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September 13, 2026

Selection Committee
Workshop on Development Economics and Networks
for Early-Career Researchers
National Bureau of Economic Research

To whom it may concern,

I am writing to apply to the Workshop on Development Economics and Networks for Early-Career Researchers. I am an Assistant Professor of Economics at the University of Southern California. I would like to submit my paper, "Endogenous Interference in Randomized Experiments," for consideration.

This paper studies the identification and inference of treatment effects in randomized controlled trials under interference, allowing the network to be endogenous and to evolve dynamically following the intervention. I construct a shift-share instrument to identify and distinguish among different causal channels. The resulting estimator is consistent in relatively sparse networks but becomes inconsistent as network density increases. I therefore propose a denoised shift-share IV estimator that restores consistency in denser networks. I illustrate the proposed methods using the randomized experiment in Prina (2015).

Participating in the workshop would provide a valuable opportunity to receive feedback on my research and to engage with researchers actively working on relevant questions in development economics and networks.

If I am shortlisted, Professor Bryan Graham can provide a recommendation on my behalf and may be contacted at bryangraham@berkeley.edu.

Thank you for considering my application. I have enclosed my paper and curriculum vitae.

Sincerely,

Mengsi Gao
Assistant Professor of Economics
University of Southern California

Mengsi Gao

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ACADEMIC APPOINTMENTS

University of Southern California

Assistant Professor of Economics — Fall 2026 -
Postdoctoral Scholar — Fall 2025 - Spring 2026

EDUCATION

Ph.D. in Economics, UC Berkeley, Berkeley, CA — 2019 - 2025
M.A. in Economics, Duke University, Durham, NC — 2017 - 2019
B.A. in Economics, Zhejiang University, Hangzhou, China — 2013 - 2017

RESEARCH INTEREST

Econometrics, Applied Econometrics

PUBLICATION

1. "Uniform Nonparametric Inference for Time Series using Stata," with Jia Li and Zhipeng Liao, **Stata Journal**, Vol.20(3), 2020, pp. 706–721.
2. "Inference under Covariate-Adaptive Randomization with Imperfect Compliance," with Federico A. Bugni (2023), **Journal of Econometrics**, Vol.237(1).
3. "Causal Inference in Network Experiments: Regression-based Analysis and Design-based Properties," with Peng Ding (2025), **Journal of Econometrics**, 252, 106119.
4. "On the Power Properties of Inference for Parameters with Interval Identified Sets," with Federico Bugni, Filip Obradović, and Amilcar Velez. Accepted at **Econometric Theory**.

WORKING PAPER

1. "Endogenous Interference in Randomized Experiments."
2. "Identification and Inference on Treatment Effects under Covariate-Adaptive Randomization and Imperfect Compliance," with Federico Bugni, Filip Obradović, and Amilcar Velez. Revise & Resubmit at **Quantitative Economics**.
3. "Coupling and Maximal Inequalities for Graph-Dependent Empirical Processes," with Demian Pouzo.
4. "Misspecified regressions with mixed regressors: robust inference and causal interpretation," with Peng Ding.

Ph.D. COURSEWORK

- Mathematical Tools for Economics (ECON 204), Economic Theory (ECON 201A and 201B), Macroeconomic Theory (ECON 202A and 202B), Econometrics (ECON 240A, 240B, 241B, and 241C), Introduction to Economic History (ECON 210A), Industrial Organization (ECON 220A

and 220B), Causal Inference (STAT 256), Theoretical Statistics (STAT 210A and 210B), Applied Econometrics (ECON 244), Experimental Design (STAT 232).

SEMINAR AND CONFERENCES

- **2027**: **Scheduled** WashU, Vanderbilt University
- **2026**: NYU, Cornell, Florida State University (virtual), UCLA, USC-UCLA mini conference, University of Chicago Booth (Econometrics & Statistics), Econometric Society Africa Meeting, EcoSta 2026 conference
  Scheduled University of Chicago Econ, SEA 96th Annual Meeting
- **2025** ASSA Annual Meeting, Emory (QTM), University of Pennsylvania, Texas A&M University, Tulane, Duke, USC, WashU, Rutgers, CAMSE-CLIMB Mini-Conference, UC Davis, Joint Statistical Meetings, World Congress of the Econometric Society, Microeconometrics Class of 2025 Conference, Brown, UC San Diego, UC Riverside, Chamberlain Online Seminar
- **2024** Zhejiang University (School of Economics & Center for Data Sciences), East China Normal University (School of Statistics), California Econometrics Conference, Peking University (GSM), UC Berkeley Econometrics Seminar
- **2023** UC Berkeley Econometrics Seminar

TEACHING

- Instructor, Department of Economics, USC
  - Economic Applications of Machine Learning (Econ 460) Spring 2026, Fall 2026
  - Econometrics Reading Group (Econ 700) Fall 2026, Spring 2027
- Teaching Assistant, Department of Statistics, UC Berkeley Spring 2025
  - Experimental Design (Stat 158)
- Tutor/Teaching Assistant, Department of Economics, UC Berkeley Fall 2019–Spring 2024
  - Graduate Econometrics (Econ 240A and B), Advanced Econometrics (Econ 143), Intermediate Econometrics (Econ 140 and 141)
- Teaching Assistant, Department of Economics, Duke University Spring 2019
  - Graduate Game Theory (Econ 620)

PROFESSIONAL ACTIVITIES

- Referee: Journal of Econometrics, Journal of Business & Economic Statistics, Journal of the American Statistical Association, Journal of Applied Statistics, Journal of Causal Inference, Biometrics, The Review of Economics and Statistics, Biometrika, Annals of Statistics, Statistics in Medicine, Economics Letters, INFORMS Journal on Computing, Econometric Theory, Journal of Applied Economics, Journal of Machine Learning Research, Journal of the Royal Statistical Society Series B, Econometrica, Bernoulli
- Program Committee: EC'26.
- Conference session chair: EcoSta 2026

HONORS & SCHOLARSHIPS

- U.C. Berkeley Dissertation Completion Fellowship 2024
- Department of Economics Master's Program Award for Academic Excellence 2019
- Duke University M.A. Merit Scholar Spring 2018, Fall 2018
- Duke Economics Master's Scholar 2017 - 2019

OTHER INFORMATION

Languages Mandarin (native), English (fluent)
Citizenship China

Research Statement

My research interests lie at the intersection of causal inference and network analysis, with a particular focus on the design and analysis of randomized experiments under interference. Network experiments have become increasingly important across economics and other social sciences, but they also raise econometric challenges that are absent in settings without social interactions. My research develops methods to address these challenges while remaining closely connected to empirical applications.

A central theme of my current work is that social networks need not be fixed or exogenous. In my paper, "Endogenous Interference in Randomized Experiments," I study identification and estimation when network formation may be endogenous and the network itself may evolve in response to an intervention. I develop a framework that distinguishes direct treatment effects from indirect effects mediated by treatment-induced network changes. The paper proposes shift–share instrumental-variable methods to address network endogeneity, studies their validity under relatively sparse networks, and develops a denoising approach for denser networks. An empirical application to Prina (2015) illustrates how an intervention can affect outcomes both directly and through changes in social connections.

More broadly, I am interested in how endogenous, dynamic, and partially observed networks should change the way researchers design and analyze experiments. One project I am developing studies optimal treatment allocation when the intervention itself may alter the network, motivated in part by empirical evidence such as Carrell et al. (2013) showing endogenous changes in social relationships after randomized treatment. Another project studies uncertainty arising from network sampling. Existing work often assumes that the network is fully observed, while in practice researchers frequently observe only a sampled or partially measured network. I am interested in understanding how alternative network sampling schemes affect identification and estimation of causal effects, and in viewing network sampling itself as part of experimental design.

The Workshop on Development Economics and Networks would provide an ideal setting in which to advance this research agenda. I would greatly value feedback from researchers working at the frontier of development economics and network econometrics, particularly on the empirical relevance and potential applications of my methods. The workshop would also help me identify substantive questions in development economics where endogenous network formation and experimental design are especially important.

At this early stage of my career, I hope to build a research agenda in which econometric methodology is closely connected to empirically relevant questions. The workshop would be a valuable opportunity to learn about recent developments in the field, refine my ongoing projects, and develop lasting connections with researchers working on related questions.

Endogenous Interference in Randomized Experiments

Mengsi Gao
University of Southern California*

September 15, 2026

Abstract

This paper studies identification and inference for treatment effects in randomized trials with social interactions. Two features introduce endogeneity: (1) latent variables that affect both networks and outcomes, and (2) treatment-induced network changes that mediate treatment effects. I develop a potential-outcome framework with post-treatment networks and define direct and network-mediated effects. For estimation, I propose a shift-share IV strategy with asymptotic theory for sparse networks, but it fails in denser settings. I then develop a denoised SSIV estimator that restores consistency in denser networks and improves the convergence rate. An application to Prina (2015) illustrates both direct and network-mediated effects.

1 Introduction

Investigating spillover effects among units is a central task in many social science problems, but identifying them can be difficult without randomized experiments. Recent studies combine peer effects with randomized controlled trials (RCTs) to improve identification in economics and the social sciences (Sobel, 2006; Banerjee et al., 2013; Cai et al., 2015; Paluck et al., 2016). These studies examine how interventions spread through social networks, potentially amplifying or weakening their effects. This phenomenon is known as interference, violating the "stable unit treatment value assumption" (SUTVA).

*E-mail: mengsi.gao@usc.edu. I am very grateful to my advisors Bryan Graham, Peng Ding, Michael Jansson and Demian Pouzo for their generous support and advice. I also thank Eric Auerbach, Kirill Borusyak, Yong Cai, Kevin Chen, Patrick Kline, Shuangning Li, Jacob Dorn, Yassine Sbai Sassi and Christopher Walters for valuable comments and discussions. I also thank seminar and conference participants where this paper was presented, for their helpful comments and discussions. All errors are my own.

The peer effects literature typically assumes that networks are exogenous and fixed, so that unobserved factors do not jointly affect network formation and outcomes, and the network does not change after the intervention. However, both assumptions can fail. First, latent traits such as ability, preferences,or socioeconomic background may affect both whom individuals connect with and the outcomes they realize. Second, the intervention itself may alter the network structure. Several studies emphasize this latter concern. Carrell et al. (2013) find that a group formation experiment intended to help low-ability students backfired, as it led them to form stronger ties with other low-performing peers. Prina (2015) find that offering savings accounts in Nepalese villages significantly changed financial relationships among households. Chetty et al. (2016) analyze the Moving to Opportunity (MTO) experiment, in which housing vouchers were intended to improve outcomes by moving families to better neighborhoods, but such moves could also disrupt social networks, especially for adolescents. Paluck et al. (2016) study a school-based anti-conflict intervention and reported a substantial turnover in the school social network, measured by students' nominations of peers with whom they spent time, with only 42.2% of nominations persisting after about nine months. Banerjee et al. (2023) show that introducing formal financial institutions in Indian villages weakened social ties by reducing reliance on informal networks. Motivated by both concerns, this paper studies the identification and inference of causal effects in RCTs with interference, explicitly accounting for both latent confounding and treatment-induced changes in network structure.

Latent variables complicate identification and can bias causal estimates. Moreover, when treatment affects network structure, it becomes difficult to separate the direct effect of treatment from indirect effects mediated through the network. Yet distinguishing these channels is crucial for uncovering intervention mechanisms and designing effective policies. Although post-intervention networks capture how treatment and latent traits jointly shape networks, using post-intervention networks introduces endogeneity even in randomized settings. Consequently, researchers often rely on pre-intervention network data to estimate treatment effects, either by assuming the network is static or, even when network changes are acknowledged, by using the pre-intervention network to control for unobserved confounders (Carter et al., 2021). However, this approach can be problematic: if outcomes depend on the post-intervention network, then using only pre-intervention data introduces measurement error. This paper proposes a novel approach that leverages panel network data. Specifically, it fits outcome regressions using post-intervention networks while constructing instruments from pre-intervention networks to correct for endogeneity and achieve consistent estimation.

A common approach to capturing interference is effective treatment, or *exposure mapping*, which maps units, treatment assignments, and network structures into some low-dimendional

exposure levels (Manski, 2013; Aronow and Samii, 2017). In my setting, to accommodate treatment-induced network change, I adopt the mediation analysis framework (Pearl, 2001; Heckman and Pinto, 2015), treating exposure mapping as a mediator to distinguish between the direct effect of treatment and the indirect effect mediated by network changes. Within this framework, I define causal parameters that separately capture these effects. Under a linear outcome model that includes both treatment and the network mediator, and allowing for flexible, additive unobserved confounding, I show that the coefficients have clear causal interpretations.

For estimation and inference, I focus on a specific mediator: the fraction of treated friends. This yields a linear model that is a special case of the linear-in-means (LIM) framework (Manski, 1993), capturing contextual peer effects, or the mean impact of peers' treatments. To address endogeneity, I use shift-share (or "Bartik") instrumental variables (SSIV) (Bartik, 1991), which combine numtiple shocks with exposure shares. I extend this approach to network settings by constructing the instrument as a combination of the random shock (others' treatment assignments) and the non-exogenous exposure (pre-intervention network), following Borusyak and Hull (2023). I examine the conditions for the validity of this IV, with particular attention to the relevance condition under varying levels of network sparsity. I show that as networks become denser, the SSIV fails the relevance condition because overlapping friendships reduce cross-unit variation in exposure. As a result, IV estimators based on SSIV are consistent only in relatively sparse networks.

To address the inconsistency that arises in denser networks, I propose a modified SSIV estimator that projects out latent traits, which are approximated using an eigen-decomposition of the pre-intervention network. This modification restores consistency in denser networks while remaining valid in sparse settings. I then establish asymptotic normality for both the SSIV and modified SSIV estimators, and provide consistent variance estimators that account for cross-unit dependence induced by the network, enabling valid inference. Finally, within the regimes where both estimators are consistent, I show that the modified SSIV delivers faster convergence rate for estimating spillover effects. While the paper focuses on a particular form of peer effects, the analysis of the relevance condition and the denoising procedure provides broader insights into the general application of SSIV.

This paper contributes to three strands of the literature. First, it builds on peer effects research, following the seminal work of Manski (1993), which examines how individual outcomes are influenced by peers' behaviors, actions, or characteristics. The literature outlines four broad strategies for identifying peer effects: random peers, random shocks, structural endogeneity, and panel data (Bramoullé et al., 2020). Existing approaches to network endogeneity include the matching method (Auerbach, 2022), the control-function method

(Johnsson and Moon, 2021), and the parametric framework of Griffith (2022), which enables the identification of counterfactual treatment effects while accounting for network-structure changes. All of these methods assume dense networks for validity. This paper contributes by developing estimation and inference tools for social interactions with treatment-induced network changes under general endogeneity. I use randomized treatment assignment and panel network data to construct valid shift–share instruments across a wide range of sparsity regimes. The SSIV approach is closely related to IVs constructed based on leave-one-out networks (Bramoullé et al., 2009; Jochmans, 2023). The work most closely addressing dynamic network evolution is Comola and Prina (2021). They allow for endogenous peer effects but assume conditional exogeneity of pre- and post-intervention networks, addressing different endogeneity sources. They use "lagged" partner characteristics as instruments and focus on deriving identification conditions rather than asymptotic properties.

Second, I contribute to the shift-share IV literature by demonstrating SSIV's relevance condition across a broad range of network sparsity regimes. Shift-share designs are widely used in many contexts, including labor, public policy, development, macroeconomics and international trade. There are two main approaches to identification in the SSIV literature: one relies on exogeneity of the share (Bartik, 1991; Goldsmith-Pinkham et al., 2020), and the other on exogeneity of the shift (Adão et al., 2019; Borusyak et al., 2022; Borusyak and Hull, 2023). However, these papers impose a crucial relevance condition: most observations must be exposed to only a small number of shocks that influence treatment. In this paper, I identify the network-sparsity regimes that guarantee the SSIV relevance condition and introduce a denoising procedure to restore consistency of IV estimates when that condition is violated. I also develop asymptotic theory by accounting for the randomness from the network generation, while Adão et al. (2019) treat the shares as fixed. I also connect my framework to Li and Wager (2022). Their estimator can be rewritten as exploiting the same shift–share variation as my SSIV estimator, but it is motivated by a design-based identification argument for indirect effects under a fixed interference graph, rather than by mediation endogeneity in a treatment-induced network. Moreover, my paper allows the network to respond endogenously to treatment, whereas Li and Wager (2022) do not. I further establish consistency of the modified SSIV over a substantially wider range of sparsity regimes than those considered in Li and Wager (2022) under weaker network assumptions.

Third, I contribute to the interference literature by allowing networks to be both stochastic and treatment-induced. The interference literature studies causal effects when one unit's treatment can affect other units' outcomes. Much of this literature is design-based (Hudgens and Halloran, 2008; Abadie et al., 2020): it avoids assumptions on outcome models and network formation, and bases inference on random treatment assignment. When net-

works are themselves stochastic, it is natural to ask how uncertainty in network formation affects identification and inference, and how to distinguish direct effects from effects operating through network-mediated channels. Recent work models network formation as realizations from an (unknown) graphon (Auerbach, 2022; Li and Wager, 2022), but does not consider how treatment-induced changes in network structure further influence causal effects. To fill this gap, I introduce a mediation framework that explicitly accounts for the treatment-induced network changes, thereby disentangling direct treatment effects from indirect, network-mediated channels.

The remainder of the paper proceeds as follows. Section 2 introduces the framework and defines the parameters of interest. Section 3 develops an SSIV estimator and establishes its asymptotic properties under relatively sparse networks. It then proposes a denoised SSIV estimator to handle cases in which the network becomes denser. Section 4 presents the results of the Monte Carlo simulations. Section 5 illustrates the results in an empirical application based on the RCT in Prina (2015). Section 6 concludes. The appendix of the paper collects all of the proofs, as well as some intermediate results.

Notation I use $O(), O_{\mathbb{P}}(), o_{\mathbb{P}}(), \asymp, \succ, \prec, \succcurlyeq, \preccurlyeq$ as follows: $a_n = O(b_n)$ if $|a_n| \leq Cb_n$ for n large enough; $X_n = O_{\mathbb{P}}(b_n)$, if for any $\delta > 0$, there exists $M, N > 0$, s.t. $\mathbb{P}[|X_n| \geq Mb_n] \leq \delta$ for any $n > N$; $X_n = o_{\mathbb{P}}(b_n)$, if $\lim \mathbb{P}[|X_n| \geq \varepsilon b_n] = 0$ for any $\varepsilon > 0$; $a_n \asymp b_n$ if there exists $k_1, k_2 > 0$ and n_0, s.t. for all $n > n_0$, $k_1 a_n \leq b_n \leq k_2 a_n$; $a_n \succ b_n$ if $\lim a_n/b_n = \infty$; $a_n \prec b_n$ if $\lim a_n/b_n = 0$; $a_n \succcurlyeq b_n$ if $b_n = O(a_n)$; $a_n \preccurlyeq b_n$ if $a_n = O(b_n)$. Let 1_n and 0_n denote n-dimensional vectors of ones and zeros. Let $\|\cdot\|_{\text{op}}$ denote the operator norm. Let X_{-i} denote the collection $\{X_j\}_{j \neq i}$.

2 Setup

2.1 Framework and Notation

Consider an RCT with n participants. For each $i \in 1, \ldots, n$, let $T_i \in \{0, 1\}$ denote the treatment assignment, with $T_i \sim \text{Bernoulli}(\pi)$ for some $0 < \pi < 1$. Let $Y_i \in \mathbb{R}$ denote the outcome of interest and w_i denote unobserved covariates. I consider a single large network that permits arbitrary interference. The network is represented by an adjacency matrix $A = \{A_{ij}\}_{i,j=1}^n$, where $A_{ij} \in \{0, 1\}$ indicates whether units i and j are connected. The matrix A is undirected, unweighted, and with no self-links. I distinguish between two network structures observed by the researchers: A^{pre} (before the intervention) and A^{post} (after the intervention). The paper adopts a super-population framework to accommodate

randomness and dynamics in the network.

Figure 1 illustrates the causal mechanism considered in this paper. I assume that the post-intervention network A^{post}, influenced by the treatment vector, plays a critical role in determining the outcome. In particular, the exposure mapping that maps $\{T_i\}_{i=1}^n$ and A^{post} into low-dimensional statistics defines a mediator M_i, through which treatment can indirectly affect the outcome. Common examples include the number or fraction of treated friends after the intervention. Specifically, T_i affects Y_i both directly and indirectly through changes in M_i, while others' treatments T_{-i} affect Y_i only through M_i. Moreover, A^{post} is shaped by both the treatments and the latent variable w_i, which may also affect the outcome, thereby confounding the mediator-outcome relationship.

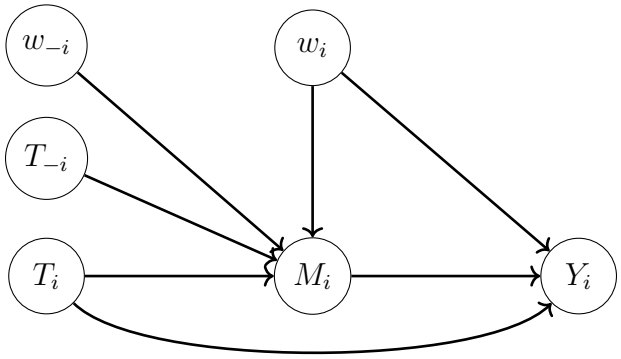


Figure 1: Causal mechanism.

Potential Value Notation To define the causal parameters of interest, I introduce the potential value notation for both the mediator and the outcome, following the mediation analysis literature (Pearl, 2001; Heckman and Pinto, 2015). First, consider a hypothetical intervention on the treatment vector t, and define the potential mediators for unit i as

$$\{M_i(t) : t \in \{0,1\}^n\}.$$

Note that $M_i(t)$ depends on t in two ways: directly, and indirectly through the network structure A^{post}, which is itself affected by t. Consider a hypothetical intervention on both t_i and m_i. Define the potential outcomes for unit i, corresponding to interventions on t_i and on the mediator $m_i = M_i(t_i, t_{-i})$, as

$$\{Y_i(t_i, m_i) : t_i \in \{0,1\},\ m_i \in \mathcal{M}\},$$

where $\mathcal{M}$ is the set of all possible mediator values. This notation indicates that the potential outcome $Y_i(t_i, m_i)$ depends both on unit i's own treatment and on the network changes

induced by the intervention (as captured by M_i), with the treatment assignments of others influencing Y_i only through M_i. I also define the nested potential outcomes corresponding to an intervention on $T_i = t_i$ and the mediator $M_i(t_i', t_{-i})$ as

$$\{Y_i(t_i, M_i(t_i', t_{-i})) : t_i \in \{0,1\},\, t_i' \in \{0,1\},\, t_{-i} \in \{0,1\}^{n-1}\}.$$

Here, $Y_i(t_i, M_i(t_i', t_{-i}))$ denotes the hypothetical outcome when unit i receives treatment level t_i while its mediator takes the potential value $M_i(t_i', t_{-i})$, corresponding to treatment t_i' for unit i and t_{-i} for the remaining units. Allowing t_i and t_i' to differ enables us to disentangle the direct and indirect channels through which the unit i's treatment affects the outcome. This definition differs from the interference framework in Hudgens and Halloran (2008), which does not account for intervention-induced network change and instead considers

$$\{Y_i(t_i, t_{-i}) : t_i \in \{0,1\},\, t_{-i} \in \{0,1\}^{n-1}\}.$$

The observed values of M_i and Y_i coincide with the corresponding potential outcomes under the realized treatment and mediator values. Specifically,

$$M_i = M_i(t_i, t_{-i}) \quad \text{if } T_i = t_i \text{ and } T_{-i} = t_{-i},$$

and

$$Y_i = Y_i(t_i, m_i) \quad \text{if } T_i = t_i \text{ and } M_i = m_i.$$

Data-Generating Process for Networks I assume that the relationship between any two units i and j, both before and after the intervention, is governed by the following random graph models:

$$A_{ij}^{\text{pre}} = 1\Big\{\eta_{ij} \le q_n^{\text{pre}}\, g^{\text{pre}}(w_i, w_j)\Big\}\, 1\{j \ne i\}, \tag{1}$$

$$A_{ij}^{\text{post}} = 1\Big\{\eta_{ij} \le q_n^{\text{post}}\, g^{\text{post}}(w_i, w_j, T_i, T_j)\Big\}\, 1\{j \ne i\}, \tag{2}$$

where $\{w_i\}_{i=1}^n$ are i.i.d. latent variables and $\{\eta_{ij}\}_{i,j=1}^n$ is a symmetric matrix of unobserved scalar disturbances with independent upper-diagonal entries. The distribution of η_{ij} is not separately identified from the graphons and sparsity parameters, so it is typically normalized to be standard uniform. Under this normalization, for $\diamond \in \{\text{pre}, \text{post}\}$, the product $q_n^{\diamond} g^{\diamond}$ gives the link probability for a given pair. The sparsity parameters q_n^{pre} and q_n^{post} are theoretical devices that allow link probabilities to vanish as $n \to \infty$, and I allow them to differ across the pre- and post-intervention networks.

The graphon-based network models in (1) and (2) take the form of inhomogeneous Erdős–Rényi graphs. Motivated by the Aldous–Hoover theorem for exchangeable arrays (Aldous, 1981; Hoover, 1979), this framework has become increasingly common in econometrics (Graham, 2020; Parise and Ozdaglar, 2023), including recent work on stochastic models of network formation (Auerbach, 2022; Cai, 2022; Li and Wager, 2022). In this paper, I use these models to capture network change following the intervention.[1] The pre-intervention graphon g^{pre} depends only on the latent pair (w_i, w_j), whereas the post-intervention graphon g^{post} may additionally depend on the treatment assignments (T_i, T_j). Because neither the idiosyncratic term η_{ij} nor the latent variables w_i is assumed to vary over time, the model also nests the case in which the network remains unchanged after the intervention, namely when $q_n^{\text{pre}} g^{\text{pre}} = q_n^{\text{post}} g^{\text{post}}$.[2] More generally, differences between g^{pre} and g^{post} capture changes in network formation unrelated to the intervention, while the dependence of g^{post} on treatment captures network responses induced by the intervention. The models in (1) and (2) therefore incorporate the two sources of endogeneity emphasized in this paper: latent variables that jointly affect network formation and outcomes, and endogenous network evolution following the intervention.

Assumption 1 below summarizes the key features of the network model used throughout the paper.

Assumption 1. *The latent variables $\{w_i\}_{i=1}^n$ are i.i.d. and independent of $\{\eta_{ij}\}_{i,j=1}^n$, where $\eta_{ij} \overset{i.i.d.}{\sim} U[0,1]$ for $j > i$ and $\eta_{ij} = \eta_{ji}$. The networks are randomly generated according to (1) and (2), where g^{pre} is a symmetric measurable function of w_i and w_j, and g^{post} is a symmetric measurable function of (T_i, w_i) and (T_j, w_j), both mapping into $[0,1]$. The sparsity parameters satisfy $n^{-1} \preccurlyeq q_n^\diamond \preccurlyeq 1$, for $\diamond \in \{\text{pre}, \text{post}\}$.*

The sparsity regimes considered in this paper are broad, covering (a) $q_n^\diamond \asymp n^{-1}$ (bounded degree): each node has about a constant number of links as n grows; (b) $\lim_{n\to\infty} q_n^\diamond = 0$ and $\lim_{n\to\infty} n q_n^\diamond = \infty$ (sparse but growing): connection probabilities are shrinking, but average degree grows with n; (c) $q_n^\diamond \asymp 1$ (dense): each node links to a nontrivial fraction of the sample; average degree is proportional to n.

[1] I impose this pairwise network formation model to obtain a sharper characterization of instrument relevance. The SSIV framework, however, is not limited to this class and can accommodate richer network formation models, including ones in which link formation depends on the full latent vector or admits multiple equilibria. See Section 3.1 for further discussion.

[2] The assumption that η_{ij} is time-invariant can be relaxed. For example, the pre- and post-intervention η_{ij} may be modeled as independent draws from the same distribution, or as draws from correlated distributions. In either case, the SSIV constructed from A^{pre} may be weaker, since additional exogenous shocks can reduce instrument relevance.

2.2 Causal Parameters of Interest

Now I define the causal parameters of interest in my setting.

Definition 1. *For $t \in \{0,1\}$, define*

(1) the own-treatment effect OTE *as*

$$\text{OTE} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[Y_i\left(1, M_i\left(1, T_{-i}\right)\right) - Y_i\left(0, M_i\left(0, T_{-i}\right)\right)\right]; \tag{3}$$

(2) the own-treatment direct effect $\text{OTDE}(t)$ *as*

$$\text{OTDE}(t) = \mathbb{E}\left[Y_i\left(1, M_i\left(t, T_{-i}\right)\right) - Y_i\left(0, M_i\left(t, T_{-i}\right)\right)\right]; \tag{4}$$

(3) the own-treatment indirect effect $\text{OTIE}(t)$ *as*

$$\text{OTIE}(t) = \mathbb{E}\left[Y_i\left(t, M_i\left(1, T_{-i}\right)\right) - Y_i\left(t, M_i\left(0, T_{-i}\right)\right)\right]; \tag{5}$$

(4) the spillover effect $\text{SE}(t)$ *as*

$$\text{SE}(t) = \mathbb{E}\left[Y_i\left(t, M_i\left(t_i = t, t_{-i} = 1_{n-1}\right)\right) - Y_i\left(t, M_i\left(t_i = t, t_{-i} = 0_{n-1}\right)\right)\right]. \tag{6}$$

The *own treatment effect* captures the overall change in unit i's outcome induced by a change in its treatment status, and it is identified under randomized assignment. I further decompose OTE into the *own-treatment direct effect* which measures the impact of changing T_i while holding the mediator at the level it would have under $T_i = t$, and the *own-treatment indirect effect* which measures the change in the outcome that operates solely through the treatment's impact on the mediator when T_i remains fixed at t. The *spillover effect* captures the effect on unit i's outcome when all other units are switched from control to treatment, keeping unit i's treatment fixed at t.[3] In all three definitions, the expectation is taken with respect to all sources of randomness, including the potential outcomes, the treatment assignments of other units, and the randomness in the graph.

A regression of Y_i on a constant and T_i identifies the *own treatment effect* under randomized assignment, but it does not distinguish the channels through which unit i's treatment

[3]The $\text{SE}(t)$ in (6) is one way to define spillover effects; alternatively, spillover effects may be defined as the change in unit i's outcome due to differences between any two distinct intervention programs via variations in the treatment of others.

T_i operates. This distinction is substantively important. For example, in the MTO experiment, Chetty et al. (2016) show that moving to a lower-poverty neighborhood at a young age improves later-life outcomes. At the same time, such moves may disrupt social networks, especially for adolescents, potentially offsetting some of these gains. Focusing only on the total effect could therefore suggest that the intervention has little or no impact, while obscuring the mechanisms through which it operates.

This distinction is also important for the identification strategy developed in Section 3. As discussed there, the shift-share instrument is constructed from peers' treatment assignments and therefore generates exogenous variation in the mediator through T_{-i}, rather than through T_i. It is thus naturally informative about outcome variation induced by peer-driven changes in the mediator, rather than about the indirect effect of unit i's own treatment. In particular, without additional structure, the SSIV does not separately identify the indirect effect defined by contrasts in T_i. At most, it can be interpreted as identifying a local spillover-type effect associated with exogenous peer-induced variation in the mediator.[4] Under additional structural assumptions, however, it can identify the common causal effect of the mediator on the outcome, which enters both the indirect effect and the spillover effect.

2.3 Under linear outcome model

These considerations motivate the benchmark outcome specification introduced in Assumption 2. Following the classical Baron–Kenny framework (Baron and Kenny, 1986), I impose a linear and additively separable potential outcome model in treatment and mediator. This assumption provides a tractable benchmark under which the direct, indirect, and spillover channels admit transparent interpretations.[5] In contrast, I do not impose linearity on the mediator M_i in T_i and w_i.

Assumption 2. *Assume the (partially) linear model for the potential outcome:*

$$Y_i(t_i, m_i) = \beta_0 + \beta_1 t_i + \beta_2 m_i + \lambda(w_i) + \varepsilon_i,^{6} \tag{7}$$

where $\lambda(\cdot)$ is an unknown measurable function with $\mathbb{E}(\lambda(w_i)) = 0$. Additionally, assume that $\{\varepsilon_i\}_{i=1}^n$ are i.i.d., $\mathbb{E}(\varepsilon_i^2) < \infty$, and $\mathbb{E}(\varepsilon_i | T_i, T_{-i}, A^{post}) = 0$.

[4] See Borusyak and Hull (2024) for discussion.

[5] More generally, one could consider alternative parametric outcome specifications. I focus on the linear model because it offers a simple and transparent benchmark for illustrating the main identification challenge and the properties of the SSIV solution.

[6] The specification $\lambda(w_i) + \varepsilon_i$ is imposed for simplicity. The SSIV framework is compatible with richer structural errors, including those that depend on latent characteristics of other units or even on the full latent vector $\{w_j\}_{j=1}^n$, provided the instrument is appropriately recentered to remain orthogonal to the relevant structural error. See Section 3.1 for further discussion.

Under Assumption 2, the observed outcome also follows the linear model:

$$Y_i = \beta_0 + \beta_1 T_i + \beta_2 M_i + \lambda(w_i) + \varepsilon_i. \tag{8}$$

Let u_i denote the error term in the outcome model: $u_i = \lambda(w_i) + \varepsilon_i$.

Remark 2.1. *Auerbach (2022) studies a partially linear model $Y_i = \beta^\top X_i + \lambda(w_i) + \varepsilon_i$, where the latent variable w_i also drives network formation, as in (1) with $q_n = 1$. They use a matching approach based on network data to identify and estimate β and $\lambda(w_i)$, with $X_i = (1, T_i, M_i)$ in this paper. Their estimation method relies on dense networks to enable codegree-based matching and on sufficient covariate variation conditional on w_i. In contrast, my approach exploits exogenous variation from randomized treatment and addresses endogeneity using an SSIV-based IV strategy, without attempting to identify $\lambda(w_i)$.*

Proposition 2.1 provides the causal interpretation of the coefficients in (7), which is analogous to the Baron–Kenny formulas for mediation (Baron and Kenny, 1986).

Proposition 2.1. *Under Assumption 2, it follows that*

(1) $\text{OTDE}(1) = \text{OTDE}(0) = \text{OTDE} = \beta_1$;

(2) $\text{OTIE}(1) = \text{OTIE}(0) = \text{OTIE} = \beta_2 \cdot \{\mathbb{E}(M_i|T_i = 1) - \mathbb{E}(M_i|T_i = 0)\}$;

(3) $\text{OTE} = \mathbb{E}(Y_i|T_i = 1) - \mathbb{E}(Y_i|T_i = 0) = \beta_1 + \beta_2 \cdot \{\mathbb{E}(M_i|T_i = 1) - \mathbb{E}(M_i|T_i = 0)\}$;

(4) $\text{SE}(t) = \beta_2 \cdot \{\mathbb{E}(M_i|T_i = t, T_{-i} = 1_{n-1}) - \mathbb{E}(M_i|T_i = t, T_{-i} = 0_{n-1})\}$.

Under Assumption 2, the treatment T_i does not interact with the mediator M_i, so neither $\text{OTDE}(t)$ nor $\text{OTIE}(t)$ depends on the value of t. Proposition 2.1 therefore implies that β_1 captures the direct effect of the treatment, whereas the indirect and spillover effects are governed by β_2, scaled by the extent to which the mediator responds to changes in T_i and T_{-i}.[7] Moreover, if the post-intervention network A^{post} is uncorrelated with treatment, then the mediator does not vary with T_i. In that case, the indirect effect is zero, so the total effect coincides with the direct effect, while the spillover effect may still be present.

If one estimates (8) by OLS, the endogeneity of the mediator M_i generally biases the estimate of β_2. Because T_i and M_i are correlated, this bias also carries over to the estimate

[7] Recall that the total effect is always identified under randomized assignment regardless of Assumption 2. If Assumption 2 holds, then OTE also admits the alternative representation given in Proposition 2.1. Thus, Assumption 2 imposes testable restrictions and can, in principle, be evaluated through specification tests. I do not pursue such specification tests here; rather, I impose Assumption 2 to provide the linear coefficients with a transparent causal interpretation and to obtain a benchmark setting for illustrating the SSIV approach.

of β_1. As a result, both β_1 and β_2 are generally biased, so the direct effect, indirect effect, and spillover effect are not identified. Empirical studies often assume that networks are fixed, or they rely on pre-intervention network data to avoid the selection bias that may arise from using post-intervention networks. However, replacing A_i^{post} with A_i^{pre} in the regression introduces measurement error and therefore does not recover the direct, indirect, or spillover effects. Define M_i^{pre} as the variable obtained by replacing A^{post} in M_i with A^{pre}, and let β^{pre} denote the population coefficients from regressing Y_i on $X_i^{\text{pre}} = (1, T_i, M_i^{\text{pre}})$. Then

$$\beta_1^{\text{pre}} = \beta_1 + \beta_2 \frac{\text{Cov}(T_i, M_i)}{\text{Var}(T_i)} = \text{OTE} \text{ and } \beta_2^{\text{pre}} = \beta_2 \frac{\text{Cov}(M_i^{\text{pre}}, M_i)}{\text{Var}(M_i^{\text{pre}})}.$$

Hence, β_1^{pre} recovers the total effect, while β_2^{pre} is an attenuated version of β_2. In other words, pre-intervention network data suffice to estimate the total effect, but they do not allow the direct and indirect effects to be disentangled or the spillover effect to be recovered.

This discussion highlights the importance of accounting for both sources of endogenous network formation when estimating causal effects in network settings, especially when the effects of interest operate through treatment-induced changes in network structure.

3 Theoretical Results

3.1 SSIV as a general solution

Endogeneity may arise because of the latent variable w_i affecting both the network formation and the outcome. To identify parameters, one needs instruments that are both uncorrelated with u_i and sufficiently relevant for M_i. Under randomized treatment assignment, a natural strategy is to exploit other units' treatment assignments together with the pre-intervention network, and consider instruments of the form: $Z_i = f_i(A^{\text{pre}}, T_{-i})$. Such an instrument exploits exogenous variation in other units' treatment assignments and leverage the variation in exposure to others' treatment. In this sense, it is a network analogue of a shift-share IV. Following Borusyak and Hull (2023), a convenient way to ensure validity is to recenter the instrument: $Z_i = f_i(A^{\text{pre}}, T_{-i}) - \mathbb{E}[f_i(A^{\text{pre}}, T_{-i}) \mid A^{\text{pre}}]$. Using A^{pre} as the share component avoids contamination from endogeneity in the post-intervention network, while recentering removes systematic dependence on latent heterogeneity. This approach guarantees exogeneity under relatively weak conditions:

$$\mathbb{E}[Z_i u_i] = \mathbb{E}\left[\left(f_i(A^{\text{pre}}, T_{-i}) - \mathbb{E}(f_i(A^{\text{pre}}, T_{-i})|A^{\text{pre}})\right) u_i\right] = 0. \tag{9}$$

Thus, for a broad class of mediators, exogeneity is relatively straightforward once the treatment assignment mechanism is known and recentering is feasible. The same logic extends beyond Bernoulli randomization and, more generally, to other moment-based settings and richer network formation models, provided the instrument is recentered to be orthogonal to the relevant structural error.

Common choices of f_i include weighted sums of peers' treatment assignments, normalized variants of such sums, and peer-of-peer constructions based on higher-order network structure (Borusyak et al., 2022; Bramoullé et al., 2009; De Giorgi et al., 2020). More generally, the framework also allows for nonlinear functions of treatment assignments and the pre-intervention network. The specific form of f matters mainly for relevance.

The more substantive issue is whether the instrument generates enough cross-sectional variation in the mediator. This becomes difficult in dense networks. As overlap in the rows of A^{pre} increases, units are exposed to increasingly similar combinations of treatment shocks, so the share component ceases to distinguish them. To illustrate, consider the linear SSIV:

$$Z_i^{\text{SSIV}} = \sum_{j=1}^{n} A_{ij}^{\text{pre}}(T_j - \pi), \tag{10}$$

Its exogeneity follows immediately from randomization and recentering:

$$\mathbb{E}[Z_i^{\text{SSIV}} u_i] = \mathbb{E}\left[\left(\sum_{j=1}^{n} A_{ij}^{\text{pre}}(T_j - \pi)\right) u_i\right] = \sum_{j=1}^{n} \mathbb{E}(A_{ij}^{\text{pre}} u_i) \cdot \mathbb{E}(T_j - \pi) = 0. \tag{11}$$

Relevance, however, can fail when the network is too dense. In the extreme case of a fully connected network, each row of A^{pre} is essentially identical, so $Z_i^{\text{SSIV}} = \sum_{j \neq i}^{n}(T_j - \pi)$ is nearly the same for all i, differing only through the negligible effect of unit i's own treatment status. More generally, greater neighborhood overlap compresses the heterogeneity that converts randomized treatment shocks into useful instrument variation. This problem is not unique to the linear specification in (10): it also affects other SSIV constructions, including peer-of-peer instruments, whenever the share component becomes too similar across units.[8]

Taken together, SSIV provides a general way to address endogeneity in network-mediated settings. Under randomized treatment, recentering delivers exogeneity under weak condi-

[8] In Appendix E, I discuss the performance of the IV estimator based on the normalized SSIV: $\frac{\sum_{j=1}^{n} A_{ij}^{\text{pre}} T_j}{\sum_{j=1}^{n} A_{ij}^{\text{pre}}}$. Rather than centering treatment around the assignment probability, this construction normalizes the total number of treated friends, thereby ensuring exogeneity. However, as the network becomes denser, this ratio converges to the assignment probability as the sample size grows. When the assignment probability is fixed, this convergence may generate a weak-IV problem. See Theorem E.1 for the corresponding consistency condition for IV estimators based on this normalized SSIV.

tions. The main challenge is relevance, which depends on whether the network share preserves enough cross-sectional heterogeneity.

3.2 A leading example of mediator: fraction of treated friends

The sparsity regimes under which the relevance condition holds depend on the specific form of M_i. In the remainder of this section, I focus on the fractional exposure measure:

$$M_i = \sum_{j=1}^{n} A_{ij}^{\text{post}} T_j / \sum_{j=1}^{n} A_{ij}^{\text{post}}, \tag{12}$$

with the convention that $0/0 = 0$. This specification reflects a widely used form of the anonymous interference assumption (Hudgens and Halloran, 2008), and with M_i as defined above, the linear outcome model in Assumption 2 becomes a special case of the linear-in-means model (Manski, 2013).[9] Additionally, with M_i in (12), $\mathbb{E}(M_i|T_i = t, T_{-i} = 1_{n-1}) - \mathbb{E}(M_i|T_i = t, T_{-i} = 0_{n-1}) = 1$, indicating that β_2 captures the spillover effect $\text{SE}(t)$ defined in (6). While the focus here is on this particular form of peer effects, the discussion of the SSIV relevance condition yields insights that extend naturally to other settings. More broadly, the paper serves as an illustration of how SSIV performs in the identification and estimation of peer effects.

As a fractional measure, M_i may exhibit limited variation and potential near-collinearity with the constant. Moreover, its cross-unit dependence can affect the convergence rate. To better understand the properties of M_i, I decompose it as $M_i = \xi_i + r_i^*$ This decomposition separates M_i into an individual-specific component ξ_i, and a remainder r_i^* that depends on all others' treatment and characteristics. For the specific M_i in (12) as an example, the term ξ_i takes the form:

$$\xi_i = \frac{\mathbb{E}(A_{ij}^{\text{post}} T_j \mid T_i, w_i)}{\mathbb{E}(A_{ij}^{\text{post}}|T_i, w_i)} = \mathbb{P}(T_j = 1|A_{ij}^{\text{post}} = 1, T_i, w_i), \tag{13}$$

and r_i^* captures deviations from this expectation. Intuitively, ξ_i reflects the conditional probability that a neighbor of unit i is treated, given unit i's own treatment and characteristics. The asymptotic analysis in later sections considers two cases, depending on whether ξ_i is constant:

(a) ξ_i is i.i.d. across i with nonzero variance;

[9] To accommodate the endogenous peer effect in the linear-in-means model, one can use the recentered treatments of peers of peers as an additional instrument, constructed using the same shift-share IV approach.

(b) $\xi_i = \pi$, implying zero variance.

The variation of ξ_i reflects whether network changes are induced by treatment. Case (b) arises when A_{ij}^{post} is mean independent of T_j given T_i and w_i, i.e., $\mathbb{E}(T_j | A_{ij}^{\text{post}}, T_i, w_i) = \pi$. This includes scenarios where the network is static ($A^{\text{pre}} = A^{\text{post}}$), evolves exogenously, or depends on T_j only in a mean-independent way. Comparing the asymptotic properties of the estimators under Cases (a) and (b) highlights how the network's responsiveness to treatment affects inference.

Case (b) has two important implications. First, in Case (b), there is no endogeneity even in the presence of unobserved confounders:

$$\mathbb{E}[M_i u_i] = \mathbb{E}\left[\frac{\sum_{j=1}^n A_{ij}^{\text{post}} T_j}{\sum_{j=1}^n A_{ij}^{\text{post}}} u_i\right] = \pi\, \mathbb{E}[u_i] = 0.$$

The second equality follows from the conditional mean independence of A_{ij}^{post} and T_j, and from Assumption 1, which ensures A_{ik}^{post} is independent of T_j for $k \neq j$. This uncorrelatedness, which arises from the row normalization of the mediator, has been exploited in the construction of the normalized SSIV (Adão et al., 2019; Goldsmith-Pinkham et al., 2020; Borusyak et al., 2022). Second, whether ξ_i is degenerate determines the variation in the regressor M_i, influencing the consistency conditions and convergence rate of the IV estimator. Treatment-induced network changes introduce additional variation in the mediator, potentially improving convergence.

3.3 Asymptotic properties of SSIV estimators

This subsection provides the asymptotic properties of the SSIV-based estimator under the linear specification of M_i in (12). For this specification, I focus on the linear SSIV defined in (10) as a natural choice. I estimate β using the instrument vector $Z_i = (1, T_i, Z_i^{\text{SSIV}})$. Stacking $\{Z_i\}_{i=1}^n$ across units yields the instrument matrix Z, and stacking the corresponding regressors $\{X_i\}_{i=1}^n$ and outcomes $\{Y_i\}_{i=1}^n$ yields the matrices X and vector Y. The resulting IV estimator is $\hat{\beta}^{\text{IV}} = (Z^\top X)^{-1} Z^\top Y$.

As discussed in Section 3.1, the relevance condition for SSIV may fail as the network becomes denser. This concern parallels the assumptions in the shift-share IV literature: Adão et al. (2019, Assumption 2), Borusyak et al. (2022, Assumption 2), and Borusyak and Hull (2023, Assumption 3) impose weak mutual dependence conditions on the instrument. In Theorem 3.1, I characterize the sparsity regimes under which SSIV remains consistent and show that both the pre- and post-treatment networks must be sufficiently sparse.

Theorem 3.1. *Under Assumptions 1 and 2,*

(a) if $\mathbb{V}(\xi_i) > 0$ *with* $\max\{q_n^{pre}, q_n^{post}\} \prec n^{-1/2}$, *then*

$$\hat{\beta}^{\text{IV}} - \beta = O_{\mathbb{P}}\left(\sqrt{n}\max\{q_n^{pre}, q_n^{post}\}\right);$$

(b) if $\mathbb{V}(\xi_i) = 0$ *with* $\max\{q_n^{pre}, q_n^{post}\} \prec \sqrt{n}q_n^{post}$, *then*

$$\hat{\beta}_1^{\text{IV}} - \beta_1 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\max\left\{\frac{\max\{q_n^{pre}, q_n^{post}\}}{\sqrt{q_n^{post}}}, 1\right\}\right).$$

Moreover, with $\max\{q_n^{pre}, q_n^{post}\} \prec n^{-1/2}$, *then*

$$\hat{\beta}_0^{\text{IV}} - \beta_0 = O_{\mathbb{P}}\left(\sqrt{n}\max\{q_n^{pre}, q_n^{post}\}\right) \text{ and } \hat{\beta}_2^{\text{IV}} - \beta_2 = O_{\mathbb{P}}\left(\sqrt{n}\max\{q_n^{pre}, q_n^{post}\}\right).$$

Theorem 3.1 characterizes the regimes under which $\hat{\beta}^{\text{IV}}$ is consistent under Cases (a) and (b). The consistency conditions and convergence rates for $\hat{\beta}_0^{\text{IV}}$ and $\hat{\beta}_2^{\text{IV}}$ are the same across the two cases: they depend on q_n^{pre} and q_n^{post}, can be slower than the standard $\sqrt{n}$ rate, and the scaling is governed by the degree in the denser network, $n\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}$. In contrast, $\hat{\beta}_1^{\text{IV}}$ converges faster in Case (b) than in Case (a), and remains consistent under weaker conditions. In Case (b), the leading component ξ_i of M_i is uncorrelated with T_i. As a result, the estimation of β_1 is less affected by cross-unit dependence in M_i and therefore converges more quickly than the estimator of β_2. Thus, although Case (b) features no endogeneity and OLS is valid, using SSIV can lead to slower convergence for $\hat{\beta}_2^{\text{IV}}$ and even inconsistency in denser networks, while $\hat{\beta}_1^{\text{IV}}$ remains consistent. Gao (2025) studies the properties of the OLS estimator in this setting and leverages them to propose a Hausman-type test for endogeneity.

Theorem 3.1 shows that the consistency of SSIV estimators fails when networks are relatively dense, i.e., $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \succcurlyeq n^{-1/2}$. This echoes the intuition: SSIV leverages the individual-level variation in the treated friends, driven by random treatment assignment. As networks become denser, this variation diminishes and weakens identification.

Corollary 3.1 simplifies the results of Theorem 3.1 under $q_n^{\text{pre}} \preccurlyeq q_n^{\text{post}}$.

Corollary 3.1. *Suppose* $q_n^{post} \succcurlyeq q_n^{pre}$. *Under Assumptions 1 and 2,*

(a) if $\mathbb{V}(\xi_i) > 0$ *with* $q_n^{post} \prec \frac{1}{\sqrt{n}}$, *then* $\hat{\beta}^{\text{IV}} - \beta = O_{\mathbb{P}}(\sqrt{n}q_n^{post})$;

(b) if $\mathbb{V}(\xi_i) = 0$ *it holds that* $\hat{\beta}_1^{\text{IV}} - \beta_1 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$. *Moreover, with* $q_n^{post} \prec \frac{1}{\sqrt{n}}$, *then* $\hat{\beta}_0^{\text{IV}} - \beta_0 = O_{\mathbb{P}}(\sqrt{n}q_n^{post})$ *and* $\hat{\beta}_2^{\text{IV}} - \beta_2 = O_{\mathbb{P}}(\sqrt{n}q_n^{post})$.

I then establish the asymptotic normality of the IV estimator $\hat{\beta}^{\text{IV}}$ in the regimes where consistency holds. Let $V^{\text{IV}}_{\text{num}} = \mathbb{V}\left(\sum_{i=1}^n Z_i u_i\right)$ denote the variance of the numerator in the centered estimator $\hat{\beta}^{\text{IV}} - \beta$. It can be shown that

$$V^{\text{IV}}_{\text{num}} = \begin{pmatrix} \sum_{i=1}^n \mathbb{E}(u_i^2) & \pi \sum_{i=1}^n \mathbb{E}(u_i^2) & 0 \\ \pi \sum_{i=1}^n \mathbb{E}(u_i^2) & \pi \sum_{i=1}^n \mathbb{E}(u_i^2) & \pi(1-\pi) \sum_{(i,j)} \mathbb{E}\left(A^{\text{pre}}_{ij} u_i u_j\right) \\ 0 & \pi(1-\pi) \sum_{(i,j)} \mathbb{E}\left(A^{\text{pre}}_{ij} u_i u_j\right) & \pi(1-\pi) \sum_{i=1}^n \mathbb{E}\left(\left(\sum_{j=1}^n A^{\text{pre}}_{ij} u_j\right)^2\right) \end{pmatrix}. \quad (14)$$

The $(2,3)$, $(3,2)$ and $(3,3)$ elements of $V^{\text{IV}}_{\text{num}}$ in (14) account for the dependence of Z_i^{SSIV} across i, even though u_i is i.i.d.. Let $\hat{u}_i^{\text{IV}}$ denote the residual from the IV fits with Z_i. Define $\hat{V}^{\text{IV}}_{\text{num}}$ as the plug-in estimator of $V^{\text{IV}}_{\text{num}}$, obtained by replacing u_i in $V^{\text{IV}}_{\text{num}}$ with $\hat{u}_i^{\text{IV}}$. The asymptotic variance of $\hat{\beta}^{\text{IV}}$ is then estimated by

$$\hat{V}^{\text{IV}} = (Z^\top X)^{-1} \hat{V}^{\text{IV}}_{\text{num}} (X^\top Z)^{-1},$$

where X and Z are the stacked matrices with ith rows $X_i^\top$ and $Z_i^\top$, respectively. Theorem 3.2 establishes the asymptotic normality of the IV estimators $\hat{\beta}^{\text{IV}}$, along with a consistent variance estimator.

Theorem 3.2. *Under Assumptions 1 and 2, if* $\max\{q_n^{pre}, q_n^{post}\} \prec n^{-1/2}$, *then*

$$(\hat{V}^{\text{IV}})^{-1/2} (\hat{\beta}^{\text{IV}} - \beta) \xrightarrow{\text{d}} \mathcal{N}(0, I_3).$$

Theorem 3.2 highlights two key insights. First, the asymptotic normality result features a self-normalization property that eliminates the need to know the sparsity rates q_n^{pre} and q_n^{post}. While the convergence rate of $\hat{\beta}^{\text{IV}}$ may depend on q_n^{pre} and q_n^{post}, this dependence is absorbed by the scaling of the variance estimator, ensuring that conventional t-tests remain valid. This robustness is particularly valuable given that q_n^{post} is typically unidentifiable in practice (Bickel et al., 2012). Similar self-normalization behavior arises in network centrality regression (Cai, 2022) and cluster-robust inference (Hansen and Lee, 2019). Moreover, the same variance estimator applies in both Case (a) and Case (b). Second, the standard heteroskedasticity-consistent IV variance estimator assumes i.i.d. data and overlooks the cross-unit dependence introduced by the shift-share instrument Z_i^{SSIV}, leading to misleading inference. In contrast, the proposed estimator $\hat{V}^{\text{IV}}_{\text{num}}$ explicitly accounts for this dependence, yielding more accurate standard errors. The estimator $\hat{V}^{\text{IV}}$ coincides with that of Adão et al.

(2019), who adopt a design-based perspective. In contrast, I establish its properties under a superpopulation framework, accounting for all sources of randomness.

There is a notable connection between SSIV and other indirect effect estimator in the interference literature. Li and Wager (2022) analyze treatment effect asymptotics under network interference in graphon-based networks with confounders. Unlike my setting, their interference graph is fixed after treatment assignment: they do not allow the network itself to respond to the intervention. They propose an unbiased estimator for the spillover effect:

$$\hat{\tau}^{\mathrm{U}}_{\mathrm{IND}} = \frac{1}{n}\sum_{i=1}^{n} Y_i \left(\frac{\sum_{j\neq i} A_{ij} T_j}{\pi} - \frac{\sum_{j\neq i} A_{ij}(1-T_j)}{1-\pi} \right) = \frac{1}{n}\sum_{i=1}^{n} Y_i \frac{\sum_{j\neq i} A_{ij}(T_j-\pi)}{\pi(1-\pi)}, \tag{15}$$

which arises from a particular difference of Horvitz–Thompson estimators for the total and direct effects, and is therefore unbiased. Rewriting the estimator as in the second equality of (15) makes clear that it exploits the same shift–share variation as the estimator in this paper. The key difference is conceptual: Li and Wager (2022)'s estimator is justified by a design-based argument for indirect effects under a fixed interference graph, whereas my SSIV approach is introduced to address endogeneity of a post-treatment network mediator when the network itself may change in response to treatment. Still, the comparison is useful because it shows that recentered shift–share constructions can unify a range of interference estimators. More generally, the IV perspective suggests that such constructions can be extended beyond the particular neighbor-treatment share and randomized assignment considered by Li and Wager (2022), and may be adapted to alternative exposure mappings and experimental designs.

3.4 Modification of SSIV

I begin with formally examining the failure of the SSIV relevance condition, which motivates the proposed modification. To simplify the argument, consider the special case where $q_n^{\mathrm{pre}} = q_n^{\mathrm{post}} = q_n$, and recall the decomposition $M_i = \xi_i + r_i^*$, where ξ_i is defined in (13). The first-stage relevance term can be decomposed as:

$$\frac{1}{n}\sum_{i=1}^{n} M_i Z_i^{\mathrm{SSIV}} = \frac{1}{n}\sum_{i=1}^{n} \xi_i Z_i^{\mathrm{SSIV}} + \frac{1}{n}\sum_{i=1}^{n} r_i^* Z_i^{\mathrm{SSIV}}.$$

Since ξ_i is a function of T_i and w_i by construction, it is uncorrelated with Z_i^{SSIV}, making the first term mean-zero and purely noise, with variance $O(nq_n^2)$ that increases with network density due to stronger cross-unit dependence. In contrast, r_i^* incorporates information from other units and can be shifted by Z_i^{SSIV}, yielding a stable signal with nonzero mean $O(1)$ and

small variance $O(1/n)$. As networks become denser, the growing noise from ξ_i overwhelms this signal, weakening the relevance and resulting in inconsistency.

To address this issue, the goal is to remove noise without attenuating the identifying signal. Ideally, one would project Z_i^{SSIV} onto ξ_i and use the residual as the new instrument, exploiting their orthogonality. Although ξ_i is unobserved, it is a function of the latent trait w_i, whose structure can be recovered from the pre-intervention graphon matrix $q_n^{\text{pre}} G_n^{\text{pre}}$, where $G_{n,ij}^{\text{pre}} = g^{\text{pre}}(w_i, w_j)$, via spectral analysis. Under the approximate low-rank condition in Assumption 3 below, it is therefore sufficient to remove only the first r eigencomponents, since the remaining spectral mass is asymptotically negligible. Accordingly, the (oracle) modified SSIV is defined as the residual from projecting Z_i^{SSIV} onto the leading r eigenvectors of $q_n^{\text{pre}} G_n^{\text{pre}}$. In practice, I approximate this projection using the top r eigenvectors of A^{pre}. Writing the eigen-decomposition as $A^{\text{pre}} = \sum_{k=1}^{n} \hat{\lambda}_k \hat{\psi}_k \hat{\psi}_k^\top$, with $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_n$, the denoised SSIV Z_i^{DN} is defined by

$$Z_i^{\text{DN}} = Z_i^{\text{SSIV}} - \sum_{k=1}^{r} \hat{\gamma}_k \, \hat{\psi}_k(w_i), \text{ where } \hat{\gamma}_k = \sum_{i=1}^{n} \hat{\psi}_k(w_i) \, Z_i^{\text{SSIV}}. \tag{16}$$

By construction, Z^{DN} is orthogonal to $\text{span}\{\hat{\psi}_1, \ldots, \hat{\psi}_r\}$. This projection removes the component of Z_i^{SSIV} most aligned with w_i, thereby weakening the dependence induced by ξ_i while preserving the variation in Z_i^{SSIV} driven by w_{-i}, which provides the identifying signal. Consequently, this residualization reduces noise in the relevance term and can restore consistency.

Assumption 3 formalizes the approximate low-rank condition for the graphon matrix.

Assumption 3. *Write the eigen-decomposition as* $q_n^{pre} G_n^{pre} = \sum_{k=1}^{n} \lambda_k^* \psi_k^* \psi_k^{*\top}$ *with* $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_n^*$. *There exists a constant* $r < n$ *such that* $\left\| \sum_{k=r+1}^{n} \lambda_k^* \psi_k^* \psi_k^{*\top} \right\|_{\text{op}} = O_{\mathbb{P}}(q_n^{pre})$.

In the network literature, particularly in work on graphon-based models (Cai, 2022; Li and Wager, 2022), it is common to impose an exact finite-rank structure on the graphon via its eigenfunctions:

$$g(w_i, w_j) = \sum_{k=1}^{r} \lambda_k \psi_k(w_i) \psi_k(w_j), \tag{17}$$

where the eigenfunctions $\{\psi_k\}$ are orthonormal in L^2, meaning that $\mathbb{E}[\psi_k^2(w_i)] = 1$ and $\mathbb{E}[\psi_k(w_i)\psi_l(w_i)] = 0$ for $k \neq l$. This restriction is satisfied by several canonical models, including the stochastic block model (Holland et al., 1983) and random dot product graphs (Young and Scheinerman, 2007). Assumption 3 relaxes this benchmark. It maintains the idea that a finite number of leading eigencomponents contain the dominant information, but requires only that the pre-intervention graphon matrix $q_n^{\text{pre}} G_n^{\text{pre}}$ be approximately low rank, in the sense that its leading eigencomponents capture most of its spectral mass and the

remaining components contribute only weakly. This is analogous to sparsity assumptions in high-dimensional statistics, where the signal is concentrated in a small number of components, and the remainder contributes only weakly. The resulting condition encompasses a broader class of models, including latent space models (Hoff et al., 2002), which are a subclass of inhomogeneous Erdős–Rényi graphs and include the homophily and beta models as special cases.

I now estimate β using the modified SSIV vector $\tilde{Z}_i^{\text{DN}} = (1,\, T_i,\, Z_i^{\text{DN}})$. Stacking $\{\tilde{Z}_i^{\text{DN}}\}_{i=1}^n$ yields the instrument matrix $\tilde{Z}^{\text{DN}}$. Let $\hat{\beta}^{\text{DN}}$ denote the resulting IV estimator:

$$\hat{\beta}^{\text{DN}} = \left((\tilde{Z}^{\text{DN}})^\top X\right)^{-1}(\tilde{Z}^{\text{DN}})^\top Y.$$

Theorem 3.3. *Suppose* $q_n^{pre} \succ \frac{\log(n)}{\log(\log(n))}/n$. *Under Assumptions 1–3,*

(a) if $\mathbb{V}(\xi_i) > 0$ *with* $\max\{q_n^{pre}, q_n^{post}\} \prec \sqrt{q_n^{pre}}$, *then*

$$\hat{\beta}^{\text{DN}} - \beta = O_{\mathbb{P}}\left(\frac{\max\{q_n^{pre}, q_n^{post}\}}{\sqrt{q_n^{pre}}}\right);$$

(b) if $\mathbb{V}(\xi_i) = 0$, *then* $\hat{\beta}_1^{\text{DN}} - \beta_1 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$. *Moreover, with* $\max\{q_n^{pre}, q_n^{post}\} \prec \sqrt{q_n^{pre}}$, *then*

$$\hat{\beta}_0^{\text{DN}} - \beta_0 \;=\; O_{\mathbb{P}}\left(\frac{\max\{q_n^{pre}, q_n^{post}\}}{\sqrt{q_n^{pre}}}\right) \text{ and } \hat{\beta}_2^{\text{DN}} - \beta_2 \;=\; O_{\mathbb{P}}\left(\frac{\max\{q_n^{pre}, q_n^{post}\}}{\sqrt{q_n^{pre}}}\right).$$

The modified SSIV reequires accurate eigenvector estimation, which can be difficult in very sparse networks. However, this requirement only demands pre-network sparsity on the order of $\frac{\log(n)}{\log\log(n)}/n$, which is far weaker than the $n^{-1/2}$ threshold at which standard SSIV breaks down. In addition, the consistency condition $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec \sqrt{q_n^{\text{pre}}}$ is weaker than $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec n^{-1/2}$. As a result, the modified SSIV preserves most of the original SSIV's consistency region and extends it to denser regimes where the original SSIV estimator becomes inconsistent. This highlights the robustness of the modified SSIV across a broader range of sparsity levels.

Theorems 3.1 and 3.3 together imply that there is a range of sparsity regimes in which both $\hat{\beta}^{\text{IV}}$ and $\hat{\beta}^{\text{DN}}$ are consistent, $\frac{\log n}{\log\log n}/n \prec q_n \prec n^{-1/2}$. These overlapping regimes naturally raise the question of which estimator performs better. Beyond consistency, Theorems 3.1 and 3.3 show that the denoised estimator $\hat{\beta}^{\text{DN}}$ converges strictly faster than the standard SSIV estimator $\hat{\beta}^{\text{IV}}$. Thus, the modification is valuable not only because it restores consistency where standard SSIV breaks down, but also because it improves convergence even in regimes where standard SSIV is already consistent.

Given the connection between Li and Wager (2022)'s spillover estimator and our IV approach, the two methods face a common challenge: consistency can break down as the network becomes denser. Under their exact low-rank condition in (17), Li and Wager (2022) propose a principal-components (PC) balancing estimator that modifies the weights in (15) to partial out the signal components associated with the leading eigenfunctions; in practice, these are implemented using eigenvectors estimated from the adjacency matrix. In finite samples, their reweighting procedure coincides with the linear projection used here. The main difference is that my consistency result applies under weaker graphon assumptions and delivers a wider admissible sparsity regime. Specifically, Li and Wager (2022) establish consistency of the PC balancing estimator for $n^{-1/2} \prec q_n \prec 1$, whereas I show under Assumption 3 that consistency continues to hold over the broader range $\frac{\log n}{\log\log n}/n \prec q_n \prec 1$.

Remark 3.1. *To clarify the condition* $\max\{q_n^{pre}, q_n^{post}\} \prec \sqrt{q_n^{pre}}$*, consider two cases:*

(1) If $q_n^{pre} \prec q_n^{post}$*, then the condition holds when* $\sqrt{q_n^{pre}} \succ q_n^{post}$*, meaning the pre-intervention network must not be too sparse relative to the post.*

(2) If $q_n^{pre} \succcurlyeq q_n^{post}$*, then the condition reduces to* $q_n^{pre} \prec 1$*, ruling out dense networks.*

Remark 3.1 suggests that the modified SSIV approach is not applicable when either network is dense. In that case, alternative strategies such as the matching method (Auerbach, 2022) or the control-function approach (Johnsson and Moon, 2021) may be appropriate, provided suitable regularity conditions hold.

Corollary 3.2 simplifies the results of Theorem 3.3 under $q_n^{\text{post}} \succcurlyeq q_n^{\text{pre}}$.

Corollary 3.2. *Suppose* $q_n^{post} \succcurlyeq q_n^{pre} \succ \frac{\log(n)}{n\log(\log(n))}$*. Under Assumptions 1–3,*

(a) if $\mathbb{V}(\xi_i) > 0$ *with* $\sqrt{q_n^{pre}} \succ q_n^{post}$*, then* $\hat{\beta}^{\text{DN}} - \beta = O_{\mathbb{P}}\left(q_n^{post}/\sqrt{q_n^{pre}}\right)$*;*

(b) if $\mathbb{V}(\xi_i) = 0$*, it holds that* $\hat{\beta}_1^{\text{DN}} - \beta_1 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$*. Moreover, with* $\sqrt{q_n^{pre}} \succ q_n^{post}$*, then* $\hat{\beta}_0^{\text{DN}} - \beta_0 = O_{\mathbb{P}}\left(q_n^{post}/\sqrt{q_n^{pre}}\right)$ *and* $\hat{\beta}_2^{\text{DN}} - \beta_2 = O_{\mathbb{P}}\left(q_n^{post}/\sqrt{q_n^{pre}}\right)$*.*

Corollary 3.2 shows that under Case (a), $\hat{\beta}^{\text{DN}}$ converges at a rate slower than $\sqrt{n}$, but faster than the SSIV estimator $\hat{\beta}^{\text{IV}}$. In Case (b), when the post-intervention network A^{post} is conditionally mean independent of treatment, $\hat{\beta}_1^{\text{DN}}$ is less affected by dependence across units and achieves the usual $\sqrt{n}$ rate. In contrast, $\hat{\beta}_0^{\text{DN}}$ and $\hat{\beta}_2^{\text{DN}}$ may still converge more slowly.

Remark 3.2. *A key takeaway is that the relevance analysis yields broader guidance for SSIV applications. In general, the endogenous regressor admits a decomposition* $M_i = \xi_i + r_i^*$*, where* ξ_i *collects the component driven by unit* i*'s own informtion while* r_i^* *aggregates information*

coming from other units. The idiosyncratic component ξ_i is uncorrelated with the SSIV and therefore enters only as noise, potentially weakening instrument relevance. The proposed denoising procedure offers a simple and broadly applicable way to remove this noise and strengthen relevance, which can in turn restore consistency and improve convergence rate.

I now establish the asymptotic normality of $\hat{\beta}^{\text{DN}}$. Define $\mu_k^u = \sum_{i=1}^n u_i \psi_k^*(w_i)$ and $\eta_i^u = u_i - \sum_{k=1}^r \mu_k^u \psi_k^*(w_i)$, where η_i^u is the residual component of u_i after projecting out the top r eigenvectors. Define $V_{\text{num}}^{\text{DN}} = \mathbb{V}\left(\sum_{i=1}^n \tilde{Z}_i^{\text{DN}} u_i\right)$:

$$V_{\text{num}}^{\text{DN}} = \begin{pmatrix} \sum_{i=1}^n \mathbb{E}(u_i^2) & \pi \sum_{i=1}^n \mathbb{E}(u_i^2) & 0 \\ \pi \sum_{i=1}^n \mathbb{E}(u_i^2) & \pi \sum_{i=1}^n \mathbb{E}(u_i^2) & \pi(1-\pi) \sum_{(i,j)} \mathbb{E}(A_{ij}^{\text{pre}} u_i \eta_j^u) \\ 0 & \pi(1-\pi) \sum_{(i,j)} \mathbb{E}(A_{ij}^{\text{pre}} u_i \eta_j^u) & \pi(1-\pi) \sum_{i=1}^n \mathbb{E}\left(\left(\sum_{j=1}^n A_{ij}^{\text{pre}} \eta_j^u\right)^2\right) \end{pmatrix}.$$

Let $\hat{u}_i^{\text{DN}}$ denote the residual from the IV regression that uses the modified instrument $\tilde{Z}_i^{\text{DN}}$. Define $\hat{\mu}_k^{\hat{u}} = \sum_{i=1}^n \hat{u}_i^{\text{DN}} \hat{\psi}_{ki}$ and $\hat{\eta}_j^{\hat{u}} = \hat{u}_j^{\text{DN}} - \sum_{k=1}^r \hat{\mu}_k^{\hat{u}} \hat{\psi}_{kj}$. Define $\hat{V}_{\text{num}}^{\text{DN}}$ as the plug-in estimator of $V_{\text{num}}^{\text{DN}}$, obtained by replacing u_i and η_j^u in $V_{\text{num}}^{\text{DN}}$ with $\hat{u}_i^{\text{DN}}$ and $\hat{\eta}_j^{\hat{u}}$, respectively. The variance estimator for $\hat{\beta}^{\text{DN}}$ is then constructed as

$$\hat{V}^{\text{DN}} = ((\tilde{Z}^{\text{DN}})^\top X)^{-1} \hat{V}_{\text{num}}^{\text{DN}} (X^\top \tilde{Z}^{\text{DN}})^{-1}.$$

Theorem 3.4. *Under Assumptions 1–3, if $q_n^{pre} \succ \frac{\log(n)}{n \log(\log(n))}$ and $\max\{q_n^{pre}, q_n^{post}\} \prec \sqrt{q_n^{pre}}$, then*

$$(\hat{V}^{\text{DN}})^{-1/2} (\hat{\beta}^{\text{DN}} - \beta) \xrightarrow{\text{d}} \mathcal{N}(0, I_3).$$

Theorem 3.4 establishes the asymptotic normality of the modified-SSIV IV estimator $\hat{\beta}^{\text{DN}}$ and shows that the associated variance estimator $\hat{V}^{\text{DN}}$ is consistent. The variance estimator $\hat{V}^{\text{DN}}$ is new to the literature and enables valid inference for $\hat{\beta}^{\text{DN}}$ using the usual t-test.

Remark 3.3. *Borusyak and Hull (2025) study asymptotically efficient instruments within the class of recentered formula IVs and propose an algorithm for constructing feasible approximations. Their Algorithm 1 includes an optional residualization step, in which the recentered instrument is residualized on or controls for functions of unobserved w that predict the error term u. Our modification is in the same spirit: we project out the components associated with the latent variable w using eigenvector-based controls. The key distinction is feasibility: we provide a practical way to approximate this unobserved variable by extracting information from the network data.*

4 Monte Carlo Simulation

In this section, I present simulation evidence to support the theoretical results. Treatment assignments are drawn independently as $T_i \overset{\text{i.i.d.}}{\sim} \text{Bern}(0.5)$, and the latent variable is drawn independently as $w_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$. I consider two graphon specifications:

$$g^{\text{SBM}}(u_i,u_j) = \tfrac{3}{5}\mathbf{1}\{(u_i,u_j)\in[0,\tfrac{1}{3}]^2\} + \tfrac{1}{3}\mathbf{1}\{(u_i,u_j)\in(\tfrac{1}{3},\tfrac{2}{3}]^2\} + \tfrac{1}{2}\mathbf{1}\{(u_i,u_j)\in(\tfrac{2}{3},1]^2\} + \tfrac{1}{5},$$

and

$$g^{\text{Homo}}(u_i,u_j) = 1-(u_i-u_j)^2.$$

The first corresponds to a stochastic block model (SBM) with rank $r=3$, while the second corresponds to a homophily graphon with rank $r=3$. Let Φ denote the CDF of $\mathcal{N}(0,1)$. Define

$$\Psi^{(A)}(w_i,T_i) = \Phi(w_i)\left(1-\tfrac{4}{5}T_i\right), \qquad \Psi^{(B)}(w_i,T_i) = \Phi\big(w_i\left(1-\tfrac{4}{5}T_i\right)\big).$$

To describe the simulation designs compactly, I write

$$g^{\text{pre}}(w_i,w_j) = g^{\text{pre},\star}\big(\Phi(w_i),\Phi(w_j)\big),$$

$$g^{\text{post}}(w_i,w_j,T_i,T_j) = g^{\text{Homo}}\big(\Psi^{(\boxplus)}(w_i,T_i),\Psi^{(\boxplus)}(w_j,T_j)\big),$$

where $\star\in\{\text{SBM},\text{Homo}\}$ indexes the pre-intervention graphon, and $\boxplus\in\{A,B\}$ determines how treatment enters the homophily-based post-intervention graphon. Crossing the two pre-intervention graphons, SBM and homophily, with the two post-intervention transformations, $\Psi^{(A)}(w,T)$ and $\Psi^{(B)}(w,T)$, generates four data-generating designs, summarized in Table 1.

Table 1: Network designs

	Post-intervention transformation	
Pre-intervention graphon	$\Psi^{(A)}(w,T)$	$\Psi^{(B)}(w,T)$
SBM	Design 1	Design 3
Homophily	Design 2	Design 4

The adjacency matrices A^{pre} and A^{post} for each design are generated according to (1) and (2), with $\eta_{ij} \overset{\text{i.i.d.}}{\sim} U[0,1]$ for $i<j$, and $\eta_{ij}=\eta_{ji}$. I set $q_n^{\text{pre}} = q_n^{\text{post}} = q_n$. To examine how different ways in which treatment enters the post-intervention network affect the properties of M_i, Table 2 reports the within-sample standard deviation of M_i across different sample sizes and sparsity levels. When $q_n = n^{-1}$, the dispersion of M_i is essentially unchanged

across designs and between $n = 200$ and $n = 800$. When $q_n = n^{-1/2}$, the dispersion declines with n in all designs, with a noticeably larger reduction under Designs 3–4. In the dense case $q_n = 1$, Designs 1–2 continues to exhibit stable dispersion, whereas Designs 3–4 display a 50% reduction. This pronounced contraction indicates that M_i becomes much less variable in finite samples under Designs 3–4 in the dense regime, making M_i nearly collinear with the constant. Motivated by this diagnostic, I use Designs 1–2 to illustrate Case (a), where ξ_i is non-degenerate. Designs 3–4 are intended to capture a Case (b)-like regime: as the network becomes denser, the within-sample variation of M_i becomes very small, so the exposure is nearly constant across units.

Table 2: Within-sample standard deviation of M_i

Design	q_n					
	n^{-1}	$n^{-1/2}$	1	n^{-1}	$n^{-1/2}$	1
	$n = 200$			$n = 800$		
1 and 2	0.422	0.170	0.085	0.424	0.134	0.085
3 and 4	0.425	0.141	0.012	0.427	0.099	0.006

The outcome is generated as follows:

$$Y_i = \beta_0 + \beta_1 T_i + \beta_2 \frac{\sum_{j=1}^n A_{ij}^{\text{post}} T_j}{\sum_{j=1}^n A_{ij}^{\text{post}}} + u_i$$

with $(\beta_0, \beta_1, \beta_2) = (1, 1, 0.5)$ and $u_i = w_i + \varepsilon_i$ where $\varepsilon_i \overset{\text{i.i.d.}}{\sim} U[-1, 1]$ and $\varepsilon_i \perp\!\!\!\perp w_i$.

Table 3b and 3c report simulation results for SSIV, modified SSIV, and OLS when $q_n = n^{-3/5}$ and $q_n = n^{-1/5}$, respectively. Each subtable represents results for $n = 200$ and $n = 800$, based on $5,000$ replications. The first column indicates the design, and the second column lists the coefficient being estimated. The SSIV panel reports the mean, standard deviation, the average estimated standard error, and the empirical coverage of the 95% confidence interval (CI) of the IV estimator using SSIV, labeled as "$\hat{\beta}^{\text{IV}}$", "std($\hat{\beta}^{\text{IV}}$)", "s.e.", and "coverage". The modified SSIV panel shows the corresponding statistics for the modified SSIV estimator, denoted "$\hat{\beta}^{\text{DN}}$." The OLS panel also reports the corresponding OLS estimates using M^{pre} and M, to illustrate the bias from different sources. Results for β_0 are omitted for brevity.

I begin with Table 3b, which considers a relatively sparse network. As discussed above, Designs 1 and 2 involve endogeneity, whereas Designs 3 and 4 do not. In this regime, Theorems 3.1 and 3.3 guarantee consistency of the SSIV estimator $\hat{\beta}^{\text{IV}}$ and the modified SSIV estimator $\hat{\beta}^{\text{DN}}$, which is supported by the simulations: the estimates cluster around the true values. As the sample size increases from 200 to 800, the standard deviation of $\hat{\beta}_1^{\text{IV}}$ roughly halves, in line with a $\sqrt{n}$ convergence rate, whereas $\hat{\beta}_2^{\text{IV}}$ converges more slowly.

Likewise, $\hat{\beta}_1^{\text{DN}}$ converges at a $\sqrt{n}$ rate, while $\hat{\beta}_2^{\text{DN}}$ converges faster than $\hat{\beta}_2^{\text{IV}}$. In line with Theorems 3.2 and 3.4, the empirical coverage of the 95% confidence intervals is close to the nominal level under the proposed variance estimators. Comparing Designs 1 and 2, and likewise Designs 3 and 4, shows that the results allow for the changes in the types of the pre- and post-intervention graphons, provided they remain sufficiently correlated through the common latent variables. Although Designs 3 and 4 feature no endogeneity, both SSIV and modified SSIV still produce consistent estimates.

Table 3c reports the simulation results for $q_n = n^{-1/5}$, a denser regime than the validity threshold for SSIV. In Designs 1 and 2, $\hat{\beta}^{\text{IV}}$ is inconsistent, as predicted by Theorem 3.1: the estimates are highly dispersed and their standard deviations do not decline with sample size. In Designs 3 and 4, $\hat{\beta}_1^{\text{IV}}$ exhibits $\sqrt{n}$-type concentration, while $\hat{\beta}_2^{\text{IV}}$ is still inconsistent. Applying the modified SSIV reduces noise and recovers consistency in all designs. In particular, $\hat{\beta}_1^{\text{DN}}$ is $\sqrt{n}$-consistent, and $\hat{\beta}_2^{\text{DN}}$ converges at a slower rate $1/\sqrt{q_n}$, with both estimators delivering nominal 95% CI coverage.

For both regimes, I also report OLS estimates from regressions using M^{pre} and M as regressors to illustrate the bias. In Designs 1 and 2, the OLS estimator based on M^{pre} is biased because of measurement error, with the bias larger in Design 1 because the pre- and post-intervention networks are generated by different graphons. The OLS estimator based on M is also biased because M is endogenous; the bias is large, can even reverse the sign of the coefficient, and leads to severe attenuation. In Designs 3 and 4, the OLS estimator of β_1 based on M^{pre} is unbiased because the total effect coincides with the direct effect. For the estimator of β_2, in Design 3, the bias in the OLS estimator of β_2 using M^{pre} persists because the pre-intervention and the post-intervention graphons differ. In contrast, the bias in Design 4 disappears as the sample size increases, since the pre- and post-intervention networks share a similar homophily structure and the post-intervention network does not respond to treatment. As a result, both M^{pre} and M are essentially random fluctuations around 1/2. The OLS estimator based on M is unbiased in Designs 3 and 4, since these designs correspond to Case (b), where no endogeneity arises.

Overall, the simulations suggest that the modified SSIV estimator is robust over a wider range of sparsity levels than the standard SSIV estimator, and delivers faster convergence in regimes where both estimators are consistent. By contrast, OLS regressions using M or M^{pre} as regressors are biased because of endogeneity or measurement error.

Table 3: Simulation results

(a) Results with $q_n = n^{-1}$

Design	β	SSIV				modified SSIV				OLS	
		$\hat{\beta}^{\text{IV}}$	std($\hat{\beta}^{\text{IV}}$)	s.e.	coverage	$\hat{\beta}^{\text{DN}}$	std($\hat{\beta}^{\text{DN}}$)	s.e.	coverage	$\hat{\beta}^{\text{OLS}}_{\text{pre}}$	$\hat{\beta}^{\text{OLS}}_{\text{post}}$
		$n = 200$									
1	$\beta_1 = 1$	0.998	0.167	0.168	0.948	0.998	0.168	0.168	0.945	1.035	1.011
	$\beta_2 = 0.5$	0.500	0.462	0.449	0.945	0.501	0.456	0.445	0.944	0.310	0.292
2	$\beta_1 = 1$	0.997	0.163	0.165	0.948	0.997	0.163	0.165	0.948	1.035	1.012
	$\beta_2 = 0.5$	0.501	0.295	0.295	0.943	0.499	0.297	0.296	0.944	0.430	0.291
3	$\beta_1 = 1$	1.002	0.167	0.169	0.948	1.002	0.167	0.169	0.948	1.009	1.002
	$\beta_2 = 0.5$	0.509	0.448	0.448	0.946	0.508	0.443	0.444	0.947	0.310	0.503
4	$\beta_1 = 1$	0.996	0.163	0.166	0.949	0.996	0.163	0.166	0.949	1.003	0.996
	$\beta_2 = 0.5$	0.492	0.272	0.274	0.945	0.492	0.275	0.275	0.947	0.460	0.496
		$n = 800$									
1	$\beta_1 = 1$	1.000	0.082	0.081	0.947	1.000	0.082	0.081	0.946	1.038	1.015
	$\beta_2 = 0.5$	0.501	0.221	0.219	0.945	0.501	0.218	0.217	0.949	0.311	0.298
2	$\beta_1 = 1$	1.000	0.081	0.081	0.950	1.000	0.081	0.081	0.951	1.038	1.015
	$\beta_2 = 0.5$	0.502	0.146	0.144	0.946	0.502	0.146	0.145	0.947	0.435	0.298
3	$\beta_1 = 1$	1.001	0.082	0.082	0.947	1.001	0.082	0.082	0.948	1.008	1.001
	$\beta_2 = 0.5$	0.504	0.223	0.219	0.943	0.503	0.219	0.218	0.947	0.310	0.500
4	$\beta_1 = 1$	1.001	0.082	0.082	0.947	1.001	0.082	0.082	0.946	1.009	1.001
	$\beta_2 = 0.5$	0.497	0.135	0.135	0.947	0.497	0.134	0.135	0.947	0.464	0.499

(b) Results with $q_n = n^{-3/5}$

Design	β	SSIV				modified SSIV				OLS	
		$\hat{\beta}^{\text{IV}}$	std($\hat{\beta}^{\text{IV}}$)	s.e.	coverage	$\hat{\beta}^{\text{DN}}$	std($\hat{\beta}^{\text{DN}}$)	s.e.	coverage	$\hat{\beta}^{\text{OLS}}_{\text{pre}}$	$\hat{\beta}^{\text{OLS}}_{\text{post}}$
		$n = 200$									
1	$\beta_1 = 1$	1.007	0.156	0.156	0.944	1.005	0.162	0.161	0.942	1.054	1.123
	$\beta_2 = 0.5$	0.509	0.874	0.870	0.945	0.518	0.799	0.787	0.946	0.143	-0.643
2	$\beta_1 = 1$	0.999	0.153	0.156	0.948	0.999	0.153	0.156	0.950	1.052	1.121
	$\beta_2 = 0.5$	0.504	0.533	0.542	0.940	0.503	0.532	0.531	0.939	0.414	-0.640
3	$\beta_1 = 1$	0.998	0.170	0.171	0.949	0.998	0.169	0.170	0.948	0.999	1.002
	$\beta_2 = 0.5$	0.484	0.921	0.918	0.947	0.495	0.840	0.834	0.949	0.124	0.486
4	$\beta_1 = 1$	1.000	0.169	0.167	0.940	1.000	0.169	0.167	0.942	0.999	1.000
	$\beta_2 = 0.5$	0.500	0.524	0.532	0.947	0.498	0.521	0.521	0.944	0.449	0.498
		$n = 800$									
1	$\beta_1 = 1$	0.999	0.076	0.076	0.946	0.999	0.080	0.079	0.948	1.052	1.211
	$\beta_2 = 0.5$	0.512	0.612	0.614	0.949	0.508	0.514	0.518	0.950	0.166	-1.461
2	$\beta_1 = 1$	0.997	0.074	0.074	0.948	0.997	0.074	0.074	0.948	1.051	1.209
	$\beta_2 = 0.5$	0.500	0.366	0.366	0.944	0.500	0.364	0.363	0.946	0.432	-1.458
3	$\beta_1 = 1$	0.999	0.083	0.083	0.950	0.999	0.082	0.083	0.949	0.999	1.002
	$\beta_2 = 0.5$	0.483	0.650	0.650	0.944	0.490	0.551	0.550	0.948	0.149	0.501
4	$\beta_1 = 1$	1.000	0.082	0.083	0.950	1.000	0.082	0.083	0.949	0.999	1.000
	$\beta_2 = 0.5$	0.502	0.358	0.359	0.950	0.502	0.357	0.355	0.951	0.458	0.500

(c) Results with $q_n = n^{-1/5}$

Design	β	SSIV				modified SSIV				OLS	
		$\hat{\beta}^{\text{IV}}$	std($\hat{\beta}^{\text{IV}}$)	s.e.	coverage	$\hat{\beta}^{\text{DN}}$	std($\hat{\beta}^{\text{DN}}$)	s.e.	coverage	$\hat{\beta}^{\text{OLS}}_{\text{pre}}$	$\hat{\beta}^{\text{OLS}}_{\text{post}}$
		$n = 200$									
1	$\beta_1 = 1$	0.821	3.675	3.766	0.959	0.996	0.248	0.247	0.944	1.052	1.727
	$\beta_2 = 0.5$	3.070	53.292	52.878	0.920	0.563	1.890	1.890	0.955	0.160	-6.509
2	$\beta_1 = 1$	0.987	0.251	0.277	0.949	1.002	0.195	0.193	0.944	1.056	1.731
	$\beta_2 = 0.5$	1.008	3.395	3.587	0.938	0.549	1.339	1.349	0.953	0.428	-6.506
3	$\beta_1 = 1$	1.000	0.205	0.209	0.957	0.999	0.167	0.169	0.948	0.994	0.996
	$\beta_2 = 0.5$	0.550	6.426	6.625	0.943	0.497	2.000	2.040	0.951	0.113	0.514
4	$\beta_1 = 1$	1.000	0.183	0.184	0.951	1.000	0.169	0.168	0.947	0.999	1.000
	$\beta_2 = 0.5$	0.450	3.097	3.252	0.940	0.467	1.315	1.313	0.948	0.401	0.469

5 Empirical Application

In this section, I revisit Prina (2015), which conducted an RCT in 19 villages in Nepal (915 households) to study the effects of offering formal savings accounts on households' financial outcomes.[10] Half of female household heads were randomly offered an account, with the remainder assigned to the control group. Outcomes are measured in Nepalese rupees over the 30 days prior to the endline survey and include spending on festivals, fish, and education-related items such as textbooks and supplies. Education-related outcomes are restricted to households with school-age children (ages 6–16). Financial links are measured using survey questions such as, "With whom did you exchange loans or gifts?" The network was reshuffled substantially even though the total number of links was nearly unchanged (328 at baseline and 329 at endline). Only 73 links persisted, while 255 links disappeared and 256 new links formed by endline.

Table 4 reports the OLS and IV estimates (using both SSIV and modified SSIV) across various household expenditure categories, with standard errors in parentheses. Results for β_0 are omitted for brevity. The divergence between OLS and SSIV estimates, whether in sign or statistical significance, highlights bias from endogeneity induced by unobserved confounders.

The SSIV estimates further suggest that the intervention operates through distinct channels. For fish consumption, the direct effect of access to a savings account is not statistically significant, whereas the indirect effect, operating through treated peers, is significantly positive. In particular, moving from having no treated friends to having all friends treated increases fish expenditure by Rs. 252.91. In contrast, for total education expenditures, textbooks, and school supplies, the direct effects of access to a savings account are positive and statistically significant, while the indirect effects via the fraction of treated friends are not. Holding peer treatment constant, access to a free savings account increases spending by Rs. 574.43 (total education), Rs. 223.26 (textbooks), and Rs. 104.47 (supplies). These patterns show that our method separates direct effects from network-mediated effects, clarifying the channels through which the intervention affects outcomes. When $\hat{\beta}_2^{\text{IV}}$ is not significant, $\hat{\beta}_1^{\text{OLS}}$ and $\hat{\beta}_1^{\text{IV}}$ are closely aligned, reflecting that random assignment mitigates the omitted-variable bias in the direct effect. Finally, the IV estimates based on SSIV and modified SSIV are highly similar, despite the sparsity of both networks: the average degree is about 0.72 links per individual in both the pre- and post-intervention networks. This similarity suggests that the eigenvector-based modification does not introduce substantial additional noise into the estimation.

[10] I integrate the panel network data released by Comola and Prina (2023) with the outcome variables from Prina (2015), and the treatment assignment is available in both datasets.

For comparison, Table 4 also reports the estimates from Prina (2015, Tables 5 and 6). They regress outcomes on treatment assignment, controlling for baseline outcomes and covariates, and interpret the treatment coefficient as the intention-to-treat (ITT) effect in the presence of non-compliance. Align with the OLS results in the top panel, for outcomes in which $\hat{\beta}_2^{\text{IV}}$ is not statistically significant, the ITT estimates from Prina (2015) and IV estimates $\hat{\beta}_1^{\text{IV}}$ are very similar, suggesting that random assignment largely mitigates bias from unobserved confounders when estimating the direct effect.

Table 4: Estimation result based on data from Prina (2015)

Method	$\hat{\beta}$	Festivals & Ceremonies	Fish	Total Exp. in Education	Exp. on Textbooks	Exp. on Sch. Supplies
OLS	$\hat{\beta}_1^{\text{OLS}}$	267.112*	48.505	562.276**	246.481**	103.767**
		(141.412)	(39.683)	(281.922)	(105.114)	(45.880)
	$\hat{\beta}_2^{\text{OLS}}$	-157.612	-55.696	87.445	-91.316	22.194
		(119.425)	(35.796)	(336.824)	(111.511)	(56.184)
SSIV	$\hat{\beta}_1^{\text{IV}}$	232.207	33.276	574.427*	223.261*	104.469**
		(152.369)	(42.002)	(299.153)	(117.319)	(48.935)
	$\hat{\beta}_2^{\text{IV}}$	549.756	252.913*	-109.182	284.441	10.823
		(961.761)	(150.221)	(1299.093)	(666.867)	(219.498)
Denoised SSIV	$\hat{\beta}_1^{\text{DN}}$	272.332	27.030	576.798*	211.983*	100.048**
		(155.618)	(42.708)	(295.529)	(114.828)	(48.519)
	$\hat{\beta}_2^{\text{DN}}$	-263.414	379.510*	-147.559	466.935	82.376
		(1314.334)	(206.481)	(1395.153)	(644.052)	(219.500)
Prina (2015)	ITT	248.070**	45.756	667.560**	213.740*	115.950**
		(123.670)	(39.055)	(320.630)	(120.220)	(49.980)

Note: Dependent variables are expressed in Nepalese rupees (the exchange rate was roughly Rs. 70 to USD 1 during the study period). The significance level of the estimators is indicated with stars in the usual manner: "***" means significant at $\alpha = 1\%$, "**" means significant at $\alpha = 5\%$, and"*" means significant at $\alpha = 10\%$.

6 Conclusion

This paper studies the identification and inference of causal effects in RCTs with network interference, focusing on two key challenges: (1) unobserved confounders affecting both network formation and outcomes, and (2) treatment-induced changes to the network structure. To account for network changes, I embed mediation analysis within a potential-outcomes framework under interference, which distinguishes direct effects from indirect effects operating through network evolution. To address endogeneity, I propose an SSIV-based IV approach for relatively sparse networks and develop a denoised modification that remains valid across a broad range of sparsity regimes, including denser networks. I develop asymptotic results and consistent variance estimators for both procedures to enable valid inference.

The easy-to-implement modification not only restores consistency in regimes where standard SSIV fails, but also delivers faster convergence even when standard SSIV is already consistent. More broadly, the paper highlights how increasing network density can weaken SSIV relevance, and the proposed modification offers guidance for other network-mediated settings. Simulation evidence supports the theoretical results, and I apply the methods to the RCT studied by Prina (2015) as an empirical illustration. Accordingly, I recommend using the modified SSIV as the default specification in empirical work.

This paper opens several avenues for future research. A natural extension is to apply the SSIV approach to more general spillover settings, including applications in trade and labor markets. In addition, a growing literature emphasizes the importance of measuring and accounting for network changes when assessing the welfare impacts of policies. Although existing work has studied optimal treatment assignment under interference, the problem of designing effective policies that account for endogenous network evolution remains unresolved. Finally, my methods rely on detailed panel network data, and it would be valuable to relax this requirement to settings with only a single network snapshot or more aggregated network measures. I leave these directions for future research.

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Supplementary Material

The supplementary materials contain the following sections.

Section A gives the proofs for the identification in Section 2.2.

Section B provides proofs for the IV results using the SSIV in Section 3.3.

Section C provides proofs for the IV results using the modified SSIV in Section 3.4.

Notation Let N_i denote the degree of unit i in network A, i.e., $N_i = \sum_{j=1}^{n} A_{ij}$. Define $g_0(i,j) = g^{\text{pre}}(w_i, w_j)$, $g_1(i,j) = g^{\text{post}}(w_i, w_j, T_i, T_j)$, and $g_0(k) = \mathbb{E}(g_0(j,k)|w_k)$. Let $X_n = \Omega_{\mathbb{P}}(b_n)$, if for any $\delta > 0$, there exists $M, N > 0$, s.t. $\mathbb{P}(|X_n| \leq Mb_n) \leq \delta$ for any $n > N$. Let $\text{diag}(a_1, \ldots, a_k)$ be the $k \times k$ diagonal matrix with diagonal entries $a_1, \ldots, a_k$. Denote by $\|\cdot\|_{\text{F}}$ the Frobenius norm. Define $X_{i0} = 1, X_{i1} = T_i, X_{i2} = M_i$. Define $\mathbb{E}_{w_i}(\cdot)$ as the expectation with respect to the marginal distribution of w_i.

A Proof of results in Section 2.2

Proposition 2.1 is proved by establishing a more general identification result that does not rely on the linearity condition in Assumption 2. To relax Assumption 2, the mediation analysis instead imposes the following alternative assumption.

Assumption 4. *For all* $i = 1, \cdots, n$,

(a) $M_i \perp\!\!\!\perp Y_i(t_i, m_i) \mid T, w_i$ *for all* $t_i \in \{0,1\}$ *and* $m_i \in \mathcal{M}$;

(b) $M_i(t_i', t_{-i}) \perp\!\!\!\perp Y_i(t_i, m_i) \mid w_i$ *for all* $t_i \in \{0,1\}$, $t_i' \in \{0,1\}$, $t_{-i} \in \{0,1\}^{n-1}$ *and* $m_i \in \mathcal{M}$.

Assumption 4(a) assumes that the variable w_i captures all the confounding factors between the mediator and outcome conditional on the treatment. Assumption 4(b) assumes the cross-world independence between the potential outcomes and potential mediators. Furthermore, randomized treatments rule out both treatment-outcome and treatment-mediator confounding.

Proposition A.1 below states the identification result under Assumption 4. It expresses the causal estimands of interest in terms of observable quantities, with dependence on the latent distribution of w_i left unspecified. Thus, the causal effects are identified up to the distribution of w_i.

Proposition A.1. *Under Assumption 4,*

(1) $\text{OTE} = \mathbb{E}(Y_i|T_i = 1) - \mathbb{E}(Y_i|T_i = 0)$;

(2) $\mathrm{DE}(t) = \mathbb{E}_{w_i}\{\mathbb{E}\left[\mathbb{E}\left(Y_i|M_i, T_i = 1, w_i\right) - \mathbb{E}\left(Y_i|M_i, T_i = 0, w_i\right)|T_i = t, w_i\right]\}$;

(3) $\mathrm{IE}(t) = \mathbb{E}_{w_i}\{\mathbb{E}\left[\mathbb{E}\left(Y_i|M_i, T_i = t, w_i\right)|T_i = 1, w_i\right] - \mathbb{E}\left[\mathbb{E}\left(Y_i|M_i, T_i = t, w_i\right)|T_i = 0, w_i\right]\}$;

(4) $\mathrm{SE}(t) = \mathbb{E}_{w_i}\left\{\begin{array}{c} \mathbb{E}\left[\mathbb{E}\left(Y_i|M_i, T_i = t, w_i\right)|T_i = t, T_{-i} = 1_{n-1}, w_i\right] \\ -\mathbb{E}\left[\mathbb{E}\left(Y_i|M_i, T_i = t, w_i\right)|T_i = t, T_{-i} = 0_{n-1}, w_i\right] \end{array}\right\}$.

If the parameter of interest is solely OTE, it can be identified, and consistent estimators can be obtained by regressing the outcome on the treatment indicator with an intercept, or equivalently, by using the difference-in-means estimator, provided the treatment is randomly assigned. However, if the focus is on distinguishing between the direct and indirect effects, or when the spillover effect is of interest, these causal effects are identified only up to the unknown distribution of w_i.

Proof of proposition A.1. The proof follows the standard mediation analysis framework. We focus on the discrete case; the continuous and mixed cases follow analogously. It suffices to show that

$$\mathbb{E}\left[Y_i\left(t_i, M_i(t_i', T_{-i})\right)|w_i = w\right] = \mathbb{E}\left[\mathbb{E}\left(Y_i|M_i, T_i = t_i, w_i = w\right)|T_i = t_i', w_i = w\right]. \tag{18}$$

To establish this, fix $T_{-i} = t_{-i}$ and $w_i = w$ and consider:

$$\begin{aligned}
&\mathbb{E}\left[Y_i\left(t_i, M_i(t_i', T_{-i})\right)|T_{-i} = t_{-i}, w_i = w\right] \\
&= \sum_{m_i \in \mathcal{M}} \mathbb{E}\left(Y_i(t_i, m_i)|M_i(t_i', t_{-i}) = m_i, T_{-i} = t_{-i}, w_i = w\right) \mathbb{P}\left(M_i(t_i', t_{-i}) = m_i|T_{-i} = t_{-i}, w_i = w\right) \\
&= \sum_{m_i \in \mathcal{M}} \mathbb{E}\left(Y_i(t_i, m_i)|T_{-i} = t_{-i}, w_i = w\right) \mathbb{P}\left(M_i(t_i', t_{-i}) = m_i|T_{-i} = t_{-i}, w_i = w\right) \\
&= \sum_{m_i \in \mathcal{M}} \mathbb{E}\left(Y_i(t_i, m_i)|w_i = w\right) \mathbb{P}\left(M_i = m_i|T_i = t_i', T_{-i} = t_{-i}, w_i = w\right) \\
&= \sum_{m_i \in \mathcal{M}} \mathbb{E}\left(Y_i|T_i = t_i, M_i = m_i, w_i = w\right) \mathbb{P}\left(M_i = m_i|T_i = t_i', T_{-i} = t_{-i}, w_i = w\right)
\end{aligned}$$

Taking expectations over $T_{-i}|w_i = w$ yields the desired conditional equality in (18). Finally, marginalizing over the distribution of w completes the proof. □

Proof of Proposition 2.1. The result follows from Proposition A.1 under Assumption 2. □

B Proof of results in Section 3.3

B.1 Taylor expansion of M_i

To simplify the notations, define

$$R_{ij} = \frac{A_{ij}^{\text{post}} T_j}{\mathbb{E}(A_{ij}^{\text{post}} T_j | T_i, w_i)} - \frac{A_{ij}^{\text{post}}}{\mathbb{E}(A_{ij}^{\text{post}} | T_i, w_i)}, \tag{19}$$

$$U_{ij} = A_{ij}^{\text{post}} T_j - \mathbb{E}(A_{ij}^{\text{post}} T_j | T_i, w_i), \tag{20}$$

$$W_{ij} = A_{ij}^{\text{post}} - \mathbb{E}(A_{ij}^{\text{post}} | T_i, w_i), \tag{21}$$

which satisfies $\mathbb{E}(R_{ij}|T_i, w_i) = \mathbb{E}(U_{ij}|T_i, w_i) = \mathbb{E}(W_{ij}|T_i, w_i) = 0$ by construction. Define $x_i = \frac{1}{n-1}\sum_{j\neq i} A_{ij}^{\text{post}} T_j$ and $y_i = \frac{1}{n-1}\sum_{j\neq i} A_{ij}^{\text{post}}$. To facilitate the proof, we expand M_i at the conditional mean of (x_i, y_i): $\theta_i = \big(\mathbb{E}(A_{ij}^{\text{post}} T_j | T_i, w_i), \mathbb{E}(A_{ij}^{\text{post}} | T_i, w_i)\big)$, we have $M_i = \xi_i + r_{0,i} + r_{1,i}$ where ξ_i is defined in (13) and

$$\begin{aligned} r_{0,i} &= \xi_i \frac{1}{n-1}\sum_{k\neq i} R_{ik}, \\ r_{1,i} &= \frac{1}{(n-1)^2}\sum_{k\neq i}\left(-\tfrac{U_{ik}W_{ik}}{\theta_i^*(y)^2} + \tfrac{2\theta_i^*(x)}{\theta_i^*(y)^3} W_{ik}^2\right) + \frac{1}{(n-1)^2}\sum_{(k,l):k\neq l}\left(-\tfrac{U_{ik}W_{il}}{\theta_i^*(y)^2} + \tfrac{2\theta_i^*(x)}{\theta_i^*(y)^3} W_{ik}W_{il}\right). \end{aligned}$$

for some $\theta_i^* = (\theta_i^*(x), \theta_i^*(y))$ between θ_i and (x_i, y_i). Under the sparse-network scaling, $\theta_i^*(y) \asymp q_n$ and $\theta_i^*(x) = O(q_n)$ with probability approaching one. Note that ξ_i is either a function of T_i and w_i, making it i.i.d. across units and independent of n, or it equals π when A^{post} is conditionally independent of T. Define $\mu_{r_1,i} = \mathbb{E}(r_{1,i}|T_i, w_i)$. By direct algebra, we can show that

$$\mu_{r_1,i} = \frac{1}{n-1}\mathbb{E}\left(-\tfrac{U_{ik}W_{ik}}{\theta_i^*(y)^2} + \tfrac{2\theta_i^*(x)}{\theta_i^*(y)^3} W_{ik}^2 \mid T_i, w_i\right) = O_{\mathbb{P}}\left(\tfrac{1}{nq_n^{\text{post}}}\right). \tag{22}$$

Lemma B.1. *Define a_i and b_i as any functions of T_i and w_i with constant variance. Under Assumption 1, we have*

$$\mathbb{E}\left[(r_{0,i} + r_{1,i} - \mu_{r_1,i})^2\right] = O\left(\tfrac{1}{nq_n^{post}}\right), \tag{23}$$

$$\text{Cov}\left(a_i(r_{0,i} + r_{1,i} - \mu_{r_1,i}), b_j(r_{0,j} + r_{1,j} - \mu_{r_1,j})\right) = O\left(\tfrac{1}{n}\right) \text{ for any } i \neq j. \tag{24}$$

Proof. See proof in Section D.1 of the Online Appendix. □

Lemma B.2. *Let $a_i \in \{1, T_i\}$. Under Assumption 1, then*

$$\frac{1}{n}\sum_{i=1}^{n} a_i(r_{0,i} + r_{1,i} - \mu_{r_1,i}) = O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{post}}}\right), \tag{25}$$

$$\frac{1}{n}\sum_{i=1}^{n} a_i M_i = \frac{1}{n}\sum_{i=1}^{n} a_i(\xi_i + \mu_{r_1,i}) + O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{post}}}\right). \tag{26}$$

Proof. The equation (25) follows directly from Lemma B.1. The equation (26) is an immediate result of (25) and the decomposition of M_i. □

B.2 Auxiliary Lemmas

Lemma B.3 is a key lemma that established the performance of the SSIV estimator.

Lemma B.3. *Define ϕ_i as an i.i.d. random variable with constant variance. Define the conditional expectation $Q_j^{\phi} = \mathbb{E}\left[\frac{A_{ij}^{pre}\phi_i}{q_n^{pre}}|w_j\right]$. Under Assumptions 1 and 2,*

$$\frac{1}{n^2 q_n^{pre}}\sum_{i=1}^{n} Z_i^{\text{SSIV}}\phi_i = \frac{1}{n}\sum_{i=1}^{n}(T_i - \pi)Q_i^{\phi} + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}\sqrt{nq_n^{pre}}}\right). \tag{27}$$

Proof. See proof in Section D.1 of the Online Appendix. □

Lemma B.4. *Under Assumption 1, we have*

$$\frac{1}{n^2 q_n^{pre}}\sum_{i=1}^{n} Z_i^{\text{SSIV}} M_i = \frac{\mathbb{E}\left[Z_i^{\text{SSIV}}(r_{0,i} + r_{1,i} - \mu_{r_1,i})\right]}{nq_n^{pre}} + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \tag{28}$$

Furthermore, we can show that

$$\frac{1}{n}\sum_{i=1}^{n} Z_i^{\text{SSIV}} M_i - \overline{Z^{\text{SSIV}}} \cdot \overline{M} = \begin{cases} O_{\mathbb{P}}\left(\frac{q_n^{pre}}{\max\{q_n^{pre}, q_n^{post}\}}\right) + O_{\mathbb{P}}\left(\frac{q_n^{pre}}{\sqrt{n}q_n^{post}}\right) & \text{if } \mathbb{V}(\xi_i) = 0; \\ O_{\mathbb{P}}\left(\frac{q_n^{pre}}{\max\{q_n^{pre}, q_n^{post}\}}\right) + O_{\mathbb{P}}\left(\sqrt{n}q_n^{pre}\right) & \text{if } \mathbb{V}(\xi_i) > 0. \end{cases} \tag{29}$$

Proof. See proof in Section D.1 of the Online Appendix. □

B.3 Consistency

Define $D_n^{\text{IV}} = \text{diag}\,(1, 1, 1/(nq_n^{\text{pre}}))$, the diagonal matrix that scales the third coordinate. The closed-form of $(D_n^{\text{IV}} Z^\top X/n)^{-1}$ is

$$\left(D_n^{\text{IV}} \frac{Z^\top X}{n}\right)^{-1} = \frac{1}{\det(D_n^{\text{IV}} Z^\top X/n)} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \tag{30}$$

where

$$\begin{aligned}
&b_{11} = \tfrac{1}{nq_n^{\text{pre}}} \left(\overline{T} \cdot \overline{MZ^{\text{SSIV}}} - \overline{TM} \cdot \overline{TZ^{\text{SSIV}}}\right),\ b_{12} = \tfrac{1}{nq_n^{\text{pre}}} \left(-\overline{T} \cdot \overline{MZ^{\text{SSIV}}} + \overline{M} \cdot \overline{TZ^{\text{SSIV}}}\right), \\
&b_{13} = \overline{T} \cdot \overline{(1-T)M},\ b_{21} = \tfrac{1}{nq_n^{\text{pre}}} \left(-\overline{T} \cdot \overline{MZ^{\text{SSIV}}} + \overline{Z^{\text{SSIV}}} \cdot \overline{TM}\right), \\
&b_{22} = \tfrac{1}{nq_n^{\text{pre}}} \left(\overline{MZ^{\text{SSIV}}} - \overline{M} \cdot \overline{Z^{\text{SSIV}}}\right),\ b_{23} = \overline{T} \cdot \overline{M} - \overline{TM}, \\
&b_{31} = \tfrac{1}{nq_n^{\text{pre}}} \left(\overline{T} \cdot \overline{TZ^{\text{SSIV}}} - \overline{T} \cdot \overline{Z^{\text{SSIV}}}\right),\ b_{32} = \tfrac{1}{nq_n^{\text{pre}}} \left(-\overline{TZ^{\text{SSIV}}} + \overline{T} \cdot \overline{Z^{\text{SSIV}}}\right),\ b_{33} = \overline{T}(1-\overline{T}),
\end{aligned}$$

and $\det(D_n^{\text{IV}} Z^\top X/n) = b_{33}b_{22} - b_{32}b_{23}$. Define $b_{22}^* = \frac{\mathbb{E}\left[Z_i^{\text{SSIV}}(r_{0,i}+r_{1,i})\right]}{nq_n^{\text{pre}}} \asymp \frac{1}{n \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}$. Below, we show the probability limit of $\left(D_n^{\text{IV}} Z^\top X/n\right)^{-1}$.

Theorem B.1. *Define $b_{13}^* = -\pi\mathbb{E}((1-T_i)(\xi_i + \mu_{r_1,i}))$. Under Assumption 1, then*
Case (a)

$$\left(D_n^{\text{IV}} \frac{Z^\top X}{n}\right)^{-1} = \frac{\begin{pmatrix} \pi b_{22}^* & -\pi b_{22}^* & b_{13}^* \\ -\pi b_{22}^* & b_{22}^* & -\text{Cov}(T_i, \xi_i + \mu_{r_1,i}) \\ 0 & 0 & \pi(1-\pi) \end{pmatrix} + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}{\pi(1-\pi)b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)};$$

Case (b)

$$\left(D_n^{\text{IV}} \frac{Z^\top X}{n}\right)^{-1} = \frac{\begin{pmatrix} \pi b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & -\pi b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & b_{13}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \\ -\pi b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}nq_n^{post}}\right) & O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{post}}}\right) \\ O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}\sqrt{nq_n^{pre}}}\right) & \pi(1-\pi) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \end{pmatrix}}{\pi(1-\pi)b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}nq_n^{post}}\right)}.$$

Proof. By Lemma B.2 and Lemma B.4, we have

$$b_{11} = \pi b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right), b_{12} = -\pi b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right), b_{13} = b_{13}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right), b_{21} = -\pi b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).$$

By Lemma B.3, we have $b_{31} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$ and $b_{32} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}\sqrt{nq_n^{\text{pre}}}}\right)$. With i.i.d. data, we have $b_{33} = \pi(1-\pi) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$. For other terms, we consider Cases (a) and (b).

Case (a) Lemmas B.2 and B.4 imply that

$$b_{22} = b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right), \quad b_{23} = -\text{Cov}(T_i, \xi_i + \mu_{r_1,i}) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right),$$

and hence $\det(D_n^{\text{IV}} Z^\top X) = \pi(1-\pi) b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$.

Case (b) Similarly, Lemmas B.2 and B.4 imply that

$$b_{22} = b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}nq_n^{\text{post}}}\right), \quad b_{23} = O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right),$$

and hence $\det(D_n^{\text{IV}} Z^\top X) = \pi(1-\pi) b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}nq_n^{\text{post}}}\right)$. □

Proof of Theorem 3.1. By Lemma B.3, we have

$$\frac{1}{n^2 q_n^{\text{pre}}} \sum_{i=1}^n Z_i^{\text{SSIV}} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \text{ and } \frac{1}{n^2 q_n^{\text{pre}}} \sum_{i=1}^n Z_i^{\text{SSIV}} u_i = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \tag{31}$$

(31) then implies that $\frac{1}{n^2 q_n^{\text{pre}}} \sum_{i=1}^n (Z_i^{\text{SSIV}} - \overline{Z^{\text{SSIV}}}) u_i = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$. Under i.i.d. data, $\frac{1}{n}\sum_{i=1}^n u_i = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$ and $\frac{1}{n}\sum_{i=1}^n (T_i - \overline{T}) u_i = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$.

Case (a) By Theorem B.1 and the closed form of $\hat{\beta}^{\text{IV}} - \beta$, we conclude:

$$\hat{\beta}_1^{\text{IV}} - \beta_1 = \frac{b_{22}^* \cdot O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) - O_{\mathbb{P}}(1) \cdot O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + O_{\mathbb{P}}\left(\frac{1}{n}\right)}{\pi(1-\pi) b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)},$$

$$\hat{\beta}_2^{\text{IV}} - \beta_2 = \frac{\pi(1-\pi) \cdot O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + O_{\mathbb{P}}\left(\frac{1}{n}\right)}{\pi(1-\pi) b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}.$$

Hence, $\hat{\beta}^{\text{IV}}$ is consistent when $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec \frac{1}{\sqrt{n}}$, with $\hat{\beta}^{\text{IV}} - \beta = O_{\mathbb{P}}\left(\sqrt{n}\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}\right)$.

Case (b) By Theorem B.1, for $\hat{\beta}_1^{\text{IV}} - \beta_1$, we have

$$\hat{\beta}_1^{\text{IV}} - \beta_1 = \frac{b_{22}^* \cdot O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + O_{\mathbb{P}}\left(\frac{1}{n\sqrt{nq_n^{\text{post}}}}\right)}{\pi(1-\pi)b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}nq_n^{\text{post}}}\right)} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}} \cdot \max\left\{\frac{\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}{\sqrt{q_n^{\text{post}}}}, 1\right\}\right),$$

where the second equality follows from $b_{22}^* \asymp \frac{1}{n\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}$ Thus, $\hat{\beta}_1^{\text{IV}}$ is consistent if $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec \sqrt{n}q_n^{\text{post}}$. For $\hat{\beta}_2^{\text{IV}} - \beta_2$, we have

$$\hat{\beta}_2^{\text{IV}} - \beta_2 = \frac{\pi(1-\pi) \cdot O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + O_{\mathbb{P}}\left(\frac{1}{n}\right)}{\pi(1-\pi)b_{22}^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}nq_n^{\text{post}}}\right)} = O_{\mathbb{P}}\left(\sqrt{n}\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}\right).$$

Therefore, $\hat{\beta}_2^{\text{IV}}$ is consistent if $\max\{q_n^{\text{pre}}, q_n^{\text{post}}\} \prec 1/\sqrt{n}$. The same condition implies consistency of $\hat{\beta}_0^{\text{IV}}$, with $\hat{\beta}_0^{\text{IV}} - \beta_0 = O_{\mathbb{P}}\left(\sqrt{n}\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}\right)$. □

B.4 Asymptotic normality

Theorem B.2 below establishes the CLT with weak dependence, which makes uses of the proof of Ross (2011, Theorem 3.6) and is the unconditional version of **?**, Theorem 4. Let $H_1, \ldots, H_n$ be real-valued random variables. Define the dependency indicator as $D_{ij} = 1\left\{A_{ij}^{\diamond} + \max_k A_{ik}^{\diamond}A_{kj}^{\diamond} + 1\{i=j\} > 0\right\}$ where $\diamond \in \{\text{pre}, \text{post}\}$. In words, two units are dependent if they are directly connected, share at least one common friend, or correspond to the same index. An $n \times n$ symmetric binary matrix D with (i,j)th entry D_{ij} is a dependency graph on $\{H_i\}_{i=1}^n$ if, for any two disjoint subsets $I_1, I_2 \subseteq \{1, \ldots, n\}$, $\{H_i : i \in I_1\} \perp\!\!\!\perp \{H_i : i \in I_2\}$ conditional on the event that $D_{ij} = 0$ for all $i \in I_1$ and $j \in I_2$. By construction, $D_{ii} = 1$ for all i. Define $\mathcal{S}_i = \{j : D_{ij} = 1\}$ and $S_i = |\mathcal{S}_i| = \sum_{j=1}^n D_{ij}$. Define $\sigma_n^2 = \mathbb{V}\left(\sum_{i=1}^n H_i\right)$. Let $(D^3)_{ij}$ denote the (i,j)th entry of the third matrix power of D.

Theorem B.2 (CLT for local dependence)**.** *Under the following assumptions, we have* $\sigma_n^{-1}\sum_{i=1}^n H_i \xrightarrow{d} \mathcal{N}(0,1)$ *as* $n \to \infty$*: (a)* $\sigma_n^2 = O(n)$*; (b)* $\max_i \mathbb{E}\left(|H_i|^4\right) = O(1)$*; (c)* $n^{-1}\sum_{i=1}^n S_i^2 = o_{\mathbb{P}}(\sqrt{n})$*,* $n^{-1}\sum_{i=1}^n S_i^3 = o_{\mathbb{P}}(n)$*, and* $n^{-1}\sum_{(i,j)}(D^3)_{ij} = o_{\mathbb{P}}(n)$*.*

Proof. Convergence in the Wasserstein distance $\Delta(U,V)$ implies convergence in the Kolmogorov metric (see Ross (2011, Proposition 1.2)), which in turn implies weak convergence. Thus, the theorem holds if $\Delta\left(\frac{1}{\sigma_n}\sum_{i=1}^n H_i, \mathcal{N}(0,1)\right) \xrightarrow{p} 0$. The distance is bounded above by

$$\Delta\left(\frac{1}{\sigma_n}\sum_{i=1}^n H_i, \mathcal{N}(0,1)\right) \leq \frac{1}{\sigma_n^3}\sum_{i=1}^n \mathbb{E}\left[\left|H_i\left(\sum_{j\in\mathcal{S}_i} H_j\right)^2\right|\right] + \frac{\sqrt{2}}{\sqrt{\pi}\sigma_n^2}\sqrt{\text{Var}\left(\sum_{i=1}^n\sum_{j\in\mathcal{S}_i} H_iH_j\right)}$$

$$\leq \max_i \mathbb{E}\left[|H_i|^3\right] \frac{n}{\sigma_n^3}\frac{1}{n}\sum_{i=1}^n S_i^2 + \frac{1}{\sqrt{n}}\frac{n}{\sigma_n^2}\sqrt{\frac{4}{\pi}\max_i \mathbb{E}\left[H_i^4\right]}\sqrt{\frac{4}{n}\sum_{i=1}^n S_i^3 + \frac{3}{n}\sum_{i=1}^n\sum_{j=1}(G^3)_{ij}},$$

which is $o_{\mathbb{P}}(1)$ under Assumptions (a)–(c). □

We verify Assumption (c) in Theorem B.2 when $nq_n^\diamond \asymp 1$ for $\diamond \in \{\text{pre}, \text{post}\}$.

Lemma B.5. *Suppose $nq_n^\diamond \asymp 1$ for $\diamond \in \{pre, post\}$. Then*

(i) $\frac{1}{n}\sum_{i=1}^n S_i^2 = o_{\mathbb{P}}(\sqrt{n})$; *(ii)* $\frac{1}{n}\sum_{i=1}^n S_i^3 = o_{\mathbb{P}}(n)$; *(iii)* $\frac{1}{n}\sum_{(i,j)}(D^3)_{ij} = o_{\mathbb{P}}(n)$.

Proof. See proof in Section D.1 of the Online Appendix. □

Proof of Theorem 3.2. We complete the proof in three steps.

Step 1: Asymptotic normality. We establish the asymptotic normality of the numerator $D_n^{\text{IV}} \cdot \frac{1}{n}\sum_{i=1}^n Z_i u_i$, considering two cases.

Case 1: $nq_n^{\text{pre}} \succ 1$. By (31), we have

$$\frac{1}{n^2 q_n^{\text{pre}}}\sum_{i=1}^n Z_i^{\text{SSIV}} u_i = \frac{1}{n}\sum_{i=1}^n (T_i - \pi)\mathbb{E}\left[\frac{A_{ij}^{\text{pre}} u_j}{q_n^{\text{pre}}}|w_i\right] + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}\sqrt{nq_n^{\text{pre}}}}\right),$$

which is the average of i.i.d. terms plus a negligible remainder $o_{\mathbb{P}}(1/\sqrt{n})$. Therefore, asymptotic normality of $D_n^{\text{IV}} \cdot \frac{1}{n}\sum_{i=1}^n Z_i u_i$ follows from a standard Cramér–Wold argument and the Lindeberg–Lévy CLT.

Case 2: $nq_n^{\text{pre}} \asymp 1$. It suffices to verify the assumptions in Theorem B.2. Assumption (a) holds since $D_n^{\text{IV}} V_{\text{num}}^{\text{IV}} D_n^{\text{IV}} = O(n)$. Assumption (b) is satisfied by boundedness of X_i. Assumption (c) follows from Lemma B.5, applied to A^{pre} under the regime $nq_n^{\text{pre}} \asymp 1$. By Theorem B.2 and the Cramér–Wold device, we conclude that $(V_{\text{num}}^{\text{IV}})^{-1/2}\sum_{i=1}^n Z_i u_i \xrightarrow{d} \mathcal{N}(0, I_3)$.

Step 2: consistent variance estimator. We show that $D_n^{\text{IV}}(\hat{V}_{\text{num}}^{\text{IV}} - V_{\text{num}}^{\text{IV}})D_n^{\text{IV}} = o_{\mathbb{P}}(1)$. We verify the consistency of each entry separately.

(1): Consistency of the $(1,1)$th element of $\hat{V}_{\text{num}}^{\text{IV}}$. We can show $\frac{1}{n}\sum_{i=1}^n(\hat{u}_i^{\text{IV}})^2 = \mathbb{E}(u_i^2) + o_{\mathbb{P}}(1)$ holds by $\hat{\beta}^{\text{IV}} - \beta = o_{\mathbb{P}}(1)$, and bounded support of T_i and M_i.

(2): Consistency of $(3,3)$th element of $\hat{V}_{\text{num}}^{\text{IV}}$. Expanding the square:

$$\frac{1}{n}\sum_{j=1}^n\left(\sum_{i\neq j} A_{ij}^{\text{pre}}\hat{u}_i^{\text{IV}}\right)^2 = \frac{1}{n}\left[\begin{array}{c}\sum_{j=1}^n\left(\sum_{i\neq j} A_{ij}^{\text{pre}} u_i\right)^2 + \sum_{j=1}^n\left(\sum_{i\neq j} A_{ij}^{\text{pre}}(\hat{\beta}^{\text{IV}} - \beta)^\top X_i\right)^2 \\ -2\sum_{j=1}^n\left(\sum_{i\neq j} A_{ij}^{\text{pre}} u_i \sum_{i\neq j} A_{ij}^{\text{pre}}(\hat{\beta}^{\text{IV}} - \beta)^\top X_i\right)\end{array}\right]. \tag{32}$$

For the first term of (32), the expectation is $\mathbb{E}\left[\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i\neq j}A_{ij}^{\text{pre}}u_i\right)^2\right] = O\left((nq_n^{\text{pre}})^2\right)$. The variance involves at most six-way dependence from adjacency terms and is

$$\text{Var}\left(\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i\neq j}A_{ij}^{\text{pre}}u_i\right)^2\right) = O\left(n^3(q_n^{\text{pre}})^4\right).$$

Hence, by Chebyshev's inequality:

$$\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i\neq j}\frac{A_{ij}^{\text{pre}}u_i}{nq_n^{\text{pre}}}\right)^2 = \frac{1}{n}\mathbb{E}\left[\sum_{j=1}^{n}\left(\sum_{i\neq j}\frac{A_{ij}^{\text{pre}}u_i}{nq_n^{\text{pre}}}\right)^2\right] + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \tag{33}$$

For the second term of (32), we apply the triangle inequality:

$$\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i\neq j}A_{ij}^{\text{pre}}(\hat{\beta}^{\text{IV}}-\beta)^\top X_i\right)^2 \leq \sum_{k=0}^{2}(\hat{\beta}_k^{\text{IV}}-\beta_k)^2\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i\neq j}A_{ij}^{\text{pre}}X_{ik}\right)^2 = o_{\mathbb{P}}\left((nq_n^{\text{pre}})^2\right) \tag{34}$$

by $\hat{\beta}^{\text{IV}}-\beta = o_{\mathbb{P}}(1)$ and $\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i\neq j}A_{ij}^{\text{pre}}X_{ik}\right)^2 = O_{\mathbb{P}}\left((nq_n^{\text{pre}})^2\right)$ for $k=0,1,2$ by boundedness of X_{ik}. For the third term of (32):

$$\begin{aligned}&\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i\neq j}A_{ij}^{\text{pre}}u_i\right)\left(\sum_{i\neq j}A_{ij}^{\text{pre}}(\hat{\beta}^{\text{IV}}-\beta)^\top X_i\right)\\ \leq &\sum_{k=0}^{2}\left|\hat{\beta}_k^{\text{IV}}-\beta_k\right|\frac{1}{n}\sum_{j=1}^{n}\left[\left(\sum_{i\neq j}A_{ij}^{\text{pre}}u_i\right)^2+\left(\sum_{i\neq j}A_{ij}^{\text{pre}}X_{ik}\right)^2\right] = o_{\mathbb{P}}\left((nq_n^{\text{pre}})^2\right),\end{aligned} \tag{35}$$

where the final equality follows from $\hat{\beta}^{\text{IV}}-\beta = o_{\mathbb{P}}(1)$, and $\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i\neq j}A_{ij}^{\text{pre}}u_i\right)^2 = O_{\mathbb{P}}\left((nq_n^{\text{pre}})^2\right)$. Combining (33), (34), and (35), we conclude:

$$\frac{1}{n(nq_n^{\text{pre}})^2}\sum_{j=1}^{n}\left(\sum_{i\neq j}A_{ij}^{\text{pre}}\hat{u}_i^{\text{IV}}\right)^2 = \frac{1}{n(nq_n^{\text{pre}})^2}\sum_{j=1}^{n}\mathbb{E}\left[\left(\sum_{i\neq j}A_{ij}^{\text{pre}}u_i\right)^2\right] + o_{\mathbb{P}}(1).$$

(3): Consistency of $(2,3)$ element of $\hat{V}_{\text{num}}^{\text{IV}}$. By expanding $\hat{u}_i^{\text{IV}}$:

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}A_{ij}^{\text{pre}}\hat{u}_i^{\text{IV}}\hat{u}_j^{\text{IV}} = \frac{1}{n}\left[\begin{array}{c}\sum_{i=1}^{n}\sum_{j\neq i}A_{ij}^{\text{pre}}u_iu_j - 2\sum_{i=1}^{n}\sum_{j\neq i}A_{ij}^{\text{pre}}u_i(\hat{\beta}^{\text{IV}}-\beta)^\top X_j\\ +\sum_{i=1}^{n}\sum_{j\neq i}A_{ij}^{\text{pre}}(\hat{\beta}^{\text{IV}}-\beta)^\top X_iX_j^\top(\hat{\beta}^{\text{IV}}-\beta)\end{array}\right]. \tag{36}$$

By analogous argument to the $(3,3)$ element of $\hat{V}^{\text{IV}}_{\text{num}}$, so we can show that

$$\frac{1}{n^2 q_n^{\text{pre}}} \sum_{i=1}^{n} \sum_{j \neq i} A_{ij}^{\text{pre}} \hat{u}_i^{\text{IV}} \hat{u}_j^{\text{IV}} = \frac{1}{n^2 q_n^{\text{pre}}} \sum_{i=1}^{n} \sum_{j \neq i} \mathbb{E}\left[A_{ij}^{\text{pre}} u_i u_j\right] + o_{\mathbb{P}}(1).$$

Step 3: Combine the results. By Steps 1–2 and CMT, $(\hat{V}^{\text{IV}}_{\text{num}})^{-1/2} \sum_{i=1}^{n} Z_i u_i \xrightarrow{d} \mathcal{N}(0, I_3)$. Applying Theorem B.1 and CMT again then completes the proof. □

C Proof of results in Section 3.4

C.1 Useful lemmas

Define $q_n^{\text{pre}} \tilde{G}_n^{\text{pre}} = \sum_{k=1}^{r} \lambda_k^* \psi_k^* \psi_k^{*\top}$, which is formed from the leading r terms of $q_n^{\text{pre}} G_n^{\text{pre}}$.

Lemma C.1. *Suppose $q_n \succ \frac{\log n}{\log\log n}/n$. Under Assumptions 1 and 3, we have*

(1) $\left\|A^{pre} - q_n^{pre} G_n^{pre}\right\|_{\text{op}} = O_{\mathbb{P}}\left(\sqrt{n q_n^{pre}} \left(\frac{\log n}{\log\log n}\right)^{1/4}\right)$;

(2) $\left\|A^{pre} - q_n^{pre} \tilde{G}_n^{pre}\right\|_{\text{op}} = O_{\mathbb{P}}\left(\sqrt{n q_n^{pre}} \left(\frac{\log n}{\log\log n}\right)^{1/4}\right)$.

Proof. See proof in Section D.2 of the Online Appendix. □

Lemma C.2. *Suppose $q_n^{pre} \succ \frac{\log n}{\log\log n}/n$. Let ψ_k^* denote the vector of $\psi_k^*(w_i)$. There exists an $r \times r$ orthogonal matrix $\hat{R}$, where r is defined in Assumption 3, such that if we write $\hat{\Psi}^R = \hat{\Psi}\hat{R}$, and let $\hat{\psi}_k^R$ be the k-th column of $\hat{\Psi}^R$ for $k \leq r$, then under Assumptions 1 and 3, for $l = 1, \cdots, n$, we have*

$$(\psi_l^*)^\top \left(\hat{\psi}_k^R - \psi_k^*\right) = O_{\mathbb{P}}\left(\frac{1}{n q_n^{pre}} \sqrt{\frac{\log n}{\log\log n}}\right). \tag{37}$$

Proof. See proof in Section D.2 of the Online Appendix. □

Let ϕ_i be any i.i.d. random variable with constant variance. Consider the projection of ϕ_i onto $\{\psi_k^*(w_i)\}_{k=1}^r$. Define the projection coefficient as $\mu_{n,k}^{\phi} = \sum_{i=1}^{n} \phi_i \psi_k^*(w_i)$ and the projection residual as $\eta_i^{\phi} = \phi_i - \sum_{k=1}^{r} \mu_{n,k}^{\phi} \psi_k^*(w_i)$.

Lemma C.3. *Suppose $q_n^{pre} \succ \frac{\log n}{\log\log n}/n$. Under Assumptions 1 and 3, we have*

$$\frac{1}{n} \sum_{i=1}^{n} Z_i^{\text{DN}} \phi_i = \frac{1}{n} \sum_{i=1}^{n} Z_i^{\text{SSIV}} \eta_i^{\phi} + o_{\mathbb{P}}\left(\sqrt{q_n^{pre}}\right). \tag{38}$$

Proof. See proof in Section D.2 of the Online Appendix. □

Define the coefficient $\mu^r_{n,k} = \sum_{i=1}^n (r_{0,i} + r_{1,i} - \mu_{r_1,i})\psi^*_k(w_i)$ and the projection residual $\eta^r_i = (r_{0,i} + r_{1,i} - \mu_{r_1,i}) - \sum_{k=1}^r \mu^r_{n,k}\psi^*_k(w_i)$.

Lemma C.4. *Suppose* $q_n^{pre} \succ \frac{\log n}{\log\log n}/n$. *Under Assumptions 1 and 3, then*

$$\frac{1}{n}\sum_{i=1}^n Z_i^{\text{DN}} M_i = \frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}} \eta_i^M + O_{\mathbb{P}}\left(\sqrt{q_n^{pre}}\right). \tag{39}$$

Proof. See proof in Section D.2 of the Online Appendix. □

C.2 Consistency

Define $D_n^{\text{DN}} = \text{diag}(1, 1, 1/\sqrt{nq_n^{\text{pre}}})$. We first show the probability limit of $(D_n^{\text{DN}}(\tilde{Z}^{\text{DN}})^\top X)^{-1}$, which has the closed-form expression:

$$\left(D_n^{\text{DN}} \frac{(\tilde{Z}^{\text{DN}})^\top X}{n}\right)^{-1} = \frac{1}{\det(D_n^{\text{DN}}(\tilde{Z}^{\text{DN}})^\top X/n)} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \tag{40}$$

where

$$\begin{aligned}
c_{11} &= \tfrac{1}{\sqrt{nq_n^{\text{pre}}}}\left(\overline{T}\cdot\overline{MZ^{\text{DN}}} - \overline{TM}\cdot\overline{TZ^{\text{DN}}}\right), \; c_{12} = \tfrac{1}{\sqrt{nq_n^{\text{pre}}}}\left(-\overline{T}\cdot\overline{MZ^{\text{DN}}} + \overline{M}\cdot\overline{TZ^{\text{DN}}}\right), \; c_{13} = b_{13}, \\
c_{21} &= \tfrac{1}{\sqrt{nq_n^{\text{pre}}}}\left(-\overline{T}\cdot\overline{MZ^{\text{DN}}} + \overline{Z^{\text{DN}}}\cdot\overline{TM}\right), \; c_{22} = \tfrac{1}{\sqrt{nq_n^{\text{pre}}}}\left(\overline{MZ^{\text{DN}}} - \overline{M}\cdot\overline{Z^{\text{DN}}}\right), \; c_{23} = b_{23}, \\
c_{31} &= \tfrac{1}{\sqrt{nq_n^{\text{pre}}}}\left(\overline{T}\cdot\overline{TZ^{\text{DN}}} - \overline{T}\cdot\overline{Z^{\text{DN}}}\right), \; c_{32} = \tfrac{1}{\sqrt{nq_n^{\text{pre}}}}\left(-\overline{TZ^{\text{DN}}} + \overline{T}\cdot\overline{Z^{\text{DN}}}\right), \; c_{33} = b_{33},
\end{aligned}$$

and $\det(D_n^{\text{DN}}(\tilde{Z}^{\text{DN}})^\top X/n) = c_{33}c_{22} - c_{32}c_{23}$. Define $c^*_{22} = \frac{\mathbb{E}\left[Z_i^{\text{SSIV}}(r_{0,i}+r_{1,i})\right]}{\sqrt{nq_n^{\text{pre}}}} \asymp \frac{\min\{q_n^{\text{pre}}, q_n^{\text{post}}\}}{q_n^{\text{post}}\sqrt{nq_n^{\text{pre}}}}$.

Theorem C.1. *Suppose* $q_n^{pre} \succ \frac{\log(n)}{\log(\log(n))}/n$. *Under Assumptions 1 and 3, then*
Case (a):

$$\left(D_n^{\text{DN}} \frac{(\tilde{Z}^{\text{DN}})^\top X}{n}\right)^{-1} = \frac{\begin{pmatrix} \pi c^*_{22} & -\pi c^*_{22} & b^*_{13} \\ -\pi c^*_{22} & c^*_{22} & -\text{Cov}(T_i, \xi_i + \mu_{r_1,i}) \\ 0 & 0 & \pi(1-\pi) \end{pmatrix} + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}{\pi(1-\pi)c^*_{22} + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}.$$

Case (b):

$$\left(D_n^{\mathrm{DN}}\frac{(\tilde{Z}^{\mathrm{DN}})^\top X}{n}\right)^{-1}=\frac{\begin{pmatrix}\pi c_{22}^*+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & -\pi c_{22}^*+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & b_{13}^*+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)\\ -\pi c_{22}^*+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & c_{22}^*+O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{post}}}\right) & O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{post}}}\right)\\ O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) & \pi(1-\pi)+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)\end{pmatrix}}{\pi(1-\pi)c_{22}^*+O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{post}}}\right)}.$$

Proof. The proof is analogous to Theorem B.1 with Lemmas C.3 and C.4. □

We show the consistency by Theorem C.1.

Proof of Theorem 3.3. By Lemma C.3, we can show that

$$\frac{1}{n\sqrt{nq_n^{\mathrm{pre}}}}\sum_{i=1}^n Z_i^{\mathrm{DN}}=\frac{1}{n\sqrt{nq_n^{\mathrm{pre}}}}\sum_{i=1}^n Z_i^{\mathrm{SSIV}}\eta_i^1+O_{\mathbb{P}}\left(\frac{\sqrt{q_n^{\mathrm{pre}}}}{\sqrt{n}}\right)=O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$$

and thus

$$\frac{1}{n\sqrt{nq_n^{\mathrm{pre}}}}\sum_{i=1}^n\left(Z_i^{\mathrm{DN}}-\frac{1}{n}\sum_{i=1}^n Z_i^{\mathrm{DN}}\right)u_i=O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).$$

Case (a): By the closed-form of $(\hat{\beta}^{\mathrm{DN}}-\beta)$ and combining Theorem C.1:

$$\hat{\beta}_1^{\mathrm{DN}}-\beta_1=\frac{\left(c_{22}^*+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)\right)O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}{\pi(1-\pi)c_{22}^*+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}=\frac{O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}{\pi(1-\pi)c_{22}^*+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)},$$

$$\hat{\beta}_2^{\mathrm{DN}}-\beta_2=\frac{\left(\pi(1-\pi)+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)\right)O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}{\pi(1-\pi)c_{22}^*+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}=\frac{O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}{\pi(1-\pi)c_{22}^*+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}.$$

Therefore, $\hat{\beta}^{\mathrm{DN}}$ is consistent when $\max\{q_n^{\mathrm{pre}},q_n^{\mathrm{post}}\}\prec\sqrt{q_n^{\mathrm{pre}}}$ with $\hat{\beta}^{\mathrm{DN}}-\beta=O_{\mathbb{P}}\left(\frac{\max\{q_n^{\mathrm{pre}},q_n^{\mathrm{post}}\}}{\sqrt{q_n^{\mathrm{pre}}}}\right)$.

Case (b): By combining the results from Theorem C.1, we have

$$\hat{\beta}_1^{\mathrm{DN}}-\beta_1=\frac{\left(c_{22}^*+O\left(\frac{1}{n\sqrt{q_n^{\mathrm{post}}}}\right)\right)O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)+O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\mathrm{post}}}}\right)O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)}{\pi(1-\pi)c_{22}^*+O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\mathrm{post}}}}\right)}.$$

The consistency of $\hat{\beta}_1^{\mathrm{DN}}$ always holds with $\hat{\beta}_1^{\mathrm{DN}} - \beta_1 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$. For $\hat{\beta}_2^{\mathrm{DN}}$, we have

$$\hat{\beta}_2^{\mathrm{DN}} - \beta_2 = \frac{\left(\pi(1-\pi)+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)\right)O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)+O_{\mathbb{P}}\left(\frac{1}{n}\right)}{\pi(1-\pi)c_{22}^{*}+O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\mathrm{post}}}}\right)}.$$

Therefore, $\hat{\beta}_2^{\mathrm{DN}} - \beta_2 = O_{\mathbb{P}}\left(\frac{\max\{q_n^{\mathrm{pre}}, q_n^{\mathrm{post}}\}}{\sqrt{q_n^{\mathrm{pre}}}}\right)$ provided $\max\{q_n^{\mathrm{pre}}, q_n^{\mathrm{post}}\} \prec \sqrt{q_n^{\mathrm{pre}}}$. Under the same condition, it follows that $\hat{\beta}_0^{\mathrm{DN}} - \beta_0 = O_{\mathbb{P}}\left(\frac{\max\{q_n^{\mathrm{pre}}, q_n^{\mathrm{post}}\}}{\sqrt{q_n^{\mathrm{pre}}}}\right)$. □

C.3 Asymptotic normality

Proof of Theorem 3.4. We complete the proof in three steps.

Step 1: asymptotic normality. By Lemma C.3, we have

$$\frac{1}{n\sqrt{nq_n^{\mathrm{pre}}}}\sum_{i=1}^{n} Z_i^{\mathrm{DN}} u_i = \frac{1}{n\sqrt{nq_n^{\mathrm{pre}}}}\sum_{i=1}^{n} Z_i^{\mathrm{SSIV}} \eta_i^u + o_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \tag{41}$$

By Theorem B.2 with each index corresponding to a pair, we have

$$\begin{aligned}
\Delta\left(\frac{1}{\sigma_n}\sum_{(i,j)} H_{ij}, \mathcal{N}(0,1)\right) &\le \frac{1}{\sigma_n^3}\sum_{(i,j)}\mathbb{E}\left[\left|H_{ij}\left(\sum_{(k,l)\in\mathcal{S}_{(i,j)}} H_{kl}\right)^2\right|\right] + \frac{\sqrt{2}}{\sqrt{\pi}\sigma_n^2}\sqrt{\mathrm{Var}\left(\sum_{(i,j)} H_{ij}\sum_{(k,l)\in\mathcal{S}_{(i,j)}} H_{kl}\right)} \\
&\le \frac{C}{\sigma_n^3}\sum_{(i,j)}\mathbb{E}\left[A_{ij}^{\mathrm{pre}}\left(\sum_{(k,l)\in\mathcal{S}_{(i,j)}} H_{kl}\right)^2\right] + \frac{\sqrt{2}}{\sqrt{\pi}\sigma_n^2}\sqrt{\mathrm{Var}\left(\sum_{(i,j)} H_{ij}\sum_{(k,l)\in\mathcal{S}_{(i,j)}} H_{kl}\right)}.
\end{aligned} \tag{42}$$

Define $H_{ij} = A_{ij}^{\mathrm{pre}}(T_j - \pi)\eta_i^u$ and $\sigma_n^2 = \mathrm{Var}\left(\sum_{(i,j)} H_{ij}\right)$. It suffices to show the asymptotic normality of $\sigma_n^{-1}\sum_{(i,j)} H_{ij}$ by showing

$$\frac{1}{\sigma_n^3}\sum_{i,j}\mathbb{E}\left[\left|H_{ij}\left(\sum_{(k,l)\in\mathcal{S}_{(i,j)}} H_{kl}\right)^2\right|\right] = O\left(\sqrt{q_n^{\mathrm{pre}}}\right), \tag{43}$$

$$\frac{\sqrt{2}}{\sqrt{\pi}\sigma_n^2}\sqrt{\mathrm{Var}\left(\sum_{(i,j)} H_{ij}\sum_{(k,l)\in\mathcal{S}_{(i,j)}} H_{kl}\right)} = O\left(\sqrt{q_n^{\mathrm{pre}}}\right). \tag{44}$$

Then by (42) with (43) and (44), we show $\sigma_n^{-1}\sum_{(i,j)} A_{ij}^{\mathrm{pre}}(T_j - \pi)\eta_i^u \xrightarrow{d} \mathcal{N}(0,1)$. Together with Cramér–Wold Theorem and (41), we can show that $(V_{\mathrm{num}}^{\mathrm{DN}})^{-1/2}\sum_{i=1}^{n}\tilde{Z}_i^{\mathrm{DN}} u_i \xrightarrow{d} \mathcal{N}(0, I_3)$.

We defer the details of (43) and (44) to Section D.2 of the Online Appendix.

Step 2: consistent variance estimator. By definition and Lemma C.2, we have

$$\begin{aligned}\hat{\mu}_k^R - \mu_k =& \sum_{i=1}^n (\hat{u}_i^{\mathrm{DN}} - u_i)(\hat{\psi}_{ki}^R - \psi_k^*(w_i)) + \sum_{i=1}^n (\hat{u}_i^{\mathrm{DN}} - u_i)\psi_k^*(w_i) + \sum_{i=1}^n u_i(\hat{\psi}_{ki}^R - \psi_k^*(w_i)) \\ =& O_{\mathbb{P}}\left(\frac{\sqrt{n}\max\{q_n^{\mathrm{pre}}, q_n^{\mathrm{post}}\}}{\sqrt{q_n^{\mathrm{pre}}}}\right),\end{aligned}$$

where each term is bounded as follows:

- For the first term,

$$\sum_{i=1}^n (\hat{u}_i^{\mathrm{DN}} - u_i)(\hat{\psi}_{ki}^R - \psi_k^*(w_i)) = (\hat{\beta}^{\mathrm{DN}} - \beta)^\top \sum_{i=1}^n X_i(\hat{\psi}_{ki}^R - \psi_k^*(w_i)) = O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{\log\log n}}\frac{1}{\sqrt{n q_n^{\mathrm{pre}}}}\right).$$

- For the second term, $\sum_{i=1}^n (\hat{u}_i^{\mathrm{DN}} - u_i)\psi_k^*(w_i) = (\hat{\beta}^{\mathrm{DN}} - \beta)^\top \sum_{i=1}^n X_i \psi_k^*(w_i) = O_{\mathbb{P}}\left(\frac{\sqrt{n}\max\{q_n^{\mathrm{pre}}, q_n^{\mathrm{post}}\}}{\sqrt{q_n^{\mathrm{pre}}}}\right)$.

- For the third term, by Lemma C.2

$$\begin{aligned}\sum_{i=1}^n u_i(\hat{\psi}_{ki}^R - \psi_k^*(w_i)) =& \sum_{i=1}^n \left(\sum_{k=1}^r \mu_k^u \psi_{ki}^*\right)(\hat{\psi}_{ki}^R - \psi_k^*(w_i)) + \sum_{i=1}^n u_i^{\mathrm{DN}}(\hat{\psi}_{ki}^R - \psi_k^*(w_i)) \\ =& O_{\mathbb{P}}\left(\frac{1}{\sqrt{q_n^{\mathrm{pre}}}}\left(\frac{\log n}{\log\log n}\right)^{1/4}\right).\end{aligned}$$

Now we show the consistency of the variance estimators one by one.

(1): Consistency of the $(1,1)$ element of $\hat{V}_{\mathrm{num}}^{\mathrm{DN}}$. The proof is analogous to that of Theorem 3.2 and follows from $\hat{\beta}^{\mathrm{DN}} - \beta = o_{\mathbb{P}}(1)$.

(2): Consistency of the $(3,3)$ element of $\hat{V}_{\mathrm{num}}^{\mathrm{DN}}$. By definition, $\hat{\eta}_i^{\hat{u}} = \hat{u}_i^{\mathrm{DN}} - \sum_{k=1}^r \hat{\mu}_k^{\hat{u}}\hat{\psi}_{ki} = \eta_i^u + (\hat{u}_i^{\mathrm{DN}} - u_i) - \Delta_i$, where $\eta_i^u = u_i - \sum_{k=1}^r \mu_k^u \psi_k^*(w_i)$ and $\Delta_i = \sum_{k=1}^r ((\hat{\mu}_k^{\hat{u}})^R \hat{\psi}_{ki}^R - \mu_k^u \psi_k^*(w_i))$. We decompose the variance estimator as:

$$\begin{aligned}\frac{1}{n^2 q_n^{\mathrm{pre}}}\sum_{i=1}^n \left(\sum_{j=1}^n A_{ij}^{\mathrm{pre}}\hat{\eta}_j^{\hat{u}}\right)^2 =& \underbrace{\frac{1}{n^2 q_n^{\mathrm{pre}}}\sum_{i=1}^n \left(\sum_{j=1}^n A_{ij}^{\mathrm{pre}}\eta_j^u\right)^2}_{S_1} + \underbrace{\frac{1}{n^2 q_n^{\mathrm{pre}}}\sum_{i=1}^n \left(\sum_{j=1}^n A_{ij}^{\mathrm{pre}}(\hat{\eta}_j^{\hat{u}} - \eta_j^u)\right)^2}_{S_2} \\ &+ \underbrace{\frac{2}{n^2 q_n^{\mathrm{pre}}}\sum_{i=1}^n \left(\sum_{j=1}^n A_{ij}^{\mathrm{pre}}\eta_j^u\right)\left(\sum_{j=1}^n A_{ij}^{\mathrm{pre}}(\hat{\eta}_j^{\hat{u}} - \eta_j^u)\right)}_{S_3}.\end{aligned}$$

We analyze each term separately.

S_1**:** Define $Q_i = \sum_{j=1}^n A_{ij}^{\mathrm{pre}}\eta_j^u$. Define $\mathbb{E}(A_{ij}^{\mathrm{pre}}) = P_{ij}^{\mathrm{pre}}$ and $R_n^{\mathrm{pre}} = \sum_{k>r}\lambda_k^*\psi_k^*\psi_k^{*\top}$. By direct

algebra, the expectation is:

$$\mathbb{E}\left[\frac{1}{n^2 q_n^{\text{pre}}}\sum_{i=1}^n Q_i^2\right] = \frac{1}{n^2 q_n^{\text{pre}}}\mathbb{E}\left[\sum_{(i,j)} P_{ij}(1-P_{ij}^{\text{pre}})\left(\eta_j^u\right)^2 + \sum_{i=1}^n\left(\sum_{j=1}^n P_{ij}^{\text{pre}}\eta_j^u\right)^2\right].$$

The first term is $O(1)$ by $\mathbb{E}((\eta_j^u)^2)$ bounded away from 0. For the second term,

$$\sum_{i=1}^n\left(\sum_{j=1}^n P_{ij}^{\text{pre}}\eta_j^u\right)^2 = \|P_n^{\text{pre}}\eta^u\|_2^2 \le \|R_n^{\text{pre}}\|_{\text{op}}^2\|\eta^u\|_2^2 = O_{\mathbb{P}}(n(q_n^{\text{pre}})^2)$$

by Assumption 3 and the projection orthogonality. Therefore, $\mathbb{E}\left[\frac{1}{n^2 q_n^{\text{pre}}}\sum_{i=1}^n Q_i^2\right] = O(1)$.

By expanding the variance: $\text{Var}\left(\sum_{i=1}^n Q_i^2\right) = \sum_{i=1}^n \text{Var}\left(Q_i^2\right) + \sum_{i\neq i'} Cov\left(Q_i^2, Q_{i'}^2\right)$.
For the diagonal term, $\sum_{i=1}^n \text{Var}\left(Q_i^2\right) \le \sum_{i=1}^n \mathbb{E}\left(Q_i^4\right)$. To bound the 4th-moment for sums of independent Bernoulli-weighted terms:

$$\mathbb{E}\left(Q_i^4\right) \le C\left(\mathbb{E}\left(\left(\sum_{j=1}^n P_{ij}^{\text{pre}}(\eta_j^u)^2\right)^2\right) + \mathbb{E}\left(\left(\sum_{j=1}^n P_{ij}^{\text{pre}}(\eta_j^u)^4\right)\right) + \mathbb{E}\left(\left(\sum_{j=1}^n P_{ij}^{\text{pre}}\eta_j^u\right)^4\right)\right).$$

Bound each term above:
- $\sum_{j=1}^n P_{ij}^{\text{pre}}(\eta_j^u)^2 = O_p(nq_n^{\text{pre}})$, so $\mathbb{E}\left(\left(\sum_{j=1}^n P_{ij}^{\text{pre}}(\eta_j^u)^2\right)^2\right) = O(n^2(q_n^{\text{pre}})^2)$
- $\sum_{j=1}^n P_{ij}^{\text{pre}}(\eta_j^u)^4 \le Cq_n^{\text{pre}}\sum_{j=1}^n(\eta_j^u)^4$, so $\mathbb{E}\left(\left(\sum_{j=1}^n P_{ij}^{\text{pre}}(\eta_j^u)^4\right)\right) = O(nq_n^{\text{pre}})$.
- $\left(\sum_{j=1}^n P_{ij}^{\text{pre}}\eta_j^u\right)^4 \le \|P_n^{\text{pre}}\eta^u\|_2^4 \le \|R_n^{\text{pre}}\|_{\text{op}}^4\|\eta^u\|_2^4 = O_p(n^2(q_n^{\text{pre}})^4)$, so $\mathbb{E}\left(\left(\sum_{j=1}^n P_{ij}\eta_j^u\right)^4\right) = O(n^2(q_n^{\text{pre}})^4)$.

So overall, $\mathbb{E}\left(Q_i^4\right) = O_p(n^2(q_n^{\text{pre}})^2)$ and $\sum_{i=1}^n \text{Var}\left(Q_i^2\right) = O_{\mathbb{P}}(n^3(q_n^{\text{pre}})^2)$.
For the off-diagonal term of the variance, the covariance is non-zero only when the index sets have overlaps. By expanding Q_i^2 and the covariance, we bound
- $\sum_{(i,i',j,l,l')}\mathbb{E}(P_{ij}^{\text{pre}}P_{il}^{\text{pre}}\eta_j^u\eta_l^u P_{i'j}^{\text{pre}}P_{i'l'}^{\text{pre}}\eta_j^u\eta_{l'}^u) = O_{\mathbb{P}}\left(n^3(q_n^{\text{pre}})^4\right)$.
- $\sum_{(i,l)}\mathbb{E}(P_{ij}^{\text{pre}}P_{il}^{\text{pre}}\eta_j^u\eta_l^u) = O_{\mathbb{P}}\left(n(q_n^{\text{pre}})^2\right)$.

Therefore, $\text{Var}\left(\sum_{i=1}^n Q_i^2\right) = O_{\mathbb{P}}(n^3(q_n^{\text{pre}})^2)$. Thus,

$$\frac{1}{n^2 q_n^{\text{pre}}}\sum_{i=1}^n\left(\sum_{j=1}^n A_{ij}^{\text{pre}}\eta_j^u\right)^2 = \mathbb{E}\left[\frac{1}{n^2 q_n^{\text{pre}}}\sum_{i=1}^n\left(\sum_{j=1}^n A_{ij}^{\text{pre}}\eta_j^u\right)^2\right] + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).$$

S_2**:** For S_2, we decompose it as:

$$\sum_{j=1}^{n} A_{ij}^{\mathrm{pre}}(\hat{\eta}_j^{\hat{u}} - \eta_j^u) = \sum_{j=1}^{n} A_{ij}^{\mathrm{pre}}(\hat{\eta}_j^{\hat{u}} - \eta_j^u) - \sum_{j=1}^{n} A_{ij}^{\mathrm{pre}}\eta_j^u = \sum_{j=1}^{n} \hat{R}_{ij}^{\mathrm{pre}}(\hat{\eta}_j^{\hat{u}} - \eta_j^u) - \sum_{j=1}^{n}(A^{\mathrm{pre}})_{ij}^r\eta_j^u.$$

For the first term, we first analyze $\|\hat{u}^{\mathrm{DN}} - u^{\mathrm{DN}}\|_2^2$. By Cauchy-Schwarz,

$$\left\|\hat{\eta}^{\hat{u}} - \eta^u\right\|_2^2 = \|(\hat{u}^{\mathrm{DN}} - u) - \Delta\|_2^2 \leq 2\left\|\hat{u}^{\mathrm{DN}} - u\right\|_2^2 + 2r\left\|(\hat{\mu}_k^{\hat{u}})^R\hat{\psi}_k^R - \mu_k^u\psi_k^*\right\|_2^2.$$

It is bounded as:

$$\|\hat{u}^{\mathrm{DN}} - u\|_2^2 \leq \sum_{k=0}^{2}(\hat{\beta}_k^{\mathrm{DN}} - \beta_k)^2\sum_{j=1}^{n} X_{jk}^2 = O_{\mathbb{P}}\left(\frac{n\max\{q_n^{\mathrm{pre}}, q_n^{\mathrm{post}}\}^2}{q_n^{\mathrm{pre}}}\right), \tag{45}$$

and

$$\begin{aligned} C\left\|(\hat{\mu}_k^{\hat{u}})^R\hat{\psi}_k^R - \mu_k^u\psi_k^*\right\|_2^2 \leq & ((\hat{\mu}_k^{\hat{u}})^R - \mu_k^u)^2\left\|\hat{\psi}_k^R - \psi_k^*\right\|_2^2 + ((\hat{\mu}_k^{\hat{u}})^R - \mu_k^u)^2 + (\mu_k^u)^2\left\|\hat{\psi}_k^R - \psi_k^*\right\|_2^2 \\ = & O_{\mathbb{P}}\left(\frac{n\max\{q_n^{\mathrm{pre}}, q_n^{\mathrm{post}}\}^2}{q_n^{\mathrm{pre}}}\right) \end{aligned}$$

using

$$(\hat{\mu}_k^{\hat{u}})^R\hat{\psi}_{ki}^R - \mu_k^u\psi_{ki}^* = ((\hat{\mu}_k^{\hat{u}})^R - \mu_k^u)(\hat{\psi}_{ki}^R - \psi_{ki}^*) + ((\hat{\mu}_k^{\hat{u}})^R - \mu_k^u)\psi_{ki}^* + \mu_k^u(\hat{\psi}_{ki}^R - \psi_{ki}^*). \tag{46}$$

Therefore,

$$\left\|\hat{\eta}^{\hat{u}} - \eta^u\right\|_2^2 = O_{\mathbb{P}}\left(\frac{n\max\{q_n^{\mathrm{pre}}, q_n^{\mathrm{post}}\}^2}{q_n^{\mathrm{pre}}}\right) = o_{\mathbb{P}}(n). \tag{47}$$

Then, by the operator norm bound,

$$\begin{aligned} \frac{1}{n^2 q_n^{\mathrm{pre}}}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\hat{R}_{ij}^{\mathrm{pre}}(\hat{\eta}_j^{\hat{u}} - \eta_j^u)\right)^2 \leq & \frac{1}{n^2 q_n^{\mathrm{pre}}}\left\|\hat{R}_{ij}^{\mathrm{pre}}\right\|_{\mathrm{op}}^2\left\|\hat{\eta}^{\hat{u}} - \eta^u\right\|_2^2 \\ = & O_{\mathbb{P}}\left(\frac{nq_n^{\mathrm{pre}}\sqrt{\frac{\log(n)}{\log(\log(n))}}}{n^2 q_n^{\mathrm{pre}}}\frac{n\max\{q_n^{\mathrm{pre}}, q_n^{\mathrm{post}}\}^2}{q_n^{\mathrm{pre}}}\right) = o_{\mathbb{P}}(1), \end{aligned}$$

which follows from

$$\left\|\hat{R}^{\mathrm{pre}}\right\|_{\mathrm{op}} = \|A - (A^{\mathrm{pre}})^r\|_{\mathrm{op}} \leq \left\|A - q_n^{\mathrm{pre}}\tilde{G}_n^{\mathrm{pre}}\right\|_{\mathrm{op}} = O_{\mathbb{P}}\left(\sqrt{nq_n^{\mathrm{pre}}}\left(\frac{\log n}{\log\log n}\right)^{1/4}\right). \tag{48}$$

For the second term, we have

$$\sum_{j=1}^{n}(A^{\text{pre}})^{r}_{ij}\eta^{u}_{j}=\sum_{k=1}^{r}\hat{\lambda}_{k}\hat{\psi}_{ki}\sum_{j=1}^{n}(\hat{\psi}_{kj}-\psi^{*}_{kj})\eta^{u}_{j}$$

by the orthogonality of η^{u}_{j} and ψ^{*}_{kj} for $k\neq r$. Given $\sum_{i=1}^{n}\eta^{u}_{j}\psi^{*}_{kj}=0$ for $k\leq r$, we expand u^{DN}_{j} in the basis $\{\psi^{*}_{kj}\}_{k=r+1}^{n}$, i.e., $\eta^{u}_{j}=\sum_{k>r}\alpha_{k}\psi^{*}_{kj}$ with $\sum_{k=r+1}^{n}\alpha_{k}^{2}=n$. Thus, $\eta^{u}_{j}=\sum_{k=r+1}^{n}\alpha_{k}\psi^{*}_{kj}$. Then we bound

$$\sum_{j=1}^{n}(\hat{\psi}_{kj}-\psi^{*}_{kj})\eta^{u}_{j}=\sum_{\ell=r+1}^{n}\alpha_{\ell}\psi^{*}_{\ell}\big(\hat{\psi}^{R}_{k}-\psi^{*}_{k}\big).$$

Therefore,

$$\left(\sum_{i=1}^{n}\hat{\eta}^{u}_{i}\big(\hat{\psi}^{R}_{ki}-\psi^{*}_{k}(w_{i})\big)\right)^{2}=\left(\sum_{\ell=r+1}^{n}\alpha_{\ell}\psi^{*}_{\ell}\big(\hat{\psi}^{R}_{k}-\psi^{*}_{k}\big)\right)^{2}=O_{\mathbb{P}}\left(n\left(\tfrac{1}{nq^{\text{pre}}_{n}}\sqrt{\tfrac{\log n}{\log\log n}}\right)^{2}\right).$$

By combining the above results, we have

$$\begin{aligned}\frac{1}{n^{2}q^{\text{pre}}_{n}}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}(A^{\text{pre}})^{r}(\hat{\eta}^{\hat{u}}_{j}-\eta^{u}_{j})\right)^{2}=&\frac{1}{n^{2}q^{\text{pre}}_{n}}\sum_{i=1}^{n}\left(\sum_{k=1}^{r}\hat{\lambda}_{k}\hat{\psi}_{ki}\sum_{j=1}^{n}(\hat{\psi}_{kj}-\psi^{*}_{kj})\eta^{u}_{j}\right)^{2}\\ \leq&\frac{1}{n^{2}q^{\text{pre}}_{n}}\sum_{i=1}^{n}\sum_{k=1}^{r}\hat{\lambda}_{k}^{2}\hat{\psi}_{ki}^{2}\left(\sum_{j=1}^{n}(\hat{\psi}_{kj}-\psi^{*}_{kj})\eta^{u}_{j}\right)^{2}\\ =&O_{\mathbb{P}}\left(\frac{n^{2}(q^{\text{pre}}_{n})^{2}}{n^{2}q^{\text{pre}}_{n}}n\left(\tfrac{1}{nq^{\text{pre}}_{n}}\sqrt{\tfrac{\log n}{\log\log n}}\right)^{2}\right)=o_{\mathbb{P}}(1).\end{aligned}\tag{49}$$

Therefore, we show that $S_{2}=o_{\mathbb{P}}(1)$.

S_{3}**:** $S_{3}=o_{\mathbb{P}}(1)$ follows from $S_{1}=O_{\mathbb{P}}(1)$ and $S_{2}=o_{\mathbb{P}}(1)$.

Therefore, S_{1}–S_{3} together imply the desired result:

$$\frac{1}{n^{2}q^{\text{pre}}_{n}}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}A^{\text{pre}}_{ij}\hat{\eta}^{\hat{u}}_{j}\right)^{2}=\mathbb{E}\left[\frac{1}{n^{2}q^{\text{pre}}_{n}}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}A^{\text{pre}}_{ij}\eta^{u}_{j}\right)^{2}\right]+o_{\mathbb{P}}(1).$$

(3): Consistency of the $(2,3)$ element of $\hat{V}^{\text{DN}}_{\text{num}}$. We decompose the term as:

$$\frac{1}{n\sqrt{nq^{\text{pre}}_{n}}}\sum_{(i,j)}A^{\text{pre}}_{ij}\hat{u}^{\text{DN}}_{i}\hat{\eta}^{\hat{u}}_{j}=\frac{1}{n\sqrt{nq^{\text{pre}}_{n}}}\left[\sum_{(i,j)}A^{\text{pre}}_{ij}u_{i}\eta^{u}_{j}+\sum_{(i,j)}A^{\text{pre}}_{ij}u_{i}(\hat{\eta}^{\hat{u}}_{j}-\eta^{u}_{j})+\sum_{(i,j)}A^{\text{pre}}_{ij}(\hat{u}^{\text{DN}}_{i}-u_{i})\hat{\eta}^{\hat{u}}_{j}\right]$$

$$=: S_1 + S_2 + S_3.$$

For S_2, similar to the proof of (49), we can show that $S_2 = o_{\mathbb{P}}(1)$. For S_3,

$$\frac{1}{n\sqrt{nq_n^{\text{pre}}}} \sum_{(i,j)} A_{ij}^{\text{pre}} (\hat{u}_i^{\text{DN}} - u_i) \hat{\eta}_j^{\hat{u}} = \frac{1}{n\sqrt{nq_n^{\text{pre}}}} \sum_{k=0}^{2} (\hat{\beta}_k^{\text{DN}} - \beta_k) \sum_{(i,j)} A_{ij}^{\text{pre}} X_i \hat{\eta}_j^{\hat{u}}.$$

Since

$$\begin{aligned}\left| \sum_{i=1}^{n} X_{ik} \sum_{j=1}^{n} A_{ij}^{\text{pre}} \hat{\eta}_j^{\hat{u}} \right| =& \left| X_k^\top \hat{R}^{\text{pre}} \hat{\eta}^{\hat{u}} \right| \leq \|X_k\|_2 \left\| A^{\text{pre}} - q_n^{\text{pre}} \tilde{G}_n^{\text{pre}} \right\|_{\text{op}} \left\| \hat{\eta}^{\hat{u}} \right\|_2 \\ =& O_{\mathbb{P}} \left(\sqrt{n} \sqrt{nq_n^{\text{pre}}} \left(\frac{\log n}{\log \log n} \right)^{1/4} \sqrt{n} \right) = O_{\mathbb{P}} \left(n \sqrt{nq_n^{\text{pre}}} \left(\frac{\log n}{\log \log n} \right)^{1/4} \right).\end{aligned}$$

Therefore, $S_3 = o_{\mathbb{P}}(1)$ by $\hat{\beta}^{\text{DN}} - \beta = o_{\mathbb{P}}(1)$.

S_1**:** We show that $S_1 = o_{\mathbb{P}}(1)$ by bounding its expectation and variance:

$$\begin{aligned}\mathbb{E}(S_1) =& \frac{1}{n\sqrt{nq_n^{\text{pre}}}} \mathbb{E} \left(\sum_{(i,j)} P_{ij} u_i \eta_j^u \right) = \frac{1}{n\sqrt{nq_n^{\text{pre}}}} \mathbb{E} \left(\sum_{(i,j)} R_{ij} u_i \eta_j^u \right) \\ =& \frac{1}{n\sqrt{nq_n^{\text{pre}}}} \mathbb{E} \left(\sum_{k>r} \lambda_k^* \sum_{i=1}^{n} \psi_{ki}^* u_i \sum_{j=1}^{n} \psi_{kj}^* \eta_j^u \right) = O\left(\sqrt{\frac{q_n^{\text{pre}}}{n}} \right) = o(1)\end{aligned}$$

and

$$\text{Var}(S_1) \leq \frac{1}{n^3 q_n^{\text{pre}}} \sum_{(i,j)} \mathbb{E}\left(A_{ij}^{\text{pre}} u_i^2 (\eta_j^u)^2 \right) + \frac{1}{n^3 q_n^{\text{pre}}} \sum_{(i,j,i',j')} \text{Cov}\left(P_{ij}^{\text{pre}} u_i \eta_j^u, P_{i'j'}^{\text{pre}} u_{i'} \eta_{j'}^u \right) = O\left(\frac{1}{n} \right).$$

Therefore, $S_1 = \frac{1}{n\sqrt{nq_n^{\text{pre}}}} \mathbb{E}\left(\sum_{(i,j)} A_{ij} u_i \eta_j^u \right) + o_{\mathbb{P}}\left(\sqrt{\frac{q_n^{\text{pre}}}{n}} \right)$. Overall, we show that

$$\frac{1}{n\sqrt{nq_n^{\text{pre}}}} \sum_{(i,j)} A_{ij}^{\text{pre}} \hat{u}_i^{\text{DN}} \hat{\eta}_j^{\hat{u}} = o_{\mathbb{P}}(1).$$

Step 3: Combine the results. By Steps 1–2 and CMT, $(\hat{V}_{\text{num}}^{\text{DN}})^{-1/2} ((\tilde{Z}^{\text{DN}})^\top u) \xrightarrow{d} \mathcal{N}(0, I_3)$. Applying Theorem C.1 and CMT again completes the proof.

□

Online Appendix to "Endogenous Interference in Randomized Experiments"

This document has two sections. Section D contains proof of lemmas omitted from the paper. Section E presents results of IV estimation with normalized SSIV.

D Proof of Auxiallary Lemmas

D.1 Proof of Lemmas in Appendix B

Proof of Lemma B.1. We first show (23). By C_r inequality, we have

$$\mathbb{E}\left[(r_{0,i} + r_{1,i} - \mu_{r_1,i})^2\right] \leq 2^3 \left[\mathbb{E}(r_{0,i}^2) + \mathbb{E}(r_{1,i}^2) + \mathbb{E}(\mu_{r_1,i}^2)\right]. \tag{D.1}$$

Bound for $\mathbb{E}[r_{0,i}^2]$: By $\mathbb{E}[R_{ij} \mid T_i, w_i] = 0$:

$$\mathbb{E}\left[r_{0,i}^2 \mid T_i, w_i\right] = \frac{\xi_i^2}{(n-1)^2}\mathbb{E}\left[\left(\sum_{j\neq i} R_{ij}\right)^2 \mid T_i, w_i\right] = \frac{\xi_i^2}{(n-1)^2}\sum_{j\neq i}\mathbb{E}\left[R_{ij}^2 \mid T_i, w_i\right].$$

Recall the definition of R_{ij} in (19) and moment bounds on the treatment and adjacency terms, we obtain $\mathbb{E}[r_{0,i}^2] \leq \frac{C}{nq_n^{\text{post}}}$.

Bound for $\mathbb{E}[r_{1,i}^2]$: Define $O_{ij} = U_{ij}W_{ij}$ and $V_{ij} = W_{ij}^2$. Expanding $r_{1,i}^2$ and applying the AM–GM inequality:

$$r_{1,i}^2 \leq \frac{2^3}{(n-1)^4}\left[\left(\frac{1}{\theta_i^*(y)^2}\sum_{j\neq i} O_{ij}\right)^2 + \left(\frac{2\theta_i^*(x)}{\theta_i^*(y)^3}\sum_{j\neq i} V_{ij}\right)^2 + \left(\frac{1}{\theta_i^*(y)^2}\sum_{(j,l)} U_{ij}W_{il}\right)^2 + \left(\frac{2\theta_i^*(x)}{\theta_i^*(y)^3}\sum_{(j,l)} W_{ij}W_{il}\right)^2\right].$$

For the first term, we expand $\mathbb{E}\left[\left(\sum_{j\neq i} O_{ij}\right)^2 \mid T_i, w_i\right] = \mathbb{E}\left[\left(\sum_{j\neq i} O_{ij}^2 + \sum_{j_1\neq j_2} O_{ij_1}O_{ij_2}\right) \mid T_i, w_i\right]$.
Each moment is bounded by $C \cdot q_n^{\text{post}}$, so: $\mathbb{E}\left[\left(\sum_{j\neq i} O_{ij}\right)^2\right] \leq C(nq_n^{\text{post}})^2$.

For the third term, we expand: $\mathbb{E}\left[\left(\sum_{j\neq l} U_{ij}W_{il}\right)^2 \mid T_i, w_i\right] = \mathbb{E}\left[\left(\sum_{j\neq l} U_{ij}^2 W_{il}^2\right) \mid T_i, w_i\right]$ by the definition of U_{ij} and W_{ij}. It yields: $\mathbb{E}\left[\left(\sum_{j\neq l} U_{ij}W_{il}\right)^2\right] \leq C(nq_n^{\text{post}})^2$. The second term and the final term are bounded analogously. Hence, $\mathbb{E}[r_{1,i}^2] \leq \frac{C}{(nq_n^{\text{post}})^2}$.

Bound for $\mathbb{E}[\mu_{r_1,i}^2]$: From the definition of $\mu_{r_1,i}$, we have $\mathbb{E}[\mu_{r_1,i}^2] = O\left(\frac{1}{(nq_n^{\text{post}})^2}\right)$. Combining the terms in (D.1), we complete showing (23).

We complete the proof of (24) by showing the following terms. To simplify presentation, we define $E_{ij} = -\frac{U_{ij}W_{ij}}{\theta_i^*(y)^2} + \frac{2\theta_i^*(x)}{\theta_i^*(y)^3}W_{ij}^2$ and $E_{ijk} = -\frac{U_{ij}W_{ik}}{\theta_i^*(y)^2} + \frac{2\theta_i^*(x)}{\theta_i^*(y)^3}W_{ij}W_{ik}$.
(a) The first term is nonzero when R_{ik} and R_{jk} share a common second index k:

$$\text{Cov}(a_i r_{0,i},\, b_j r_{0,j}) = \frac{1}{(n-1)^2}\sum_{k\neq i,j} \text{Cov}(a_i\xi_i R_{ik},\, b_j\xi_j R_{jk}) = O\left(\tfrac{1}{n}\right).$$

(b) The second term is nonzero when the two sides share a common index:

$$\text{Cov}(a_i r_{0,i},\, b_j r_{1,j}) = \frac{1}{(n-1)^3}\sum_{k\neq i,j}\left[\ \text{Cov}\left(a_i\xi_i R_{ik},\, b_j E_{jk}\right) + \text{Cov}\left(a_i\xi_i R_{ik},\, b_j E_{jki}\right)\ \right] = O\left(\tfrac{1}{n^2 q_n^{\text{post}}}\right).$$

(c) The last term is nonzero when the two sides share a common index:

$$\begin{aligned}\text{Cov}(a_i r_{1,i},\, b_j r_{1,j}) &= \frac{1}{(n-1)^4}\left[\begin{array}{c}\sum_{k\neq i,j}\text{Cov}\left(a_i E_{ik}, b_j E_{jk}\right) + \sum_{k\neq i,j}\text{Cov}\left(a_i E_{ik}, b_j E_{jki}\right)\\ +\sum_{k\neq i,j}\text{Cov}\left(a_i E_{ikj}, b_j E_{jk}\right) + \sum_{(k,l):k\neq l}\text{Cov}\left(a_i E_{ikl}, b_j E_{jkl}\right)\end{array}\right]\\ &= O\left(\tfrac{1}{n^3(q_n^{\text{post}})^2}\right) + O\left(\tfrac{1}{n^2}\right).\end{aligned}$$

We omit the proof for the remaining terms involving $\mu_{r_1,i}$, which can be shown by (22). Combining results from parts (a)–(c) above, we conclude (24).

□

Proof of Lemma B.3. By index reordering,

$$\frac{1}{n^2 q_n^{\text{pre}}}\sum_{i=1}^n Z_i^{\text{SSIV}}\phi_i = \frac{1}{n^2 q_n^{\text{pre}}}\sum_{i=1}^n\sum_{j\neq i} A_{ij}^{\text{pre}}(T_j-\pi)\phi_i = \frac{1}{n}\sum_{j=1}^n (T_j-\pi)\frac{1}{nq_n^{\text{pre}}}\sum_{i\neq j} A_{ij}^{\text{pre}}\phi_i.$$

Given w_j, the terms $\frac{A_{ij}^{\text{pre}}\phi_i}{q_n^{\text{pre}}}$ are i.i.d.. Then we have

$$\mathbb{E}\left[\left(\sum_{i\neq j}\frac{A_{ij}^{\text{pre}}\phi_i}{(n-1)q_n^{\text{pre}}}-Q_j^{\phi}\right)^2\right]=\frac{1}{n-1}\mathbb{E}\left[\left(\frac{A_{ij}^{\text{pre}}\phi_i}{q_n^{\text{pre}}}-Q_j^{\phi}\right)^2\right]\leq\frac{1}{(n-1)q_n^{\text{pre}}}\mathbb{E}\left[\frac{A_{ij}^{\text{pre}}\phi_i^2}{q_n^{\text{pre}}}\right]\leq\frac{C}{(n-1)q_n^{\text{pre}}}.$$

We complete the proof by showing that

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{j=1}^{n}(T_j-\pi)\left(\sum_{i\neq j}\frac{A_{ij}^{\text{pre}}\phi_i}{nq_n^{\text{pre}}}-Q_j^{\phi}\right)\right)^2\right]=\frac{\pi(1-\pi)}{n^2}\sum_{j=1}^{n}\mathbb{E}\left[\left(\sum_{i\neq j}\frac{A_{ij}^{\text{pre}}\phi_i}{nq_n^{\text{pre}}}-Q_j^{\phi}\right)^2\right]\leq\frac{C}{n^2q_n^{\text{pre}}}.$$

□

Proof of Lemma B.4. To show (28), we decompose it into

$$\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}M_i=\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}(\xi_i+\mu_{r_1,i})+\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}(r_{0,i}+r_{1,i}-\mu_{r_1,i}).$$

For the first term, we can show by Lemma B.3 and (22) that

$$\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}\xi_i=O_{\mathbb{P}}\left(\sqrt{n}q_n^{\text{pre}}\right)\text{ and }\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}\mu_{r_1,i}=O_{\mathbb{P}}\left(\frac{q_n^{\text{pre}}}{\sqrt{n}q_n^{\text{post}}}\right).\tag{D.2}$$

For $\underline{\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}r_{0,i}}$, the expectation satisfies

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}r_{0,i}\right]=\frac{1}{n(n-1)}\sum_{i=1}^{n}\mathbb{E}\left[\xi_i\sum_{k\neq i}A_{ik}^{\text{pre}}(T_k-\pi)R_{ik}\right]\leq C\cdot\frac{q_n^{\text{pre}}}{\max\{q_n^{\text{pre}},q_n^{\text{post}}\}}.$$

The variance is $\text{Var}\left(\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}r_{0,i}\right)=\frac{1}{n^2}\sum_{i=1}^{n}\text{Var}\left(Z_i^{\text{SSIV}}r_{0,i}\right)+\frac{1}{n^2}\sum_{i\neq j}\text{Cov}\left(Z_i^{\text{SSIV}}r_{0,i},Z_j^{\text{SSIV}}r_{0,j}\right)$. For the diagonal terms,

$$\begin{aligned}\text{Var}\left(Z_i^{\text{SSIV}}r_{0,i}\right)&\leq\mathbb{E}\left[(Z_i^{\text{SSIV}})^2r_{0,i}^2\right]=\frac{\xi_i^2}{(n-1)^2}\mathbb{E}\left[\left(\sum_{j\neq i}A_{ij}^{\text{pre}}(T_j-\pi)\right)^2\left(\sum_{j\neq i}R_{ij}\right)^2\right]\\&=\frac{\xi_i^2}{(n-1)^2}\mathbb{E}\left[\sum_{(j,k)}A_{ij}^{\text{pre}}A_{ik}^{\text{pre}}(T_j-\pi)(T_k-\pi)\sum_{(\ell,m)}R_{i\ell}R_{im}\right]\leq C\cdot\frac{q_n^{\text{pre}}}{q_n^{\text{post}}}.\end{aligned}\tag{D.3}$$

For the off-diagonal terms,

$$\text{Cov}\left(Z_i^{\text{SSIV}}r_{0,i},Z_j^{\text{SSIV}}r_{0,j}\right)=\frac{1}{(n-1)^2}\text{Cov}\left(\xi_i\sum_{k\neq i}A_{ik}^{\text{pre}}(T_k-\pi)\sum_{k\neq i}R_{ik},\ \xi_j\sum_{k\neq j}A_{jk}^{\text{pre}}(T_k-\pi)\sum_{k\neq j}R_{jk}\right)$$

$$= O\left(\frac{(q_n^{\text{pre}})^2}{n\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}^2}\right). \tag{D.4}$$

Combining both terms: $\text{Var}\left(\frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}} r_{0,i}\right) = O\left(\frac{q_n^{\text{pre}}}{nq_n^{\text{post}}}\right)$. Therefore,

$$\frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}} r_{0,i} = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}} r_{0,i}\right] + O_{\mathbb{P}}\left(\frac{\sqrt{q_n^{\text{pre}}}}{\sqrt{nq_n^{\text{post}}}}\right). \tag{D.5}$$

For $\frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}} r_{1,i}$, the expectation satisfies

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}} r_{1,i}\right] = \frac{1}{n(n-1)^2}\sum_{i=1}^n \mathbb{E}\left[\sum_{k\neq i} A_{ik}^{\text{pre}}(T_k - \pi)\left(-\frac{U_{ik}W_{ik}}{\theta_i^*(y)^2} + \frac{2\theta_i^*(x)}{\theta_i^*(y)^3}W_{ik}^2\right)\right] \leq C\cdot\frac{\min\{q_n^{\text{pre}}, q_n^{\text{post}}\}}{n(q_n^{\text{post}})^2}.$$

The variance is $\text{Var}\left(\frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}} r_{1,i}\right) = \frac{1}{n^2}\sum_{i=1}^n \text{Var}\left(Z_i^{\text{SSIV}} r_{1,i}\right) + \frac{1}{n^2}\sum_{i\neq j}\text{Cov}\left(Z_i^{\text{SSIV}} r_{1,i}, Z_j^{\text{SSIV}} r_{1,j}\right)$. For the diagonal terms, similar to the argument of (D.3), $\text{Var}\left(Z_i^{\text{SSIV}} r_{1,i}\right) = O\left(\frac{q_n^{\text{pre}}}{n(q_n^{\text{post}})^2}\right)$. For the off-diagonal terms, similar to the argument of (D.4), $\text{Cov}\left(Z_i^{\text{SSIV}} r_{1,i}, Z_j^{\text{SSIV}} r_{1,j}\right) = O\left(\frac{(q_n^{\text{pre}})^2}{n(q_n^{\text{post}})^2}\right)$. Therefore, $\text{Var}\left(\frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}} r_{1,i}\right) = O\left(\frac{q_n^{\text{pre}}}{n^2(q_n^{\text{post}})^2}\right)$. Hence,

$$\frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}} r_{1,i} = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}} r_{1,i}\right] + O_{\mathbb{P}}\left(\frac{\sqrt{q_n^{\text{pre}}}}{nq_n^{\text{post}}}\right) + O_{\mathbb{P}}\left(\frac{q_n^{\text{pre}}}{\sqrt{n}q_n^{\text{post}}}\right). \tag{D.6}$$

Combining (D.5) and (D.6), we obtain

$$\frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}}(r_{0,i} + r_{1,i}) = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}}(r_{0,i} + r_{1,i})\right] + O_{\mathbb{P}}\left(\frac{\sqrt{q_n^{\text{pre}}}}{\sqrt{nq_n^{\text{post}}}}\right) + O_{\mathbb{P}}\left(\frac{q_n^{\text{pre}}}{\sqrt{n}q_n^{\text{post}}}\right). \tag{D.7}$$

Therefore, (D.2) and (D.7) together imply (28).

To show (29), we can rewrite the expression as

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}} M_i - \overline{Z^{\text{SSIV}}}\cdot\overline{M} = {} & \frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}}(\xi_i + \mu_{r_1,i}) - \overline{Z^{\text{SSIV}}}\frac{1}{n}\sum_{i=1}^n(\xi_i + \mu_{r_1,i}) \\ & + \frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}}(r_{0,i} + r_{1,i} - \mu_{r_1,i}) - \overline{Z^{\text{SSIV}}}\frac{1}{n}\sum_{i=1}^n(r_{0,i} + r_{1,i} - \mu_{r_1,i}).\end{aligned}$$

By the argument in the proof of Lemma B.4 (see (D.7) for details), we show that

$$\frac{1}{n}\sum_{i=1}^n Z_i^{\text{SSIV}}(r_{0,i} + r_{1,i} - \mu_{r_1,i}) = \mathbb{E}\left[Z_i^{\text{SSIV}}(r_{0,i} + r_{1,i} - \mu_{r_1,i})\right] + O_{\mathbb{P}}\left(\frac{q_n^{\text{pre}}}{\sqrt{n}q_n^{\text{post}}}\right)$$

where $\mathbb{E}[Z_i^{\text{SSIV}}(r_{0,i}+r_{1,i}-\mu_{r_1,i})] \asymp \frac{q_n^{\text{pre}}}{\max\{q_n^{\text{pre}},q_n^{\text{post}}\}}$. By Lemmas B.2 and B.3, we have

$$\overline{Z^{\text{SSIV}}}\frac{1}{n}\sum_{i=1}^{n}(r_{0,i}+r_{1,i}-\mu_{r_1,i}) = O_{\mathbb{P}}\left(\frac{nq_n^{\text{pre}}}{\sqrt{n}}\right)O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right) = O_{\mathbb{P}}\left(\frac{q_n^{\text{pre}}}{\sqrt{nq_n^{\text{post}}}}\right).$$

For the sum of the first two terms, we consider Cases (a) and (b), respectively.
Case (a) By Lemma B.3, we have

$$\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}(\xi_i+\mu_{r_1,i}) - \overline{Z^{\text{SSIV}}}\frac{1}{n}\sum_{i=1}^{n}(\xi_i+\mu_{r_1,i}) = O_{\mathbb{P}}\left(\sqrt{n}q_n^{\text{pre}}\right).$$

Case (b) By Lemma B.3 and (22), we have

$$\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}(\xi_i+\mu_{r_1,i}) - \overline{Z^{\text{SSIV}}}\frac{1}{n}\sum_{i=1}^{n}(\xi_i+\mu_{r_1,i}) = O_{\mathbb{P}}\left(\frac{q_n^{\text{pre}}}{\sqrt{n}q_n^{\text{post}}}\right).$$

By combining these results, we complete the proof.

□

Proof of Lemma B.5. We omit the proof for (i) since it is analogous to that of (ii). For (ii), we bound it in L_1 norm: $\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}S_i^3\right) \le C\cdot\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}(O_i^\diamond+H_i^\diamond)^3\right]$, where $O_i^\diamond=\sum_j A_{ij}^\diamond$ is the degree of node i, and $H_i^\diamond=\sum_{(j,k)}A_{ik}^\diamond A_{kj}^\diamond$ counts 2-hop paths through neighbors. By the inequality $(a+b)^3\le 4(a^3+b^3)$, we have:

$$\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}S_i^3\right) \le 4\cdot\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}(O_i^\diamond)^3\right] + 4\cdot\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}(H_i^\diamond)^3\right]$$

where for the first term, we have $\mathbb{E}[(O_i^\diamond)^3]=\mathbb{E}[(\sum_j A_{ij}^\diamond)^3]\le C$ since $A_{ij}^\diamond\in\{0,1\}$ and entries are sparse, and for the second term, $\mathbb{E}[(H_i^\diamond)^3]=\mathbb{E}[(\sum_{(j,k)}A_{ik}^\diamond A_{kj}^\diamond)^3]\le C$ by expanding and bounding cross-products. Thus, $\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}S_i^3\right)\le C$.
For (iii), by definition: $(D^3)_{ij}=\sum_{k=1}^{n}(D^2)_{ik}D_{kj}=\sum_{(k,l)}D_{il}D_{lk}D_{kj}$. We show that

$$\sum_{i=1}^{n}\sum_{j\neq i}(D^3)_{ij} \le \sum_{i=1}^{n}\sum_{j\neq i}\sum_{(k,l)}\left(A_{il}^\diamond+\max_h A_{ih}^\diamond A_{hl}^\diamond\right)\left(A_{lk}^\diamond+\max_h A_{lh}^\diamond A_{hk}^\diamond\right)\left(A_{kj}^\diamond+\max_h A_{kh}^\diamond A_{hj}^\diamond\right)$$
$$\le \sum_{i=1}^{n}\sum_{j\neq i}\sum_{(k,l)}\left(A_{il}^\diamond+\sum_h A_{ih}^\diamond A_{hl}^\diamond\right)\left(A_{lk}^\diamond+\sum_h A_{lh}^\diamond A_{hk}^\diamond\right)\left(A_{kj}^\diamond+\sum_h A_{kh}^\diamond A_{hj}^\diamond\right) \le C(nq_n^\diamond)^6 \le C,$$

where the first inequality follows from $D_{il}\le A_{il}^\diamond+\max_h A_{ih}^\diamond A_{hl}^\diamond$, the second inequality follows from $\max_h X_h\le\sum_h X_h$ for nonnegative X_h and the third inequality follows from the fact

that each term contains six adjacency indicators. The other cases are handled similarly, so we omit them. By combining these results, we can show that $\frac{1}{n}\sum_{(i,j)}(D^3)_{ij} = o_{\mathbb{P}}(\sqrt{n})$. □

D.2 Proof of Lemmas in Appendix C

Proof of Lemma C.1. We first show (1). Suppose $q_n^{\text{pre}} \succcurlyeq \log(n)/n$. By Theorem 5.2 in Lei and Rinaldo (2015) with $d = nq_n^{\text{pre}}$, there exists some constant C such that $\|A^{\text{pre}} - q_n^{\text{pre}} G_n^{\text{pre}}\|_{\text{op}} \leq C\sqrt{nq_n^{\text{pre}}}$ with probability approaching to 1.

Suppose instead $\frac{\log n}{\log\log n}/n \prec q_n^{\text{pre}} \prec \frac{\log(n)}{n}$. By Corollary 3.3 in Benaych-Georges et al. (2020) and by setting their $\varepsilon^2 = \sqrt{\frac{\log n}{\log\log n}}/(nq_n^{\text{pre}})$, we have that with probability approaching to 1, $\|A^{\text{pre}} - q_n^{\text{pre}} G_n^{\text{pre}}\|_{\text{op}} \leq k\sqrt{nq_n^{\text{pre}}}\left(\frac{\log n}{\log\log n}\right)^{1/4}$, where k is a universal constant. For (2), by the triangle inequality, we have

$$\left\|A^{\text{pre}} - q_n^{\text{pre}}\tilde{G}_n^{\text{pre}}\right\|_{\text{op}} \leq \|A^{\text{pre}} - q_n^{\text{pre}} G_n^{\text{pre}}\|_{\text{op}} + \left\|\sum_{k=r+1}^{n} \lambda_k^* \psi_k^{*\top}\psi_k^*\right\|_{\text{op}} = O_{\mathbb{P}}\left(\sqrt{nq_n^{\text{pre}}}\left(\frac{\log n}{\log\log n}\right)^{1/4}\right).$$

□

Lemma D.1. *Suppose $q_n^{pre} \succ \frac{\log n}{\log\log n}/n$. Under Assumptions 1 and 3, for $k \in \{1, \cdots, r\}$, then*

$$\left|\hat{\lambda}_k - \lambda_k^*\right| = O_{\mathbb{P}}\left(\sqrt{nq_n^{pre}}\left(\frac{\log n}{\log\log n}\right)^{1/4}\right).$$

Proof. For (1), by Weyl's Inequality and Lemma C.1, we have

$$\left|\hat{\lambda}_k - \lambda_k^*\right| \leq \|A^{\text{pre}} - q_n^{\text{pre}} G_n^{\text{pre}}\|_{\text{op}} = O_{\mathbb{P}}\left(\sqrt{nq_n^{\text{pre}}}\left(\frac{\log n}{\log\log n}\right)^{1/4}\right)$$

By the proof of Lemma C.1, we can also show that when $q_n^{\text{pre}} \succcurlyeq \log(n)/n$,

$$\left|\hat{\lambda}_k - \lambda_k^*\right| = O_{\mathbb{P}}\left(\sqrt{nq_n^{\text{pre}}}\right).$$

□

Proof of Lemma C.2. By Lemma C.1 and Theorem 2 in Yu et al. (2015), we get that there exists an $r \times r$ orthogonal matrix $\hat{R}$ such that

$$\left\|\hat{\Psi}\hat{R} - \Psi^*\right\|_{\text{F}} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{nq_n^{\text{pre}}}}\left(\frac{\log n}{\log\log n}\right)^{1/4}\right). \tag{D.8}$$

Define $\hat{\Psi}^R = \hat{\Psi}\hat{R}$. Let $\hat{\psi}_k^R$ and ψ_k^* denote the k-th columns of $\hat{\Psi}^R$ and Ψ^*, respectively. Then

we have

$$\left\|\hat{\psi}_k^R - \psi_k^*\right\| = O_{\mathbb{P}}\left(\frac{1}{\sqrt{nq_n^{\text{pre}}}}\left(\frac{\log n}{\log\log n}\right)^{1/4}\right). \tag{D.9}$$

For $l \le r$, we will show that $\left\|(\Psi^*)^\top(\hat{\Psi}^R - \Psi^*)\right\|_{\text{F}}$ is small. Let $\hat{V}^\top V = O_1 D O_2^\top$ be the singular value decomposition of $\hat{V}^\top V$, then $\hat{O}$ is constructed by taking $\hat{O} = O_1 O_2^\top$. The construction of $\hat{O}$ ensures that $(\hat{V}\hat{O})^\top V$ is symmetric, i.e., $(\hat{V}\hat{O})^\top V = O_2 O_1^\top O_1 D O_2^\top = O_2 D O_2^\top$. This implies that $(\hat{\Psi}^R)^\top \Psi^*$ is symmetric. By definition,

$$\|\hat{V}\hat{O} - V\|_{\text{F}}^2 = \text{tr}\left(\hat{O}^\top\hat{V}^\top\hat{V}\hat{O} - (\hat{V}\hat{O})^\top V - V^\top\hat{V}\hat{O} + V^\top V\right) = 2\,\text{tr}(I_r - D).$$

If we write $(\hat{\Psi}^R)^\top\Psi^* = O_\Psi D_\Psi O_\Psi^\top$, then by (D.8), we have

$$\left\|\hat{\Psi}\hat{R} - \Psi^*\right\|_{\text{F}}^2 = \text{tr}\,(I_r - D_\Psi) = O_{\mathbb{P}}\left(\frac{1}{nq_n^{\text{pre}}}\sqrt{\frac{\log n}{\log\log n}}\right).$$

Note that $(\Psi^*)^\top(\hat{\Psi}^R - \Psi^*) = (\Psi^*)^\top\hat{\Psi}^R - I_r = O_\Psi D_\Psi O_\Psi^\top - I_r$. Therefore,

$$\left\|(\Psi^*)^\top\left(\hat{\Psi}^R - \Psi^*\right)\right\|_{\text{F}}^2 = \text{tr}\left(\left(O_\Psi D_\Psi O_\Psi^\top - I_r\right)^2\right) = \text{tr}\left((D_\Psi - I_r)^2\right) = O_{\mathbb{P}}\left(\left(\frac{1}{nq_n^{\text{pre}}}\sqrt{\frac{\log n}{\log\log n}}\right)^2\right).$$

It implies that (37) holds for any $k, l \in \{1, \ldots, r\}$.

For $l > r$, we have $(\psi_l^*)^\top(\hat{\Psi}^R - \Psi^*) = (\psi_l^*)^\top\hat{\Psi}^R$ by orthogonality. Note that

$$\left\|(\psi_l^*)^\top\hat{\Psi}_r^R\right\| = \left\|(\psi_l^*)^\top\hat{\Psi}_r\hat{R}\right\| = \left\|(\psi_l^*)^\top\hat{\Psi}_r\hat{\Lambda}_r\hat{R}\hat{R}^\top\hat{\Lambda}_r^{-1}\hat{R}\right\|$$

where $\hat{\Lambda}_r$ is the $r \times r$ diagonal matrix with $\hat{\lambda}_1, \ldots, \hat{\lambda}_r$ on its diagonal. We complete the proof by bounding $\|(\psi_l^*)^\top\hat{\Psi}_r\hat{\Lambda}_r\hat{R}\|$. Note that

$$(\psi_l^*)^\top\hat{\Psi}_r\hat{\Lambda}_r\hat{R} = (\psi_l^*)^\top A^{\text{pre}}\hat{\Psi}_r^R = (\psi_l^*)^\top(A^{\text{pre}} - q_n^{\text{pre}}\tilde{G}_n^{\text{pre}})(\hat{\Psi}_r^R - \Psi_r^*) + (\psi_l^*)^\top(A^{\text{pre}} - q_n^{\text{pre}}\tilde{G}_n^{\text{pre}})\Psi_r^*.$$

The first term can be easily bounded by

$$\begin{aligned}\left\|(\psi_l^*)^\top(A^{\text{pre}} - q_n^{\text{pre}}\tilde{G}_n^{\text{pre}})\left(\hat{\Psi}_r^R - \Psi_r^*\right)\right\| &\le \|\psi_l^*\|\left\|A^{\text{pre}} - q_n^{\text{pre}}\tilde{G}_n^{\text{pre}}\right\|_{\text{op}}\left\|\hat{\Psi}_r^R - \Psi_r^*\right\| \\ &= O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{\log\log n}}\right),\end{aligned} \tag{D.10}$$

where the last inequality follows from Lemma C.1 and (D.8). For the second term, we rewrite

it in the summation form:

$$(\psi_l^*)^\top \left(A^{\text{pre}} - q_n^{\text{pre}} \tilde{G}_n^{\text{pre}}\right) \psi_k^* = (\psi_l^*)^\top \left(A^{\text{pre}} - q_n^{\text{pre}} G_n^{\text{pre}}\right) \psi_k^* + (\psi_l^*)^\top \left(\sum_{h=r+1}^{n} \lambda_h^* \psi_h^{*\top} \psi_h^*\right) \psi_k^*$$
$$= \sum_{(i,j)} \psi_{li}^* \psi_{kj}^* \left(A_{ij}^{\text{pre}} - q_n^{\text{pre}} g^{\text{pre}}(w_i, w_j)\right).$$

As A_{ij}^{pre} are independent given w, we have:

$$\mathbb{E}\left[\left(\sum_{(i,j)} \psi_{li}^* \psi_{kj}^* \left(A_{ij}^{\text{pre}} - q_n^{\text{pre}} g_0(i,j)\right)\right)^2\right] = \sum_{(i,j)} \mathbb{E}\left[(\psi_{li}^*)^2 (\psi_{kj}^*)^2 \, \mathbb{E}\left[\left(A_{ij}^{\text{pre}} - q_n^{\text{pre}} g_0(i,j)\right)^2 | w\right]\right]$$
$$\leq \sum_{(i,j)} \mathbb{E}\left[(\psi_{li}^*)^2 (\psi_{kj}^*)^2 q_n^{\text{pre}} g_0(i,j)\right] \leq q_n^{\text{pre}} \mathbb{E}\left[\sum_{(i,j)} (\psi_{li}^*)^2 (\psi_{kj}^*)^2\right] \leq C q_n^{\text{pre}}. \tag{D.11}$$

Combining (D.10) and (D.11), we can show that $\| (\psi_l^*)^\top \hat{\Psi}_r \hat{\Lambda}_r \hat{R} \| = O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{\log\log n}}\right)$. Moreover, $\hat{\lambda}_k = \Omega_{\mathbb{P}}(n q_n^{\text{pre}})$ for $k \leq r$ by Lemma D.1. Thus

$$\left\|(\psi_l^*)^\top \hat{\Psi}_r^R\right\| = \left\|(\psi_l^*)^\top \hat{\Psi}_r \hat{R}\right\| = \left\|(\psi_l^*)^\top \hat{\Psi}_r \hat{\Lambda}_r \hat{R} \hat{R}^\top \hat{\Lambda}_r^{-1} \hat{R}\right\| = O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{\log\log n}} / (n q_n^{\text{pre}})\right).$$

It implies that (37) holds for $k \in \{1, \ldots, r\}$ and $l > r$, and we complete the proof.

□

Lemma D.2. *Suppose $q_n^{pre} \succ \frac{\log n}{\log\log n}/n$. Define $\hat{\psi}_k^R$ as in Lemma C.2. Under Assumptions 1 and 3, then $\left\|A^{pre}\left(\hat{\psi}_k^R - \psi_k^*\right)\right\| = O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{\log\log n}}\right)$.*

Proof of Lemma D.2. By decomposing the target expression:

$$\left\|A^{\text{pre}}\left(\hat{\psi}_k^R - \psi_k^*\right)\right\| \leq \left\|q_n^{\text{pre}} \tilde{G}_n^{\text{pre}}\left(\hat{\psi}_k^R - \psi_k^*\right)\right\| + \left\|\left(A^{\text{pre}} - q_n^{\text{pre}} \tilde{G}_n^{\text{pre}}\right)\left(\hat{\psi}_k^R - \psi_k^*\right)\right\|$$
$$\leq \left\|\sum_{l=1}^{r} \lambda_l^* \psi_l^* \psi_l^{*\top}\left(\hat{\psi}_k^R - \psi_k^*\right)\right\| + \left\|A^{\text{pre}} - q_n^{\text{pre}} \tilde{G}_n^{\text{pre}}\right\|_{\text{op}} \left\|\hat{\psi}_k^R - \psi_k^*\right\|$$
$$= O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{\log\log n}}\right)$$

by (D.9) and Lemma C.1 and Lemma C.2.

□

Lemma D.3. *Suppose $q_n^{pre} \succ \frac{\log n}{\log\log n}/n$. Define $\hat{R}$ and $\hat{\Psi}^R$ as in Lemma C.2, and let $\hat{\gamma}^R = \hat{R}^\top \hat{\gamma}$. Under Assumptions 1 and 3, it holds that $\sum_{k=1}^r \hat{\gamma}_k \hat{\psi}_{ki} = \sum_{k=1}^r \hat{\gamma}_k^R \hat{\psi}_{ki}^R$ and $\hat{\gamma}_k^R = O_{\mathbb{P}}(n q_n^{pre})$.*

Proof of Lemma D.3. The first result holds by construction. Multiplying both sides by $\hat{R}$, we obtain

$$\hat{\gamma}_k^R = \sum_{i=1}^n \hat{\psi}_{ki}^R Z_i^{\text{SSIV}} = \sum_{i=1}^n \psi_k^*(w_i) Z_i^{\text{SSIV}} + \sum_{i=1}^n \left(\hat{\psi}_{ki}^R - \psi_k^*(w_i)\right) Z_i^{\text{SSIV}} =: S_1 + S_2.$$

We analyze two terms separately. Rewrite $S_1 = \sum_{(i,j)} \psi_k^*(w_i) A_{ij}^{\text{pre}} (T_j - \pi)$. We bound S_1 in L_2 norm:

$$\mathbb{E}[S_1^2] = \pi(1-\pi) \sum_{j=1}^n \left(\sum_{i=1}^n \mathbb{E}[\psi_k^*(w_i)^2 A_{ij}^{\text{pre}}] + \sum_{(i_1,i_2)} \mathbb{E}[\psi_k^*(w_{i_1}) \psi_k^*(w_{i_2}) A_{i_1 j}^{\text{pre}} A_{i_2 j}^{\text{pre}}] \right) \leq C(n q_n^{\text{pre}})^2.$$

Hence,

$$\sum_{i=1}^n \psi_k^*(w_i) Z_i^{\text{SSIV}} = O_{\mathbb{P}}(n q_n^{\text{pre}}). \tag{D.12}$$

For $S_2 = \sum_{j=1}^n (T_j - \pi) \sum_{i \neq j} A_{ij}^{\text{pre}} (\hat{\psi}_{ki}^R - \psi_{ki}^*)$, we bound it in L_2 norm: $\mathbb{E}[S_2^2] = \pi(1-\pi) \cdot \mathbb{E}\left[\left\| A^{\text{pre}} (\hat{\psi}_k^R - \psi_k^*) \right\|^2 \right] = O\left(\frac{\log n}{\log \log n} \right)$ by Lemma D.2. Therefore,

$$\sum_{i=1}^n (\hat{\psi}_{ki}^R - \psi_{ki}^*) Z_i^{\text{SSIV}} = O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{\log \log n}} \right). \tag{D.13}$$

Combining (D.12) and (D.13) gives $\hat{\gamma}_k^R = O_{\mathbb{P}}(n q_n^{\text{pre}})$ as claimed.

□

By construction, the residual is orthogonal to each basis function: $\sum_{i=1}^n \eta_i^\phi \psi_k^*(w_i) = 0$ for $k \leq r$. Furthermore, we have $\mu_{n,l}^\phi = O_{\mathbb{P}}(\sqrt{n})$, which follows from the bound on its second moment: $\mathbb{E}\left((\mu_{n,l}^\phi)^2 \right) = \sum_{i=1}^n \mathbb{E}\left(\phi_i^2 (\psi_{li}^*)^2 \right) + \sum_{(i,j)} \mathbb{E}\left(\phi_i \psi_{li}^* \phi_j \psi_{lj}^* \right) = O(n)$.

Proof of Lemma C.3. Define $\hat{\mu}_k^\phi = \sum_{i=1}^n \phi_i \hat{\psi}_{ki}$ and $\hat{\eta}_i^\phi = \phi_i - \sum_{k=1}^r \hat{\mu}_k^\phi \hat{\psi}_{ki}$. By direct algebra,

$$\begin{aligned}
\frac{1}{n} \sum_{i=1}^n Z_i^{\text{DN}} \phi_i =& \frac{1}{n} \sum_{i=1}^n \left(Z_i^{\text{SSIV}} - \sum_{k=1}^r \hat{\gamma}_k \hat{\psi}_{ki} \right) \phi_i \\
=& \frac{1}{n} \sum_{i=1}^n \left(Z_i^{\text{SSIV}} - \sum_{k=1}^r \hat{\gamma}_k \hat{\psi}_{ki} \right) \left(\hat{\eta}_i^\phi + \sum_{k=1}^r \hat{\mu}_k^\phi \hat{\psi}_{ki} \right) \\
=& \frac{1}{n} \sum_{i=1}^n Z_i^{\text{SSIV}} \hat{\eta}_i^\phi + \frac{1}{n} \sum_{i=1}^n \left(Z_i^{\text{SSIV}} - \sum_{k=1}^r \hat{\gamma}_k \hat{\psi}_{ki} \right) \sum_{k=1}^r \hat{\mu}_k^\phi \hat{\psi}_{ki} - \frac{1}{n} \sum_{i=1}^n \left(\sum_{k=1}^r \hat{\gamma}_k \hat{\psi}_{ki} \right) \hat{\eta}_i^\phi.
\end{aligned}$$

By the orthogonality of projection, we can show that the second and third terms are zero.:

$$\frac{1}{n}\sum_{i=1}^{n}\left(Z_i^{\text{SSIV}}-\sum_{k=1}^{r}\hat{\gamma}_k\hat{\psi}_{ki}\right)\sum_{k=1}^{r}\hat{\mu}_k^{\phi}\hat{\psi}_{ki}=\frac{1}{n}\sum_{k=1}^{r}\hat{\mu}_k^{\phi}\sum_{i=1}^{n}\left(Z_i^{\text{SSIV}}-\sum_{k=1}^{r}\hat{\gamma}_k\hat{\psi}_{ki}\right)\hat{\psi}_{ki}=0$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{r}\hat{\gamma}_k\hat{\psi}_{ki}\right)\hat{\eta}_i^{\phi}=\frac{1}{L}\sum_{k=1}^{r}\hat{\gamma}_k\sum_{i=1}^{n}\hat{\psi}_{ki}\left(\phi_i-\sum_{k=1}^{r}\hat{\mu}_k^{\phi}\hat{\psi}_{ki}\right)=0.$$

Therefore, $\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{DN}}\phi_i=\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}\hat{\eta}_i^{\phi}$. Then we show $\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}\hat{\eta}_i^{\phi}=\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}\phi_i^{\text{DN}}+o_{\mathbb{P}}\left(\sqrt{q_n^{\text{pre}}}\right)$. We can rewrite the expression as:

$$\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}\hat{\eta}_i^{\phi}=\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}\phi_i^{\text{DN}}+\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}\sum_{k=1}^{r}\left(\mu_k^{\phi}\psi_k^*(w_i)-(\hat{\mu}_k^{\phi})^R\hat{\psi}_{ki}^R\right).$$

For the first term, since T_j is mean-zero after centering and ϕ_i^{DN} is mean-zero and independent of T_j, the expectation is zero. Its variance is given by

$$\begin{aligned}\text{Var}\left(\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}\phi_i^{\text{DN}}\right)=&\frac{\pi(1-\pi)}{n^2}\sum_{(i,j)}^{n}\mathbb{E}\left(P_{ij}^{\text{pre}}(1-P_{ij}^{\text{pre}})(\phi_i^{\text{DN}})^2\right)+\frac{\pi(1-\pi)}{n^2}\sum_{j=1}^{n}\mathbb{E}\left(\left(\sum_{i=1}^{n}P_{ij}^{\text{pre}}\phi_i^{\text{DN}}\right)^2\right)\\=&O\left(q_n^{\text{pre}}\right),\end{aligned}$$

which follows from the fact that $\mathbb{E}\left((\phi_i^{\text{DN}})^2\right)$ is bounded and

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{n}P_{ij}^{\text{pre}}\phi_i^{\text{DN}}\right)^2\leq\|R_n^{\text{pre}}\|_{\text{op}}^2\|\phi^{\text{DN}}\|_2^2=O_p\left(n(q_n^{\text{pre}})^2\right).$$

Thus, we have $\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}\phi_i^{\text{DN}}=O_{\mathbb{P}}\left(\sqrt{q_n^{\text{pre}}}\right)$. Then we bound the remainder term. By definition,

$$(\hat{\mu}_k^{\phi})^R\hat{\psi}_{ki}^R-\mu_k^{\phi}\psi_{ki}^*=((\hat{\mu}_k^{\phi})^R-\mu_k^{\phi})(\hat{\psi}_{ki}^R-\psi_{ki}^*)+((\hat{\mu}_k^{\phi})^R-\mu_k^{\phi})\psi_{ki}^*+\mu_k^{\phi}(\hat{\psi}_{ki}^R-\psi_{ki}^*).\quad\text{(D.14)}$$

For the first term and the third term on the right-hand side of (D.14), we have,

$$\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}\sum_{k=1}^{r}((\hat{\mu}_k^{\phi})^R-\mu_k^{\phi})(\hat{\psi}_{ki}^R-\psi_{ki}^*)=\sum_{k=1}^{r}((\hat{\mu}_k^{\phi})^R-\mu_k^{\phi})\frac{1}{n}\sum_{i=1}^{n}Z_i^{\text{SSIV}}(\hat{\psi}_{ki}^R-\psi_{ki}^*)=O_{\mathbb{P}}\left(\frac{q_n^{\text{pre}}}{\sqrt{n}}\right)$$

and

$$\frac{1}{n}\sum_{i=1}^{n} Z_i^{\text{SSIV}} \sum_{k=1}^{r} \mu_k^{\phi}(\hat{\psi}_{ki}^{R} - \psi_{ki}^{*}) = \sum_{k=1}^{r} \mu_k^{\phi} \frac{1}{n}\sum_{i=1}^{n} Z_i^{\text{SSIV}}(\hat{\psi}_{ki}^{R} - \psi_{ki}^{*}) = O_{\mathbb{P}}\left(\frac{q_n^{\text{pre}}}{\sqrt{n}}\right)$$

by (D.13). For the second term, we have

$$\frac{1}{n}\sum_{i=1}^{n} Z_i^{\text{SSIV}} \sum_{k=1}^{r} ((\hat{\mu}_k^{\phi})^R - \mu_k^{\phi})\psi_{ki}^{*} = \sum_{k=1}^{r} ((\hat{\mu}_k^{\phi})^R - \mu_k^{\phi}) \frac{1}{n}\sum_{i=1}^{n} Z_i^{\text{SSIV}} \psi_{ki}^{*} = O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{\log\log n}/n}\right).$$

Therefore, the leading term is $\frac{1}{n}\sum_{i=1}^{n} Z_i^{\text{SSIV}} \hat{\eta}_i^{\phi} = O_{\mathbb{P}}(\sqrt{q_n^{\text{pre}}})$, and we can show (38).

□

Proof of Lemma C.4. By the decomposition of M_i, we have

$$\frac{1}{n}\sum_{i=1}^{n}(Z_i^{\text{SSIV}} + \hat{\delta}_i)M_i = \frac{1}{n}\sum_{i=1}^{n}(Z_i^{\text{SSIV}} + \hat{\delta}_i)(\xi_i + \mu_{r_1,i}) + \frac{1}{n}\sum_{i=1}^{n}(Z_i^{\text{SSIV}} + \hat{\delta}_i)(r_{0,i} + r_{1,i} - \mu_{r_1,i}).$$

First, by Lemma C.3, we can show

$$\frac{1}{n}\sum_{i=1}^{n}(Z_i^{\text{SSIV}} + \hat{\delta}_i)(\xi_i + \mu_{r_1,i}) = \frac{1}{n}\sum_{i=1}^{n} Z_i^{\text{SSIV}} \eta_i^{(\xi_i+\mu_{r_1,i})} + O_{\mathbb{P}}\left(q_n^{\text{pre}}\right) = O_{\mathbb{P}}\left(\sqrt{q_n^{\text{pre}}}\right).$$

Second, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}(Z_i^{\text{SSIV}} + \hat{\delta}_i)(r_{0,i} + r_{1,i} - \mu_{r_1,i}) = \frac{1}{n}\sum_{i=1}^{n}\left(Z_i^{\text{SSIV}} + \sum_{k=1}^{r}\hat{\gamma}_k\hat{\psi}_{ki}\right)\left(\sum_{l=1}^{r}\mu_{n,l}^{r}\psi_{li}^{*} + \eta_i^{(\xi_i+\mu_{r_1,i})}\right) \\
=&\frac{1}{n}\sum_{i=1}^{n}\left(Z_i^{\text{SSIV}} + \sum_{k=1}^{r}\hat{\gamma}_k\hat{\psi}_{ki}\right)\eta_i^{(\xi_i+\mu_{r_1,i})} + \frac{1}{n}\sum_{l=1}^{r}\mu_{n,l}^{r}\sum_{i=1}^{n}\left(Z_i^{\text{SSIV}} + \sum_{k=1}^{r}\hat{\gamma}_k\hat{\psi}_{ki}\right)\left(\psi_{li}^{*} - \hat{\psi}_{li}^{R}\right) \\
=:&S_1 + S_2.
\end{aligned}$$

For S_1, we have $S_1 = S_{11} + S_{12}$. For S_{11}, we have

$$\mathbb{E}(S_{11}) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i} A_{ij}^{\text{pre}}(T_j - \pi)\eta_i^{(\xi_i+\mu_{r_1,i})}\right) \asymp \frac{\min\{q_n^{\text{pre}}, q_n^{\text{post}}\}}{q_n^{\text{post}}}$$

and

$$\mathrm{Var}(S_{11}) = \frac{1}{n^2}\left[\begin{array}{c} \sum\limits_{(i,j)} \mathrm{Var}\left(A_{ij}^{\mathrm{pre}}(T_j-\pi)\eta_i^{(\xi_i+\mu_{r_1,i})}\right) \\ + \sum\limits_{(i,j)} \mathrm{Cov}\left(A_{ij}^{\mathrm{pre}}(T_j-\pi)\eta_i^{(\xi_i+\mu_{r_1,i})}, A_{ji}^{\mathrm{pre}}(T_i-\pi)r_j^{\mathrm{DN}}\right) \\ + \sum\limits_{(i,j,k)} \mathrm{Cov}\left(A_{ij}^{\mathrm{pre}}(T_j-\pi)\eta_i^{(\xi_i+\mu_{r_1,i})}, A_{ik}^{\mathrm{pre}}(T_k-p)\eta_i^{(\xi_i+\mu_{r_1,i})}\right) \\ + \sum\limits_{(i,j,k)} \mathrm{Cov}\left(A_{ij}^{\mathrm{pre}}(T_j-\pi)\eta_i^{(\xi_i+\mu_{r_1,i})}, A_{kj}^{\mathrm{pre}}(T_j-\pi)r_k^{\mathrm{DN}}\right) \\ + \sum\limits_{(i,j,k,l)} \mathrm{Cov}\left(A_{ij}^{\mathrm{pre}}(T_j-\pi)\eta_i^{(\xi_i+\mu_{r_1,i})}, A_{kl}^{\mathrm{pre}}(T_l-p)r_k^{\mathrm{DN}}\right) \end{array}\right] = O\left(\frac{q_n^{\mathrm{pre}}}{nq_n^{\mathrm{post}}}\right).$$

Therefore,

$$S_{11} = O_{\mathbb{P}}\left(\frac{q_n^{\mathrm{pre}}}{\max\{q_n^{\mathrm{pre}}, q_n^{\mathrm{post}}\}}\right) + O_{\mathbb{P}}\left(\sqrt{\frac{q_n^{\mathrm{pre}}}{nq_n^{\mathrm{post}}}}\right).$$

For S_{12}, we can show that

$$\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{r}\hat{\gamma}_k\hat{\psi}_{ki}\right)\eta_i^{(\xi_i+\mu_{r_1,i})} = \sum_{k=1}^{r}\frac{\hat{\gamma}_k}{n}\left(\sum_{i=1}^{n}\eta_i^{(\xi_i+\mu_{r_1,i})}(\hat{\psi}_{ki}-\psi_{ki}^*)\right) = O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\mathrm{post}}}}\sqrt{\frac{\log n}{\log\log n}/n}\right)$$

by Lemma C.2 and Lemma D.3. Therefore,

$$S_1 = \frac{1}{n}\sum_{i=1}^{n}\left(Z_i^{\mathrm{SSIV}} + \sum_{k=1}^{r}\hat{\gamma}_k\hat{\psi}_{ki}\right)\eta_i^{(\xi_i+\mu_{r_1,i})} = O_{\mathbb{P}}\left(\frac{q_n^{\mathrm{pre}}}{\max\{q_n^{\mathrm{pre}}, q_n^{\mathrm{post}}\}}\right) + O_{\mathbb{P}}\left(\sqrt{\frac{q_n^{\mathrm{pre}}}{nq_n^{\mathrm{post}}}}\right).$$

For S_2, we apply the same proof in Lemma C.3. By definition and Lemma B.2,

$$\mu_{n,k}^r = \sum_{i=1}^{n}(r_{0,i}+r_{1,i}-\mu_{r_1,i})\psi_{ki}^* = O_{\mathbb{P}}\left(\frac{1}{\sqrt{nq_n^{\mathrm{post}}}}\right),$$

and this implies that $S_2 = O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\mathrm{post}}}}\sqrt{\frac{\log n}{\log\log n}/n}\right)$. To conclude,

$$\frac{1}{n}\sum_{i=1}^{n}(Z_i^{\mathrm{SSIV}}+\hat{\delta}_i)(r_{0,i}+r_{1,i}-\mu_{r_1,i}) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}Z_i^{\mathrm{SSIV}}\eta_i^r\right) + O_{\mathbb{P}}\left(\frac{\sqrt{q_n^{\mathrm{pre}}}}{\sqrt{nq_n^{\mathrm{post}}}}\right).$$

□

Below we provide the proof of (43) and (44).

Proof of Theorem 3.4 (Step 1). By definition of $\mathcal{S}_{(i,j)}$, we have

$$\sum_{(k,l)\in\mathcal{S}_{(i,j)}} H_{kl} = H_{ij} + H_{ji} + \sum_{k\neq i,j} H_{ik} + \sum_{k\neq i,j} H_{ki} + \sum_{k\neq i,j} H_{jk} + \sum_{k\neq i,j} H_{kj}. \tag{D.15}$$

Then we bound the first term of (42) by considering the terms in (D.15).
(1): $H_{ij}+H_{ji}$. By definition,

$$\mathbb{E}\left[A_{ij}^{\text{pre}}\left(H_{ij}+H_{ji}\right)^2\right]=\mathbb{E}\left[A_{ij}^{\text{pre}}\left((T_j-\pi)\eta_i^u+(T_i-\pi)\eta_j^u\right)^2\right]\leq Cq_n^{\text{pre}}.$$

(2): $\sum_{k\neq i,j}H_{ik}$. We bound it by independence between $(T_{k_1}-\pi)$ and $(T_{k_2}-\pi)$:

$$\mathbb{E}\left[A_{ij}^{\text{pre}}\left(\sum_{k\neq i,j}H_{ik}\right)^2\right]=\mathbb{E}\left[P_{ij}^{\text{pre}}\sum_{k\neq i,j}P_{ik}^{\text{pre}}(T_k-\pi)^2(\eta_i^u)^2\right]\leq Cn(q_n^{\text{pre}})^2$$

(3): $\sum_{k\neq i,j}H_{ki}$. It can be bounded by $\mathbb{E}\left[A_{ij}^{\text{pre}}\left(\sum_{k\neq i,j}H_{ki}\right)^2\right]\leq Cn(q_n^{\text{pre}})^2$, which follows from

$$\begin{aligned}&\mathbb{E}\left[A_{ij}^{\text{pre}}(T_i-\pi)^2\sum_{(k_1,k_2)}A_{k_1i}^{\text{pre}}\eta_{k_1}^uA_{k_2i}^{\text{pre}}\eta_{k_2}^u\right]\\=&\pi(1-\pi)q_n^{\text{pre}}\mathbb{E}\left[f_0(i,j)\sum_{k_1}\sum_{l_1>r}\lambda_{l_1}^*\psi_{l_1}^*(w_{k_1})\eta_{k_1}^u\psi_{l_1}^*(w_i)\sum_{k_2}\sum_{l_2>r}\lambda_{l_2}^*\psi_{l_2}^*(w_{k_2})\eta_{k_2}^u\psi_{l_2}^*(w_i)\right]\leq Cn(q_n^{\text{pre}})^3.\end{aligned}$$

By the variance calculation, we have $\sigma_n^2=O(n^2q_n^{\text{pre}})$. By combining these three cases, we show (43). To show (44), we decompose it into several pieces in terms of (D.15), and consider these terms one by one.
(1): $H_{ij}+H_{ji}$. By definition,

$$\begin{aligned}&\text{Var}\left(\sum_{(i,j)}H_{ij}\left(H_{ij}+H_{ji}\right)\right)=\text{Var}\left(\sum_{(i,j)}A_{ij}^{\text{pre}}(T_j-\pi)\eta_i^u\left((T_j-\pi)\eta_i^u+(T_i-\pi)\eta_j^u\right)\right)\\=&\left[\begin{array}{l}\sum_{(i,j)}\text{Var}\left(A_{ij}^{\text{pre}}(T_j-\pi)\eta_i^u\left((T_j-\pi)\eta_i^u+(T_i-\pi)\eta_j^u\right)\right)\\+\sum_{(i,j,k)}\text{Cov}\left(A_{ij}^{\text{pre}}(T_j-\pi)\eta_i^u\left((T_j-\pi)\eta_i^u+(T_i-\pi)\eta_j^u\right),A_{ik}^{\text{pre}}(T_k-\pi)\eta_i^u\left((T_k-\pi)\eta_i^u+(T_i-\pi)\eta_k^u\right)\right)\end{array}\right]\\\leq&Cn^3(q_n^{\text{pre}})^2.\end{aligned}$$

(2): $\sum_{k\neq i,j}H_{ik}$. We bound the variance:

$$\text{Var}\left(\sum_{(i,j)}H_{ij}\sum_{k\neq i,j}H_{ik}\right)\leq\text{Var}\left(\sum_{(i,j,k)}A_{ij}^{\text{pre}}A_{ik}^{\text{pre}}(T_j-\pi)(T_k-\pi)(\eta_i^u)^2\right)$$

$$= \left[\begin{array}{l} \sum_{(i,j,k)} \mathrm{Var}\left(A^{\mathrm{pre}}_{ij} A^{\mathrm{pre}}_{ik} (T_j - \pi)(T_k - \pi)(\eta^u_i)^2\right) \\ + \sum_{(i_1,i_2,j,k)} \mathbb{E}\left[A^{\mathrm{pre}}_{i_1 j} A^{\mathrm{pre}}_{i_1 k} A^{\mathrm{pre}}_{i_2 j} A^{\mathrm{pre}}_{i_2 k} (T_j - \pi)^2 (T_k - \pi)^2 (\eta^u_{i_1})^2 (\eta^u_{i_2})^2\right] \end{array} \right] \leq C_1 n^3 (q^{\mathrm{pre}}_n)^2 + C_2 n^4 (q^{\mathrm{pre}}_n)^4.$$

(3): $\sum_{k \neq i,j} H_{ki}$. We bound the variance:

$$\mathrm{Var}\left(\sum_{(i,j)} H_{ij} \sum_{k \neq i,j} H_{ki}\right) = \left[\begin{array}{l} \sum_{(i,j,k)} \mathrm{Var}\left(A^{\mathrm{pre}}_{ij} A^{\mathrm{pre}}_{ki} (T_j - \pi)(T_i - \pi)\eta^u_i \eta^u_k\right) \\ + \sum_{(i,j,k_1,k_2)} \mathbb{E}\left[A^{\mathrm{pre}}_{ij} A^{\mathrm{pre}}_{k_1 i} A^{\mathrm{pre}}_{k_2 i} (T_j - \pi)^2 (T_i - \pi)^2 (\eta^u_i)^2 \eta^u_{k_1} \eta^u_{k_2}\right] \end{array} \right] \leq C_1 n^3 (q^{\mathrm{pre}}_n)^2.$$

(4): $\sum_{k \neq i,j} H_{kj}$. Recall that $\sum_{i=1}^n \eta^u_i \psi^*_k(w_i) = 0$ for $k \leq r$.

$$\begin{aligned} &\mathrm{Var}\left(\sum_{(i,j)} H_{ij} \sum_{k \neq i,j} H_{kj}\right) \leq \mathbb{E}\left[\left(\sum_{(i,j,k)} A^{\mathrm{pre}}_{ij} (T_j - \pi)^2 \eta^u_i A^{\mathrm{pre}}_{kj} \eta^u_k\right)^2\right] \\ =& \sum_{\substack{(i_1,j_1,k_1) \\ (i_2,j_2,k_2)}} \mathbb{E}\left[(T_{j_1} - \pi)^2 (T_{j_2} - \pi)^2 A^{\mathrm{pre}}_{i_1 j_1} \eta^u_{i_1} A^{\mathrm{pre}}_{k_1 j_1} \eta^u_{k_1} A^{\mathrm{pre}}_{i_2 j} \eta^u_{i_2} A^{\mathrm{pre}}_{k_2 j_2} \eta^u_{k_2}\right] \\ =& \pi^2 (1-\pi)^2 \left[\sum_{(i,j_1,j_2,k)} \mathbb{E}\left[A^{\mathrm{pre}}_{ij_1} A^{\mathrm{pre}}_{kj_1} A^{\mathrm{pre}}_{kj_2} (\eta^u_i)^2 (\eta^u_k)^2\right] + \sum_{\substack{(i,j_1,j_2) \\ \text{all distinct}}} \mathbb{E}\left[A^{\mathrm{pre}}_{ij_1} A^{\mathrm{pre}}_{ij_2} (\eta^u_i)^4\right] \right] \\ \leq& C_1 n^4 (q^{\mathrm{pre}}_n)^3 + C_2 n^3 (q^{\mathrm{pre}}_n)^2. \end{aligned}$$

By combining these results, we show (44). □

E Normalized SSIV

Borusyak et al. (2022) suggest the following normalized SSIV for M_i:

$$Z^{\mathrm{alt}}_i = \sum_{j=1}^n A^{\mathrm{pre}}_{ij} T_j \Big/ \sum_{j=1}^n A^{\mathrm{pre}}_{ij}. \tag{E.1}$$

The exclusion restriction holds by the randomness of the shocks and the normalization of the shares: $\mathbb{E}\left[Z^{\mathrm{alt}}_i u_i\right] = \pi \cdot \mathbb{E}\left[\frac{\sum_{j=1}^n A^{\mathrm{pre}}_{ij}}{\sum_{j=1}^n A^{\mathrm{pre}}_{ij}} u_i\right] = 0$. Borusyak et al. (2022) state that the relevance condition holds when individual units are mostly exposed to only a small number of shocks. In this section, we quantify the regime where the normalized SSIV in (E.1) yields consistent estimators.

E.1 Consistency

For estimation, we consider the IV estimation with the IV vector: $\tilde{Z}_i^{\text{alt}} = (1, T_i, Z_i^{\text{alt}})$. Let $\hat{\beta}^{\text{alt}}$ denote the vector of coefficients obtained from the above IV regression. Define $\tilde{Z}^{\text{alt}}$ as the $n \times 3$ matrix obtained by stacking $\tilde{Z}_i^{\text{alt}}$.

Theorem E.1. *Under Assumptions 1 and 2,*

(a) if $\text{Var}(\xi_i) > 0$ *with* $\max\{q_n^{pre}, q_n^{post}\} \prec 1/\sqrt{n}$, *then* $\hat{\beta}^{alt} - \beta = O_{\mathbb{P}}\left(\sqrt{n}\max\{q_n^{pre}, q_n^{post}\}\right)$;

(b) if $\text{Var}(\xi_i) = 0$ *with* $\max\{q_n^{pre}, q_n^{post}\} \prec \sqrt{n} q_n^{post}$, *then* $\hat{\beta}_1^{alt} - \beta_1 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}} \max\left\{\frac{\max\{q_n^{pre}, q_n^{post}\}}{\sqrt{q_n^{post}}}, 1\right\}\right)$. *Moreover, with* $\max\{q_n^{pre}, q_n^{post}\} \prec 1/\sqrt{n}$, *then* $\hat{\beta}_0^{alt} - \beta_0 = O_{\mathbb{P}}\left(\sqrt{n}\max\{q_n^{pre}, q_n^{post}\}\right)$ *and* $\hat{\beta}_2^{alt} - \beta_2 = O_{\mathbb{P}}\left(\sqrt{n}\max\{q_n^{pre}, q_n^{post}\}\right)$.

Theorem E.1 shows that the consistency regime for the normalized SSIV in (E.1) coincides with that of the original SSIV Z_i^{SSIV} in Theorem 3.1. This is unsurprising, since both instruments rely on the same core component, $\sum_{j=1}^n A_{ij}^{\text{post}} T_j$, which is endogenous because it correlates with the error term u_i. Normalizing or centering this sum by its expectation effectively removes that endogeneity. Corollary S1 simplifies Theorem E.1 to the special case where $q_n^{\text{pre}} \preccurlyeq q_n^{\text{post}}$.

Corollary S1. *Suppose* $q_n^{pre} \preccurlyeq q_n^{post}$. *Under Assumptions 1 and 2,*

(a) if $\text{Var}(\xi_i) > 0$ *with* $q_n^{post} \prec \frac{1}{\sqrt{n}}$, *then* $\hat{\beta}^{alt} - \beta = O_{\mathbb{P}}(\sqrt{n} q_n^{post})$;

(b) if $\text{Var}(\xi_i) = 0$ *with* $q_n^{post} \prec \frac{1}{\sqrt{n}}$, *then* $\hat{\beta}_0^{alt} - \beta_0 = O_{\mathbb{P}}(\sqrt{n} q_n^{post})$ *and* $\hat{\beta}_2^{alt} - \beta_2 = O_{\mathbb{P}}(\sqrt{n} q_n^{post})$. *Moreover, it always holds that* $\hat{\beta}_1^{alt} - \beta_1 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$.

E.2 Proof of Theorem E.1

We prove Theorem E.1 by first showing Lemma E.1 and Lemma E.2.

Lemma E.1. *Define* ϕ_i *as an i.i.d. random variable with constant variance. Define the conditional expectation* $Q_j^{\phi} = \mathbb{E}\left[\frac{A_{ij}^{pre}\phi_i}{q_n^{pre} g_0(i)} | w_j\right]$. *Under Assumptions 1 and 2,*

$$\frac{1}{n}\sum_{i=1}^n (Z_i^{alt} - \pi)\phi_i = \frac{1}{n}\sum_{i=1}^n (T_i - \pi) Q_i^{\phi} + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}\sqrt{n q_n^{pre}}}\right). \tag{E.2}$$

Proof of Lemma E.1. By reordering the index, we have

$$\frac{1}{n}\sum_{i=1}^n (Z_i^{\text{alt}} - \pi)\phi_i = \frac{1}{n}\sum_{i=1}^n \frac{\sum_{j=1}^n A_{ij}^{\text{pre}}(T_j - \pi)}{\sum_{j=1}^n A_{ij}^{\text{pre}}}\phi_i = \frac{1}{n}\sum_{j=1}^n (T_j - \pi)\sum_{i \neq j} \frac{A_{ij}^{\text{pre}}\phi_i}{N_i}$$

where

$$\sum_{i\neq j}\frac{A_{ij}^{\mathrm{pre}}\phi_i}{N_i}=\sum_{i\neq j}\frac{A_{ij}^{\mathrm{pre}}\phi_i}{(n-1)q_n^{\mathrm{pre}}g_0(i)}-\sum_{i\neq j}\frac{A_{ij}^{\mathrm{pre}}\phi_i(N_i-(n-1)q_n^{\mathrm{pre}}g_0(i))}{(n-1)q_n^{\mathrm{pre}}g_0(i)N_i}. \tag{E.3}$$

First term in (E.3): Given w_j, the terms $\frac{A_{ij}^{\mathrm{pre}}\phi_i}{q_n^{\mathrm{pre}}g_0(i)}$ are i.i.d. across i, so

$$\mathbb{E}\left[\left(\frac{1}{n-1}\sum_{i\neq j}\frac{A_{ij}^{\mathrm{pre}}\phi_i}{q_n^{\mathrm{pre}}g_0(i)}-Q_j^{\phi}\right)^2\right]\leq\frac{1}{n-1}\mathbb{E}\left[\frac{A_{ij}^{\mathrm{pre}}\phi_i^2}{(q_n^{\mathrm{pre}}g_0(i))^2}\right]\leq\frac{C}{(n-1)q_n^{\mathrm{pre}}}. \tag{E.4}$$

This implies that $\frac{1}{n-1}\sum_{i\neq j}\frac{A_{ij}^{\mathrm{pre}}\phi_i}{q_n^{\mathrm{pre}}g_0(i)}$ approximates Q_j^{ϕ} with an error of order $O_{\mathbb{P}}\left(\frac{1}{nq_n^{\mathrm{pre}}}\right)$.

Second term in (E.3): We start by noting that

$$\frac{A_{ij}^{\mathrm{pre}}\phi_i(N_i-(n-1)q_n^{\mathrm{pre}}g_0(i))}{q_n^{\mathrm{pre}}g_0(i)N_i}=\frac{A_{ij}^{\mathrm{pre}}\phi_i((N_i-A_{ij}^{\mathrm{pre}}+1)-(n-1)q_n^{\mathrm{pre}}g_0(i))}{q_n^{\mathrm{pre}}g_0(i)(N_i-A_{ij}^{\mathrm{pre}}+1)}. \tag{E.5}$$

Conditional on w, the term $N_i-A_{ij}^{\mathrm{pre}}$ follows Binomial$(n-1,q_n^{\mathrm{pre}}g_0(i))$. For a binomial random variable $X\sim\text{Binomial}(n,p)$, it holds that $\mathbb{E}\left(\frac{1}{X+1}\right)=\frac{1-(1-p)^{n+1}}{(n+1)p}$. Applying this identity, we compute

$$\begin{aligned}
&\mathbb{E}\left[\tfrac{(N_i-A_{ij}^{\mathrm{pre}}+1)-(n-1)q_n^{\mathrm{pre}}g_0(i)}{(n-1)q_n^{\mathrm{pre}}g_0(i)(N_i-A_{ij}^{\mathrm{pre}}+1)}|w\right]=\frac{1}{(n-1)q_n^{\mathrm{pre}}g_0(i)}-\mathbb{E}\left[\tfrac{1}{N_i-A_{ij}^{\mathrm{pre}}+1}|w\right]\\
&=\frac{1}{(n-1)q_n^{\mathrm{pre}}g_0(i)}-\frac{1-(1-q_n^{\mathrm{pre}}g_0(i))^n}{nq_n^{\mathrm{pre}}g_0(i)}=\frac{1}{n(n-1)q_n^{\mathrm{pre}}g_0(i)}+\frac{(1-q_n^{\mathrm{pre}}g_0(i))^n}{nq_n^{\mathrm{pre}}g_0(i)}\\
&\leq\frac{1}{n(n-1)q_n^{\mathrm{pre}}g_0(i)}+\frac{e^{-Cnq_n^{\mathrm{pre}}}}{nq_n^{\mathrm{pre}}g_0(i)}\leq C\cdot\frac{1}{(nq_n^{\mathrm{pre}})^2}.
\end{aligned} \tag{E.6}$$

By the Cauchy–Schwarz inequality and Li and Wager (2022, Lemma 15), we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\tfrac{(N_i-A_{ij}^{\mathrm{pre}}+1)-(n-1)q_n^{\mathrm{pre}}g_0(i)}{(n-1)q_n^{\mathrm{pre}}g_0(i)(N_i-A_{ij}^{\mathrm{pre}}+1)}\right)^2|w\right]\\
&\leq\tfrac{1}{(n-1)^2(q_n^{\mathrm{pre}}g_0(i))^2}\sqrt{\mathbb{E}\left(\left((N_i-A_{ij}^{\mathrm{pre}}+1)-(n-1)q_n^{\mathrm{pre}}g_0(i)\right)^4\right)}\sqrt{\mathbb{E}\left(\tfrac{1}{(N_i-A_{ij}^{\mathrm{pre}}+1)^4}\right)}\\
&\leq C\frac{1}{((n-1)q_n^{\mathrm{pre}})^2}\frac{nq_n^{\mathrm{pre}}}{(nq_n^{\mathrm{pre}})^2}\leq C\cdot\frac{1}{(nq_n^{\mathrm{pre}})^3}.
\end{aligned} \tag{E.7}$$

Let $B_{ij}=\frac{A_{ij}^{\mathrm{pre}}\phi_i(N_i-(n-1)q_n^{\mathrm{pre}}g_0(i))}{(n-1)q_n^{\mathrm{pre}}g_0(i)N_i}$. We bound the second moment as $\mathbb{E}\left[\left(\sum_{i\neq j}B_{ij}\right)^2\right]=\sum_{i\neq j}\mathbb{E}[B_{ij}^2]+\sum_{i\neq k}\mathbb{E}[B_{ij}B_{kj}]$.

For the diagonal term: By using (E.7),

$$\mathbb{E}[B_{ij}^2] = \mathbb{E}\left[A_{ij}^{\text{pre}}\phi_i^2\left(\frac{(N_i - A_{ij}^{\text{pre}}+1)-(n-1)q_n^{\text{pre}}g_0(i)}{(n-1)q_n^{\text{pre}}g_0(i)(N_i-A_{ij}^{\text{pre}}+1)}\right)^2\right] \leq \frac{C}{n(nq_n^{\text{pre}})^2}.$$

For the off-diagonal term: Conditional on w, A_{ij}^{pre}, A_{jk}^{pre}, $\frac{N_j-(n-1)q_n^{\text{pre}}g_0(w_j)}{N_j}$ and $\frac{N_k-(n-1)q_n^{\text{pre}}g_0(w_k)}{N_k}$ are all independent. By using (E.6):

$$\begin{aligned}\mathbb{E}[B_{ij}B_{ik}] =& \mathbb{E}\left[(q_n^{\text{pre}})^2 g_0(i,j)g_0(i,k)\phi_j\phi_k \mathbb{E}\left[\frac{N_j-(n-1)q_n^{\text{pre}}g_0(w_j)}{(n-1)q_n^{\text{pre}}g_0(w_j)N_j}|w\right]\mathbb{E}\left[\frac{N_k-(n-1)q_n^{\text{pre}}g_0(w_k)}{(n-1)q_n^{\text{pre}}g_0(w_k)N_k}|w\right]\right] \\ \leq& C\cdot\frac{1}{n^2(nq_n^{\text{pre}})^2}.\end{aligned}$$

Combining the diagonal and off-diagonal terms, we conclude:

$$\mathbb{E}\left[\left(\sum_{i\neq j}\frac{A_{ij}^{\text{pre}}\phi_i(N_i-(n-1)q_n^{\text{pre}}g_0(i))}{(n-1)q_n^{\text{pre}}g_0(i)N_i}\right)^2\right] \leq \frac{C}{(nq_n^{\text{pre}})^2}. \tag{E.8}$$

Combining (E.4) and (E.8), we get $\mathbb{E}\left[\left(\sum_{i\neq j}\frac{A_{ij}^{\text{pre}}}{N_i}\phi_i - Q_j^{\phi}\right)^2\right] \leq \frac{C}{nq_n^{\text{pre}}}$, which yields:

$$\frac{1}{n}\sum_{j=1}^n(T_j-\pi)\left(\sum_{i\neq j}\frac{A_{ij}^{\text{pre}}\phi_i}{N_i} - Q_j^{\phi}\right) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}\sqrt{nq_n^{\text{pre}}}}\right)$$

and thus completes the proof.

□

Define

$$S_1^0 = \frac{1}{n}\sum_{j=1}^n(T_j-\pi)\sum_{i\neq j}\frac{A_{ij}^{\text{pre}}r_{0,i}}{(n-1)q_n^{\text{pre}}g_0(i)}. \tag{E.9}$$

Lemma E.2. *Under Assumptions 1 and 2, we have*

$$\frac{1}{n}\sum_{i=1}^n(Z_i^{alt}-\pi)M_i \;= \mathbb{E}(S_1^0) + O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{post}}}\right) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \tag{E.10}$$

Proof of Lemma E.2. By the decomposition of M_i, we have

$$\frac{1}{n}\sum_{i=1}^n(Z_i^{\text{alt}}-\pi)M_i = \frac{1}{n}\sum_{i=1}^n(Z_i^{\text{alt}}-\pi)\xi_i + \frac{1}{n}\sum_{i=1}^n(Z_i^{\text{alt}}-\pi)r_{0,i} + \frac{1}{n}\sum_{i=1}^n(Z_i^{\text{alt}}-\pi)r_{1,i}.$$

First term: By Lemma E.1, we have

$$\frac{1}{n}\sum_{i=1}^{n}(Z_i^{\text{alt}}-\pi)\xi_i = \frac{1}{n}\sum_{i=1}^{n}(T_i-\pi)\mathbb{E}\left[\frac{A_{ij}^{\text{pre}}\xi_j}{q_n^{\text{pre}}g_0(j)}|w_i\right] + O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{pre}}}}\right) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \tag{E.11}$$

Second term: Rewrite $\frac{1}{n}\sum_{i=1}^{n}(Z_i^{\text{alt}}-\pi)r_{0,i} = S_1^0 - S_2^0$ where S_1^0 defined in (E.9) and

$$S_2^0 = \frac{1}{n}\sum_{j=1}^{n}(T_j-\pi)\sum_{i\neq j}\frac{A_{ij}^{\text{pre}}r_{0,i}\left(N_i-(n-1)q_n^{\text{pre}}g_0(i)\right)}{(n-1)q_n^{\text{pre}}g_0(i)N_i}. \tag{E.12}$$

For S_1^0, the expectation is

$$\mathbb{E}[S_1^0] = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left[(T_j-\pi)\sum_{i\neq j}\frac{A_{ij}^{\text{pre}}\xi_i R_{ij}}{(n-1)^2 q_n^{\text{pre}}g_0(i)}\right] \asymp \frac{1}{n\max\{q_n^{\text{pre}},q_n^{\text{post}}\}}.$$

The variance is

$$\mathbb{V}[S_1^0] = \frac{1}{(n-1)^2(nq_n^{\text{pre}})^2}\left[\begin{array}{c}\sum_{j=1}^{n}\sum_{i\neq j}^{n}\text{Var}\left(\frac{(T_j-\pi)A_{ij}^{\text{pre}}r_{0,i}}{g_0(i)}\right)+\sum_{j=1}^{n}\sum_{(i_1,i_2)}\text{Cov}\left(\frac{(T_j-\pi)A_{i_1j}^{\text{pre}}r_{0,i_1}}{g_0(i_1)},\frac{(T_j-\pi)A_{i_2j}^{\text{pre}}r_{0,i_2}}{g_0(i_2)}\right)\\ +\sum_{(j_1,j_2)}\sum_{i\neq j_1,j_2}\text{Cov}\left(\frac{(T_{j_1}-\pi)A_{ij_1}^{\text{pre}}r_{0,i}}{g_0(i)},\frac{(T_{j_2}-\pi)A_{ij_2}^{\text{pre}}r_{0,i}}{g_0(i)}\right)\\ +\sum_{(j_1,j_2)}\sum_{(i_1,i_2)}\text{Cov}\left(\frac{(T_{j_1}-\pi)A_{i_1j_1}^{\text{pre}}r_{0,i_1}}{g_0(i_1)},\frac{(T_{j_2}-\pi)A_{i_2j_2}^{\text{pre}}r_{0,i_2}}{g_0(i_2)}\right)\end{array}\right].$$

Bounding each component, we find:
- Variance term is $O(n^2)$ terms of order $O\left(\frac{q_n^{\text{pre}}}{nq_n^{\text{post}}}\right)$;
- Within-j and across-i covariance is $O(n^3)$ terms of order $O\left(\frac{(q_n^{\text{pre}})^2}{n}\right)$;
- Across-j and within-i covariances is $O(n^3)$ terms of order $O\left(\frac{(q_n^{\text{pre}})^2}{n^2\max\{q_n^{\text{pre}},q_n^{\text{post}}\}^2}\right)$;
- Across-j and i covariances is $O(n^4)$ terms of order $O\left(\frac{(q_n^{\text{pre}})^2}{n^2\max\{q_n^{\text{pre}},q_n^{\text{post}}\}^2}\right)$ and $O\left(\frac{(q_n^{\text{pre}})^2}{n^2q_n^{\text{post}}}\right)$

Combining these bounds, we conclude:

$$S_1^0 = \mathbb{E}[S_1^0] + O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right). \tag{E.13}$$

Recall (E.5). Define $B_{ij} = \frac{A_{ij}^{\text{pre}}r_{0,i}\left(N_i-(n-1)q_n^{\text{pre}}g_0(i)\right)}{(n-1)q_n^{\text{pre}}g_0(i)N_i}$, so that $S_2^0 = \frac{1}{n}\sum_{j=1}^{n}(T_j-\pi)\sum_{i\neq j}B_{ij}$. We bound the second moment of S_2^0 as follows:

$$\mathbb{E}\left[(S_2^0)^2\right] = \frac{1}{n^2}\left[\begin{array}{c}\sum_{j=1}^{n}\mathbb{E}\left[(T_j-\pi)^2\left(\sum_{i\neq j}B_{ij}^2+\sum_{(i_1,i_2)}B_{i_1j}B_{i_2j}\right)\right]\\ +\sum_{(j,k)}\mathbb{E}\left[(T_j-\pi)(T_k-\pi)\left(\sum_{i\neq j,k}B_{ij}B_{ik}+\sum_{(i_1,i_2)}B_{i_1j}B_{i_2k}\right)\right]\end{array}\right].$$

We analyze the components of the moment bound for S_2^0 in turn. Conditional on w, A_{ij}^{pre} and $\frac{N_i-(n-1)q_n^{\text{pre}}g_0(i)}{N_i}$ are independent.
For $\mathbb{E}\left[(T_j-\pi)^2B_{ij}^2\right]$:

$$\begin{aligned}\mathbb{E}\left[(T_j-\pi)^2B_{ij}^2\right] &= \mathbb{E}\left[\mathbb{E}\left[(T_j-\pi)^2A_{ij}^{\text{pre}}r_{0,i}^2|w\right]\mathbb{E}\left[\left(\frac{(N_i-A_{ij}^{\text{pre}}+1)-(n-1)q_n^{\text{pre}}g_0(i)}{(n-1)q_n^{\text{pre}}g_0(i)(N_i-A_{ij}^{\text{pre}}+1)}\right)^2|w\right]\right]\\ &\le \frac{C}{(nq_n^{\text{pre}})^3}\cdot\mathbb{E}\left[\frac{(T_j-\pi)^2}{(n-1)^2}A_{ij}^{\text{pre}}\xi_i^2\sum_{k\neq i}R_{ik}^2\right] = O\left(\frac{1}{n^4(q_n^{\text{pre}})^2q_n^{\text{post}}}\right).\end{aligned}$$

For $\mathbb{E}\left[(T_j-\pi)^2B_{i_1j}B_{i_2j}\right]$: Since $A_{i_1j}^{\text{pre}}$, $A_{i_2j}^{\text{pre}}$, and the $\frac{N_{i_1}-(n-1)q_n^{\text{pre}}g_0(i_1)}{(n-1)q_n^{\text{pre}}g_0(i_1)N_{i_1}}$ and $\frac{N_{i_2}-(n-1)q_n^{\text{pre}}g_0(i_2)}{(n-1)q_n^{\text{pre}}g_0(i_2)N_{i_2}}$ are independent conditional on w, we have

$$\begin{aligned}&\mathbb{E}\left[(T_j-\pi)^2B_{i_1j}B_{i_2j}\right]\\ =&\mathbb{E}\left[\mathbb{E}\left[(T_j-\pi)^2A_{i_1j}^{\text{pre}}A_{i_2j}^{\text{pre}}r_{0,i_1}r_{0,i_2}|w\right]\mathbb{E}\left[\frac{N_{i_1}-(n-1)q_n^{\text{pre}}g_0(i_1)}{(n-1)q_n^{\text{pre}}g_0(i_1)N_{i_1}}|w\right]\mathbb{E}\left[\frac{N_{i_2}-(n-1)q_n^{\text{pre}}g_0(i_2)}{(n-1)q_n^{\text{pre}}g_0(i_2)N_{i_2}}|w\right]\right]\\ =&\mathbb{E}\left[\mathbb{E}\left[\frac{(T_j-\pi)^2}{(n-1)^2}\xi_{i_1}\xi_{i_2}A_{i_1j}^{\text{pre}}A_{i_2j}^{\text{pre}}R_{i_1j}R_{i_2j}|w\right]\mathbb{E}\left[\frac{N_{i_1}-(n-1)q_n^{\text{pre}}g_0(i_1)}{(n-1)q_n^{\text{pre}}g_0(i_1)N_{i_1}}|w\right]\mathbb{E}\left[\frac{N_{i_2}-(n-1)q_n^{\text{pre}}g_0(i_2)}{(n-1)q_n^{\text{pre}}g_0(i_2)N_{i_2}}|w\right]\right]\\ \le&C\cdot\frac{\min\{q_n^{\text{pre}},q_n^{\text{post}}\}^2}{(nq_n^{\text{post}})^2}\cdot\frac{1}{(nq_n^{\text{pre}})^4} = O\left(\frac{1}{n^6(q_n^{\text{pre}})^2\max\{q_n^{\text{pre}},q_n^{\text{post}}\}^2}\right).\end{aligned}$$

For $\mathbb{E}((T_j-\pi)(T_k-\pi)B_{ij}B_{ik})$: By (E.7),

$$\begin{aligned}&\mathbb{E}((T_j-\pi)(T_k-\pi)B_{ij}B_{ik})\\ =&\mathbb{E}\left[\mathbb{E}\left[(T_j-\pi)(T_k-\pi)A_{ij}^{\text{pre}}A_{ik}^{\text{pre}}r_{0,i}^2|w\right]\mathbb{E}\left[\left(\frac{(N_i-(n-1)q_n^{\text{pre}}g_0(i))}{(n-1)q_n^{\text{pre}}g_0(i)N_i}\right)^2|w\right]\right]\\ =&\mathbb{E}\left[\mathbb{E}\left[\frac{(T_j-\pi)(T_k-\pi)}{(n-1)^2}\xi_i^2A_{ij}^{\text{pre}}A_{ik}^{\text{pre}}R_{ij}R_{ik}|w\right]\mathbb{E}\left[\left(\frac{(N_i-(n-1)q_n^{\text{pre}}g_0(i))}{(n-1)q_n^{\text{pre}}g_0(i)N_i}\right)^2|w\right]\right] = O\left(\frac{1}{n^5q_n^{\text{pre}}\max\{q_n^{\text{pre}},q_n^{\text{post}}\}^2}\right)\end{aligned}$$

For $\mathbb{E}((T_j-\pi)(T_k-\pi)B_{i_1j}B_{i_2k})$: By (E.6),

$$\begin{aligned}&\mathbb{E}((T_j-\pi)(T_k-\pi)B_{i_1j}B_{i_2k})\\ =&\mathbb{E}\left[\mathbb{E}\left[(T_j-\pi)(T_k-\pi)A_{i_1j}^{\text{pre}}A_{i_2k}^{\text{pre}}r_{0,i_1}r_{0,i_2}|w\right]\mathbb{E}\left[\frac{(N_{i_1}-(n-1)q_n^{\text{pre}}g_0(i_1))}{(n-1)q_n^{\text{pre}}g_0(i_1)N_{i_1}}|w\right]\mathbb{E}\left[\frac{(N_{i_2}-(n-1)q_n^{\text{pre}}g_0(i_2))}{(n-1)q_n^{\text{pre}}g_0(i_2)N_{i_2}}|w\right]\right]\\ =&\mathbb{E}\left[\mathbb{E}\left[\frac{(T_j-\pi)(T_k-\pi)}{(n-1)^2}\xi_{i_1}\xi_{i_2}A_{i_1j}^{\text{pre}}A_{i_2k}^{\text{pre}}R_{i_1j}R_{i_2k}|T,w\right]\mathbb{E}\left[\frac{(N_{i_1}-(n-1)q_n^{\text{pre}}g_0(i_1))}{(n-1)q_n^{\text{pre}}g_0(i_1)N_{i_1}}|w\right]\mathbb{E}\left[\frac{(N_{i_2}-(n-1)q_n^{\text{pre}}g_0(i_2))}{(n-1)q_n^{\text{pre}}g_0(i_2)N_{i_2}}|w\right]\right]\\ \le&C\frac{1}{(nq_n^{\text{pre}})^4}\frac{\min\{q_n^{\text{pre}},q_n^{\text{post}}\}^2}{n^2(q_n^{\text{post}})^2} = O\left(\frac{1}{n^6(q_n^{\text{pre}})^2\max\{q_n^{\text{pre}},q_n^{\text{post}}\}^2}\right).\end{aligned}$$

By combining these results, we have

$$\mathbb{E}\left[(S_2^0)^2\right] = O\left(\frac{1}{n^4(q_n^{\text{pre}})^2q_n^{\text{post}}}\right)+O\left(\frac{1}{n^4(q_n^{\text{pre}})^2\max\{q_n^{\text{pre}},q_n^{\text{post}}\}^2}\right)$$

and thus

$$S_2^0 = O_{\mathbb{P}}\left(\frac{1}{n^2 q_n^{\text{pre}}\sqrt{q_n^{\text{post}}}}\right) + O_{\mathbb{P}}\left(\frac{1}{n^2 q_n^{\text{pre}} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}\right). \tag{E.14}$$

Therefore, by combining (E.13) and (E.14), we have

$$\frac{1}{n}\sum_{i=1}^n (Z_i^{\text{alt}} - \pi) r_{0,i} = \mathbb{E}[S_1^0] + O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right) + O_{\mathbb{P}}\left(\frac{1}{n^2 q_n^{\text{pre}} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}\right). \tag{E.15}$$

Third term: Rewrite $\frac{1}{n}\sum_{i=1}^n (Z_i^{\text{alt}} - \pi) r_{1,i} = S_1^1 - S_2^1$ where

$$S_1^1 = \frac{1}{n}\sum_{j=1}^n (T_j - \pi) \sum_{i\neq j}^n \frac{A_{ij}^{\text{pre}} r_{1,i}}{(n-1) q_n^{\text{pre}} g_0(i)} \text{ and } S_2^1 = \frac{1}{n}\sum_{j=1}^n (T_j - \pi) \sum_{i\neq j}^n \frac{A_{ij}^{\text{pre}} r_{1,i} (N_i - (n-1) q_n^{\text{pre}} g_0(i))}{(n-1) q_n^{\text{pre}} g_0(i) N_i}.$$

Again, for S_1^1, the expectation is

$$\mathbb{E}[S_1^1] = \frac{1}{n(n-1)^3 q_n^{\text{pre}}} \sum_{j=1}^n \mathbb{E}\left[(T_j - \pi) \sum_{i\neq j}^n \frac{A_{ij}^{\text{pre}}}{g_0(i)} \left(-\frac{U_{ij} W_{ij}}{\theta_i^*(y)^2} + \frac{2\theta_i^*(x)}{\theta_i^*(y)^3} W_{ij}^2\right)\right] \asymp \frac{1}{n^2 q_n^{\text{post}} \max\{q_n^{\text{pre}}, q_n^{\text{post}}\}},$$

and the variance is

$$\begin{aligned}
\text{Var}[S_1^1] =& \frac{1}{(n-1)^2 (n q_n^{\text{pre}})^2} \left[\begin{array}{l} \sum_{j=1}^n \sum_{i\neq j}^n \text{Var}\left(\frac{(T_j-\pi) A_{ij}^{\text{pre}} r_{1,i}}{g_0(i)}\right) + \sum_{j=1}^n \sum_{(i_1,i_2)} \text{Cov}\left(\frac{(T_j-\pi) A_{i_1 j}^{\text{pre}} r_{1,i_1}}{g_0(i_1)}, \frac{(T_j-\pi) A_{i_2 j}^{\text{pre}} r_{1,i_2}}{g_0(i_2)}\right) \\ \quad + \sum_{(j_1,j_2)} \sum_{i\neq j_1,j_2} \text{Cov}\left(\frac{(T_{j_1}-\pi) A_{i j_1}^{\text{pre}} r_{1,i}}{g_0(i)}, \frac{(T_{j_2}-\pi) A_{i j_2}^{\text{pre}} r_{1,i}}{g_0(i)}\right) \\ \quad + \sum_{(j_1,j_2)} \sum_{(i_1,i_2)} \text{Cov}\left(\frac{(T_{j_1}-\pi) A_{i_1 j_1}^{\text{pre}} r_{1,i_1}}{g_0(i_1)}, \frac{(T_{j_2}-\pi) A_{i_2 j_2}^{\text{pre}} r_{1,i_2}}{g_0(i_2)}\right) \end{array}\right] \\
=& \frac{1}{(n-1)^2 (n q_n^{\text{pre}})^2} \left(S_{11}^1 + S_{12}^1 + S_{13}^1 + S_{14}^1\right).
\end{aligned}$$

Recall $E_{ik} = -\frac{U_{ik} W_{ik}}{\theta_i^*(y)^2} + \frac{2\theta_i^*(x)}{\theta_i^*(y)^3} W_{ik}^2$ and $E_{ikl} = -\frac{U_{ik} W_{il}}{\theta_i^*(y)^2} + \frac{2\theta_i^*(x)}{\theta_i^*(y)^3} W_{ik} W_{il}$. For S_{11}^1, we bound the variance of the term:

$$\text{Var}\left(\frac{(T_j-\pi) A_{ij}^{\text{pre}} r_{1,i}}{g_0(i)}\right) \leq \frac{1}{(n-1)^4} \mathbb{E}\left[\frac{(T_j-\pi)^2}{g_0(i)^2} A_{ij}^{\text{pre}} \left(\sum_{k\neq i} E_{ik} + \sum_{(k,l)} E_{ikl}\right)^2\right] \asymp \frac{q_n^{\text{pre}}}{(n q_n^{\text{post}})^2}. \tag{E.16}$$

For S_{12}^1,

$$\begin{aligned}
&\text{Cov}\left(\frac{(T_j-\pi) A_{i_1 j}^{\text{pre}} r_{1,i_1}}{g_0(i_1)}, \frac{(T_j-\pi) A_{i_2 j}^{\text{pre}} r_{1,i_2}}{g_0(i_2)}\right) \\
=& \frac{1}{(n-1)^4} \text{Cov}\left(\frac{(T_j-\pi) A_{i_1 j}^{\text{pre}}}{g_0(i_1)} \left(\sum_{k\neq i_1} E_{i_1 k} + \sum_{(k,l)} E_{i_1 kl}\right), \frac{(T_j-\pi) A_{i_2 j}^{\text{pre}}}{g_0(i_2)} \left(\sum_{k\neq i_2} E_{i_2 k} + \sum_{(k,l)} E_{i_2 kl}\right)\right) \asymp \frac{(q_n^{\text{pre}})^2}{(n q_n^{\text{post}})^2}
\end{aligned} \tag{E.17}$$

For S_{13}^1, we have

$$\begin{aligned}&\mathrm{Cov}\left(\frac{(T_{j_1}-\pi)A_{ij_1}^{\mathrm{pre}}r_{1,i}}{g_0(i)},\frac{(T_{j_2}-\pi)A_{ij_2}^{\mathrm{pre}}r_{1,i}}{g_0(i)}\right)\\ =&\frac{1}{(n-1)^4}\mathrm{Cov}\left(\frac{(T_{j_1}-\pi)A_{ij_1}^{\mathrm{pre}}}{g_0(i)}\left(\sum_{k\neq i}E_{ik}+\sum_{(k,l)}E_{ikl}\right),\frac{(T_{j_2}-\pi)A_{ij_2}^{\mathrm{pre}}}{g_0(i)}\left(\sum_{k\neq i}E_{ik}+\sum_{(k,l)}E_{ikl}\right)\right)\asymp\frac{(q_n^{\mathrm{pre}})^2}{n^3q_n^{\mathrm{post}}\max\{q_n^{\mathrm{pre}},q_n^{\mathrm{post}}\}^2}.\end{aligned}\tag{E.18}$$

For S_{14}^1, we have

$$\begin{aligned}&\mathrm{Cov}\left(\frac{(T_{j_1}-\pi)A_{i_1j_1}^{\mathrm{pre}}r_{1,i_1}}{g_0(i_1)},\frac{(T_{j_2}-\pi)A_{i_2j_2}^{\mathrm{pre}}r_{1,i_2}}{g_0(i_2)}\right)\\ =&\frac{1}{(n-1)^4}\mathrm{Cov}\left(\frac{(T_{j_1}-\pi)A_{i_1j_1}^{\mathrm{pre}}}{g_0(i_1)}\left(\sum_{k\neq i_1}E_{i_1k}+\sum_{(k,l)}E_{i_1kl}\right),\frac{(T_{j_2}-\pi)A_{ij_2}^{\mathrm{pre}}}{g_0(i)}\left(\sum_{k\neq i}E_{ik}+\sum_{(k,l)}E_{ikl}\right)\right)\asymp\frac{(q_n^{\mathrm{pre}})^2}{n^3q_n^{\mathrm{post}}\max\{q_n^{\mathrm{pre}},q_n^{\mathrm{post}}\}}.\end{aligned}\tag{E.19}$$

By combining these results, we have $\mathrm{Var}(S_1^1)=O\left(\frac{1}{n^3(q_n^{\mathrm{post}})^2}\right)$. In particular, this implies that

$$S_1^1=O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}nq_n^{\mathrm{post}}}\right)+O_{\mathbb{P}}\left(\frac{1}{n^2q_n^{\mathrm{post}}\max\{q_n^{\mathrm{pre}},q_n^{\mathrm{post}}\}}\right).\tag{E.20}$$

Next, for S_2^1, define $C_{ij}=\frac{A_{ij}^{\mathrm{pre}}r_{1,i}(N_i-(n-1)q_n^{\mathrm{pre}}g_0(i))}{(n-1)q_n^{\mathrm{pre}}g_0(i)N_i}$. Then we bound S_2^1 in L_2 norm:

$$\mathbb{E}[(S_2^1)^2]=\frac{1}{n^2}\left[\begin{array}{c}\sum_{j=1}^n\mathbb{E}\left[(T_j-\pi)^2\sum_{i\neq j}^nC_{ij}^2\right]+\sum_{j=1}^n\mathbb{E}\left[(T_j-\pi)^2\sum_{(i_1,i_2)}C_{i_1j}C_{i_2j}\right]\\ +\sum_{(j,k)}\sum_{i\neq j}\mathbb{E}\left[(T_j-\pi)(T_k-\pi)C_{ij}C_{ik}\right]+\sum_{(j,k)}\sum_{(i_1,i_2)}\mathbb{E}\left[(T_j-\pi)(T_k-\pi)C_{i_1j}C_{i_2k}\right].\end{array}\right]$$

For the first term, by (E.7) and analogous argument to (E.16), we have

$$\begin{aligned}\mathbb{E}\left[(T_j-\pi)^2C_{ij}^2\right]=&\mathbb{E}\left[\mathbb{E}\left[(T_j-\pi)^2A_{ij}^{\mathrm{pre}}r_{1,i}^2|w\right]\mathbb{E}\left[\left(\frac{(N_i-(n-1)q_n^{\mathrm{pre}}g_0(i))}{(n-1)q_n^{\mathrm{pre}}N_i}\right)^2|w\right]\right]\\ \leq&\frac{1}{(nq_n^{\mathrm{pre}})^3}\mathbb{E}\left[(T_j-\pi)^2A_{ij}^{\mathrm{pre}}r_{1,i}^2\right]=O\left(\frac{1}{n^5(q_n^{\mathrm{post}})^2(q_n^{\mathrm{pre}})^2}\right).\end{aligned}$$

For the second term, by (E.6) and analogous argument to (E.17), we have

$$\mathbb{E}\left[(T_j-\pi)^2C_{i_1j}C_{i_2j}\right]=O\left(\frac{1}{n^6(q_n^{\mathrm{post}})^2(q_n^{\mathrm{pre}})^2}\right).$$

For the third term, by (E.7) and analogous argument to (E.18), we have

$$\mathbb{E}\left[(T_j-\pi)(T_k-\pi)C_{ij}C_{ik}\right]=O\left(\frac{\min\{q_n^{\mathrm{pre}},q_n^{\mathrm{post}}\}^2}{n^6(q_n^{\mathrm{pre}})^3(q_n^{\mathrm{post}})^3}\right).$$

For the fourth term, by (E.6) and analogous argument to (E.19), we have

$$\mathbb{E}\left[(T_j-\pi)(T_k-\pi)C_{i_1j}C_{i_2k}\right] = O\left(\frac{\min\{q_n^{\text{pre}},q_n^{\text{post}}\}^2}{n^8 q_n^{\text{post}}(q_n^{\text{pre}})^4}\right) + O\left(\frac{\min\{q_n^{\text{pre}},q_n^{\text{post}}\}}{n^7(q_n^{\text{post}})^2(q_n^{\text{pre}})^3}\right).$$

By combining these results, we have $\mathbb{E}\left[(S_2^1)^2\right] = O\left(\frac{1}{n^5(q_n^{\text{post}})^2(q_n^{\text{pre}})^2}\right)$ and thus

$$S_2^1 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}n^2 q_n^{\text{pre}} q_n^{\text{post}}}\right). \tag{E.21}$$

In particular, combining (E.20) and (E.21) implies that

$$\frac{1}{n}\sum_{i=1}^{n}(Z_i^{\text{alt}}-\pi)r_{1,i} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}nq_n^{\text{post}}}\right) + O_{\mathbb{P}}\left(\frac{1}{n^2 q_n^{\text{post}}\max\{q_n^{\text{pre}},q_n^{\text{post}}\}}\right). \tag{E.22}$$

By combining (E.15) and (E.22), we have

$$\frac{1}{n}\sum_{i=1}^{n}(Z_i^{\text{alt}}-\pi)(r_{0,i}+r_{1,i}) = \mathbb{E}[S_1^0] + O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right). \tag{E.23}$$

Combining (E.11), (E.15) and (E.23) together completes the proof.

□

Proof of Theorem E.1. The proof is analogous to that of Theorem 3.2 but by Lemma E.1 and Lemma E.2. We start with deriving the probability limit of $\left(\tilde{Z}^{\text{alt}\top}X\right)^{-1}$:

$$\left(\tilde{Z}^{\text{alt}\top}X\right)^{-1} = \frac{1}{\det(\tilde{Z}^{\text{alt}\top}X)}\begin{pmatrix} * & * & * \\ * & d_{22} & d_{23} \\ * & d_{32} & d_{33}\end{pmatrix}$$

where $d_{22} = \overline{MZ^{\text{alt}}} - \overline{M}\cdot\overline{Z^{\text{alt}}}, d_{23} = b_{23}, d_{32} = -\overline{TZ^{\text{alt}}} + \overline{T}\cdot\overline{Z^{\text{alt}}}, d_{33} = b_{33}$ and $\det(\tilde{Z}^{\text{alt}\top}X) = d_{33}d_{22} - d_{32}d_{23}$. Let $*$ denote the terms that are not relevant. By Lemma E.1, we have $d_{32} = O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{pre}}}}\right)$ and $\frac{1}{n}\sum_{i=1}^{n}(Z_i^{\text{alt}} - \overline{Z^{\text{alt}}})u_i = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$.
We consider the remaining terms under Cases (a) and (b), respectively.
Case (a) By Lemma E.2, we have $d_{22} = \mathbb{E}(S_1^0) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + o_{\mathbb{P}}\left(\frac{1}{n\max\{q_n^{\text{pre}},q_n^{\text{post}}\}}\right)$. By Lemma B.2, we have $d_{23} = -\text{Cov}(T_i, \xi_i + \mu_{r_1,i}) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) = O_{\mathbb{P}}(1)$. By combining the results above, we have

$$\det(\tilde{Z}^{\text{alt}\top}X) = \pi(1-\pi)\mathbb{E}(S_1^0) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + o_{\mathbb{P}}\left(\frac{1}{n\max\{q_n^{\text{pre}},q_n^{\text{post}}\}}\right).$$

Case (b) By Lemma E.2, we have $d_{22} = \mathbb{E}(S_1^0) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}nq_n^{\text{post}}}\right) + o_{\mathbb{P}}\left(\frac{1}{n\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}\right)$. By Lemma B.2, we have $d_{23} = O_{\mathbb{P}}\left(\frac{1}{n\sqrt{q_n^{\text{post}}}}\right)$. Hence,

$$\det(\tilde{Z}^{\text{alt}\top} X) = \pi(1-\pi)\mathbb{E}(S_1^0) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}nq_n^{\text{post}}}\right) + o_{\mathbb{P}}\left(\frac{1}{n\max\{q_n^{\text{pre}}, q_n^{\text{post}}\}}\right).$$

The convergence rate of $\hat{\beta}^{\text{alt}}$ follows from the above results.

□

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