225
1982	899.30	1172.00	72.031	34.017	1.562	7.991	22.053	1.303
1983	973.80	1307.80	73.331	36.106	1.577	8.422	22.973	1.343
1984	1063.70	1428.60	74.718	37.985	1.590	8.951	23.648	1.343
1985	1137.60	1575.20	75.917	39.843	1.605	9.339	24.640	1.385
1986	1195.60	1690.70	75.562	41.480	1.619	9.771	25.169	1.414
1987	1256.30	1820.00	77.992	42.726	1.634	9.858	26.069	1.449
1988	1337.30	1992.70	80.048	44.769	1.649	10.132	26.994	1.490
1989	1423.80	2153.00	83.128	46.880	1.665	10.290	27.591	1.512
1990	1491.30	2317.70	86.532	49.029	1.683	10.237	28.081	1.554
1991	1497.40	2446.00	88.647	50.946	1.706	9.901	28.141	1.633
1992	1563.30	2634.30	89.717	52.758	1.730	10.073	28.864	1.685
1993	1642.30	2809.60	90.496	54.582	1.753	10.353	29.366	1.711
1994	1746.60	2974.40	91.417	56.066	1.775	10.767	29.897	1.703
1995	1815.50	3147.10	92.271	57.632	1.796	10.957	30.409	1.733
1996	1917.70	3326.90	93.285	59.214	1.817	11.315	30.925	1.735
1997	2006.50	3530.30	93.177	60.883	1.839	11.711	31.535	1.759
1998	2108.40	3768.80	91.777	62.172	1.860	12.349	32.585	1.788
1999	2287.10	3996.70	92.258	63.409	1.882	13.174	33.494	1.747
2000	2453.20	4314.00	94.089	65.210	1.903	13.702	34.766	1.759
2001	2525.60	4548.20	94.018	67.292	1.922	13.978	35.169	1.801
2002	2598.80	4750.10	93.122	69.033	1.940	14.387	35.474	1.828
2003	2722.60	5018.20	93.003	71.336	1.956	14.963	35.956	1.843
2004	2902.00	5329.90	94.311	73.528	1.975	15.583	36.709	1.837
2005	3082.90	5686.10	96.203	75.998	1.993	16.080	37.542	1.844
2006	3239.70	6037.60	97.494	78.750	2.012	16.514	38.101	1.864
2007	3367.00	6379.60	98.576	81.388	2.031	16.814	38.585	1.895
2008	3363.20	6686.90	101.524	83.783	2.051	16.153	38.918	1.988
2009	3180.00	6711.20	99.084	84.432	2.069	15.513	38.421	2.110
2010	3317.80	6942.40	100.533	86.077	2.086	15.820	38.663	2.092
2011	3518.10	7180.70	104.325	87.742	2.101	16.048	38.947	2.041
2012	3637.70	7409.60	105.620	89.648	2.117	16.271	39.046	2.037
2013	3742.20	7646.10	105.049	91.659	2.131	16.713	39.137	2.043
2014	3886.60	7987.90	104.542	93.795	2.147	17.315	39.663	2.055
2015	3955.10	8342.30	101.350	95.462	2.163	18.041	40.401	2.109
2016	4033.00	8693.80	99.710	97.629	2.179	18.564	40.871	2.156
2017	4212.20	9078.40	100.000	100.000	2.193	19.211	41.405	2.155
2018	4414.20	9520.20	100.811	102.626	2.204	19.865	42.087	2.157
2019	4532.80	9904.70	100.426	104.965	2.214	20.384	42.616	2.185
2020	4708.60	9522.70	99.666	107.088	2.236	21.131	39.773	2.022
2021	5499.40	10620.20	104.599	111.044	2.239	23.479	42.709	1.931
2022	5938.70	11751.30	113.639	117.130	2.248	23.252	44.638	1.979
2023	6115.70	12717.50	115.037	123.091	2.259	23.538	45.743	2.079
2024	6261.50	13634.50	114.538	128.119	2.305	23.715	46.166	2.178