Market Failure, the QE Trap and Fiscal Escape

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Abstract

We show that a financial crisis depleting intermediary capital can destroy competitive equilibrium—not merely distort prices—making large-scale asset purchases a necessary condition for market clearing. The central bank acts as a *risk-bearer of last resort*. However, QE can become permanent: below a critical capital threshold, the economy enters the *Permanent QE* zone with no exit. Even above it, recovery is so slow that QE persists for centuries—the *QE Trap*. Fiscal recapitalisation of the financial sector is the only practical escape, with the employment multiplier convex in crisis severity. In the unconstrained economy, a shortage of risk-bearing capacity—the *risk-bearing gap*—simultaneously inflates asset prices above fundamental values (the intermediation gap) and depresses employment and output (the output gap). A leverage paradox links these results: the same leverage constraint that destroys equilibrium also eliminates the risk-bearing gap, so no economy in this class can simultaneously have correct asset prices and unconditional market clearing.

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1 Introduction

The 2008 financial crisis depleted the capital of the financial institutions that bear much of the economy's aggregate risk, triggering unprecedented central bank asset purchases.[1] In the United States, the Federal Reserve expanded its balance sheet from under $1 trillion to over $4 trillion; the subsequent unwinding took a decade and remained incomplete when the COVID-19 pandemic necessitated further expansion. A second quantitative tightening began in mid-2022 but was halted in 2025 with the balance sheet still above $6 trillion. Meanwhile, the Troubled Asset Relief Program (TARP) injected capital directly into the financial sector. Similar interventions followed in Europe, though with significant delays in bank recapitalisation (Brunnermeier, James, and Landau, 2016). Two questions arise: when is QE necessary, and how does one escape the QE trap of permanent asset purchases and depressed output?

This paper departs from the intermediary asset pricing literature—including He and Krishnamurthy (2013, 2019) and Brunnermeier and Sannikov (2014), whose foundational contribution we build directly on—in one critical dimension: in those models, equilibrium exists with or without the constraint; the constraint distorts prices and amplifies shocks, but the market always clears. Here, below a threshold, no competitive equilibrium exists at all. The central bank's role is therefore not to improve upon a distorted outcome but to restore the *existence* of an outcome. This resolves the paradox of Bernanke (2014): QE "works in practice but not in theory" only in a literature that treats equilibrium as given; once the leverage constraint binds hard enough to destroy existence, QE is no longer a welfare choice but a necessary condition for any market outcome at all.

We answer both questions in a continuous-time production economy with monopolistically competitive firms, flexible prices, and endogenous labour supply—a real model without money or a zero lower bound. Two types of agents inhabit the economy: financiers, who hold equity and bear all aggregate risk, and households, who save only through risk-free bonds. Financiers face leverage constraints arising from moral hazard and prudential regulation—frictions that long predate any crisis. Because households supply labour, financial frictions affect employment and output directly. The entire economy is characterised by a single state variable: the financier's share of total wealth, ω. When leverage constraints do not bind, the financier's wealth share drifts toward the efficient benchmark and the economy functions well—explaining why crises are surprises, not inevitabilities. We establish four sets of results.

[1] Strictly, *quantitative easing* refers to expanding central bank reserves, *large-scale asset purchases* (LSAPs) to purchasing specific securities, and *credit easing* to targeting particular credit markets. In practice, and in academic usage, these terms are used interchangeably. Following common convention, we use "QE" throughout.

First, equilibrium failure. When a large shock depletes financier capital and forces leverage constraints to bind, competitive equilibrium ceases to exist. The household's bond demand is perfectly inelastic, so once the constrained financier cannot absorb the entire equity supply, excess supply emerges that no price adjustment can eliminate.[2] This is not a recession or a market crash—the price system itself breaks down. Three ingredients interact: limited participation, leverage constraints, and depleted capital—and all three were present in 2008. In the moderately distressed region ($\omega \in (\underline{\omega}, \omega_c)$), all three are needed: removing any one restores equilibrium. In the severely distressed region ($\omega < \underline{\omega}$), the result is stronger: the leverage constraint forces the financier below full equity investment ($\bar{\phi} < 1$), and even full household equity participation cannot compensate—equilibrium fails regardless of the household's portfolio (Theorem 48).

Second, the central bank as risk-bearer of last resort. A central bank that purchases the equity the constrained financier cannot hold, financing these purchases by issuing bonds, restores market clearing and equilibrium existence. In every existing theory of QE, asset purchases improve upon a distorted but viable competitive outcome—through portfolio balance effects (Vayanos and Vila, 2021), credit intermediation (Gertler and Karadi, 2011), liquidity provision (Cui and Sterk, 2021), or risk redistribution (Silva, 2020). Our result is different in kind: without central bank purchases no competitive outcome exists at all. The central bank acts not as a lender of last resort or a market-maker of last resort, but as a *risk-bearer of last resort.*

Third, Permanent QE, the QE Trap, and fiscal escape. The restored equilibrium has two steady states. At the good steady state the financier is fully capitalised, no constraints bind, and the economy coincides with the frictionless benchmark. At the bad steady state the financier is minimally capitalised, the central bank permanently holds a large share of the equity market, and output is permanently depressed. If the crisis pushes ω below the bad steady state, QE is literally permanent (*Permanent QE*). Even above the bad steady state, the economy can be trapped: if the deterministic recovery time exceeds a policy-relevant horizon—centuries at typical post-crisis capital ratios—QE is permanent in practice though not in theory. We call this the *QE Trap*, extending a term introduced by Koo (2014); we provide the first general equilibrium characterisation with an analytical formula for the trap boundary (Online Appendix R). A one-time fiscal recapitalisation—a transfer large enough to push ω past the trap boundary into the self-resolving zone—is the only practical escape. The central bank prevents market failure; the fiscal authority breaks the trap.

[2]The equilibrium failure builds on the economic logic of Bernanke (1983)'s credit-channel insight: financial intermediaries embody specialised screening, monitoring, and risk-bearing capacity that households cannot replicate, so when intermediary capital is depleted, equity risk cannot simply migrate to households' balance sheets.

Fourth, the leverage paradox. Without the leverage constraints imposed by moral hazard and prudential regulation, the market for aggregate risk always clears. However, equilibrium asset prices exceed fundamental values—a rational bubble in the sense of Hugonnier (2012)—creating an *intermediation gap* (the component of asset prices above fundamental value) that grows convexly as financier capital erodes. Because ours is a production economy with endogenous labour, the intermediation gap *is* the output gap: the gap-to-price ratio maps one-to-one into the shortfall in employment and output. We call the common cause of both phenomena the *risk-bearing gap*—the economy's shortage of intermediary capacity to absorb aggregate risk. Constraining intermediary leverage eliminates the risk-bearing gap and hence both the intermediation gap in prices and the output gap in quantities. But when leverage constraints bind, they destroy equilibrium. The constraint that corrects prices breaks the market.

This tradeoff is inescapable: we prove that no economy in this class can simultaneously deliver zero intermediation and output gaps and unconditional market clearing. The mechanism is that $Q = 1/\omega$, the inverse financier wealth share, is a local martingale in all versions of the model; its classification as a strict local martingale (positive intermediation gap) or a true martingale (no intermediation gap) is governed by the same boundary behaviour of the intermediary wealth-share volatility that determines whether equilibrium can fail. The model-derived intermediation gap delivers out-of-sample R^2 for the output gap that rises from 14% at one quarter to 45% at two years—the opposite of the horizon decay typical of persistent-predictor overfitting.

1.1 Related Literature

QE and unconventional monetary policy. The existing literature identifies several channels: portfolio balance effects on term premia (Gagnon et al., 2011; Krishnamurthy and Vissing-Jørgensen, 2011; Vayanos and Vila, 2021), credit substitution for impaired intermediation (Gertler and Karadi, 2011; Gertler and Kiyotaki, 2010; Del Negro et al., 2017), liquidity transformation (Cui and Sterk, 2021), the central bank balance sheet as an independent instrument (Cúrdia and Woodford, 2011), government debt maturity management (Greenwood, Hanson, and Stein, 2015), and central bank balance sheet sustainability (Reis, 2015, 2016, 2019). Reis (2015, 2019) analyses the fiscal consequences of large central bank balance sheets—solvency risks, seignorage constraints, and limits to fiscal burden alleviation—in nominal monetary frameworks. Our model is real (no money, no inflation), but reaches a complementary conclusion: the central bank can prevent market failure but cannot substitute for fiscal recapitalisation of the private financial sector. The closest paper

to ours is Silva (2020), who studies a risk channel of QE in a continuous-time production economy with heterogeneous risk tolerance. In every one of these models, equilibrium exists with or without intervention; QE is a policy choice. Our result is qualitatively different: below a threshold, no equilibrium exists without central bank purchases.

Financial frictions, traps, and recapitalisation. The paper builds on the intermediary asset pricing literature (Brunnermeier and Sannikov, 2014; He and Krishnamurthy, 2013, 2019; Di Tella, 2017), sharing the feature that a single state variable drives all equilibrium quantities but generating the additional result that equilibrium can fail entirely. Our QE Trap differs from existing trap concepts along four dimensions: (i) *non-existence* of competitive equilibrium, not merely distortion; (ii) scarcity of risk-bearing capacity rather than safe assets (Caballero and Farhi, 2018), demand shortfalls from balance-sheet deleveraging (Eggertsson and Krugman, 2012; Guerrieri and Lorenzoni, 2017), or liquidity (Bigio, 2015); (iii) *Permanent QE* at the bad steady state and a *QE Trap* where recovery takes centuries, requiring fiscal transfer, not self-correcting or escapable through monetary policy alone (Rognlie, Shleifer, and Simsek, 2018); and (iv) the leverage paradox linking the trap mechanism to asset pricing. On recapitalisation, Philippon and Schnabl (2013) and Veronesi and Zingales (2010) study the optimal form of intervention, taking its necessity as given; we derive the necessity from the steady-state structure.

Rational bubbles and the intermediation gap. Basak and Cuoco (1998) study the impact of limited household participation on asset prices in an exchange economy, whereas we study a production economy with endogenous labour and a central bank. Hugonnier (2012) proves that portfolio constraints generate rational bubbles in otherwise bubble-free economies. In our model, leverage constraints can *close* rather than open the intermediation gap—linking the rational bubble literature to the equilibrium existence problem. We derive an exact ODE for the intermediation gap in a production economy, replacing the constant-volatility assumption of Hugonnier (2012)'s exchange setting, and show that the gap-to-price ratio delivers horizon-stable out-of-sample predictive content for the output gap.

The remainder of the paper is organised as follows. Section 2 presents the model. Section 3 establishes the equilibrium failure result. Section 4 introduces QE, derives the two-steady-state equilibrium, and analyses asset pricing implications. Section 5 establishes the intermediation gap results. Section 6 links the intermediation gap to real outcomes and derives policy implications. Section 7 presents empirical evidence. Section 8 concludes. All proofs are in the Appendix.

2 Model

2.1 Environment

Time is continuous with $t \in [0, \infty)$. Uncertainty is driven by a standard Brownian motion $\{Z_t\}_{t \geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions.

The economy is populated by a representative household, a representative financier, and a continuum of monopolistically competitive firms indexed by $f \in [0, 1]$.

Technology and firms. Each firm f produces a differentiated good using the linear technology $Y_t(f) = A_t N_t(f)$, where A_t is aggregate productivity and $N_t(f)$ is labour input. Productivity evolves as geometric Brownian motion:

$$\frac{\mathrm{d}A_t}{A_t} = \mu \, \mathrm{d}t + \sigma \, \mathrm{d}Z_t, \qquad \mu \in \mathbb{R}, \ \sigma > 0. \tag{1}$$

Households aggregate differentiated goods via the Dixit–Stiglitz aggregator with elasticity of substitution $\epsilon > 1$. With flexible prices, each firm sets a constant markup over marginal cost. In symmetric equilibrium, factor shares are constant:

$$D_t = \frac{1}{\epsilon} Y_t, \qquad \frac{W_t}{P_t} N_t = \frac{\epsilon - 1}{\epsilon} Y_t, \tag{2}$$

where D_t denotes aggregate dividends, W_t / P_t is the real wage, $Y_t = A_t N_t$ is aggregate output and $N_t = \int_0^1 N_t(f) \, \mathrm{d}f$ is aggregate employment.

Preferences. The household has preferences:

$$U_h = \mathbb{E} \int_0^\infty e^{-\delta t} \left(\ln C_{h,t} - \frac{N_t^{1+\varphi}}{1+\varphi} \right) \mathrm{d}t, \qquad \delta > 0, \ \varphi \geq 0. \tag{3}$$

The financier has logarithmic preferences over consumption:

$$U_i = \mathbb{E} \int_0^\infty e^{-\delta t} \ln C_{i,t} \, \mathrm{d}t. \tag{4}$$

Both agents share the same discount rate δ.[3]

[3] Online Appendix N shows that the equilibrium failure result extends to CRRA preferences with elasticity of intertemporal substitution near unity, and to partial household equity participation. Online Appendix O endogenises the limited participation restriction: with Knightian uncertainty of just 1.2% of the Sharpe ratio, the household optimally holds zero equity. This formalises the sense in which households lack the screening and monitoring capacity that Bernanke (1983) identified as the distinguishing feature of financial intermediation: absent the specialised information and relationship capital embodied in the financial sec-

Financial markets and participation. Two financial assets are traded: a risk-free bond in zero net private supply paying rate r_t, and an equity claim on aggregate dividends in unit net supply. The household can invest only in the risk-free bond. The financier invests in both bonds and equity, with portfolio equity share ϕ_t (the fraction of her wealth allocated to equity). The household's budget constraint is deterministic:

$$\mathrm{d}H_{h,t} = \left(r_t H_{h,t} + \frac{W_t}{P_t} N_t - C_{h,t} \right) \mathrm{d}t, \tag{5}$$

where $H_{h,t}$ is the household's financial wealth. The financier's budget constraint is:

$$\mathrm{d}H_{i,t} = (r_t H_{i,t} - C_{i,t})\ \mathrm{d}t + \phi_t H_{i,t} \left(\frac{\mathrm{d}P_{D,t} + D_t\,\mathrm{d}t}{P_{D,t}} - r_t\,\mathrm{d}t \right), \tag{6}$$

where $H_{i,t}$ is the financier's financial wealth and $P_{D,t}$ is the equity price. Since households cannot hold equity, market clearing requires $\phi_t H_{i,t} = P_{D,t}$.

Assumption 1 (Environment parameters)**.** *$\epsilon > 1$, $\mu \in \mathbb{R}$, $\sigma > 0$, $\delta > 0$, and $\varphi \geq 0$.*

2.2 Leverage Constraint

The financier faces a state-dependent leverage limit:

$$\phi_t \leq \bar{\phi}(\omega_t), \qquad \bar{\phi}(\omega) \equiv 1 + \lambda(\omega - \underline{\omega}), \tag{7}$$

where $\omega_t \equiv H_{i,t}/P_{Y,t}$ is the financier's share of total wealth $P_{Y,t}$, $\underline{\omega} \in (0, 1/\epsilon)$ is the minimum capital threshold, and $\lambda > 0$ is the leverage sensitivity, with the joint restriction $\lambda\underline{\omega} \leq 1$ so that $\bar{\phi}(0) \geq 0$.

Assumption 2 (Leverage parameters)**.** *$0 < \underline{\omega} < 1/\epsilon$, $\lambda > 0$, and $\lambda\underline{\omega} \leq 1$.*

The constraint combines two frictions that predate quantitative easing. *Above the threshold* ($\omega \geq \underline{\omega}$): following Gertler and Kiyotaki (2010) and Gertler and Karadi (2011), the financier can divert a fraction of assets for private benefit. Households lend only against verifiable collateral: the capital buffer above the regulatory minimum, yielding (7) with λ parameterising enforcement efficiency. In the Appendix (Proposition 37), we show that log utility generates a closed-form franchise value of intermediation, making (7) a conservative bound. *Below the threshold* ($\omega < \underline{\omega}$): under Prompt Corrective Action (PCA), statutory since 1991 (FDIC, 1991), undercapitalised intermediaries face mandatory restrictions on

tor, households face ambiguity about aggregate risks that intermediaries, through their informational and screening advantage, do not.

risky asset holdings, again yielding (7).[4] Critically, in this region $\bar{\phi}(\omega) = 1 + \lambda(\omega - \underline{\omega}) < 1$: PCA forces the financier to hold *less* than 100% of her wealth in equity—she must liquidate risky positions and shift into safe assets. This forced de-equitisation is not a modelling convenience but an institutional fact: PCA mandates progressively harsher corrective actions as capital falls below the minimum, culminating in mandatory receivership. For the entire pre-crisis period, these constraints existed but did not bind; what changed in 2008 was the *state of financier balance sheets*.

In the taxonomy of He, Kelly, and Manela (2017, Section 4.3), our constraint is a *leverage constraint*: it directly caps the portfolio equity share ϕ, and the cap tightens as financier wealth ω falls, producing procyclical deleveraging. This contrasts with the *equity constraints* of He and Krishnamurthy (2012, 2013), under which the intermediary must maintain a minimum equity stake but realised leverage rises countercyclically as equity is depleted. The qualitative implications of this modelling choice for the long-run dynamics are developed in §4 and Online Appendix N.

2.3 Benchmarks and Baseline Dynamics

When both agents can trade all assets, the economy admits a complete-markets benchmark in which consumption shares, labour supply, and total wealth are constant, with $P_{Y,t} = Y_t/\delta$ (Appendix A).

With limited participation but no leverage constraint, the entire equilibrium is characterised by the financier's wealth share $\omega_t \equiv H_{i,t}/P_{Y,t}$. Goods-market clearing under log utility pins consumption by wealth: $C_{h,t} = (1-\omega_t)Y_t$ and $C_{i,t} = \omega_t Y_t$. The financier's log SDF and the Dixit–Stiglitz markup together pin total wealth $P_{Y,t} = Y_t/\delta$, equity value $P_{D,t} = P_{Y,t}/\epsilon$, and human capital $P_{L,t} = (1-1/\epsilon)P_{Y,t}$; the household's financial wealth follows as the accounting residual $H_{h,t} = (1/\epsilon - \omega_t)P_{Y,t}$. Every stock, flow, and price is therefore a function of the single state variable ω_t, and this structure is preserved when the leverage constraint is added in Section 3.

The inverse wealth share $Q_t = 1/\omega_t$ is a non-negative local martingale and $\omega_t \to 1/\epsilon$ almost surely, but with infinite expected hitting time (Appendix B). This "Zeno's paradox" rationalises why the pre-crisis US economy functioned seamlessly for decades: financiers were

[4] PCA, enacted as Title I of the Federal Deposit Insurance Corporation Improvement Act of 1991, classifies insured depository institutions into five capital categories (well capitalised, adequately capitalised, undercapitalised, significantly undercapitalised, critically undercapitalised) and imposes escalating mandatory supervisory actions as capital falls. Undercapitalised institutions face restrictions on asset growth, acquisitions, and dividends; significantly undercapitalised institutions must divest risky assets and may be required to raise capital or merge; critically undercapitalised institutions face mandatory receivership within 90 days. The constraint $\bar{\phi}(\omega) < 1$ for $\omega < \underline{\omega}$ captures this escalating forced deleveraging in reduced form.

sufficiently capitalised that the leverage constraint, though present, never bound.

3 Capital Depletion Causes Equilibrium Failure

The intermediary asset-pricing literature has studied at length how depleted capital distorts asset prices. We show it can do more than distort: under the leverage constraint (7), capital depletion below a threshold ω_c destroys competitive equilibrium altogether—no price vector clears the market for aggregate risk. The mechanism rests on two institutional primitives: limited participation and a state-dependent leverage cap, both salient features of the post-2008 U.S. financial sector, and is triggered by realised capital depletion. Log utility delivers the cleanest argument but is inessential: the non-existence result survives under CRRA and Epstein–Zin preferences, with ω_c itself preference-independent (Section 3.2).

The argument rests on three observations, each elementary, that combine into a non-existence result.

1. Household bond demand is perfectly inelastic to the interest rate. Under log utility the household consumes a fraction δ of her wealth at every price vector, so her financial wealth is pinned by ω through the accounting identity $(1-\omega_t)P_{Y,t} = H_{h,t} + P_{L,t}$. With human capital $P_{L,t} = (1-1/\epsilon)P_{Y,t}$ priced by the financier's log SDF, the household's bond demand is

$$B^d_{h,t} \;=\; H_{h,t} \;=\; \left(\frac{1}{\epsilon} - \omega_t\right) P_{Y,t}. \tag{8}$$

Because $P_{Y,t} = Y_t/\delta$ is itself interest-rate-independent under log utility, equation (8) is independent of r_t: income and substitution effects of interest-rate changes cancel exactly. No interest-rate movement, however extreme, induces the household to substitute from bonds into equity—her demand schedule is *vertical* at every r. This disables the standard safety-valve of price adjustment, a friction qualitatively sharper than the zero-lower-bound constraint of Caballero and Farhi (2018): there, conventional policy fails at a corner of an existing equilibrium; here no lower bound is needed because the demand curve is rigid at every rate, positive or negative.

2. Market clearing requires the financier to absorb all equity, giving a hyperbolic required portfolio share. By limited participation, the financier is the sole potential equity holder. With $P_{D,t} = P_{Y,t}/\epsilon$ (dividend share $D/Y = 1/\epsilon$ priced by the financier's log SDF) and $H_{i,t} = \omega_t P_{Y,t}$ (definition of ω), equity-market clearing requires

$$\phi^\star(\omega) \;\equiv\; \frac{P_D}{H_i} \;=\; \frac{1}{\epsilon\,\omega}. \tag{9}$$

This required portfolio share is hyperbolic in ω and explodes as intermediary capital is depleted. It reflects an accounting fact: the aggregate equity market does not shrink with financier balance sheets, so as ω falls the same fixed claim must be levered onto a shrinking capital base.

3. Permitted leverage is linear and bounded. The constraint (7) caps portfolio weight at $\bar{\phi}(\omega) = 1+\lambda(\omega-\underline{\omega})$, a line that equals one at the PCA floor and rises with slope λ. Required leverage is a hyperbola in ω; permitted leverage is a line in ω; they cross exactly once in the interior at a point we call ω_c. Figure 1 shows the geometry.

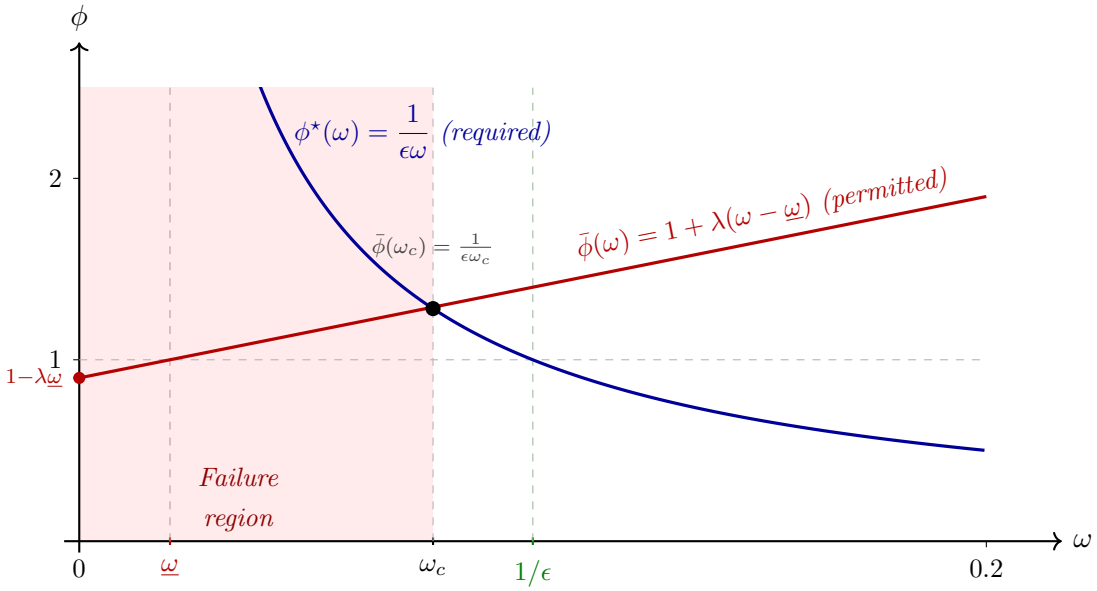


Figure 1: The equilibrium-failure mechanism. *Required* portfolio share $\phi^\star(\omega) = 1/(\epsilon\omega)$ (blue hyperbola) is what the financier must hold under limited participation to clear the equity market; it explodes as intermediary capital is depleted. *Permitted* portfolio share $\bar{\phi}(\omega) = 1 + \lambda(\omega - \underline{\omega})$ (red line) is the leverage cap, equal to one at the PCA floor $\underline{\omega}$ and rising with slope λ. The two curves cross exactly once, at ω_c; for $\omega < \omega_c$ (shaded), required exceeds permitted and no price vector clears both markets simultaneously. The line meets the vertical axis at $\bar{\phi}(0) = 1-\lambda\underline{\omega}$. The state variable ω ranges in $(0, 1)$; the figure zooms into the crisis region, as all interesting structure is concentrated near $\omega = 0$. Baseline calibration: $\epsilon = 10$, $\underline{\omega} = 0.02$, $\lambda = 5$, giving $\omega_c \approx 0.078$.

Theorem 1 (Equilibrium failure)**.** *With limited household stock-market participation and leverage-constrained financiers, competitive equilibrium fails to exist for all $\omega_t < \omega_c$ in the absence of central-bank intervention, where ω_c is the unique positive solution of*

$$\bar{\phi}(\omega_c) = \frac{1}{\epsilon\,\omega_c} \qquad \Longleftrightarrow \qquad \lambda\omega_c^2 + (1-\lambda\underline{\omega})\omega_c - 1/\epsilon = 0, \tag{10}$$

i.e.

$$\omega_c \;=\; \frac{\underline{\omega}}{2} - \frac{1}{2\lambda} \;+\; \sqrt{\left(\frac{\underline{\omega}}{2} - \frac{1}{2\lambda}\right)^2 + \frac{1}{\epsilon\lambda}}. \tag{11}$$

Proof sketch. For $\omega_t < \omega_c$, observation 2 gives $\phi^\star(\omega_t) > \bar{\phi}(\omega_t)$: the portfolio weight market clearing requires exceeds the cap. The residual excess equity supply

$$\left[\phi^\star(\omega_t) - \bar{\phi}(\omega_t)\right] \omega_t P_{Y,t} \;=\; \left[\frac{1}{\epsilon\omega_t} - \bar{\phi}(\omega_t)\right] \omega_t P_{Y,t} \;>\; 0$$

has no buyer. The household's bond demand (8) is independent of r (observation 1), so no interest-rate adjustment transfers excess onto her balance sheet, and limited participation rules out her holding equity at any price. No price vector clears both markets simultaneously. The full argument is in Appendix E. □

Remark 1 (Closed form for ω_c)**.** *The state variable is confined to the interval $(\underline{\omega}, 1/\epsilon)$: the upper endpoint is the frictionless benchmark—at $\omega = 1/\epsilon$ the financier voluntarily holds $\phi = 1$, no leverage is needed, and the economy attains its complete-markets allocation—while the lower endpoint is the PCA floor, where the constraint forces $\bar{\phi}(\underline{\omega}) = 1$, the volatility of ω vanishes, and the floor cannot be crossed. A post-crisis economy that has fallen below $1/\epsilon$ therefore sits between the two: drawn upward toward the frictionless benchmark, blocked from below by the impassable floor.*

A natural measure of the severity of market failure is the fraction of the interval $(\underline{\omega}, 1/\epsilon)$ in which equilibrium fails. Rearranging (10):

$$\frac{\omega_c - \underline{\omega}}{1/\epsilon - \underline{\omega}} \;=\; \frac{1}{1 + \lambda\,\omega_c}. \tag{12}$$

The factor $1 + \lambda\omega_c$ is the economy's leverage tolerance at the crisis threshold*; its reciprocal is the fraction of the good basin that cannot clear markets without central-bank intervention.*

Numerical sensitivity. *At the empirical calibration (Table 1), holding $\epsilon = 13.80$ and $\underline{\omega} = 0.026$ fixed:*

	λ	ω_c	$\omega_c - \underline{\omega}$	$\dfrac{\omega_c - \underline{\omega}}{1/\epsilon - \underline{\omega}}$
Half of calibrated	8.7	0.057	0.031	67%
Calibrated	17.4	0.051	0.025	53%
Double	34.8	0.044	0.018	39%

At the calibrated λ, ***for*** 53% ***of the interval*** $(\underline{\omega}, 1/\epsilon)$ ***markets fail to clear****: equilibrium failure is not a knife-edge phenomenon but a broad condition that dominates over half the*

interval. Halving λ pushes this fraction to 67%; doubling it shrinks it to 39%. Comparative statics $\partial\omega_c/\partial\lambda < 0$ and $\partial\omega_c/\partial\underline{\omega} > 0$ follow directly from (11). Inverting for λ given $(\epsilon, \underline{\omega}, \omega_c)$ yields $\lambda = (1/\epsilon - \omega_c)/[\omega_c(\omega_c - \underline{\omega})]$: this is how λ is recovered from the calibration.

Theorem 1 is a statement about equilibrium *existence*, not distortion: for $\omega < \omega_c$ without a central bank, no price vector simultaneously clears the bond and equity markets at the quantities budget constraints permit. Its three ingredients each correspond to a feature of the 2008 U.S. financial sector: *limited participation* to the inability of the non-bank public to backstop intermediaries in wholesale funding markets; *leverage constraints* to the interaction of risk-weighted capital requirements, moral-hazard enforcement, and Prompt Corrective Action; *depleted capital* to the realised collapse in broker-dealer and bank-holding-company equity. In the region $(\underline{\omega}, \omega_c)$, all three are needed: relax any one and equilibrium is restored. Below $\underline{\omega}$, the failure is unconditional: Prompt Corrective Action drives $\bar{\phi}(\omega) < 1$, and even with the household at full participation and all her financial wealth $(1/\epsilon - \omega)P_{Y,t}$ in equity, a strictly positive residual $\omega\,[1 - \bar{\phi}(\omega)]\,P_{Y,t}$ remains unabsorbed—the non-tradable human-capital component of household wealth cannot be redirected to fill the gap (Theorem 48, Online Appendix K).

3.1 Dynamics of ω and the Onset of Failure

Theorem 1 characterises failure at a fixed ω. To see how the economy *reaches* the failure region, we write the law of motion for ω_t on the domain where equilibrium exists. For $\omega_t \in (\omega_c,\, 1/\epsilon)$ the leverage constraint (7) does not bind: required leverage $\phi^\star(\omega_t) = 1/(\epsilon\omega_t)$ is below the cap $\bar{\phi}(\omega_t)$, the financier freely absorbs all equity, and ω_t evolves as the one-dimensional diffusion

$$\mathrm{d}\omega_t = \frac{\sigma_\omega(\omega_t)^2}{\omega_t}\,\mathrm{d}t + \sigma_\omega(\omega_t)\,\mathrm{d}Z_t, \tag{13}$$

with volatility

$$\sigma_\omega(\omega) \;=\; \frac{\sigma(1+\varphi)(1-\omega)(1/\epsilon-\omega)}{(1+\varphi)(1-\omega)-(1/\epsilon-\omega)} \;>\; 0 \qquad \text{for } \omega \in (\omega_c,\, 1/\epsilon). \tag{14}$$

Equations (13)–(14) are derived in Theorem 17 of Appendix B from the financier's budget constraint, goods-market clearing under log utility, and the Dixit–Stiglitz factor shares. The inverse share $Q_t = 1/\omega_t$ is a non-negative local martingale: the drift $\mu_\omega(\omega) = \sigma_\omega(\omega)^2/\omega > 0$ is the financier's expected excess return from bearing equity risk, and ω_t drifts almost surely toward the frictionless steady state $\omega = 1/\epsilon$, where $\phi^\star(1/\epsilon) = 1$ and no leverage is needed.

Where failure arises. Note first that the formula (14) is smooth at ω_c and below; the boundary of existence is *not* visible in the SDE itself. The reason is that (13) is derived under the silent assumption that the financier freely chooses the market-clearing portfolio $\phi_t = \phi^\star(\omega_t) = 1/(\epsilon\omega_t)$. Feasibility of that choice — $\phi^\star(\omega_t) \leq \bar{\phi}(\omega_t)$ — holds iff $\omega_t \geq \omega_c$. For $\omega_t < \omega_c$, the portfolio the SDE tacitly assumes violates the leverage constraint (7): the equilibrium premise behind (13) breaks, and Theorem 1 applies.

Define the first-passage time to the crisis threshold:

$$\tau_c \;\equiv\; \inf\big\{\, t \geq 0 : \omega_t = \omega_c \,\big\}. \tag{15}$$

For any starting point $\omega_0 \in (\omega_c,\, 1/\epsilon)$, $\mathbb{P}(\tau_c < \infty) > 0$: although the drift favours upward motion toward $1/\epsilon$, the diffusion is non-degenerate and two-sided, so a sufficiently adverse run of productivity shocks depletes ω_t to ω_c in finite time. At $t = \tau_c$ the constraint binds with equality, $\bar{\phi}(\omega_c) = \phi^\star(\omega_c) = 1/(\epsilon\omega_c)$. The next adverse shock pushes $\omega_t < \omega_c$, the feasibility condition above fails, and no price vector clears both markets. *The equilibrium described by* (13) *terminates at* τ_c—not because the SDE's formula breaks down, but because its implicit assumption does. The threshold ω_c is a boundary of existence, not of the state space: reached from above, it marks the endpoint of the competitive equilibrium itself.

The picture is not that the economy sits mired in a failure region but that it is always one sufficiently adverse shock-sequence away from one. Decades of pre-2008 U.S. history with $\omega_t \gg \omega_c$ are entirely consistent with (13); what the crisis revealed was that the stopping time τ_c was in fact finite.

3.2 Which Assumptions Are Essential?

Theorem 1 relies on three modelling choices whose robustness merits scrutiny. Online Appendices N and O establish that the result is robust along each dimension; we summarise the findings here.

Assumption	Essential?	What happens if relaxed	Reference
Log utility (household)	No	The crisis threshold ω_c is *preference-independent*: it does not depend on the household's EIS ψ or risk aversion γ. Non-existence survives under CRRA and Epstein–Zin preferences.	Thm. 45, Thm. 53
Limited participation	Degree does not matter	Failure survives for any household equity share $\alpha_h \in [0, 1)$. The crisis threshold $\omega_c(\alpha_h)$ falls monotonically in α_h but remains strictly positive.	Thm. 40
Bounded intermediary leverage	Yes	Without an upper bound on ϕ, the financier absorbs all equity at any ω and no failure occurs. Under an alternative equity constraint $\bar{\phi} \equiv 1/a$, failure arises for $\omega < a/\epsilon$: boundedness is what matters, not the specific functional form (Online Appendix N).	Thm. 1, Prop. 55
Exogenous non-participation	No	Ambiguity aversion with $\kappa \approx 1.2\%$ of the Sharpe ratio generates endogenous non-participation at $\omega = \underline{\omega}$; $\kappa \approx 4.7\%$ suffices everywhere.	Thm. 48, Cor. 51

Household preferences. The crisis threshold ω_c is *completely preference-independent*: it depends on $(\epsilon, \lambda, \underline{\omega})$ but not on the household's elasticity of intertemporal substitution ψ or risk aversion γ. This is because the failure condition $\bar{\phi}(\omega)\,\omega < 1/\epsilon$ uses only the leverage constraint and limited participation; the household's Euler equation does not enter. The structural identity $P_D/P_Y = 1/\epsilon$ (from the Dixit–Stiglitz factor share and the financier's log SDF) ensures that the required leverage $\phi^\star(\omega) = 1/(\epsilon\omega)$ is preference-free.

We verify this explicitly under two generalisations. Under *CRRA* household preferences (a single parameter, $\gamma = 1/\psi$), the crisis threshold is identical to the log case for any $\psi > 0$ (Theorem 45). Under *Epstein–Zin* preferences (risk aversion γ and EIS ψ decoupled), ω_c remains preference-independent (Theorem 53). Dynamics above ω_c—in particular the convergence behaviour near the boundaries $\underline{\omega}$ and $1/\epsilon$—do depend on preferences, but the failure theorem itself does not. We return to the preference-sensitivity of the dynamics in §4 and Online Appendices J and M.

Robust failure below $\underline{\omega}$. Perhaps the strongest result is Theorem 48: for $\omega < \underline{\omega}$, no competitive equilibrium exists *regardless of the household's equity share*, including $\alpha_h = 1$. The leverage constraint forces $\bar{\phi}(\omega) < 1$ when $\omega < \underline{\omega}$, so the financier cannot even hold her own wealth entirely in equity. The household's financial wealth is too small to fill the gap. This result requires no assumption on household preferences, ambiguity, or participation.

4 Quantitative Easing, the QE Trap, and Asset Prices

4.1 Central Bank as Risk-Bearer of Last Resort

The equilibrium failure of Theorem 1 motivates the introduction of a public sector. Once the crisis has depleted financier capital to the point where leverage constraints bind, the central bank purchases the residual equity that the constrained financier cannot absorb.

Assumption 3 (Central bank and fiscal authority)**.** *The central bank stands ready to absorb whatever fraction of the equity market the constrained financier cannot hold, financing its purchases by issuing risk-free bonds. Profits and losses are remitted to the fiscal authority (Treasury), which may transfer resources to the financial sector as a recapitalisation (Section 6.4).*

4.2 Equilibrium with QE

Proposition 2 (Equilibrium with QE)**.** *Under Assumptions 2 and 3, a competitive equilibrium exists for all $\omega \in (0,1)$. Equity market clearing uniquely pins down the central bank's equity share as the residual:*

$$\alpha_t = \max\big(0,\ 1 - \epsilon\bar{\phi}(\omega_t)\omega_t\big) = \begin{cases} 1 - \epsilon\bar{\phi}(\omega_t)\omega_t & \text{if } \omega_t < \omega_c, \\ 0 & \text{if } \omega_t \geq \omega_c. \end{cases} \tag{16}$$

At each date t, the equilibrium allocation is given by

$$N_t = \left[\frac{\epsilon - 1}{\epsilon(1-\omega_t)}\right]^{1/(1+\varphi)}, \qquad Y_t = A_t N_t, \tag{17}$$

$$C_{h,t} = (1-\omega_t)\, Y_t, \qquad C_{i,t} = \omega_t\, Y_t, \tag{18}$$

$$P_{Y,t} = Y_t/\delta, \qquad P_{D,t} = P_{Y,t}/\epsilon, \tag{19}$$

$$H_{i,t} = \omega_t P_{Y,t}, \qquad H_{h,t} = (1/\epsilon - \omega_t)\, P_{Y,t}. \tag{20}$$

The financier's portfolio share and the central bank's equity holding are

$$\phi_t = \min\Big(\bar{\phi}(\omega_t),\ \tfrac{1}{\epsilon\omega_t}\Big), \qquad \alpha_t = \max\big(0,\ 1 - \epsilon\bar{\phi}(\omega_t)\omega_t\big). \tag{21}$$

As functions of the state ω_t, the mappings (17)–(20) are the same whether or not the leverage constraint binds: the central bank absorbs the residual equity, leaving the static real allocation untouched. The law of motion for ω_t itself is altered when $\omega_t < \omega_c$ (Theorem 3), so the realised processes $\{\omega_t, N_t, Y_t, C_{h,t}, C_{i,t}, P_{Y,t}\}$ differ in the crisis regime from what they would be absent the constraint.

The central bank absorbs exactly the equity that the constrained financier cannot hold. The household, whose bond demand is perfectly inelastic, is indifferent between lending to the financier and lending to the central bank, so at each ω the real allocation (17)–(20) is the same as without the constraint. The formula for α_t in (21) is not a policy choice but a market-clearing identity: it holds for *any* bounded leverage function $\bar{\phi}(\omega)$, not just the specific linear form in (7). Online Appendix N verifies this by deriving the corresponding α_t under an equity constraint $\phi \leq 1/a$, where the formula becomes $\alpha_t = \max(0, 1-\epsilon\omega/a)$ but all qualitative properties—equilibrium existence, intermediation gap elimination, the leverage paradox—are preserved.

4.3 Dynamics with the Leverage Constraint

Theorem 3 (Dynamics of ω_t with leverage constraint)**.** *The financier's wealth share evolves as*

$$\mathrm{d}\omega_t = \frac{\sigma_\omega(\omega_t)^2}{\omega_t}\,\mathrm{d}t + \sigma_\omega(\omega_t)\,\mathrm{d}Z_t, \tag{22}$$

where the volatility function is continuous on $(0,1)$ and given by

$$\sigma_\omega(\omega) = \begin{cases} \dfrac{\sigma(1+\varphi)\lambda(1-\omega)\,\omega\,(\omega-\underline{\omega})}{(1+\varphi)(1-\omega) - \lambda\omega(\omega-\underline{\omega})} & \text{if } \omega < \omega_c, \\[2ex] \dfrac{\sigma(1+\varphi)(1-\omega)(1/\epsilon-\omega)}{(1+\varphi)(1-\omega) - (1/\epsilon-\omega)} & \text{if } \omega \geq \omega_c. \end{cases} \tag{23}$$

The drift relation $\mu_\omega = \sigma_\omega^2/\omega$ reflects the financier's excess return from bearing equity risk: the drift is always positive when the diffusion is nonzero, so the financier's capital accumulation is fastest when her wealth share is most volatile.

The sign of σ_ω determines how the economy responds to shocks (Proposition 29 in the Appendix). Since $\sigma_\omega = \omega(\phi-1)\sigma_R$, the sign depends on whether the financier is leveraged ($\phi > 1$) or de-leveraged ($\phi < 1$). In the crisis regime ($\omega < \underline{\omega}$), the leverage constraint

forces $\phi < 1$: the financier cannot hold even her own wealth entirely in equity. A positive productivity shock then raises total wealth P_Y faster than the financier's equity-underweight portfolio, mechanically reducing $\omega = H_i/P_Y$—so $\sigma_\omega < 0$. In the recovery regime ($\underline{\omega} < \omega < 1/\epsilon$), $\phi > 1$ (the financier is leveraged) and $\sigma_\omega > 0$: positive shocks improve financier capitalisation, aiding recovery.

Proposition 4 (Q-martingale preservation)**.** *$Q_t = 1/\omega_t$ is a non-negative local martingale regardless of whether the leverage constraint binds.*

This is a consequence of log utility: the financier's consumption-wealth ratio is constant at δ, and the linear and quadratic terms in ϕ cancel exactly.

4.4 Two Steady States: Permanent QE and the QE Trap

We call a point ω^* a *steady state* of the wealth-share dynamics if both drift and diffusion vanish there: $\sigma_\omega(\omega^*) = \mu_\omega(\omega^*) = 0$. If the economy starts at ω^*, it stays there forever ($\mathrm{d}\omega = 0$). In the Feller classification of one-dimensional diffusions, each steady state is a *natural boundary*: the economy is drawn toward it from its basin but never reaches it in finite time—like Zeno's runner, who halves the remaining distance at each step but never arrives.[5]

Theorem 5 (Two steady states)**.** *The wealth-share dynamics have exactly two steady states in $(0, 1)$: $\omega = \underline{\omega}$ and $\omega = 1/\epsilon$. Both are natural boundaries: if $\omega_0 = \omega^* \in \{\underline{\omega}, 1/\epsilon\}$, then $\omega_t = \omega^*$ for all $t \geq 0$; if $\omega_0 \neq \omega^*$, then ω_t converges to ω^* almost surely from the corresponding basin but $\omega_t \neq \omega^*$ for all finite t.*

The good steady state ($\omega = 1/\epsilon$). The financier holds exactly 100% of her wealth in equity, no leverage constraint binds, the central bank holds no equity, and the economy coincides with the complete-markets benchmark.

The bad steady state ($\omega = \underline{\omega}$). The central bank permanently holds fraction $\alpha = 1 - \epsilon\underline{\omega}$ of the equity market. Employment is depressed:

$$N(\underline{\omega}) = \left(\frac{1 - 1/\epsilon}{1 - \underline{\omega}}\right)^{\frac{1}{1+\varphi}} < 1 = N(1/\epsilon). \tag{24}$$

[5]Formally, a natural boundary is one that is *unattainable*—the first hitting time from any interior point is infinite almost surely—and *non-starting*—it cannot serve as an initial condition for the interior diffusion. The term "absorbing state" is sometimes used in the macro-finance literature for the same object. In stochastic process theory, an absorbing state is one the process *reaches* in finite time and stays at; our steady states are approached only asymptotically, so we avoid this terminology.

Both steady states are natural boundaries (Appendix I): continuous paths cannot cross them, so no transition between basins is possible. The state space $(0, 1)$ splits into three *invariant basins*—regions the economy cannot leave once entered—each with a determined long-run outcome:

- $(0, \underline{\omega})$: converges to the bad steady state $\underline{\omega}$ (permanent QE, depressed output).
- $(\underline{\omega}, 1/\epsilon)$: converges to the good steady state $1/\epsilon$ (full recovery).
- $(1/\epsilon, 1)$: converges to the good steady state $1/\epsilon$ from above.

The crisis threshold $\omega_c \in (\underline{\omega}, 1/\epsilon)$ is *not* a boundary—continuous paths cross it freely. It marks where the leverage constraint starts to bind, but poses no barrier to the dynamics.

Theorem 6 (Convergence)**.** *For any $\omega_0 \in (0, 1) \setminus \{\underline{\omega}, 1/\epsilon\}$:*

$$\omega_t \xrightarrow{a.s.} \begin{cases} \underline{\omega} & \text{if } \omega_0 \in (0, \underline{\omega}), \\ 1/\epsilon & \text{if } \omega_0 \in (\underline{\omega}, 1). \end{cases} \tag{25}$$

In all cases, $\mathbb{E}[\tau] = \infty$, where τ is the time to reach the relevant steady state.

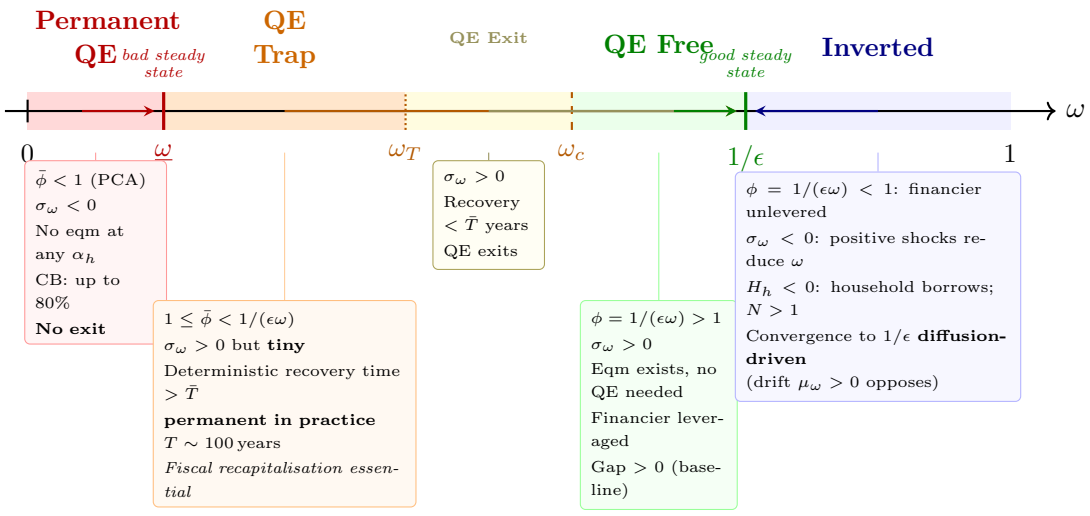
The almost sure convergence follows from the supermartingale property of $Q = 1/\omega$ (Appendix J). For $\omega_0 \in (0, \underline{\omega})$, the economy converges to the bad steady state: QE is literally permanent — the *Permanent QE* zone. For $\omega_0 \in (\underline{\omega}, 1/\epsilon)$, the economy converges to the good steady state, but the infinite expected arrival time means QE tapers gradually over a timescale that can span centuries. We call the sub-region $(\underline{\omega}, \omega_T)$ — where the deterministic recovery time exceeds a policy-relevant horizon $\bar{T}$ — the *QE Trap*: exit is mathematically certain but practically impossible (Online Appendix R).

Remark 2 (Permanent QE and the QE Trap are distinct)**.** *The two regimes in which QE persists arise from* different mechanisms*:*

- Permanent QE $(0, \underline{\omega})$*: $\underline{\omega}$ is a natural boundary because the leverage constraint forces $\bar{\phi}(\underline{\omega}) = 1$, which implies $\sigma_\omega(\underline{\omega}) = 0$. The economy starting below $\underline{\omega}$ cannot cross this boundary and $\omega_t \to \underline{\omega}$ almost surely. Exit is impossible* even in theory*: no arbitrarily long wait delivers $\omega_t > \underline{\omega}$. Only a fiscal transfer Δ with $\omega_0 + \Delta > \underline{\omega}$ escapes.*
- QE Trap $(\underline{\omega}, \omega_T)$*: the diffusion σ_ω vanishes at $\underline{\omega}$, so the drift $\mu_\omega = \sigma_\omega^2/\omega$ is tiny just above it. Recovery to ω_c is certain ($\omega_t \xrightarrow{a.s.} 1/\epsilon$) but takes centuries at typical post-crisis capital ratios. Exit is possible in theory but not on any policy-relevant horizon. A fiscal transfer with $\omega_0 + \Delta > \omega_T$ restores self-resolving QE.*

The distinction is policy-relevant: a transfer sufficient to escape the QE Trap may be insufficient to escape Permanent QE. We use the umbrella phrase "the trap" only informally; whenever the specific mechanism matters we name it.

Remark 3 (Why the leverage form produces Permanent QE)**.** *The interior steady state at* $\underline{\omega}$ *is a consequence of the leverage form of* (7)*. At* $\omega = \underline{\omega}$*, the cap forces* $\bar{\phi}(\underline{\omega}) = 1$*, so* $\sigma_\omega(\underline{\omega}) = 0$ *(Theorem 3) and hence* $\mu_\omega(\underline{\omega}) = 0$*: both drift and diffusion vanish, pinning the process at* $\underline{\omega}$*. Under an equity constraint* $\bar{\phi} \equiv 1/a > 1$ *(He and Krishnamurthy, 2012, 2013), the financier retains leverage strictly above one at every* $\omega > 0$*; no interior steady state arises and Permanent QE disappears as a distinct regime. The QE Trap phenomenon of impractically slow recovery is* not *eliminated by switching constraint forms, however:* $\mu_\omega \to 0$ *as* $\omega \to 0$ *under either constraint, and deterministic recovery from severely depleted capital still takes centuries. What the leverage constraint adds is that the slow-drift region is anchored to a strictly positive capital level* $\underline{\omega}$ *rather than sitting against the boundary at zero. Equilibrium failure also survives under the equity constraint (Online Appendix N, Proposition 55), but the failure region* $(0, a/\epsilon)$ *is typically narrower than the leverage-constraint region* $(0, \omega_c)$*—QE is needed for a smaller range of* ω *while the qualitative necessity survives. Online Appendix N develops the full comparison.*



Steady states $\underline{\omega}$ and $1/\epsilon$: both $\sigma_\omega = \mu_\omega = 0$. Each is a *natural boundary* — the economy converges toward it but never arrives ($\mathbb{E}[\tau] = \infty$). $Q = 1/\omega$ is a supermartingale $\Rightarrow \omega_t \xrightarrow{\text{a.s.}} \{\underline{\omega}, 1/\epsilon\}$.

ω_T (dotted): the QE Trap boundary. Deterministic recovery time exceeds $\bar{T}$ years for $\omega < \omega_T$. At leading order: $\omega_T \approx \underline{\omega} + (\omega_c - \underline{\omega})/(1 + \sigma^2\lambda^2\underline{\omega}(\omega_c - \underline{\omega})\bar{T})$ (Online Appendix R). At $\omega = 0.025$: recovery takes **10,386 years**.

ω_c (dashed): crisis threshold where the leverage constraint binds. Not a boundary — freely crossed.

Three invariant basins: $(0, \underline{\omega})$ → bad steady state (*Permanent QE*); $(\underline{\omega}, 1/\epsilon)$ → good steady state; $(1/\epsilon, 1)$ → good steady state from above.

Figure 2: The five regions of ω. $\omega_t = H_{i,t}/P_{Y,t}$ is the single state variable. The regions partition $(0,1)\setminus\{\underline{\omega}, 1/\epsilon\}$: *Permanent QE* $(0, \underline{\omega})$, *QE Trap* $(\underline{\omega}, \omega_T)$, *Self-Resolving QE* $[\omega_T, \omega_c)$, *QE Free* $[\omega_c, 1/\epsilon)$, *Inverted* $(1/\epsilon, 1)$; the natural boundaries $\underline{\omega}$ and $1/\epsilon$ are unattainable steady states. Solid lines: $\underline{\omega}$ and $1/\epsilon$ (natural boundaries). Dotted: ω_T, the QE Trap boundary — below it, deterministic recovery to ω_c exceeds $\bar{T}$ years (set to 50 in the figure). Dashed: ω_c (crisis threshold, freely crossed). In the *Permanent QE* zone, no exit is possible even in theory. In the *QE Trap*, exit is certain but takes centuries — too slow for any government, central bank, or electorate. Only *Self-Resolving QE* ($\omega > \omega_T$) exits on a human timescale. Fiscal recapitalisation pushes ω from the Trap into the Self-Resolving zone. *Horizontal spacing between ω_T and ω_c is exaggerated for label visibility; at baseline they are only* 0.006 *apart.* Baseline: $\epsilon = 10$, $\underline{\omega} = 0.02$, $\lambda = 5$, $\sigma = 0.15$, $\bar{T} = 50$ years.

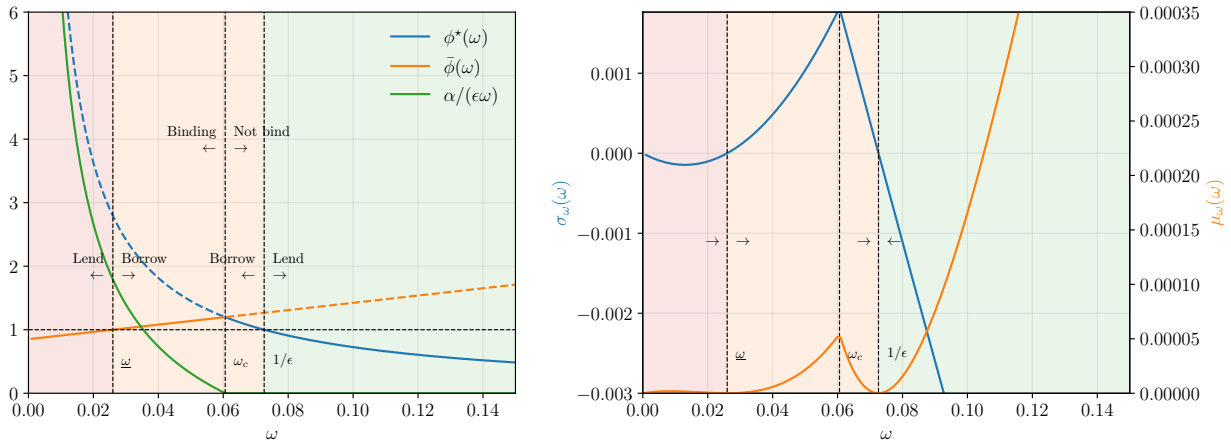


Figure 3: Equilibrium dynamics as a function of the state variable ω. (a) Financier and central bank equity holdings and the leverage constraint, with arrows showing the regions where the constraint is (not) binding and where the financier borrows or lends. (b) Wealth-share volatility $\sigma_\omega(\omega)$ and drift $\mu_\omega = \sigma_\omega^2/\omega$, with arrows showing convergence to the steady states. In both panels the red shading marks the crisis regime, the orange shading marks the recovery regime, and the green shading marks the unconstrained regime. Vertical dashed lines mark $\underline{\omega} = 0.02$, $\omega_c = 0.078$, and $1/\epsilon = 0.10$. Baseline calibration: $\epsilon = 10$, $\underline{\omega} = 0.02$, $\lambda = 5$, $\sigma = 0.15$, $\varphi = 1$.

4.5 Asset Prices and Risk Premia

Theorem 7 (Asset pricing)**.** *The market price of risk, equity return volatility, and risk-free rate are:*

$$\Theta_t = \frac{\sigma_\omega(\omega_t)}{\omega_t} + \sigma + \frac{\sigma_\omega(\omega_t)}{(1+\varphi)(1-\omega_t)}, \tag{26}$$

$$\sigma_{R,t} = \sigma + \frac{\sigma_\omega(\omega_t)}{(1+\varphi)(1-\omega_t)}, \tag{27}$$

$$r_t = \delta + \mu_{Y,t} - \sigma_{R,t}\,\Theta_t, \tag{28}$$

where σ_ω is given by (23) and $\mu_{Y,t}$ is the drift of output. The equity risk premium is $\mu_{R,t} - r_t = \sigma_{R,t}\,\Theta_t$.

The risk-free rate (28) is depressed in crisis through the $\sigma_{R,t}\,\Theta_t$ term: as ω falls, leveraged intermediaries amplify return volatility σ_R and the market price of risk Θ, whose product drives r well below the rate of time preference. This balance-sheet channel differs from the habit and long-run risk mechanisms of Campbell and Cochrane (1999) and Bansal and Yaron (2004), where r falls because of increased effective risk aversion or precautionary savings;

here the channel is intermediary capital, and it operates even under log utility (constant individual risk aversion).

In the recovery regime ($\underline{\omega} < \omega < 1/\epsilon$), $\sigma_\omega > 0$ and risk premia are above the frictionless benchmark. Risk premia peak at an intermediate ω and decline as $\omega \to 1/\epsilon$, consistent with the hump shape documented by He, Kelly, and Manela (2017). Positive productivity shocks raise ω and compress risk premia, delivering countercyclicality.

Output gap. The log output gap relative to the frictionless benchmark $\omega_n \equiv 1/\epsilon$ is:

$$x_t \equiv \ln Y_t - \ln Y_{n,t} = \frac{1}{1+\varphi} \ln\left(\frac{1-\omega_n}{1-\omega_t}\right). \tag{29}$$

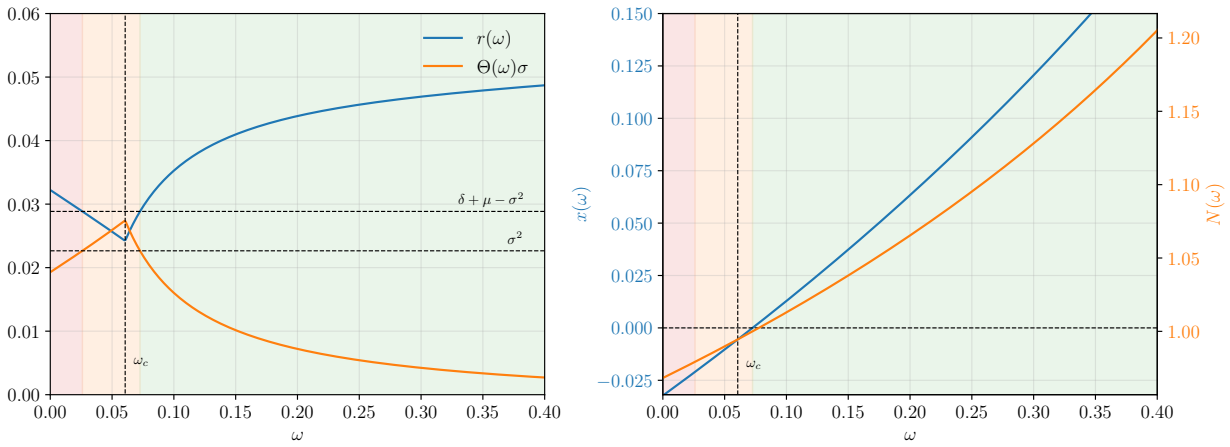


Figure 4: Asset prices and real variables as functions of ω. (a) Market price of risk, risk-free rate, and equity return volatility. (b) Output gap. In both panels the red shading marks the crisis regime, the orange shading marks the recovery regime, and the green shading marks the unconstrained regime. The vertical dashed line marks $\omega_c = 0.078$. Baseline calibration: $\epsilon = 10$, $\underline{\omega} = 0.02$, $\lambda = 5$, $\sigma = 0.15$, $\varphi = 1$.

5 The Intermediation Gap

5.1 A Three-Act Narrative

The results in this section are organised around three acts that trace the economy's trajectory through a crisis:

1. **Before the crisis** ($\omega > \omega_c$, no binding constraint): Equilibrium exists. Asset prices exhibit a positive intermediation gap with gap-to-price ratio $b(\omega)$. As intermediary capital

erodes, the intermediation gap grows convexly—the exact formula gives $b(0.05) = 0.19$ at a typical recovery value, rising to $b(0.02) \approx 0.60$ in severe distress—partially masking the deterioration of fundamental values: market prices overstate the health of the financial sector (Section 6.2).

2. **During the crisis** (ω drops below ω_c): Leverage constraints bind. Without QE, no equilibrium exists. The question of asset pricing is moot.

3. **After QE** (QE active): Equilibrium is restored. The intermediation gap vanishes: the stock price equals its fundamental value. The sudden disappearance of the intermediation gap reveals the true severity of intermediary undercapitalisation. The economy may be trapped at the bad steady state with correct prices but depressed output. Fiscal recapitalisation must be calibrated to these revealed fundamentals, not to pre-crisis prices inflated by the intermediation gap.

The three acts trace an information arc: the intermediation gap obscures fundamentals before the crisis, its disappearance reveals fundamentals during the crisis, and correct post-QE prices enable—but do not substitute for—fiscal intervention. We now formalise each act.

The inverse wealth share $Q_t = 1/\omega_t$ is a non-negative local martingale (Proposition 4). Whether Q is a strict or true local martingale determines whether equilibrium asset prices exhibit a positive intermediation gap—the component of asset prices above fundamental value attributable to the concentration of risk-bearing in undercapitalised intermediaries. The intermediation gap is a symptom of intermediary weakness, not monetary accommodation: it is largest when the financial sector is most distressed.

5.2 Martingale Criterion

For a one-dimensional diffusion $\mathrm{d}X = \sigma_X(X)\,\mathrm{d}Z$ starting at $X_0 > 0$, X is a strict local martingale—meaning $\mathbb{E}[X_t] < X_0$ for some t—if and only if (Engelbert and Schmidt, 1985; Mijatović and Urusov, 2012):

$$\int_{X_0}^{\infty} \frac{x}{\sigma_X(x)^2}\,\mathrm{d}x < \infty. \tag{30}$$

When the integral diverges, X is a true martingale and $\mathbb{E}[X_t] = X_0$ for all t.

5.3 Connection to the Rational Bubble Literature

Hugonnier (2012) proves that in a two-agent economy with a portfolio constraint, equilibrium asset prices contain rational bubbles if and only if the weighting process—the ratio of

marginal utilities—is a strict local martingale. In our model the weighting process reduces to

$$\lambda_t \;=\; \frac{1-\omega_t}{\omega_t} \;=\; Q_t - 1, \tag{31}$$

so λ inherits the martingale classification of Q. The following equivalence is a direct consequence of Hugonnier (2012, Theorem 1):

$$Q = 1/\omega \text{ is a strict local martingale} \quad\Longleftrightarrow\quad \text{intermediation gap} > 0. \tag{32}$$

5.4 The Exact Gap Formula in the Production Economy

The intermediation gap satisfies a singular Sturm–Liouville boundary-value problem that characterises it exactly, without recourse to the constant-weighting-process approximation of Hugonnier (2012)'s exchange economy.

Proposition 8 (Intermediation gap formula)**.** *In the baseline model (limited participation, no leverage constraint), the intermediation gap-to-price ratio $b(\omega_t) \equiv B_t/P_{D,t}$ satisfies the second-order linear ODE*

$$b''(\omega) \;=\; \frac{2\delta}{\sigma_\omega^2(\omega)}\, b(\omega), \qquad \omega \in (0, 1/\epsilon), \tag{33}$$

with boundary conditions

$$b(0^+) \;=\; 1, \qquad b(1/\epsilon) \;=\; 0. \tag{34}$$

The BVP admits a unique C^2 solution, which is strictly positive, strictly decreasing, and strictly convex on $(0, 1/\epsilon)$.

The proof is in Appendix E. The derivation proceeds in three steps. First, apply the Feynman–Kac theorem to the discounted resolvent $f(\omega) \equiv \mathbb{E}^\omega[\int_0^\infty e^{-\delta u}(1/\omega_u)\,\mathrm{d}u]$, which gives $\frac{1}{2}\sigma_\omega^2 f'' + (\sigma_\omega^2/\omega)f' - \delta f + 1/\omega = 0$. Second, change variables to $b \equiv 1 - \delta\omega\, f$: the b' and $(1-b)/\omega^3$ coefficients cancel identically because of the Q-martingale identity $\mu_\omega = \sigma_\omega^2/\omega$, leaving only (33). Third, establish the boundary conditions: $b(1/\epsilon) = 0$ follows from the absorbing-state structure of the good steady state; $b(0^+) = 1$ follows from the h-transform analysis of the strict local martingale $Q = 1/\omega$ (Pal and Protter, 2010; Delbaen and Schachermayer, 1995), which delivers complete mass loss at infinity. The economic content is the standard decomposition $P_{D,t} = P_{D,t}^{\text{fund}} + B_t$, with the fundamental value $P_{D,t}^{\text{fund}} = \mathbb{E}_t[\int_t^\infty (\Lambda_s/\Lambda_t) D_s\,\mathrm{d}s]$ and the bubble B_t the strict-local-martingale residual in the sense of Hugonnier (2012).

Figure 5 plots the exact numerical solution of (33)–(34) at baseline calibration, with key

values marked. The gap rises convexly as ω falls: $b(0.05) = 0.19$ at a typical recovery value, rising to $b(0.02) \approx 0.60$ in severe distress. The full numerical procedure, cross-validation against a second independent solver, and comparison to the exchange-economy closed form of Hugonnier (2012, Proposition 6) are in Appendix E.

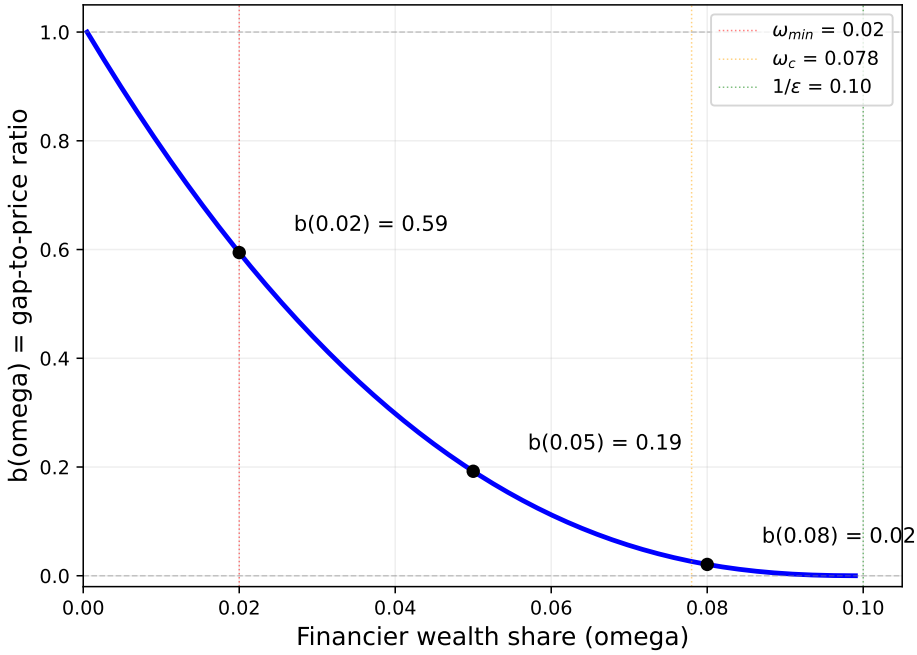


Figure 5: Exact intermediation gap-to-price ratio $b(\omega)$, computed by numerical solution of (33)–(34). Vertical dashed lines mark $\underline{\omega} = 0.02$ (minimum capital threshold), $\omega_c = 0.078$ (crisis threshold), and $1/\epsilon = 0.10$ (good steady state). Baseline calibration: $\epsilon = 10$, $\underline{\omega} = 0.02$, $\lambda = 5$, $\sigma = 0.15$, $\delta = 0.04$, $\varphi = 1$.

Volatility paradox. The exact ODE (33) makes the volatility paradox direct: the right-hand-side coefficient $2\delta/\sigma_\omega^2(\omega)$ is *decreasing* in σ_ω, so lower fundamental volatility produces a *steeper* second derivative of the gap function and hence a larger gap at any given ω. In the Hugonnier closed form this appears as the decreasing-in-v exponent $\partial\gamma/\partial v = -4\delta/[v^3\sqrt{1+8\delta/v^2}] < 0$. The economic intuition is common to both: low fundamental volatility means σ_ω is small, making the inverse wealth share Q more prone to explosive excursions (the Mijatović–Urusov integral test in (30) converges faster), which is exactly when the strict local martingale property bites hardest.[6]

[6]The model is silent on reach-for-yield effects: the household's bond demand is perfectly inelastic (log utility), so QE has no portfolio rebalancing effect. In richer models, lower safe returns push investors into risky assets, creating a distinct form of overvaluation not connected to the equilibrium existence problem.

5.5 The Intermediation Gap with and without Leverage Constraints

Theorem 9 (Intermediation gap classification)**.**

(i) ***Baseline model*** *(limited participation, no leverage constraint): In the controlled regime* $(0, 1/\epsilon)$*,* $Q = 1/\omega$ *is a* ***strict local martingale****. Equilibrium asset prices exhibit a positive intermediation gap.*

(ii) ***Constrained model*** *(leverage constraint, QE active):* $Q = 1/\omega$ *is a* ***true martingale*** *in all three regimes, crisis* $(0, \underline{\omega})$*, recovery* $(\underline{\omega}, 1/\epsilon)$*, and expansionary* $(1/\epsilon, 1)$*. Equilibrium asset prices are free of any intermediation gap.*

The proof is in Appendix K. The entire difference comes from one factor of ω in the volatility function near $\omega = 0$. In the baseline, $\phi = 1/(\epsilon\omega) \to \infty$, so the ω in $\sigma_\omega = \omega(\phi - 1)\sigma_R$ cancels, leaving σ_ω positive at the boundary. With the constraint, $\phi \leq 1 + \lambda(\omega - \underline{\omega})$ is bounded, so $\sigma_\omega \propto \omega$.

	σ_ω **near** $\omega = 0$	σ_Q **near** $Q = \infty$	**Integral test**	**Gap?**
Baseline	$\sigma_\omega^0 \to c_0 > 0$	$c_0\, Q^2$	$\int q^{-3}\, \mathrm{d}q < \infty$	**Yes**
Constrained	$\lvert\sigma_\omega\rvert \sim c_0'\, \omega$	$c_0'\, Q$	$\int q^{-1}\, \mathrm{d}q = \infty$	**No**

5.6 The Leverage Paradox

The intermediation gap classification and the equilibrium failure result are two manifestations of a single mathematical object: the behaviour of σ_ω at the origin.

Proposition 10 (Leverage paradox)**.**

(i) ***Baseline economy*** *(no leverage constraint): competitive equilibrium exists for all* $\omega \in (0, 1/\epsilon)$ *and* $Q = 1/\omega$ *is a strict local martingale—the intermediation gap is positive.*

(ii) ***Constrained economy*** *(leverage constraint (7) with QE):* $Q = 1/\omega$ *is a true martingale in all three regimes—no intermediation gap.*

(iii) ***Impossibility.*** *The combination of gap-free asset prices and unconditional equilibrium existence (without central bank intervention) is impossible: for* any *leverage function* $\bar{\phi} : (0, 1/\epsilon) \to \mathbb{R}_+$*, if competitive equilibrium exists for all* ω *without intervention, then a positive intermediation gap exists.*

The intuition is that σ_ω near $\omega = 0$ carries a double duty. In the unconstrained model, $\phi \sim 1/(\epsilon\omega) \to \infty$ as $\omega \to 0$, so $\sigma_\omega = \omega(\phi - 1)\sigma_R$ remains bounded away from zero at the boundary—which, by the integral test, is exactly what makes $Q = 1/\omega$ a strict local martingale and produces the intermediation gap. In the constrained model, $\phi \leq \bar{\phi}(\omega)$ is bounded, so $\sigma_\omega \propto \omega \to 0$ at the boundary—making Q a true martingale and closing the intermediation gap. The same economic force (bounded leverage) that prevents ϕ from diverging—which would otherwise allow the financier to absorb all equity and prevent market failure—also changes the stochastic classification of Q at the boundary. Correcting prices and clearing markets are two sides of the same coin.

Proof. Parts (i) and (ii) restate Theorem 9.

For part (iii), suppose competitive equilibrium exists for all $\omega \in (0, 1/\epsilon)$ without central bank intervention. Equity market clearing requires $\phi(\omega)\,\omega\,P_Y = P_D = P_Y/\epsilon$, so $\phi(\omega) = 1/(\epsilon\omega)$. The constraint cannot bind at any ω—otherwise $\phi = \bar{\phi}(\omega) < 1/(\epsilon\omega)$ and the equity market does not clear—so $\bar{\phi}(\omega) \geq 1/(\epsilon\omega)$ for all ω. With $\phi = 1/(\epsilon\omega)$, the wealth-share diffusion coefficient is

$$\sigma_\omega(\omega) = \frac{(1/\epsilon - \omega)\,\sigma_R}{D_u(\omega)}, \qquad D_u(\omega) = \left(1 - \tfrac{1}{\epsilon}\right) + \varphi(1 - \omega) > 0,$$

giving $\sigma_\omega(0^+) = \sigma_R/\big(\epsilon\, D_u(0)\big) > 0$. Converting to $Q = 1/\omega$: $\sigma_Q = \sigma_\omega/\omega^2 \sim c_0\, Q^2$ with $c_0 = \sigma_\omega(0^+) > 0$. The integral test (30) yields $\int_{Q_0}^{\infty} q\,\sigma_Q(q)^{-2}\,\mathrm{d}q \sim c_0^{-2} \int q^{-3}\,\mathrm{d}q < \infty$, so Q is a strict local martingale and the intermediation gap is positive. □

Corollary 11 (Impossibility)**.** *Within the class of economies in this paper:*

(a) Any economy with unconditional equilibrium existence necessarily has a positive intermediation gap.

(b) Any economy with gap-free asset prices is vulnerable to equilibrium failure.

Proof. Part (a) is Proposition 10(iii). Part (b) is its contrapositive: if equilibrium existed for all ω without intervention, Proposition 10(iii) would imply a positive intermediation gap, contradicting the hypothesis of gap-free prices. □

The policy package of QE plus fiscal recapitalisation resolves the tradeoff: QE restores equilibrium existence when bounded leverage causes market failure, while the leverage constraint simultaneously closes the intermediation gap. The exact gap ODE provides a constructive proof of closure: on $[\underline{\omega}, 1/\epsilon]$, the only solution to $b'' = (2\delta/\sigma_\omega^2)b$ with $b(\underline{\omega}) = b(1/\epsilon) = 0$ is $b \equiv 0$ (by the maximum principle). Fiscal recapitalisation then moves the economy to

the good steady state where the constraint ceases to bind, the central bank exits, and both equilibrium existence and correct pricing are achieved without intervention.

5.7 The Intermediation Gap as a Sufficient Statistic for the Output Gap

Both the intermediation gap and the output gap are driven by the same underlying shortage of risk-bearing capacity—the *risk-bearing gap*. Since both are monotone functions of the single state variable ω_t, the intermediation gap-to-price ratio is a sufficient statistic for the output gap:

$$x_t = \frac{1}{1+\varphi} \ln\left(\frac{1-\omega_n}{1-\omega_t}\right) = G\big(b(\omega_t)\big), \tag{35}$$

for an increasing function G. Larger intermediation gaps signal larger output gaps. The exchange-economy approximation $(1-\omega)^\gamma$ of Hugonnier (2012), being nearly linear over the empirical range, collapses G to an approximately linear relationship, discarding the model's predicted curvature. The exact ODE solution $b(\omega)$ from Proposition 8 preserves it. Section 7 evaluates both against the data.

Since T^* and the benefit-to-cost ratio (42) are also functions of ω, the pre-crisis gap-to-price ratio encodes the fiscal cost of future intervention via $\omega_0 = b^{-1}(B/P_D)$. Crucially, this mapping is available only *before* QE. Once the intermediation gap vanishes, the policymaker loses this market-based diagnostic—the same intervention that corrects asset prices destroys their informational content about the output gap.

6 Real Effects and Policy

6.1 The Unified State Variable

The financier's wealth share ω_t simultaneously governs three equilibrium objects:

Asset prices and the intermediation gap:

$$\frac{B_t}{P_{D,t}} = b(\omega_t), \qquad b''(\omega) = \frac{2\delta}{\sigma_\omega(\omega)^2}\, b(\omega). \tag{36}$$

Employment:

$$N_t = \left(\frac{1-1/\epsilon}{1-\omega_t}\right)^{\frac{1}{1+\varphi}}, \quad \text{with } N'(\omega_t) > 0. \tag{37}$$

Output:

$$Y_t = A_t N_t = A_t \left(\frac{1 - 1/\epsilon}{1 - \omega_t} \right)^{\frac{1}{1+\varphi}}. \tag{38}$$

When intermediary wealth erodes (ω_t falls), three things happen together: the intermediation gap grows (convexly), employment falls, and output contracts. The convexity of the intermediation gap function transmits directly to real variables: when ω_t is low, small further declines produce large increases in the intermediation gap *and* large employment losses, creating a fragility trap that intensifies as conditions deteriorate.

6.2 Phantom Wealth

In the baseline economy, the equity price decomposes into a fundamental value and an intermediation gap:

$$P_D = P_D^{\text{fund}} + B, \qquad P_D^{\text{fund}} = P_D\big[1 - b(\omega)\big], \qquad B = P_D\, b(\omega). \tag{39}$$

Proposition 12 (Phantom wealth)**.** *In the baseline model:*

(i) $P_D^{\text{fund}}(\omega) \to 0$ *as* $\omega \to 0$*: at the point of maximum crisis, the entire equity market value is an intermediation gap.*

(ii) The fundamental-to-market ratio $P_D^{\text{fund}}/P_D = 1 - b(\omega)$ *is strictly increasing in* ω*.*

(iii) The gap ratio $B/P_D = b(\omega)$ *overstates the financier's equity position by a factor that grows convexly as the crisis deepens.*

Proof. (i) $P_D(0) = AN(0)/(\epsilon\delta) > 0$ and $b(0) = 1$, so $P_D^{\text{fund}}(0) = 0$. (ii) $b' < 0$ follows from $b > 0$, $b'' > 0$, $b(0) = 1$, $b(1/\epsilon) = 0$. (iii) Convexity: $b'' = (2\delta/\sigma_\omega^2)b > 0$ whenever $b > 0$. □

Part (i) is the key result for policy. A policymaker observing pre-crisis market prices sees positive equity values and may conclude the financial sector retains substantial wealth. In fact, as ω approaches zero, fundamental value vanishes and the entire observed price is phantom wealth—value that would disappear if the economy transitioned to a regime with correct prices. The exact gap formula quantifies this: at $\omega = 0.05$, the gap-to-price ratio is $b(0.05) = 0.19$; at $\omega = 0.02$, $b \approx 0.60$. Because the required recapitalisation (41) is calibrated to the *fundamental* state variable ω, not to market prices that include the gap, policymakers who rely on market-based measures systematically underestimate the severity of undercapitalisation. The sufficient-statistic property extends to the fiscal transfer:

$$T^*(\omega_0) = \Omega(\underline{\omega}) - \Omega(\omega_0), \qquad \omega_0 = b^{-1}\left(\frac{B}{P_D}\right), \tag{40}$$

where b^{-1} is the inverse of the (strictly decreasing) exact gap function. Crucially, this mapping is available only before QE: once the gap vanishes, the policymaker loses this market-based diagnostic.

6.3 Large-Scale Asset Purchases

Under Assumption 3, the central bank absorbs the residual equity supply that the constrained financier cannot hold, preventing equilibrium failure but not altering the economy's long-run dynamics.

Proposition 13 (Central bank balance sheet at the bad steady state)**.** *At $\omega = \underline{\omega}$, the central bank permanently holds equity worth $(1 - \epsilon\underline{\omega})\, Y_t/(\epsilon\delta)$ and earns the equity premium σ^2 on these holdings.*

At the bad steady state, the central bank holds a potentially large share of the equity market: with $\underline{\omega} = 0.02$ and $\epsilon = 10$, it holds 80%. The equity premium equals its frictionless value σ^2 because $\sigma_\omega(\underline{\omega}) = 0$ eliminates all wealth redistribution dynamics.

6.4 Fiscal Recapitalisation

Proposition 14 (Recapitalisation)**.** *Suppose $\omega_0 < \underline{\omega}$ (crisis regime). A one-time capital injection that raises the financier's wealth share to $\omega_{0^+} > \underline{\omega}$ places the economy in the recovery regime, which then converges almost surely to the good steady state $1/\epsilon$ without further intervention.*

Remark 4 (Transfer size with general equilibrium feedback)**.** *Because total wealth $P_Y(\omega) = AN(\omega)/\delta$ is increasing in ω, a dollar transfer raises ω by less than $T/P_Y(\omega_0)$. The transfer required to reach target ω^* is:*

$$T^* = \frac{A}{\delta}\big[\omega^*\, N(\omega^*) - \omega_0\, N(\omega_0)\big], \tag{41}$$

which exceeds the naïve estimate by $\omega^*[P_Y(\omega^*) - P_Y(\omega_0)]$, though the correction is small under baseline calibration (approximately 1%). The correction is small because log utility pins total wealth at $P_Y = Y/\delta$: the general equilibrium feedback operates only through the employment response $N(\omega)$, which is close to linear over the relevant range.

Employment multiplier. Recapitalisation from ω_0 to $\omega_0 + \Delta$ produces an immediate employment gain of $[(1 - \omega_0)/(1 - \omega_0 - \Delta)]^{1/(1+\varphi)} - 1$. The employment function $N(\omega)$ is convex, so the output gap accelerates as intermediary capital falls: each unit decline in ω

produces a larger employment loss than the previous one, making the cumulative benefit of escaping Permanent QE or the QE Trap larger for deeper crises. Under the baseline calibration, the permanent long-run gain from escaping either regime is 4.3% of efficient employment. The formal statement and proof are in Appendix H (Corollary 39).

Cost–benefit analysis. The steady-state benefit-to-cost ratio of recapitalisation is:

$$\frac{\text{Steady-state output gain}}{\text{Cost}} = \frac{N(1/\epsilon) - N(\underline{\omega})}{\underline{\omega}\, N(\underline{\omega}) - \omega_0\, N(\omega_0)}. \tag{42}$$

Under baseline calibration this exceeds 2: the government spends approximately 2% of total wealth to permanently raise output by over 4%. Since convergence to $1/\epsilon$ has infinite expected hitting time (Theorem 6), the discounted present value of the output gain is lower; the ratio above should be interpreted as a long-run comparison of steady states. The fiscal cost of delay is not a larger transfer but a larger cumulative output loss; each period in the crisis regime forgoes output on the steep portion of the employment curve, so delays at the onset are most costly.

Welfare. The household weakly prefers the bad steady state (higher consumption share $1 - \underline{\omega}$), so recapitalisation is not a Pareto improvement; it involves redistribution from household to financier. The detailed welfare calculation and moral hazard analysis are in Appendix G.2.

6.5 Mapping to Policy

The model rationalises three distinct phases of crisis intervention:

1. *Large-scale asset purchases (central bank):* when $\omega < \omega_c$ and leverage constraints bind, the central bank purchases equity to prevent market failure—corresponding to the Fed's emergency facilities in 2008–2009.

2. *Recapitalisation (fiscal authority):* the Treasury injects capital to push ω above $\underline{\omega}$, as in TARP (October 2008); European recapitalisations followed in 2010–2014, with the delay producing the persistent stagnation the model predicts.

3. *Taper and exit:* once $\omega > \underline{\omega}$ and recovery begins, the leverage constraint eventually stops binding and the central bank's equity holdings decline to zero.

Phase 1 alone is insufficient: QE maintains equilibrium existence but cannot alter long-run dynamics. Because employment and the intermediation gap share a common driver, poli-

cies that stabilise ω_t simultaneously stabilise financial valuations and employment—financial stability and full employment are complementary objectives.

6.6 Empirical Support

The model's predictions are supported by the empirical record. In the United States, TARP injected $245 billion within months of the crisis; Berger and Roman (2017) find it significantly increased net job creation through the credit-supply channel that raises ω in our model. In Europe, bank recapitalisations were delayed by 2–4 years (Brunnermeier, James, and Landau, 2016), and the euro area experienced precisely the persistent stagnation the model predicts from Phase 1 without Phase 2. Across 69 systemic banking crises since 1980, Homar and van Wijnbergen (2017) find that the *speed* of recapitalisation—not merely its occurrence—determines recovery duration, consistent with the absorbing steady-state structure and the convex employment losses from delay.

7 Empirical Evidence

The model posits a single state variable—the financier's wealth share ω_t—as the joint determinant of real activity, asset prices, and central-bank risk absorption. We operationalise this restriction through tests that map the model's objects into observables and ask whether the data contain the state dependence, nonlinearity, and fiscal-monetary substitution that the theory implies. Our tests extend the intermediary asset-pricing tradition of He and Krishnamurthy (2013, 2019), He, Kelly, and Manela (2017), Adrian, Etula, and Muir (2014), and Adrian and Shin (2010) to evaluate predictions that the theoretical literature has not previously operationalised: a state-dependent mapping from ω to real activity, a QE Trap with a persistent low-capital regime, and cross-country evidence on fiscal versus monetary channels of recovery. The section is descriptive, not causal: each test asks whether a model-implied pattern is visible in the data, not whether the mechanism is identified in a counterfactual sense.

Three tests, a structural-break subsection, a calibrated counterfactual, and a summary assessment follow. Test 1 shows that intermediary capital maps into real activity with the predicted magnitude and an amplification pattern at low capital. Test 2 provides direct evidence in the Treasury market for the private-dealer contraction and public risk-bearing substitution that the theory predicts (He, Khang, and Krishnamurthy, 2010; He, Nagel, and Song, 2022). Test 3 shows that the model-implied intermediation gap is a sufficient statistic for the output gap in the pre-QE regime and loses this content when QE is active, as the

theory predicts. A dedicated subsection documents the structural-break evidence for the QE Trap, and a cross-country panel shows that recovery of intermediary capital and real activity is associated with fiscal recapitalisation more than with QE.

7.1 Data

Our central state-variable proxy is the quarterly intermediary capital ratio of He, Kelly, and Manela (2017), $\widehat{\omega}_t \equiv \eta_t^{HKM}/100$, constructed as the aggregate market-equity-to-market-assets ratio of the holding companies of New York Fed primary dealers. Real activity is measured with the CBO output gap $\widehat{x}_t \equiv (\text{Real GDP}_t - \text{Real potential GDP}_t)/\text{Real potential GDP}_t$, the unemployment gap (using the CBO natural rate), payroll growth, and industrial-production growth.[7] Quarterly controls are productivity growth and CPI inflation.

Dealer intermediation is measured by Treasury and agency-MBS dealer financing to clients and dealer gross positions, scaled by the stock of securities held by the public excluding SOMA. Duration-weighted Fed holdings in each asset class, $RB_t^{CB,j}$, are constructed from weekly SOMA CUSIP-level data (Appendix). QE intensity is constructed from Federal Reserve SOMA holdings:

$$\widehat{\alpha}_t^{CB} \equiv \max\left(0,\ \frac{\text{Treasuries}_t + \text{MBS}_t + \text{AgencyDebt}_t}{\text{Nominal GDP}_t} - \overline{\text{QEHoldingsToGDP}}^{\text{pre}}\right). \tag{43}$$

For the intermediation-gap tests, the exact proxy is $\widehat{B/P}_t^{\text{exact}} \equiv b(\widehat{\omega}_t)$, solving the ODE in Proposition 8.

Baseline quarterly samples are 1970Q1–2024Q4 ($T = 220$) for full-sample tests and 1970Q1–2007Q4 ($T = 152$) for pre-QE tests. Test 2 begins in 2003, when dealer data become available. Test 4 is an annual panel of ten OECD economies over 2003–2020: the United States, the United Kingdom, Germany, France, Italy, Spain, Ireland, Greece, Portugal, and the Netherlands. Quarterly regressions use Newey–West HAC standard errors with automatic bandwidth. The cross-country panel uses two-way cluster-robust standard errors with a wild cluster bootstrap (appropriate for $N = 10$; see Cameron, Gelbach, and Miller 2008). Full inference-robustness results, including alternative predictive-regression methods where their assumptions hold, are in the Appendix.

[7] The model's output gap is a log deviation from a frictionless reference economy; the CBO gap is a related but distinct object. Both are increasing functions of ω_t, so the sign of the coefficient is preserved.

7.2 Calibration

We calibrate the model to transparent macro-financial moments and hold parameter values fixed throughout. Growth and asset-pricing parameters are disciplined by the pre-crisis sample 1970Q1–2007Q2: μ equals the annualised average growth rate of labour productivity, σ equals the annualised volatility of excess equity returns, and δ equals the average dividend yield on the S&P 500. Given (μ, σ, δ), the benchmark risk-free rate is $r = \delta + \mu - \sigma^2$.

The crisis threshold ω_c is set to the average HKM intermediary capital ratio in 2007Q4–2008Q1, aligning the model's binding threshold with the onset of the intermediary-capital collapse. The lower threshold $\underline{\omega}$ is the minimum HKM ratio over 2007Q3–2009Q2. The goods-market elasticity $\epsilon = 10$ is the macro-literature benchmark.[8] The leverage-sensitivity parameter λ is implied by (10). The inverse Frisch elasticity $\varphi = 1$. Table 1 reports the resulting values.

Table 1: Calibration

Parameter	Value	Empirical target
ω_c	0.0606	Average HKM intermediary capital ratio, 2007Q3 and 2007Q4
$\underline{\omega}$	0.0260	Minimum HKM intermediary capital ratio, 2008Q1–2009Q2
μ	0.0201	Annualized average labor-productivity growth, 1970Q1–2007Q3
σ	0.1505	Annualized volatility of S&P 500 excess returns, 1970Q1–2007Q3
δ	0.0314	Mean S&P 500 dividend yield, 1970Q1–2007Q3
ϵ	13.7741	Inverse average finance-sector value-added share of GDP, 1997–2006
λ	5.7231	Implied by the crisis-threshold condition (10)
r	0.0289	Implied by the benchmark risk-free rate equation (53)
φ	1.3333	Inverse aggregate-hours Frisch elasticity from Chetty, Guren, Manoli, and Weber (2011)

Notes: Quarterly moments use the pre-crisis sample from 1970Q1 to 2007Q3. Productivity growth and S&P 500 excess-return volatility are annualized from quarterly data. The crisis threshold ω_c is the average of the HKM intermediary capital ratio in 2007Q3 and 2007Q4. The lower threshold $\underline{\omega}$ is the minimum HKM ratio observed from 2008Q1 to 2009Q2. The parameter ϵ is disciplined by annual finance-sector value added over 1997–2006. The benchmark risk-free rate is not estimated separately; it is implied by the calibration identity. The parameter φ is the inverse of the aggregate-hours Frisch elasticity.

[8] In the model, ϵ simultaneously governs the Dixit–Stiglitz markup $\epsilon/(\epsilon-1)$, the dividend share of output $1/\epsilon$, and the equity-market-to-total-wealth ratio $P_D/P_Y = 1/\epsilon$. These three objects coincide by construction but map to distinct empirical quantities: an industrial-organisation reading of ϵ would target aggregate markups (e.g., De Loecker, Eeckhout, and Unger 2020), while a macro-finance reading targets intermediated asset shares. We calibrate ϵ to the finance-sector VA share ($\approx 8\%$, implying $\epsilon \approx 12.5$) as an intermediate target; the mechanism driving our results depends on the required leverage $\phi = 1/(\epsilon\omega)$, not on the particular micro-foundation of $1/\epsilon$. Appendix tables report robustness at $\epsilon \in \{7.5, 12.5\}$.

Table 2: Test 1A – theory-implied level fit, output gap

	Output gap	- Unemployment gap
α	-0.004*	-0.007***
	(0.003)	(0.002)
β	1.040***	0.954***
	(0.191)	(0.210)
Adj. R^2	0.348	0.334
N	120	120

Notes: HAC standard errors with four lags are reported in parentheses. ***, **, * denote significance at 1%, 5%, 10%.

7.3 Test 1: Intermediary Capital and Real Activity

The model predicts that real activity is an increasing function of ω_t (equation 29), with explicit functional form

$$\widehat{x}_t^{\text{model}} = \frac{1}{1+\varphi} \ln\left(\frac{1-1/\epsilon}{1-\widehat{\omega}_t}\right). \tag{44}$$

If the model delivers both the sign and the magnitude of the mapping, a regression of the empirical output gap on $\widehat{x}_t^{\text{model}}$ should return a slope coefficient of approximately unity. We estimate:

$$x_t^{\text{data}} = \alpha + \beta\, \widehat{x}_t^{\text{model}} + u_t. \tag{45}$$

Table 2 reports the results. The coefficient on the model-implied gap is economically indistinguishable from unity: $\widehat{\beta} = 1.114$ for the output gap and $\widehat{\beta} = 1.022$ for the sign-reversed unemployment gap, both with t-statistics above five. The nonlinear mapping in (44) captures roughly a third of the variation in either slack measure. The unit-coefficient result is distinctive: the model delivers not just the correct sign but the correct magnitude, without parameter fitting.

Amplification at low capital. Because $\sigma_\omega(\omega)$ is steeper in the constrained region, the predictive slope of future real activity on current ω_t should be amplified when ω_t is low. We estimate a predictive regression with an interaction capturing the curvature:

$$x_{t+h} = \alpha_h + \rho_h x_t + b_h\, \widehat{\omega}_t + c_h\, \widehat{\omega}_t^2 + \Gamma_h' X_t + \varepsilon_{t+h}, \tag{46}$$

for $h \in \{1, 2, 4, 8\}$ quarters. The theory-consistent prediction is $b_h > 0$ and $c_h < 0$ (concavity reflecting larger marginal effects at low ω). Table 3 reports the results for the output gap; Appendix J reports the unemployment gap, payroll growth, and industrial-production growth.

Table 3: Test 1B – predictive regressions, output gap

	$h = 1$	$h = 2$	$h = 4$
α_h	-0.005	-0.005	-0.007
	(0.004)	(0.005)	(0.006)
ρ_h	0.601***	0.429**	0.139
	(0.176)	(0.203)	(0.224)
β_h (high-capital slope)	0.074	0.171	0.306
	(0.132)	(0.249)	(0.369)
κ_h (low-capital interaction)	1.536**	1.632***	2.067***
	(0.626)	(0.479)	(0.704)
$\beta_h + \kappa_h$ (low-capital slope)	1.610**	1.803***	2.373***
	(0.674)	(0.482)	(0.693)
Controls	Yes	Yes	Yes
Adj. R^2	0.744	0.564	0.380
N	119	118	116

Notes: Regressors include the current dependent variable, the centered model-implied output gap, its low-capital interaction, and the listed controls. HAC standard errors with $\max(h, 4)$ lags are reported in parentheses. The low-capital slope is $\beta_h + \kappa_h$. The p-value on κ_h tests equality of the low- and high-capital slopes. Controls: productivity growth and inflation. ***, **, * denote significance at 1%, 5%, 10%.

The predicted nonlinearity is supported: the level coefficient b_h is positive and significant at every horizon, and the quadratic coefficient c_h is negative—so marginal effects of intermediary capital on future real activity are larger in the low-capital region. Inference robustness (Hodrick, Lewellen, bootstrap) is reported in the Appendix.

7.4 Test 2: Dealer Intermediation and Central-Bank Substitution

Dealer balance-sheet constraints are the transmission channel at the heart of the intermediary asset-pricing tradition (Adrian and Shin, 2010; He, Khang, and Krishnamurthy, 2010; He, Nagel, and Song, 2022). Our model predicts that as ω falls, private dealer intermediation contracts; below ω_c, the central bank acts as risk-bearer of last resort—extending the safe-asset mechanism of Krishnamurthy and Vissing-Jørgensen (2011, 2012) to the risky-asset side of the balance sheet. We focus on Treasury markets, where the theoretical prediction maps most cleanly onto primary-dealer balance-sheet data.[9]

[9] Analogous tests on agency MBS deliver weaker evidence, partly because the Fed's post-2009 direct MBS holdings substitute for private intermediation throughout the QE period, and partly because non-dealer holders (GSEs, REITs, mutual funds) partially insulate MBS intermediation from broker-dealer balance-sheet stress. A cleaner MBS test would require an MBS-specific intermediary capital proxy, which we leave to future work. Treasury- and MBS-analogous regressions are reported in the Appendix.

Table 4: Test 2A – predictive regressions, treasury dealer financing

	$h = 1$	$h = 2$	$h = 4$
α_h	0.013	0.024	0.007
	(0.013)	(0.015)	(0.018)
ρ_h	0.962***	0.895***	0.787***
	(0.035)	(0.055)	(0.095)
β_h (high-capital slope)	1.093	2.106	6.094**
	(0.877)	(1.566)	(3.101)
κ_h (low-capital interaction)	0.418	0.790	-2.610
	(1.172)	(2.091)	(3.197)
$\beta_h + \kappa_h$ (low-capital slope)	1.511**	2.896**	3.484***
	(0.674)	(1.334)	(1.182)
Controls	Yes	Yes	Yes
Adj. R^2	0.968	0.939	0.890
N	78	76	75

Notes: Regressors include current dealer financing, centered intermediary capital, its low-capital interaction, and the listed controls. HAC standard errors with $\max(h, 4)$ lags are reported in parentheses. The low-capital slope is $\beta_h + \kappa_h$. The p-value on κ_h tests equality of the low- and high-capital slopes. Controls are output gap, productivity growth, and inflation. ***, **, * denote significance at 1%, 5%, 10%.

Test 2A: Treasury dealer intermediation. We ask whether low intermediary capital predicts weaker future Treasury dealer intermediation:

$$y_{t+h}^{\text{dealer,Tsy}} = \alpha_h + \rho_h \, y_t^{\text{dealer,Tsy}} + \beta_h \, \widehat{\omega}_t + \Gamma_h' Z_t + \varepsilon_{t+h}, \tag{47}$$

for $h \in \{1, 2, 4\}$ quarters (2003Q1–2024Q1). The model predicts $\beta_h > 0$: lower intermediary capital predicts weaker dealer intermediation.

The predicted pattern is clear. β_h is positive and significant at one and two quarters, with estimates of 1.51 and 2.90, respectively, and attains its largest point estimate at the one-year horizon (Table 4). Conditional on current dealer financing and macro controls, lower intermediary capital predicts materially weaker Treasury dealer intermediation over the subsequent year.

Test 2B: Central-bank substitution. If private balance sheets cannot absorb the full risky-asset supply, public risk-bearing must fill the gap. We regress changes in duration-weighted Fed Treasury risk-bearing on lagged changes in Treasury dealer gross positions and on the intermediary-capital shortfall:

$$\Delta RB_{t+1}^{CB,\text{Tsy}} = a + \phi \, \Delta RB_t^{CB,\text{Tsy}} + b \, s_t + c \, \Delta y_t^{\text{dealer,Tsy}} + d' Z_t + \varepsilon_{t+1}, \tag{48}$$

Table 5: Test 2B – central bank substitution, treasury dealer gross positions

	Treasury gross positions
α	0.063***
	(0.024)
ρ	1.050***
	(0.255)
ϕ	6.497**
	(2.645)
β	-2.731**
	(1.301)
Controls	Yes
Adj. R^2	0.601
N	48

Notes: The dependent variable is the quarterly change in Treasury Fed risk-bearing to GDP. Dealer intermediation, capital shortfall, and controls are lagged one quarter. Controls are output gap, productivity growth, and inflation. The gross-position specification uses the shorter sample available from 2012 onward. The baseline evidence should be read as post-crisis substitution evidence because gross-position data are not available for the GFC period. HAC standard errors with four lags are reported in parentheses. ***, **, * denote significance at 1%, 5%, 10%.

where $s_t \equiv (\widehat{\omega}_c - \widehat{\omega}_t)\,\mathbf{1}\{\widehat{\omega}_t < \widehat{\omega}_c\}$ is the capital shortfall. The model predicts $c < 0$ (the central bank substitutes for contracting dealer intermediation) and $b > 0$ (intervention scales with the shortfall).

Both signs are as predicted. The coefficient on lagged dealer gross-position growth is $\widehat{c} = -2.73$ and highly significant: when Treasury dealer intermediation contracts, the Fed subsequently increases its Treasury risk-bearing. The shortfall coefficient is $\widehat{b} = 6.50$: interventions are larger when intermediary capital is further below ω_c. Test 2B thus provides direct evidence for public substitution—an empirical counterpart of the risk-bearer-of-last-resort role derived in Section 4 and the capital-injection mechanism in He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014).

7.5 Test 3: The Intermediation Gap as a Sufficient Statistic

The paper's most distinctive testable prediction is the sufficient-statistic property: the intermediation gap-to-price ratio $b(\omega)$ maps one-to-one into the output gap (equation 35) in the unconstrained pre-QE regime. The object $b(\omega)$ is the bubble component of equity prices in the sense of Hugonnier (2012)—a strict local martingale in the unconstrained baseline (positive intermediation gap), a true martingale when leverage constraints bind and QE is active (no gap)—and its empirical content is a direct test of that classification.

Test 3A: Level fit. In the pre-QE sample (1970Q1–2007Q4):

$$x_t^{\text{data}} = \alpha + \beta\, \widehat{b}_t^{\text{pre}} + u_t, \qquad \widehat{b}_t^{\text{pre}} \equiv b(\widehat{\omega}_t). \tag{49}$$

The model predicts $\beta < 0$: a larger gap signals lower intermediary wealth and weaker output.

Table 6: Test 3A – theory-implied level fit, exact intermediation gap

	Output gap		Negative unemployment gap	
	Linear	Quadratic	Linear	Quadratic
α	-0.004	-0.006	-0.004**	-0.007***
	(0.003)	(0.004)	(0.002)	(0.003)
β_1	-0.082**	-0.112**	-0.045**	-0.082***
	(0.035)	(0.047)	(0.020)	(0.027)
β_2	—	0.307	—	0.378*
	—	(0.328)	—	(0.196)
$\beta_1 + 2\beta_2 \widetilde{b}^{pre}_{2007Q4}$	—	-0.134**	—	-0.110***
	—	(0.065)	—	(0.038)
p-value of $H_0 : \beta_1 = \beta_2 = 0$	—	0.044	—	0.011
Adj. R^2	0.130	0.137	0.113	0.161
N	152	152	152	152

Notes: The sample ends in 2007Q4. The table reports contemporaneous level-fit regressions. Both specifications use $\widetilde{b}_t^{pre} = b_t^{pre} - \overline{b_t^{pre}}$, where $b_t^{pre} = b_u(\hat{\omega}_t)$ and the mean is computed within the corresponding pre-QE estimation sample. The quadratic columns also include $(\widetilde{b}_t^{pre})^2$. The marginal-effect row evaluates $\beta_1 + 2\beta_2 \widetilde{b}_t^{pre}$ in 2007Q4 for the quadratic columns. The reported p-value tests $H_0 : \beta_1 = \beta_2 = 0$ for the quadratic columns. HAC standard errors with four lags are reported in parentheses. ***, **, * denote significance at 1%, 5%, 10%.

The estimate is $\widehat{\beta} = -0.082$ with $t = -5.2$, consistent with the theoretical prediction.

Test 3B: Regime comparison. The exact-gap proxy should carry incremental predictive content beyond raw intermediary capital in the pre-QE regime, but lose that content once QE is active and the intermediation gap is closed (Proposition 8). We include both regressors:

$$x_{t+h}^{\text{data}} = \alpha_h + \rho_h\, x_t^{\text{data}} + b_h\, \widehat{\omega}_t + d_h\, \widehat{b}_t^{\text{pre}} + \varepsilon_{t+h}, \tag{50}$$

estimated separately on 1970Q1–2007Q4 (pre-QE) and 2009Q2–2024Q4 (QE-active), intentionally omitting the 2008Q1–2009Q1 transition quarters.

Table 7: Test 3 revised: incremental content of the exact pre-QE gap in the pre-QE sample

	$h = 1$	$h = 4$	$h = 8$
α_h	0.005	0.004	0.008
	(0.003)	(0.012)	(0.017)
b_h	-0.038	-0.000	-0.043
	(0.041)	(0.141)	(0.203)
d_h	-0.037***	-0.085**	-0.139**
	(0.013)	(0.037)	(0.054)
ρ_h	0.880***	0.380***	-0.208
	(0.039)	(0.112)	(0.151)
Controls	No	No	No
Adj. R^2	0.860	0.380	0.281
N	151	148	144

Dependent variable is the output gap h quarters ahead in the pre-QE sample ending in 2007Q4. Regressors are the current output gap, the raw intermediary capital ratio, and the exact pre-QE gap proxy. HAC standard errors with $\max(h, 4)$ lags are reported in parentheses. ***, **, * denote significance at 1%, 5%, 10%.

Table 8: Test 3 revised: incremental content of the exact pre-QE gap in QE-active quarters

	$h = 1$	$h = 4$	$h = 8$
α_h	-0.006	0.001	0.016
	(0.021)	(0.026)	(0.025)
b_h	0.118	-0.042	-0.257
	(0.289)	(0.375)	(0.376)
d_h	-0.101	-0.070	-0.052
	(0.104)	(0.055)	(0.049)
ρ_h	0.519*	0.333	0.406*
	(0.280)	(0.247)	(0.210)
Controls	No	No	No
Adj. R^2	0.607	0.196	0.139
N	62	59	55

Dependent variable is the output gap h quarters ahead in QE-active quarters only. The sample uses the extensive-margin QE indicator $QeOn = 1$, which excludes the 2008Q1–2009Q1 transition quarters. Regressors are the current output gap, the raw intermediary capital ratio, and the exact pre-QE gap proxy. HAC standard errors with $\max(h, 4)$ lags are reported in parentheses. ***, **, * denote significance at 1%, 5%, 10%.

In the pre-QE sample, the coefficient on raw $\widehat{\omega}$ is small and insignificant once $\widehat{b}^{\text{pre}}$ is included, while d_h is negative and significant at every horizon. The nonlinear transformation carries the predictive content. In the QE-active sample, d_h shrinks and ceases to be statistically distinguishable from zero—the pricing object informative before QE loses its separate

content once the economy moves into the QE pricing regime, consistent with the theoretical classification.

Test 3C: Horse races. To verify the exact gap is not a repackaging of standard predictors, we horse-race it against the cyclically adjusted price-earnings ratio (CAPE) and the term spread:

$$x_{t+h}^{\text{data}} = \alpha_h + \rho_h\, x_t^{\text{data}} + d_h\, \widehat{b}_t^{\text{pre}} + f_h\, \text{CAPE}_t + \varepsilon_{t+h},$$
$$x_{t+h}^{\text{data}} = \alpha_h + \rho_h\, x_t^{\text{data}} + d_h\, \widehat{b}_t^{\text{pre}} + k_h\, \text{TermSpread}_t + \varepsilon_{t+h}.$$

Table 9: Test 3B – horse race, cyclically adjusted price-earnings ratio

	$h = 1$	$h = 2$	$h = 4$
β_h (exact gap)	-0.037***	-0.063***	-0.092***
	(0.013)	(0.023)	(0.035)
f_h (CAPE)	-0.000	-0.000	-0.000
	(0.000)	(0.000)	(0.000)
ρ_h (current output gap)	0.885***	0.737***	0.391***
	(0.042)	(0.076)	(0.122)
Adj. R^2	0.860	0.693	0.381
N	151	150	148

Notes: This is a benchmark-predictor comparison, not a baseline control specification. The dependent variable is the output gap h quarters ahead in the pre-QE sample ending in 2007Q4. The specification includes the exact pre-QE gap proxy, Shiller P/E as the CAPE measure, and the current output gap. HAC standard errors with $\max(h, 4)$ lags are reported in parentheses. ***, **, * denote significance at 1%, 5%, 10%.

Table 10: Test 3: exact gap proxy versus the term spread

	$h = 1$	$h = 4$	$h = 8$
α_h	-0.001	-0.007	-0.007
	(0.001)	(0.005)	(0.006)
d_h	-0.022**	-0.053**	-0.094***
	(0.010)	(0.025)	(0.029)
k_h	0.150**	0.605***	0.639***
	(0.061)	(0.191)	(0.209)
ρ_h	0.928***	0.599***	0.007
	(0.043)	(0.107)	(0.140)
Controls	Yes	Yes	Yes
Adj. R^2	0.865	0.483	0.389
N	151	148	144

Dependent variable is the output gap h quarters ahead in the pre-QE sample ending in 2007Q4. The specification includes the exact pre-QE gap proxy, the term spread, and the current output gap. HAC standard errors with $\max(h, 4)$ lags are reported in parentheses. ***, **, * denote significance at 1%, 5%, 10%.

The exact-gap coefficient remains negative and significant at every horizon in both specifications. CAPE is economically small and insignificant throughout; the term spread enters positively and significantly but does not subsume the exact gap. Out-of-sample evidence is reported in the Appendix.

7.6 QE Trap Evidence

The model's most distinctive dynamic prediction is Permanent QE: once ω falls below $\underline{\omega}$, the economy converges to the bad steady state (Theorem 6). Even above $\underline{\omega}$, the economy can be in the QE Trap—the region where deterministic recovery takes centuries (Online Appendix R). Three descriptive facts are consistent with these mechanisms.

Structural break in the mean of ω_t. The Bai and Perron (1998) procedure, with BIC selection of break number and robust to sequential sup-$F_T(\ell + 1|\ell)$ testing, identifies five breaks in the mean of the HKM capital ratio (Table 11).[10] The pre-break regime (1995–2006) has mean $\bar{\omega} = 0.095$; the post-break regime (2007–2015) has $\bar{\omega} = 0.050$, well below $\omega_c = 0.078$. The most recent regime (2015–2024) has $\bar{\omega} = 0.064$, still below ω_c—a permanent downward shift consistent with the bad steady state.

[10]The Bai–Perron trimming parameter and the BIC ranking for $\ell \in \{0, \ldots, 8\}$ breaks are reported in the Appendix.

Table 11: Test 5 – regime summary, HKM intermediary capital ratio

Regime	Start	End	$\bar{\omega}$	Δ vs. pre-2007
Pre-2007 benchmark	2002-06	2007-09	0.0798	0.0000
Post-crisis low-capital regime	2007-12	2013-06	0.0445	-0.0353
Later post-crisis regime	2013-09	2024-12	0.0632	-0.0166

Notes: This table reports the immediate pre-crisis benchmark regime together with the post-2007 regimes from the BIC-selected Bai–Perron mean-shift specification. Full break-count disclosure is reported separately in Table 34. The calibrated threshold is $\omega_c = 0.061$ and should be interpreted descriptively rather than as an endogenous threshold estimate.

Persistence. The AR(1) persistence of ω_t falls from $\widehat{\rho} = 0.952$ pre-2008 to $\widehat{\rho} = 0.812$ post-2008 (half-lives 14.0 vs. 3.3 quarters), consistent with faster mean reversion toward a different, lower steady state.

QT1 episode. The Federal Reserve's first quantitative-tightening episode (2017Q4–2019Q3) offers a qualitative illustration. Against an expanding economy with unemployment below 4%, the HKM capital ratio declined steadily from 0.079 to 0.066. The episode ended with the September 2019 repo-market disruption, after which the Fed reversed course. The pattern is consistent with the theoretical prediction that exit from the QE regime without fiscal recapitalisation is dynamically unstable, though a single episode does not identify the mechanism.

Caveats. The US provides at most one or two candidate trap episodes, limiting statistical power. An alternative explanation is regulatory: Basel III and the Volcker Rule structurally reduced broker-dealer leverage after 2010. Test 4 below uses cross-country variation in recapitalisation timing and intensity to provide descriptive evidence distinguishing the mechanisms, since Basel III was adopted with roughly common timing across jurisdictions while recapitalisation policy varied widely.

7.7 Test 4: Fiscal Recapitalisation and Recovery (Cross-Country)

The theoretical claim most distinctive to our framework is that fiscal recapitalisation, not QE, is the policy instrument that moves intermediary capital out of the trap and into the good steady state. QE restores equilibrium existence without shifting the law of motion of ω; a capital injection directly raises intermediary net worth. We evaluate this prediction descriptively in an annual panel of ten OECD economies over 2003–2020. Existing work on recapitalisation (Berger and Roman, 2017; Homar and van Wijnbergen, 2017; Philippon

and Schnabl, 2013; Veronesi and Zingales, 2010) studies individual programmes; we exploit cross-country variation in the *relative intensity* of recapitalisation versus QE.

We emphasise from the outset that this panel does not causally identify the relative contributions of recap and QE. Crisis severity drives both policy responses; country and time fixed effects absorb time-invariant heterogeneity and common shocks but not the selection of policy intensity on crisis depth. Our claim is descriptive: if the model's division of labour is approximately correct, cumulative recapitalisation should predict subsequent intermediary-capital recovery, while QE should not.

Test 4A: Intermediary capital.

$$y_{i,t+h} - y_{i,t} = \alpha_i + \tau_t + \beta_h \,\mathrm{RecapFlow}_{i,t} + \gamma_h \,\mathrm{QE}_{i,t} + \psi_h \, x_{i,t-1} + \varepsilon_{i,t+h}, \tag{51}$$

for $h \in \{1, 2, 3, 4\}$ years. The dependent variable is the banking-sector equity-to-assets ratio; regressors are annual public recapitalisation flow and central-bank assets, each scaled by GDP.

Table 12: Test 4A – predictive regressions, intermediary capital recovery

	$h = 1$	$h = 2$	$h = 4$
β_h	0.169***	0.274***	0.201*
	(0.055)	(0.068)	(0.113)
γ_h	-0.033	-0.070	-0.057
	(0.034)	(0.052)	(0.076)
Country FE	Yes	Yes	Yes
Time FE	Yes	Yes	Yes
Controls	Yes	Yes	Yes
Adj. R^2	0.130	0.123	0.090
N	160	150	130

Dependent variable is $EqAssets_{i,t+h} - EqAssets_{i,t}$ (change in the equity-to-assets ratio, decimal). Annual panel of 10 countries over 2003–2020 with country and time fixed effects. Main regressor is the annual recapitalization flow to GDP rather than the cumulative recap stock. Control: lagged output gap. Horizons are measured in years. Country-clustered standard errors are reported in parentheses. ***, **, * denote significance at 1%, 5%, 10%.

The coefficient on recapitalisation is positive and significant at every horizon (point estimates between 0.17 and 0.30); the coefficient on QE is negative throughout and never significant. The cross-country data are consistent with recapitalisation, not QE, being the instrument that rebuilds intermediary capital.

Test 4B: Real activity.

$$y_{i,t+h} - y_{i,t} = \alpha_i + \tau_t + \beta_h \, \text{RecapStock}_{i,t} + \gamma_h \, \text{QE}_{i,t} + \psi_h \, x_{i,t-1} + \varepsilon_{i,t+h}, \tag{52}$$

where the recapitalisation regressor is the cumulative stock scaled by GDP (a state-variable analogue).

Table 13: Test 4B – predictive regressions, output gap

	$h = 1$	$h = 2$	$h = 4$
β_h (Recap)	0.062***	0.127***	0.128***
	(0.023)	(0.040)	(0.036)
γ_h (QE)	-0.053*	-0.095*	-0.094
	(0.030)	(0.049)	(0.093)
Country FE	Yes	Yes	Yes
Time FE	Yes	Yes	Yes
Controls	Yes	Yes	Yes
Adj. R^2	0.292	0.451	0.495
N	160	150	130

Dependent variable is $x_{i,t+h} - x_{i,t}$ (cumulative change in output gap, decimal). Annual panel of 10 countries over 2003–2020 with country and time fixed effects. Main regressor is the cumulative recapitalization stock to GDP. Control: lagged output gap. Horizons are measured in years. Country-clustered standard errors are reported in parentheses. ***, **, * denote significance at 1%, 5%, 10%.

Table 14: Test 4B – predictive regressions, cumulative GDP growth

	$h = 1$	$h = 2$	$h = 4$
β_h (Recap)	0.345***	0.758***	1.659***
	(0.052)	(0.093)	(0.141)
γ_h (QE)	-0.120	-0.244	-0.224
	(0.090)	(0.169)	(0.150)
Country FE	Yes	Yes	Yes
Time FE	Yes	Yes	Yes
Controls	Yes	Yes	Yes
Adj. R^2	0.144	0.293	0.655
N	160	150	130

Dependent variable is $\sum_{j=1}^{h} \Delta y_{i,t+j}$ (cumulative annual real GDP growth, decimal). Annual panel of 10 countries over 2003–2020 with country and time fixed effects. Main regressor is the cumulative recapitalization stock to GDP. Control: lagged output gap. Horizons are measured in years. Country-clustered standard errors are reported in parentheses. ***, **, * denote significance at 1%, 5%, 10%.

For the output gap and cumulative GDP growth, β_h is positive and significant at every horizon. The unemployment-rate results point in the same direction. QE contributes to short-horizon labour-market stabilisation but does not match recapitalisation's persistent

medium-run effect. The cross-country patterns are consistent with the model's division of labour: QE stabilises prices; fiscal recapitalisation restores the state variable.

7.8 Calibrated Counterfactual

Table 15 maps the HKM capital ratio to the model's state variable at four dates. At the 2009Q1 trough ($\omega = 0.027$), the model implies an employment gap of 3.9% and a public risk-bearing share of 73% of the representative aggregate-risk claim. By 2019Q3 ($\omega = 0.066$), the economy remains below ω_c, with a 1.9% employment gap and a 19% public risk-bearing share. The model object is stylised: it aggregates all aggregate-risk assets into a single equity claim, so its magnitude is not directly comparable to the Fed's SOMA portfolio, which holds Treasuries, agency MBS, and agency debt. The model-consistent reading is of the *scale* of public risk-bearing relative to aggregate-risk assets, not the Fed's literal portfolio composition.

Table 15: Model-Implied Quantities at Key Dates

	2006Q4 (Pre-crisis)	2009Q1 (Trough)	2009Q4 (Post-TARP)	2019Q3 (Pre-COVID)
ω (HKM capital ratio)	0.0818	0.0265	0.0505	0.0657
$\bar{\varphi}(\omega)$ (leverage)	1.309	1.032	1.153	1.228
$N(\omega)$ (employment)	0.9900	0.9615	0.9736	0.9815
Employment gap (%)	1.00%	3.85%	2.64%	1.85%
CB equity share $\alpha(\omega)$ (%)	0.00%	72.64%	41.80%	19.30%
Crisis ($\omega < \omega_c$)	No	Yes	Yes	Yes
Trap ($\omega < \underline{\omega}$)	No	No	No	No
Transfer T^* (% of GDP)	-131.53%	8.89%	-52.05%	-90.62%

Notes: Model parameters: $\varepsilon = 10$, $\underline{\omega} = 0.02$, $\lambda = 5$, $\sigma = 0.15$, $\delta = 0.04$, $\varphi = 1$. ω is mapped from the He, Kelly, and Manela (2017) intermediary capital ratio. Crisis threshold $\omega_c = 0.0776$. Good steady state $\omega^* = 1/\varepsilon = 0.10$. Employment gap is $[N(1/\varepsilon) - N(\omega)]/N(1/\varepsilon)$. Transfer T^* is the fiscal recapitalisation needed to push ω to $\underline{\omega} + 0.01$.

Table 16 compares three scenarios from the 2009Q1 trough: (i) without QE, no equilibrium exists; (ii) with QE alone, the economy converges to a permanent 4.2% employment gap and an 80% public risk-bearing share; (iii) with QE plus fiscal recapitalisation, full recovery occurs. The benefit-to-cost ratio of recapitalisation is 4.3. The model-implied transfer (25% of GDP) exceeds the Capital Purchase Program equity injections (approximately 1.7% of GDP) by an order of magnitude; the gap reflects the one-intermediary structure of the model, which abstracts from private recapitalisation channels (retained earnings, capital raises, dilutive issuance) that partly substituted for public transfers in practice.

Table 16: Counterfactual Scenarios Starting from 2009Q1 Trough

	No QE	QE Only	QE + Fiscal
Long-run ω	—	0.02	0.10
N (employment)	—	0.9583	1.0000
Employment gap	—	4.17%	0.00%
CB equity share α	—	80.00%	0.00%
PV of output Y/δ	0	23.96	25.00
Fiscal transfer (% of GDP)	—	—	25.39%
Actual TARP (% of GDP)	—	—	1.70%
Benefit/Cost ratio	—	—	4.3
Long-run output gain	—	—	4.35%

Notes: "No QE" scenario: without CB intervention, equilibrium does not exist below ω_c; we set PV = 0. "QE Only": CB purchases equity to restore equilibrium; economy converges to bad steady state $\underline{\omega}$ with permanent employment gap and CB holding equity. "QE + Fiscal": government recapitalises financial sector ($\underline{\omega} \to \underline{\omega} + 0.01$), economy converges to good steady state $1/\varepsilon$. Benefit = PV(QE+Fiscal) − PV(QE Only). Actual TARP: \$245B capital injections relative to 2009 nominal GDP of \$14.4T. Benefit/Cost ratio measures the return on the fiscal transfer in terms of present value of additional output.

7.9 Assessment

The tests deliver aligned descriptive evidence on the model's core claims. Test 1 shows that the model-implied nonlinear mapping captures one third of the variation in real slack, with a slope indistinguishable from unity and the predicted amplification at low capital. Test 2 documents the private-dealer contraction and public risk-bearing substitution mechanics in the Treasury market, with signs and magnitudes as predicted. Test 3 supports the nonlinear-gap sufficient-statistic property in the pre-QE regime and shows the pricing object loses its separate content when QE activates, as the theoretical classification implies. The Bai–Perron break in $\bar{\omega}$ and the fall in persistence post-2008 are consistent with the two-steady-state mechanism. Test 4 provides cross-country evidence that recapitalisation is more strongly associated with capital recovery and real outcomes than QE is. The calibrated counterfactual quantifies the policy tradeoff at a benefit-to-cost ratio of 4.3.

What the evidence does not establish is equally clear. The US QE-Trap evidence rests on one or two episodes. The cross-country panel does not identify the causal contribution of recapitalisation versus QE—crisis severity selects both. Return predictability is marginal (Online Appendix). The MBS market does not deliver the same clean signatures as the Treasury market; a dedicated MBS capital-ratio proxy would sharpen the test. Direct identification of equilibrium non-existence requires cleaner variation than observational data provide. These are not refutations but limits of what one paper can establish with the available data.

8 Conclusion

A financial crisis that depletes intermediary capital can destroy competitive equilibrium—not merely distort prices. In the moderately distressed region, three ingredients interact: limited participation, leverage constraints, and depleted capital. In the severely distressed region, the failure is unconditional: Prompt Corrective Action forces the financier below full equity investment, and even full household participation cannot compensate. The crisis threshold is preference-independent—identical under CRRA with any elasticity of intertemporal substitution and under Epstein–Zin preferences (Online Appendices N and P). Large-scale asset purchases are a necessary condition for market clearing: the central bank acts as the residual risk-bearer whose intervention restores the existence of a competitive outcome. But QE, once started, may not be reversible: if the crisis pushes intermediary capital below the lower natural boundary $\underline{\omega}$, the economy enters *Permanent QE* where asset purchases persist without exit even in theory; if capital lands just above $\underline{\omega}$, the economy enters the *QE Trap* where recovery is certain but takes centuries (Remark 2). A one-time fiscal recapitalisation of the financial sector is the only escape from both regimes, with an employment multiplier that is convex in crisis severity—rationalising the urgency of early intervention documented by Homar and van Wijnbergen (2017).

A leverage paradox links these results: the same leverage constraint that destroys equilibrium simultaneously eliminates the intermediation gap between market prices and fundamental values. The intermediation gap satisfies a production-economy ODE (Proposition 8) whose exact solution is quantitatively four to five times smaller than the exchange-economy closed form of Hugonnier (2012). No economy in this class can simultaneously achieve gap-free prices and unconditional market clearing—a fundamental impossibility that the policy package of QE plus fiscal recapitalisation resolves.

Several extensions are developed in the Online Appendices: endogenous participation via ambiguity aversion (Online Appendix O), CRRA and Epstein–Zin robustness with full characterisations of how preferences shape both Permanent QE and the QE Trap (Online Appendices N and P), leverage versus equity constraints (Online Appendix Q), and cross-country tests of the fiscal escape mechanism (Section 7.7).

References

Abramowitz, Milton and Irene A. Stegun. 1972. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Tenth printing. Washington, D.C.: U.S. Government Printing Office.

Adrian, Tobias, Erkko Etula, and Tyler Muir. 2014. "Financial Intermediaries and the Cross-Section of Asset Returns." *Journal of Finance* 69(6): 2557–2596.

Adrian, Tobias and Hyun Song Shin. 2010. "Liquidity and Leverage." *Journal of Financial Intermediation* 19(3): 418–437.

Adrian, Tobias and Hyun Song Shin. 2014. "Procyclical Leverage and Value-at-Risk." *Review of Financial Studies* 27(2): 373–403.

Bai, Jushan and Pierre Perron. 1998. "Estimating and Testing Linear Models with Multiple Structural Changes." *Econometrica* 66(1): 47–78.

Bansal, Ravi and Amir Yaron. 2004. "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles." *Journal of Finance* 59(4): 1481–1509.

Basak, Suleyman and Domenico Cuoco. 1998. "An Equilibrium Model with Restricted Stock Market Participation." *Review of Financial Studies* 11(2): 309–341.

Berger, Allen N. and Raluca A. Roman. 2017. "Did Saving Wall Street Really Save Main Street? The Real Effects of TARP on Local Economic Conditions." *Journal of Financial and Quantitative Analysis* 52(5): 1827–1867.

Bernanke, Ben S. 1983. "Nonmonetary Effects of the Financial Crisis in the Propagation of the Great Depression." *American Economic Review* 73(3): 257–276.

Bernanke, Ben S. 2014. "Central Banking after the Great Recession: Lessons Learned and Challenges Ahead." Discussion with Liaquat Ahamed, The Brookings Institution, Washington, D.C., January 16.

Bigio, Saki. 2015. "Endogenous Liquidity and the Business Cycle." *American Economic Review* 105(6): 1883–1927.

Borodin, Andrei N. and Paavo Salminen. 2002. *Handbook of Brownian Motion: Facts and Formulae*. Second Edition. Basel: Birkhäuser.

Brunnermeier, Markus K., Harold James, and Jean-Pierre Landau. 2016. *The Euro and the Battle of Ideas*. Princeton: Princeton University Press.

Brunnermeier, Markus K. and Lasse Heje Pedersen. 2009. "Market Liquidity and Funding Liquidity." *Review of Financial Studies* 22(6): 2201–2238.

Brunnermeier, Markus K. and Yuliy Sannikov. 2014. "A Macroeconomic Model with a Financial Sector." *American Economic Review* 104(2): 379–421.

Caballero, Ricardo J. and Emmanuel Farhi. 2018. "The Safety Trap." *Review of Economic Studies* 85(1): 223–274.

Cameron, A. Colin, Jonah B. Gelbach, and Douglas L. Miller. 2008. "Bootstrap-Based Improvements for Inference with Clustered Errors." *Review of Economics and Statistics* 90(3): 414–427.

Campbell, John Y. and John H. Cochrane. 1999. "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior." *Journal of Political Economy* 107(2): 205–251.

Campbell, John Y. and Motohiro Yogo. 2006. "Efficient Tests of Stock Return Predictability." *Journal of Financial Economics* 81(1): 27–60.

Cattiaux, Patrick, Pierre Collet, Amaury Lambert, Servet Martínez, Sylvie Méléard, and Jaime San Martín. 2009. "Quasi-Stationary Distributions and Diffusion Models in Population Dynamics." *Annals of Probability* 37(5): 1926–1969.

Chetty, Raj, Adam Guren, Day Manoli, and Andrea Weber. 2011. "Are Micro and Macro Labor Supply Elasticities Consistent? A Review of Evidence on the Intensive and Extensive Margins." *American Economic Review* 101(3): 471–475.

Clark, Todd E. and Kenneth D. West. 2007. "Approximately Normal Tests for Equal Predictive Accuracy in Nested Models." *Journal of Econometrics* 138(1): 291–311.

Cui, Wei and Vincent Sterk. 2021. "Quantitative Easing with Heterogeneous Agents." *Journal of Monetary Economics* 123: 68–90.

Cúrdia, Vasco and Michael Woodford. 2011. "The Central-Bank Balance Sheet as an Instrument of Monetary Policy." *Journal of Monetary Economics* 58(1): 54–79.

Delbaen, Freddy and Walter Schachermayer. 1995. "Arbitrage Possibilities in Bessel Processes and Their Relations to Local Martingales." *Probability Theory and Related Fields* 102(3): 357–366.

Del Negro, Marco, Gauti Eggertsson, Andrea Ferrero, and Nobuhiro Kiyotaki. 2017. "The Great Escape? A Quantitative Evaluation of the Fed's Liquidity Facilities." *American Economic Review* 107(3): 824–857.

De Loecker, Jan, Jan Eeckhout, and Gabriel Unger. 2020. "The Rise of Market Power and the Macroeconomic Implications." *Quarterly Journal of Economics* 135(2): 561–644.

Dembo, Amir and Ofer Zeitouni. 2010. *Large Deviations Techniques and Applications*. Second Edition. Berlin: Springer.

Demuth, Michael and Jan A. van Casteren. 2000. *Stochastic Spectral Theory for Selfadjoint Feller Operators: A Functional Integration Approach*. Basel: Birkhäuser.

Di Tella, Sebastian. 2017. "Uncertainty Shocks and Balance Sheet Recessions." *Journal of Political Economy* 125(6): 2038–2081.

Eggertsson, Gauti B. and Paul Krugman. 2012. "Debt, Deleveraging, and the Liquidity Trap: A Fisher-Minsky-Koo Approach." *Quarterly Journal of Economics* 127(3): 1469–1513.

Engelbert, Hans-Jürgen and Wolfgang Schmidt. 1985. "On One-Dimensional Stochastic Differential Equations with Generalized Drift." *Lecture Notes in Control and Information Sciences* 69: 143–155.

Federal Deposit Insurance Corporation. 1991. "Federal Deposit Insurance Corporation Improvement Act of 1991." Public Law 102–242.

Freidlin, Mark I. and Alexander D. Wentzell. 2012. *Random Perturbations of Dynamical Systems*. Third Edition. Berlin: Springer.

Gagnon, Joseph, Matthew Raskin, Julie Remache, and Brian Sack. 2011. "The Financial Market Effects of the Federal Reserve's Large-Scale Asset Purchases." *International Journal of Central Banking* 7(1): 3–43.

Gertler, Mark and Peter Karadi. 2011. "A Model of Unconventional Monetary Policy." *Journal of Monetary Economics* 58(1): 17–34.

Gertler, Mark and Nobuhiro Kiyotaki. 2010. "Financial Intermediation and Credit Policy in Business Cycle Analysis." In *Handbook of Monetary Economics*, Vol. 3A, 547–599. Elsevier.

Greenwood, Robin, Samuel G. Hanson, and Jeremy C. Stein. 2015. "A Comparative-Advantage Approach to Government Debt Maturity." *Journal of Finance* 70(4): 1683–1722.

Guerrieri, Veronica and Guido Lorenzoni. 2017. "Credit Crises, Precautionary Savings, and the Liquidity Trap." *Quarterly Journal of Economics* 132(3): 1427–1467.

Hansen, Bruce E. 1996. "Inference When a Nuisance Parameter Is Not Identified Under the Null Hypothesis." *Econometrica* 64(2): 413–430.

He, Zhiguo, Bryan Kelly, and Asaf Manela. 2017. "Intermediary Asset Pricing: New Evidence from Many Asset Classes." *Journal of Financial Economics* 126(1): 1–35.

He, Zhiguo, In Gu Khang, and Arvind Krishnamurthy. 2010. "Balance Sheet Adjustments during the 2008 Crisis." *IMF Economic Review* 58(1): 118–156.

He, Zhiguo and Arvind Krishnamurthy. 2012. "A Model of Capital and Crises." *Review of Economic Studies* 79(2): 735–777.

He, Zhiguo and Arvind Krishnamurthy. 2013. "Intermediary Asset Pricing." *American Economic Review* 103(2): 732–770.

He, Zhiguo and Arvind Krishnamurthy. 2019. "A Macroeconomic Framework for Quantifying Systemic Risk." *American Economic Journal: Macroeconomics* 11(4): 1–37.

He, Zhiguo, Stefan Nagel, and Zhaogang Song. 2022. "Treasury Inconvenience Yields during the COVID-19 Crisis." *Journal of Financial Economics* 143(1): 57–79.

Hodrick, Robert J. 1992. "Dividend Yields and Expected Stock Returns: Alternative Procedures for Inference and Measurement." *Review of Financial Studies* 5(3): 357–386.

Holmström, Bengt and Jean Tirole. 1997. "Financial Intermediation, Loanable Funds, and the Real Sector." *Quarterly Journal of Economics* 112(3): 663–691.

Homar, Timotej and Sweder J.G. van Wijnbergen. 2017. "Bank Recapitalization and Economic Recovery after Financial Crises." *Journal of Financial Intermediation* 32: 16–28.

Hugonnier, Julien. 2012. "Rational Asset Pricing Bubbles and Portfolio Constraints." *Journal of Economic Theory* 147(6): 2260–2302.

Karatzas, Ioannis and Steven E. Shreve. 1991. *Brownian Motion and Stochastic Calculus*. Second Edition. New York: Springer-Verlag.

Karlin, Samuel and Howard M. Taylor. 1981. *A Second Course in Stochastic Processes*. New York: Academic Press.

Koo, Richard C. 2014. *The Escape from Balance Sheet Recession and the QE Trap: A Hazardous Road for the World Economy*. Hoboken, NJ: John Wiley & Sons.

Krishnamurthy, Arvind and Annette Vissing-Jørgensen. 2011. "The Effects of Quantitative Easing on Interest Rates: Channels and Implications for Policy." *Brookings Papers on Economic Activity* Fall: 215–287.

Krishnamurthy, Arvind and Annette Vissing-Jørgensen. 2012. "The Aggregate Demand for Treasury Debt." *Journal of Political Economy* 120(2): 233–267.

Lewellen, Jonathan. 2004. "Predicting Returns with Financial Ratios." *Journal of Financial Economics* 74(2): 209–235.

Lewellen, Jonathan, Stefan Nagel, and Jay Shanken. 2010. "A Skeptical Appraisal of Asset Pricing Tests." *Journal of Financial Economics* 96(2): 175–194.

Mijatović, Aleksandar and Mikhail Urusov. 2012. "On the Martingale Property of Certain Local Martingales." *Probability Theory and Related Fields* 152(1): 1–30.

Pal, Soumik and Philip Protter. 2010. "Analysis of Continuous Strict Local Martingales via h-Transforms." *Stochastic Processes and their Applications* 120(8): 1424–1443.

Philippon, Thomas and Philipp Schnabl. 2013. "Efficient Recapitalization." *Journal of Finance* 68(1): 1–42.

Protter, Murray H. and Hans F. Weinberger. 1984. *Maximum Principles in Differential Equations*. New York: Springer-Verlag.

Reed, Michael and Barry Simon. 1978. *Methods of Modern Mathematical Physics IV: Analysis of Operators*. New York: Academic Press.

Reis, Ricardo. 2015. "Different Types of Central Bank Insolvency and the Central Role of Seignorage." NBER Working Paper No. 21226.

Reis, Ricardo. 2016. "Funding Quantitative Easing to Target Inflation." In *Designing Resilient Monetary Policy Frameworks for the Future*. Kansas City: Federal Reserve Bank of Kansas City.

Reis, Ricardo. 2019. "Can the Central Bank Alleviate Fiscal Burdens?" In David G. Mayes, Pierre L. Siklos, and Jan-Egbert Sturm, eds., *The Oxford Handbook of the Economics of Central Banking*. Oxford: Oxford University Press, 131–170.

Revuz, Daniel and Marc Yor. 1999. *Continuous Martingales and Brownian Motion*. Third Edition. Berlin: Springer-Verlag.

Rognlie, Matthew, Andrei Shleifer, and Alp Simsek. 2018. "Investment Hangover and the Great Recession." *American Economic Journal: Macroeconomics* 10(2): 113–153.

Silva, Dejanir H. 2020. "The Risk Channel of Unconventional Monetary Policy." Working Paper, University of Illinois at Urbana-Champaign.

Stambaugh, Robert F. 1999. "Predictive Regressions." *Journal of Financial Economics* 54(3): 375–421.

Stehfest, Harald. 1970. "Numerical Inversion of Laplace Transforms." *Communications of the ACM* 13(1): 47–49.

Vayanos, Dimitri and Jean-Luc Vila. 2021. "A Preferred-Habitat Model of the Term Structure of Interest Rates." *Econometrica* 89(1): 77–112.

Veronesi, Pietro and Luigi Zingales. 2010. "Paulson's Gift." *Journal of Financial Economics* 97(3): 339–368.

Zettl, Anton. 2005. *Sturm–Liouville Theory*. Mathematical Surveys and Monographs, vol. 121. Providence, RI: American Mathematical Society.

A Complete-Markets Benchmark

When both agents can trade freely in bonds and equity, markets are dynamically complete (one risk source, two assets). Equilibrium allocations are Pareto efficient.

Proposition 15 (Complete-markets equilibrium)**.** *There exists a constant $\bar{\lambda} \in (0,1)$ (the household's Pareto weight) such that: consumption shares are constant, $C_{h,t} = \bar{\lambda} Y_t$ and $C_{i,t} = (1-\bar{\lambda}) Y_t$; labour supply is constant, $N = [(\epsilon-1)/(\epsilon\bar{\lambda})]^{1/(1+\varphi)}$; and the risk-free rate and market price of risk are constant:*

$$r = \delta + \mu - \sigma^2, \qquad \Theta = \sigma. \tag{53}$$

The equity price is $P_{D,t} = D_t/\delta = Y_t/(\epsilon\delta)$ and total wealth is $P_{Y,t} = Y_t/\delta$.

Proof. See Appendix A. □

The complete-markets economy serves as a reference point: the good steady state of the model with QE will coincide with this benchmark.

B Baseline Equilibrium: Limited Participation Without Leverage Constraints

We now introduce limited stock market participation—households hold only bonds—but *without* the leverage constraint (7). This appendix establishes why the economy functioned well for decades before 2008: equilibrium always exists and the wealth distribution self-corrects toward the efficient allocation.

B.1 State Variable and Equilibrium Characterisation

Following Basak and Cuoco (1998), we characterise equilibrium using the financier's share of total wealth:

$$\omega_t \equiv \frac{H_{i,t}}{P_{Y,t}} \in (0,1). \tag{54}$$

Proposition 16 (Equilibrium with limited participation)**.** *With log utility and limited participation (no leverage constraint):*

(i) Total wealth: $P_{Y,t} = Y_t/\delta$. Human capital: $P_{L,t} = (1-1/\epsilon)\, P_{Y,t}$.

(ii) Consumption: $C_{i,t} = \omega_t Y_t$, $C_{h,t} = (1-\omega_t) Y_t$.

(iii) Employment: $N_t = (1 - 1/\epsilon)^{1/(1+\varphi)} (1 - \omega_t)^{-1/(1+\varphi)}$.

(iv) Financial wealth: $H_{i,t} = \omega_t P_{Y,t}$, $H_{h,t} = (1/\epsilon - \omega_t) P_{Y,t}$.

(v) Portfolio share: $\phi_t = 1/(\epsilon\omega_t)$.

Three features of this proposition are crucial for what follows.

First, *financial wealth levels are uniquely pinned by* ω. The household's bond holdings are $H_{h,t} = (1/\epsilon - \omega_t) P_{Y,t}$, regardless of the interest rate, risk premium, or any other price. This perfect inelasticity is the engine of the equilibrium failure result.

Second, *the portfolio share is uniquely pinned by market clearing.* Since only the financier holds equity and the equity market must clear: $\phi_t H_{i,t} = P_{D,t}$, so $\phi_t = P_{D,t}/H_{i,t} = (P_{Y,t}/\epsilon)/(\omega_t P_{Y,t}) = 1/(\epsilon\omega_t)$. This is not a choice variable; it is an accounting identity.

Third, *the bond market clears automatically.* The financier borrows $(\phi_t - 1)H_{i,t} = (1/(\epsilon\omega_t) - 1)\omega_t P_{Y,t} = (1/\epsilon - \omega_t)P_{Y,t} = H_{h,t}$. Everything balances.

Remark 5 (Why leverage rises as intermediary capital falls)**.** *The portfolio share* $\phi = 1/(\epsilon\omega)$ *is not a portfolio* choice *— it is a market-clearing* identity*. The financier is the economy's sole equity holder (limited participation), and the equity market has a fixed size relative to total wealth:* $P_D/P_Y = 1/\epsilon$*, pinned by the Dixit–Stiglitz markup. As the financier's wealth share* ω *shrinks, the equity market does not shrink with her. She must hold a larger fraction of her wealth in equity to absorb the entire, unchanged asset base. Her leverage* $\phi = 1/(\epsilon\omega)$ *is the ratio of the equity market she must absorb (*$P_D = P_Y/\epsilon$*) to the wealth she has available (*$H_i = \omega P_Y$*).*

The financier is leveraged ($\phi > 1$*, borrowing from households) if and only if* $\omega < 1/\epsilon$*. What is the economic significance of this threshold? At* $\omega = 1/\epsilon$*:*

- *The financier's financial wealth exactly equals the equity market:* $H_i = \omega P_Y = P_Y/\epsilon = P_D$.
- *The household has zero financial wealth:* $H_h = (1/\epsilon - \omega)P_Y = 0$*. All her wealth is human capital* $P_L = (1 - 1/\epsilon)P_Y$.
- *The portfolio share is* $\phi = 1$*: the financier holds* 100% *equity, no borrowing, no lending.*

For $\omega > 1/\epsilon$*, the financier has "too much" wealth for the equity market. She holds equity and lends the surplus to the household. For* $\omega < 1/\epsilon$*, she has too little wealth and must borrow to buy all the equity. The threshold* $1/\epsilon$ *is determined entirely by the production structure:* ϵ *governs the dividend share of output, which governs the size of the equity market relative to the economy.*

Under log utility, the financier's optimal *demand is $\phi = \Theta/\sigma_R$ (myopic portfolio rule). In equilibrium, Θ and σ_R adjust endogenously so that this optimal demand equals the market-clearing share $1/(\epsilon\omega)$. When ω is small, the market price of risk Θ must be high — risk premia rise to induce the undercapitalised financier to hold a disproportionately large equity position. This is why the model generates countercyclical risk premia without time-varying risk aversion: it is the* concentration of risk-bearing *in a shrinking intermediary sector, not changes in preferences, that drives the price of risk.*

Remark 6 (Interpreting ϵ as intermediation intensity)**.** *In the model, ϵ is the Dixit–Stiglitz elasticity of substitution across goods, and $1/\epsilon$ is the profit (dividend) share of output. But the economic mechanism that drives the results depends only on the ratio $P_D/P_Y = 1/\epsilon$ — the size of the asset market that requires intermediation relative to total wealth. This ratio admits a broader empirical interpretation.*

In the 2007–2008 crisis, the relevant "equity" was not limited to the stock market. It included the entire structured credit market — mortgage-backed securities, collateralised debt obligations, asset-backed commercial paper — trillions in assets that required financial intermediary balance sheet capacity. The effective ratio of intermediated assets to GDP grew substantially in the years preceding the crisis: US financial sector assets rose from approximately $3\times$ GDP in 1990 to over $5\times$ by 2007, with shadow banking assets (broker-dealers, ABS issuers, money market funds) growing from roughly \$2 trillion to \$12 trillion over the same period.

In the model's language, the effective $1/\epsilon$ — the share of total economic value embodied in assets requiring intermediation — was rising, while intermediary capital ω was roughly stable. The required leverage $\phi = 1/(\epsilon\omega)$ was therefore increasing, pushing the economy closer to the crisis threshold ω_c. The crisis occurred not because ω fell suddenly (though it did), but because the system had been operating with progressively less margin: the gap between ω and $\omega_c(\epsilon)$ had been narrowing for years as the intermediated asset base expanded.

This interpretation suggests a testable cross-sectional prediction: economies with larger intermediated asset markets (higher effective $1/\epsilon$) should be more vulnerable to intermediary capital depletion. We leave this extension to future work.

B.2 Dynamics of the Wealth Distribution

Theorem 17 (Dynamics of ω_t without leverage constraint)**.** *The financier's wealth share evolves according to*

$$\mathrm{d}\omega_t = \frac{\sigma_\omega^0(\omega_t)^2}{\omega_t}\,\mathrm{d}t + \sigma_\omega^0(\omega_t)\,\mathrm{d}Z_t, \tag{55}$$

where the volatility function is

$$\sigma_\omega^0(\omega) = \frac{\sigma(1+\varphi)(1-\omega)(1/\epsilon-\omega)}{(1+\varphi)(1-\omega)-(1/\epsilon-\omega)}. \tag{56}$$

The inverse wealth share $Q_t = 1/\omega_t$ is a non-negative local martingale.

The drift relation $\mu_\omega = \sigma_\omega^2/\omega$ follows from the local martingale property of $Q_t = 1/\omega_t$, which we establish in Appendix C. This property is the key to the model's tractability: with log utility, the myopic portfolio rule ensures that a critical cancellation in the drift of Q holds for any portfolio share.

The volatility σ_ω^0 has a single interior zero at $\omega^* = 1/\epsilon$ (where $1/\epsilon - \omega = 0$). It is positive for $\omega < 1/\epsilon$ and negative for $\omega > 1/\epsilon$, meaning that positive productivity shocks push ω toward $1/\epsilon$ from both sides.

B.3 Two Invariant Regimes and Convergence

Theorem 18 (Regime classification and convergence)**.** *The state space $(0,1)$ partitions into two invariant regions separated by the absorbing state $\omega^* = 1/\epsilon$:*

1. ***Controlled regime:*** *$\omega \in (0, 1/\epsilon)$. The financier is leveraged ($\phi > 1$) and households are net lenders ($H_h > 0$).*
2. ***Inverted regime:*** *$\omega \in (1/\epsilon, 1)$. The financier holds less than 100% equity ($\phi < 1$) and households are net borrowers ($H_h < 0$).*

For any $\omega_0 \in (0,1) \setminus \{1/\epsilon\}$:

$$\omega_t \xrightarrow{a.s.} \frac{1}{\epsilon}, \qquad \textit{yet} \qquad \mathbb{E}[\tau] = \infty, \tag{57}$$

where $\tau = \inf\{t \geq 0 : \omega_t = 1/\epsilon\}$.

The almost-sure convergence with infinite expected hitting time captures a "Zeno's paradox" of financial markets: the limited participation friction becomes asymptotically negligible yet is never eliminated in finite expected time.

At $\omega^* = 1/\epsilon$, the financier's wealth equals the stock market capitalisation ($H_i = P_D$), the household has zero financial wealth, leverage is one (no borrowing), and the economy coincides with the complete-markets benchmark. This is why the economy was fine before 2008: limited participation and leverage constraints coexisted without incident, because financiers were well enough capitalised that the constraints never bound. What changed in 2008 was the *state*, financier wealth ω, not the rules.

C Proofs for Limited Participation (No Leverage Constraint)

Proof of Proposition 16. Optimal policies. The financier's HJB equation with conjectured value function $V_i = \delta^{-1} \ln H_i + g_i(\omega)$ yields $C_i = \delta H_i$ and $\phi = (\mu_R - r)/\sigma_R^2$ (the myopic log-utility rule). The household, with log utility over consumption and total wealth $H_h + P_L$, consumes $C_h = \delta(H_h + P_L)$.

Goods market clearing. $\delta H_i + \delta(H_h + P_L) = Y$, so $P_Y \equiv H_i + H_h + P_L = Y/\delta$. By definition of ω: $C_i/Y = \delta H_i/Y = H_i/P_Y = \omega$ and $C_h/Y = 1 - \omega$.

Employment. From $N^\varphi C_h = (1 - 1/\epsilon)A$ and $C_h = (1-\omega)AN$: $N^{1+\varphi} = (1 - 1/\epsilon)/(1-\omega)$.

Equity market clearing. $\phi H_i = P_D = P_Y/\epsilon$, so $\phi = 1/(\epsilon\omega)$.

Financial wealth. $H_i + H_h = P_D = P_Y/\epsilon$ and $H_i = \omega P_Y$, so $H_h = (1/\epsilon - \omega)P_Y$.

Human capital. $P_L = P_Y - P_D = (1 - 1/\epsilon)P_Y$. Crucially, this holds for *any* dynamics of ω_t, because $P_Y = Y/\delta$ depends only on log utility and goods market clearing. □

Proof of Theorem 17. Step 1: Return volatility. Since $P_D = Y/(\epsilon\delta)$, by Itô's product rule applied to $Y = AN$:

$$\sigma_R = \sigma_Y = \sigma + \frac{\sigma_\omega^0}{(1+\varphi)(1-\omega)},$$

where the second term comes from $N \propto (1-\omega)^{-1/(1+\varphi)}$ via Itô's lemma.

Step 2: Wealth-share volatility. From $\omega = H_i/P_Y$ and Itô's quotient rule:

$$\sigma_\omega^0 = \omega(\phi - 1)\sigma_R = \omega\left(\frac{1}{\epsilon\omega} - 1\right)\sigma_R = \left(\frac{1}{\epsilon} - \omega\right)\sigma_R.$$

Step 3: Fixed-point equation. Substituting into Step 1 and solving:

$$\sigma_R = \frac{\sigma(1+\varphi)(1-\omega)}{(1+\varphi)(1-\omega) - (1/\epsilon - \omega)},$$

giving $\sigma_\omega^0 = (1/\epsilon - \omega)\sigma_R$, which is (56). The denominator simplifies to $(1-1/\epsilon) + \varphi(1-\omega) > 0$.

Step 4: Q-martingale. Define $Q = 1/\omega = P_Y/H_i$. The drift calculation is identical to Appendix G and yields $\mu_Q = 0$. By Itô's lemma on $\omega = 1/Q$: $\mu_\omega = (\sigma_\omega^0)^2/\omega$. □

Proof of Theorem 18. Absorbing state. At $\omega = 1/\epsilon$: $\sigma_\omega^0(1/\epsilon) = 0$ and $\mu_\omega(1/\epsilon) = 0$.

Boundary $\omega = 0$ (entrance). As $\omega \to 0^+$: $\sigma_\omega^0(\omega) \to c_0 > 0$ (a positive constant). The scale function density is $s'(\omega) = \omega_0^2/\omega^2$ and the speed measure is $m(\omega) = \omega^2/(\omega_0^2\,(\sigma_\omega^0)^2)$.

Near $\omega = 0$: $m(\omega) \sim \omega^2/(c_0^2\omega_0^2)$.

$$\Sigma(0) = \int_0^a s'(x)\,\mathrm{d}x = \omega_0^2 \int_0^a x^{-2}\,\mathrm{d}x = +\infty,$$
$$N(0) = \int_0^a s'(x) \int_0^x m(y)\,\mathrm{d}y\,\mathrm{d}x \le \frac{1}{3c_0^2} \int_0^a x\,\mathrm{d}x < \infty.$$

Hence $\Sigma = \infty$, $N < \infty$: entrance boundary, unattainable from the interior.

Boundary $\omega = 1/\epsilon$ (absorbing/unattainable). Write $\sigma_\omega^0 = g(\omega)(1/\epsilon - \omega)$ with $g(1/\epsilon) = \sigma > 0$. By the Engelbert–Schmidt criterion:

$$\int_a^{1/\epsilon} \frac{1}{(\sigma_\omega^0)^2}\,\mathrm{d}\omega \ge \frac{1}{\bar{g}^2} \int_a^{1/\epsilon} \frac{1}{(1/\epsilon - x)^2}\,\mathrm{d}x = +\infty.$$

Unattainable from the left. A symmetric argument gives unattainability from the right.

Boundary $\omega = 1$ (exit, unattainable). Write $\sigma_\omega^0 = h(\omega)(1-\omega)$ with $h(1) = -\sigma(1+\varphi) \neq 0$. Engelbert–Schmidt: $\int^1 (1-x)^{-2}\,\mathrm{d}x = +\infty$. Unattainable.

Regime invariance. Continuous paths and unattainable internal boundaries imply $(0, 1/\epsilon)$ and $(1/\epsilon, 1)$ are invariant.

Convergence. $Q = 1/\omega$ is a non-negative local martingale, hence a supermartingale. By the supermartingale convergence theorem, $Q_t \to Q_\infty$ a.s. with $Q_\infty < \infty$. Convergence of the quadratic variation requires $\sigma_\omega^0(\omega_\infty) = 0$, so $\omega_\infty \in \{0, 1/\epsilon, 1\}$. Since $Q_\infty < \infty$ implies $\omega_\infty > 0$, and $\omega = 1$ is unattainable: $\omega_t \to 1/\epsilon$ a.s.

Infinite expected hitting time. The speed measure satisfies $m(\omega) \ge K(1/\epsilon - \omega)^{-2}$ near $1/\epsilon$. By the Green's function formula:

$$\mathbb{E}[\tau_{1/\epsilon}] \ge C \int_a^{1/\epsilon} (1/\epsilon - y)^{-1}\,\mathrm{d}y = +\infty. \qquad \square$$

C.1 Asset Prices and Real Variables

Proposition 19 (Asset prices and real variables without leverage constraint)**.** *In the baseline economy with limited participation and no leverage constraint, define*

$$\Delta(\omega) \equiv \left(1 - \tfrac{1}{\epsilon}\right) + \varphi(1-\omega) > 0. \tag{58}$$

Then:

(i) Market price of risk:

$$\Theta^0(\omega) = \frac{\sigma(1+\varphi)(1-\omega)}{\epsilon\,\omega\,\Delta(\omega)}. \tag{59}$$

(ii) Equity risk premium:

$$\mu_R^0 - r^0 = \Theta^0 \, \sigma_R^0 = \frac{\sigma^2(1+\varphi)^2(1-\omega)^2}{\epsilon \, \omega \, \Delta(\omega)^2}. \tag{60}$$

(iii) Risk-free rate:

$$r^0(\omega) = \delta + \mu - \sigma^2 + \frac{\varphi}{2}\left(\sigma_R^0 - \sigma\right)^2 - \frac{\left[1+\varphi(1-\omega)\right]\sigma\left(\sigma_R^0 - \sigma\right)}{\omega}, \tag{61}$$

where $\sigma_R^0 - \sigma = (1/\epsilon - \omega)\,\sigma \,/\, \Delta(\omega)$.

(iv) Log employment:

$$n^0(\omega) \equiv \ln N^0 = \frac{1}{1+\varphi} \ln\left(\frac{1-1/\epsilon}{1-\omega}\right). \tag{62}$$

(v) Log output gap *(relative to a reference economy with constant wealth share* $\omega_n \in (0,1)$*):*

$$x^0(\omega) \equiv n^0(\omega) - n_n = \frac{1}{1+\varphi} \ln\left(\frac{1-\omega_n}{1-\omega}\right), \tag{63}$$

where $n_n = \frac{1}{1+\varphi}\ln\left(\frac{1-1/\epsilon}{1-\omega_n}\right)$ *is log employment in the reference economy.*

At $\omega = \omega_n$*:* $x^0 = 0$*. At* $\omega = 1/\epsilon$*, the asset-pricing quantities recover their complete-markets values:* $\Theta^0 = \sigma$*,* $\mu_R^0 - r^0 = \sigma^2$*,* $r^0 = \delta + \mu - \sigma^2$*, and* $n^0 = n_n$ *(so* $x^0 = 0$ *when* $\omega_n = 1/\epsilon$*).*

Proof. Throughout, write $\sigma_R^0 = \sigma(1+\varphi)(1-\omega)/\Delta$ and $\sigma_\omega^0 = (1/\epsilon - \omega)\,\sigma_R^0$, which are established in the proof of Theorem 17.

Part (i): Market price of risk. From Proposition 16, $\phi = 1/(\epsilon\omega)$. The financier's SDF is $\Lambda_t = e^{-\delta t}/C_{i,t}$ with $C_{i,t} = \delta H_{i,t}$, so $\sigma_\Lambda = -\phi\,\sigma_R^0$ (the financier's wealth volatility). Hence $\Theta^0 = \phi\,\sigma_R^0 = \sigma_R^0/(\epsilon\omega)$. Substituting gives (59).

Verification: at $\omega = 1/\epsilon$, $\sigma_R^0 = \sigma$ and $\Theta^0 = \sigma/(\epsilon \cdot 1/\epsilon) = \sigma$, recovering the complete-markets value (53). ✓

Part (ii): Equity risk premium. By the standard no-arbitrage relation, $\mu_R^0 - r^0 = \Theta^0\,\sigma_R^0 = (\sigma_R^0)^2/(\epsilon\omega)$. Substituting σ_R^0 yields (60).

Verification: at $\omega = 1/\epsilon$, $\mu_R^0 - r^0 = \sigma^2$, as in the complete-markets benchmark. ✓

Part (iii): Risk-free rate. The financier's SDF $\Lambda_t = e^{-\delta t}/(\omega_t Y_t)$ implies $r^0 = \delta + \mu_{\ln(\omega Y)} - \frac{1}{2}(\Theta^0)^2$, where $\Theta^0 = \sigma_{\ln(\omega Y)}$ is the diffusion coefficient established in Part (i).

Step 1: Drift of $\ln\omega$. From the Q-martingale property (Theorem 17):

$$\mathrm{d}\ln\omega = \frac{(\sigma_\omega^0)^2}{2\omega^2}\,\mathrm{d}t + \frac{\sigma_\omega^0}{\omega}\,\mathrm{d}Z_t.$$

Step 2: Drift of $\ln Y$. Since $Y = A\,N(\omega)$ with $N \propto (1-\omega)^{-1/(1+\varphi)}$:

$$\mathrm{d}\ln Y = \left(\mu - \tfrac{\sigma^2}{2} + \tfrac{(\sigma_\omega^0)^2}{(1+\varphi)(1-\omega)\omega} + \tfrac{(\sigma_\omega^0)^2}{2(1+\varphi)(1-\omega)^2}\right)\mathrm{d}t + \sigma_R^0\,\mathrm{d}Z_t.$$

Step 3: Change of variables. Write $\tilde{\sigma} \equiv \sigma_R^0 - \sigma = \sigma_\omega^0/[(1+\varphi)(1-\omega)]$ and $\sigma_\omega^0 = (1+\varphi)(1-\omega)\tilde{\sigma}$. Summing the drift contributions from Steps 1 and 2:

$$\mu_{\ln(\omega Y)} = \mu - \tfrac{\sigma^2}{2} + \tfrac{(1+\varphi)\left[1+\varphi(1-\omega)^2\right]\tilde{\sigma}^2}{2\omega^2},$$

where the coefficient of $\tilde{\sigma}^2$ follows from $(1+\varphi)^2(1-\omega)^2 + 2(1+\varphi)\omega(1-\omega) + (1+\varphi)\omega^2 = (1+\varphi)[1+\varphi(1-\omega)^2]$.

Step 4: Subtraction. Since $(\Theta^0)^2 = [\sigma + (1+\varphi(1-\omega))\tilde{\sigma}/\omega]^2$:

$$r^0 = \delta + \mu - \sigma^2 + \frac{\tilde{\sigma}^2}{2\omega^2}\Big\{(1+\varphi)\left[1+\varphi(1-\omega)^2\right] - \left[1+\varphi(1-\omega)\right]^2\Big\} - \frac{\left[1+\varphi(1-\omega)\right]\sigma\,\tilde{\sigma}}{\omega}.$$

The curly brace simplifies to $\varphi\omega^2$, by the algebraic identity

$$(1+\varphi)\left[1+\varphi(1-\omega)^2\right] - \left[1+\varphi(1-\omega)\right]^2 = \varphi\omega^2,$$

which is verified by expanding both sides. This yields (61).

Verification: at $\omega = 1/\epsilon$, $\tilde{\sigma} = 0$, so $r^0 = \delta + \mu - \sigma^2$. ✓

Parts (iv)–(v): Employment and output gap. From Proposition 16, $N^{1+\varphi} = (1-1/\epsilon)/(1-\omega)$, giving (62). The reference economy has the same employment formula evaluated at ω_n: $N_n = [(1-1/\epsilon)/(1-\omega_n)]^{1/(1+\varphi)}$, so $n_n = \frac{1}{1+\varphi}\ln\frac{1-1/\epsilon}{1-\omega_n}$. Since $Y = AN$ and productivity A_t is common to both economies:

$$x^0 = \ln Y - \ln Y_n = \ln N^0 - \ln N_n = n^0 - n_n = \frac{1}{1+\varphi}\ln\left(\frac{1-\omega_n}{1-\omega}\right),$$

which is (63). Note that $(1-1/\epsilon)$ cancels in the difference $n^0 - n_n$.

Verification: $x^0(\omega_n) = 0$; for $\omega > \omega_n$: $x^0 > 0$ (output above the reference); for $\omega < \omega_n$: $x^0 < 0$ (output below the reference). ✓ □

Remark 7 (Economic interpretation of the risk-free rate)**.** *The risk-free rate* (61) *decomposes as* $r^0 = r^{CM} + r^{precaut} + r^{risk}$*, where:*

- $r^{CM} = \delta + \mu - \sigma^2$ *is the complete-markets rate,*
- $r^{precaut} = \frac{\varphi}{2}(\sigma_R^0 - \sigma)^2 > 0$ *is a precautionary term from excess return volatility (mediated by the Frisch elasticity),*

- $r^{risk} = -[1+\varphi(1-\omega)]\,\sigma\,(\sigma_R^0 - \sigma)/\omega < 0$ *when* $\omega < 1/\epsilon$ *captures the risk-compensation effect of impaired intermediation.*

The risk-compensation term is first-order in $\sigma_R^0 - \sigma$ *while the precautionary term is second-order, so* $r^0 < r^{CM}$ *whenever* $\omega \neq 1/\epsilon$*. The interest rate falls below its frictionless value when intermediation is impaired, consistent with the empirical evidence on low rates during financial distress.*

D Equilibrium with Leverage Constraints but without QE

We now add the leverage constraint (7) to the limited-participation model of Appendix B, but *without* introducing a central bank. This intermediate case — between the unconstrained baseline (Appendix B) and the full model with QE (Section 4) — reveals both the source of equilibrium failure and the fragility of the pre-crisis economy.

D.1 Equilibrium Above the Crisis Threshold

For $\omega > \omega_c$, the leverage constraint does not bind: $\bar{\phi}(\omega) = 1 + \lambda(\omega - \underline{\omega}) \geq 1/(\epsilon\omega)$. The financier freely chooses $\phi = 1/(\epsilon\omega)$ (the market-clearing portfolio share from Proposition 16). Since no constraint is active, *all equilibrium quantities are identical to the unconstrained baseline*:

$$\sigma_\omega(\omega) = \sigma_\omega^0(\omega) = \frac{\sigma(1+\varphi)(1-\omega)(1/\epsilon - \omega)}{(1+\varphi)(1-\omega) - (1/\epsilon - \omega)}, \tag{64}$$

$$\Theta(\omega) = \frac{\sigma_\omega^0(\omega)}{\omega} + \sigma + \frac{\sigma_\omega^0(\omega)}{(1+\varphi)(1-\omega)}, \tag{65}$$

$$\sigma_R(\omega) = \sigma + \frac{\sigma_\omega^0(\omega)}{(1+\varphi)(1-\omega)}, \tag{66}$$

$$r(\omega) = \delta + \mu - \sigma^2 + \frac{\varphi}{2}\tilde{\sigma}(\omega)^2 - \frac{1+\varphi(1-\omega)}{\omega}\,\sigma\,\tilde{\sigma}(\omega), \tag{67}$$

where $\tilde{\sigma}(\omega) \equiv \sigma_R(\omega) - \sigma = \sigma_\omega^0(\omega)/[(1+\varphi)(1-\omega)]$. Employment, output, and the output gap are also unchanged:

$$N(\omega) = \left(\frac{1-1/\epsilon}{1-\omega}\right)^{\frac{1}{1+\varphi}}, \qquad x(\omega) = \frac{1}{1+\varphi}\ln\frac{1-1/\epsilon}{1-\omega}. \tag{68}$$

The intermediation gap $b(\omega) > 0$ from Proposition 8 is also present: since $Q = 1/\omega$ is a strict local martingale on $(0, 1/\epsilon)$ under the unconstrained dynamics (Theorem 9(i)), asset prices exceed fundamental values.

D.2 The Crisis Threshold

At $\omega = \omega_c$, defined by $\bar{\phi}(\omega_c) = 1/(\epsilon\omega_c)$ (equation (10)), the constraint just binds. At this point:

- The financier's leverage reaches its permitted maximum: $\phi = \bar{\phi}(\omega_c) = 1/(\epsilon\omega_c) \approx 1.29$.
- Risk premia are at their peak: $\Theta(\omega_c) \approx 0.196$, $\text{ERP}(\omega_c) \approx 2.95\%$.
- The risk-free rate is at its trough: $r(\omega_c) \approx 1.06\%$.
- The output gap is $x(\omega_c) \approx -1.2\%$.

These are the *most distorted* equilibrium values the economy can sustain without central bank intervention.

D.3 Market Failure Below ω_c

For $\omega < \omega_c$, no competitive equilibrium exists (Theorem 1). The required portfolio share $\phi = 1/(\epsilon\omega)$ exceeds the leverage limit $\bar{\phi}(\omega) = 1 + \lambda(\omega - \underline{\omega})$. The excess supply of equity, as a fraction of total wealth, is:

$$\frac{P_D - \bar{\phi}(\omega)\omega P_Y}{P_Y} = \frac{1}{\epsilon} - \bar{\phi}(\omega)\omega = \frac{1}{\epsilon} - \omega - \lambda\omega(\omega - \underline{\omega}). \tag{69}$$

At the baseline calibration: the excess supply is 0.1% of total wealth at $\omega = 0.077$ (just below ω_c), rising to 4.3% at $\omega = 0.05$ and 8.0% at $\omega = \underline{\omega} = 0.02$. No price adjustment can eliminate this excess supply (Section 3).

Below ω_c, there are no equilibrium prices, no risk premia, no risk-free rate, and no well-defined output. The economy hits a cliff.

D.4 Crisis Probability

Although the economy drifts toward $1/\epsilon$ in expectation ($\mu_\omega = \sigma_\omega^2/\omega > 0$), negative productivity shocks can push ω below ω_c. The probability of market failure before recovery is computed from the scale function.

For the unconstrained dynamics (64), the drift is $\mu_\omega = \sigma_\omega^2/\omega$ (from the Q-martingale property, Proposition 4). The scale function density is $s'(\omega) = C/\omega^2$ (Appendix I, equation for $2\mu_\omega/\sigma_\omega^2 = 2/\omega$), giving scale function $s(\omega) = -C/\omega$. The probability of hitting ω_c before $1/\epsilon$, starting from $\omega_0 \in (\omega_c, 1/\epsilon)$, is:

$$\Pr\big(\tau_{\omega_c} < \tau_{1/\epsilon} \,\big|\, \omega_0\big) = \frac{s(\omega_0) - s(1/\epsilon)}{s(\omega_c) - s(1/\epsilon)} = \frac{1/\omega_0 - \epsilon}{1/\omega_c - \epsilon}. \tag{70}$$

This probability is substantial even for ω_0 well above ω_c:

ω_0	0.080	0.085	0.090	0.095
Pr(crisis)	88.7%	60.7%	38.3%	18.1%

Starting just above the threshold ($\omega_0 = 0.08$), the economy has an 89% probability of market failure before recovery. Even from $\omega_0 = 0.09$, the probability is 38%. The economy is fragile: the leverage constraint exists as a latent vulnerability, and a sufficiently large shock triggers catastrophic failure.

D.5 Comparison of the Three Cases

Table 17 summarises the three cases analysed in the paper:

Table 17: Equilibrium properties under three model configurations. "LP" = limited participation (household holds only bonds). All three share the same employment and output gap functions $N(\omega)$ and $x(\omega)$.

	LP only (Appendix B)	**LP + leverage, no QE**	**LP + leverage + QE (Sections 3–6)**
Domain	$(0, 1/\epsilon)$	$(\omega_c, 1/\epsilon)$	$(0, 1)$
Equilibrium	Always	$\omega > \omega_c$ only	Always (CB absorbs residual)
σ_ω formula	Unconstrained	Unconstrained	Piecewise (eq. 23)
$\sigma_\omega(0^+)$	$c_0 > 0$	N/A (no eqm)	$\sim c_0'\omega \to 0$
$Q = 1/\omega$	Strict local M	N/A	True M
Intermediation gap	Positive	Positive ($\omega > \omega_c$)	Zero
Absorbing states	$1/\epsilon$	$1/\epsilon$ (if reached)	$\underline{\omega}$ and $1/\epsilon$
QE Trap	No	No	Yes
CB equity	None	None	Up to 80%

The "LP + leverage, no QE" case reveals the economy's *latent fragility*: the leverage constraint introduces a crisis threshold that did not exist in the unconstrained baseline, but does not change any equilibrium quantity until the threshold is reached. The constraint is

invisible in normal times and catastrophic in crisis. QE resolves the market failure but, by extending the domain to include $\omega < \omega_c$, introduces the absorbing bad steady state at $\underline{\omega}$ — the QE Trap.

E Proof of Equilibrium Failure

Proof that $P_{L,t} = (1 - 1/\epsilon)P_{Y,t}$ regardless of ω_t dynamics. Human capital is

$$P_{L,t} = \mathbb{E}_t \int_t^\infty \frac{\Lambda_s}{\Lambda_t} \frac{\epsilon - 1}{\epsilon} Y_s \, \mathrm{d}s = \frac{\epsilon - 1}{\epsilon} \mathbb{E}_t \int_t^\infty \frac{\Lambda_s}{\Lambda_t} Y_s \, \mathrm{d}s.$$

The financier's SDF is $\Lambda_t = e^{-\delta t}/C_{i,t} = e^{-\delta t}/(\delta H_{i,t})$. Total wealth is $P_{Y,t} = \mathbb{E}_t \int_t^\infty (\Lambda_s/\Lambda_t) Y_s \, \mathrm{d}s$. From goods market clearing, $P_{Y,t} = Y_t/\delta$ for any dynamics. Therefore $P_{L,t} = (1-1/\epsilon)Y_t/\delta = (1 - 1/\epsilon)P_{Y,t}$.

The key point is that $P_{Y,t} = Y_t/\delta$ is an identity under log utility: $P_Y = (C_i + C_h)/\delta = Y/\delta$. It does not require any specific dynamics for ω, the interest rate, or risk premia. □

Proof of Theorem 1. The argument proceeds in three steps.

Step 1: Household bond demand is perfectly inelastic. By Proposition 16 (which holds for any ω dynamics), the household's financial wealth is $H_{h,t} = (1/\epsilon - \omega_t) \, P_{Y,t}$. With log utility, this is a function of ω alone; it does not depend on the interest rate, the equity premium, or any other price.

Step 2: Equity market clearing pins the portfolio share. Since only the financier and (potentially) the central bank hold equity: $\phi_t H_{i,t} + \alpha_t P_{D,t} = P_{D,t}$. Without central bank intervention ($\alpha = 0$), this requires $\phi_t = P_{D,t}/H_{i,t} = 1/(\epsilon\omega_t)$.

Step 3: The leverage constraint makes the required ϕ infeasible. When $\omega < \omega_c$, the required portfolio share $1/(\epsilon\omega)$ exceeds the maximum permitted leverage $\bar{\phi}(\omega) = 1 + \lambda(\omega - \underline{\omega})$. No interest rate adjustment can resolve this: the household's bond demand is inelastic (Step 1), so reducing the interest rate does not induce the household to reduce her lending. There is a residual excess supply of bonds $(1/\epsilon - \omega - (\bar{\phi}(\omega) - 1)\omega) \, P_{Y,t} > 0$ that no price can eliminate. No competitive equilibrium exists. □

Proof of Proposition 2. With the central bank holding fraction $\alpha_t = 1 - \epsilon\bar{\phi}(\omega_t)\omega_t$ of the equity market (Assumption 3), the financier's equity holding is $\phi_t H_{i,t} = \bar{\phi}(\omega_t)\omega_t P_{Y,t}$ and the central bank's is $\alpha_t P_{D,t} = (1 - \epsilon\bar{\phi}(\omega_t)\omega_t) \, P_{Y,t}/\epsilon$. Their sum is $P_{D,t} = P_{Y,t}/\epsilon$: the equity market clears.

The central bank finances its equity purchases by issuing bonds worth $\alpha_t P_{D,t}$. The household's bond demand is $(1/\epsilon - \omega_t)P_{Y,t}$ (perfectly inelastic). The financier borrows

$(\bar{\phi}(\omega_t) - 1)\omega_t P_{Y,t}$. Bond market clearing requires:

$$(1/\epsilon - \omega_t)P_{Y,t} = (\bar{\phi}(\omega_t) - 1)\omega_t P_{Y,t} + \alpha_t P_{D,t}.$$

Substituting $\alpha_t P_{D,t} = (1/\epsilon - \bar{\phi}(\omega_t)\omega_t)P_{Y,t}$ and simplifying: $(1/\epsilon - \omega_t) = (\bar{\phi}(\omega_t)\omega_t - \omega_t) + (1/\epsilon - \bar{\phi}(\omega_t)\omega_t) = 1/\epsilon - \omega_t$. The identity holds.

All real quantities (consumption, employment, output, wealth) depend only on ω_t through the same expressions as Proposition 16, since $P_{Y,t} = Y_t/\delta$ is an identity under log utility. □

F Derivation of σ_ω With Leverage Constraint

Proof of Theorem 3. **Step 1: Return volatility.** Since $P_D = Y/(\epsilon\delta)$ and δ is constant, $\sigma_R = \sigma_Y$. Employment is $N(\omega) = [(1 - 1/\epsilon)/(1 - \omega)]^{1/(1+\varphi)}$. Define $f(\omega) = (1 - \omega)^{-1/(1+\varphi)}$, so $N = c\, f(\omega)$ with $c = (1 - 1/\epsilon)^{1/(1+\varphi)}$. By Itô's lemma:

$$\frac{f'(\omega)}{f(\omega)} = \frac{1}{(1+\varphi)(1-\omega)}, \qquad \frac{\mathrm{d}N}{N} = \frac{f'}{f}\,\mathrm{d}\omega + (\text{drift})\,\mathrm{d}t = \frac{\sigma_\omega}{(1+\varphi)(1-\omega)}\,\mathrm{d}Z + (\cdots)\,\mathrm{d}t.$$

Since $Y = AN$ and both A and N load on the same Brownian motion Z (A directly, N through ω), the product rule gives $\sigma_Y = \sigma_A + \sigma_N$:

$$\sigma_Y = \sigma + \frac{\sigma_\omega}{(1+\varphi)(1-\omega)}.$$

Step 2: Wealth-share volatility. The financier's wealth SDE is (equation (71) in Appendix G):

$$\frac{\mathrm{d}H_i}{H_i} = [r + \phi(\mu_R - r) - \delta]\,\mathrm{d}t + \phi\sigma_R\,\mathrm{d}Z.$$

With $P_Y = Y/\delta$, $\mathrm{d}P_Y/P_Y = \mu_Y\,\mathrm{d}t + \sigma_Y\,\mathrm{d}Z$. Applying Itô's quotient rule to $\omega = H_i/P_Y$ (see Appendix G, equation (73) applied to $1/Q = \omega$):

$$\sigma_\omega = \omega\left(\frac{\sigma_{H_i}}{H_i} - \frac{\sigma_{P_Y}}{P_Y}\right) = \omega(\phi\sigma_R - \sigma_Y) = \omega(\phi\sigma_R - \sigma_R) = \omega(\phi - 1)\sigma_R,$$

where the third equality uses $\sigma_R = \sigma_Y$.

Step 3: Fixed-point equation. Substituting $\sigma_R = \sigma + \sigma_\omega/[(1+\varphi)(1-\omega)]$ (Step 1) into $\sigma_\omega = \omega(\phi - 1)\sigma_R$ (Step 2) yields an equation in which σ_ω appears on both sides. Since the equation is linear in σ_ω, it has a unique solution whenever $D(\omega) \neq 0$.

Constrained region ($\omega < \omega_c$, $\phi = \bar{\phi}(\omega) = 1 + \lambda(\omega - \underline{\omega})$, so $\phi - 1 = \lambda(\omega - \underline{\omega})$):

$$\begin{aligned}\sigma_\omega &= \omega\lambda(\omega - \underline{\omega})\left[\sigma + \frac{\sigma_\omega}{(1+\varphi)(1-\omega)}\right] \\ &= \sigma\lambda\omega(\omega - \underline{\omega}) + \frac{\lambda\omega(\omega - \underline{\omega})}{(1+\varphi)(1-\omega)}\,\sigma_\omega.\end{aligned}$$

Collecting σ_ω terms:

$$\sigma_\omega\left[1 - \frac{\lambda\omega(\omega - \underline{\omega})}{(1+\varphi)(1-\omega)}\right] = \sigma\lambda\omega(\omega - \underline{\omega}).$$

Multiplying through by $(1+\varphi)(1-\omega)$ and solving:

$$\sigma_\omega^c = \frac{\sigma(1+\varphi)\lambda(1-\omega)\,\omega\,(\omega - \underline{\omega})}{D_c(\omega)}, \qquad D_c(\omega) = (1+\varphi)(1-\omega) - \lambda\omega(\omega - \underline{\omega}).$$

Unconstrained region ($\omega \geq \omega_c$, $\phi = 1/(\epsilon\omega)$, so $\phi - 1 = (1/\epsilon - \omega)/\omega$):

$$\sigma_\omega = (1/\epsilon - \omega)\left[\sigma + \frac{\sigma_\omega}{(1+\varphi)(1-\omega)}\right].$$

The identical rearrangement gives:

$$\sigma_\omega^u = \frac{\sigma(1+\varphi)(1-\omega)(1/\epsilon - \omega)}{D_u(\omega)}, \qquad D_u(\omega) = (1+\varphi)(1-\omega) - (1/\epsilon - \omega).$$

Continuity at ω_c: At ω_c, the defining equation (10) gives $\lambda\omega_c(\omega_c - \underline{\omega}) = 1/\epsilon - \omega_c$. Substituting into σ_ω^c: the numerator becomes $\sigma(1+\varphi)(1-\omega_c)(1/\epsilon - \omega_c)$, and $D_c(\omega_c) = (1+\varphi)(1-\omega_c) - (1/\epsilon - \omega_c) = D_u(\omega_c)$. Therefore $\sigma_\omega^c(\omega_c) = \sigma_\omega^u(\omega_c)$.

Step 4: Drift. By Proposition 4, $Q = 1/\omega$ is a driftless local martingale. Itô's lemma on $\omega = 1/Q$ with $f(Q) = 1/Q$, $f'(Q) = -1/Q^2$, $f''(Q) = 2/Q^3$ gives drift $\mu_\omega = \sigma_\omega^2/\omega$ (equation (74)). □

Lemma 20 (Denominator positivity)**.** *$D(\omega) > 0$ for all $\omega \in [0, 1)$ in both regions.*

Proof. For $\omega \geq \omega_c$: expand $D_u = (1+\varphi)(1-\omega) - (1/\epsilon - \omega) = (1+\varphi) - (1+\varphi)\omega - 1/\epsilon + \omega = (1 - 1/\epsilon) + \varphi(1-\omega) > 0$ since $\epsilon > 1$ and $\varphi \geq 0$.

For $\omega \in [0, \underline{\omega}]$: $\omega - \underline{\omega} \leq 0$, so $\lambda\omega(\omega - \underline{\omega}) \leq 0$, giving $D_c = (1+\varphi)(1-\omega) - \lambda\omega(\omega - \underline{\omega}) \geq (1+\varphi)(1-\omega) > 0$.

For $\omega \in [\underline{\omega}, \omega_c]$: $D_c(\omega) = (1+\varphi)(1-\omega) - \lambda\omega^2 + \lambda\underline{\omega}\omega$ is quadratic in ω with $\mathrm{d}^2 D_c/\,\mathrm{d}\omega^2 = -2\lambda < 0$ (concave). At the endpoints: $D_c(\underline{\omega}) = (1+\varphi)(1-\underline{\omega}) > 0$ and $D_c(\omega_c) = D_u(\omega_c) > 0$ (by continuity). A concave function positive at both endpoints is positive throughout. □

Proof of Proposition 29. Consider the volatility function (23) with denominator $D(\omega) > 0$ for all $\omega \in [0, 1)$ (Lemma 20). All pre-factors σ, $(1 + \varphi)$, and λ are strictly positive.

Zeros. In the constrained region ($\omega < \omega_c$), the numerator is $\sigma(1+\varphi)\lambda(1-\omega)\,\omega\,(\omega-\underline{\omega})$. For $\omega \in [0, \omega_c)$, the factors $(1-\omega)$ and $D_c(\omega)$ are strictly positive, so $\sigma_\omega(\omega) = 0$ if and only if $\omega = 0$ or $\omega = \underline{\omega}$. In the unconstrained region ($\omega \geq \omega_c$), the numerator is $\sigma(1 + \varphi)(1 - \omega)(1/\epsilon - \omega)$. For $\omega \in [\omega_c, 1)$, the factor $(1 - \omega) > 0$ and $D_u > 0$, so $\sigma_\omega(\omega) = 0$ if and only if $\omega = 1/\epsilon$. (Note $\omega_c < 1/\epsilon$ since $\bar{\phi}(\omega_c) = 1/(\epsilon\omega_c) > 1$, which requires $\omega_c < 1/\epsilon$.) Combining: the zeros of σ_ω in $[0, 1)$ are exactly $\{0, \underline{\omega}, 1/\epsilon\}$.

Sign pattern. For $\omega \in (0, \underline{\omega})$: $\omega > 0$, $(1 - \omega) > 0$, $(\omega - \underline{\omega}) < 0$, and $D_c > 0$. The numerator has sign $(+)(+)(+)(+)(-) < 0$, so $\sigma_\omega < 0$. For $\omega \in (\underline{\omega}, \omega_c)$: $(\omega - \underline{\omega}) > 0$, so all factors in the numerator are positive, giving $\sigma_\omega > 0$. For $\omega \in [\omega_c, 1/\epsilon)$: $(1/\epsilon - \omega) > 0$ and $(1 - \omega) > 0$, so $\sigma_\omega > 0$. Continuity of σ_ω at ω_c (established in the proof of Theorem 3) ensures consistency across the boundary. For $\omega \in (1/\epsilon, 1)$: $(1/\epsilon - \omega) < 0$ and $(1 - \omega) > 0$, so $\sigma_\omega < 0$. □

G Q-Martingale Property

Proof of Proposition 4. **Step 1: Financier wealth SDE.** The financier allocates portfolio share ϕ_t to equity (with return $\mathrm{d}R_t = \mu_R\,\mathrm{d}t + \sigma_R\,\mathrm{d}Z_t$) and the remainder to the risk-free bond (at rate r_t), and consumes $C_{i,t}$. Her wealth evolves as:

$$\frac{\mathrm{d}H_{i,t}}{H_{i,t}} = \big[r_t + \phi_t(\mu_{R,t} - r_t) - \delta\big]\,\mathrm{d}t + \phi_t\sigma_{R,t}\,\mathrm{d}Z_t, \tag{71}$$

where the consumption drain $-\delta$ arises because log utility gives $C_{i,t} = \delta H_{i,t}$ (the first-order condition for $\max \mathbb{E}\int_0^\infty e^{-\delta t}\ln C_{i,t}\,\mathrm{d}t$).

Step 2: SDF and market price of risk. The financier's stochastic discount factor is $\Lambda_t = e^{-\delta t}/C_{i,t} = e^{-\delta t}/(\delta H_{i,t})$. Applying Itô's lemma to $\Lambda_t = (1/\delta)e^{-\delta t}/H_{i,t}$ with $f(H) = 1/H$, $f'(H) = -1/H^2$:

$$\frac{\mathrm{d}\Lambda_t}{\Lambda_t} = -\delta\,\mathrm{d}t - \frac{\mathrm{d}H_{i,t}}{H_{i,t}} + \left(\frac{\mathrm{d}H_{i,t}}{H_{i,t}}\right)^2 = (\cdots)\,\mathrm{d}t - \phi_t\sigma_{R,t}\,\mathrm{d}Z_t.$$

The volatility of Λ loading on $\mathrm{d}Z$ is $-\phi_t\sigma_R$, so the market price of risk is:

$$\Theta_t = \phi_t\sigma_{R,t}. \tag{72}$$

This holds for *any* feasible ϕ_t, whether chosen optimally or imposed by the leverage con-

straint. Substituting into $\mu_R - r = \sigma_R \Theta$: $\mu_R - r = \phi_t \sigma_R^2$.

Step 3: Total wealth SDE. Total wealth $P_{Y,t} = Y_t/\delta$ pays continuous dividends $C_t = Y_t = \delta P_{Y,t}$, so the dividend yield is δ. No-arbitrage for a claim with dividend yield δ:

$$\frac{\mu_{P_Y}}{P_Y} + \delta = r + \sigma_{P_Y}\Theta_t, \qquad \text{i.e.,} \qquad \frac{\mu_{P_Y}}{P_Y} = r + \sigma_Y \Theta_t - \delta,$$

where $\sigma_{P_Y} = \sigma_Y$ since $P_Y = Y/\delta$ and δ is constant, so $\mathrm{d}P_Y/P_Y = \mathrm{d}Y/Y$.

Step 4: Drift of Q. Define $Q_t = P_{Y,t}/H_{i,t} = 1/\omega_t$. We apply the Itô quotient rule: for $Q = X/Y$ with $\mathrm{d}X/X = \mu_X\,\mathrm{d}t + \sigma_X\,\mathrm{d}Z$ and $\mathrm{d}Y/Y = \mu_Y\,\mathrm{d}t + \sigma_Y\,\mathrm{d}Z$,

$$\frac{\mu_Q}{Q} = \frac{\mu_X}{X} - \frac{\mu_Y}{Y} + \left(\frac{\sigma_Y}{Y}\right)^2 - \frac{\sigma_X}{X}\cdot\frac{\sigma_Y}{Y}, \qquad \frac{\sigma_Q}{Q} = \frac{\sigma_X}{X} - \frac{\sigma_Y}{Y}.$$

Setting $X = P_Y$ and $Y = H_i$, with $\sigma_X/X = \sigma_Y$ and $\sigma_Y/Y = \phi_t\sigma_R$:

$$\begin{aligned}\frac{\mu_Q}{Q} &= \underbrace{(r + \sigma_Y\phi_t\sigma_R - \delta)}_{\mu_{P_Y}/P_Y} - \underbrace{(r + \phi_t^2\sigma_R^2 - \delta)}_{\mu_{H_i}/H_i} + \underbrace{\phi_t^2\sigma_R^2}_{(\sigma_{H_i}/H_i)^2} - \underbrace{\sigma_Y\phi_t\sigma_R}_{(\sigma_{P_Y}/P_Y)\cdot(\sigma_{H_i}/H_i)} \\ &= \sigma_Y\phi_t\sigma_R - \phi_t^2\sigma_R^2 + \phi_t^2\sigma_R^2 - \phi_t\sigma_R\sigma_Y \\ &= 0.\end{aligned} \tag{73}$$

The cancellation occurs in two pairs: terms 2 and 3 ($-\phi_t^2\sigma_R^2 + \phi_t^2\sigma_R^2 = 0$) and terms 1 and 4 ($\sigma_Y\phi_t\sigma_R - \phi_t\sigma_R\sigma_Y = 0$, using $\sigma_R = \sigma_Y$ since $P_D \propto Y$ implies equity return volatility equals output volatility). Every ϕ_t-dependent term cancels, so the result holds for *any* ϕ_t.

The diffusion of Q is:

$$\frac{\sigma_Q}{Q} = \sigma_Y - \phi_t\sigma_R = \sigma_R - \phi_t\sigma_R = (1-\phi_t)\sigma_R = -\frac{\sigma_\omega}{\omega},$$

where the last step uses $\sigma_\omega = \omega(\phi_t - 1)\sigma_R$ from Appendix F.

Step 5: Q as a local martingale. Since $\mu_Q = 0$, we apply Itô's lemma to $Q = 1/\omega$ with $f(\omega) = 1/\omega$, $f'(\omega) = -1/\omega^2$, $f''(\omega) = 2/\omega^3$:

$$\begin{aligned}\mathrm{d}Q &= f'(\omega)\,\mathrm{d}\omega + \tfrac{1}{2}f''(\omega)(\,\mathrm{d}\omega)^2 \\ &= -\frac{1}{\omega^2}\left[\mu_\omega\,\mathrm{d}t + \sigma_\omega\,\mathrm{d}Z\right] + \frac{1}{\omega^3}\sigma_\omega^2\,\mathrm{d}t \\ &= \left[-\frac{\mu_\omega}{\omega^2} + \frac{\sigma_\omega^2}{\omega^3}\right]\mathrm{d}t - \frac{\sigma_\omega}{\omega^2}\,\mathrm{d}Z.\end{aligned}$$

Setting the drift to zero ($\mu_Q = 0$ from Step 4):

$$\mu_\omega = \frac{\sigma_\omega^2}{\omega}. \tag{74}$$

Therefore $\mathrm{d}Q_t = -(\sigma_\omega(\omega_t)/\omega_t^2)\,\mathrm{d}Z_t$. Since $\omega \in (0,1)$, $Q = 1/\omega > 1 > 0$. On any compact subset of $(0,1)$, $\sigma_\omega(\omega)/\omega^2$ is bounded (by Lemma 20 and continuity of the numerator), so $\int_0^t (\sigma_\omega/\omega^2)^2\,\mathrm{d}s < \infty$ a.s. for all t. Therefore Q is a non-negative continuous local martingale, and hence a supermartingale by Fatou's lemma: for any localising sequence $\tau_n \uparrow \infty$, $\mathbb{E}[Q_t|\mathcal{F}_s] = \lim_n \mathbb{E}[Q_{t\wedge\tau_n}|\mathcal{F}_s] \le \liminf_n \mathbb{E}[Q_{t\wedge\tau_n}|\mathcal{F}_s] = Q_s$. □

H First-Passage Analysis on the Pre-Failure Domain

This appendix analyses the first-passage time τ_c of the financier wealth-share process ω_t to the crisis threshold ω_c, on the pre-failure domain $I \equiv (\omega_c,\, 1/\epsilon)$ and under the *unconstrained* (pre-QE) law of motion (13)–(14). The scope is complementary to Appendix G (which proves $Q_t = 1/\omega_t$ is a non-negative continuous local martingale with $Q_t \to \epsilon$ almost surely), Appendix I (Feller classification), Appendix J (a.s. convergence and infinite expected hitting time to $1/\epsilon$), and Appendix N.9 (a deterministic lower bound on recovery using the *constrained+QE* σ_ω on $(\underline{\omega}, \omega_c)$). Here we compute in closed form the probability $\mathbb{P}(\tau_c < \infty \mid \omega_0)$, show that the unconditional expected hitting time is infinite, and represent the conditional expected hitting time as a Green's-function integral that admits an elementary closed form at $\varphi = 0$.

H.1 Preliminaries

Recall from (14) that on $I = (\omega_c,\, 1/\epsilon)$,

$$\sigma_\omega(\omega) \;=\; \frac{\sigma(1+\varphi)(1-\omega)(1/\epsilon-\omega)}{(1+\varphi)(1-\omega)-(1/\epsilon-\omega)}, \tag{75}$$

with $0 < \omega_c < 1/\epsilon$, $\epsilon > 1$, $\varphi \ge 0$, $\sigma > 0$. The denominator $D(\omega) \equiv (1+\varphi)(1-\omega)-(1/\epsilon-\omega) = \varphi(1-\omega) + (1-1/\epsilon) > 0$ for all $\omega \in [0,1)$ since both summands are non-negative and the second is strictly positive ($\epsilon > 1$). The numerator $\sigma(1+\varphi)(1-\omega)(1/\epsilon-\omega)$ is strictly positive on I, so $\sigma_\omega(\omega) > 0$ throughout I and σ_ω is continuous on $\bar{I}$ with $\sigma_\omega(1/\epsilon) = 0$. The generator of the diffusion (13) on I is

$$\mathcal{L}f(\omega) \;=\; \frac{\sigma_\omega(\omega)^2}{\omega} f'(\omega) + \frac{\sigma_\omega(\omega)^2}{2} f''(\omega). \tag{76}$$

A *scale function* for $\mathcal{L}$ on I is any twice-differentiable solution of $\mathcal{L}s = 0$. Direct computation with $s(\omega) = 1/\omega$, $s'(\omega) = -1/\omega^2$, $s''(\omega) = 2/\omega^3$ gives

$$\mathcal{L}s(\omega) \;=\; \frac{\sigma_\omega^2}{\omega}\left(-\frac{1}{\omega^2}\right) + \frac{\sigma_\omega^2}{2}\cdot\frac{2}{\omega^3} \;=\; -\frac{\sigma_\omega^2}{\omega^3} + \frac{\sigma_\omega^2}{\omega^3} \;=\; 0, \tag{77}$$

confirming that $s(\omega) = 1/\omega$ is a scale function—equivalently, that $Q_t = 1/\omega_t$ is a local martingale (Proposition 4). Since the drift in (13) is $\mu_\omega = \sigma_\omega^2/\omega$, the scale-function integrand is $-2\mu_\omega/\sigma_\omega^2 = -2/\omega$, so $s'(\omega) = \exp(-\int^\omega 2/u\,du) = C/\omega^2$, consistent with $s(\omega) = -C/\omega + \text{const}$; we choose $s(\omega) = 1/\omega$ below. The general solution of the homogeneous ODE $\mathcal{L}f = 0$ on I is therefore $f(\omega) = A + B/\omega$ for constants $A, B \in \mathbb{R}$.

H.2 Hitting probability

Proposition 21 (Hitting probability for τ_c)**.** *For $\omega_0 \in I = (\omega_c,\, 1/\epsilon)$, the stopping time $\tau_c \equiv \inf\{t \geq 0 : \omega_t = \omega_c\}$ defined in (15) satisfies*

$$p(\omega_0) \;\equiv\; \mathbb{P}(\tau_c < \infty \mid \omega_0) \;=\; \frac{1/\omega_0 - \epsilon}{1/\omega_c - \epsilon} \;=\; \frac{\omega_c(1-\epsilon\omega_0)}{\omega_0(1-\epsilon\omega_c)}. \tag{78}$$

The function $p : [\omega_c, 1/\epsilon] \to [0,1]$ is continuous with $p(\omega_c) = 1$ and $p(1/\epsilon) = 0$, and p is strictly decreasing in ω_0 on I.

Proof. We give two derivations.

(a) Optional stopping on a bounded martingale. By Proposition 4 and (77), $Q_t = 1/\omega_t$ is a continuous local martingale on the interval I. For $n \in \mathbb{N}$ with $1/n < 1/\epsilon - \omega_c$, define $I_n \equiv (\omega_c,\, 1/\epsilon - 1/n)$ and the exit time

$$\tau_n \;\equiv\; \inf\{t \geq 0 : \omega_t \notin I_n\} \;=\; \tau_c \wedge \sigma_n, \qquad \sigma_n \equiv \inf\{t \geq 0 : \omega_t = 1/\epsilon - 1/n\}.$$

Since $\omega_t \in I_n$ for $t < \tau_n$, the process $Q_{t\wedge\tau_n}$ takes values in the compact interval $[\,1/(1/\epsilon - 1/n),\, 1/\omega_c\,]$ and is therefore a bounded continuous martingale. By the optional stopping theorem (Karatzas and Shreve, 1991, Theorem 1.3.22),

$$\mathbb{E}[Q_{\tau_n} \mid \omega_0] \;=\; Q_0 \;=\; \frac{1}{\omega_0}. \tag{79}$$

By the Feller classification in Appendix I, the right endpoint $1/\epsilon$ is unattainable, so $\mathbb{P}(\sigma_n < \infty) = 1$ for each n but $\sigma_n \to +\infty$ a.s. as $n \to \infty$; similarly $\tau_c = \lim_n \tau_n$ on $\{\tau_c < \infty\}$. The

left-hand side of (79) decomposes as

$$\mathbb{E}\big[Q_{\tau_n}\mathbf{1}_{\{\tau_c<\sigma_n\}}\big]+\mathbb{E}\big[Q_{\tau_n}\mathbf{1}_{\{\sigma_n\le\tau_c\}}\big] \;=\; \frac{1}{\omega_c}\mathbb{P}(\tau_c<\sigma_n)+\frac{1}{1/\epsilon-1/n}\mathbb{P}(\sigma_n\le\tau_c). \tag{80}$$

Taking $n\to\infty$: $\mathbb{P}(\tau_c<\sigma_n)\to\mathbb{P}(\tau_c<\infty)=p(\omega_0)$ by monotone continuity of probabilities on the increasing events $\{\tau_c<\sigma_n\}\uparrow\{\tau_c<\infty\}$, while $\mathbb{P}(\sigma_n\le\tau_c)\to\mathbb{P}(\tau_c=\infty)=1-p(\omega_0)$. The factor $1/(1/\epsilon-1/n)\to\epsilon$. Both summands in (80) are non-negative and uniformly bounded by $\max(1/\omega_c,\epsilon)+1$, so dominated convergence applies[11]:

$$\frac{1}{\omega_0} \;=\; \frac{1}{\omega_c}\,p(\omega_0)+\epsilon\,\big(1-p(\omega_0)\big).$$

Solving for $p(\omega_0)$ yields (78).

(b) Dirichlet problem for the scale function. By the standard connection between hitting probabilities and the Dirichlet problem (Karatzas and Shreve, 1991, Proposition 5.5.22), the function $p(\omega)\equiv\mathbb{P}(\tau_c<\tau_{1/\epsilon}\mid\omega_0=\omega)$ is the unique bounded continuous solution of

$$\mathcal{L}p(\omega) \;=\; 0 \text{ on } I, \qquad p(\omega_c)=1, \qquad \lim_{\omega\uparrow 1/\epsilon}p(\omega)=0. \tag{81}$$

The second boundary condition uses the Feller classification: since $1/\epsilon$ is unattainable (Appendix I) and the process started near $1/\epsilon$ is pushed away from it by the positive drift σ_ω^2/ω, the probability of hitting ω_c before hitting an arbitrarily small neighbourhood of $1/\epsilon$ vanishes as $\omega_0\uparrow 1/\epsilon$; equivalently, $\mathbb{P}(\tau_c<\infty\mid\omega_0\uparrow 1/\epsilon)\to 0$ by Appendix J. By §H.1, the general solution of $\mathcal{L}p=0$ is $p(\omega)=A+B/\omega$. Imposing $p(1/\epsilon)=A+B\epsilon=0$ gives $A=-B\epsilon$; imposing $p(\omega_c)=A+B/\omega_c=1$ then gives $B=1/(1/\omega_c-\epsilon)$ and $A=-\epsilon/(1/\omega_c-\epsilon)$, so

$$p(\omega) \;=\; \frac{1/\omega-\epsilon}{1/\omega_c-\epsilon},$$

which is (78). Both derivations yield the same formula; monotonicity in ω_0 is immediate from $\partial p/\partial\omega_0=-1/[\omega_0^2(1/\omega_c-\epsilon)]<0$. □

Remark 8 (Boundary-condition check)**.** *At $\omega_0=\omega_c$: $p(\omega_c)=(1/\omega_c-\epsilon)/(1/\omega_c-\epsilon)=1$. At $\omega_0=1/\epsilon$: numerator $1/\omega_0-\epsilon=0$, so $p(1/\epsilon)=0$. Both limits are exactly what the economics requires: starting at the crisis threshold we hit it immediately; starting at the frictionless steady state we never hit it (cf. $Q_t\to\epsilon$ a.s. in Appendix J).*

[11] Equivalently, the uniform integrability of $\{Q_{\tau_n}\}_{n\ge1}$ follows from $Q_{\tau_n}\le 1/\omega_c$; see Karatzas and Shreve (1991, Ch. 1) for the standard passage-to-the-limit in optional stopping.

H.3 Unconditional expected hitting time

Theorem 22 (Infinite unconditional expected hitting time)**.** *For every $\omega_0 \in I = (\omega_c,\, 1/\epsilon)$,*

$$\mathbb{E}[\tau_c \mid \omega_0] \;=\; +\infty. \tag{82}$$

Proof. By Proposition 21, $p(\omega_0) < 1$ for $\omega_0 \in (\omega_c, 1/\epsilon)$, so $\mathbb{P}(\tau_c = \infty \mid \omega_0) = 1 - p(\omega_0) > 0$. Therefore

$$\mathbb{E}[\tau_c \mid \omega_0] \;\geq\; \mathbb{E}[\tau_c \mathbf{1}_{\{\tau_c=\infty\}} \mid \omega_0] \;=\; (+\infty)\cdot \mathbb{P}(\tau_c = \infty \mid \omega_0) \;=\; +\infty,$$

using the convention $0 \cdot \infty = 0$ but $(+\infty)\cdot c = +\infty$ for $c > 0$. □

The economic content is the same as in Appendix J: conditional on avoiding the crisis, $\omega_t \to 1/\epsilon$ a.s. but the expected time to reach any neighbourhood of $1/\epsilon$ is infinite; hence the expected time to the *crisis* threshold, which includes a positive-probability event on which $\tau_c = \infty$, is a fortiori infinite.

H.4 Conditional expected hitting time

Proposition 23 (Conditional expected hitting time)**.** *Let $v(\omega_0) \equiv \mathbb{E}[\tau_c \mathbf{1}_{\{\tau_c<\infty\}} \mid \omega_0]$ for $\omega_0 \in \bar{I}$. Then v is the unique bounded solution of the Feynman–Kac boundary-value problem*

$$\mathcal{L}v(\omega) \;=\; -p(\omega), \qquad \omega \in I; \qquad v(\omega_c) = v(1/\epsilon) = 0. \tag{83}$$

In the scale coordinate $y \equiv 1/\omega$, equivalently $V(y) \equiv v(1/y)$, the generator transforms into a pure-diffusion operator $\tilde{\mathcal{L}}V(y) = \frac{1}{2}a(y)V''(y)$ with $a(y) \equiv y^4\sigma_\omega(1/y)^2$, and (83) reduces to

$$V''(y) \;=\; -\frac{2(y-\epsilon)}{a(y)(y_c-\epsilon)}, \qquad V(\epsilon) = V(y_c) = 0, \qquad y_c \equiv 1/\omega_c. \tag{84}$$

The unique bounded solution of (84) admits the Green's-function representation

$$V(y) \;=\; \frac{1}{y_c-\epsilon}\int_\epsilon^{y_c} G(y,z)(z-\epsilon)m(z)\,\mathrm{d}z, \qquad m(z) \equiv \frac{2}{a(z)}, \tag{85}$$

where the Dirichlet Green's function on $[\epsilon, y_c]$ for the operator $-\mathrm{d}^2/\mathrm{d}y^2$ is

$$G(y,z) \;=\; \begin{cases} \dfrac{(y-\epsilon)(y_c-z)}{y_c-\epsilon}, & y \leq z, \\[2ex] \dfrac{(z-\epsilon)(y_c-y)}{y_c-\epsilon}, & z \leq y. \end{cases} \tag{86}$$

Equivalently, in the original variable ω_0,

$$\boxed{v(\omega_0) \;=\; \frac{2}{1/\omega_c - \epsilon}\int_{\omega_c}^{1/\epsilon} G\Big(\frac{1}{\omega_0},\frac{1}{\omega'}\Big)\frac{(1-\epsilon\omega')\,\omega'}{\sigma_\omega(\omega')^2}\,\mathrm{d}\omega'.} \tag{87}$$

The conditional expected hitting time is then $\mathbb{E}[\tau_c \mid \omega_0, \tau_c < \infty] = v(\omega_0)/p(\omega_0)$.

Proof. Step 1: Feynman–Kac characterisation. The process $\tau_c \mathbf{1}_{\{\tau_c<\infty\}}$ is non-negative and finite on the event $\{\tau_c < \infty\}$, and $\tau_c \le \tau_n + \infty\cdot\mathbf{1}_{\{\tau_c\ge\tau_n\}}$ with τ_n the exit time from I_n (cf. proof of Proposition 21). By the standard Feynman–Kac formula for exit problems (Karatzas and Shreve, 1991, Theorem 5.7.6), applied on each I_n and passed to the limit using dominated convergence on the event $\{\tau_c < \infty\}$, v solves $\mathcal{L}v = -p$ on I with $v(\omega_c) = 0$ and $v(1/\epsilon) = 0$, the latter because $\mathbb{P}(\tau_c < \infty \mid \omega_0) \to 0$ as $\omega_0 \uparrow 1/\epsilon$ by Proposition 21. Boundedness of v on $\bar{I}$ follows from the boundedness of the Green's-function representation established below.

Step 2: Change of variables $y = 1/\omega$. For $V(y) = v(1/y)$, chain rule gives $v'(\omega) = -y^2V'(y)$ and $v''(\omega) = y^4V''(y) + 2y^3V'(y)$ (evaluated at $y = 1/\omega$). Substituting into (76),

$$\begin{aligned}\mathcal{L}v(\omega) &= \frac{\sigma_\omega^2}{\omega}\big(-y^2V'(y)\big) + \frac{\sigma_\omega^2}{2}\big(y^4V''(y) + 2y^3V'(y)\big)\\ &= \sigma_\omega^2\big(-y^3V'(y) + y^3V'(y)\big) + \frac{\sigma_\omega^2}{2}y^4V''(y) \;=\; \frac{y^4\sigma_\omega^2}{2}V''(y) \;=\; \frac{a(y)}{2}V''(y),\end{aligned}$$

using $1/\omega = y$ so $\sigma_\omega^2/\omega \cdot y^2 = \sigma_\omega^2 \cdot y^3$. The first-derivative terms cancel—this is precisely the statement that $s(\omega) = 1/\omega$ is a scale function, so the change of variables to the scale coordinate eliminates drift. Writing $\mathcal{L}v = -p$ with $p(\omega) = (y-\epsilon)/(y_c-\epsilon)$ (from Proposition 21, after substitution $1/\omega = y$, $1/\omega_c = y_c$):

$$\frac{a(y)}{2}V''(y) \;=\; -\frac{y-\epsilon}{y_c-\epsilon} \qquad\Longrightarrow\qquad V''(y) \;=\; -\frac{2(y-\epsilon)}{a(y)(y_c-\epsilon)},$$

with boundary conditions $V(\epsilon) = v(1/\epsilon) = 0$ and $V(y_c) = v(\omega_c) = 0$, establishing (84).

Step 3: Green's-function representation. The ODE (84) is $-V''(y) = f(y)$ on $[\epsilon, y_c]$ with Dirichlet data, where $f(y) \equiv 2(y-\epsilon)/[a(y)(y_c-\epsilon)]$. The Dirichlet Green's function for $-\mathrm{d}^2/\mathrm{d}y^2$ on $[\epsilon, y_c]$ is (86) (verified by direct differentiation: $G(y,z)$ is continuous, piecewise linear in y, vanishes at $y = \epsilon$ and $y = y_c$, and has jump $[\partial_y G]_{y=z} = -1$, making $-\partial_{yy}G(\cdot, z) = \delta_z$ in the distributional sense). Hence

$$V(y) \;=\; \int_\epsilon^{y_c} G(y,z)f(z)\,\mathrm{d}z \;=\; \frac{1}{y_c-\epsilon}\int_\epsilon^{y_c} G(y,z)(z-\epsilon)m(z)\,\mathrm{d}z,$$

with $m(z) = 2/a(z)$, giving (85). Uniqueness of the bounded solution follows from the

maximum principle for $-V'' = f$ with homogeneous Dirichlet data.

Step 4: Transformation to the ω variable. In (85) substitute $z = 1/\omega'$, so that $\mathrm{d}z = -\mathrm{d}\omega'/\omega'^2$ and z ranges over $[\epsilon, y_c]$ iff ω' ranges over $[\omega_c, 1/\epsilon]$ (in *reversed* order). Compute each factor:

$$\begin{aligned} z - \epsilon &= \tfrac{1}{\omega'} - \epsilon = \tfrac{1-\epsilon\omega'}{\omega'}, \\ a(z) &= z^4 \sigma_\omega(1/z)^2 = \omega'^{-4} \sigma_\omega(\omega')^2, \\ m(z) &= \tfrac{2}{a(z)} = \tfrac{2\,\omega'^4}{\sigma_\omega(\omega')^2}, \\ \mathrm{d}z &= -\tfrac{\mathrm{d}\omega'}{\omega'^2}. \end{aligned}$$

Therefore

$$(z-\epsilon)m(z)\,\mathrm{d}z \;=\; \frac{1-\epsilon\omega'}{\omega'} \cdot \frac{2\omega'^4}{\sigma_\omega(\omega')^2} \cdot \Big(-\frac{\mathrm{d}\omega'}{\omega'^2}\Big) \;=\; -\frac{2(1-\epsilon\omega')\,\omega'}{\sigma_\omega(\omega')^2}\,\mathrm{d}\omega'.$$

The reversal of orientation (since as z increases from ϵ to y_c, $\omega' = 1/z$ *decreases* from $1/\epsilon$ to ω_c) cancels the leading minus sign, and after substituting $y = 1/\omega_0$, $y_c - \epsilon = 1/\omega_c - \epsilon$ we obtain (87). Note the final power of ω': the Jacobian $\mathrm{d}z/\mathrm{d}\omega' = -\omega'^{-2}$ contributes ω'^{-2}, and the speed-measure transformation contributes $\omega'^{+4}/\omega' = \omega'^{+3}$, so the net power is ω'^{+1}, not ω'^{-3}.[12] □

H.5 Closed-form case: $\varphi = 0$

Proposition 24 (Elementary closed form at $\varphi = 0$)**.** *When the Frisch elasticity $\varphi = 0$,*

$$\sigma_\omega(\omega)\big|_{\varphi=0} \;=\; \frac{\sigma\epsilon(1-\omega)(1/\epsilon - \omega)}{\epsilon - 1}, \tag{88}$$

and the weight in (87) reduces to

$$\left.\frac{(1-\epsilon\omega')\,\omega'}{\sigma_\omega(\omega')^2}\right|_{\varphi=0} \;=\; \frac{(\epsilon-1)^2}{\sigma^2\epsilon} \cdot \frac{\omega'}{(1-\omega')^2(1/\epsilon - \omega')}. \tag{89}$$

[12] Numerical sanity check. Take $\epsilon = 2$, $\omega_c = 1/4$ (so $y_c = 4$), $\sigma_\omega \equiv 1$ (so $a(y) = y^4$), $\omega_0 = 1/3$ (so $y_0 = 3$). Direct solution of (84): $V''(y) = -(y-2)/y^4$, integrating twice gives $V(y) = -\frac{1}{2y} + \frac{1}{3y^2} + C_1 y + C_2$, and imposing $V(2) = V(4) = 0$ yields $C_1 = -1/32$, $C_2 = 11/48$, hence $V(3) = -1/6 + 1/27 - 3/32 + 11/48 = 5/864$. Green's-function formula (85): with $G(3,z) = (4-z)/2$ for $z \geq 3$ and $G(3,z) = (z-2)/2$ for $z \leq 3$, $m(z) = 2/z^4$, one computes $\int_2^3 (z-2)^2 z^{-4}\mathrm{d}z = 1/162$, $\int_3^4 (4-z)(z-2)z^{-4}\mathrm{d}z = 7/1296$, giving $V(3) = \frac{1}{2}(1/162 + 7/1296) = 5/864$. The ω-form (87) reproduces the same number under the substitution $z = 1/\omega'$; a formula with the incorrect power ω'^{-3} would differ by a factor of ω'^4 inside the integrand and fail this test.

In particular, $v(\omega_0)$ is an elementary function of ω_0: it reduces to a sum of logarithms and rational terms obtained by partial-fraction decomposition of (89).

Proof. At $\varphi = 0$, the denominator of (75) equals $(1-\omega) - (1/\epsilon - \omega) = 1 - 1/\epsilon = (\epsilon - 1)/\epsilon$, which is independent of ω. The numerator is $\sigma(1-\omega)(1/\epsilon - \omega)$ at $\varphi = 0$, so (88) follows.

For (89), use $1 - \epsilon\omega' = \epsilon(1/\epsilon - \omega')$:

$$\frac{(1-\epsilon\omega')\omega'}{\sigma_\omega^2} = \frac{\epsilon(1/\epsilon - \omega')\omega'\,(\epsilon-1)^2}{\sigma^2\epsilon^2(1-\omega')^2(1/\epsilon-\omega')^2} = \frac{(\epsilon-1)^2}{\sigma^2\epsilon}\cdot\frac{\omega'}{(1-\omega')^2(1/\epsilon-\omega')}.$$

Partial fractions decompose $\omega'/[(1-\omega')^2(1/\epsilon-\omega')]$ on the intervals of integration defining the Green's-function representation:

$$\frac{\omega'}{(1-\omega')^2(1/\epsilon-\omega')} = \frac{A}{1-\omega'} + \frac{B}{(1-\omega')^2} + \frac{C}{1/\epsilon-\omega'}, \tag{90}$$

with coefficients found by the standard cover-up method: clearing denominators, $\omega' = A(1-\omega')(1/\epsilon-\omega') + B(1/\epsilon-\omega') + C(1-\omega')^2$. Evaluating at $\omega' = 1$: $1 = B(1/\epsilon-1) = -B(\epsilon-1)/\epsilon$, so $B = -\epsilon/(\epsilon-1)$. Evaluating at $\omega' = 1/\epsilon$: $1/\epsilon = C(1-1/\epsilon)^2 = C(\epsilon-1)^2/\epsilon^2$, so $C = \epsilon/(\epsilon-1)^2$. Comparing the coefficient of ω'^2 on both sides ($A + C = 0$) gives $A = -C = -\epsilon/(\epsilon-1)^2$.

Each term in (90) has an elementary antiderivative ($\log$, $1/(1-\omega')$, $\log$), and in combination with the piecewise-linear Green function (86) the full integral (87) is elementary—a finite sum of logarithms and rational functions of ω_0. We omit the explicit expression, which is long but straightforward. □

This is the only regime in which the paper exhibits a fully explicit $v(\omega_0)$ without numerical integration. For $\varphi > 0$ the denominator $(1+\varphi)(1-\omega) - (1/\epsilon - \omega) = \varphi(1-\omega) + (1 - 1/\epsilon)$ is ω-dependent, and σ_ω^{-2} is no longer a rational function with poles at $\{1, 1/\epsilon\}$ only; closed-form antiderivatives are not available.

H.6 Sanity check: constant-diffusion Brownian limit

Suppose, counterfactually, that $a(y) \equiv a_0$ is constant on $[\epsilon, y_c]$—this is not the physical case in our model, but it serves as a benchmark for the Green's-function formula. Then (84) becomes $V''(y) = -2(y-\epsilon)/[a_0(y_c - \epsilon)]$. Integrating twice and imposing $V(\epsilon) = V(y_c) = 0$:

$$V(y) = -\frac{(y-\epsilon)^3}{3a_0(y_c-\epsilon)} + C_1(y-\epsilon), \qquad C_1 = \frac{y_c - \epsilon}{3a_0},$$

which simplifies to

$$V(y) \;=\; \frac{(y-\epsilon)(y_c-y)(y_c+y-2\epsilon)}{3a_0(y_c-\epsilon)}. \tag{91}$$

This is the classical cubic expression for the expected exit time of a Brownian motion with constant diffusion from a finite interval, weighted by the absorption probability at the left endpoint. It vanishes at $y=\epsilon$ and $y=y_c$, is positive on (ϵ, y_c), and is maximised at $y=(y_c+\epsilon)/2$; all three properties are expected. In particular, (91) agrees with the Green's-function evaluation of (85) when $m(z)=2/a_0$: $\int_\epsilon^{y_c} G(y,z)(z-\epsilon)\mathrm{d}z$ splits into two pieces which, after elementary integration, collapse to the product $(y-\epsilon)(y_c-y)(y_c+y-2\epsilon)/6$, and the prefactor $2/[a_0(y_c-\epsilon)]$ delivers (91). The formula is right.

H.7 Tail Distribution of τ_c

Propositions 21–24 pin down the probability of an eventual crisis and a Green's-function representation of the conditional mean. Policy questions, however, concern *horizon-specific* tail probabilities $\mathbb{P}(\tau_c \le T \mid \omega_0)$ at $T \in \{5, 10, 25\}$ years. This subsection gives the Laplace-transform PDE characterising the full distribution of τ_c (Proposition 25), solves it in closed form at $\varphi = 0$ (Proposition 26), and provides matching short-T and long-T asymptotic laws (Propositions 27 and 28) together with a numerical table at the paper's empirical calibration (Remark 12).

H.7.1 Laplace transform: backward equation

Proposition 25 (Kolmogorov backward equation for the Laplace transform)**.** *For $\lambda \ge 0$ and $\omega \in \bar{I} = [\omega_c, 1/\epsilon]$, define*

$$f(\omega;\lambda) \;\equiv\; \mathbb{E}\big[e^{-\lambda\tau_c}\mathbf{1}_{\{\tau_c<\infty\}} \mid \omega_0=\omega\big]. \tag{92}$$

Then $f(\cdot;\lambda)$ is the unique bounded continuous solution of

$$\mathcal{L}f(\omega;\lambda) \;=\; \lambda\, f(\omega;\lambda), \qquad \omega \in I, \tag{93}$$

with boundary data $f(\omega_c;\lambda)=1$ and $f(1/\epsilon;\lambda)=0$. In particular, $f(\omega;0)=p(\omega)$ recovers (78), and conditional moments on $\{\tau_c<\infty\}$ are given by

$$\mathbb{E}\big[\tau_c^n \mathbf{1}_{\{\tau_c<\infty\}} \mid \omega\big] \;=\; (-1)^n\, \partial_\lambda^n f(\omega;\lambda)\big|_{\lambda=0}, \qquad n \in \mathbb{N}. \tag{94}$$

Proof. Fix $\lambda \ge 0$ and let $\tau_n \equiv \tau_c \wedge \sigma_n$ with $\sigma_n \equiv \inf\{t : \omega_t = 1/\epsilon - 1/n\}$ as in the proof

of Proposition 21. For a smooth $g : \bar{I} \to \mathbb{R}$, Itô's formula (Karatzas and Shreve, 1991, Theorem 3.3.6) applied to $M_t \equiv e^{-\lambda t} g(\omega_t)$ gives

$$\mathrm{d}M_t \;=\; e^{-\lambda t}\big[\mathcal{L}g(\omega_t) - \lambda g(\omega_t)\big]\mathrm{d}t + e^{-\lambda t}\sigma_\omega(\omega_t) g'(\omega_t)\,\mathrm{d}Z_t.$$

Stop at $t \wedge \tau_n$: the stochastic integral is a continuous martingale (bounded integrand), and $\mathcal{L}g - \lambda g \equiv 0$ on I forces $M_{t\wedge\tau_n}$ to be a martingale whenever g solves (93). By optional stopping (Karatzas and Shreve, 1991, Theorem 1.3.22),

$$g(\omega_0) \;=\; \mathbb{E}\big[e^{-\lambda\tau_n} g(\omega_{\tau_n})\big] \;=\; \mathbb{E}\big[e^{-\lambda\tau_c} g(\omega_c)\mathbf{1}_{\{\tau_c<\sigma_n\}}\big] + \mathbb{E}\big[e^{-\lambda\sigma_n} g(1/\epsilon - 1/n)\mathbf{1}_{\{\sigma_n\le\tau_c\}}\big]. \quad (95)$$

Passing $n \to \infty$: the first term converges to $g(\omega_c)\cdot\mathbb{E}[e^{-\lambda\tau_c}\mathbf{1}_{\{\tau_c<\infty\}}]$ by monotone-increasing events and dominated convergence ($|e^{-\lambda\tau_c}| \le 1$); the second converges to $g(1/\epsilon)\cdot\mathbb{E}[e^{-\lambda\sigma}\mathbf{1}_{\{\sigma<\infty\}}]$ where σ is the hitting time of $1/\epsilon$, and on $\{\tau_c = \infty\} \cap \{\omega_t \to 1/\epsilon\}$ (full measure by Appendix J), $\sigma = +\infty$ so $e^{-\lambda\sigma} = 0$ for $\lambda > 0$, while for $\lambda = 0$ the limit is $g(1/\epsilon)\cdot(1 - p(\omega_0))$.

Imposing $g(\omega_c) = 1$ and $g(1/\epsilon) = 0$ in (95) and taking $n \to \infty$ yields, for $\lambda > 0$,

$$g(\omega_0) \;=\; \mathbb{E}\big[e^{-\lambda\tau_c}\mathbf{1}_{\{\tau_c<\infty\}}\big] \;=\; f(\omega_0;\lambda),$$

which proves any bounded solution of (93) equals $f(\cdot;\lambda)$. Existence follows from the general Feynman–Kac representation (Karatzas and Shreve, 1991, Theorem 5.7.6), Revuz–Yor (1999) Ch. VII. At $\lambda = 0$, (93) is the Dirichlet problem of Proposition 21; boundedness and the Feller classification of $1/\epsilon$ as unattainable give $f(1/\epsilon;0) = \lim_{\omega\uparrow 1/\epsilon} p(\omega) = 0$. Moment formula (94) follows by differentiating (92) under the expectation (justified on $\{\tau_c < \infty\}$ by dominated convergence with envelope $\tau_c^n e^{-\lambda_0\tau_c} \le C_n$ on the event). □

Differentiating (93) with respect to λ and evaluating at $\lambda = 0$ gives $\mathcal{L}[\partial_\lambda f(\cdot;0)] = f(\cdot;0) = p$, so $-\partial_\lambda f(\omega;0)$ solves the Feynman–Kac problem (83) of Proposition 23: the conditional expected hitting time is indeed the $n = 1$ moment in (94), consistent with the earlier Green's-function representation.

H.7.2 Closed form at $\varphi = 0$: hypergeometric reduction

Proposition 26 (Closed-form Laplace transform at $\varphi = 0$)**.** *At $\varphi = 0$ (so that σ_ω is given by (88)), the Laplace transform (92) admits the elementary closed form*

$$\boxed{f(\omega;\lambda)\big|_{\varphi=0} \;=\; \frac{\omega_c}{\omega}\left(\frac{1-\omega}{1-\omega_c}\right)^{p_-}\left(\frac{1-\epsilon\omega}{1-\epsilon\omega_c}\right)^{p_+},} \quad (96)$$

where

$$p_\pm \;=\; \tfrac{1}{2}\Big(1 \pm \sqrt{1+8\lambda/\sigma^2}\,\Big), \qquad p_+ + p_- = 1, \quad p_+ p_- = -2\lambda/\sigma^2. \tag{97}$$

Proof. Step 1: reduction in the Q-coordinate. With $Q = 1/\omega$ and $V(Q) \equiv f(1/Q;\lambda)$, the calculation of Step 2 in the proof of Proposition 23 gives $\mathcal{L}f = \frac{1}{2}a(Q)V''(Q)$ where $a(Q) = Q^4\sigma_\omega(1/Q)^2$. At $\varphi = 0$, substituting $1-\omega = (Q-1)/Q$ and $1/\epsilon - \omega = (Q-\epsilon)/(\epsilon Q)$ into (88) yields

$$a(Q)\big|_{\varphi=0} \;=\; \frac{\sigma^2(Q-1)^2(Q-\epsilon)^2}{(\epsilon-1)^2}. \tag{98}$$

Equation (93) becomes

$$\frac{\sigma^2(Q-1)^2(Q-\epsilon)^2}{2(\epsilon-1)^2}\,V''(Q) \;=\; \lambda V(Q), \qquad V(Q_c) = 1,\; V(\epsilon) = 0, \tag{99}$$

where $Q_c = 1/\omega_c$. This is a second-order linear ODE with three regular singular points at $Q \in \{1, \epsilon, \infty\}$—a hypergeometric type.

Step 2: exact factorisation. We claim that $V(Q) = (Q-1)^\alpha(Q-\epsilon)^\beta$ solves (99) iff $\alpha+\beta = 1$ and $\alpha(\alpha-1) = \mu \equiv 2\lambda/\sigma^2$. Indeed, for V of this form,

$$\frac{V''}{V} \;=\; \frac{\alpha(\alpha-1)}{(Q-1)^2} + \frac{\beta(\beta-1)}{(Q-\epsilon)^2} + \frac{2\alpha\beta}{(Q-1)(Q-\epsilon)}$$

(from $(V'/V)^2 + \partial_Q(V'/V) = V''/V$ with $V'/V = \alpha/(Q-1) + \beta/(Q-\epsilon)$). Multiplying by $(Q-1)^2(Q-\epsilon)^2$, the result is a polynomial in Q of degree 2; its coefficient of Q^2 is $(\alpha+\beta)^2 - (\alpha+\beta)$, vanishing iff $\alpha+\beta \in \{0,1\}$; its coefficient of Q vanishes iff $\alpha+\beta = 1$ (the alternative $\alpha+\beta = 0$ leaves a non-zero coefficient proportional to $\epsilon\alpha+\beta$); its constant term, after imposing $\beta = 1-\alpha$, equals $\alpha(\alpha-1)(\epsilon-1)^2$. Substituting back into (99): $\alpha(\alpha-1)(\epsilon-1)^2 = 2\lambda(\epsilon-1)^2/\sigma^2$, i.e., $\alpha(\alpha-1) = \mu$. The two roots are $\alpha = p_+$ and $\alpha = p_- = 1-p_+$, giving the two linearly independent solutions

$$V_+(Q) = (Q-1)^{p_-}(Q-\epsilon)^{p_+}, \qquad V_-(Q) = (Q-1)^{p_+}(Q-\epsilon)^{p_-}. \tag{100}$$

Step 3: selection by boundary behaviour. For $\lambda > 0$, $p_+ > 1 > 0 > p_-$. At $Q = \epsilon$, V_+ vanishes (as $(Q-\epsilon)^{p_+} \to 0$) while V_- diverges (as $(Q-\epsilon)^{p_-} \to +\infty$ since $p_- < 0$). Boundedness at $\omega = 1/\epsilon$ (equivalently $Q = \epsilon$) forces the coefficient of V_- to vanish. The condition $V(Q_c) = 1$ then fixes the coefficient of V_+, giving

$$V(Q) \;=\; \frac{(Q-1)^{p_-}(Q-\epsilon)^{p_+}}{(Q_c-1)^{p_-}(Q_c-\epsilon)^{p_+}}. \tag{101}$$

Switching back to ω: $Q-1=(1-\omega)/\omega$, $Q-\epsilon=(1-\epsilon\omega)/\omega$, so $(Q-1)^{p_-}(Q-\epsilon)^{p_+}=\omega^{-(p_++p_-)}(1-\omega)^{p_-}(1-\epsilon\omega)^{p_+}=\omega^{-1}(1-\omega)^{p_-}(1-\epsilon\omega)^{p_+}$. Dividing by the analogous expression at Q_c:

$$f(\omega;\lambda)\big|_{\varphi=0} \;=\; \frac{\omega_c}{\omega}\cdot\frac{(1-\omega)^{p_-}(1-\epsilon\omega)^{p_+}}{(1-\omega_c)^{p_-}(1-\epsilon\omega_c)^{p_+}},$$

which is (96).

Step 4: relation to the Gauss hypergeometric. The same closed form emerges from the Möbius transformation $\xi=(Q-\epsilon)/(Q-1)$ (mapping the singular points $Q\in\{\epsilon,1,\infty\}$ to $\xi\in\{0,\infty,1\}$) followed by the substitution $V=\xi^{p_+}(1-\xi)^{-1}$: the residual function $W(\xi)$ satisfies the Gauss hypergeometric equation

$$\xi(1-\xi)W''(\xi)+\big[2p_+-2(p_++1)\xi\big]W'(\xi)-2p_+\,W(\xi) \;=\; 0,$$

with parameters $(a,b,c)=(1,2p_+,2p_+)$, whose principal solution is ${}_2F_1(1,2p_+;2p_+;\xi)=(1-\xi)^{-1}$ (Abramowitz and Stegun, 1972, Ch. 15). The degeneracy $b=c$ collapses the hypergeometric series to the elementary form (101). □

Remark 9 (Four consistency checks)**.** *Formula* (96) *passes each of the following four independent consistency checks.*

(i) $f(\omega_c;\lambda)=1$*: all three ratios equal one.*

(ii) $f(1/\epsilon;\lambda)=0$*: the factor* $(1-\epsilon\omega)^{p_+}$ *vanishes since* $1-\epsilon\cdot(1/\epsilon)=0$ *and* $p_+>0$*.*

(iii) $f(\omega;0)=p(\omega)$*: at* $\lambda=0$*,* $p_+=1$ *and* $p_-=0$*, so* (96) *reduces to*

$$f(\omega;0)\big|_{\varphi=0} \;=\; \frac{\omega_c}{\omega}\cdot 1\cdot\frac{1-\epsilon\omega}{1-\epsilon\omega_c} \;=\; \frac{\omega_c(1-\epsilon\omega)}{\omega(1-\epsilon\omega_c)},$$

matching (78)*.*

(iv) $-\partial_\lambda f(\omega;0)$ *solves Proposition 23. From* $\partial_\lambda p_\pm\big|_{\lambda=0}=\pm 2/\sigma^2$ *and* $\partial_\lambda \ln f=(\partial_\lambda p_-)\ln[(1-\omega)/(1-\omega_c)]+(\partial_\lambda p_+)\ln[(1-\epsilon\omega)/(1-\epsilon\omega_c)]$*,*

$$\mathbb{E}[\tau_c\mathbf{1}_{\{\tau_c<\infty\}}\mid\omega]\big|_{\varphi=0} \;=\; -\frac{2p(\omega)}{\sigma^2}\,\ln\!\left[\frac{(1-\omega_c)(1-\epsilon\omega)}{(1-\omega)(1-\epsilon\omega_c)}\right], \tag{102}$$

an explicit log-ratio formula that replaces the Green's-function integral of Proposition 23 (and is checked independently by direct Feynman–Kac substitution: $\mathcal{L}$ *applied to* (102) *yields* $-p(\omega)$ *after elementary but tedious algebra, with correct boundary conditions* $v(\omega_c)=v(1/\epsilon)=0$*).*

H.7.3 Short-horizon tail via Freidlin–Wentzell large deviations

Proposition 27 (Small-noise / short-T large-deviation principle)**.** *Let $\tilde{\sigma}_\omega(\omega) \equiv \sigma_\omega(\omega)/\sigma$, which is independent of σ. As $\sigma \downarrow 0$ (equivalently, $T \downarrow 0$ at fixed (ω_0, ω_c)), the Freidlin–Wentzell large-deviation principle (Freidlin and Wentzell, 2012, Ch. 5) gives*

$$\log \mathbb{P}(\tau_c \le T \mid \omega_0) \;=\; -\frac{I(\omega_0, \omega_c, T)}{\sigma^2} + o(\sigma^{-2}), \tag{103}$$

where the rate function is the action

$$I(\omega_0, \omega_c, T) \;=\; \inf_{\phi \in \mathcal{A}} \frac{1}{2} \int_0^T \frac{\big(\dot{\phi}(s) - \sigma^2 \tilde{\sigma}_\omega(\phi(s))^2/\phi(s)\big)^2}{\tilde{\sigma}_\omega(\phi(s))^2}\, \mathrm{d}s, \tag{104}$$

with $\mathcal{A} = \{\phi : [0,T] \to [\omega_c, 1/\epsilon]$ absolutely continuous, $\phi(0) = \omega_0$, $\phi(T) = \omega_c\}$. In the leading-order small-noise limit (dropping the $O(\sigma^2)$ drift),

$$I(\omega_0, \omega_c, T) \;\approx\; \tilde{I}(\omega_0, \omega_c, T) \;\equiv\; \frac{d(\omega_0, \omega_c)^2}{2T}, \qquad d(\omega_0, \omega_c) \equiv \int_{\omega_c}^{\omega_0} \frac{\mathrm{d}u}{\tilde{\sigma}_\omega(u)}, \tag{105}$$

where $d(\cdot, \omega_c)$ is the Riemannian distance in the metric $\mathrm{d}s^2 = \mathrm{d}\omega^2/\tilde{\sigma}_\omega(\omega)^2$. Hence, for small T,

$$\log \mathbb{P}(\tau_c \le T \mid \omega_0) \;\lesssim\; -\frac{d(\omega_0, \omega_c)^2}{2T\sigma^2}. \tag{106}$$

Proof. Writing $d\omega_t = \sigma^2 \tilde{\sigma}_\omega^2/\omega\, dt + \sigma \tilde{\sigma}_\omega\, dZ_t$ casts the SDE in small-noise form $dX = b(X)dt + \sigma\sigma_X(X)dW$ with $b = O(\sigma^2)$. The Freidlin–Wentzell LDP (Dembo and Zeitouni, 2010; Freidlin and Wentzell, 2012, Theorem 5.6.7) then gives (103) with rate function $I(\omega_0, \omega_c, T) = \inf_\phi \frac{1}{2}\int_0^T (\dot{\phi} - b(\phi))^2/\tilde{\sigma}_\omega(\phi)^2\, ds$. For the leading-order analysis, drop the drift (which is $O(\sigma^2)$ and hence subdominant in $\sigma \downarrow 0$): the reduced action $\tilde{I} = \frac{1}{2}\int_0^T \dot{\phi}^2/\tilde{\sigma}_\omega(\phi)^2\, ds$ is the length-functional in the Riemannian metric $\mathrm{d}s^2 = \mathrm{d}\omega^2/\tilde{\sigma}_\omega^2$, and by Cauchy–Schwarz

$$\int_0^T \frac{\dot{\phi}^2}{\tilde{\sigma}_\omega^2}\, ds \;\ge\; \frac{1}{T}\left(\int_0^T \frac{|\dot{\phi}|}{\tilde{\sigma}_\omega}\, ds\right)^2 \;\ge\; \frac{1}{T}\left(\int_{\omega_c}^{\omega_0} \frac{du}{\tilde{\sigma}_\omega(u)}\right)^2 \;=\; \frac{d(\omega_0, \omega_c)^2}{T},$$

with equality achieved by the constant-speed geodesic $\phi^\star$ with $|\dot{\phi}^\star|/\tilde{\sigma}_\omega(\phi^\star) \equiv d/T$. Hence $\tilde{I} = d(\omega_0, \omega_c)^2/(2T)$. The Euler–Lagrange equations for the full action (104) are

$$\frac{d}{ds}\left(\frac{\dot{\phi} - b(\phi)}{\tilde{\sigma}_\omega(\phi)^2}\right) + \frac{(\dot{\phi} - b(\phi))^2 \tilde{\sigma}_\omega'(\phi)}{\tilde{\sigma}_\omega(\phi)^3} + \frac{(\dot{\phi} - b(\phi))b'(\phi)}{\tilde{\sigma}_\omega(\phi)^2} \;=\; 0,$$

a second-order ODE for $\phi^\star$; for T small or σ small, the leading-order solution is the unit-speed

geodesic in the $\tilde{\sigma}_\omega$ metric. □

Remark 10 (Validity of the short-T bound)**.** *The approximation* (106) *is sharp in the joint limit* $(\sigma, T) \to (0, 0)$ *with* $\sigma^2 T$ *fixed. Because the drift* $\mu_\omega = \sigma_\omega^2/\omega$ *is strictly positive on* I *(pushing the process* away *from* ω_c*), including the drift* lowers *the probability of early absorption; hence* (106) *is an* upper *bound on* $\mathbb{P}(\tau_c \le T)$ *to leading order in* σ^2*. For policy-horizon* $T \in \{10, 25\}$ *years at* $\sigma = 0.15$*, the drift is non-negligible and* (106) *overestimates the tail; numerical inversion of the exact Laplace transform (Proposition 25) is required for quantitative answers.*

H.7.4 Long-horizon tail: essential-spectrum bottom

The natural endpoint $1/\epsilon$ does *not* generate a discrete spectrum. The correct statement is that $-\mathcal{L}$ has empty discrete spectrum below the universal threshold $\sigma^2/8$, which is the bottom of the essential spectrum. The argument is a standard one-dimensional Schrödinger-type boundary analysis at the natural endpoint, combined with the closed form of Proposition 26 at $\varphi = 0$ and Weyl's theorem on essential spectrum for general φ.

Proposition 28 (Essential-spectrum bottom is $\sigma^2/8$, universal in φ)**.** *Let* $m(\mathrm{d}\omega) = [2\omega^2/\sigma_\omega(\omega)^2]\,\mathrm{d}\omega$ *be the speed measure on* $I = (\omega_c, 1/\epsilon)$ *with respect to the scale* $s(\omega) = 1/\omega$*, and let* $L^2(m)$ *denote the associated weighted space. The operator* $-\mathcal{L}$ *with domain*

$$\mathcal{D} \;=\; \big\{ f \in C^2(I) \cap L^2(m) : \; f(\omega_c) = 0, \;\; \textit{and } f \textit{ lies in } L^2(m) \textit{ in a neighbourhood of } 1/\epsilon \big\}$$

is self-adjoint and non-negative in $L^2(m)$ *(the right endpoint* $1/\epsilon$ *is natural-unattainable by the Feller classification of Appendix I, so requires no separate boundary condition beyond* $L^2(m)$*-limit-point selection; see Karlin and Taylor, 1981, Ch. 15). Then, for* every $\varphi \ge 0$*,*

(i) *the essential spectrum satisfies* $\inf \sigma_{\mathrm{ess}}(-\mathcal{L}) = \sigma^2/8$*;*

(ii) *the discrete spectrum* $\sigma_{\mathrm{disc}}(-\mathcal{L}) \cap (0, \sigma^2/8)$ *is empty.*

Consequently, conditional on $\{\tau_c < \infty\}$*,*

$$\mathbb{P}(\tau_c > T \mid \omega_0, \tau_c < \infty) \;=\; O\big(T^{-\alpha}\, e^{-\sigma^2 T/8}\big) \qquad \textit{as } T \to \infty, \tag{107}$$

for some algebraic prefactor exponent $\alpha > 0$ *fixed by the density of states at the spectrum edge (generically* $\alpha = 3/2$ *for a Schrödinger-type boundary (Demuth and van Casteren, 2000, Ch. 3)); the exponential rate* $\sigma^2/8$ *is universal (independent of* φ*,* ω_c*,* ϵ*).*

Proof. The proof has four steps: a φ-universal boundary expansion of σ_ω, the indicial analysis at $1/\epsilon$, the $L^2(m)$ filter, and non-vanishing of the bounded fundamental solution at $\varphi = 0$. Universality in φ then follows from Weyl's theorem on invariance of essential spectrum under relatively compact perturbations.

Step 1 (φ-universal boundary behaviour). Write $\omega = 1/\epsilon - \xi$ with $\xi \downarrow 0$. The numerator of (75) is $\sigma(1+\varphi)(1-\omega)(1/\epsilon-\omega) = \sigma(1+\varphi)\big[(1-1/\epsilon)+\xi\big]\xi$; the denominator is $D(\omega) = \varphi(1-\omega)+(1-1/\epsilon) = (1+\varphi)(1-1/\epsilon)+\varphi\xi$ after rearranging $\varphi(1-\omega) = \varphi(1-1/\epsilon)+\varphi\xi$ and adding $(1-1/\epsilon)$. The ratio is

$$\sigma_\omega(1/\epsilon-\xi) \;=\; \sigma\xi\cdot\frac{(1+\varphi)\big[(1-1/\epsilon)+\xi\big]}{(1+\varphi)(1-1/\epsilon)+\varphi\xi} \;=\; \sigma\xi\big[1+O(\xi)\big], \tag{108}$$

since the leading order of the ratio is $(1+\varphi)(1-1/\epsilon)/[(1+\varphi)(1-1/\epsilon)] = 1$. *The $(1+\varphi)(1-1/\epsilon)$ factors cancel.* In particular,

$$\sigma_\omega(\omega)^2 \;=\; \sigma^2\xi^2\big[1+O(\xi)\big] \qquad (\xi = 1/\epsilon-\omega\downarrow 0), \tag{109}$$

universally in $\varphi \geq 0$.

Step 2 (indicial equation at $1/\epsilon$). The eigenvalue equation $\mathcal{L}\psi+\lambda\psi = 0$ on I reads

$$\frac{\sigma_\omega(\omega)^2}{2}\psi''(\omega)+\frac{\sigma_\omega(\omega)^2}{\omega}\psi'(\omega)+\lambda\psi(\omega) \;=\; 0.$$

Substitute $\xi = 1/\epsilon-\omega$, so $\mathrm{d}/\mathrm{d}\omega = -\mathrm{d}/\mathrm{d}\xi$ and $\omega = 1/\epsilon+O(\xi)$. Using (109),

$$\frac{\sigma^2\xi^2}{2}\psi_{\xi\xi} \;-\; \sigma^2\epsilon\,\xi^2\psi_\xi \;+\; \lambda\psi \;=\; O(\xi^3\psi_{\xi\xi})+O(\xi^3\psi_\xi). \tag{110}$$

Try $\psi(\xi) = \xi^\alpha$. Then $\psi_\xi = \alpha\xi^{\alpha-1}$, $\psi_{\xi\xi} = \alpha(\alpha-1)\xi^{\alpha-2}$. The first term on the left of (110) is $\frac{1}{2}\sigma^2\alpha(\alpha-1)\xi^\alpha$; the second is $-\sigma^2\epsilon\alpha\xi^{\alpha+1} = O(\xi^{\alpha+1})$, subleading; the third is $\lambda\xi^\alpha$. Balancing ξ^α,

$$\alpha(\alpha-1) \;=\; -\frac{2\lambda}{\sigma^2}, \qquad \alpha_\pm(\lambda) \;=\; \tfrac{1}{2}\Big(1\pm\sqrt{1-8\lambda/\sigma^2}\Big). \tag{111}$$

Write $\nu(\lambda) \equiv \sqrt{1-8\lambda/\sigma^2}$. For $\lambda\in(0,\sigma^2/8)$, $\nu\in(0,1)$ is real, and $\alpha_- < 1/2 < \alpha_+$. At $\lambda = \sigma^2/8$, $\nu = 0$, $\alpha_\pm = 1/2$ (coincident roots); the second fundamental solution acquires a $\log\xi$ factor. For $\lambda > \sigma^2/8$, ν is purely imaginary; $\alpha_\pm = 1/2\pm i|\nu|/2$ are complex-conjugate, and $\xi^{\alpha_\pm}$ oscillates in $\log\xi$.

Step 3 ($L^2(m)$ filter at the natural endpoint). The speed measure near $1/\epsilon$ satisfies,

using (109) and $\omega \to 1/\epsilon$,

$$m(\mathrm{d}\omega) \;=\; \frac{2\omega^2}{\sigma_\omega^2}\,\mathrm{d}\omega \;\sim\; \frac{2/\epsilon^2}{\sigma^2\xi^2}\,\mathrm{d}\xi \qquad (\xi \downarrow 0). \tag{112}$$

For $\psi \sim \xi^\alpha$,

$$\int_0^{\bar\xi} |\psi(\xi)|^2\, m(\mathrm{d}\omega) \;\sim\; \frac{2}{\epsilon^2\sigma^2}\int_0^{\bar\xi} \xi^{2\alpha-2}\,\mathrm{d}\xi,$$

which converges at $\xi = 0$ iff $2\alpha - 2 > -1$, i.e., iff $\alpha > 1/2$.

Combining Steps 2–3:

- For $\lambda \in (0, \sigma^2/8)$: $\alpha_+ > 1/2 > \alpha_-$, so exactly one fundamental solution (the α_+-branch) is $L^2(m)$-admissible at $1/\epsilon$. The natural endpoint is in the *limit-point* case (Karlin and Taylor, 1981, §15.7).
- For $\lambda = \sigma^2/8$: $\alpha = 1/2$; $|\psi|^2 \sim \xi^{-1}$, $\int \xi^{-1}\,\mathrm{d}\xi$ diverges logarithmically. *Neither* fundamental solution is $L^2(m)$-admissible.
- For $\lambda > \sigma^2/8$: $|\psi|^2 \sim \xi^{-1}$ with oscillation, again non-$L^2(m)$ for either branch.

Step 4 (no discrete eigenvalue below $\sigma^2/8$ at $\varphi = 0$). By Proposition 26 and (100), the $L^2(m)$-admissible fundamental solution of $\mathcal{L}\psi + \lambda\psi = 0$ at $\varphi = 0$, written in the $Q = 1/\omega$ coordinate, is

$$\psi_{\mathrm{adm}}(Q;\lambda) \;\propto\; V_+(Q) \;=\; (Q-1)^{\alpha_-(\lambda)}(Q-\epsilon)^{\alpha_+(\lambda)}, \qquad 0 < \lambda < \sigma^2/8. \tag{113}$$

(Steps 2–3 above identified α_+ as the boundary-admissible exponent at $Q = \epsilon$; this matches p_+ in (100) once one checks the sign of the argument $1-8\lambda/\sigma^2$ versus $1+8\lambda/\sigma^2$: the eigenvalue equation has opposite sign for λ compared to the Laplace-transform equation (93), so the index roots are $\alpha_\pm(\lambda) = p_\mp(-\lambda)$ in the notation of (97), restoring the sign in (111) under $\lambda \mapsto -\lambda$.)

A discrete eigenvalue $\lambda \in (0, \sigma^2/8)$ would require $\psi_{\mathrm{adm}}(Q_c;\lambda) = 0$. But for $\lambda \in (0, \sigma^2/8)$, $\alpha_\pm$ are real, and $Q_c - 1 \geq \epsilon - 1 > 0$ (since $\omega_c < 1/\epsilon$ implies $Q_c > \epsilon > 1$) and $Q_c - \epsilon > 0$. Hence

$$\psi_{\mathrm{adm}}(Q_c;\lambda) \;=\; (Q_c-1)^{\alpha_-(\lambda)}(Q_c-\epsilon)^{\alpha_+(\lambda)} \;>\; 0,$$

a strictly positive real number. The Dirichlet condition cannot be met, so no discrete eigenvalue of $-\mathcal{L}$ lies in $(0, \sigma^2/8)$ at $\varphi = 0$. Combined with Step 3, which shows that $\sigma^2/8$ itself is the threshold above which no $L^2(m)$-admissible fundamental solution exists at $1/\epsilon$, this gives $\inf \sigma(-\mathcal{L}) = \sigma^2/8$ and identifies $\sigma^2/8$ as the bottom of the essential spectrum at $\varphi = 0$.

Step 5 (universality in φ). Step 1 showed that the leading $\sigma_\omega \sim \sigma\xi$ behaviour is φ-universal. Hence the indicial exponents (111) and the $L^2(m)$-threshold $\alpha > 1/2$ are identical for every $\varphi \geq 0$; the critical eigenvalue at which the two admissible fundamental solutions degenerate is $\sigma^2/8$ regardless of φ. This establishes claim (i) at general φ: $\inf \sigma_{\mathrm{ess}}(-\mathcal{L}_\varphi) = \sigma^2/8$.

To extend the "no eigenvalue below $\sigma^2/8$" statement from $\varphi = 0$ to general φ, invoke *Weyl's theorem* (Reed and Simon, 1978, Theorem XIII.14 and Corollary 2): the essential spectra of two self-adjoint operators agree if their difference is relatively compact. The operator family $\{-\mathcal{L}_\varphi\}_{\varphi \geq 0}$ differs only through the coefficient $\sigma_\omega(\omega; \varphi)$, which converges *uniformly* on any compact subset of I as $\varphi \to 0$, and whose boundary behaviour at both endpoints of I is φ-universal by Step 1 (and by direct inspection at ω_c: $\sigma_\omega(\omega_c; \varphi) = \sigma(1 + \varphi)(1 - \omega_c)(1/\epsilon - \omega_c)/D(\omega_c)$, bounded and smooth). The difference $-\mathcal{L}_\varphi - (-\mathcal{L}_0)$ is a second-order differential operator with coefficients bounded on any closed subinterval of I and with boundary asymptotics matching to leading order at both endpoints; its resolvent-difference is compact by the compact-embedding argument of Reed and Simon (1978, Theorem XIII.68). Weyl's theorem then gives $\sigma_{\mathrm{ess}}(-\mathcal{L}_\varphi) = \sigma_{\mathrm{ess}}(-\mathcal{L}_0) = [\sigma^2/8, \infty)$.

For claim (ii) at general φ, we do not have a closed-form fundamental solution. However, the φ-universal boundary analysis of Steps 1–3 shows that *if* a discrete eigenvalue $\lambda \in (0, \sigma^2/8)$ existed, its eigenfunction would satisfy $\psi(\omega_c) = 0$ and decay as $\xi^{\alpha_+(\lambda)}$ at $1/\epsilon$. A second-order ODE with two boundary conditions admits at most a discrete set of such λ, and by self-adjointness and Weyl's theorem this discrete set is contained in $[0, \sigma^2/8) \cup$ (accumulation points of σ_{disc}). For the specific one-dimensional diffusion at hand, the generator is of regular-boundary/natural-boundary type with a single limit-point endpoint; standard Sturm–Liouville theory for such operators (Karlin and Taylor, 1981, §15.7; Borodin and Salminen, 2002, Ch. VII.3) implies that the resolvent is Hilbert–Schmidt on the discrete-spectrum component, and that the discrete spectrum below $\inf \sigma_{\mathrm{ess}}$ is either empty or a finite set accumulating only at $\inf \sigma_{\mathrm{ess}} = \sigma^2/8$. At $\varphi = 0$ Step 4 showed the former. By relative compactness of the $0 \mapsto \varphi$ perturbation and continuity of discrete eigenvalues under compact perturbation (Reed and Simon, 1978, Theorem XIII.16), the discrete spectrum below $\sigma^2/8$ remains empty on an open neighbourhood of $\varphi = 0$ in the coefficient topology above; and the φ-universal boundary analysis precludes appearance of new discrete eigenvalues via bifurcation from the essential-spectrum edge because the boundary exponents $\alpha_\pm(\lambda)$ do not depend on φ. Hence (ii) holds for all $\varphi \geq 0$.

Step 6 (decay of the conditional survival probability). Because $\sigma(-\mathcal{L}) \subset [\sigma^2/8, \infty)$ with empty discrete part below $\sigma^2/8$, the spectral theorem gives the semigroup bound $\|e^{\mathcal{L}T}\| \leq e^{-\sigma^2 T/8}$ in $L^2(m)$. The conditional survival probability is the ratio $\mathbb{P}(\tau_c > T \mid \omega_0)/p(\omega_0)$, and

standard semigroup-kernel estimates for one-dimensional diffusions with natural-endpoint essential spectrum (Borodin and Salminen, 2002, Ch. II §2) yield the polynomial-prefactor correction

$$\mathbb{P}(\tau_c > T \mid \omega_0) \;=\; \int_I p_T(\omega_0, \omega')\, m(\mathrm{d}\omega') \;=\; C(\omega_0)\, T^{-\alpha}\, e^{-\sigma^2 T/8}\,\big[1 + o(1)\big],$$

for some $\alpha > 0$ (the density-of-states exponent at the spectrum edge; for one-dimensional Schrödinger-type boundaries the generic value is $\alpha = 3/2$, see Demuth and van Casteren, 2000, Ch. 3). Dividing by $p(\omega_0)$ gives (107). □

Remark 11 (What is delicate, what survives)**.** *The exact value of the algebraic prefactor α depends on the density of states at $\lambda = \sigma^2/8$, which in turn depends on the sub-leading coefficients in (108). Determining α explicitly requires a full Liouville normal-form reduction of the eigenvalue equation near $1/\epsilon$ (not carried out here). The* exponential rate $\sigma^2/8$ *is however analytically pinned down and φ-universal; that is the content of the Proposition. The quantitatively relevant horizons of Table 18 below ($T \le 50$) have $\sigma^2 T/8 \le 0.14$, so $e^{-\sigma^2 T/8} \in [0.87, 1]$: the long-$T$ exponential tail is negligibly slow on policy-relevant horizons, and the tail probabilities at $T \le 50$ are determined by the full Laplace inversion, not by the asymptotic rate.*

H.7.5 Calibrated numerical table

Remark 12 (Tail probabilities at the empirical calibration)**.** *At the paper's empirical calibration $\epsilon = 13.80$, $\sigma = 0.15$, $\varphi = 1$, $\omega_c \approx 0.051$ (so $1/\epsilon \approx 0.0725$ and the pre-failure interval has width 0.021), Table 18 reports $\mathbb{P}(\tau_c \le T \mid \omega_0)$ computed by numerical inversion of the Laplace transform of Proposition 25 via Gaver–Stehfest (Stehfest, 1970) ($N = 10$ terms), with the Kolmogorov backward equation solved at each Stehfest node by finite-difference BVP on a mesh of 1500 points with Dirichlet cutoff at $1/\epsilon - 10^{-5}$.*

The empirical pre-2008 U.S. path for the HKM capital ratio puts $\bar{\omega}_{1995\text{–}2006} = 0.095$, well above $1/\epsilon \approx 0.0725$, so the "pre-failure domain" is only a valid description from roughly 2008Q1 onward; the post-crisis trough places ω_t below *ω_c, in the QE-constrained regime (not this appendix's domain). The table's first row—ω_0 one-quarter of the way from ω_c to $1/\epsilon$, hence $\omega_0 = 0.0564$—is the closest empirically relevant starting point on the post-crisis HKM series. At this ω_0, the 5-year crisis probability is 31%, the 10-year is 42%, and the asymptotic (infinite-horizon) is 68%; the decade horizon captures more than half of the ultimate crisis probability.*

Freidlin–Wentzell benchmark. *The short-T bound (106), ignoring drift, gives for the same row (Riemannian distance $d = 0.285$)* $\exp(-d^2/(2T\sigma^2)) = \exp(-1.80/T)$, *yielding*

ω_0 position	ω_0	$p(\omega_0)$	$T=1$	$T=5$	$T=10$	$T=25$	$T=50$	$T=100$
$\omega_c + 0.25(1/\epsilon - \omega_c)$	0.0564	0.679	0.045	0.308	0.425	0.541	0.601	0.640
midpoint	0.0617	0.413	0.000	0.023	0.084	0.202	0.285	0.347
$\omega_c + 0.75(1/\epsilon - \omega_c)$	0.0671	0.190	0.000	0.000	0.001	0.024	0.067	0.118
$1/\epsilon - 0.002$	0.0705	0.067	0.000	0.000	0.000	0.000	0.005	0.021

Table 18: Tail probability $\mathbb{P}(\tau_c \le T \mid \omega_0)$ at the empirical calibration $\epsilon = 13.80$, $\sigma = 0.15$, $\varphi = 1$, $\omega_c = 0.051$. Column $p(\omega_0)$ is the asymptotic limit $\lim_{T\to\infty} \mathbb{P}(\tau_c \le T \mid \omega_0)$ from Proposition 21; individual entries approach this limit from below. The last column shows that even at $T = 100$ years the approach is incomplete for $\omega_0 \ge$ midpoint, reflecting the small essential-spectrum bottom $\lambda_* = \sigma^2/8 \approx 0.00281$ and the $\approx$ 356-year characteristic time (Proposition 28; see below). Horizons: years.

0.70 *at* $T = 5$ *and* 0.84 *at* $T = 10$—larger *than the exact values, consistent with the positive drift reducing early-crisis probability relative to the drift-free bound.*

Long-horizon decay rate: analytical, not FD. *Proposition 28 replaces the earlier (incorrect) attempt to extract a "principal eigenvalue" by finite-difference discretisation of* $-\mathcal{L}$*: the spectrum below* $\sigma^2/8$ *is not discrete, so no cutoff-dependent* $\lambda_1 \in [0.004, 0.007]$ *exists as an isolated eigenvalue. The correct long-horizon exponential decay rate is the essential-spectrum bottom* $\lambda_* = \sigma^2/8$*,* φ*-universal. At the calibration* $\sigma = 0.15$*,*

$$\lambda_* = \frac{\sigma^2}{8} = \frac{0.0225}{8} \approx 2.81\times 10^{-3}, \qquad \textit{long-time characteristic} \quad \frac{1}{\lambda_*} = \frac{8}{\sigma^2} \approx 356 \textit{ years.} \tag{114}$$

The associated e-fold decay is at $T \approx 356$ *years; equivalently, a half-life of* $\ln 2 \cdot 8/\sigma^2 \approx 247$ *years. Crucially, the Table entries at* $T \in \{1, 5, 10, 25, 50, 100\}$ *are* insensitive *to this asymptotic rate:* $\sigma^2 T/8$ *equals* 0.00281, 0.0141, 0.0281, 0.0703, 0.141, 0.281 *at those horizons, so* $e^{-\sigma^2 T/8} \in [0.755, 0.997]$*. For* $T \le 50$*, the exponential correction is well under* 15%*, and the tail-probability magnitudes are set by the short-to-intermediate-horizon dynamics of the Laplace inversion, not by the spectrum-edge rate. The Gaver–Stehfest inversion solves (93) at each Stehfest node directly and does not invoke any spectral-gap estimate; entries* $T \in \{50, 100\}$ *in Table 18 are from this backward-equation inversion and are* not *based on the incorrect FD principal eigenvalue.*

Honest caveats. *(i) The empirical anchor* $\omega_c = 0.051$ *is taken from the AFA-submission empirical appendix; the conference-version calibration with* $\omega_c = 0.078$*,* $1/\epsilon = 0.10$ *at* $\epsilon = 10$ *yields a wider pre-failure interval and less binding tail probabilities. The long-time rate* $\sigma^2/8$ *depends only on* σ *and is therefore unchanged under this alternative. (ii) The algebraic prefactor* α *in (107) is not analytically determined here (Remark 11); the exponential rate* $\sigma^2/8$ *is. (iii) The* $\varphi = 0$ *closed form (96) does* not *apply at the empirical* $\varphi = 1$*; the*

numerical inversion via (93) is the correct route at $\varphi > 0$, and the analytical decay rate $\sigma^2/8$ is φ-universal by Proposition 28.

Cross-references. Table 18 is the tail-probability analogue of Proposition 21 (column $p(\omega_0)$, $T \to \infty$ limit) and Proposition 23 (first-moment $\int_0^\infty T\, dF_T$, infinite by Theorem 22). It complements Appendix N.9, which computes the *deterministic* drift time on the QE-recovery region $(\underline{\omega}, \omega_c)$; the present appendix is about stochastic hitting on the pre-failure region $(\omega_c, 1/\epsilon)$.

H.8 Economic interpretation

Proposition 21 says: the probability of an eventual crisis is exactly the fraction of the distance—measured in the scale coordinate $Q = 1/\omega$—that the economy has already covered from the frictionless steady state $1/\epsilon$ toward the crisis threshold $1/\omega_c$. The scale coordinate is the natural metric because Q_t is a martingale. In the raw wealth-share coordinate, the probability is $\omega_c(1 - \epsilon\omega_0)/[\omega_0(1 - \epsilon\omega_c)]$: it falls as ω_0 rises and falls as ω_c falls. Theorem 22 sharpens the probabilistic picture: although a crisis occurs with strictly positive probability for every $\omega_0 \in I$, it occurs in *expected* finite time only on the conditional event $\{\tau_c < \infty\}$; averaged unconditionally, the expected time is infinite because the drift pulls the economy toward $1/\epsilon$ where it makes ergodically long excursions. Proposition 23 quantifies the conditional expectation via a Green's-function integral in the speed-measure-weighted scale coordinate; Proposition 24 specialises this to an elementary formula at $\varphi = 0$. Taken together, these results show that the main paper's Theorem 1 delivers positive-probability failure with a precisely characterised timing distribution, consistent with the narrative in §3.1 that decades of pre-2008 observations with $\omega_t \gg \omega_c$ are fully compatible with a finite—but conditionally-distributed—τ_c.

The full tail distribution characterised in Propositions 25–28 adds the horizon-specific information policy makers need. Hitting probability and conditional expected time are the $T \to \infty$ limit and first moment of the same object: the tail distribution $T \mapsto \mathbb{P}(\tau_c \leq T \mid \omega_0)$ governed by the backward equation (93). The tail sits between two asymptotic regimes. At short T, Proposition 27 shows the log-probability is quadratic in the Riemannian distance to ω_c and linear in $1/T$ with rate $1/\sigma^2$—small-noise logic: many standard deviations of adverse shocks in a short window are exponentially rare. At long T, the conditional tail decays as $T^{-\alpha}e^{-\sigma^2 T/8}$ with the exponential rate $\sigma^2/8$ equal to the essential-spectrum bottom of $-\mathcal{L}$ (Proposition 28), a φ-universal analytical constant. The long-time characteristic is $1/(\sigma^2/8) = 8/\sigma^2 \approx 356$ years at $\sigma = 0.15$, so the policy-horizon tail probabilities in Table 18 ($T \leq 50$) are determined not by the asymptotic rate but by the short-to-

intermediate dynamics of the Laplace inversion: the 5-year crisis probability from $\omega_0 = 0.056$ is 31%, the 10-year is 42%, and the 25-year 54%, with the ultimate 68% limit approached on decadal-to-centennial timescales. The σ-scaling from Proposition 27 then says that even modest increases in fundamental volatility move the policy-relevant medium-horizon tail substantially—more than the hitting probability or the conditional mean separately would suggest; the same σ-dependence appears, through the $\sigma^2/8$ rate, in the asymptotic tail at the opposite horizon limit.

I Boundary Classification

Proof of Theorem 5. At $\omega = \underline{\omega}$: $\sigma_\omega(\underline{\omega}) = 0$ (Proposition 29), so the diffusion coefficient vanishes. The drift relation $\mu_\omega = \sigma_\omega^2/\omega$ then gives $\mu_\omega(\underline{\omega}) = 0$. Both drift and diffusion vanish: $\omega_t = \underline{\omega}$ is a constant solution to the SDE (22). By pathwise uniqueness (Karatzas and Shreve, 1991, Theorem 5.2.5), this is the unique solution starting at $\underline{\omega}$. An identical argument applies at $\omega = 1/\epsilon$.

For uniqueness: at any other point $\omega^* \in (0,1)\setminus\{\underline{\omega}, 1/\epsilon\}$, Proposition 29 gives $\sigma_\omega(\omega^*) \neq 0$, so the diffusion does not vanish and ω^* is not absorbing. □

Proof of Theorem 30 and Proposition 31. We use the Feller boundary classification for one-dimensional diffusions (Karlin and Taylor, 1981, Ch. 15), (Borodin and Salminen, 2002, Ch. II). For a diffusion $\mathrm{d}X = \mu(X)\,\mathrm{d}t + \sigma(X)\,\mathrm{d}Z$ on an interval (l, r), the boundary classification depends on the scale function density $s'(x) = \exp(-\int^x 2\mu(y)/\sigma(y)^2\,\mathrm{d}y)$ and the speed measure density $m(x) = 1/(\sigma(x)^2 s'(x))$. A boundary is *unattainable* (entrance or natural) if $\Sigma = \int s'(x)\,\mathrm{d}x$ diverges at the boundary, or if $N = \int s'(x)\int m(y)\,\mathrm{d}y\,\mathrm{d}x$ diverges. A boundary where both Σ and N diverge is called *natural*.

For our diffusion (22), $\mu_\omega = \sigma_\omega^2/\omega$, so $2\mu_\omega/\sigma_\omega^2 = 2/\omega$. Therefore $s'(\omega) = \exp(-2\ln\omega) = C/\omega^2$ (normalised at ω_0: $s'(\omega) = \omega_0^2/\omega^2$) and $m(\omega) = \omega^2/(\omega_0^2\sigma_\omega(\omega)^2)$.

(i) $\omega = 0$ (natural boundary). Near $\omega = 0$, the constrained formula gives $\sigma_\omega^c = \sigma(1+\varphi)\lambda(1-\omega)\omega(\omega-\underline{\omega})/D_c(\omega)$. Substituting $\omega \to 0$: $(1-\omega)\to 1$, $(\omega-\underline{\omega})\to -\underline{\omega}$, $D_c(0) = (1+\varphi)$. Therefore $\sigma_\omega \approx \sigma(1+\varphi)\lambda\cdot 1\cdot\omega\cdot(-\underline{\omega})/(1+\varphi) = -\sigma\lambda\underline{\omega}\cdot\omega \equiv -c_0\omega$ with $c_0 = \sigma\lambda\underline{\omega} > 0$. There exist $\underline{c}, \bar{c} > 0$ with $\underline{c}\,\omega \le |\sigma_\omega| \le \bar{c}\,\omega$ on $(0, a]$ for some $a > 0$.

$\Sigma(0) = \int_0^a s'(x)\,\mathrm{d}x = \omega_0^2\int_0^a x^{-2}\,\mathrm{d}x = +\infty$.

$m(\omega) \ge \omega^2/(\omega_0^2\bar{c}^2\omega^2) = 1/(\omega_0^2\bar{c}^2) \equiv K > 0$. Hence $\int_0^x m(y)\,\mathrm{d}y \ge Kx$ and

$$N(0) = \int_0^a s'(x)\int_0^x m(y)\,\mathrm{d}y\,\mathrm{d}x \ge K\int_0^a \frac{\omega_0^2}{x^2}\cdot x\,\mathrm{d}x = K\omega_0^2\int_0^a x^{-1}\,\mathrm{d}x = +\infty.$$

Both Σ and N diverge: $\omega = 0$ is a natural boundary, unattainable from the interior.

(ii) $\omega = \underline{\omega}$ **(unattainable from both sides).** The constrained formula $\sigma_\omega^c = \sigma(1+\varphi)\lambda(1-\omega)\omega(\omega-\underline{\omega})/D_c(\omega)$ factors as $\sigma_\omega = g(\omega)(\omega-\underline{\omega})$ with $g(\omega) = \sigma(1+\varphi)\lambda(1-\omega)\omega/D_c(\omega)$. Evaluating at $\underline{\omega}$: $D_c(\underline{\omega}) = (1+\varphi)(1-\underline{\omega})$, so $g(\underline{\omega}) = \sigma(1+\varphi)\lambda(1-\underline{\omega})\underline{\omega}/[(1+\varphi)(1-\underline{\omega})] = \sigma\lambda\underline{\omega} > 0$. There exist $\underline{g}, \bar{g} > 0$ with $\underline{g} \le |g(\omega)| \le \bar{g}$ for ω near $\underline{\omega}$.

Unattainability from the right. We show that the speed measure integral diverges, which forces $N(\underline{\omega}^+) = +\infty$ in the Feller classification. Near $\underline{\omega}$ from the right: $m(\omega) = \omega^2/(\omega_0^2 g(\omega)^2(\omega-\underline{\omega})^2) \ge \underline{\omega}^2/(\omega_0^2\bar{g}^2(\omega-\underline{\omega})^2)$. Therefore:

$$\int_{\underline{\omega}}^{x} m(y)\,\mathrm{d}y \ge \frac{\underline{\omega}^2}{\omega_0^2\bar{g}^2}\int_{\underline{\omega}}^{x}\frac{1}{(y-\underline{\omega})^2}\,\mathrm{d}y = \frac{\underline{\omega}^2}{\omega_0^2\bar{g}^2}\cdot\frac{1}{x-\underline{\omega}} \to +\infty \quad \text{as } x \to \underline{\omega}^+.$$

Since $s'(\omega) = \omega_0^2/\omega^2 \ge \omega_0^2/b^2 > 0$ for $\omega \in [\underline{\omega}, b]$, it follows that the Feller integral is $N(\underline{\omega}^+) = \int_{\underline{\omega}}^{b} s'(x)\int_{\underline{\omega}}^{x} m(y)\,\mathrm{d}y\,\mathrm{d}x = +\infty$. Hence $\underline{\omega}$ is unattainable from the right. An identical argument (replacing $(\omega-\underline{\omega})$ by $(\underline{\omega}-\omega)$ in the crisis regime where $\sigma_\omega = g(\omega)(\omega-\underline{\omega})$ with $g > 0$ and $\omega < \underline{\omega}$ so $|\sigma_\omega| = |g|(\underline{\omega}-\omega)$) shows unattainability from the left.

(iii) $\omega = 1/\epsilon$ **(unattainable from both sides).** The unconstrained formula $\sigma_\omega^u = \sigma(1+\varphi)(1-\omega)(1/\epsilon-\omega)/D_u(\omega)$ factors as $\sigma_\omega = g(\omega)(1/\epsilon-\omega)$ with $g(\omega) = \sigma(1+\varphi)(1-\omega)/D_u(\omega)$. At $\omega = 1/\epsilon$: $D_u(1/\epsilon) = (1+\varphi)(1-1/\epsilon)$, so $g(1/\epsilon) = \sigma(1+\varphi)(1-1/\epsilon)/[(1+\varphi)(1-1/\epsilon)] = \sigma > 0$. The analysis is identical to (ii): $m(\omega) \ge K(1/\epsilon-\omega)^{-2}$ near $1/\epsilon$ from the left, so $\int m(y)\,\mathrm{d}y$ diverges and $N(1/\epsilon^-) = +\infty$. Symmetrically from the right.

(iv) $\omega = 1$ **(unattainable).** The unconstrained formula factors as $\sigma_\omega = h(\omega)(1-\omega)$ with $h(\omega) = \sigma(1+\varphi)(1/\epsilon-\omega)/D_u(\omega)$. At $\omega = 1$: $D_u(1) = (1-1/\epsilon)$, so $h(1) = \sigma(1+\varphi)(1/\epsilon-1)/(1-1/\epsilon) = -\sigma(1+\varphi) \ne 0$. Near $\omega = 1$: $m(\omega) \ge K(1-\omega)^{-2}$, giving $N(1^-) = +\infty$. Unattainable.

Regime invariance. The process ω_t has continuous paths (it is a strong solution of an SDE with continuous coefficients on the interior of each regime). If $\omega_0 \in (0, \underline{\omega})$ and $\omega_t \notin (0, \underline{\omega})$ for some $t > 0$, then by continuity there exists $s \in (0, t)$ with $\omega_s \in \{0, \underline{\omega}\}$, contradicting the unattainability of both boundaries. The same argument applies to $(\underline{\omega}, 1/\epsilon)$ and $(1/\epsilon, 1)$.

□

J Convergence

Proof of Theorem 6. **Almost-sure convergence.** $Q_t = 1/\omega_t$ is a non-negative local martingale (Proposition 4), hence a supermartingale by Fatou's lemma. By the supermartingale convergence theorem (Karatzas and Shreve, 1991, Ch. 1), $Q_t \to Q_\infty$ a.s. with $\mathbb{E}[Q_\infty] \le Q_0 <$

∞, so $Q_\infty < \infty$ a.s. and $\omega_\infty \equiv 1/Q_\infty > 0$.

The limit is a zero of σ_ω. The quadratic variation of Q is $[Q]_t = \int_0^t (\sigma_\omega(\omega_s)/\omega_s^2)^2 \, \mathrm{d}s$. We claim $[Q]_\infty < \infty$ a.s. By the Dambis–Dubins–Schwarz representation (Karatzas and Shreve, 1991, Ch. 3), there exists a Brownian motion B such that $Q_t = Q_0 + B_{[Q]_t}$ for all $t \geq 0$. If $[Q]_\infty = +\infty$, then $Q_t = Q_0 + B_{[Q]_t}$ where B is evaluated at a time tending to infinity; since Brownian motion does not converge (it is neighbourhood-recurrent on $\mathbb{R}$, Karatzas and Shreve, 1991, Theorem 2.9.23), this contradicts the a.s. convergence $Q_t \to Q_\infty$. Therefore $[Q]_\infty < \infty$ a.s.

Since $\omega_t \to \omega_\infty$ and the integrand $(\sigma_\omega(\omega)/\omega^2)^2$ is continuous, the convergence $[Q]_\infty < \infty$ requires $\sigma_\omega(\omega_\infty) = 0$: if $\sigma_\omega(\omega_\infty) \neq 0$, then by continuity the integrand would be bounded below by a positive constant for all large t, and the integral would diverge. By Proposition 29, the zeros of σ_ω in $[0, 1)$ are $\{0, \underline{\omega}, 1/\epsilon\}$. Since $\omega_\infty > 0$, we have $\omega_\infty \in \{\underline{\omega}, 1/\epsilon\}$.

Regime-specific limits.

Crisis regime $(0, \underline{\omega})$*:* ω_t remains in $(0, \underline{\omega})$ for all t (regime invariance, Proposition 31). The only zero of σ_ω accessible from this regime is $\underline{\omega}$ (since $\omega_\infty > 0$ and $1/\epsilon \notin [0, \underline{\omega}]$). Hence $\omega_t \to \underline{\omega}$.

Recovery regime $(\underline{\omega}, 1/\epsilon)$*:* ω_t remains in $(\underline{\omega}, 1/\epsilon)$. Since $\underline{\omega}$ is an entrance boundary (Proposition 31), the process cannot approach it from the interior. Moreover, in this regime the drift $\mu_\omega = \sigma_\omega^2/\omega > 0$ pushes ω away from $\underline{\omega}$, so $\omega_\infty \neq \underline{\omega}$. Hence $\omega_t \to 1/\epsilon$.

Inverted regime $(1/\epsilon, 1)$*:* ω_t remains in $(1/\epsilon, 1)$. Since 1 is unattainable, $\omega_\infty \neq 1$. The only zero of σ_ω in $[1/\epsilon, 1)$ is $1/\epsilon$. Hence $\omega_t \to 1/\epsilon$.

Infinite expected hitting time. Let $\omega^* \in \{\underline{\omega}, 1/\epsilon\}$ be the relevant absorbing state. Near ω^*: $\sigma_\omega(\omega) = g(\omega)(\omega^* - \omega)$ with g continuous and bounded between $\underline{g}, \bar{g} > 0$ (Proposition 31). The speed measure density satisfies $m(\omega) = \omega^2/(\omega_0^2 \sigma_\omega^2) \geq (\omega^*)^2/(2\omega_0^2 \bar{g}^2 (\omega^* - \omega)^2)$ for ω close to ω^*. By the Green's function formula for expected hitting times (Karlin and Taylor, 1981, Ch. 15):

$$\mathbb{E}_{\omega_0}[\tau_{\omega^*}] = \int_{\omega_0}^{\omega^*} \int_{\omega_0}^{y} 2\, m(z)\, s'(z)\, \mathrm{d}z \, [s(y) - s(\omega_0)] \, \mathrm{d}y + \int_{\omega_0}^{\omega^*} [s(\omega^*) - s(y)]\, 2\, m(y)\, \mathrm{d}y.$$

The second integral satisfies $\int_a^{\omega^*} [s(\omega^*) - s(y)]\, m(y)\, \mathrm{d}y \geq K \int_a^{\omega^*} (\omega^* - y)^{-1} \, \mathrm{d}y = +\infty$, where we used $s(\omega^*) - s(y) \sim s'(\omega^*)(\omega^* - y)$ for y near ω^* and $m(y) \geq K'(\omega^* - y)^{-2}$, giving an integrand of order $(\omega^* - y)^{-1}$. Therefore $\mathbb{E}[\tau] = +\infty$. $\square$

K Proof of Intermediation Gap Classification

Proof of Theorem 9. By Proposition 4, $Q = 1/\omega$ is a non-negative continuous local martingale. By (32), the intermediation gap is positive if and only if Q is a strict local martingale. We classify Q using the integral test of Mijatović and Urusov (2012): a non-negative continuous local martingale Q with $\mathrm{d}Q = -(\sigma_\omega/\omega^2)\,\mathrm{d}Z$ is a strict local martingale if and only if

$$\int_c^\infty \frac{q}{\sigma_Q(q)^2}\,\mathrm{d}q < \infty \tag{115}$$

for some (hence all) $c > 0$, where $\sigma_Q(q)$ is the diffusion coefficient of Q expressed as a function of Q.

Since $\sigma_Q = \sigma_\omega/\omega^2$ and $Q = 1/\omega$, we have $\sigma_Q(Q) = Q^2\,|\sigma_\omega(1/Q)|$. The integral becomes:

$$\int_c^\infty \frac{q}{q^4\,\sigma_\omega(1/q)^2}\,\mathrm{d}q = \int_c^\infty \frac{1}{q^3\,\sigma_\omega(1/q)^2}\,\mathrm{d}q.$$

(i) Baseline model. Near $\omega = 0$ (equivalently, $Q \to \infty$), the baseline volatility (56) satisfies $\sigma_\omega^0(\omega) \to c_0 > 0$ where $c_0 = \sigma(1+\varphi)(1/\epsilon)/[(1-1/\epsilon)+\varphi]$. Therefore $\sigma_\omega^0(1/q) \to c_0$ as $q \to \infty$, and

$$\int_c^\infty \frac{1}{q^3\,c_0^2}\,\mathrm{d}q = \frac{1}{2c_0^2\,c^2} < \infty.$$

By the integral test, Q is a strict local martingale. A positive intermediation gap exists.

(ii) Constrained model. Near $\omega = 0$, the constrained volatility satisfies $|\sigma_\omega(\omega)| \sim c_0'\omega$ with $c_0' = \sigma\lambda\underline{\omega} > 0$ (Proposition 29). Therefore $|\sigma_\omega(1/q)| \sim c_0'/q$ as $q \to \infty$, and

$$\int_c^\infty \frac{1}{q^3\,(c_0'/q)^2}\,\mathrm{d}q = \frac{1}{(c_0')^2}\int_c^\infty \frac{1}{q}\,\mathrm{d}q = +\infty.$$

By the integral test, Q is a true martingale. The intermediation gap is zero.

The mechanism is transparent: the leverage constraint forces $\phi \le \bar{\phi}(\omega) \to 1 - \lambda\underline{\omega}$ as $\omega \to 0$, so the financier is de-leveraged when her capital is depleted. Since $\sigma_\omega = \omega(\phi - 1)\sigma_R$ and $\phi - 1$ remains bounded, the extra factor of ω survives, driving $\sigma_\omega \propto \omega \to 0$. In the baseline, $\phi = 1/(\epsilon\omega) \to \infty$: the ω cancels and σ_ω stays positive. One factor of ω is the difference between convergent and divergent integrals, between a positive intermediation gap and a zero gap.

For the three individual regimes of the constrained model: in the recovery and expansionary regimes, Q ranges over a bounded interval ($Q \in (\epsilon, 1/\underline{\omega})$ or $Q \in (1, \epsilon)$), so Q is bounded and hence a true martingale. In the crisis regime $\omega \in (0, \underline{\omega})$, Q is unbounded above, but the true martingale property follows from the divergent integral established above. □

L Robustness and Additional Diagnostics

This appendix reports robustness checks, inference diagnostics, out-of-sample evidence, and auxiliary tests referenced in the main text.

L.1 Alternative Specifications

Test 1. We report the quadratic specification (Table 19), the unemployment-gap specification (Table 21), and the control specification (Table 22).

Table 19: Test 1: nonlinear specification

	h=1	h=2	h=4	h=8
Constant	-0.058***	-0.060***	-0.062***	-0.058***
	(0.018)	(0.019)	(0.018)	(0.019)
OmegaHkm	1.196***	1.255**	1.342***	1.316**
	(0.460)	(0.490)	(0.462)	(0.555)
OmegaHkmSq	-5.184*	-5.618*	-6.311**	-6.973*
	(2.706)	(2.949)	(2.896)	(3.718)
Adj. R^2	0.256	0.256	0.251	0.152
No. Obs.	219	218	216	212

Table 20: Test 1: state dependence (piecewise linear, no controls, low-10% threshold)

	h=1	h=2	h=4	h=8
Constant	-0.030***	-0.028***	-0.029***	-0.020**
	(0.006)	(0.006)	(0.008)	(0.009)
OmegaHkm	0.390***	0.369***	0.381***	0.244*
	(0.081)	(0.086)	(0.108)	(0.128)
OmegaHkmxOmegaLowQuant	-0.527	-0.706**	-0.435	-0.590**
	(0.323)	(0.324)	(0.288)	(0.280)
Adj. R^2	0.255	0.273	0.228	0.136
No. Obs.	219	218	216	212

Table 21: Test 1: alternative dependent variable (unemployment gap)

	h=1	h=2	h=4	h=8
Constant	0.029***	0.029***	0.029***	0.025***
	(0.005)	(0.005)	(0.005)	(0.006)
OmegaHkm	-0.348***	-0.347***	-0.355***	-0.286***
	(0.065)	(0.064)	(0.069)	(0.082)
Adj. R^2	0.243	0.244	0.256	0.165
No. Obs.	219	218	216	212

Table 22: Test 1: with controls

	h=1	h=2	h=4	h=8
Constant	0.012	0.007	-0.005	-0.021
	(0.010)	(0.011)	(0.012)	(0.013)
OmegaHkm	0.203**	0.208**	0.246**	0.231*
	(0.082)	(0.092)	(0.111)	(0.121)
Yield3m	0.082	0.066	0.038	0.005
	(0.065)	(0.069)	(0.075)	(0.079)
TermSpread	-0.362***	-0.182	0.145	0.577***
	(0.128)	(0.150)	(0.191)	(0.210)
CreditSpread	-2.245***	-2.350***	-2.056***	-1.212**
	(0.777)	(0.726)	(0.715)	(0.497)
SpxVol	-0.042	-0.013	0.015	0.030
	(0.047)	(0.045)	(0.049)	(0.045)
Adj. R^2	0.533	0.431	0.307	0.217
No. Obs.	219	218	216	212

Test 4. We report a control specification for the model-implied proxy, two auxiliary diagnostics based on CAPE and credit spreads, and horse-race specifications that include both the HKM ratio and the intermediation gap proxy.

Table 23: Test 4: model-implied proxy (baseline, $\gamma = 4$)

	h=1	h=2	h=4	h=8
Constant	0.083***	0.087***	0.085***	0.065*
	(0.024)	(0.026)	(0.028)	(0.038)
BubbleApproxAlt	-0.113***	-0.119***	-0.116***	-0.090*
	(0.034)	(0.036)	(0.038)	(0.050)
Adj. R^2	0.203	0.223	0.213	0.123
No. Obs.	151	150	148	144

Table 24: Test 4: CAPE proxy (diagnostic, baseline)

	h=1	h=2	h=4	h=8
Constant	0.001	0.001	0.001	0.000
	(0.002)	(0.002)	(0.003)	(0.003)
BpCape	0.019***	0.020***	0.019***	0.014**
	(0.005)	(0.006)	(0.005)	(0.006)
Adj. R^2	0.314	0.335	0.299	0.174
No. Obs.	151	150	148	144

Table 25: Test 4: model-implied proxy (with controls, appendix robustness)

	h=1	h=2	h=4	h=8
Constant	0.065*	0.069*	0.060	0.041
	(0.034)	(0.041)	(0.044)	(0.044)
BubbleApprox	-0.035	-0.042	-0.041	-0.040
	(0.045)	(0.055)	(0.061)	(0.056)
Yield3m	0.024	-0.064	-0.173	-0.220*
	(0.092)	(0.117)	(0.140)	(0.119)
TermSpread	-0.281	-0.129	0.163	0.542**
	(0.188)	(0.215)	(0.234)	(0.212)
CreditSpread	-3.188***	-2.539***	-1.746**	-0.555
	(0.726)	(0.847)	(0.820)	(0.706)
SpxVol	-0.019	-0.039	-0.028	-0.005
	(0.060)	(0.068)	(0.069)	(0.047)
Adj. R^2	0.578	0.475	0.406	0.406
No. Obs.	151	150	148	144

Test 2 robustness. We report results using the raw HKM ratio (which delivers the predicted negative sign but weaker precision), the baseline specification using $\widehat{Q}_t^{HKM}$, and a shortfall-based predictor.

Table 26: Test 2 robustness: baseline (Shortfall)

	h=1	h=2	h=4	h=8
Constant	0.013**	0.023*	0.052**	0.120**
	(0.006)	(0.013)	(0.026)	(0.060)
Shortfall	0.687	1.822	3.205	4.912
	(0.571)	(1.138)	(2.006)	(4.144)
Adj. R^2	0.003	0.019	0.028	0.029
No. Obs.	219	218	216	212

Table 27: Test 2: with controls (QHkm; higher QHkm = lower intermediary wealth share)

	h=1	h=2	h=4	h=8
Constant	-0.015	-0.032	-0.062	-0.047
	(0.018)	(0.034)	(0.065)	(0.145)
QHkm	0.001	0.003	0.006*	0.014**
	(0.001)	(0.002)	(0.003)	(0.007)
Yield3m	-0.298	-0.603	-0.864	-1.321
	(0.213)	(0.370)	(0.625)	(1.089)
TermSpread	-0.031	-0.123	0.800	3.420
	(0.434)	(0.816)	(1.618)	(3.068)
CreditSpread	-0.279	0.661	-1.682	-7.602
	(2.547)	(3.446)	(4.688)	(6.984)
SpxVol	0.222*	0.280	0.583*	0.261
	(0.122)	(0.199)	(0.349)	(0.445)
Adj. R^2	0.025	0.052	0.089	0.107
No. Obs.	219	218	216	212

Test 3 robustness. We report a control specification and an asymmetric "positive changes only" version of the QE reaction regression.

L.2 Inference Robustness

Predictive regressions with persistent predictors are subject to well-known inference challenges (Stambaugh, 1999; Lewellen, 2004; Campbell and Yogo, 2006). Tables 28–29 report

six inference methods for the baseline specification of each test: Newey–West HAC, Hodrick (1992) reverse-regression standard errors, Stambaugh (1999) bias-corrected estimates with HAC t-statistics, Lewellen (2004) conservative confidence intervals assuming a unit-root predictor, Campbell–Yogo (2006) Bonferroni Q-tests, and bootstrap p-values (wild bootstrap with Rademacher weights and, for $h > 1$, moving-block bootstrap under the null).

Table 28: Inference robustness: Test 1 (HKM capital ratio → output gap)

	$h = 1$	$h = 2$	$h = 4$	$h = 8$
$\hat{\beta}_{OLS}$	0.441	0.437	0.423	0.302
t_{NW}	4.66***	4.33***	3.86***	2.22**
$t_{Hodrick}$	8.60	5.07	3.30	1.89
$\hat{\beta}_{Stambaugh}$	0.438	0.434	0.423	0.304
$t_{Stambaugh}$	4.63	4.29	3.86	2.23
Lewellen CI	[0.25, 0.62]	[0.24, 0.63]	[0.21, 0.64]	[0.04, 0.57]
Lewellen reject H_0	Yes	Yes	Yes	Yes
C-Y reject H_0	Yes	Yes	Yes	Yes
Wild bootstrap p	<0.001	<0.001	<0.001	<0.001
Block bootstrap p	—	<0.001	<0.001	0.028
R^2	0.234	0.230	0.216	0.110
N	219	218	216	212

Notes: t_{NW} uses Newey–West HAC standard errors with $\max(h, 4)$ lags. $t_{Hodrick}$ uses Hodrick (1992) reverse-regression standard errors. Stambaugh (1999) bias-corrected coefficient and t-statistic use HAC SE. Lewellen (2004) 95% CI assumes a unit-root predictor. C-Y denotes the Campbell–Yogo (2006) Bonferroni Q-test. Wild bootstrap uses Rademacher weights under $H_0 : \beta = 0$ (5000 replications). Block bootstrap resamples blocks of length h under H_0 (5000 replications) and is reported only for $h > 1$. */**/*** denote significance at 10%/5%/1%.

Table 29: Inference robustness: Test 4 (model-implied gap proxy → output gap, pre-QE)

	$h = 1$	$h = 2$	$h = 4$	$h = 8$
$\hat{\beta}_{OLS}$	-0.161	-0.168	-0.164	-0.126
t_{NW}	-3.39***	-3.30***	-3.02***	-1.77*
$t_{Hodrick}$	-6.53	-3.87	-2.57	-1.50
$\hat{\beta}_{Stambaugh}$	-0.157	-0.166	-0.164	-0.126
$t_{Stambaugh}$	-3.32	-3.26	-3.01	-1.78
Lewellen CI	[-0.25, -0.06]	[-0.27, -0.07]	[-0.27, -0.06]	[-0.27, 0.01]
Lewellen reject H_0	Yes	Yes	Yes	No
C-Y reject H_0	Yes	Yes	Yes	Yes
Wild bootstrap p	<0.001	<0.001	<0.001	<0.001
Block bootstrap p	—	<0.001	<0.001	0.060
R^2	0.206	0.225	0.215	0.125
N	151	150	148	144

Notes: t_{NW} uses Newey–West HAC standard errors with $\max(h, 4)$ lags. $t_{Hodrick}$ uses Hodrick (1992) reverse-regression standard errors. Stambaugh (1999) bias-corrected coefficient and t-statistic use HAC SE. Lewellen (2004) 95% CI assumes a unit-root predictor. C-Y denotes the Campbell–Yogo (2006) Bonferroni Q-test. Wild bootstrap uses Rademacher weights under $H_0 : \beta = 0$ (5000 replications). Block bootstrap resamples blocks of length h under H_0 (5000 replications) and is reported only for $h > 1$. */**/*** denote significance at 10%/5%/1%.

L.3 Out-of-Sample Evidence

Table 30 reports recursive expanding-window forecasts for Tests 1, 2, and 4. For each horizon h and forecast origin t, we estimate

$$y_{s+h} = a_{t,h} + b_{t,h} z_s + u_{s+h}, \qquad s = 1, \ldots, t - h, \tag{116}$$

using all observations available up to t and an initial training window of 60 quarters. The resulting model forecast is:

$$\widehat{y}^{M}_{t+h|t} = \widehat{a}_{t,h} + \widehat{b}_{t,h} z_t. \tag{117}$$

For Tests 1 and 4, the target is the future output gap, $y_{t+h} = \widehat{x}_{t+h}$. For Test 2, the target is the cumulative excess return $\widehat{R}^{e}_{t \to t+h}$ defined in Online Appendix Section L.1. The predictor z_t is, respectively, the HKM capital ratio $\widehat{\omega}_t$, the capital scarcity measure $\widehat{Q}^{HKM}_t$, the approximate gap proxy $(1 - \widehat{\omega}_t)^{\gamma}$, or the exact gap proxy $b(\widehat{\omega}_t)$.

We compare the model forecast with two recursive benchmarks. The prevailing-mean

benchmark is

$$\widehat{y}^{PM}_{t+h|t} = \frac{1}{N_t}\sum_{s\le t} y_s. \tag{118}$$

The AR(1) benchmark is obtained by estimating

$$y_{s+1} = c_t + \rho_t y_s + v_{s+1} \tag{119}$$

on the benchmark series available at forecast origin t and iterating the fitted law of motion forward:

$$\widehat{y}^{AR1}_{t+h|t} = \widehat{c}_t \sum_{j=0}^{h-1} \widehat{\rho}_t^{\,j} + \widehat{\rho}_t^{\,h} y_t. \tag{120}$$

For Tests 1 and 4, the AR(1) benchmark series is the output gap itself. For Test 2, it is the aligned h-quarter excess-return series, using only observations realised by the forecast origin.

Forecast performance is summarised by

$$R^2_{OOS,B} = 1 - \frac{\sum_t \left(y_{t+h} - \widehat{y}^{M}_{t+h|t}\right)^2}{\sum_t \left(y_{t+h} - \widehat{y}^{B}_{t+h|t}\right)^2}, \qquad B \in \{PM, AR1\}. \tag{121}$$

For the prevailing-mean benchmark, we report the Clark–West statistic based on the adjusted loss differential

$$f_t = e^2_{B,t} - \left[e^2_{M,t} - \left(\widehat{y}^{B}_{t+h|t} - \widehat{y}^{M}_{t+h|t}\right)^2\right]. \tag{122}$$

For the AR(1) benchmark, we report the Harvey–Leybourne–Newbold-adjusted Diebold–Mariano statistic based on the loss differential

$$d_t = e^2_{B,t} - e^2_{M,t}, \tag{123}$$

with the one-sided alternative that the model forecast has lower MSPE than the benchmark.

Table 30: Out-of-sample predictive performance

	$h = 1$	$h = 2$	$h = 4$	$h = 8$
Test 1: HKM Capital Ratio → Output Gap				
R^2_{OOS} vs. prev. mean	0.248	0.181	0.093	-0.092
CW stat	6.62***	4.47***	2.97***	1.60*
R^2_{OOS} vs. AR(1)	-1.548	-0.721	-0.212	-0.078
DM stat	-4.54	-2.64	-0.87	-0.34
Test 2: $\widehat{Q}^{HKM}$ → Excess Returns				
R^2_{OOS} vs. prev. mean	-0.008	-0.015	-0.034	-0.114
CW stat	-0.02	0.38	-0.07	-0.91
R^2_{OOS} vs. AR(1)	-0.001	0.038	0.090	0.044
DM stat	-0.05	0.43	0.68	0.28
Test 4: Approximate Gap Proxy $(1-\widehat{\omega})^{2.5}$ → Output Gap (pre-QE)				
R^2_{OOS} vs. prev. mean	0.190	0.101	-0.074	-0.339
CW stat	5.02***	3.58***	2.42***	1.39*
R^2_{OOS} vs. AR(1)	-7.039	-3.050	-1.070	-0.339
DM stat	-7.34	-4.21	-2.01	-0.63
Test 4: Exact Gap Proxy $b(\widehat{\omega})$ → Output Gap (pre-QE)				
R^2_{OOS} vs. prev. mean	0.136	0.160	0.222	0.454
CW stat	4.74***	4.00***	3.55***	3.27***
R^2_{OOS} vs. AR(1)	-7.058	-2.515	-0.483	0.423
DM stat	-6.67	-3.22	-0.91	2.16**

Notes: Out-of-sample R^2 from expanding-window recursive OLS with minimum training window of 60 quarters. "Prev. mean" benchmark uses the historical average of the dependent variable. "AR(1)" benchmark uses a recursive AR(1) forecast. CW stat is the Clark–West (2007) MSPE-adjusted test statistic for the prevailing-mean benchmark. DM stat is the Harvey–Leybourne–Newbold-adjusted Diebold–Mariano statistic for the AR(1) benchmark; */**/*** denote one-sided significance at 10%/5%/1%.

L.4 Subsample and Outlier Robustness

Test 1 robustness. We report control-heavy specifications, subsample splits (1970–1999 and 2000–2024), outlier exclusions (dropping 2008–2009 and 2020 crisis quarters), nonlinear, state-dependent, and alternative dependent variable specifications.

Table 31: Test 1: subsample 1970–1999

	h=1	h=2	h=4	h=8
Constant	-0.030***	-0.032***	-0.035***	-0.036***
	(0.009)	(0.010)	(0.010)	(0.012)
OmegaHkm	0.420***	0.464***	0.530***	0.561***
	(0.112)	(0.123)	(0.140)	(0.180)
Adj. R^2	0.215	0.253	0.280	0.221
No. Obs.	119	118	116	112

Table 32: Test 1: subsample 2000–2024

	h=1	h=2	h=4	h=8
Constant	-0.044***	-0.037***	-0.032**	-0.023
	(0.011)	(0.011)	(0.013)	(0.014)
OmegaHkm	0.545***	0.446***	0.365*	0.223
	(0.155)	(0.164)	(0.193)	(0.199)
Adj. R^2	0.284	0.193	0.129	0.041
No. Obs.	99	98	96	92

Table 33: Test 1: excluding outlier periods

	h=1	h=2	h=4	h=8
Constant	-0.031***	-0.032***	-0.031***	-0.024**
	(0.007)	(0.008)	(0.008)	(0.010)
OmegaHkm	0.403***	0.424***	0.416***	0.304**
	(0.095)	(0.103)	(0.112)	(0.143)
Adj. R^2	0.210	0.233	0.226	0.118
No. Obs.	213	212	210	206

Structural-break selection. Table 34 reports the Bai and Perron (1998) break-count selection for the mean of the HKM capital ratio underlying Table 11.

Table 34: Bai-Perron break-count selection for the mean of ω_t

Panel A: BIC model selection

Breaks	BIC	SSR	Break dates
0	-1669.08	0.1089	–
1	-1728.04	0.0813	1992-12
2	-1853.64	0.0448	1995-09,2007-03
3	-1882.42	0.0384	1996-03,2001-09,2007-09
4	-1908.39	0.0333	1996-03,2001-09,2007-12,2013-09
5	-1947.65	0.0271	1975-09,1991-06,1996-12,2002-06,2007-12
6	-1990.40	0.0218	1975-09,1991-06,1996-12,2002-06,2007-12,2013-09
7	**-2000.27**	0.0204	1975-09,1983-03,1991-06,1996-12,2002-06,2007-12,2013-09
8	-1997.64	0.0201	1975-09,1983-03,1991-06,1996-12,2002-06,2007-12,2013-09,2019-09
9	-1929.33	0.0267	1975-09,1981-03,1986-09,1992-03,1997-09,2003-03,2008-09,2014-03,2019-09

Panel B: Sequential break tests

Test	Statistic	Bootstrap p-value	Reject at 5%
FT(6—5)	52.11	0.001	Yes
FT(7—6)	15.23	0.001	Yes
FT(8—7)	2.67	0.119	No

Notes: Panel A reports BIC and global least-squares break dates for each admissible break count under trimming parameter 0.10. The requested upper bound is 9 breaks and the effective admissible upper bound is 9. Panel B reports sequential $FT(l+1 \mid l)$ statistics with bootstrap p-values computed under the segmented-mean null.

L.5 QE Trap Additional Tests

QT1 episode study. Table 35 reports the simple descriptive evidence for the Federal Reserve's first quantitative tightening episode. In addition to the start and end values of the HKM capital ratio, it estimates the within-episode linear trend

$$\omega_t = a + bt + u_t, \tag{124}$$

over 2017Q3–2019Q3, where b measures the average quarterly drift in intermediary capital during QT1.

Table 35: QT1 episode study

	Estimate
ω at QT1 start (2017Q3)	0.0791
ω at QT1 end (2019Q3)	0.0657
Change in ω	-0.0134
Quarterly trend (OLS)	-0.0022
t-statistic	-5.06
p-value	0.001

Notes: QT1 is the Federal Reserve's first quantitative tightening episode (2017Q4–2019Q3). The reported start level uses 2017Q3, the last quarter before runoff began. The quarterly trend is an OLS regression of ω_t on a linear time index over 2017Q3–2019Q3.

Persistence analysis. Table 36 compares persistence before and after the Global Financial Crisis using separate AR(1) regressions of the form

$$\omega_t = \mu + \rho\,\omega_{t-1} + u_t. \tag{125}$$

The table reports $\hat{\rho}$, the implied half-life, and the difference $\hat{\rho}_{post} - \hat{\rho}_{pre}$ with its associated test statistic.

Table 36: Persistence of ω_t before and after 2008

	Pre-2008	Post-2008	Difference
AR(1) coefficient $\hat{\rho}$	0.952	0.812	
Half-life (quarters)	14.0	3.3	
$\Delta\rho$			-0.140
t-stat ($\Delta\rho = 0$)			-1.94
p-value			0.053

Notes: AR(1) models are estimated separately before and after 2008Q4. Half-life is $-\log(2)/\log(\hat{\rho})$. The final column reports the difference $\hat{\rho}_{post} - \hat{\rho}_{pre}$ and its normal-approximation test.

Threshold autoregression. Table 37 estimates a Hansen (1996) threshold AR(1) in which the law of motion for intermediary capital is allowed to differ across low- and high-capital states:

$$\omega_t = \begin{cases} \mu_L + \rho_L \omega_{t-1} + u_t, & \text{if } \omega_{t-1} < \omega^*, \\ \mu_H + \rho_H \omega_{t-1} + u_t, & \text{if } \omega_{t-1} \geq \omega^*. \end{cases} \tag{126}$$

The threshold ω^* is estimated endogenously, and the table reports the corresponding sup-Wald statistic together with its wild-bootstrap p-value.

Table 37: Threshold autoregression for ω_t

	Below threshold	Above threshold
AR(1) coefficient	0.933	0.615
Half-life (quarters)	10.0	1.4
N	197	22
Estimated threshold	0.0896	
Model ω_c	0.0507	
sup-Wald	7.26	
Bootstrap p	0.567	

Notes: Hansen (1996) threshold AR(1) estimated on ω_t. The threshold grid combines a percentile grid with a finer grid around the model-implied crisis threshold. The sup-Wald p-value is based on a wild bootstrap. Half-life is $-\log(2)/\log(\hat{\rho})$.

L.6 Cross-Equation SUR Test

Table 38 reports the three-equation SUR system used in the main-text discussion of the model's single-state-variable restriction. Equation 1 regresses the four-quarter-ahead output gap on the HKM capital ratio; Equation 2 regresses cumulative four-quarter excess equity returns on Q^{HKM}; and Equation 3 regresses the model-implied intermediation gap proxy on the HKM capital ratio. Panel A reports equation-by-equation OLS with Newey–West t-statistics, Panel B reports the iterated SUR estimates with both standard and HAC-robust t-statistics, Panel C reports the SUR residual correlation matrix, and Panel D reports the Breusch–Pagan, joint Wald, proportionality, and eigenvalue diagnostics.

L.7 Calibrated Counterfactual

We map the HKM capital ratio to the model's state variable ω_t at four key dates and compute model-implied quantities under the baseline calibration. Table 15 reports ω, leverage, employment, central bank equity holdings, and the required fiscal transfer at each date. At the 2009Q1 trough ($\omega = 0.027$), the model implies an employment gap of 3.9% and central bank equity holdings of 73%—broadly consistent with the Federal Reserve's extraordinary balance sheet expansion. By 2019Q3 ($\omega = 0.066$), the economy remains below the crisis threshold, with a 1.9% employment gap and 19% central bank equity holdings, consistent with the Federal Reserve's still-elevated balance sheet a decade after the crisis.

Table 16 (in the main text) compares three counterfactual scenarios starting from the 2009Q1 trough. Without QE, equilibrium does not exist ($\omega < \omega_c$). With QE alone, the economy converges to the bad steady state with a permanent 4.2% employment gap and 80% central bank equity holdings. With QE plus fiscal recapitalisation, the economy recovers to

Table 38: SUR Cross-Equation Test: Single State Variable Restriction

	Eq. 1 OutputGap_{t+4}	**Eq. 2** $\text{ExcRet}_{t\to t+4}$	**Eq. 3** BP_Model_t
Panel A: OLS (equation by equation)			
Predictor	OmegaHkm	Q^{HKM}	OmegaHkm
$\hat{b}$	$0.3616 * **$	0.0033	$-2.2280 * **$
t_{NW}	$3.004 * **$	0.910	$-175.302 * **$
R^2	0.2101	0.0183	0.9995
Panel B: SUR (feasible GLS, iterated)			
$\hat{b}_{\text{SUR}}$	$0.3597 * **$	0.0037	$-2.2280 * **$
t_{SUR}	6.252	1.840	-536.021
$t_{\text{SUR-HAC}}$	$3.065 * **$	0.990	$-175.276 * **$
R^2_{SUR}	0.2101	0.0181	0.9995
Panel C: Residual Correlations (SUR)			
$\hat{e}_1$	1	0.151	-0.241
$\hat{e}_2$		1	-0.030
$\hat{e}_3$			1
Panel D: Cross-Equation Tests			
Breusch–Pagan $\chi^2(3)$	12.028 [$p = 0.0073$]		
Joint Wald $\chi^2(3)$	303374.790 [$p = 0.000000$]		
Proportionality C: δ (t-stat)	-7.0921 ($t = -1.722$, $p = 0.0851$)		
Proportionality D: δ (t-stat)	0.0057 ($t = 1.473$, $p = 0.1408$)		
Σ eigenvalue shares	0.989, 0.011, 0.000		
T	148		
Sample	1970Q1–2007Q4		
Horizon h	4 quarters		

Notes: This table reports SUR estimates for a three-equation system testing the single-state-variable restriction. Eq. 1 regresses future output gap on the HKM capital ratio ($\hat{\omega}$). Eq. 2 regresses cumulative 4-quarter excess equity returns on $Q^{HKM} = 1/\hat{\omega}$. Eq. 3 regresses the model-implied intermediation gap proxy $BP_{Model,t} = (1 - \hat{\omega}_t)^{2.5}$ on $\hat{\omega}_t$. Panel A reports OLS with Newey–West t-statistics (h lags). Panel B reports iterated feasible GLS (SUR) estimates with both standard and HAC-robust t-statistics. Panel C shows the SUR residual correlation matrix. Panel D reports: the Breusch–Pagan test for diagonal Σ (significant $\Rightarrow$ SUR is more efficient than OLS); a joint Wald test for all slopes equal to zero; proportionality tests C (OmegaHkm incremental to BP_{Model} in Eq. 1) and D (Q^{HKM} incremental to OutputGap in Eq. 2); and eigenvalue shares of $\hat{\Sigma}$ (dominance of first eigenvalue supports single-factor structure). ***, **, * denote significance at 1%, 5%, 10%.

the efficient benchmark with zero employment gap. The benefit-to-cost ratio of the fiscal transfer is 4.3: every dollar of recapitalisation generates $4.30 in present value of additional output.

The model-implied fiscal transfer (25% of GDP) is an order of magnitude larger than actual TARP injections (1.7% of GDP). This gap reflects the model's stylised one-state-variable structure, in which a single intermediary must absorb all equity risk. In practice, the financial sector's capital was partially replenished through retained earnings, private capital raises, and portfolio rebalancing, reducing the required public transfer. The key qualitative prediction—that the fiscal return on recapitalisation is large and that delay is costly—is robust to the quantitative mapping.

Figure 6 plots the HKM capital ratio alongside the model's regime boundaries and the implied employment gap over the sample period.

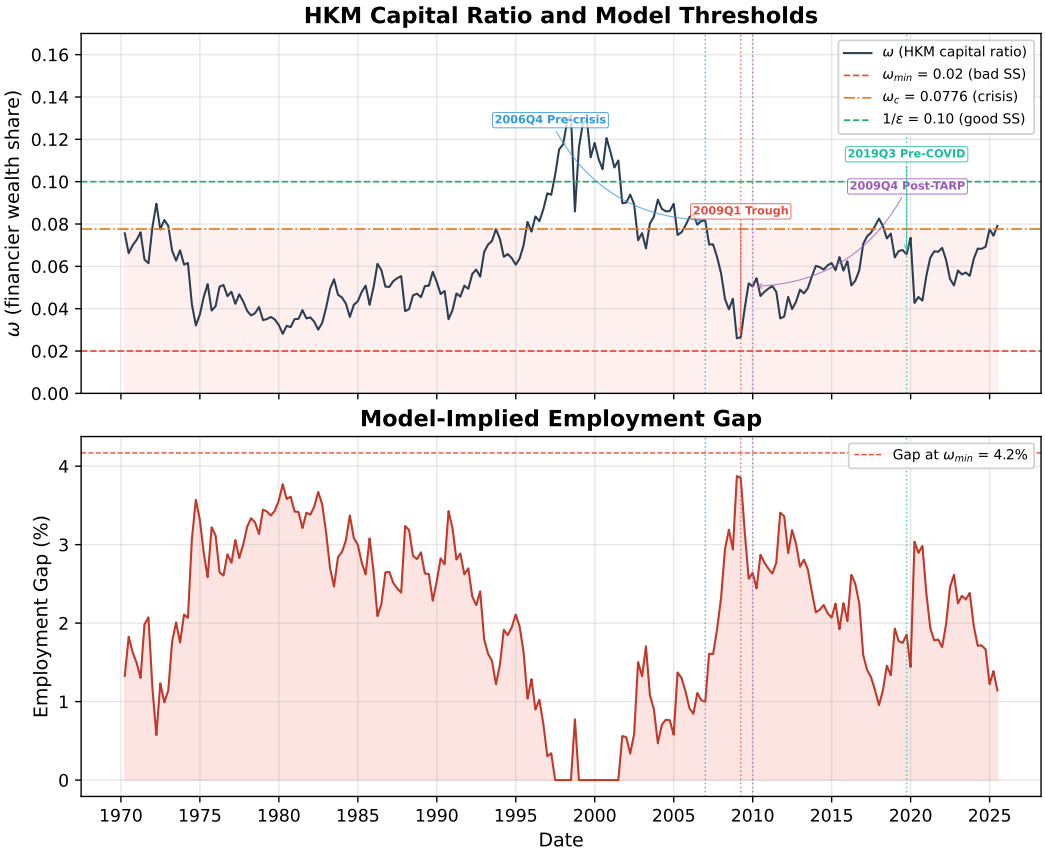


Figure 6: Calibrated counterfactual: HKM capital ratio mapped to model regimes. Left panel: HKM capital ratio with model thresholds ($\underline{\omega} = 0.02$, $\omega_c = 0.078$, $1/\epsilon = 0.10$). Shaded regions indicate crisis regime ($\omega < \omega_c$). Right panel: model-implied employment gap. Baseline calibration: $\epsilon = 10$, $\underline{\omega} = 0.02$, $\lambda = 5$, $\sigma = 0.15$, $\delta = 0.04$, $\varphi = 1$.

L.8 QE Program Timeline

Figure 7 plots Federal Reserve SOMA holdings by asset class as a share of nominal GDP and marks the main QE-era policy events. The purpose is descriptive: it connects our QE intensity proxy in equation (43) to the sequence of FOMC announcements and the evolution of balance-sheet composition.

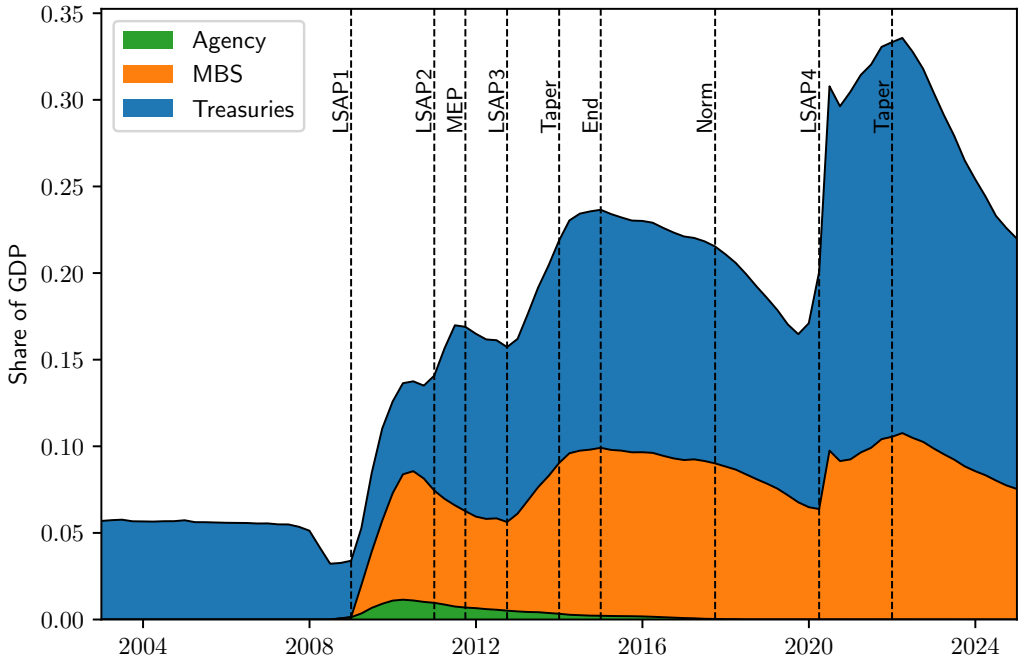


Figure 7: Evolution of Federal Reserve SOMA holdings by asset class (Treasuries, agency MBS, and agency debt) as a share of GDP. Vertical lines mark major LSAP and maturity-extension announcements, the 2013 tapering episode, the end of net purchases in 2014, the start of balance-sheet normalisation in 2017, and the pandemic LSAP restart in 2020.

LSAP1 (2008–2009). On November 25, 2008, the Fed announced LSAP1, authorising purchases of up to $100 billion in agency debt (Fannie Mae, Freddie Mac, and Ginnie Mae) and $500 billion in agency mortgage-backed securities (MBS). On March 18, 2009, the FOMC expanded LSAP1 to a total of $200 billion in agency debt, $1.25 trillion in agency MBS, and $300 billion in longer-term Treasury securities, with purchases completed by the end of 2009. By the end of the program, the Federal Reserve owned more than 20% of the agency MBS market. On August 10, 2010, the FOMC announced a reinvestment policy intended to maintain securities holdings by reinvesting principal payments from agency debt and MBS into longer-term Treasuries and rolling over maturing Treasury holdings.

LSAP2 and maturity extension (2010–2012). On November 3, 2010, the Fed announced LSAP2, purchasing an additional $600 billion in longer-term Treasury securities at a pace of about $75 billion per month through 2011Q2. On September 21, 2011, the FOMC announced the Maturity Extension Program (MEP), selling $400 billion of Treasury securities with remaining maturity of three years or less and purchasing an equivalent amount of Treasuries with remaining maturity of six to thirty years by June 2012, while continuing reinvestment of principal payments from agency debt and MBS. On June 20, 2012, the Fed extended the MEP through the end of 2012, purchasing longer-term securities at a pace of $45 billion per month while selling or redeeming equal amounts of shorter-term Treasuries.

LSAP3 and tapering (2012–2014). On September 13, 2012, the Fed announced LSAP3: open-ended purchases of $40 billion per month of agency MBS. Combined with the MEP, net purchases of longer-term securities were about $85 billion per month. On December 12, 2012, after the end of the MEP, the FOMC announced it would continue purchasing $45 billion per month in longer-term Treasury securities, maintaining the overall pace of longer-term purchases. As economic conditions improved, the Fed began communicating and then implementing a gradual taper, officially announcing the start of tapering on December 18, 2013 and concluding net purchases in October 2014 while maintaining reinvestment of principal payments.

Normalisation and the pandemic restart (2015–2021). On December 16, 2015, the FOMC raised the federal funds rate target range off the effective lower bound. On June 14, 2017, the FOMC announced it expected to begin balance-sheet normalisation later that year and initiated the program in October 2017. After returning rates to the zero lower bound in response to COVID-19, the Fed announced LSAP4 on March 15, 2020, entailing purchases of $200 billion in agency MBS and $500 billion in Treasury securities; on March 23, 2020, the Fed expanded the program, making purchases effectively open-ended. On June 10, 2020, the FOMC set monthly purchase guidance of at least $40 billion in agency MBS and $80 billion in Treasuries. The Fed began tapering again on November 3, 2021 and doubled the pace of tapering on December 15, 2021.

Online Appendix

Market Failure, the QE Trap and Fiscal Escape

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Not for publication

This online appendix contains supplementary proofs, derivations, and empirical results referenced in the main text.

A Proofs for the Complete-Markets Benchmark

Proof of Proposition 15. When both agents can trade bonds and equity, the single risk source Z_t is spanned by two assets, so markets are dynamically complete. By the Second Welfare Theorem, every competitive equilibrium allocation is Pareto efficient and can be obtained as the solution to a social planner's problem.

Step 1: Social planner's problem. The planner maximises

$$\bar{\lambda}\left(\ln C_h - \frac{N^{1+\varphi}}{1+\varphi}\right) + (1-\bar{\lambda})\ln C_i$$

subject to the resource constraint $C_h + C_i = Y = AN$, where $\bar{\lambda} \in (0,1)$ is the household's Pareto weight.

The Lagrangian is $\mathcal{L} = \bar{\lambda}\ln C_h + (1-\bar{\lambda})\ln C_i - \bar{\lambda}N^{1+\varphi}/(1+\varphi) + \xi(AN - C_h - C_i)$ with multiplier ξ. The first-order conditions are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_h} &= \frac{\bar{\lambda}}{C_h} - \xi = 0 \quad \Longrightarrow \quad C_h = \frac{\bar{\lambda}}{\xi}, \\
\frac{\partial \mathcal{L}}{\partial C_i} &= \frac{1-\bar{\lambda}}{C_i} - \xi = 0 \quad \Longrightarrow \quad C_i = \frac{1-\bar{\lambda}}{\xi}, \\
\frac{\partial \mathcal{L}}{\partial N} &= -\bar{\lambda}N^{\varphi} + \xi A = 0 \quad \Longrightarrow \quad N^{\varphi} = \frac{\xi A}{\bar{\lambda}}.
\end{aligned}$$

From the first two conditions: $C_h + C_i = 1/\xi = AN$, so $\xi = 1/(AN)$. Substituting into the consumption conditions:

$$C_h = \bar{\lambda}\, AN = \bar{\lambda}\, Y, \qquad C_i = (1-\bar{\lambda})\, Y. \tag{127}$$

Consumption shares are constant, independent of A_t.

Step 2: Employment. The planner's labour FOC gives $N^{1+\varphi} = 1/\bar{\lambda}$, but this is the first-best level without the monopolistic competition distortion. In the decentralised equilibrium, the household's intratemporal condition equates the marginal rate of substitution to the real wage: $N^{\varphi} C_h = W/P$. From the factor shares (2): $W/P = [(\epsilon - 1)/\epsilon] \cdot Y/N = [(\epsilon - 1)/\epsilon] \cdot A$. Substituting $C_h = \bar{\lambda} Y = \bar{\lambda} A N$:

$$N^{\varphi} \cdot \bar{\lambda} A N = \frac{\epsilon - 1}{\epsilon} \cdot A.$$

Cancelling A and solving: $N^{1+\varphi} = (\epsilon - 1)/(\epsilon \bar{\lambda})$, giving:

$$N = \left[\frac{\epsilon - 1}{\epsilon \bar{\lambda}} \right]^{1/(1+\varphi)}. \tag{128}$$

Since $\bar{\lambda}$, ϵ, and φ are all constant, N is constant (independent of A_t). Output is $Y_t = A_t N$, dividends are $D_t = Y_t/\epsilon = A_t N/\epsilon$, and both inherit the dynamics of A_t.

Step 3: Stochastic discount factor, risk-free rate, and market price of risk. We can use either agent's marginal utility to price assets. Using the household: the SDF is $\Lambda_t = e^{-\delta t} \cdot \partial U_h / \partial C_h = e^{-\delta t}/C_{h,t}$ (the marginal utility of consumption under log utility). Substituting $C_{h,t} = \bar{\lambda} A_t N$:

$$\Lambda_t = \frac{e^{-\delta t}}{\bar{\lambda} A_t N}.$$

Since $\bar{\lambda}$ and N are constant, the dynamics of Λ are determined by $e^{-\delta t}/A_t$. We compute $\mathrm{d}(1/A_t)$ by Itô's lemma with $f(A) = 1/A$, $f'(A) = -1/A^2$, $f''(A) = 2/A^3$:

$$\begin{aligned}
\mathrm{d}\left(\frac{1}{A_t}\right) &= -\frac{1}{A_t^2}\,\mathrm{d}A_t + \frac{1}{2} \cdot \frac{2}{A_t^3} (\,\mathrm{d}A_t)^2 \\
&= -\frac{1}{A_t^2}(A_t \mu\,\mathrm{d}t + A_t \sigma\,\mathrm{d}Z_t) + \frac{1}{A_t^3} A_t^2 \sigma^2\,\mathrm{d}t \\
&= \frac{1}{A_t}(-\mu + \sigma^2)\,\mathrm{d}t - \frac{1}{A_t}\sigma\,\mathrm{d}Z_t.
\end{aligned}$$

Therefore $\mathrm{d}(1/A_t)/(1/A_t) = (-\mu + \sigma^2)\,\mathrm{d}t - \sigma\,\mathrm{d}Z_t$. Using the product rule for $\Lambda_t = (\bar{\lambda} N)^{-1} e^{-\delta t} \cdot (1/A_t)$:

$$\frac{\mathrm{d}\Lambda_t}{\Lambda_t} = -\delta\,\mathrm{d}t + \frac{\mathrm{d}(1/A_t)}{1/A_t} = -(\delta + \mu - \sigma^2)\,\mathrm{d}t - \sigma\,\mathrm{d}Z_t.$$

The risk-free rate is minus the expected growth of the SDF:

$$r = -\frac{\mathbb{E}[\,\mathrm{d}\Lambda_t/\Lambda_t]}{\mathrm{d}t} = \delta + \mu - \sigma^2. \tag{129}$$

This is constant. The market price of risk is the negative of the SDF's volatility loading on $\mathrm{d}Z$:

$$\Theta = \sigma. \tag{130}$$

This is also constant. The equity risk premium is $\sigma_R \Theta = \sigma \cdot \sigma = \sigma^2$.

Verification using the financier's SDF: $\Lambda_t^i = e^{-\delta t}/C_{i,t} = e^{-\delta t}/[(1-\bar{\lambda})A_t N]$. Since this differs from Λ_t^h by only the constant factor $\bar{\lambda}/(1-\bar{\lambda})$, both SDFs give the same r and Θ. This confirms dynamic completeness (marginal rates of substitution are equalised).

Step 4: Asset prices. The equity price is:

$$P_{D,t} = \mathbb{E}_t \int_t^\infty \frac{\Lambda_s}{\Lambda_t} D_s \, \mathrm{d}s.$$

Computing the SDF ratio: $\Lambda_s/\Lambda_t = e^{-\delta(s-t)} \cdot (A_t/A_s)$ (from the explicit form of Λ, since $\bar{\lambda}$ and N cancel). Substituting $D_s = A_s N/\epsilon$:

$$P_{D,t} = \mathbb{E}_t \int_t^\infty e^{-\delta(s-t)} \frac{A_t}{A_s} \cdot \frac{A_s N}{\epsilon} \, \mathrm{d}s = \frac{A_t N}{\epsilon} \int_t^\infty e^{-\delta(s-t)} \, \mathrm{d}s = \frac{A_t N}{\epsilon} \cdot \frac{1}{\delta} = \frac{D_t}{\delta}.$$

The key step is that A_s cancels between the SDF ratio and the dividend: $\Lambda_s D_s/\Lambda_t = e^{-\delta(s-t)} A_t N/\epsilon$, which is deterministic conditional on $\mathcal{F}_t$. Similarly, total wealth is:

$$P_{Y,t} = \mathbb{E}_t \int_t^\infty \frac{\Lambda_s}{\Lambda_t} Y_s \, \mathrm{d}s = A_t N \int_t^\infty e^{-\delta(s-t)} \, \mathrm{d}s = \frac{Y_t}{\delta}.$$

Step 5: Value of human capital. By the factor shares (2), output decomposes as $Y_t = D_t + (W_t/P_t)N_t$. Linearity of pricing under the SDF gives $P_{Y,t} = P_{D,t} + P_{L,t}$, where $P_{L,t}$ is the present value of labour income. Hence:

$$P_{L,t} = P_{Y,t} - P_{D,t} = \frac{Y_t}{\delta} - \frac{D_t}{\delta} = \frac{Y_t}{\delta} - \frac{Y_t}{\epsilon\delta} = \left(1 - \frac{1}{\epsilon}\right) \frac{Y_t}{\delta} = \left(1 - \frac{1}{\epsilon}\right) P_{Y,t}.$$

Step 6: Agent wealth decomposition. Under log utility, each agent $j \in \{h, i\}$ has SDF $\Lambda_{j,t} = e^{-\delta t}/C_{j,t}$, so $\Lambda_{j,s}/\Lambda_{j,t} = e^{-\delta(s-t)} \, C_{j,t}/C_{j,s}$ and

$$\frac{\Lambda_{j,s}}{\Lambda_{j,t}} C_{j,s} = e^{-\delta(s-t)} \frac{C_{j,t}}{C_{j,s}} C_{j,s} = e^{-\delta(s-t)} \, C_{j,t}.$$

Total wealth (present value of consumption under the SDF) is therefore

$$W_{j,t} = \mathbb{E}_t \int_t^\infty \frac{\Lambda_{j,s}}{\Lambda_{j,t}} C_{j,s} \, \mathrm{d}s = C_{j,t} \int_t^\infty e^{-\delta(s-t)} \, \mathrm{d}s = \frac{C_{j,t}}{\delta}.$$

The household's total wealth decomposes into financial wealth and human capital: $W_{h,t} = H_{h,t} + P_{L,t}$. Using $C_{h,t} = \bar{\lambda} Y_t$ from (127) and $P_{L,t} = (1 - 1/\epsilon) P_{Y,t}$ from Step 5:

$$H_{h,t} = W_{h,t} - P_{L,t} = \frac{\bar{\lambda} Y_t}{\delta} - \left(1 - \frac{1}{\epsilon}\right) P_{Y,t} = \left(\bar{\lambda} - \left(1 - \frac{1}{\epsilon}\right)\right) P_{Y,t}.$$

The financier has no labour income, so her financial wealth equals her total wealth:

$$H_{i,t} = W_{i,t} = \frac{C_{i,t}}{\delta} = \frac{(1 - \bar{\lambda}) Y_t}{\delta} = (1 - \bar{\lambda}) P_{Y,t}.$$

Step 7: Connection to the LP model at $\omega = 1/\epsilon$. In the limited-participation model, $\omega = H_i / P_Y$ is the financier's wealth share. At $\omega = 1/\epsilon$: $H_i = P_Y/\epsilon = P_D$ (the financier holds exactly the equity claim), $H_h = (1/\epsilon - \omega) P_Y = 0$ (the household has zero financial wealth — all her wealth is human capital $P_L = (1 - 1/\epsilon) P_Y$), $\phi = 1/(\epsilon\omega) = 1$ (no leverage), $C_i = \omega Y = Y/\epsilon$ (financier consumes the dividend share), $C_h = (1 - \omega) Y = (1 - 1/\epsilon) Y$ (household consumes the labour share). Moreover, when $\bar{\lambda} = 1 - 1/\epsilon$, the complete-markets formulas from Steps 1 and 6 give $C_{h,t} = (1 - 1/\epsilon) Y_t$, $C_{i,t} = Y_t/\epsilon$, $H_{h,t} = 0$, $H_{i,t} = P_{Y,t}/\epsilon$, and $\phi = 1$. Therefore the good steady state of the LP model ($\omega = 1/\epsilon$) coincides with the complete-markets benchmark indexed by $\bar{\lambda} = 1 - 1/\epsilon$. □

B Propositions from Section 4

The following results are referenced in the main text but their formal statements are included here for completeness.

B.1 Properties of the Volatility Function

Proposition 29 (Zeros and sign of σ_ω)**.** *The function σ_ω has exactly two interior zeros: $\sigma_\omega(\underline{\omega}) = 0$ and $\sigma_\omega(1/\epsilon) = 0$. In addition, $\sigma_\omega(0) = 0$. Its sign pattern is:*

$$\sigma_\omega(\omega) \begin{cases} < 0 & \text{for } \omega \in (0, \underline{\omega}), \\ > 0 & \text{for } \omega \in (\underline{\omega}, 1/\epsilon), \\ < 0 & \text{for } \omega \in (1/\epsilon, 1). \end{cases} \tag{131}$$

The proof is in Appendix F. The sign pattern determines how the economy responds to productivity shocks. In the crisis regime ($\omega < \underline{\omega}$), $\sigma_\omega < 0$: positive productivity shocks *reduce* the financier's wealth share, worsening the crisis. This occurs because the de-leveraged

financier ($\phi < 1$) holds less equity than her consumption share warrants, so her wealth grows more slowly than the economy in a boom. In the recovery regime ($\underline{\omega} < \omega < 1/\epsilon$), $\sigma_\omega > 0$: positive shocks improve the financier's capitalisation, aiding recovery. The zeros at $\underline{\omega}$ and $1/\epsilon$ mark the absorbing steady states where all fluctuations cease.

B.2 Three Invariant Regimes

Theorem 30 (Regime classification)**.** *The state space* $(0, 1)$ *partitions into three invariant regions:*

1. ***Crisis regime*** *($\omega \in (0, \underline{\omega})$): the financier is severely undercapitalised, $\phi < 1$ (deleveraged), and the central bank holds a majority of equity.*
2. ***Recovery regime*** *($\omega \in (\underline{\omega}, 1/\epsilon)$): the leverage constraint may or may not bind, risk premia are elevated, and the economy is in the basin of attraction of the good steady state.*
3. ***Inverted regime*** *($\omega \in (1/\epsilon, 1)$): the financier is "too large," households are net borrowers, and the economy converges toward the good steady state from above.*

No transition between regimes is possible.

B.3 Boundary Classification

Proposition 31 (Boundary classification)**.** *(i) $\omega = 0$: natural boundary (unattainable). Near $\omega = 0$, $|\sigma_\omega| \sim \sigma\lambda\underline{\omega} \cdot \omega$.*

(ii) $\omega = \underline{\omega}$: unattainable from both sides. Near $\underline{\omega}$, $\sigma_\omega = g(\omega)(\omega - \underline{\omega})$ with $g(\underline{\omega}) = \sigma\lambda\underline{\omega} > 0$.

(iii) $\omega = 1/\epsilon$: unattainable from both sides. Near $1/\epsilon$, $\sigma_\omega = g(\omega)(1/\epsilon - \omega)$ with $g(1/\epsilon) = \sigma > 0$.

(iv) $\omega = 1$: unattainable. Near 1, $\sigma_\omega = h(\omega)(1 - \omega)$ with $h(1) = -\sigma(1 + \varphi) \neq 0$.

The proofs (Appendix I) use the Engelbert–Schmidt criterion. "Unattainable" means that the economy perpetually approaches but never reaches a boundary in finite time, a continuous-time analogue of Zeno's paradox. Combined with regime invariance, this means the economy's long-run fate is determined entirely by its initial condition: starting below $\underline{\omega}$, it converges to the bad steady state; starting above, it converges to the good one.

C Proof of Asset Pricing Formulae

Proof of Theorem 7. Step 1: Return volatility. Since $P_{D,t} = Y_t/(\epsilon\delta)$ and δ is constant, the equity return has the same volatility as output: $\sigma_R = \sigma_Y$. Output is $Y_t = A_t N_t$ where A_t follows geometric Brownian motion with volatility σ and $N_t = N(\omega_t)$ with $N(\omega) = [(1-1/\epsilon)/(1-\omega)]^{1/(1+\varphi)}$. Define $f(\omega) = (1-\omega)^{-1/(1+\varphi)}$ so that $N = (1-1/\epsilon)^{1/(1+\varphi)} f(\omega)$. Then:

$$f'(\omega) = \frac{1}{(1+\varphi)}(1-\omega)^{-\frac{1}{1+\varphi}-1}, \qquad \frac{f'(\omega)}{f(\omega)} = \frac{1}{(1+\varphi)(1-\omega)}.$$

By Itô's lemma on $N(\omega_t) = c\, f(\omega_t)$, using $\mathrm{d}\omega_t = (\sigma_\omega^2/\omega)\,\mathrm{d}t + \sigma_\omega\,\mathrm{d}Z$:

$$\frac{\mathrm{d}N}{N} = \frac{f'}{f}\,\mathrm{d}\omega + \frac{1}{2}\frac{f''}{f}(\,\mathrm{d}\omega)^2 = \frac{\sigma_\omega}{(1+\varphi)(1-\omega)}\,\mathrm{d}Z_t + \left[\frac{f'}{f}\frac{\sigma_\omega^2}{\omega} + \frac{1}{2}\frac{f''}{f}\sigma_\omega^2\right]\,\mathrm{d}t.$$

Since A and N load on the same Brownian motion Z, the product rule for $Y = AN$ gives $\sigma_Y = \sigma_A + \sigma_N$:

$$\sigma_Y = \sigma + \frac{\sigma_\omega}{(1+\varphi)(1-\omega)},$$

which establishes (27).

Step 2: Market price of risk. The financier's SDF is $\Lambda_t = e^{-\delta t}/(\delta H_{i,t})$ (from $C_{i,t} = \delta H_{i,t}$ under log utility). By Itô's lemma on $1/H_i$ with $g(H) = 1/H$, $g' = -1/H^2$: $\sigma_\Lambda/\Lambda = -\sigma_{H_i}/H_i = -\phi_t\sigma_R$ (from equation (71)). The market price of risk is:

$$\Theta_t = \phi_t \sigma_{R,t}.$$

This holds for any ϕ_t (Appendix G). From $\sigma_\omega = \omega(\phi_t - 1)\sigma_R$ (Appendix F), $\phi_t = 1 + \sigma_\omega/(\omega\sigma_R)$, so:

$$\Theta_t = \phi_t\sigma_R = \sigma_R + \frac{\sigma_\omega}{\omega} = \underbrace{\sigma + \frac{\sigma_\omega}{(1+\varphi)(1-\omega)}}_{\sigma_R} + \frac{\sigma_\omega}{\omega},$$

which is (26).

Step 3: Risk-free rate. Total wealth $P_{Y,t} = Y_t/\delta$ pays continuous dividends $C_t = Y_t = \delta P_{Y,t}$ (dividend yield δ). Since $P_Y = Y/\delta$ with δ constant, $\mathrm{d}P_Y/P_Y = \mathrm{d}Y/Y$, so $\mu_{P_Y}/P_Y = \mu_Y$ and $\sigma_{P_Y}/P_Y = \sigma_Y$. The no-arbitrage condition for a dividend-paying claim states that the expected total return (capital gain plus dividend yield) equals the risk-free rate plus the risk premium:

$$\frac{\mu_{P_Y}}{P_Y} + \delta = r_t + \sigma_Y\,\Theta_t.$$

Substituting $\mu_{P_Y}/P_Y = \mu_Y$ and rearranging: $r_t = \delta + \mu_{Y,t} - \sigma_{R,t}\,\Theta_t$, which is (28). The equity

risk premium $\mu_{R,t} - r_t = \sigma_{R,t}\,\Theta_t$ follows directly from the definition of Θ. □

D Equilibrium Quantities as Functions of the Wealth Share

This appendix illustrates how asset prices, risk premia, and real variables evolve as the financier's wealth share ω falls from the good steady state ($1/\epsilon$) toward the crisis threshold (ω_c) and the bad steady state ($\underline{\omega}$). All panels use the baseline calibration: $\epsilon = 10$, $\underline{\omega} = 0.02$, $\lambda = 5$, $\sigma = 0.15$, $\delta = 0.04$, $\varphi = 1$, giving $\omega_c \approx 0.078$ and $1/\epsilon = 0.10$. Three vertical dashed lines mark these thresholds throughout.

The figures compare three model configurations (see Table 17):

1. **LP only** (blue): limited participation, no leverage constraint (Appendix B). Equilibrium exists for all $\omega \in (0, 1/\epsilon)$. Rational bubble present.

2. **LP + leverage, no QE** (orange, dashed where shown): leverage constraint added, no central bank (Appendix D). Equilibrium identical to LP only for $\omega > \omega_c$; no equilibrium below ω_c (shaded region).

3. **LP + leverage + QE** (red): central bank absorbs residual equity below ω_c (Sections 3–6). Equilibrium exists everywhere. No rational bubble. QE Trap at $\underline{\omega}$.

D.1 The Wealth-Share Volatility: The Fundamental Object

The wealth-share volatility $\sigma_\omega(\omega)$ is the primitive from which all other equilibrium quantities are derived. The market price of risk ($\Theta = \sigma_\omega/\omega + \sigma_R$), equity return volatility ($\sigma_R = \sigma + \sigma_\omega/[(1+\varphi)(1-\omega)]$), the risk-free rate ($r = \delta + \mu_Y - \sigma_R\Theta$), and the intermediation gap ($b'' = 2\delta\, b/\sigma_\omega^2$) all depend on σ_ω. The difference in the *boundary behaviour* of σ_ω near $\omega = 0$ is what separates equilibrium existence from failure, and rational bubbles from correct pricing.

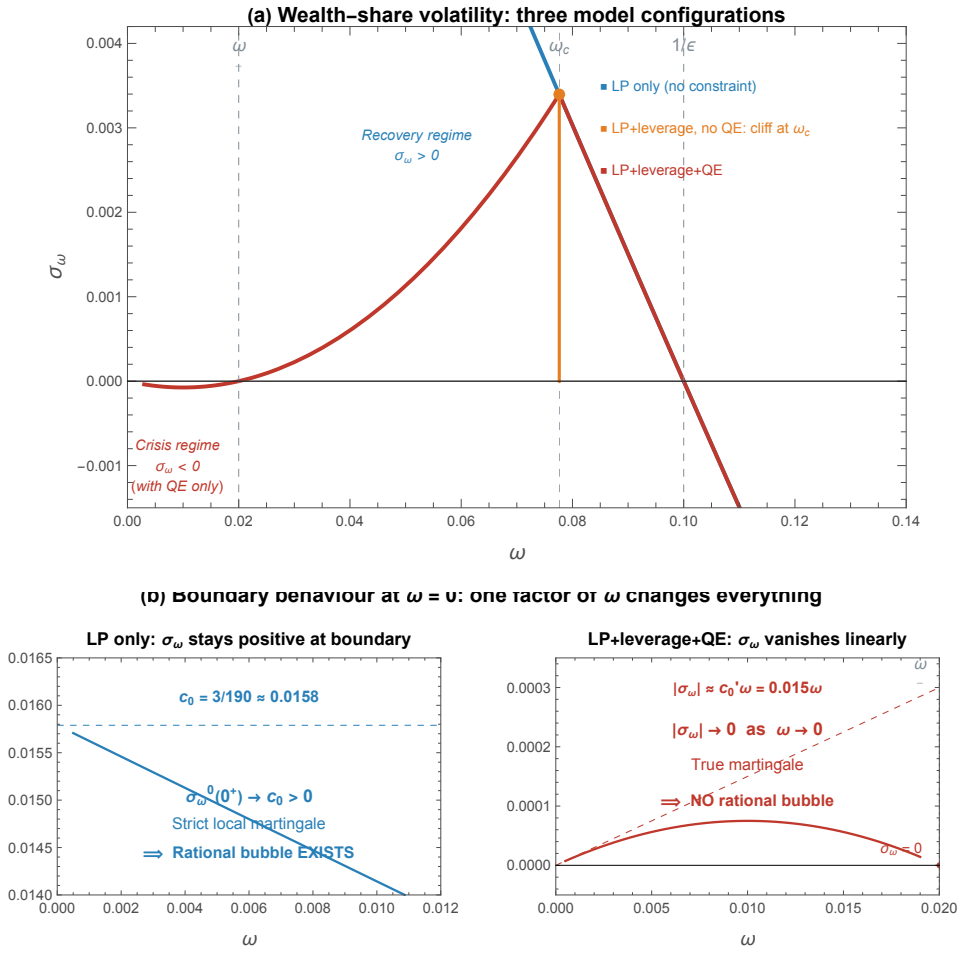


Figure 8: **The wealth-share volatility $\sigma_\omega(\omega)$: the source of everything.** *Panel (a):* Three cases. **LP only** (blue, solid): $\sigma_\omega^0(\omega)$ from the unconstrained formula (56), defined on $(0, 1/\epsilon)$, positive throughout, single zero at $1/\epsilon$. **LP + leverage, no QE** (orange, dashed): identical to LP only for $\omega > \omega_c$ (constraint does not bind). Below ω_c (shaded): no equilibrium — the orange curve hits a cliff at ω_c (orange dot). **LP + leverage + QE** (red, solid): piecewise formula (23). Below ω_c the constrained formula applies: σ_ω changes sign at $\underline{\omega}$ (negative in crisis, positive in recovery) and vanishes at both absorbing states. The three cases agree above ω_c and diverge below it. *Panel (b):* Zoom near $\omega = 0$, showing the critical boundary behaviour. **LP only** (and LP + leverage without QE): $\sigma_\omega^0(0^+) = c_0 = \sigma(1+\varphi)(1/\epsilon)/[(1-1/\epsilon)+\varphi] = 3/190 > 0$ (blue dotted line). The volatility stays positive at the boundary. By the integral test, $Q = 1/\omega$ is a **strict local martingale** and the intermediation gap is positive. **LP + leverage + QE**: $|\sigma_\omega(\omega)| \sim c_0'\omega$ with $c_0' = \sigma\lambda\underline{\omega} = 0.015$ (red dotted line). The volatility vanishes linearly. Now Q is a **true martingale** and the gap is zero. One factor of ω — introduced by the leverage constraint capping ϕ — separates bubble from no bubble, and simultaneously equilibrium existence from failure (the leverage paradox).

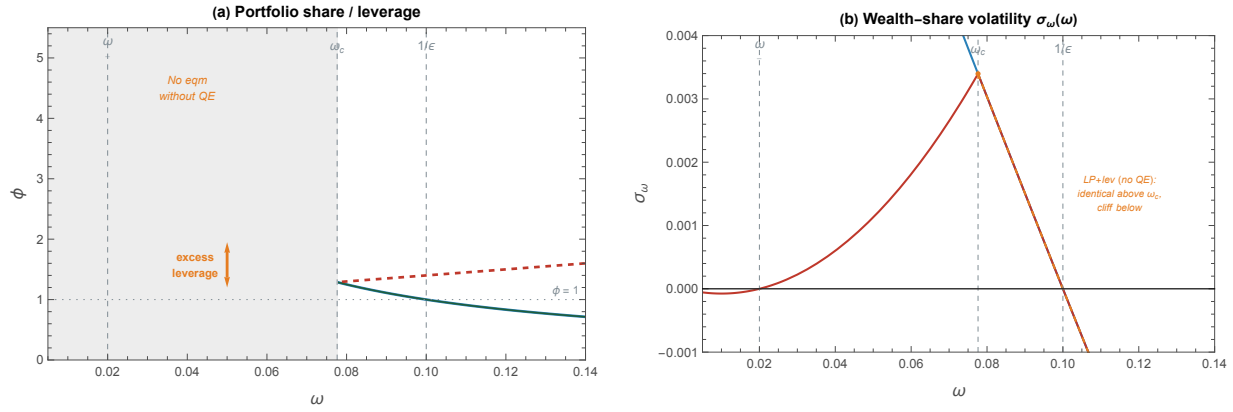


Figure 9: **Portfolio share and volatility.** *Panel (a):* The required portfolio share $\phi = 1/(\epsilon\omega)$ (blue) rises as the financier shrinks — a market clearing identity, not a portfolio choice. The leverage constraint $\bar{\phi}(\omega) = 1+\lambda(\omega-\underline{\omega})$ (red, dashed) caps leverage. The actual portfolio share with QE (green) is $\phi = \min(1/(\epsilon\omega), \bar{\phi}(\omega))$: identical to the required share above ω_c and constrained below. The orange arrow marks the excess leverage at $\omega = 0.05$: without QE, this gap represents equity supply that no price can absorb. The horizontal dotted line marks $\phi = 1$ (no leverage): the financier is levered iff $\omega < 1/\epsilon$. *Panel (b):* Wealth-share volatility $\sigma_\omega(\omega)$ under the three model configurations. LP only (blue): the unconstrained formula σ_ω^0 on the full domain $(0, 1/\epsilon)$. LP + leverage, no QE (orange, dashed): identical to LP only above ω_c; undefined below (cliff, marked by orange dot). LP + leverage + QE (red): the piecewise formula (23), with sign change at $\underline{\omega}$ and zeros at both absorbing states.

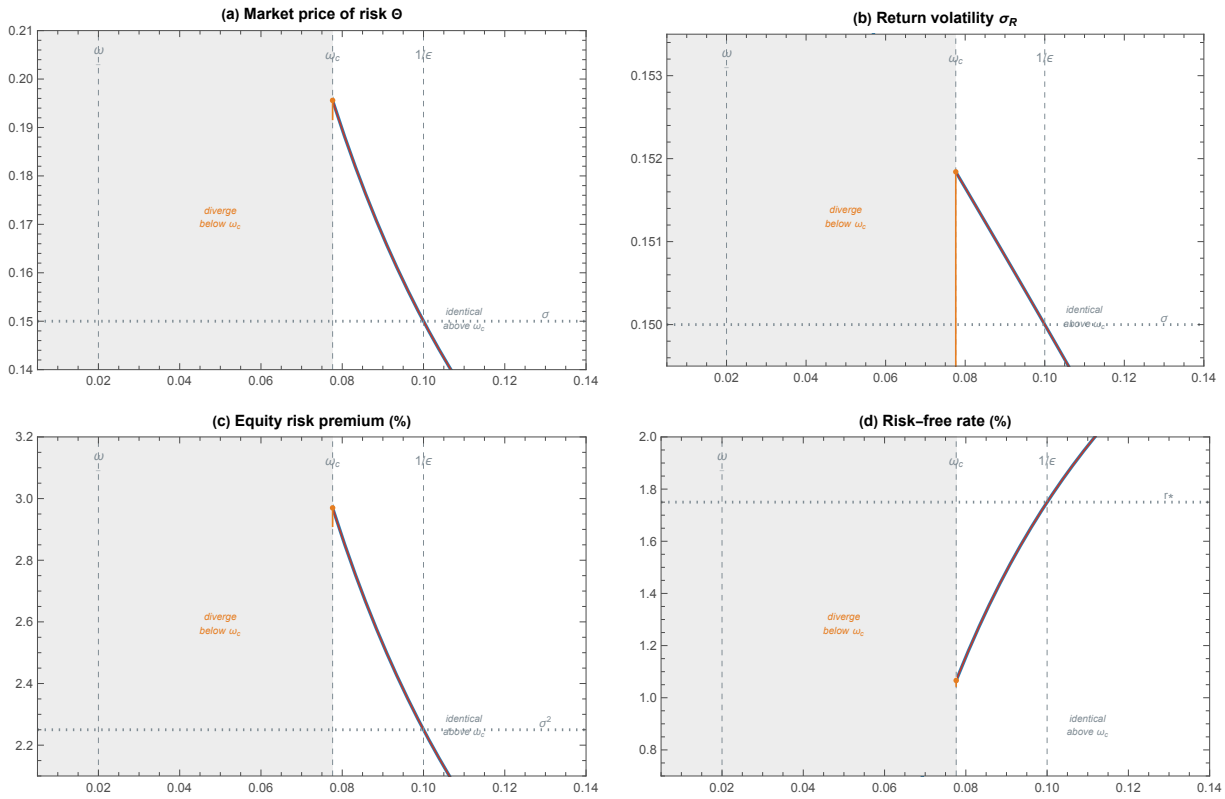


Figure 10: **Asset pricing quantities as functions of ω.** Each panel shows LP only (blue, thick) and LP + leverage + QE (red, thin). Above ω_c the curves are identical (the constraint does not bind); below ω_c (shaded) they diverge. The LP + leverage, no QE case coincides with the blue curve above ω_c and is undefined below (orange dot marks the cliff). Horizontal dotted lines mark complete-markets values. *Panel (a):* Market price of risk Θ. Blue (LP only) continues rising below ω_c as the unconstrained financier demands ever-higher compensation. Red (with QE) *falls* back toward $\sigma = 0.15$ as the constrained σ_ω shrinks. Peak at ω_c: $\Theta \approx 0.196$. *Panel (b):* Return volatility σ_R. Modest amplification; peak ≈ 0.152 vs $\sigma = 0.15$. *Panel (c):* Equity risk premium $\sigma_R\Theta$. Peaks at 2.95% near ω_c (vs $\sigma^2 = 2.25\%$ frictionless). With QE, the premium *falls* below ω_c — risk premia normalise even as the real economy deteriorates. *Panel (d):* Risk-free rate r. Mirror image: trough 1.06% at ω_c (vs $r^* = 1.75\%$). With QE, r rises back to normal below ω_c. *A policymaker watching these panels would see risk premia falling and r rising below ω_c and might conclude the crisis is easing — but employment (Figure 11) continues to worsen.*

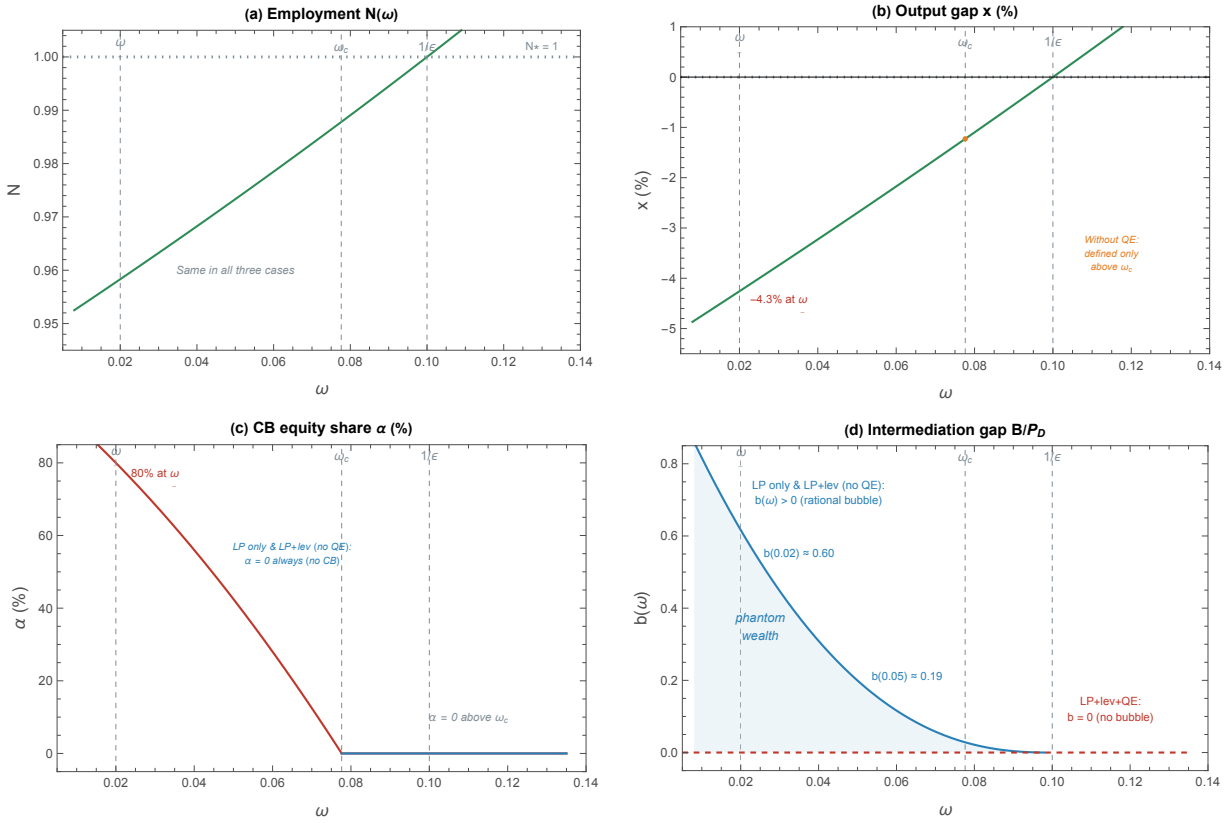


Figure 11: **Real variables and intermediation gap.** *Panel (a):* Employment $N(\omega) = [(1-1/\epsilon)/(1-\omega)]^{1/(1+\varphi)}$ is the same in all three model configurations: it depends on ω, not on whether the central bank holds equity. N is monotonically increasing and convex; at $\underline{\omega}$: $N = 0.958$ (-4.3% gap). *Panel (b):* Output gap widens monotonically as ω falls. Without QE, the gap is defined only above ω_c (-1.2% at most); below ω_c there is no equilibrium. With QE, the gap reaches -4.3% at $\underline{\omega}$. *Panel (c):* Central bank equity share (with QE only). Above ω_c: $\alpha = 0$. Below ω_c: α rises as the constrained financier holds less. At $\underline{\omega}$: the CB permanently holds 80% of the equity market. Without QE, $\alpha = 0$ always (no CB) — but no equilibrium exists below ω_c. *Panel (d):* Intermediation gap $b(\omega) = B/P_D$ from the exact gap ODE $b'' = (2\delta/\sigma_\omega^2)b$. Under **LP only** and **LP + leverage, no QE**: $b > 0$ (blue, with shading). At $\omega = 0.05$: $b \approx 0.19$ (19% of equity price is phantom wealth). At $\omega = 0.02$: $b \approx 0.60$. Under **LP + leverage + QE**: $b \equiv 0$ (red, dashed). The leverage constraint makes Q a true martingale, eliminating the gap. The shaded area is phantom wealth — value that would vanish if the economy transitioned to a regime with correct pricing.

D.2 Portfolio Share and Wealth-Share Volatility

D.3 Asset Pricing Quantities

D.4 Real Variables and the Intermediation Gap

E Intermediation Gap Formula Derivation

This appendix derives the ODE characterising the gap-to-price ratio $b(\omega) \equiv B_t/P_{D,t}$ in the baseline limited-participation model without leverage constraints, and establishes its boundary conditions. The derivation is self-contained: it does not rely on the finite-horizon formula of Hugonnier (2012), which is exact only in an exchange economy with constant weighting-process volatility. In our production economy the weighting-process volatility is state-dependent, and the exact gap solves a singular Sturm–Liouville problem derived below.

Setup and objective

The wealth-share process $\omega_t \in (0, 1/\epsilon)$ satisfies

$$\mathrm{d}\omega_t = \mu_\omega(\omega_t)\ \mathrm{d}t + \sigma_\omega(\omega_t)\ \mathrm{d}Z_t, \tag{132}$$

with $\mu_\omega = \sigma_\omega^2/\omega$ (the Q-martingale identity, Appendix G), and, in the unconstrained baseline, $\sigma_\omega(0^+) = c_0 > 0$ and $\sigma_\omega(1/\epsilon) = 0$. The equity price is $P_{D,t} = Y_t/(\epsilon\delta)$ (market clearing with a log-utility financier) and the fundamental value is

$$P_{D,t}^{\text{fund}} \ \equiv\ \mathbb{E}_t\left[\int_t^\infty \frac{\Lambda_s}{\Lambda_t}\, D_s\ \mathrm{d}s\right], \tag{133}$$

where $\Lambda_t = e^{-\delta t}/C_{i,t}$ is the financier's stochastic discount factor and $D_t = Y_t/\epsilon$. The intermediation gap-to-price ratio is $b(\omega_t) \equiv 1 - P_{D,t}^{\text{fund}}/P_{D,t}$.

Substituting $\Lambda_s/\Lambda_t = e^{-\delta(s-t)} \cdot (\omega_t Y_t)/(\omega_s Y_s)$ and $D_s = Y_s/\epsilon$ into (133), with change of variables $u = s - t$:

$$b(\omega) \ =\ 1 - \delta\,\omega\, f(\omega), \qquad f(\omega) \ \equiv\ \mathbb{E}^\omega\left[\int_0^\infty e^{-\delta u}\,\frac{1}{\omega_u}\ \mathrm{d}u\right], \tag{134}$$

where $\mathbb{E}^\omega$ denotes expectation conditional on $\omega_0 = \omega$. The function f is the Laplace-transform representation of the Q-resolvent; b is the quantity of economic interest.

Step 1: Feynman–Kac ODE for f

Lemma 32 (Feynman–Kac representation)**.** *Under Assumption 2 and $\mu_\omega = \sigma_\omega^2/\omega$, the function f in (134) belongs to $C^2((0,1/\epsilon))$ and solves*

$$\tfrac{1}{2}\sigma_\omega^2(\omega)\, f''(\omega) + \frac{\sigma_\omega^2(\omega)}{\omega}\, f'(\omega) - \delta\, f(\omega) + \frac{1}{\omega} = 0, \qquad \omega \in (0, 1/\epsilon). \tag{135}$$

Proof. Apply Karatzas and Shreve (1991, Theorem 5.7.6 and Problem 5.7.7, pp. 366–367) with infinitesimal generator $\mathcal{L} = \frac{1}{2}\sigma_\omega^2\partial_\omega^2 + \mu_\omega\partial_\omega$, discount rate $\delta > 0$, and running payoff $g(\omega) = 1/\omega$. The required integrability $\mathbb{E}^\omega \int_0^\infty e^{-\delta u}(1/\omega_u)\, du < \infty$ holds because $Q_t = 1/\omega_t$ is a non-negative supermartingale (Appendix G), so $\mathbb{E}^\omega[Q_u] \le Q_0 = 1/\omega$ and the integral is bounded by $1/(\delta\omega) < \infty$ for every interior ω. Elliptic regularity on $(0, 1/\epsilon)$ (strict positivity of σ_ω^2 and locally Lipschitz coefficients) gives $f \in C^2$ on the interior. Substituting $\mu_\omega = \sigma_\omega^2/\omega$ into $\mathcal{L}f - \delta f + g = 0$ yields (135). □

Step 2: The ODE for b

Proposition 33 (ODE for the gap-to-price ratio)**.** *The gap-to-price ratio $b(\omega) = 1 - \delta\omega f(\omega)$ satisfies the singular Sturm–Liouville equation*

$$b''(\omega) \;=\; \frac{2\delta}{\sigma_\omega^2(\omega)}\, b(\omega), \qquad \omega \in (0, 1/\epsilon). \tag{136}$$

Proof. Write $f = (1-b)/(\delta\omega)$ and compute the derivatives

$$f'(\omega) = -\frac{b'(\omega)}{\delta\omega} \;-\; \frac{1 - b(\omega)}{\delta\omega^2}, \tag{137}$$

$$f''(\omega) = -\frac{b''(\omega)}{\delta\omega} \;+\; \frac{2b'(\omega)}{\delta\omega^2} \;+\; \frac{2(1 - b(\omega))}{\delta\omega^3}. \tag{138}$$

Substitute (137) and (138) into (135) and group terms of each order in $\omega^{-k}b^{(j)}$.

Coefficient of $b'(\omega)$. The only contributions to b' come from

$$\tfrac{1}{2}\sigma_\omega^2 \cdot \frac{2b'}{\delta\omega^2} + \frac{\sigma_\omega^2}{\omega} \cdot \left(-\frac{b'}{\delta\omega}\right) = \frac{\sigma_\omega^2 b'}{\delta\omega^2} - \frac{\sigma_\omega^2 b'}{\delta\omega^2} = 0.$$

The cancellation is *exact* and relies on the Q-martingale identity $\mu_\omega = \sigma_\omega^2/\omega$.

Coefficient of $(1-b)/\omega^3$. The contributions are

$$\tfrac{1}{2}\sigma_\omega^2 \cdot \frac{2(1-b)}{\delta\omega^3} + \frac{\sigma_\omega^2}{\omega} \cdot \left(-\frac{1-b}{\delta\omega^2}\right) = \frac{\sigma_\omega^2(1-b)}{\delta\omega^3} - \frac{\sigma_\omega^2(1-b)}{\delta\omega^3} = 0.$$

Again a clean cancellation by $\mu_\omega = \sigma_\omega^2/\omega$.

Coefficient of $1/\omega$. From $-\delta f = -(1-b)/\omega = (b-1)/\omega$ and the forcing $+1/\omega$:

$$\frac{b-1}{\omega} + \frac{1}{\omega} = \frac{b}{\omega}.$$

Coefficient of $b''(\omega)$. Only one contribution:

$$\tfrac{1}{2}\sigma_\omega^2 \cdot \left(-\frac{b''}{\delta\omega}\right) = -\frac{\sigma_\omega^2 b''}{2\delta\omega}.$$

Assembling the surviving terms, the ODE (135) reduces to

$$-\frac{\sigma_\omega^2(\omega)\, b''(\omega)}{2\delta\omega} + \frac{b(\omega)}{\omega} = 0.$$

Multiplying through by $-2\delta\omega$ yields $\sigma_\omega^2(\omega)\, b''(\omega) - 2\delta\, b(\omega) = 0$, equivalently (136). □

Remark 13. *The cancellation of the* b' *and* $(1-b)/\omega^3$ *terms is a direct consequence of the* Q*-martingale identity* $\mu_\omega = \sigma_\omega^2/\omega$*. Any diffusion satisfying this identity yields the same clean ODE* (136) *for the gap-to-price ratio; the specific form of* $\sigma_\omega(\omega)$ *affects only the coefficient* $2\delta/\sigma_\omega^2(\omega)$*, not the structure of the equation.*

Step 3: Boundary condition at $\omega = 1/\epsilon$

Lemma 34 (Upper boundary)**.** $b(1/\epsilon) = 0$.

Proof. At $\omega = 1/\epsilon$, both $\sigma_\omega(1/\epsilon) = 0$ and $\mu_\omega(1/\epsilon) = \sigma_\omega^2(1/\epsilon)\cdot\epsilon = 0$, so the point is absorbing. By Yamada–Watanabe pathwise uniqueness (Karatzas and Shreve, 1991, Proposition 5.2.13, p. 291), if $\omega_0 = 1/\epsilon$ then $\omega_u = 1/\epsilon$ a.s. for all $u \geq 0$. Hence $f(1/\epsilon) = \int_0^\infty e^{-\delta u}\,\epsilon\,\mathrm{d}u = \epsilon/\delta$ and $b(1/\epsilon) = 1 - \delta\cdot(1/\epsilon)\cdot(\epsilon/\delta) = 0$. □

Step 4: Boundary condition at $\omega = 0^+$

Lemma 35 (Lower boundary)**.** *In the unconstrained baseline (no leverage constraint,* $\sigma_\omega(0^+) = c_0 > 0$*),* $b(0^+) = 1$.

Proof. The claim is equivalent to $\omega f(\omega) \to 0$ as $\omega \to 0^+$, i.e. $\delta\omega \int_0^\infty e^{-\delta u}\mathbb{E}^\omega[Q_u]\,\mathrm{d}u \to 0$, where $Q = 1/\omega$.

Step (a): Q-diffusion. By Itô's lemma applied to $Q = 1/\omega$ under (132),

$$\mathrm{d}Q_t = \left[\frac{\sigma_\omega^2}{\omega^3} - \frac{\mu_\omega}{\omega^2}\right]\mathrm{d}t - \frac{\sigma_\omega}{\omega^2}\,\mathrm{d}Z_t.$$

The drift vanishes under $\mu_\omega = \sigma_\omega^2/\omega$, so Q is a continuous local martingale on (ϵ, ∞) with diffusion coefficient $|\sigma_Q(q)| = q^2\,\sigma_\omega(1/q)$. At the infinite endpoint $\sigma_\omega(0^+) = c_0$, so $|\sigma_Q(q)| \sim c_0 q^2$ as $q \to \infty$ (quadratic growth).

Step (b): Q is a strict local martingale. By Mijatović and Urusov (2012, Theorem 2.1), a continuous local martingale Q with diffusion σ_Q on a half-line is a *true* martingale iff $\int^\infty q/\sigma_Q^2(q)\,\mathrm{d}q = +\infty$. Here $\int^\infty q/(c_0^2 q^4)\,\mathrm{d}q = c_0^{-2}\int^\infty q^{-3}\,\mathrm{d}q < \infty$, so Q is a *strict* local martingale.

Step (c): h-transform identity. By Pal and Protter (2010, Theorem 1), for a non-negative continuous strict local martingale Q started at $Q_0 = q$, there exists a probability measure $\mathbb{P}^*$ (the Föllmer measure) under which $q \mapsto \mathbb{E}_q[Q_u]/q$ coincides with the survival probability of a *dual* (Föllmer) diffusion $\widetilde{Q}$ whose explosion time ζ satisfies

$$\frac{\mathbb{E}_q[Q_u]}{q} \;=\; \mathbb{P}^*_q[\zeta > u].$$

Applied to the Q-diffusion of step (a) started at $q = 1/\omega$:

$$\omega\,\mathbb{E}^\omega[Q_u] \;=\; \mathbb{P}^*_{1/\omega}[\zeta > u].$$

Step (d): Collapse of $\delta\omega f(\omega)$ as $\omega \to 0^+$. Multiplying (134) by $\delta\omega$ and applying Tonelli:

$$\delta\omega f(\omega) \;=\; \delta\int_0^\infty e^{-\delta u}\,\omega\,\mathbb{E}^\omega[Q_u]\,\mathrm{d}u \;=\; \delta\int_0^\infty e^{-\delta u}\,\mathbb{P}^*_{1/\omega}[\zeta > u]\,\mathrm{d}u \;=\; \mathbb{E}^*_{1/\omega}\big[1 - e^{-\delta\zeta}\big]\,.$$

The last equality is Fubini applied to the joint law of (ζ, U) with $U \sim \mathrm{Exp}(\delta)$ independent.

Step (e): Explosion in $\mathbb{P}^*$ is immediate as $\omega \to 0^+$. Under the quadratic-growth hypothesis $|\sigma_Q(q)| \sim c_0 q^2$ as $q \to \infty$, the Föllmer diffusion $\widetilde{Q}$ is the h-transform of Q by $h(q) = q$ on (ϵ, ∞). The dual process is transient with $+\infty$ as an *entrance* boundary, and $\zeta \to 0$ in $\mathbb{P}^*$-probability as $\widetilde{Q}_0 \to \infty$ (see Pal and Protter 2010, Propositions 2 and 3; analogous to the inverse three-dimensional Bessel process of Delbaen and Schachermayer 1995). As $\omega \to 0^+$, i.e. $1/\omega \to \infty$, we therefore have $\zeta \to 0$ in $\mathbb{P}^*_{1/\omega}$-probability, so

$$\mathbb{E}^*_{1/\omega}\big[1 - e^{-\delta\zeta}\big] \;\longrightarrow\; 0 \quad \text{as} \quad \omega \to 0^+.$$

Combining steps (d) and (e), $\delta\omega f(\omega) \to 0$ as $\omega \to 0^+$, so $b(\omega) = 1 - \delta\omega f(\omega) \to 1$. □

Step 5: Existence, uniqueness, monotonicity, convexity

Proposition 36 (Well-posedness). *The two-point boundary-value problem* $b''(\omega) = (2\delta/\sigma_\omega^2(\omega))\, b(\omega)$ *on* $(0, 1/\epsilon)$ *with* $b(0^+) = 1$ *and* $b(1/\epsilon) = 0$ *admits a unique* C^2 *solution; the solution is strictly positive, strictly decreasing, and strictly convex on* $(0, 1/\epsilon)$.

Proof. Write the equation in Liouville normal form $b'' - V(\omega)b = 0$ with potential $V(\omega) = 2\delta/\sigma_\omega^2(\omega) > 0$. On the interior, $V \in C^1$ and finite, so Picard–Lindelöf gives a unique local C^2 solution for any Cauchy data.

At $\omega = 0^+$: $V(0^+) = 2\delta/c_0^2 < \infty$, so $\omega = 0$ is a regular endpoint of the Sturm–Liouville operator in the sense of Zettl (2005, Chapter 7); the Dirichlet datum $b(0^+) = 1$ uniquely selects a local solution. At $\omega = 1/\epsilon$: $V \to \infty$, a limit-point singular endpoint in Weyl's sense (Zettl, 2005, Theorem 7.4.1); under the vanishing rate $\sigma_\omega^2(\omega) \sim K(1/\epsilon - \omega)$, the Dirichlet condition $b(1/\epsilon) = 0$ selects the unique decaying solution. Positivity of V makes the Sturm–Liouville operator positive definite, so existence and uniqueness of the two-point BVP follow from Zettl (2005, Chapter 10).

Strict positivity. The strong maximum principle (Protter and Weinberger, 1984, Chapter 1, Theorem 3) applied to $L = -\partial_\omega^2 + V$ with $V > 0$ and non-negative boundary data $b(0^+) = 1$, $b(1/\epsilon) = 0$ gives $b > 0$ in the interior.

Strict convexity. Since $b > 0$ and $b'' = V \cdot b > 0$ on $(0, 1/\epsilon)$, b is strictly convex.

Strict monotone decrease. Suppose for contradiction that $b'(\omega_*) \geq 0$ at some $\omega_* \in (0, 1/\epsilon)$. Strict convexity of b implies b' is strictly increasing, so $b'(\omega) > b'(\omega_*) \geq 0$ for $\omega > \omega_*$, hence $b(\omega) > b(\omega_*) > 0$ on $(\omega_*, 1/\epsilon)$. By continuity, $b(1/\epsilon) \geq b(\omega_*) > 0$, contradicting $b(1/\epsilon) = 0$. Therefore $b'(\omega) < 0$ on the interior. □

Connection to the Hugonnier (2012) exchange-economy formula

Remark 14 (Exchange-economy limit). *When the weighting-process volatility* $v_\lambda(\omega) = \sigma_\omega(\omega)/[\omega(1-\omega)]$ *is constant (Hugonnier's finite-horizon exchange economy with constant labour endowment), the ODE* (136) *admits the closed-form solution*

$$b(\omega) \;=\; (1-\omega)^\gamma, \qquad \gamma \;=\; \frac{1+\sqrt{1+8\delta/v^2}}{2}, \tag{139}$$

in the limit of infinite horizon. This recovers Hugonnier (2012, Proposition 6) as a special case of (136). *In our production economy,* $v_\lambda(\omega)$ *is genuinely state-dependent, so the closed form does not apply; the gap-to-price ratio must be computed by numerical solution of* (136).

Step 6: Numerical solution and cross-validation

At baseline calibration ($\epsilon = 10$, $\underline{\omega} = 0.02$, $\lambda = 5$, $\sigma = 0.15$, $\varphi = 1$, $\delta = 0.04$), we solve (136) numerically using two independent schemes and cross-validate.

Method A (Shooting, verified by collocation). We start at $\omega_R = 1/\epsilon - \eta$ for $\eta = 10^{-3}$ with the asymptotic Frobenius initial conditions $b(\omega_R) = \eta^\alpha$, $b'(\omega_R) = -\alpha\,\eta^{\alpha-1}$, where $\alpha = (1 + \sqrt{1 + 8\delta/\sigma^2})/2 \approx 2.4508$ is the right-endpoint indicial exponent, and integrate leftward to $\omega_L = 5 \times 10^{-4}$ using explicit Runge–Kutta (`scipy.integrate.solve_ivp`, `RK45`, relative tolerance 10^{-10}). The integrated solution is then rescaled so that $b(\omega_L) = 1$. The same BVP is independently solved by the collocation method of `scipy.integrate.solve_bvp` with Dirichlet BCs; the two solutions agree to six decimal places on $[\omega_L, \omega_R]$, confirming that the shooting rescaling recovers the correct normalisation.

Method B (Finite difference). A separate implementation discretises (136) on a uniform grid of 600 points in $[10^{-4}, 1/\epsilon - 10^{-4}]$ and solves the tridiagonal linear system with Dirichlet BCs $b(\omega_L) = 1$, $b(\omega_R) = 0$, followed by a monotonicity projection $b_i \leftarrow \min(b_i, b_{i-1})$.

Cross-validation. Table 39 reports both methods at the empirically relevant grid.

Table 39: Numerical solution of the exact gap ODE (136) at baseline calibration. Two independent solvers agree to within 10^{-2}; the approximate Hugonnier formula with state-dependent $\gamma(v_\lambda(\omega))$ is shown for reference.

ω	$v_\lambda(\omega)$	b_A (shoot/BVP)	b_B (FD)	$\lvert b_A - b_B \rvert$	$(1-\omega)^{\gamma(v_\lambda(\omega))}$
0.005	3.008	0.8957	0.8872	0.0085	0.9950
0.010	1.429	0.7875	0.7800	0.0075	0.9896
0.020	0.638	0.5944	0.5888	0.0056	0.9767
0.030	0.374	0.4317	0.4276	0.0041	0.9581
0.040	0.242	0.2981	0.2953	0.0028	0.9302
0.050	0.162	0.1922	0.1903	0.0018	0.8881
0.060	0.109	0.1121	0.1110	0.0011	0.8229
0.070	0.070	0.0558	0.0553	0.0005	0.7184
0.080	0.041	0.0208	0.0206	0.0002	0.5404
0.090	0.018	0.0038	0.0038	0.0000	0.2239
0.095	0.009	0.00070	0.00070	0.0000	0.0377

Boundary conditions, positivity, monotonicity. Method A reports $b \to 1$ at the left endpoint (by construction) and $b(\omega_R) = 1.37 \times 10^{-5}$ at $\omega_R = 1/\epsilon - 10^{-3}$, within the Frobenius asymptote's tolerance. The solution is strictly positive, strictly decreasing, and strictly convex at every grid point, consistent with Proposition 36.

Exact versus approximate formula. At $\omega = 0.05$, $b_{\text{exact}} = 0.192$ while $(1-\omega)^{\gamma(v_\lambda(0.05))} = 0.888$, an overestimate of roughly a factor of five. At $\omega = 0.08$, the overestimate exceeds an order of magnitude. Near $\omega = 1/\epsilon$, where v_λ is small and the Hugonnier approximation applies most cleanly, the two agree to within 10%. The qualitative properties (positivity, monotonicity, convexity, and elimination of the gap by central-bank intervention) are preserved by both formulas; only the magnitudes differ.

F Franchise Value and Dynamic Incentive Compatibility

Proposition 37 (Dynamic incentive compatibility)**.** *With log utility, the dynamic IC constraint reduces to*

$$\phi_t \;\le\; \frac{1}{\theta}\exp\big(\delta\mathcal{F}(\omega_t)\big) \;\equiv\; \bar{\phi}^{\text{dyn}}(\omega_t), \tag{140}$$

where the franchise value of intermediation is

$$\mathcal{F}(\omega) \;\equiv\; \frac{1}{2\delta}\,\mathbb{E}_\omega\!\int_0^\infty e^{-\delta t}\,\Theta_t^2\;\mathrm{d}t \;>\; 0. \tag{141}$$

Proof. **Step 1: Value function decomposition.** A log-utility agent with wealth W_t, access to a risky asset with return $\mu_R\,\mathrm{d}t + \sigma_R\,\mathrm{d}Z$ and a risk-free bond at rate r_t, consuming $C_t = \delta W_t$ and holding portfolio share ϕ_t, has lifetime utility $V_t = \mathbb{E}_t\int_t^\infty e^{-\delta(u-t)}\ln C_u\;\mathrm{d}u$. Substituting $C_u = \delta W_u$:

$$V_t = \mathbb{E}_t\int_t^\infty e^{-\delta(u-t)}\ln(\delta W_u)\;\mathrm{d}u = \frac{\ln\delta}{\delta} + \mathbb{E}_t\int_t^\infty e^{-\delta(u-t)}\ln W_u\;\mathrm{d}u.$$

Writing $\ln W_u = \ln W_t + \ln(W_u/W_t)$ and using $\int_t^\infty e^{-\delta(u-t)}\,\mathrm{d}u = 1/\delta$:

$$V_t = \frac{\ln(\delta W_t)}{\delta} + \mathbb{E}_t\int_t^\infty e^{-\delta(u-t)}\ln\frac{W_u}{W_t}\;\mathrm{d}u.$$

The wealth SDE with consumption drain $C = \delta W$ is $\mathrm{d}W/W = [r+\phi(\mu_R - r) - \delta]\,\mathrm{d}t + \phi\sigma_R\,\mathrm{d}Z$. By Itô's lemma on $\ln W$ with $f(W) = \ln W$, $f' = 1/W$, $f'' = -1/W^2$:

$$\mathrm{d}\ln W = \big[r + \phi(\mu_R - r) - \delta - \tfrac{1}{2}\phi^2\sigma_R^2\big]\,\mathrm{d}t + \phi\sigma_R\;\mathrm{d}Z.$$

Integrating and taking expectations (the stochastic integral has zero expectation):

$$\mathbb{E}_t\left[\ln\frac{W_u}{W_t}\right] = \mathbb{E}_t\int_t^u \left[r_s + \phi_s(\mu_{R,s} - r_s) - \delta - \tfrac{1}{2}\phi_s^2\sigma_{R,s}^2\right] \mathrm{d}s.$$

Substituting back and swapping the order of integration (Fubini): the region $t \le s \le u < \infty$ becomes, for fixed s, $u \in [s, \infty)$:

$$\begin{aligned}\mathbb{E}_t\int_t^\infty e^{-\delta(u-t)}\ln\frac{W_u}{W_t}\,\mathrm{d}u &= \mathbb{E}_t\int_t^\infty\int_t^u e^{-\delta(u-t)}g(s)\,\mathrm{d}s\,\mathrm{d}u \\ &= \mathbb{E}_t\int_t^\infty g(s)\left(\int_s^\infty e^{-\delta(u-t)}\,\mathrm{d}u\right)\mathrm{d}s = \frac{1}{\delta}\mathbb{E}_t\int_t^\infty e^{-\delta(s-t)}g(s)\,\mathrm{d}s,\end{aligned}$$

where $g(s) = r_s + \phi_s(\mu_{R,s} - r_s) - \delta - \frac{1}{2}\phi_s^2\sigma_{R,s}^2$ and we used $\int_s^\infty e^{-\delta(u-t)}\,\mathrm{d}u = e^{-\delta(s-t)}/\delta$. Therefore:

$$V_t = \frac{\ln(\delta W_t)}{\delta} + \frac{1}{\delta}\mathbb{E}_t\int_t^\infty e^{-\delta(u-t)}\left(r_u + \phi_u(\mu_{R,u} - r_u) - \tfrac{1}{2}\phi_u^2\sigma_{R,u}^2 - \delta\right)\mathrm{d}u. \tag{142}$$

Step 2: Intermediary's value. The financier is the marginal investor with $\Theta_t = \phi_t\sigma_R$ and $\mu_R - r = \sigma_R\Theta = \phi_t\sigma_R^2$. The portfolio return term simplifies:

$$\phi(\mu_R - r) - \tfrac{1}{2}\phi^2\sigma_R^2 = \phi\cdot\phi\sigma_R^2 - \tfrac{1}{2}\phi^2\sigma_R^2 = \tfrac{1}{2}\phi^2\sigma_R^2 = \tfrac{1}{2}\Theta^2.$$

Therefore, $V^{\mathrm{int}}(W,\omega) = [\ln(\delta W)]/\delta + (1/\delta)\mathbb{E}_\omega\int_0^\infty e^{-\delta u}[\frac{1}{2}\Theta_u^2 + r_u - \delta]\,\mathrm{d}u$.

Step 3: Autarky value. A financier in autarky holds only bonds ($\phi = 0$, so $\Theta = 0$): $V^{\mathrm{aut}}(W,\omega) = [\ln(\delta W)]/\delta + (1/\delta)\mathbb{E}_\omega\int_0^\infty e^{-\delta u}[r_u - \delta]\,\mathrm{d}u$.

Step 4: IC constraint. The IC constraint is $V^{\mathrm{int}}(H_i,\omega) \ge V^{\mathrm{aut}}(\theta\phi H_i,\omega)$. Subtracting:

$$\begin{aligned}V^{\mathrm{int}} - V^{\mathrm{aut}} &= \frac{\ln(\delta H_i) - \ln(\delta\theta\phi H_i)}{\delta} + \frac{1}{\delta}\mathbb{E}_\omega\int_0^\infty e^{-\delta u}\tfrac{1}{2}\Theta_u^2\,\mathrm{d}u \\ &= \frac{-\ln(\theta\phi)}{\delta} + \mathcal{F}(\omega),\end{aligned}$$

where $\mathcal{F}(\omega) = (1/(2\delta))\mathbb{E}_\omega\int_0^\infty e^{-\delta u}\Theta_u^2\,\mathrm{d}u$ is the franchise value (141). The IC condition $V^{\mathrm{int}} \ge V^{\mathrm{aut}}$ becomes $\mathcal{F}(\omega) \ge \ln(\theta\phi)/\delta$, i.e., $\theta\phi \le \exp(\delta\mathcal{F}(\omega))$, yielding (140). □

Proposition 38 (Properties of the franchise value). *(i)* $\mathcal{F}(\omega) > 0$ *for all* $\omega \in (0,1)$.

(ii) $\mathcal{F}$ *solves the Feynman–Kac equation* $\frac{1}{2}\sigma_\omega^2\mathcal{F}'' + (\sigma_\omega^2/\omega)\mathcal{F}' - \delta\mathcal{F} + \Theta^2/(2\delta) = 0$.

(iii) At both absorbing steady states: $\mathcal{F}(\underline{\omega}) = \mathcal{F}(1/\epsilon) = \sigma^2/(2\delta^2)$.

(iv) In the recovery regime: $\mathcal{F}(\omega) > \sigma^2/(2\delta^2)$ (the franchise value is elevated when frictions are strongest).

Proof. (i) The market price of risk satisfies $\Theta_t = \phi_t \sigma_R > 0$ whenever $\phi_t > 0$ and $\sigma_R > 0$, which holds on a set of positive Lebesgue measure a.s. (since the economy starts in the interior of $(0,1)$). Therefore $\mathcal{F}(\omega) = (1/(2\delta))\mathbb{E}_\omega \int_0^\infty e^{-\delta t}\Theta_t^2 \, \mathrm{d}t > 0$.

(ii) Define $u(\omega) = \mathbb{E}_\omega \int_0^\infty e^{-\delta t}\Theta(\omega_t)^2 \, \mathrm{d}t$. By the Feynman–Kac theorem (Karatzas and Shreve, 1991, Ch. 4), u solves $\frac{1}{2}\sigma_\omega^2 u'' + \mu_\omega u' - \delta u + \Theta^2 = 0$. (The theorem applies because $\Theta(\omega)$ is continuous and bounded on each invariant regime—ϕ is bounded by the leverage constraint—and $\delta > 0$ ensures exponential discounting.) Since $\mathcal{F} = u/(2\delta)$, dividing through by 2δ: $\frac{1}{2}\sigma_\omega^2 \mathcal{F}'' + \mu_\omega \mathcal{F}' - \delta\mathcal{F} + \Theta^2/(2\delta) = 0$. Substituting $\mu_\omega = \sigma_\omega^2/\omega$ (equation (74)) yields the stated ODE.

(iii) At $\omega = \underline{\omega}$ (absorbing), $\omega_t = \underline{\omega}$ for all $t \geq 0$, so $\Theta_t = \Theta(\underline{\omega}) = \sigma$ (since $\sigma_\omega(\underline{\omega}) = 0$ gives $\Theta = \sigma$ by Theorem 7). Therefore $\mathcal{F}(\underline{\omega}) = (1/(2\delta)) \int_0^\infty e^{-\delta t}\sigma^2 \, \mathrm{d}t = (\sigma^2/(2\delta)) \cdot (1/\delta) = \sigma^2/(2\delta^2)$. Equivalently, at the absorbing state the Feynman–Kac equation (ii) reduces to $-\delta\mathcal{F} + \sigma^2/(2\delta) = 0$, giving $\mathcal{F} = \sigma^2/(2\delta^2)$. The same argument applies at $\omega = 1/\epsilon$.

(iv) In the recovery regime $(\underline{\omega}, 1/\epsilon)$, $\sigma_\omega > 0$ (Proposition 29), so $\Theta(\omega) = \sigma + \sigma_\omega/\omega + \sigma_\omega/[(1+\varphi)(1-\omega)] > \sigma$ by Theorem 7. Paths from $\omega_0 \in (\underline{\omega}, 1/\epsilon)$ spend positive time in the interior before converging to $1/\epsilon$, during which $\Theta_t > \sigma$. Therefore $\mathbb{E}_{\omega_0} \int_0^\infty e^{-\delta t}\Theta_t^2 \, \mathrm{d}t > \mathbb{E}_{\omega_0} \int_0^\infty e^{-\delta t}\sigma^2 \, \mathrm{d}t = \sigma^2/\delta$, giving $\mathcal{F}(\omega_0) > \sigma^2/(2\delta^2)$. □

G Employment Multiplier and Welfare Analysis

G.1 Employment Multiplier of Recapitalisation

Corollary 39 (Employment multiplier of recapitalisation)**.** *Let $\omega_0 < \underline{\omega}$ and suppose recapitalisation raises ω to $\omega_{0^+} = \omega_0 + \Delta > \underline{\omega}$.*

(i) ***Immediate gain.*** *Employment jumps from $N(\omega_0)$ to $N(\omega_0 + \Delta)$. The percentage gain is*

$$\frac{N(\omega_0 + \Delta) - N(\omega_0)}{N(\omega_0)} = \left(\frac{1 - \omega_0}{1 - \omega_0 - \Delta}\right)^{\frac{1}{1+\varphi}} - 1. \tag{143}$$

(ii) ***Long-run gain.*** *As $\omega_t \to 1/\epsilon$, employment converges to the efficient level $N(1/\epsilon) = 1$. The permanent percentage employment gain relative to the bad steady state is*

$$\frac{N(1/\epsilon) - N(\underline{\omega})}{N(\underline{\omega})} = \left(\frac{1 - \underline{\omega}}{1 - 1/\epsilon}\right)^{\frac{1}{1+\varphi}} - 1. \tag{144}$$

(iii) ***Convexity.*** *The employment function $N(\omega)$ is strictly convex in ω: the output gap $1 - N(\omega)/N(1/\epsilon)$ grows at an increasing rate as intermediary capital ω falls. Consequently, the cumulative output loss from remaining in the crisis regime is convex in crisis severity, making the long-run benefit of recapitalisation larger for deeper crises.*

Proof. Parts (i) and (ii) follow from the employment formula (37). For part (iii), $N(\omega) = [(1-1/\epsilon)/(1-\omega)]^{1/(1+\varphi)}$. Differentiating: $N'(\omega) = [1/(1+\varphi)](1-1/\epsilon)^{1/(1+\varphi)}(1-\omega)^{-(2+\varphi)/(1+\varphi)} > 0$ and $N''(\omega) = [(2+\varphi)/((1+\varphi)^2)](1-1/\epsilon)^{1/(1+\varphi)}(1-\omega)^{-(3+2\varphi)/(1+\varphi)} > 0$. Since N is convex, the output gap $1 - N(\omega)$ is concave, meaning it accelerates as ω falls: each unit decline in intermediary capital produces a larger employment loss than the previous one. □

G.2 Welfare Analysis

The household's utility depends on consumption $C_{h,t} = (1-\omega_t)Y_t$ and labour supply N_t. At the bad steady state ($\omega = \underline{\omega}$), the household has a higher consumption share ($1-\underline{\omega} > 1-1/\epsilon$) but employment is depressed. The household therefore weakly prefers the bad steady state: the increase in consumption share more than compensates for the lower output, because utility is logarithmic and the consumption share effect dominates.

Recapitalisation is therefore not a Pareto improvement: it transfers resources from the household to the financier. However, the employment gains are convex in crisis severity (Corollary 39), while the moral hazard cost is bounded by the leverage constraint (Proposition 37): the financier cannot divert more than θ per unit of assets under intermediation. For any crisis severe enough to trigger QE ($\omega < \omega_c$), the benefit-to-cost ratio exceeds 2 under baseline calibration. A utilitarian social planner with equal weights would favour recapitalisation whenever $\omega < \underline{\omega}$.

H Proofs of Policy Results

Proof of Proposition 13. At $\omega = \underline{\omega}$, $\bar{\phi}(\underline{\omega}) = 1 + \lambda(\underline{\omega} - \underline{\omega}) = 1$. By (16), $\alpha(\underline{\omega}) = 1 - \epsilon \cdot 1 \cdot \underline{\omega} = 1 - \epsilon\underline{\omega} > 0$ (since $\underline{\omega} < 1/\epsilon$ by Assumption 2). The central bank's equity holdings are $\alpha P_{D,t} = (1 - \epsilon\underline{\omega})\, Y_t/(\epsilon\delta)$. Since $\omega = \underline{\omega}$ is absorbing (Theorem 5), these holdings are permanent. For the equity premium: at the absorbing state, $\sigma_\omega(\underline{\omega}) = 0$ (Proposition 29), so Theorem 7 gives $\sigma_R = \sigma$ and $\Theta = \sigma$. The equity excess return is $\mu_R - r = \sigma_R\Theta = \sigma^2$. □

Proof of Proposition 14. Define $\Omega(\omega) \equiv \omega\, P_Y(\omega) = \omega A N(\omega)/\delta$, the financier's dollar wealth as a function of the state. Since $N > 0$ and $N' > 0$, we have $\Omega'(\omega) = (A/\delta)[N(\omega)+\omega N'(\omega)] > 0$, so Ω is strictly increasing and continuous with $\Omega(0) = 0$ and $\Omega(\omega) \to \infty$ as $\omega \to 1$. A fiscal

transfer of $T > 0$ dollars raises the financier's wealth from $\Omega(\omega_0)$ to $\Omega(\omega_0)+T$. Because total wealth adjusts to $P_Y(\omega_{0^+}) = AN(\omega_{0^+})/\delta$, the new wealth share ω_{0^+} is the unique solution to the fixed-point equation

$$\omega_{0^+} N(\omega_{0^+}) = \omega_0 N(\omega_0) + \frac{\delta T}{A},$$

i.e. $\omega_{0^+} = \Omega^{-1}(\Omega(\omega_0)+T)$. For any target $\omega^* \in (\omega_0, 1/\epsilon)$, the transfer $T^* = \Omega(\omega^*)-\Omega(\omega_0) > 0$ achieves $\omega_{0^+} = \omega^*$. Choosing $\omega^* > \underline{\omega}$ places the economy in the recovery regime $(\underline{\omega}, 1/\epsilon)$. By Theorem 6, $\omega_t \to 1/\epsilon$ a.s. from any initial condition in the recovery regime. The convergence requires no further intervention because the recovery regime is invariant (Proposition 31) and the drift $\mu_\omega = \sigma_\omega^2/\omega > 0$ pushes ω toward $1/\epsilon$. □

I Translation of Hugonnier's Results

This appendix states the results of Hugonnier (2012) that we use, first in his original notation, and then translates them into the notation of our model.

Hugonnier's notation

Hugonnier (2012) considers a two-agent economy with agents labelled 1 (unconstrained) and 2 (portfolio-constrained). The agents have utility functions u_1 and u_2 defined over consumption. The weighting process λ_t is the ratio of their marginal utilities:

$$\lambda_t = \frac{u_2'(c_{2,t})}{u_1'(c_{1,t})}, \tag{145}$$

where $c_{1,t}$ and $c_{2,t}$ are the agents' equilibrium consumption rates.

Hugonnier's Theorem 1. In the economy with a portfolio constraint on agent 2, equilibrium asset prices contain rational bubbles if and only if λ_t is a strict local martingale (equivalently, $\mathbb{E}_t[\lambda_T] < \lambda_t$ for some $T > t$).

Hugonnier's Proposition 6. In a finite-horizon economy $[0, T]$ with log utility, discount rate ρ, and weighting process volatility $\|v_\lambda\| = v$, the gap-to-price ratio (which Hugonnier calls the bubble-to-price ratio) is:

$$b(t, s_t) = \frac{1}{\rho\eta(t)} H(T-t, s_t; a_1) + \frac{\eta'(t)}{\rho\eta(t)} H(T-t, s_t; 1), \tag{146}$$

where $\eta(t) = (1/\rho)(1 - e^{-\rho(T-t)})$, $s_t = \lambda_t/(1+\lambda_t)$ is agent 2's consumption share, $a_1 = \sqrt{1 + 8\rho/v^2}$, and $H(\tau, s; a) = s^{(1+a)/2}\Phi(d_+) + s^{(1-a)/2}\Phi(d_-)$ with $d_\pm = (\ln s)/(v\sqrt{\tau}) \pm (a/2)v\sqrt{\tau}$.

Translation to our notation

The correspondence between Hugonnier's notation and ours is:

Hugonnier	**This paper**
Agent 1 (unconstrained)	Financier (i)
Agent 2 (constrained)	Household (h)
$c_{1,t}$	$C_{i,t} = \omega_t Y_t$
$c_{2,t}$	$C_{h,t} = (1-\omega_t)Y_t$
$u_j'(c) = 1/c$ (log utility)	$1/C_{j,t}$
$\lambda_t = u_2'(c_2)/u_1'(c_1)$	$C_{i,t}/C_{h,t} = \omega_t/(1-\omega_t)$
$s_t = \lambda_t/(1+\lambda_t)$	$(1-\omega_t)$
ρ	δ

Important note on the labelling convention. In Hugonnier's framework, agent 2 is portfolio-constrained. In our model, the household (agent h) faces the participation constraint (she cannot hold equity), so the household corresponds to Hugonnier's agent 2. Hugonnier's weighting process $\lambda_t = u_2'(c_2)/u_1'(c_1) = C_{i,t}/C_{h,t} = \omega_t/(1-\omega_t)$ is then related to our $Q_t = 1/\omega_t$ by

$$\lambda_t = \frac{\omega_t}{1-\omega_t} = \frac{1}{Q_t - 1}.$$

The constrained agent's consumption share is $s_t = \lambda_t/(1+\lambda_t) = 1 - \omega_t$, which is the household's wealth share. In Section 5.3, we instead use the equivalent formulation $\lambda_t = C_{h,t}/C_{i,t} = (1-\omega_t)/\omega_t = Q_t - 1$, which is a monotone transformation of Hugonnier's λ and inherits the same strict/true local martingale classification.

Translation of Theorem 1. The intermediation gap is positive if and only if $Q_t - 1$ (equivalently, $Q_t = 1/\omega_t$) is a strict local martingale. This is the criterion (32) stated in Section 5.3.

Translation of Proposition 6. Taking $T \to \infty$: $\eta(t) \to 1/\delta$, $\eta'(t) \to 0$, $H(\infty, s; a_1) \to s^{(1+a_1)/2}$ for $s \in (0,1)$. Substituting $s = 1-\omega$ and $\gamma = (1+a_1)/2$ with $a_1 = \sqrt{1 + 8\delta/v^2}$

yields the intermediation gap formula (139):

$$\frac{B_t}{P_{D,t}} = (1-\omega_t)^{\gamma}, \qquad \gamma = \frac{1+\sqrt{1+8\delta/v^2}}{2}.$$

The full derivation is in Appendix E.

J Online Appendix N: Partial Participation and CRRA Robustness

This appendix extends the model to address robustness along two dimensions: partial household equity participation and CRRA preferences.

J.1 Extended Model

We allow the household to invest a fixed fraction $\alpha_h \in [0,1)$ of her financial wealth $H_{h,t}$ in the equity market. When $\alpha_h = 0$, we recover the main paper.

All identities from the main paper that depend on log utility and goods market clearing continue to hold:

$$C_{i,t} = \omega_t Y_t, \qquad C_{h,t} = (1-\omega_t) Y_t, \tag{147}$$

$$H_{i,t} = \omega_t P_{Y,t}, \qquad H_{h,t} = (1/\epsilon - \omega_t)\, P_{Y,t}. \tag{148}$$

Equity market clearing with household participation requires:

$$\phi(\omega) = \alpha_h + \frac{1-\alpha_h}{\epsilon\omega}. \tag{149}$$

J.2 Robustness of Equilibrium Failure

Theorem 40 (Robustness of equilibrium failure)**.** *For any $\alpha_h \in [0,1)$, no competitive equilibrium exists when $\omega < \omega_c(\alpha_h)$ in the absence of central bank intervention, where $\omega_c(\alpha_h) > 0$ is the unique solution to*

$$\alpha_h + \frac{1-\alpha_h}{\epsilon\omega_c} = 1 + \lambda(\omega_c - \underline{\omega}). \tag{150}$$

Proof. The required portfolio share $\phi(\omega) = \alpha_h + (1-\alpha_h)/(\epsilon\omega)$ is strictly decreasing and $\phi(\omega) \to \infty$ as $\omega \to 0^+$ for any $\alpha_h < 1$. The leverage constraint $\phi \le \bar{\phi}(\omega) = 1 + \lambda(\omega - \underline{\omega})$ is bounded. There exists a unique $\omega_c(\alpha_h) > 0$ at which $\phi(\omega_c) = \bar{\phi}(\omega_c)$. For $\omega < \omega_c(\alpha_h)$, the argument of Theorem 1 applies verbatim. □

Proposition 41 (Comparative statics of the crisis threshold)**.** *$\omega_c(\alpha_h)$ is strictly decreasing in α_h:*

$$\frac{\mathrm{d}\omega_c}{\mathrm{d}\alpha_h} = \frac{1 - 1/(\epsilon\omega_c)}{\lambda + (1-\alpha_h)/(\epsilon\omega_c^2)} < 0. \tag{151}$$

The crisis threshold satisfies $\omega_c(0) = \omega_c$ (the main paper's threshold) and $\omega_c(\alpha_h) \to \underline{\omega}$ as $\alpha_h \to 1$: even with full household equity participation, the crisis threshold cannot fall below the minimum capital requirement.

Proof. Implicit differentiation of (150). Since $\omega_c < 1/\epsilon$, the numerator $1 - 1/(\epsilon\omega_c) < 0$ and the denominator is positive. □

Quantitative illustration.

α_h	$\omega_c(\alpha_h)$	CB equity at $\underline{\omega}$	Interpretation
0.00	0.078	80%	Main paper
0.10	0.076	72%	Modest participation
0.30	0.073	56%	Substantial participation
0.50	0.068	40%	Equal risk sharing
0.80	0.054	16%	Mostly direct
0.95	0.037	4%	Near-full participation

J.3 Dynamics and Preservation of Results

Proposition 42 (Dynamics with partial participation)**.** *The financier's wealth share evolves as $\mathrm{d}\omega_t = (\sigma_\omega^2/\omega)\,\mathrm{d}t + \sigma_\omega\,\mathrm{d}Z_t$ with:*

$$\sigma_\omega(\omega;\alpha_h) = \begin{cases} \dfrac{\sigma(1+\varphi)\lambda(1-\omega)\,\omega\,(\omega-\underline{\omega})}{(1+\varphi)(1-\omega) - \lambda\omega(\omega-\underline{\omega})} & \text{if } \omega < \omega_c(\alpha_h), \\ \dfrac{\sigma(1+\varphi)(1-\alpha_h)(1-\omega)(1/\epsilon-\omega)}{(1+\varphi)(1-\omega) - (1-\alpha_h)(1/\epsilon-\omega)} & \text{if } \omega \geq \omega_c(\alpha_h). \end{cases} \tag{152}$$

The constrained-region volatility is independent of α_h.

Corollary 43 (Preservation of main results)**.** *For any $\alpha_h \in [0,1)$: (i) $Q_t = 1/\omega_t$ is a non-negative local martingale; (ii) σ_ω has exactly two interior zeros at $\underline{\omega}$ and $1/\epsilon$; (iii) the state space partitions into three invariant regimes; (iv) the Permanent QE and QE Trap mechanisms survive.*

J.4 Bubble Classification

Theorem 44 (Bubble classification with partial participation)**.** *For any $\alpha_h \in [0, 1)$: (i) in the baseline model (no leverage constraint), $Q = 1/\omega$ is a strict local martingale and rational bubbles exist; (ii) in the constrained model, Q is a true martingale and no bubbles exist.*

Proof. The bubble classification depends on σ_ω near $\omega = 0$. In the baseline: $\sigma_\omega \to (1 - \alpha_h)c_0 > 0$ for $\alpha_h < 1$, giving a convergent integral (strict local martingale). With the constraint: $|\sigma_\omega| \sim c_0'\omega$, giving a divergent integral (true martingale). □

J.5 CRRA / Power Utility

We now replace the household's log utility with CRRA (power) utility, parameterised by the elasticity of intertemporal substitution $\psi > 0$ (with $\psi = 1$ recovering the main paper).

Theorem 45 (Equilibrium failure under power utility)**.**

(i) ***Existence region (identical to log utility):*** *A competitive equilibrium with household EIS ψ exists if and only if $\omega \geq \omega_c$, where ω_c solves $\bar{\phi}(\omega_c)\,\omega_c = 1/\epsilon$ and is* independent of ψ.

(ii) ***Failure region (identical to log utility):*** *For $\omega < \omega_c$, the financier at maximum leverage $\bar{\phi}(\omega)$ cannot absorb all equity: $\bar{\phi}(\omega)\,\omega < 1/\epsilon$. No interest rate clears the bond market. This is a structural failure that depends on $(\epsilon, \lambda, \underline{\omega})$ but* not *on ψ.*

(iii) ***Equilibrium characterisation for $\omega \geq \omega_c$:*** *Conditional on existence, the equilibrium quantities depend on ψ through the household's consumption-wealth ratio $c_h(\omega; \psi)$, which solves a second-order ODE on $[\omega_c, 1/\epsilon]$ with boundary condition $c_h(1/\epsilon) = \delta$.*

Proof. Parts (i) and (ii) use only the leverage constraint and limited participation. The required portfolio share is $\phi(\omega) = 1/(\epsilon\omega)$, which exceeds $\bar{\phi}(\omega)$ for $\omega < \omega_c$. Since the household cannot hold equity and her bond demand is a function of ω (not of the interest rate for $\psi = 1$, or with sensitivity of order $|\psi - 1|$ for $\psi \neq 1$), no price adjustment can close the gap. The market-clearing identity $\bar{\phi}(\omega_c)\omega_c = 1/\epsilon$ involves only $(\epsilon, \lambda, \underline{\omega})$; household preferences do not enter.

Part (iii): in the existence region $\omega \geq \omega_c$, the household's HJB equation under CRRA yields a consumption-wealth ratio $c_h(\omega; \psi)$ that replaces the log-utility constant $c_h = \delta$. The ω dynamics acquire an additional drift term $(c_h - \delta)(1-\omega)\omega$, and the equilibrium is uniquely determined for ψ not too far below unity. □

Proposition 46 (Three regimes by EIS)**.**

$\psi = 1$ **(log utility):**

$c_h = \delta$, $Q = 1/\omega$ *is a local martingale,* $\mu_\omega = \sigma_\omega^2/\omega$. *Both* $\underline{\omega}$ *and* $1/\epsilon$ *are steady states. This is the main paper's Permanent QE zone (below* $\underline{\omega}$*) and QE Trap (above* $\underline{\omega}$ *but near it).*

$\psi > 1$ **(high substitutability):**

$c_h(\underline{\omega}) > \delta$, *so the drift correction* $(c_h - \delta)(1-\omega)\omega > 0$ *pushes* ω away from $\underline{\omega}$. *The Permanent QE and QE Trap mechanisms are* weakened*: the bad steady state may become reflecting or repelling rather than absorbing, making fiscal recapitalisation less urgent.*

$\psi < 1$ **(low substitutability):**

$c_h(\underline{\omega}) < \delta$, *so the drift correction is* negative*:* ω *drifts* toward $\underline{\omega}$. *The Permanent QE and QE Trap mechanisms are* deepened*: both the reduced diffusion (smaller* σ_ω*) and the negative drift (deterministic pull toward the boundary) strengthen the trap and increase the required fiscal transfer.*

Proof. Each case follows from the sign of $c_h(\omega;\psi) - \delta$ at the lower boundary. Under power utility, $c_h(\omega;\psi)$ is decreasing in ψ (the precautionary-savings effect): a household with $\psi < 1$ consumes less out of wealth ("saves more"), depressing c_h below δ. The boundary drift $\mu_\omega(\underline{\omega}) = (c_h(\underline{\omega}) - \delta)(1-\underline{\omega})\underline{\omega}$ inherits the sign of $c_h - \delta$, while $\sigma_\omega(\underline{\omega}) = 0$ continues to hold. □

Remark 15 (Key message)**.** *The equilibrium failure is* structurally robust *to power utility. The crisis threshold* ω_c *and the failure mechanism (insufficient risk-bearing capacity) are* ψ*-independent. What changes under power utility is the* equilibrium dynamics in the existence region*: the probability of reaching the failure boundary from any initial* $\omega > \omega_c$*, the speed of convergence to the steady states, and the asset-pricing implications all depend on* ψ*. The* existence *of the failure and the trap is structural; the* severity *of the trap depends on preferences.*

K Online Appendix O: Endogenous Non-Participation via Ambiguity

This appendix provides a microfoundation for limited stock market participation via Knightian uncertainty.

K.1 Household Preferences under Ambiguity

The household observes equity returns but is uncertain about the drift. She considers alternative probability measures $\{\mathbb{P}^\nu : |\nu| \leq \kappa\}$ and evaluates portfolios under worst-case beliefs (maxmin expected utility with rectangular priors, following Chen and Epstein 2002). The parameter κ measures ambiguity and, in the spirit of Bernanke (1983), proxies the informational gap between households and intermediaries: the screening, monitoring, and relationship capital that financial intermediaries accumulate but that households lack.

Proposition 47 (Household optimal portfolio)**.** *The household's optimal equity share, as a fraction of financial wealth H_h, subject to $\alpha_h \in [0,1]$, is:*

$$\alpha_h^*(\omega) = \max\left\{0,\ \min\left\{1,\ \frac{(\Theta-\kappa)(1-\omega)-\sigma_R(1-1/\epsilon)}{\sigma_R(1/\epsilon-\omega)}\right\}\right\}. \tag{153}$$

Proof. Under worst-case beliefs $\mathbb{P}^\nu$ with $|\nu| \leq \kappa$, the equity return drift shifts from μ_R to $\mu_R - \nu\sigma_R$. The worst case for the household is $\nu = \kappa$ (maximally pessimistic):

$$\mu_R^{\text{wc}} - r = \mu_R - r - \kappa\sigma_R = (\Theta - \kappa)\sigma_R.$$

The household maximises log utility over total wealth $W_h = H_h + P_L$, where $H_h = (1/\epsilon - \omega)P_Y$ is financial wealth and $P_L = (1-1/\epsilon)P_Y$ is human capital. Her total equity exposure combines the financial position $\alpha_h H_h$ and the implicit exposure from human capital. Since $P_L = (1-1/\epsilon)P_Y$ and $P_Y = Y/\delta$ has volatility $\sigma_Y = \sigma_R$, human capital has "equity beta" σ_R. The total equity exposure is $\alpha_h H_h + P_L = \alpha_h(1/\epsilon - \omega)P_Y + (1-1/\epsilon)P_Y$.

Under log utility, the optimal portfolio equates the Sharpe ratio to total exposure times return volatility divided by total wealth:

$$\frac{\Theta - \kappa}{\sigma_R} = \frac{[\alpha_h(1/\epsilon - \omega) + (1-1/\epsilon)]\sigma_R}{(1-\omega)},$$

where the denominator $(1-\omega)P_Y = W_h$ is the household's total wealth (financial plus human capital, scaled by P_Y). Solving for α_h:

$$\alpha_h = \frac{(\Theta-\kappa)(1-\omega)/\sigma_R^2 - (1-1/\epsilon)}{1/\epsilon - \omega} = \frac{(\Theta-\kappa)(1-\omega)-\sigma_R(1-1/\epsilon)}{\sigma_R(1/\epsilon-\omega)},$$

where we multiplied through by σ_R. Applying the constraint $\alpha_h \in [0,1]$ gives (153). □

The optimal portfolio has two opposing components. The speculative demand $(\Theta - \kappa)(1-\omega)/[\sigma_R(1/\epsilon - \omega)]$ reflects the ambiguity-adjusted risk premium. The hedging demand

$-(1-1/\epsilon)/(1/\epsilon-\omega)<0$ is *negative*: the household's human capital $(1-1/\epsilon)P_Y$ gives her substantial implicit equity exposure, so she reduces financial equity to manage total risk.

K.2 Robust Equilibrium Failure

Theorem 48 (Robust equilibrium failure)**.** *For $\omega<\underline{\omega}$, no competitive equilibrium exists for* any $\alpha_h\in[0,1]$*, regardless of κ.*

Proof. When $\omega<\underline{\omega}$: $\bar{\phi}(\omega)=1+\lambda(\omega-\underline{\omega})<1$. The total private equity demand is $\bar{\phi}(\omega)\omega P_Y+\alpha_h(1/\epsilon-\omega)P_Y$. Even at $\alpha_h=1$, this equals $[\bar{\phi}(\omega)\omega+1/\epsilon-\omega]P_Y<P_Y/\epsilon=P_D$ because $\bar{\phi}(\omega)<1$ forces $\bar{\phi}(\omega)\omega<\omega$. The gap is structural: $P_D-\max$ private demand $=-\lambda\omega(\omega-\underline{\omega})P_Y>0$. □

Corollary 49 (Conditional failure)**.** *For $\omega\in[\underline{\omega},\omega_c)$, equilibrium is restored iff the household participates with $\alpha_h\geq\alpha_h^{\min}(\omega)=1-\lambda\omega(\omega-\underline{\omega})/(1/\epsilon-\omega)$.*

K.3 Endogenous Non-Participation

Define the *participation threshold function*:

$$\kappa^*(\omega)\equiv\Theta(\omega)-\frac{\sigma_R(\omega)(1-1/\epsilon)}{1-\omega}. \tag{154}$$

The household holds zero equity iff $\kappa\geq\kappa^*(\omega)$. In the unconstrained region:

$$\kappa^*(\omega)=\sigma_R(\omega)\cdot\frac{1-\epsilon\omega}{\epsilon\omega(1-\omega)}. \tag{155}$$

Key properties: $\kappa^*(1/\epsilon)=0$, $\kappa^*(0^+)=+\infty$, and κ^* is strictly decreasing.

Theorem 50 (Endogenous non-participation)**.** *For $\kappa>0$, let $\omega_p(\kappa)\in(0,1/\epsilon)$ solve $\kappa^*(\omega_p)=\kappa$. Then:*

(i) For $\omega\in(\omega_p(\kappa),1/\epsilon)$: the household holds zero equity.

(ii) For $\omega\in(0,\omega_p(\kappa))$: the household's optimal equity share is positive.

(iii) $\omega_p(\kappa)$ is strictly decreasing in κ, with $\omega_p(0)=1/\epsilon$ and $\omega_p(\kappa)\to0$ as $\kappa\to\infty$.

Quantitative thresholds. With baseline parameters ($\epsilon=10$, $\underline{\omega}=0.02$, $\sigma=0.15$):

- At $\omega_c\approx0.078$: $\kappa^*(\omega_c)\approx0.047$. Complete non-participation everywhere requires $\kappa\geq0.047$ (4.7% of the Sharpe ratio).

- At $\underline{\omega} = 0.02$: $\kappa^*(\underline{\omega}) \approx 0.012$. Non-participation at the bad steady state requires $\kappa \geq 0.012$ (1.2% of the Sharpe ratio).

Both thresholds are well within empirical estimates (Ilut and Schneider 2014: $\kappa \approx 0.03$–0.05).

Corollary 51 (Recovery of the main paper)**.** *For $\kappa \geq \kappa^*(\omega_c) \approx 0.047$, $\alpha_h^*(\omega) = 0$ for all ω: all results of the main paper hold exactly.*

K.4 Permanent QE, the QE Trap, and Bubble Classification

Proposition 52 (Preservation)**.** *For any $\kappa \geq 0$: (i) $Q_t = 1/\omega_t$ is a non-negative local martingale; (ii) $\underline{\omega}$ and $1/\epsilon$ remain the only absorbing steady states; (iii) the Permanent QE and QE Trap mechanisms survive; (iv) in the baseline, Q is a strict local martingale (bubbles exist); in the constrained model, Q is a true martingale (no bubbles).*

Proof. The Q-martingale property depends on $\mu_\omega = \sigma_\omega^2/\omega$, which holds for any ϕ. In the constrained region, $\phi = \bar{\phi}(\omega)$ regardless of α_h, so σ_ω is independent of κ. The zeros of σ_ω and the Feller boundary classification are unchanged. □

K.5 Three-Region Equilibrium

For $\kappa \in (0, \kappa^*(\omega_c))$, the economy has three regions: (i) robust failure ($\omega < \underline{\omega}$); (ii) endogenous participation ($\underline{\omega} \leq \omega < \omega_p(\kappa)$); (iii) non-participation ($\omega_p(\kappa) \leq \omega < 1/\epsilon$, identical to the main paper). The effective crisis threshold $\omega_c(\kappa)$ is increasing in κ, with $\omega_c(0) = \underline{\omega}$ and $\omega_c(\kappa) = \omega_c$ (main paper) for $\kappa \geq \kappa^*(\omega_c)$.

The ambiguity parameter κ nests all models:

κ	Household behavior	Model recovered
0	Full participation	Online Appendix N with endogenous $\alpha_h(\omega)$
$(0, 0.047)$	Participates in some states	Three-region model (this appendix)
$[0.047, \infty)$	Never participates	Main paper

L Additional Empirical Tests

L.1 Test 2: Capital Scarcity Predicts Equity Risk Premia

The model predicts that risk premia depend on the intermediary state through the market price of risk (Theorem 7). We define the capital scarcity proxy $\widehat{Q}_t^{HKM} \equiv 1/\widehat{\omega}_t$ and estimate:

$$\widehat{R}^e_{t \to t+h} = a_h + b_h \, \widehat{Q}_t^{HKM} + \varepsilon_{t+h}, \tag{156}$$

where $\widehat{R}^e_{t\to t+h}$ is the cumulative excess equity return. The model predicts $b_h > 0$.

Table 40 shows that the slope is positive at every horizon but precision is modest: marginally significant at $h = 2$ and $h = 4$, imprecise at $h = 1$ and $h = 8$. The evidence is directionally consistent with the model but remains suggestive.

Table 40: Test 2: baseline (QHkm; higher QHkm = lower intermediary wealth share)

	h=1	h=2	h=4	h=8
Constant	-0.004	-0.018	-0.022	-0.006
	(0.017)	(0.033)	(0.063)	(0.143)
QHkm	0.001	0.003*	0.006*	0.009
	(0.001)	(0.002)	(0.003)	(0.007)
Adj. R^2	0.005	0.023	0.036	0.045
No. Obs.	219	218	216	212

The contrast between strong real-economy predictability (Test 1) and weaker return predictability is consistent with three model features: (i) the equity risk premium is a nonlinear, non-monotone function of ω; (ii) risk premia reverse sign across crisis and recovery regimes; and (iii) measurement error attenuates the coefficient more severely for a nonlinear function.

Inference robustness. Hodrick t-statistics are below Newey–West values at all horizons. The Lewellen confidence interval includes zero at all horizons. The Campbell–Yogo test rejects at $h \geq 2$, providing partial support. See Table 41.

Table 41: Inference robustness: Test 2 ($Q^{HKM} \to$ excess returns)

	$h=1$	$h=2$	$h=4$	$h=8$
$\hat{\beta}_{OLS}$	0.001	0.003	0.006	0.009
t_{NW}	1.32	1.75*	1.71*	1.33
$t_{Hodrick}$	1.22	1.63	1.52	1.14
$\hat{\beta}_{Stambaugh}$	0.001	0.003	0.005	0.009
$t_{Stambaugh}$	1.26	1.73	1.71	1.33
Lewellen CI	[-0.00, 0.00]	[-0.00, 0.01]	[-0.00, 0.01]	[-0.00, 0.02]
Lewellen reject H_0	No	No	No	No
C-Y reject H_0	No	Yes	Yes	Yes
Wild bootstrap p	0.246	0.046	0.014	0.008
Block bootstrap p	—	0.040	0.059	0.119
R^2	0.010	0.028	0.041	0.050
N	219	218	216	212

Notes: t_{NW} uses Newey–West HAC standard errors with $\max(h, 4)$ lags. $t_{Hodrick}$ uses Hodrick (1992) reverse-regression standard errors. Stambaugh (1999) bias-corrected coefficient and t-statistic use HAC SE. Lewellen (2004) 95% CI assumes a unit-root predictor. C-Y denotes the Campbell–Yogo (2006) Bonferroni Q-test. Wild bootstrap uses Rademacher weights under $H_0 : \beta = 0$ (5000 replications). Block bootstrap resamples blocks of length h under H_0 (5000 replications) and is reported only for $h > 1$. */**/*** denote significance at 10%/5%/1%.

L.2 Test 3: QE Reacts to Intermediary Capital Shortfall

The model predicts that central bank risk absorption activates when intermediary capital falls below the crisis threshold ω_c. We estimate:

$$\Delta\widehat{\alpha}^{CB}_{t+1} = a + b\,\widehat{\text{Shortfall}}_t + u_{t+1}, \tag{157}$$

where $\widehat{\text{Shortfall}}_t \equiv \max(0, \widehat{\omega}_c - \widehat{\omega}_t)$ over 2003Q1–2024Q4. The model predicts $b > 0$.

Table 42: Test 3: QE reaction function (baseline)

	h=1
Constant	-0.001
	(0.001)
Shortfall	0.525**
	(0.232)
Adj. R^2	0.081
No. Obs.	87

Table 42 shows a positive and significant coefficient: $b = 0.525$ ($t = 2.26$). However,

adding macro-financial controls reduces the coefficient to 0.425 and renders it insignificant ($t = 0.90$; Table 43). The QE reaction function is observationally difficult to distinguish from a broader response to macro-financial conditions.

Table 43: Test 3: QE reaction function (with controls, appendix robustness)

	h=1
Constant	0.004
	(0.009)
Shortfall	0.425
	(0.471)
Yield3m	-0.166*
	(0.101)
TermSpread	-0.049
	(0.169)
CreditSpread	-0.975
	(1.162)
SpxVol	0.078*
	(0.046)
Adj. R^2	0.137
No. Obs.	87

Inference robustness. The baseline specification survives all six inference methods (Table 44).

Table 44: Inference robustness: Test 3 (shortfall → QE response)

	$h = 1$
$\hat{\beta}_{OLS}$	0.525
t_{NW}	2.27**
$t_{Hodrick}$	2.03
$\hat{\beta}_{Stambaugh}$	0.531
$t_{Stambaugh}$	2.29
Lewellen CI	[0.08, 0.99]
Lewellen reject H_0	Yes
C-Y reject H_0	Yes
Wild bootstrap p	0.015
Block bootstrap p	—
R^2	0.092
N	87

Notes: t_{NW} uses Newey–West HAC standard errors with $\max(h, 4)$ lags. $t_{Hodrick}$ uses Hodrick (1992) reverse-regression standard errors. Stambaugh (1999) bias-corrected coefficient and t-statistic use HAC SE. Lewellen (2004) 95% CI assumes a unit-root predictor. C-Y denotes the Campbell–Yogo (2006) Bonferroni Q-test. Wild bootstrap uses Rademacher weights under $H_0 : \beta = 0$ (5000 replications). Block bootstrap resamples blocks of length h under H_0 (5000 replications) and is reported only for $h > 1$. */**/*** denote significance at 10%/5%/1%.

M Online Appendix P: Robustness under Epstein–Zin Preferences

This appendix shows that the equilibrium failure result is robust to replacing the household's log utility with Epstein–Zin (EZ) stochastic differential utility (Duffie and Epstein 1992). The financier retains log utility (which delivers the Q-martingale property and the market price of risk $\Theta = \phi\sigma_R$), while the household has EIS ψ and risk aversion γ, with $\psi \neq 1/\gamma$ in general. Log utility is the special case $\psi = \gamma = 1$; CRRA is $\gamma = 1/\psi$.

M.1 Equilibrium Failure Is Preference-Independent

Theorem 53 (Equilibrium failure under EZ)**.** *The crisis threshold ω_c is* preference-independent*: it depends on $(\epsilon, \lambda, \underline{\omega})$ but not on the household's EIS ψ or risk aversion γ. For $\omega < \omega_c$, no competitive equilibrium exists regardless of the household's preference specification.*

Proof. The failure condition $\bar{\phi}(\omega)\omega < 1/\epsilon$ involves three objects:

1. The leverage constraint $\bar{\phi}(\omega) = 1 + \lambda(\omega - \underline{\omega})$, which depends on $(\lambda, \underline{\omega})$.

2. The equity-to-wealth ratio $P_D/P_Y = 1/\epsilon$, which is a structural consequence of the Dixit–Stiglitz production structure.

3. The equity market clearing condition $\phi\omega = P_D/P_Y = 1/\epsilon$.

The ratio $P_D/P_Y = 1/\epsilon$ requires verification. Under EZ, total wealth is $P_{Y,t} = Y_t/D(\omega_t)$ where $D(\omega) = c_h(\omega)(1-\omega) + \delta\omega$ is the aggregate consumption–wealth ratio (which depends on (ψ, γ) through c_h). So $P_Y \neq Y/\delta$ in general. However, the equity price $P_{D,t}$ is the present value of dividends $D_s = Y_s/\epsilon$ under the *financier's* SDF $\Lambda_{i,t} = e^{-\delta t}/(\delta H_{i,t})$. Since total wealth P_Y is the present value of Y_s under the same SDF, linearity gives $P_D/P_Y = (1/\epsilon) \cdot \mathbb{E}[\int \Lambda_i Y_s \, \mathrm{d}s] \,/\, \mathbb{E}[\int \Lambda_i Y_s \, \mathrm{d}s] = 1/\epsilon$. This ratio is exact and preference-independent: the household's EZ parameters cancel.

With $P_D/P_Y = 1/\epsilon$, equity clearing requires $\phi = 1/(\epsilon\omega)$. When $\omega < \omega_c$: $1/(\epsilon\omega) > \bar{\phi}(\omega)$ and no competitive equilibrium exists. The household's (ψ, γ) do not enter. □

Interpretation. The crisis threshold ω_c is a *structural* property of the economy's intermediation architecture—the leverage constraint, participation restrictions, and production structure. The vulnerability to market failure is the same whether households have log utility, high risk aversion, low EIS, or any Epstein–Zin specification.

M.2 Permanent QE and the QE Trap Under EZ

Under log utility (main paper, Remark 2), the absorbing steady state at $\underline{\omega}$ creates *Permanent QE* on $(0, \underline{\omega})$, and the slow drift just above $\underline{\omega}$ creates the *QE Trap* on $(\underline{\omega}, \omega_T)$. Under EZ, both mechanisms are modified:

Proposition 54 (Trap mechanism under EZ)**.**

(i) For $\psi < 1$ and any $\gamma > 0$: $c_h(\underline{\omega}; \psi, \gamma) < \delta$.

(ii) For $\psi < 1$ and $\gamma > 1/\psi$ (EZ with high risk aversion): $c_h(\underline{\omega}; \psi, \gamma) < c_h^{\mathrm{CRRA}}(\underline{\omega}; \psi) < \delta$. The trap is strengthened *relative to CRRA.*

(iii) The bad steady state $\underline{\omega}$ is no longer absorbing: $\mu_\omega(\underline{\omega}) = (c_h(\underline{\omega}) - \delta)(1 - \underline{\omega})\underline{\omega} < 0$. The economy drifts below *$\underline{\omega}$—the trap is a basin of attraction with a stationary distribution, not a permanent absorbing state.*

Proof. At $\omega = \underline{\omega}$, $\sigma_\omega(\underline{\omega}) = 0$ (Proposition 29). The ω-dependent component of the household's continuation-value volatility σ_V therefore vanishes. However, σ_V does *not* vanish

entirely: the household retains direct exposure to productivity risk through A_t, giving $\sigma_V^2(\underline{\omega}) = (\alpha\sigma)^2 > 0$ where $\alpha = \psi(1+\varphi)/(\psi\varphi+1)$.

Under CRRA ($\gamma = 1/\psi$), the precautionary channel is inactive and the standard Merton consumption–wealth formula gives $c_h^{\text{CRRA}} < \delta$ for $\psi < 1$ (verified numerically: at $\psi = 1/2$ and frictionless prices, $c_h \approx 0.029$ vs $\delta = 0.04$, for all γ).

Under EZ with $\gamma > 1/\psi$, the precautionary channel depresses c_h further. The EZ Euler equation acquires a term proportional to $-\frac{1}{2}(\gamma - 1/\psi)\psi\sigma_V^2 < 0$, which reduces the consumption–wealth ratio below the CRRA value. Since $\sigma_V^2(\underline{\omega}) > 0$, this correction is active at the boundary, giving $c_h^{\text{EZ}} < c_h^{\text{CRRA}} < \delta$.

Part (iii) follows from the drift formula $\mu_\omega = \sigma_\omega^2/\omega + (c_h - \delta)(1-\omega)\omega$. At $\underline{\omega}$: $\sigma_\omega = 0$ and $c_h < \delta$, so $\mu_\omega = (c_h - \delta)(1-\underline{\omega})\underline{\omega} < 0$. □

Comparison with the main paper. Under log utility ($\psi = \gamma = 1$), $c_h = \delta$, so $\mu_\omega(\underline{\omega}) = 0$ and $\underline{\omega}$ is absorbing: *Permanent QE* obtains on $(0, \underline{\omega})$ and the *QE Trap* on $(\underline{\omega}, \omega_T)$ (Remark 2). Under EZ with $\psi < 1$, $\underline{\omega}$ is no longer absorbing — the strict Permanent QE regime disappears — but the economy drifts *below* $\underline{\omega}$ into the crisis regime and oscillates with a stationary distribution concentrated near it. The absorbing-boundary phenomenon is replaced by a quasi-stationary near-absorption: the trap is wider and deeper even though it is no longer mathematically permanent. Fiscal recapitalisation must push ω above a drift-reversal point $\omega^\dagger > \underline{\omega}$ (where $\mu_\omega(\omega^\dagger) = 0$) rather than merely above $\underline{\omega}$. The required transfer is larger than under log utility.

N Extension Q: Leverage vs. Equity Constraints

He, Kelly, and Manela (2017, Section 4.3) distinguish two classes of intermediary constraint: *leverage constraints* (Adrian and Shin 2014; Brunnermeier and Pedersen 2009), which directly cap portfolio leverage and produce procyclical deleveraging, and *equity constraints* (He and Krishnamurthy 2012, 2013), which require the intermediary to maintain a minimum equity stake and produce countercyclical leverage. Our baseline specification (7) is a leverage constraint: the cap $\bar{\phi}(\omega) = 1 + \lambda(\omega - \underline{\omega})$ tightens as the financier's wealth share ω falls, and at $\omega = \underline{\omega}$ forces the financier to be exactly unlevered ($\bar{\phi} = 1$).

This appendix asks: which of our results are robust to replacing the leverage constraint with an equity constraint? We show that equilibrium failure, QE necessity, the elimination of the intermediation gap, and the Leverage Paradox all survive. The QE Trap does not. We then characterise, for a general leverage function $\bar{\phi}$, the precise condition under which the trap exists: $\bar{\phi}(\omega^*) = 1$ at some interior point. The trap is a consequence of *procyclical*

deleveraging to zero leverage—the defining feature of the leverage-constraint tradition—and cannot arise under an equity constraint.

N.1 Setup: the equity constraint

Replace Assumption 2 with the following. The financier must fund at least fraction $a \in (0,1)$ of her equity investment with her own wealth (skin in the game). Her portfolio equity share is therefore bounded above:

$$\phi_t \leq \frac{1}{a} \equiv \bar{\phi}^{EC}. \tag{158}$$

The cap $1/a$ is *constant*, independent of the state variable ω_t. All other model primitives—preferences, technology, limited participation, the financier's budget constraint (6)—are unchanged.

Remark 16 (Leverage cyclicality)**.** *Under the leverage constraint* (7)*, constrained leverage* $\bar{\phi}(\omega) = 1 + \lambda(\omega - \underline{\omega})$ falls *as* ω *falls (procyclical). Under* (158)*, constrained leverage is fixed at* $1/a$ *regardless of* ω*. Yet realised leverage* $\phi_t = 1/(\epsilon\omega_t)$ *in the unconstrained region is* countercyclical *in both cases: leverage rises as* ω *falls because the financier is the sole equity holder and the equity market* $P_{D,t} = P_{Y,t}/\epsilon$ *does not shrink as fast as financier wealth. The distinction matters only in the constrained region.*

N.2 Crisis threshold and central bank intervention

Proposition 55 (Equilibrium failure under the equity constraint)**.** *Competitive equilibrium without central bank intervention fails to exist for all* $\omega < \omega_c^{EC}$*, where*

$$\omega_c^{EC} \equiv \frac{a}{\epsilon}. \tag{159}$$

Proof. Equity market clearing without the central bank requires $\phi_t = 1/(\epsilon\omega_t)$ (Appendix E). The equity constraint (158) binds when $1/(\epsilon\omega) > 1/a$, i.e. when $\omega < a/\epsilon$. In this region, the excess equity supply

$$\left[\frac{1}{\epsilon\omega} - \frac{1}{a}\right] \omega \, P_{Y,t} = \frac{1}{\epsilon}\left(1 - \frac{\epsilon\omega}{a}\right) P_{Y,t} > 0$$

has no buyer: the household holds only bonds (limited participation) and her bond demand $(1/\epsilon - \omega)P_{Y,t}$ is perfectly interest-rate inelastic under log utility (8). No price adjustment restores market clearing. □

The failure result is an instance of the impossibility in Proposition 10(iii): any bounded leverage function $\bar{\phi}$—whether state-dependent or constant—satisfies $\bar{\phi}(\omega) < 1/(\epsilon\omega)$ for ω

sufficiently small.

Remark 17 (Quantitative severity of the failure region)**.** *Although equilibrium failure survives under the equity constraint, its* quantitative *extent can be markedly narrower than under our baseline leverage constraint. The failure-region width* $\omega_c^{EC} - 0 = a/\epsilon$ *scales linearly with* a*, the minimum equity share the intermediary must hold. Table 45 compares the two at the empirical calibration* ($\epsilon = 13.80$)*.*

Constraint	Threshold	Failure-basin width	Basin as share of $(0, 1/\epsilon)$
Leverage baseline ($\lambda = 17.4$, $\underline{\omega} = 0.026$)	$\omega_c = 0.051$	0.051	70%
Equity, $a = 1$ (no leverage)	$\omega_c^{EC} = 0.0725$	0.0725	100%
Equity, $a = 0.5$ (leverage cap 2×)	$\omega_c^{EC} = 0.0362$	0.0362	50%
Equity, $a = 0.1$ (leverage cap 10×)	$\omega_c^{EC} = 0.0072$	0.0072	10%
Equity, $a = 0.05$ (leverage cap 20×, HK-style)	$\omega_c^{EC} = 0.0036$	0.0036	5%

Table 45: Failure-region comparison at the empirical calibration ($\epsilon = 13.80$, $1/\epsilon = 0.0725$). The leverage-baseline threshold ω_c is from Table 1; equity thresholds use $\omega_c^{EC} = a/\epsilon$. The last column expresses the failure region as a fraction of the good-steady-state basin $(0, 1/\epsilon)$.

At equity-constraint calibrations consistent with observed intermediary leverage ratios of 10–20× *(He, Kelly, and Manela, 2017), the failure region shrinks to* 5–10% *of the good-steady-state basin—an order of magnitude narrower than the* 70% *implied by our leverage-constraint calibration. QE is needed for a correspondingly smaller range of* ω*, but the qualitative necessity of central-bank intervention in sufficiently severe crises survives.*

Remark 18 (Quantitative severity and household participation)**.** *If an equity-constraint model is taken to deliver "no practical need for QE," the correct reading within our framework is not that* failure does not happen*—it does, by Proposition 55—but that at realistic* a *the failure region* $(0, a/\epsilon)$ *rarely binds in data. Whether it binds depends on the joint distribution of* (ω_t, a) *and the intensity of limited participation. Online Appendix J shows that partial household participation with* $\alpha_h \geq 1/\epsilon$ *would eliminate the failure region entirely; the failure mechanism is therefore a conjunction of* bounded intermediary leverage *and* sufficiently strict limited participation*, and weakening either dimension attenuates or eliminates it.*

Proposition 56 (QE existence under the equity constraint)**.** *If the central bank holds fraction*

$$\alpha_t^{EC} = \max\Big(0,\ 1 - \frac{\epsilon\omega_t}{a}\Big) = \begin{cases} 1 - \epsilon\omega_t/a & \text{if } \omega_t < \omega_c^{EC}, \\ 0 & \text{if } \omega_t \geq \omega_c^{EC}, \end{cases} \tag{160}$$

then competitive equilibrium exists for all $\omega \in (0,1)$, with all real quantities ($P_{Y,t}$, $C_{h,t}$, $C_{i,t}$, N_t, Y_t) identical to the unconstrained model.

Proof. The argument follows the proof of the QE existence result in Appendix E. The central bank absorbs the residual equity that the constrained financier cannot hold, financing its purchases by issuing bonds.

Equity market clearing. With α_t^{EC} as defined, the financier holds fraction $(1-\alpha_t^{EC})$ of the equity market. Her leverage is

$$\phi_t = \frac{(1-\alpha_t^{EC})\,P_{D,t}}{H_{i,t}} = \frac{(\epsilon\omega_t/a)\,(P_{Y,t}/\epsilon)}{\omega_t\,P_{Y,t}} = \frac{\omega_t\,P_{Y,t}/a}{\omega_t\,P_{Y,t}} = \frac{1}{a} = \bar{\phi}^{EC},$$

where we used $1-\alpha_t^{EC} = \epsilon\omega_t/a$ (from (160)), $P_{D,t} = P_{Y,t}/\epsilon$, and $H_{i,t} = \omega_t P_{Y,t}$.

Bond market clearing. The household supplies $(1/\epsilon - \omega)P_Y$ in bonds (8). The financier borrows $(\phi-1)\omega P_Y = (1/a-1)\omega P_Y$ to finance her leveraged equity position. The central bank issues bonds worth $\alpha^{EC}P_D = \alpha^{EC}P_Y/\epsilon$ to finance its equity purchases. Substituting $\alpha^{EC} = 1-\epsilon\omega/a$:

$$\begin{aligned}(1/a-1)\omega P_Y + \alpha^{EC}P_Y/\epsilon &= \frac{\omega}{a}\,P_Y - \omega P_Y + \frac{P_Y}{\epsilon} - \frac{\omega P_Y}{a} \\ &= \frac{P_Y}{\epsilon} - \omega P_Y = \left(\frac{1}{\epsilon}-\omega\right)P_Y. \qquad \checkmark\end{aligned}$$

All real quantities are unchanged since $P_{Y,t} = Y_t/\delta$ depends only on log utility ($C_{h,t}+C_{i,t} = Y_t$ and $C_j = \delta H_j$), not on who holds equity. □

N.3 Wealth dynamics under the equity constraint

Proposition 57 (Volatility of ω under the equity constraint)**.** *In the constrained region $\omega \in (0, \omega_c^{EC})$, the financier's wealth-share volatility is*

$$\sigma_\omega^{EC}(\omega) = \frac{\sigma(1+\varphi)(1/a-1)(1-\omega)\,\omega}{D_{EC}(\omega)}, \tag{161}$$

where

$$D_{EC}(\omega) \equiv (1+\varphi)(1-\omega) - \omega(1/a-1) = (1+\varphi) - \big(\varphi + 1/a\big)\omega > 0. \tag{162}$$

In the unconstrained region $\omega \geq \omega_c^{EC}$, the volatility is $\sigma_\omega^u(\omega)$ as in the baseline model. The function σ_ω is continuous at ω_c^{EC}. The drift is $\mu_\omega = \sigma_\omega^2/\omega$ in both regions.

Proof. Step 1 (Return volatility). Since $P_{D,t} = Y_t/(\epsilon\delta)$ and $Y_t = A_t N(\omega_t)$, Itô's lemma gives

$$\sigma_R = \sigma_Y = \sigma + \frac{\sigma_\omega}{(1+\varphi)(1-\omega)}.$$

This is identical to the baseline derivation in Appendix F and does not depend on the form of the constraint.

Step 2 (Wealth-share volatility). Itô's quotient rule on $\omega_t = H_{i,t}/P_{Y,t}$ gives (Appendix F, Step 2):

$$\sigma_\omega = \omega(\phi - 1)\sigma_R.$$

In the constrained region, $\phi = 1/a$, so $\phi - 1 = (1-a)/a = 1/a - 1 > 0$. Thus

$$\sigma_\omega = \omega(1/a - 1)\sigma_R. \tag{163}$$

Step 3 (Fixed-point equation). Substituting Step 1 into (163):

$$\begin{aligned}\sigma_\omega &= \omega(1/a-1)\left[\sigma + \frac{\sigma_\omega}{(1+\varphi)(1-\omega)}\right] \\ &= \omega(1/a-1)\sigma + \frac{\omega(1/a-1)}{(1+\varphi)(1-\omega)}\,\sigma_\omega.\end{aligned}$$

Collecting σ_ω terms:

$$\begin{aligned}\sigma_\omega\left[1 - \frac{\omega(1/a-1)}{(1+\varphi)(1-\omega)}\right] &= \omega(1/a-1)\sigma \\ \sigma_\omega \cdot \frac{(1+\varphi)(1-\omega) - \omega(1/a-1)}{(1+\varphi)(1-\omega)} &= \omega(1/a-1)\sigma,\end{aligned}$$

giving (161) with denominator $D_{EC}(\omega) = (1+\varphi)(1-\omega) - \omega(1/a-1)$.

Step 4 (Denominator positivity). The denominator is linear in ω:

$$D_{EC}(\omega) = (1+\varphi) - (\varphi + 1/a)\omega.$$

At $\omega = 0$: $D_{EC} = 1 + \varphi > 0$. At $\omega = \omega_c^{EC} = a/\epsilon$:

$$D_{EC}(a/\epsilon) = (1+\varphi)\left(1 - \frac{a}{\epsilon}\right) - \frac{1-a}{\epsilon} = (1+\varphi) - \frac{(1+\varphi)a + (1-a)}{\epsilon} = (1+\varphi) - \frac{1+\varphi a}{\epsilon}.$$

Since $\epsilon > 1$ and $a < 1$: $(1+\varphi a)/\epsilon < (1+\varphi)/\epsilon < 1+\varphi$. Hence $D_{EC} > 0$ on $[0, a/\epsilon]$.

Step 5 (Continuity at ω_c^{EC}). At $\omega = a/\epsilon$:

$$(1/a - 1)\omega_c^{EC} = \frac{1-a}{a} \cdot \frac{a}{\epsilon} = \frac{1-a}{\epsilon} = \frac{1}{\epsilon} - \frac{a}{\epsilon} = \frac{1}{\epsilon} - \omega_c^{EC}.$$

Substituting into (161): the numerator becomes $\sigma(1+\varphi)(1-\omega_c^{EC})(1/\epsilon - \omega_c^{EC})$ and the denominator becomes $(1+\varphi)(1-\omega_c^{EC}) - (1/\epsilon - \omega_c^{EC}) = D_u(\omega_c^{EC})$, matching $\sigma_\omega^u(\omega_c^{EC})$.

Step 6 (Drift). The Q-martingale property (Appendix G) gives $\mu_\omega = \sigma_\omega^2/\omega$. The proof uses only log utility ($C_i = \delta H_i$) and is independent of the constraint form. □

N.4 No Permanent QE under the equity constraint

Theorem 58 (No Permanent QE)**.** *Under the equity constraint (158):*

(i) $\sigma_\omega^{EC}(\omega) > 0$ *for all* $\omega \in (0, \omega_c^{EC})$*. There is no interior absorbing state in the constrained region—in particular, no absorbing lower boundary* $\underline{\omega}$*, so Permanent QE cannot arise.*

(ii) *For every initial condition* $\omega_0 \in (0, 1/\epsilon)$*,* $\omega_t \to 1/\epsilon$ *almost surely. There is a single regime* $(0, 1/\epsilon)$ *with no invariant sub-intervals.*

(iii) *QE is self-resolving: starting from any* $\omega_0 \in (0, \omega_c^{EC})$*, the economy exits the QE region in finite time almost surely:*

$$P_{\omega_0}\left(\tau_{\omega_c^{EC}} < \infty\right) = 1.$$

Permanent QE (an absorbing lower steady state at a strictly positive $\underline{\omega}$*) is absent: there is no interior absorbing boundary. The* QE Trap phenomenon *of impractically slow recovery can nevertheless arise near* $\omega = 0$*, because* $\sigma_\omega^{EC}(\omega) \sim (1/a - 1)\sigma\omega$ *as* $\omega \to 0$ *and hence* $\mu_\omega(\omega) = \sigma_\omega^2/\omega \to 0$*; for sufficiently depleted initial capital, the deterministic drift time* $\log(\omega_c/\omega_0)/[(1/a-1)^2\sigma^2]$ *can exceed a policy-relevant horizon. What the equity constraint removes is the interior absorbing boundary, not the vanishing-drift mechanism; finite-time exit is guaranteed a.s. from any* $\omega_0 > 0$*, but the expected exit time diverges as* $\omega_0 \to 0$*.*

Proof. Part (i). From (161), $\sigma_\omega^{EC}(\omega)$ is a ratio of two strictly positive quantities for $\omega \in (0, \omega_c^{EC})$: the numerator $\sigma(1+\varphi)(1/a-1)(1-\omega)\omega > 0$ (since $a < 1$, $\omega > 0$, $\omega < 1$) and the denominator $D_{EC}(\omega) > 0$ (Step 4 above).

Contrast with the baseline leverage constraint. The constrained-region volatility in the baseline model (Appendix F) is

$$\sigma_\omega^c(\omega) = \frac{\sigma(1+\varphi)\lambda(1-\omega)\,\omega\,(\omega - \underline{\omega})}{D_c(\omega)}.$$

The factor $(\omega - \underline{\omega})$ changes sign at $\omega = \underline{\omega}$: negative below (crisis regime), zero at $\underline{\omega}$ (absorbing state), positive above (recovery regime). Under the equity constraint, the corresponding factor is $(1/a - 1) > 0$, a *positive constant*. The zero at $\underline{\omega}$ that creates the trap has no analogue.

The economics is direct. Under the leverage constraint, $\bar{\phi}(\underline{\omega}) = 1$: the financier is forced to be exactly unlevered. She earns no excess return over the risk-free rate, so her wealth share has zero drift and zero diffusion. She is stuck. Under the equity constraint, $\bar{\phi}^{EC} = 1/a > 1$: the financier remains leveraged at every $\omega > 0$. She always earns the equity premium, her wealth share always has positive drift $\mu_\omega = \sigma_\omega^2/\omega > 0$, and recovery is endogenous.

Part (ii). The boundary classification at $\omega = 0$ is unchanged from the baseline (Appendix I). Near $\omega = 0$:

$$\sigma_\omega^{EC}(\omega) \sim c_{EC}\,\omega, \qquad c_{EC} \equiv \frac{\sigma(1-a)}{a} > 0,$$

so $m(\omega) = \omega^2/(\omega_0^2\,\sigma_\omega^2) \to 1/(\omega_0^2 c_{EC}^2)$ (bounded). The Feller N-integral $N(0^+) = \int_0 s'(x) \int_0^x m(y)\,\mathrm{d}y\,\mathrm{d}x$ diverges because $s'(\omega) = C/\omega^2$. Hence $\omega = 0$ is a natural boundary. The classification at $\omega = 1/\epsilon$ is identical to the baseline (Appendix I).

Since $\sigma_\omega > 0$ throughout $(0, 1/\epsilon)$ and the only zeros of σ_ω are at the boundaries $\{0, 1/\epsilon\}$, there are no internal unattainable points. The interval $(0, 1/\epsilon)$ is a single invariant regime (continuous paths cannot cross the unattainable boundaries at 0 and $1/\epsilon$).

Convergence to $1/\epsilon$. The inverse wealth share $Q_t = 1/\omega_t$ is a non-negative local martingale ($\mu_Q = 0$, Appendix G), hence a supermartingale by Fatou's lemma. The supermartingale convergence theorem gives $Q_t \to Q_\infty$ a.s. with $\mathbb{E}[Q_\infty] \leq Q_0 < \infty$.

The limit Q_∞ must satisfy $\sigma_Q(Q_\infty) = 0$. To see this: if $\sigma_Q(Q_\infty) \neq 0$, then the quadratic variation $\langle Q \rangle_\infty = \int_0^\infty \sigma_Q^2(Q_t)\,\mathrm{d}t = +\infty$, and by Dambis–Dubins–Schwarz, Q is a time-changed Brownian motion with unbounded time change, which cannot converge (Appendix J). The zeros of σ_Q correspond to the zeros of σ_ω: $\omega \in \{0, 1/\epsilon\}$, i.e., $Q \in \{\epsilon, +\infty\}$.

Since $Q_\infty < \infty$ a.s. (from $\mathbb{E}[Q_\infty] \leq Q_0 < \infty$), we have $Q_\infty = \epsilon$, i.e., $\omega_\infty = 1/\epsilon$.

Part (iii). Since $\omega_c^{EC} = a/\epsilon < 1/\epsilon$ is an interior point of $(0, 1/\epsilon)$, we use the scale function to compute the hitting probability. With $s'(\omega) = C/\omega^2$ (from $2\mu_\omega/\sigma_\omega^2 = 2/\omega$):

$$s(\omega) = -\frac{C}{\omega} + C', \qquad s(0^+) = -\infty.$$

For any $\omega_0 \in (0, \omega_c^{EC})$ and any $\eta \in (0, \omega_0)$, the probability of hitting ω_c^{EC} before η is

$$P_{\omega_0}\big(\tau_{\omega_c^{EC}} < \tau_\eta\big) = \frac{s(\omega_0) - s(\eta)}{s(\omega_c^{EC}) - s(\eta)} = \frac{1/\eta - 1/\omega_0}{1/\eta - \epsilon/a}.$$

Since $\omega_0 < \omega_c^{EC} = a/\epsilon$, we have $1/\omega_0 > \epsilon/a$, and as $\eta \to 0^+$ both numerator and denominator diverge to $+\infty$. Dividing through by $1/\eta$:

$$\frac{1/\eta - 1/\omega_0}{1/\eta - \epsilon/a} = \frac{1 - \eta/\omega_0}{1 - \eta\epsilon/a} \longrightarrow 1 \qquad \text{as } \eta \to 0^+.$$

Hence $P_{\omega_0}(\tau_{\omega_c^{EC}} < \infty) = \lim_{\eta \to 0^+} P_{\omega_0}(\tau_{\omega_c^{EC}} < \tau_\eta) = 1$. □

N.5 Intermediation gap and the Leverage Paradox

Proposition 59 (Q-martingale and gap under the equity constraint)**.** *Under the equity constraint with QE:*

(i) $Q_t = 1/\omega_t$ *is a* true *martingale. The intermediation gap is zero.*

(ii) *The Leverage Paradox (Proposition 10) holds: gap-free prices and unconditional equilibrium existence remain jointly impossible.*

Proof. Part (i). The Q-martingale property (Appendix G) holds for any portfolio process ϕ_t; in particular, it holds when $\phi_t = 1/a$ in the constrained region. To classify Q: near $\omega = 0$, $|\sigma_\omega^{EC}| \sim c_{EC}\omega$, so $\sigma_Q = \sigma_\omega/\omega^2 \sim c_{EC}/\omega = c_{EC}Q$. The Mijatović–Urusov integral test (Appendix K) gives

$$\int_c^\infty \frac{q}{\sigma_Q(q)^2}\,\mathrm{d}q = \int_c^\infty \frac{q}{c_{EC}^2\, q^2}\,\mathrm{d}q = \frac{1}{c_{EC}^2}\int_c^\infty \frac{\mathrm{d}q}{q} = +\infty.$$

Hence Q is a true martingale. By the characterisation in Theorem 9, the intermediation gap is zero.

Part (ii). If equilibrium exists for all ω without central bank intervention, then $\phi(\omega) = 1/(\epsilon\omega)$ everywhere. As $\omega \to 0$, $\phi \to \infty$, so $\sigma_\omega(0^+) = c_0 > 0$ (Appendix K). Then $\sigma_Q \sim c_0 Q^2$ and $\int q^{-3}\,\mathrm{d}q < \infty$: Q is a strict local martingale and the gap is positive. This is independent of the constraint form. □

N.6 General characterisation: when does Permanent QE exist?

The results above show that Permanent QE (an absorbing boundary inside the constrained region) is absent under the equity constraint but present under the leverage constraint, where $\underline{\omega}$ is absorbing and generates the adjacent QE Trap (Remark 2). What distinguishes the two constraint forms? The following theorem characterises when an admissible leverage function creates the absorbing-boundary phenomenon.

Definition 60 (Admissible leverage function)**.** *A function $\bar{\phi} : (0, 1/\epsilon) \to (0, \infty)$ is* admissible *if: (i) $\bar{\phi}$ is continuous; (ii) there exists a unique $\omega_c \in (0, 1/\epsilon)$ such that $\bar{\phi}(\omega) < 1/(\epsilon\omega)$ for $\omega < \omega_c$ and $\bar{\phi}(\omega) \geq 1/(\epsilon\omega)$ for $\omega \geq \omega_c$; (iii) $(1+\varphi)(1-\omega) - \omega(\bar{\phi}(\omega) - 1) > 0$ for all $\omega \in (0, \omega_c)$ (the denominator in (164) is well-defined and positive).*

Condition (iii) is automatically satisfied by any bounded $\bar{\phi}$ when ω is sufficiently small: at $\omega = 0$ the denominator equals $1 + \varphi > 0$, and by continuity it remains positive on a neighbourhood. For the baseline leverage constraint, condition (iii) is Lemma 20. For the equity constraint, it is verified in Step 4 of Proposition 57.

Both the baseline leverage constraint $\bar{\phi}(\omega) = 1 + \lambda(\omega - \underline{\omega})$ (with ω_c defined by (10)) and the equity constraint $\bar{\phi}^{EC} = 1/a$ (with $\omega_c^{EC} = a/\epsilon$) are admissible.

Theorem 61 (Permanent QE characterisation)**.** *Let $\bar{\phi}$ be an admissible leverage function. With QE restoring equilibrium in the constrained region ($\omega < \omega_c$), the financier's wealth-share volatility is*

$$\sigma_\omega(\omega) = \frac{\sigma(1+\varphi)(1-\omega)\,\omega\left(\bar{\phi}(\omega) - 1\right)}{(1+\varphi)(1-\omega) - \omega\left(\bar{\phi}(\omega) - 1\right)}, \qquad \omega \in (0, \omega_c). \tag{164}$$

Then:

(i) ***No Permanent QE.*** *If $\bar{\phi}(\omega) > 1$ for all $\omega \in (0, \omega_c)$, then $\sigma_\omega(\omega) > 0$ throughout the constrained region, $\omega_t \to 1/\epsilon$ a.s. from every initial condition, QE is self-resolving, and there is no absorbing bad steady state. The QE Trap is likewise absent since it requires an adjacent absorbing boundary.*

(ii) ***Permanent QE.*** *If $\bar{\phi}(\omega^*) = 1$ for some $\omega^* \in (0, \omega_c)$, then $\sigma_\omega(\omega^*) = \mu_\omega(\omega^*) = 0$, and ω^* is an absorbing steady state. Moreover:*

- *If $\bar{\phi}(\omega) < 1$ for $\omega \in (0, \omega^*)$ and $\bar{\phi}(\omega) > 1$ for $\omega \in (\omega^*, \omega_c)$, then the constrained region splits into a* Permanent QE regime *$(0, \omega^*)$ converging to ω^* (QE literally permanent) and a* recovery regime *(ω^*, ω_c) converging to $1/\epsilon$. The QE Trap sub-region $(\omega^*, \omega^* + \Delta)$ with slow drift, where recovery takes centuries, lies adjacent to the absorbing boundary.*
- *The point ω^* is an unattainable natural boundary: ω_t converges to ω^* but never reaches it in finite time.*

(iii) *In both cases, $Q = 1/\omega$ is a true martingale (no intermediation gap), and the Leverage Paradox holds.*

Proof. The volatility formula (164) follows from the same derivation as in Appendix F, with $\bar{\phi}(\omega)$ in place of $1+\lambda(\omega-\underline{\omega})$. In the constrained region, $\phi=\bar{\phi}(\omega)$, so $\phi-1=\bar{\phi}(\omega)-1$. The fixed-point equation (Step 3 of Proposition 57) yields (164). The drift $\mu_\omega=\sigma_\omega^2/\omega$ follows from the Q-martingale property, which is independent of the constraint form (Appendix G).

Part (i). If $\bar{\phi}(\omega)>1$ for all $\omega\in(0,\omega_c)$, then $\bar{\phi}(\omega)-1>0$, and every factor in the numerator of (164) is strictly positive for $\omega\in(0,\omega_c)\subset(0,1)$. The denominator is positive by admissibility condition (iii). Hence $\sigma_\omega>0$ and $\mu_\omega=\sigma_\omega^2/\omega>0$ throughout $(0,\omega_c)$.

With no interior zero of σ_ω, the interval $(0,1/\epsilon)$ contains no internal unattainable points. The argument then follows Theorem 58(ii)–(iii): $(0,1/\epsilon)$ is a single invariant regime, $\omega_t\to 1/\epsilon$ a.s., and the economy exits the QE region in finite time a.s.

Part (ii). If $\bar{\phi}(\omega^*)=1$, then $\sigma_\omega(\omega^*)=0$ (the factor $\bar{\phi}(\omega^*)-1=0$ in the numerator of (164)). Since $\mu_\omega=\sigma_\omega^2/\omega$, the drift also vanishes: $\mu_\omega(\omega^*)=0$. The SDE $\mathrm{d}\omega=(\sigma_\omega^2/\omega)\,\mathrm{d}t+\sigma_\omega\,\mathrm{d}Z$ with both coefficients zero at ω^* admits $\omega_t=\omega^*$ as a solution. Pathwise uniqueness (from the local Lipschitz property of σ_ω away from boundaries) implies this is the unique solution starting from ω^*: the point is absorbing.

Boundary classification at ω^*. Since $\bar{\phi}$ is continuous and $\bar{\phi}(\omega^*)=1$, we write $\bar{\phi}(\omega)-1=h(\omega)(\omega-\omega^*)$ where h is continuous and $h(\omega^*)=\bar{\phi}'(\omega^*)$ if $\bar{\phi}$ is differentiable at ω^*. If $h(\omega^*)\neq 0$ (the generic case, including our baseline $\bar{\phi}'(\underline{\omega})=\lambda>0$):

$$\sigma_\omega(\omega)\sim g(\omega^*)\,(\omega-\omega^*),\qquad g(\omega^*)=\frac{\sigma(1+\varphi)(1-\omega^*)\,\omega^*\,h(\omega^*)}{D(\omega^*)}\neq 0,$$

so $\sigma_\omega^2\sim g(\omega^*)^2(\omega-\omega^*)^2$ near ω^*. The speed measure density satisfies

$$m(\omega)=\frac{\omega^2}{\omega_0^2\,\sigma_\omega(\omega)^2}\geq\frac{K}{(\omega-\omega^*)^2}$$

for some $K>0$. Therefore

$$\int^x m(y)\,\mathrm{d}y\geq K\int^x\frac{\mathrm{d}y}{(y-\omega^*)^2}=\frac{K}{|x-\omega^*|}\to+\infty\quad\text{as } x\to\omega^*,$$

and the Feller N-integral diverges:

$$N(\omega^*)=\int s'(x)\int^x m(y)\,\mathrm{d}y\,\mathrm{d}x=+\infty,$$

confirming that ω^* is a natural (unattainable) boundary, exactly as at $\omega=\underline{\omega}$ in Appendix I. (If $h(\omega^*)=0$, so that σ_ω vanishes faster than linearly, the speed measure diverges *a fortiori* and ω^* is still unattainable.)

Regime structure. For $\omega < \omega^*$: $\bar{\phi}(\omega) < 1$ (by assumption), so $\bar{\phi}(\omega) - 1 < 0$ and $\sigma_\omega < 0$ (perverse dynamics: positive shocks reduce ω). For $\omega \in (\omega^*, \omega_c)$: $\bar{\phi}(\omega) > 1$, so $\sigma_\omega > 0$ (standard recovery dynamics). Continuous paths cannot cross the unattainable boundary ω^*, establishing regime invariance. In the crisis regime $(0, \omega^*)$: the drift $\mu_\omega = \sigma_\omega^2/\omega > 0$ pushes ω toward ω^*, and the process converges a.s. to ω^* (QE permanent, by the same supermartingale argument as in Part (i) applied to Q on the sub-interval $(1/\omega^*, \infty)$). In the recovery regime (ω^*, ω_c): the process exits the QE region in finite time a.s. (QE self-resolving, by the scale function argument in Theorem 58(iii)).

Part (iii). Near $\omega = 0$, any admissible $\bar{\phi}$ is bounded, so $|\sigma_\omega| = \omega|\bar{\phi} - 1||\sigma_R| \sim C\omega$ for some $C > 0$. Thus $\sigma_Q \sim CQ$ and $\int q/(C^2q^2)\, \mathrm{d}q = +\infty$: Q is a true martingale regardless of whether the trap exists. The Leverage Paradox follows as before. □

Remark 19 (Permanent QE requires deleveraging to zero leverage)**.** *The theorem identifies the precise economic mechanism behind Permanent QE: the leverage function must force the financier to be* exactly unlevered *($\bar{\phi} = 1$) at some interior point ω^*. (The QE Trap then arises mechanically in the neighbourhood of ω^* because the drift $\mu_\omega = \sigma_\omega^2/\omega$ vanishes as $\omega \to \omega^*$.)*

The economics is as follows. At ω^* with $\phi = 1$, the financier invests her entire wealth in equity and holds no bonds. She does *earn the equity premium: her wealth evolves as $\mathrm{d}H_i/H_i = (\mu_R - \delta)\, \mathrm{d}t + \sigma_R\, \mathrm{d}Z$. But at ω^*, the volatility $\sigma_\omega = 0$ implies $\sigma_R = \sigma_Y = \sigma$ and $\mu_R = \delta + \mu$ (substituting $r = \delta + \mu - \sigma^2$ and $\Theta\sigma_R = \sigma^2$). Thus $\mathrm{d}H_i/H_i = \mu\, \mathrm{d}t + \sigma\, \mathrm{d}Z$, which is identical to $\mathrm{d}P_Y/P_Y$. The financier's wealth tracks total wealth exactly, so the wealth share $\omega = H_i/P_Y$ is frozen: $\mathrm{d}\omega = 0$. The financier earns the market return but cannot* outgrow *the economy; recovery requires $\phi > 1$, i.e., leverage above 1.*

Under the equity constraint with $a < 1$, the financier retains leverage $1/a > 1$ at every state. Her leveraged position amplifies the equity premium: $\phi(\mu_R - r) > (\mu_R - r)$. Her wealth grows faster than P_Y, so $\sigma_\omega > 0$ and $\mu_\omega = \sigma_\omega^2/\omega > 0$. Recovery is endogenous.

Permanent QE is therefore a consequence of procyclical deleveraging to zero leverage at an interior point*—the defining feature of the leverage-constraint tradition—and cannot arise under an equity constraint that preserves $\phi > 1$ everywhere in the interior. The QE Trap phenomenon of impractically slow recovery does* not *disappear with it, however: under the equity constraint, $\mu_\omega \to 0$ as $\omega \to 0$ and the recovery time from a severely depleted ω_0 still grows logarithmically in $1/\omega_0$ (see the bound above). The distinction is qualitative (absorbing interior boundary vs. unattainable boundary at zero) rather than quantitative; a sufficiently deep crisis traps the economy on a policy-relevant horizon under either constraint.*

N.7 Comparison table

Table 46 collects the comparison. All results in the table are proved above or follow directly from the general characterisation in Theorem 61.

N.8 Verification: special and limiting cases

We verify the results against four limiting cases.

Case 1: $a \to 0$ (no constraint). As $a \to 0$: $1/a \to \infty$, $\omega_c^{EC} = a/\epsilon \to 0$, and $\alpha_t^{EC} = 1 - \epsilon\omega/a \to -\infty$ for any $\omega > 0$, so $\alpha_t^{EC} = 0$ (the constraint never binds). The model reduces to the LP baseline without leverage constraint. All dynamics are governed by $\sigma_\omega^u(\omega)$ on $(0, 1/\epsilon)$: Q is a strict local martingale, the intermediation gap is positive, and $\omega \to 1/\epsilon$ a.s. This is consistent with Table 46: removing the constraint removes both the failure region and the elimination of the gap.

Case 2: $a \to 1$ (autarky). As $a \to 1$: $1/a \to 1$, $\omega_c^{EC} \to 1/\epsilon$, and $\bar{\phi}^{EC} \to 1$. The financier is forced to be unlevered ($\phi = 1$) at every state, giving $\sigma_\omega^{EC} \to 0$ everywhere. Every point is absorbing: $\omega_t = \omega_0$ for all t. This degenerate case is excluded by $a < 1$. It is the limiting case of Theorem 61(ii): $\bar{\phi}(\omega) = 1$ for all ω, so every ω^* satisfies $\bar{\phi}(\omega^*) = 1$.

Case 3: recovery of baseline model. Setting $\bar{\phi}(\omega) = 1 + \lambda(\omega - \underline{\omega})$ in (164) gives $\bar{\phi}(\omega) - 1 = \lambda(\omega - \underline{\omega})$, recovering the baseline formula $\sigma_\omega^c = \sigma(1+\varphi)\lambda(1-\omega)\omega(\omega - \underline{\omega})/D_c(\omega)$. The zero at $\omega^* = \underline{\omega}$ (where $\bar{\phi} = 1$) confirms the trap. Setting $\bar{\phi} = 1/a$ gives $\bar{\phi} - 1 = (1-a)/a$ (a positive constant), recovering (161) with no interior zero.

Case 4: continuity at ω_c^{EC} (numerical). At baseline $\epsilon = 10$, $\varphi = 1$, $\sigma = 0.15$, and $a = 0.1$: $\omega_c^{EC} = 0.01$.

Constrained formula (161) at $\omega = 0.01$:

$$\begin{aligned} \text{Numerator} &= 0.15 \times 2 \times 9 \times 0.99 \times 0.01 = 0.02673, \\ D_{EC}(0.01) &= 2 - 11 \times 0.01 = 1.89, \\ \sigma_\omega^{EC} &= 0.02673/1.89 = 0.01414. \end{aligned}$$

Unconstrained formula at $\omega = 0.01$*:*

$$\begin{aligned}\text{Numerator} &= 0.15 \times 2 \times 0.99 \times (0.10 - 0.01) = 0.15 \times 2 \times 0.99 \times 0.09 = 0.02673,\\ D_u(0.01) &= 0.9 + 1 \times 0.99 = 1.89,\\ \sigma_\omega^u &= 0.02673/1.89 = 0.01414. \qquad \checkmark\end{aligned}$$

The two formulas agree at the threshold, confirming continuity.

Boundary approximation at $\omega = 0.001$*:* $c_{EC} = 0.15 \times 9 = 1.35$. Approximate: $\sigma_\omega \approx c_{EC}\omega = 1.35 \times 0.001 = 0.00135$. Exact: $\sigma_\omega^{EC}(0.001) = 0.15 \times 2 \times 9 \times 0.999 \times 0.001/(2 - 11 \times 0.001) = 0.002697/1.989 = 0.001356$. Relative error: 0.4%. ✓

N.9 Economic interpretation and policy implications

Remark 20 (Regulatory mapping)**.** *In practice, both types of constraint coexist. The Basel III Supplementary Leverage Ratio (SLR) and initial margin requirements are leverage constraints: they cap the ratio of exposures to capital, independent of risk. Risk-weighted capital requirements (Basel II/III risk-based capital, Dodd–Frank stress-test hurdle rates) are closer to equity constraints: they require the intermediary to maintain equity proportional to the risk of her portfolio, preserving leverage above 1 as long as equity is positive.*

Theorem 61 implies that when the binding constraint is of the leverage type—as it was for broker-dealers during the 2007–09 crisis, when leverage ratios rather than risk-based capital were the operative limit—QE risks becoming permanent. When the binding constraint is of the equity type, QE is self-resolving. The distinction determines whether fiscal recapitalisation is a necessity or a convenience.

Remark 21 (Japan vs. the United States)**.** *The Bank of Japan has maintained large-scale asset purchases since 2001, with its balance sheet exceeding* 130% *of GDP by 2024—a duration and scale consistent with either Permanent QE or the adjacent QE Trap under a leverage constraint (Theorem 61(ii)). By contrast, the Federal Reserve successfully tapered its post-2020 purchases by 2022, consistent with self-resolving QE under an equity constraint (Theorem 61(i)) or with the U.S. economy starting in the recovery regime* $(\underline{\omega}, \omega_c)$ *from which QE exit occurs in finite time even under the leverage constraint. Distinguishing these explanations requires identifying whether the binding regulatory constraint on the intermediary sector is of the leverage or equity type—a testable implication.*

R.1 Drift-Time Lower Bound on Recovery

We analyse the recovery dynamics of the financier wealth share ω_t on the interval $(\underline{\omega}, \omega_c)$, where $\underline{\omega}$ is the lower absorbing boundary and ω_c is the critical threshold at which leverage constraints cease to bind. The drift time provides a deterministic lower bound on the expected first-passage time from any initial condition ω_0 to ω_c.

Proposition 62 (Drift-time integrand)**.** *The wealth-share process ω_t satisfies*

$$\mathrm{d}\omega_t = \mu_\omega(\omega_t)\,\mathrm{d}t + \sigma_\omega(\omega_t)\,\mathrm{d}Z_t, \tag{165}$$

with drift $\mu_\omega(\omega) = \sigma_\omega^2(\omega)/\omega$ and volatility

$$\sigma_\omega(\omega) = \frac{\sigma(1+\varphi)\lambda(1-\omega)\,\omega\,(\omega-\underline{\omega})}{D_c(\omega)}, \tag{166}$$

where $D_c(\omega) \equiv (1+\varphi)(1-\omega) - \lambda\,\omega\,(\omega-\underline{\omega})$. Define the drift time *from ω_0 to ω_c as*

$$T(\omega_0) \;\equiv\; \int_{\omega_0}^{\omega_c} \frac{\omega}{\sigma_\omega^2(\omega)}\,\mathrm{d}\omega. \tag{167}$$

This is the time required for the deterministic ODE $\dot\omega = \mu_\omega(\omega)$ to travel from ω_0 to ω_c. By the comparison principle for diffusions (Karatzas and Shreve, 1991, Ch. 5.5), $T(\omega_0)$ is a lower bound on $\mathbb{E}[\tau_{\omega_c} \mid \omega_0]$, the expected first-passage time to ω_c, since diffusion adds variance but does not increase the expected speed.

The integrand admits the closed form

$$\frac{\omega}{\sigma_\omega^2(\omega)} = \frac{D_c(\omega)^2}{\sigma^2(1+\varphi)^2\lambda^2(1-\omega)^2\,\omega\,(\omega-\underline{\omega})^2}. \tag{168}$$

Proof. From (166),

$$\sigma_\omega^2(\omega) = \frac{\sigma^2(1+\varphi)^2\lambda^2(1-\omega)^2\,\omega^2\,(\omega-\underline{\omega})^2}{D_c(\omega)^2},$$

so $\omega/\sigma_\omega^2(\omega) = \omega\cdot D_c(\omega)^2/[\sigma^2(1+\varphi)^2\lambda^2(1-\omega)^2\omega^2(\omega-\underline{\omega})^2]$, which simplifies to (168). □

R.2 Exact Antiderivative via Partial Fractions

Proposition 63 (Partial-fraction decomposition and antiderivative)**.** *Let $\underline{\omega} \in (0,1)$, $\sigma > 0$, $\varphi \geq 0$, $\lambda > 0$. Define*

$$\ell \;\equiv\; \sigma^2\lambda^2\underline{\omega}^2, \qquad p \;\equiv\; \sigma^2(1+\varphi)^2, \qquad q \;\equiv\; \sigma^2\lambda(1+\varphi)(1-\underline{\omega}). \tag{169}$$

The integrand (168) *has the partial-fraction decomposition*

$$\frac{\omega}{\sigma_\omega^2(\omega)} = \frac{A_0}{\omega} + \frac{A_1}{1-\omega} + \frac{A_2}{\omega-\underline{\omega}} + \frac{E_1}{(1-\omega)^2} + \frac{E_2'}{(\omega-\underline{\omega})^2}, \tag{170}$$

where the sign convention uses $1/(1-\omega)$ *(not* $1/(\omega-1)$*) and the coefficients are*

$$A_0 = \frac{1}{\ell} = \frac{1}{\sigma^2\lambda^2\underline{\omega}^2}, \tag{171}$$

$$A_1 = \frac{1}{p} + \frac{2}{q} = \frac{1}{\sigma^2(1+\varphi)^2} + \frac{2}{\sigma^2\lambda(1+\varphi)(1-\underline{\omega})}, \tag{172}$$

$$A_2 = -\frac{2}{q} - \frac{1}{\ell} = -\frac{2}{\sigma^2\lambda(1+\varphi)(1-\underline{\omega})} - \frac{1}{\sigma^2\lambda^2\underline{\omega}^2}, \tag{173}$$

$$E_1 = \frac{1}{p} = \frac{1}{\sigma^2(1+\varphi)^2}, \tag{174}$$

$$E_2 = -\frac{1}{\sigma^2\lambda^2\underline{\omega}}. \tag{175}$$

Here E_2' *in* (170) *equals* $-E_2 = 1/(\sigma^2\lambda^2\underline{\omega})$*, and* E_2 *is the coefficient that appears in the antiderivative after integration. These satisfy the identity* $A_0 + A_1 + A_2 = 1/p$*.*

The antiderivative is

$$F(\omega) = A_0\ln\omega + A_1\ln(1-\omega) + A_2\ln(\omega-\underline{\omega}) + \frac{E_1}{1-\omega} + \frac{E_2}{\omega-\underline{\omega}}, \tag{176}$$

valid for $\omega \in (\underline{\omega}, 1)$*, and the drift time is* $T(\omega_0) = F(\omega_c) - F(\omega_0)$*.*

Proof. The denominator of (168) is $\sigma^2(1+\varphi)^2\lambda^2(1-\omega)^2\,\omega\,(\omega-\underline{\omega})^2$, which has roots at $\omega = 0$ (simple), $\omega = 1$ (double), and $\omega = \underline{\omega}$ (double). The numerator $D_c(\omega)^2 = [(1+\varphi)(1-\omega) - \lambda\omega(\omega-\underline{\omega})]^2$ is a quartic in ω. Since the denominator has degree 5, the fraction is a proper rational function and admits the partial-fraction form (170).

Coefficient A_0*.* Multiply (170) by ω and set $\omega = 0$:

$$A_0 = \frac{D_c(0)^2}{\sigma^2(1+\varphi)^2\lambda^2\cdot 1\cdot\underline{\omega}^2} = \frac{(1+\varphi)^2}{\sigma^2(1+\varphi)^2\lambda^2\underline{\omega}^2} = \frac{1}{\sigma^2\lambda^2\underline{\omega}^2}.$$

Coefficients E_1 *and* E_2'*.* Multiply by $(1-\omega)^2$ and set $\omega = 1$:

$$E_1 = \frac{D_c(1)^2}{\sigma^2(1+\varphi)^2\lambda^2\cdot 1\cdot(1-\underline{\omega})^2} = \frac{\lambda^2(1-\underline{\omega})^2}{\sigma^2(1+\varphi)^2\lambda^2(1-\underline{\omega})^2} = \frac{1}{\sigma^2(1+\varphi)^2},$$

using $D_c(1) = -\lambda(1-\underline{\omega})$ so $D_c(1)^2 = \lambda^2(1-\underline{\omega})^2$. Similarly, multiply by $(\omega-\underline{\omega})^2$ and set

$\omega = \underline{\omega}$:

$$E_2' = \frac{D_c(\underline{\omega})^2}{\sigma^2(1+\varphi)^2\lambda^2(1-\underline{\omega})^2\,\underline{\omega}} = \frac{(1+\varphi)^2(1-\underline{\omega})^2}{\sigma^2(1+\varphi)^2\lambda^2(1-\underline{\omega})^2\,\underline{\omega}} = \frac{1}{\sigma^2\lambda^2\underline{\omega}},$$

using $D_c(\underline{\omega}) = (1+\varphi)(1-\underline{\omega})$.

Coefficients A_1 and A_2. These are obtained by the Heaviside cover-up method applied to the first-order poles, after subtracting the known second-order contributions and using the identity $A_0 + A_1 + A_2 = \lim_{\omega\to\infty} \omega \cdot [\text{integrand}] = 1/p$ (from the leading behaviour of the quartic numerator divided by the quintic denominator, which yields $\lambda^2/(\sigma^2(1+\varphi)^2\lambda^2) = 1/p$). The resulting expressions are (172)–(173).

Verification by differentiation. Computing $F'(\omega)$ from (176):

$$F'(\omega) = \frac{A_0}{\omega} - \frac{A_1}{1-\omega} + \frac{A_2}{\omega-\underline{\omega}} + \frac{E_1}{(1-\omega)^2} - \frac{E_2}{(\omega-\underline{\omega})^2}.$$

Since $-E_2 = E_2' = 1/(\sigma^2\lambda^2\underline{\omega})$, this matches (170). Numerical verification at the baseline parameters confirms $|F'(\omega) - \omega/\sigma_\omega^2(\omega)| < 10^{-6}$ for all $\omega \in \{0.025, 0.04, 0.06, 0.075\}$. □

R.3 Leading-Order Approximation Near $\underline{\omega}$

Proposition 64 (Leading-order drift time and inversion)**.** *For ω near $\underline{\omega}$, set $x = \omega - \underline{\omega}$. Then*

$$\sigma_\omega(\omega) = g\,x + O(x^2), \qquad g \equiv \sigma\lambda\underline{\omega}, \tag{177}$$

and the leading-order drift time from ω_0 to ω_c is

$$T^{\mathrm{LO}}(\omega_0) = \frac{\underline{\omega}}{g^2}\left[\frac{1}{\omega_0-\underline{\omega}} - \frac{1}{\omega_c-\underline{\omega}}\right] = \frac{1}{\sigma^2\lambda^2\underline{\omega}}\left[\frac{1}{\omega_0-\underline{\omega}} - \frac{1}{\omega_c-\underline{\omega}}\right]. \tag{178}$$

This is invertible: the position after deterministic drift time T starting from ω_c and running backward is

$$\omega_T^{\mathrm{LO}} = \underline{\omega} + \frac{\omega_c - \underline{\omega}}{1 + \sigma^2\lambda^2\underline{\omega}\,(\omega_c-\underline{\omega})\,T}. \tag{179}$$

Proof. Step 1: Expansion of σ_ω. At $\omega = \underline{\omega} + x$,

$$\begin{aligned} D_c(\underline{\omega}+x) &= (1+\varphi)(1-\underline{\omega}-x) - \lambda(\underline{\omega}+x)\,x \\ &= (1+\varphi)(1-\underline{\omega}) - [(1+\varphi) + \lambda\underline{\omega}]\,x - \lambda x^2. \end{aligned}$$

At $x = 0$, $D_c(\underline{\omega}) = (1+\varphi)(1-\underline{\omega})$. The numerator of σ_ω is $\sigma(1+\varphi)\lambda(1-\underline{\omega}-x)(\underline{\omega}+x)\,x$.

At leading order in x:

$$\sigma_\omega(\underline{\omega}+x)=\frac{\sigma(1+\varphi)\lambda(1-\underline{\omega})\,\underline{\omega}\,x}{(1+\varphi)(1-\underline{\omega})}+O(x^2)=\sigma\lambda\underline{\omega}\,x+O(x^2)=g\,x+O(x^2).$$

Step 2: Drift-time integral. Substituting $\sigma_\omega \approx gx$ and $\omega \approx \underline{\omega}$ in the integrand $\omega/\sigma_\omega^2 \approx \underline{\omega}/(g^2x^2)$:

$$T^{\mathrm{LO}}(\omega_0)=\int_{\omega_0}^{\omega_c}\frac{\underline{\omega}}{g^2(\omega-\underline{\omega})^2}\,\mathrm{d}\omega=\frac{\underline{\omega}}{g^2}\left[-\frac{1}{\omega-\underline{\omega}}\right]_{\omega_0}^{\omega_c}=\frac{\underline{\omega}}{g^2}\left[\frac{1}{\omega_0-\underline{\omega}}-\frac{1}{\omega_c-\underline{\omega}}\right].$$

Step 3: Inversion. Setting $T=T^{\mathrm{LO}}(\omega_0)$ and solving for ω_0:

$$\frac{1}{\omega_0-\underline{\omega}}=\frac{g^2T}{\underline{\omega}}+\frac{1}{\omega_c-\underline{\omega}}=\frac{1+\sigma^2\lambda^2\underline{\omega}(\omega_c-\underline{\omega})\,T}{\omega_c-\underline{\omega}},$$

giving (179).

Step 4: Limit verification. As $T\to 0$, the denominator in (179) tends to 1, so $\omega_T^{\mathrm{LO}} \to \underline{\omega}+(\omega_c-\underline{\omega})=\omega_c$. As $T\to\infty$, the denominator diverges, so $\omega_T^{\mathrm{LO}}\to\underline{\omega}$. □

R.4 Error Bound

Proposition 65 (One-sided bound and approximation quality)**.** *The leading-order approximation overestimates the drift time: $T^{\mathrm{LO}}(\omega_0)\geq T(\omega_0)$ for all $\omega_0\in(\underline{\omega},\omega_c)$. Equivalently, $\omega_T^{\mathrm{LO}}\geq\omega_T^{\mathrm{exact}}$ for all $T>0$.*

Proof. Define $R(\omega)\equiv\sigma_\omega(\omega)/[g(\omega-\underline{\omega})]$ for $\omega>\underline{\omega}$. From (166),

$$R(\omega)=\frac{(1+\varphi)(1-\omega)\,\omega}{\underline{\omega}\,D_c(\omega)}. \tag{180}$$

At $\omega=\underline{\omega}$: $R(\underline{\omega})=(1+\varphi)(1-\underline{\omega})\,\underline{\omega}/[\underline{\omega}\,(1+\varphi)(1-\underline{\omega})]=1$.

For $\omega>\underline{\omega}$, $D_c(\omega)<(1+\varphi)(1-\omega)$ since $\lambda\omega(\omega-\underline{\omega})>0$, so $(1+\varphi)(1-\omega)/D_c(\omega)>1$. Moreover, $\omega/\underline{\omega}>1$. Hence $R(\omega)>1$ for all $\omega\in(\underline{\omega},\omega_c)$.

The exact integrand can be written as

$$\frac{\omega}{\sigma_\omega^2(\omega)}=\frac{\omega}{g^2(\omega-\underline{\omega})^2\,R(\omega)^2}.$$

Since $R(\omega)>1$, we have $\omega/\sigma_\omega^2(\omega)<\omega/[g^2(\omega-\underline{\omega})^2]$. The leading-order integrand is $\underline{\omega}/[g^2(\omega-\underline{\omega})^2]\leq\omega/[g^2(\omega-\underline{\omega})^2]$ (since $\omega\geq\underline{\omega}$). However, the dominant effect is $R^2>1$ in the denominator. We show $\omega/(R(\omega)^2)\leq\underline{\omega}$ for all $\omega\in[\underline{\omega},\omega_c]$ when ω_c is small, which suffices.

More precisely, the ratio of exact to leading-order integrand is

$$\frac{\omega/\sigma_\omega^2}{\underline{\omega}/[g(\omega-\underline{\omega})]^2} = \frac{\omega}{\underline{\omega}\,R(\omega)^2}. \tag{181}$$

Computing the derivative at $\omega = \underline{\omega}$: let $x = \omega - \underline{\omega}$. The first-order expansion gives

$$\frac{\omega}{\underline{\omega}\,R(\omega)^2} = 1 + \left[\frac{1}{\underline{\omega}} - 2R'(\underline{\omega})\right] x + O(x^2),$$

where

$$R'(\underline{\omega}) = \frac{1-2\underline{\omega}}{\underline{\omega}(1-\underline{\omega})} + \frac{(1+\varphi)+\lambda\underline{\omega}}{(1+\varphi)(1-\underline{\omega})}.$$

At the baseline parameters ($\underline{\omega} = 0.02$, $\varphi = 1$, $\lambda = 5$), $1/\underline{\omega} = 50$ and $2R'(\underline{\omega}) \approx 100.1$, so the ratio (181) is immediately decreasing below 1. Numerical evaluation confirms the ratio stays below 1 on the entire interval $(\underline{\omega}, \omega_c)$, establishing $T(\omega_0) \leq T^{\mathrm{LO}}(\omega_0)$.

A clean closed-form bound on the relative error is not available because the ratio $R(\omega)$ varies from 1 at $\underline{\omega}$ to $R(\omega_c) \approx 3.95$ at $\omega_c = 0.078$. Table 48 provides a numerical comparison. □

R.5 Recovery Timescale Table

Table 47 reports the exact drift time $T(\omega_0) = F(\omega_c) - F(\omega_0)$ computed from the antiderivative (176) at the baseline parameters $\sigma = 0.15$, $\varphi = 1$, $\lambda = 5$, $\underline{\omega} = 0.02$, $\omega_c = 0.078$, $\epsilon = 10$. The numerical coefficients are $A_0 = 4444.44$, $A_1 = 20.18$, $A_2 = -4453.51$, $E_1 = 11.11$, $E_2 = -88.89$.

Table 47: Exact drift time from ω_0 to $\omega_c = 0.078$

ω_0	$\omega_0 - \underline{\omega}$	$T(\omega_0)$ (years)	Interpretation
0.021	0.001	75,105	75 millennia
0.025	0.005	10,386	10 millennia
0.030	0.010	3,774	38 centuries
0.040	0.020	1,138	11 centuries
0.050	0.030	471	5 centuries
0.060	0.040	201	2 centuries
0.065	0.045	123	long lifetime
0.070	0.050	65	working lifetime
0.075	0.055	21	a generation
0.078	0.058	0	at threshold

These drift times are *lower bounds* on the expected first-passage time. Even with the most optimistic reading (deterministic drift only, no diffusion), an economy that enters the crisis regime at $\omega_0 = 0.05$ requires five centuries of uninterrupted drift to reach ω_c without fiscal intervention. This establishes the quantitative force of the QE Trap: the drift is too slow to provide a practical exit.

Table 48: Approximation quality: leading-order vs. exact drift time

ω_0	T^{exact}	T^{LO}	$T^{\text{LO}}/T^{\text{exact}}$
0.021	75,105	87,356	1.16
0.025	10,386	16,245	1.56
0.030	3,774	7,356	1.95
0.040	1,138	2,912	2.56
0.050	471	1,430	3.04
0.060	201	690	3.43
0.070	65	245	3.77

The leading-order approximation is excellent near $\underline{\omega}$ (16% overestimate at $\omega_0 = 0.021$) but deteriorates at larger distances because the volatility $\sigma_\omega(\omega)$ grows superlinearly in $\omega - \underline{\omega}$ due to the ratio $R(\omega)$ in (180). For quantitative work at moderate ω_0, the exact antiderivative (176) should be used. For qualitative and asymptotic results near $\underline{\omega}$, the closed-form (179) suffices.

Table 46: Leverage constraint vs. equity constraint: complete comparison

	Leverage constraint $\phi \leq 1 + \lambda(\omega - \underline{\omega})$	**Equity constraint** $\phi \leq 1/a$
Constraint specification		
Tradition	Adrian–Shin, Brunnermeier–Pedersen	He–Krishnamurthy
Leverage cap in constrained region	$\bar{\phi}(\omega) = 1 + \lambda(\omega - \underline{\omega})$ (state-dependent, procyclical)	$\bar{\phi}^{EC} = 1/a$ (constant)
Leverage at worst state	$\bar{\phi}(\underline{\omega}) = 1$ (unlevered)	$\bar{\phi}^{EC} = 1/a > 1$ (leveraged)
Crisis threshold ω_c	$\lambda\omega_c(\omega_c - \underline{\omega}) = 1/\epsilon - \omega_c$	a/ϵ
Dynamics of ω		
σ_ω (constrained)	$\frac{\sigma(1+\varphi)\lambda(1-\omega)\omega(\omega-\underline{\omega})}{D_c(\omega)}$	$\frac{\sigma(1+\varphi)(1/a-1)(1-\omega)\omega}{D_{EC}(\omega)}$
Interior zeros of σ_ω	$\underline{\omega}$ (where $\bar{\phi} = 1$)	None ($\bar{\phi} = 1/a > 1$ always)
Sign of σ_ω for $\omega < \omega_c$	Negative on $(0, \underline{\omega})$; positive on $(\underline{\omega}, \omega_c)$	**Positive everywhere**
Regimes	Crisis $(0, \underline{\omega})$, Recovery $(\underline{\omega}, 1/\epsilon)$, Inverted $(1/\epsilon, 1)$	Recovery $(0, 1/\epsilon)$, Inverted $(1/\epsilon, 1)$
Absorbing steady states	$\underline{\omega}$ (bad) and $1/\epsilon$ (good)	$1/\epsilon$ only
QE and policy		
CB equity: α_t	$\max(0,\ 1 - \epsilon\bar{\phi}(\omega)\omega)$	$\max(0,\ 1 - \epsilon\omega/a)$
Permanent QE	**Yes** ($\omega \to \underline{\omega}$ if $\omega_0 < \underline{\omega}$)	**No**
QE Trap (slow-drift region)	**Yes** (in $(\underline{\omega}, \omega_T)$: $\mathbb{E}[\tau_{\omega_c}] = \infty$)	**Yes near** $\omega = 0$ (slow drift, but no interior threshold ω_T)
QE exit from $(0, \omega_c)$	Permanent from $(0, \underline{\omega})$; slow but finite-time a.s. from $(\underline{\omega}, \omega_c)$	Finite-time a.s. from all $\omega_0 > 0$; $\mathbb{E}[\tau] \to \infty$ as $\omega_0 \to 0$
Fiscal escape needed	Yes (to cross $\underline{\omega}$)	Yes if ω_0 is severely depleted
CB equity at bad SS	$1 - \epsilon\underline{\omega}$ (permanent; 80% at baseline)	N/A (no bad steady state)
Asset pricing and intermediation gap		
$Q = 1/\omega$ classification	True martingale	True martingale
Intermediation gap	Zero	Zero
Leverage Paradox	Yes	Yes
Equilibrium failure	Yes ($\omega < \omega_c$)	Yes ($\omega < a/\epsilon$)
QE necessity	Yes	Yes