Incontestable Assignments*

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Abstract

We develop a refinement of stability for school choice based on the information students can use to justify appeals. It accommodates any type of partial information about schools' choice functions, and about the realized assignment. A student has a legitimate complaint if, in every compatible problem and assignment, she can block with some more-preferred school, and an assignment is incontestable when no student has such a complaint. Legitimate complaints are characterized by a set-version of a safety school. When students observe only their own assignment but may know schools' choice functions, student-proposing deferred acceptance, Top Trading Cycle always produce incontestable assignment. Pareto improvements of incontestable assignments are incontestable, too. Under a mild consistency condition and non-wastefulness, all incontestable mechanism induce for each student the same attainable set of schools.

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1 Introduction

Stability is a desirable objective when designing matching or assignment mechanisms because it prevents eventual disruptions by blocking pairs, a cause of market failure—see Roth (1991) and Kagel and Roth (2000).[1,2] While this criterion is unquestionably needed in many cases, we contend that stability may be unnecessarily demanding in settings like school choice. In such problems, students typically only have coarse information about how schools select students and about the realized assignment. For instance, students may know that for, say, 50% of the seats low-income students have a higher priority, but because of privacy protection laws they are unlikely to know who qualifies as low-income among those assigned to a specific school. What this means is that students may not have the information to identify every school with which they can form a blocking pair, so not all blocking pairs are a threat to an assignment. This problem becomes more critical because assignments in such markets are exclusively determined by a central authority. Blocking pairs are not free to form independently, but arise through the actions of individual students who can *appeal* their assignment.[3]

Appealing one's assignment differs from the standard blocking concept because appeals can only be made when justified.[4] For instance, a student could justify an appeal on the basis that, given her priority ranking and school capacities, she should have gotten into one of her top three schools, independently of the preferences of other students. In contrast, a student who is unhappy with her assignment but cannot provide a valid reason why her assignment should be different has no recourse. To sum up, stability requires the absence of justified envy, but for a student to identify justified envy requires her to know the assignment of other students and how schools select students. This is the point of departure for our paper: *whether a school choice assignment is appeal-proof hinges on the information available to*

[1] It should be noted, though, that stability does not always prevent market failure. See for instance McKinney et al. (2005).

[2] Another common justification for stability is fairness. However, in school choice problems this normative criterion comes at the cost of efficiency (Ergin, 2002).

[3] A standard justification for stability relies on the possibility of decentralized rematching. In a labor market, for example, a worker may directly approach a more-preferred firm after the matching is announced, and the firm may accept. This argument does not apply when only the central authority can alter assignments.

[4] See for instance https://www.nysed.gov/essa/new-york-state-essa-funded-programs-complaint-procedures or https://education.nsw.gov.au/schooling/parents-and-carers/going-to-school/astp/appeals.

students.

We use this perspective to revisit what it means for a student to justify a blocking claim in school choice: what can a student certify, given the information available to her, when she challenges her assignment? To accommodate the widest possible range of informational environments, we impose minimal structure on students' information. In particular, a student may know only part of the profile of schools' choice functions. For instance, a student may know between school s and s', only one gives higher priority to a specific set of students (but does not know which one), or that schools give a higher priority to a specific set of students but without knowing who (all) these students are. Similarly, we allow students to have coarse information about the assignment (e.g., there are between, say, 50 and 80 students assigned to school s). This flexibility regarding the structure of students' information is simply captured by equating student's information with the set of all schools' choice functions and assignments that are deemed plausible for the student.

We first ask the following question: given the information available for an appeal, when can a student establish that she can block with a more preferred school? Our answer this question does not rely on students' beliefs. We instead let the student consider all schools' choice functions and assignments compatible with her information. If, for every such choice functions and assignment she can block with some school that she prefers to her assignment, then we say that she has a *legitimate complaint*, and we say that an assignment is *incontestable* if no student has a legitimate complaint. Our first result (Theorem 1) shows that the universal requirement implicit in a legitimate complaint can be summarized by a simple object, an *ex-post safety-set*. A student has a legitimate complaint given the information she has if, and only if she is assigned to a school outside her ex-post safety-set. The following example illustrates the notion of ex-post safety-set.

Example 1 *Consider a student i who is looking for an assignment between four schools, s_1, s_2, s_3 and s_4, each of which has one seat available. The student ranks the schools so that*

$$s_1 \succ s_2 \succ s_3 \succ s_4.$$

There are two other students, j and k, and student i knows that at s_1 and s_2, the only student with higher priority than i is j, and at s_3 and s_4, the only student with higher priority than i is k. Then based on her information, i knows that if she applies to s_1 and s_2, she must be accepted by one of them, and if she applies to s_3 and s_4, she must be accepted by one of them. Therefore $\{s_1, s_2\}$ and $\{s_3, s_4\}$ are what we call "high-priority sets" for i: if i applies to all

schools in such a set she is guaranteed a seat in one of those schools. In contrast, i cannot say that she must get into either s_1 or s_2, so $\{s_1\}$ and $\{s_2\}$ are not high-priority sets. A key property is that if a set of schools $\widehat{S}$ is a high-priority set for a student, then any superset of $\widehat{S}$ is also a high-priority set.

As this demonstrates, high-priority sets are not necessarily nested. However, if we consider i's upper contour sets of all schools that are better than some school, they are

$$\{s_1\}, \{s_1, s_2\}, \{s_1, s_2, s_3\}, \{s_1, s_2, s_3, s_4\}.$$

These are nested, and among these, the smallest high priority upper contour set is $\{s_1, s_2\}$. This is what we call a safety-set. Based on i's information, i should be able to get into either s_1 or s_2. Knowing that $\{s_1, s_2, s_3\}$ is a also high-priority set is irrelevant for i since she can guarantee admission to one of s_1 and s_2. Therefore, i knows that given her preferences, s_2 is the worst school she should legitimately be assigned to, and we call s_2 her safety cutoff. In other words, i knows she has a legitimate complaint if she is assigned to a school that is worse for her than s_2, and has no legitimate complaint if she is assigned to a school that is weakly preferred to s_2. □

A student's ex-post safety-set captures a worst-case guarantee and is akin to a set version of the notion of safety school. It differs, however, from the usual notion of a safety school (a school to which the student is certain to be admitted) in two respects. First, it is an ex-post notion as it is defined given some assignment. Second, under partial information, the guarantee does not generally attach to a particular school. Rather, the student can establish that at least one school in her safety-set must sustain her complaint, without necessarily knowing which one.

Under complete information about the assignment and schools' choice functions, our safety-set condition reduces to stability. If the assignment is stable, a student's safety-set contains only her assignment. If instead she can profitably block with a school s, then s belongs to her safety-set, whereas her assignment does not.

We then turn to the study of incontestable assignment mechanism, that is, mechanism that produce assignments where no student has a legitimate complaint. Because legitimate complaints depend on the information available to students, incontestability cannot be assessed from a mechanism alone. We capture the informational environment through an *information rule*. For every problem, an information rule specifies each student's interim

information about schools' choice functions and, for every realized assignment, her ex-post information about the choice functions and the assignment. We study classes of information rules, which can represent the different disclosure and appeal procedures under which the same mechanism is used, and given a class of information rules Π we say that a mechanism is Π-incontestable if it produces no legitimate complaint under any information in Π.

An intuitive but useful feature of this formulation is that information rules can be partially ordered by informativeness. Safety-sets are nested according to this order, which translates into a nested sequence of safety-sets: a more informative rule generates a weakly smaller safety-set. Hence, when a class contains a most informative rule, Π-incontestability can be checked under that rule alone.

Among the informational environments covered by our framework, we focus on *school-side information rules*. Under such a rule, students may have some information about schools' choice functions but, once the assignment is realized, observe no information about the assignment beyond their own assignment.[5] These rules capture what we view as the most common situation in school choice: students may know capacities, priority rules, or other features of schools' choice procedures, while individual-level information about the realized assignment is generally unavailable when appeals must be filed.[6]

School-side information rules are also analytically useful because they allow us to express an inherently ex-post requirement in interim terms. Students appeal after learning their own assignment, so verifying incontestability directly would require computing each student's ex-post safety-set for every possible outcome of the mechanism. For school-side information rules, we instead introduce an *interim safety-set*, which depends only on the student's information about schools' choice functions and is computed before the assignment is realized. Theorem 2 shows that a student's realized assignment belongs to her ex-post safety-set if and only if it belongs to her interim safety-set. Consequently, a mechanism is Π-incontestable if and only if it always assigns each student within her interim safety-set. This replaces an outcome-by-outcome ex-post test with a simpler interim condition.

The interim characterization becomes sharper still for the largest class of school-side

[5] We discuss at the end of the paper other classes of of information rules.

[6] For example, in England admission authorities must specify a deadline for lodging an appeal, and Gloucestershire County Council states that information about other children or their families is not made available to appellants. See https://www.gov.uk/government/publications/school-admissions-appeals-code and https://www.gloucestershire.gov.uk/media/3qpjmdmq/school-admission-appeal-guidance-notes-for-parents.pdf.

information rules, denoted Π^C, which contains every such rule. This class contains a most informative rule, under which students know the full profile of schools' choice functions. This allows us to obtain the following characterization: a mechanism is Π^C-incontestable if, and only if it satisfied a novel property that we call *respects safety-sets*. This property simply requires that for every problem, each student receives a school in her interim safety-set computed under full information about schools' choice functions.

A key question, though, remains: what does this buy us? Crucially, incontestability is compatible with efficiency in a way that stability is not: stable assignments may be Pareto dominated by unstable ones, whereas Π^C-incontestability (or, equivalently, respect for safety-sets) is preserved by Pareto improvements. We show that that any mechanism that, at every problem, Pareto dominates a Π^C-incontestable mechanism is itself Π^C-incontestable. Since every stable mechanism is Π^C-incontestable, any mechanism that Pareto improves the student-optimal stable mechanism (SOSM) remains Π^C-incontestable. In particular, Kesten's (2010) Efficiency Adjusted Deferred Acceptance Mechanism (EADAM) is Π^C-incontestable. We also show that the Top Trading Cycles (TTC) mechanism is Π^C-incontestable, even though its outcome need not be Pareto comparable to that of SOSM. Thus, our result provides an efficiency rationale for mechanisms that may depart from stability while retaining the appeal-proofness captured by incontestability. The Boston mechanism shows, however, that efficiency alone is not sufficient: it is neither stable nor Π^C-incontestable, even though it may select efficient assignments.

The characterization by respect for safety-sets also raises a natural question about what the interim safety-set represents. It tells us that a Π^C-incontestable mechanism never assigns a student outside her interim safety-set, so the set provides an outer bound on the schools she may receive. Is this bound tight? That is, can every school in the interim safety-set arise at some problem compatible with her information? To answer this question, we impose the standard assumptions of substitutability and consistency on schools' choice functions.[7] We consider mechanisms that are non-wasteful and satisfy a mild consistency condition, which we call *top–top consistency*.[8]

[7] Consistency is also called *irrelevance of rejected contracts* in the matching with contracts literature.

[8] In short, this condition says that if a student ranks a school first and would be chosen by that school from any set of applicants, then, whenever the mechanism assigns her to that school, removing her and one seat at the school does not change the assignments of the remaining students. The familiar mechanisms considered above satisfy these requirements: SOSM and, when schools have responsive priorities, TTC and EADAM.

One of our main results establishes that this outer bound is exact: for every Π^C-incontestable, non-wasteful, and top–top consistent mechanism, the student's set of attainable schools coincides with her interim safety-set (Theorem 3). That every attainable school belongs to the safety-set follows from incontestability. The substantive part is the converse: every school in the interim safety-set is assigned to the student at some compatible problem.

Thus, the interim safety-set has two complementary interpretations. It determines whether a realized assignment can be contested, and it describes exactly the schools that the student may receive across problems compatible with her information. Consequently, any two mechanisms satisfying these conditions may select different assignments at a particular problem but, from the student's interim perspective, offer the same range of possible schools. This equivalence concerns attainable sets rather than the mechanisms' pointwise outcomes.

Related literature Our paper intersects with several branches of the matching and assignment literature. The first studies mechanisms that Pareto improve the student-optimal stable mechanism. A key contribution is EADAM (Kesten, 2010). Related rationales are provided by Ehlers and Morrill (2020) and Reny (2022), and Dur et al. (2019) proposes a family of mechanisms that allow for certain forms of justified envy. Our approach differs from these contributions in that the relevant notion of blocking is based on the information available to students. It is the lack of complete information on students' side that allows for Pareto improvements over stable assignments, and not a tolerance for justified envy per se.

The paper also relates to the literature on matching and assignment under partial information. First, the use of information sets to model partial information is not new. It appears in work on regret-free truth-telling (Fernandez, 2018; Chen and Möller, 2024), transparent mechanisms (Möller, 2022), and auditability (Grigoryan and Möller, 2023, 2024), among others. In those papers, as in ours, agents' information determines a set of compatible scenarios. But rather than asking whether a student regrets a submitted preference list, or whether she can verify that an announced mechanism was used, we ask whether she can justify an appeal. The informational environment is also different. In Fernandez (2018) and Chen and Möller (2024), schools have responsive priorities and the priority structure is fully available to students; in Möller (2022), agents have access to object priorities in a one-to-one assignment problem. We allow general choice functions and partial information about choice functions, priorities, capacities, preferences, and assignments.

Our paper is also related to work on stability under incomplete information. Chakraborty

et al. (2010) studies a setting where agents infer the types of potential partners from the observed matching, Yenmez (2013) considers ex-ante and interim stability conditions,[9] and Liu et al. (2014) studies stability when firms do not observe workers' types. We share with this literature the idea that blocking must be evaluated relative to agents' information, but our focus is on one-sided appeals in school choice.

Finally, our results connect to the literature on obvious manipulations. Troyan and Morrill (2020) introduce non-obvious manipulability and identify, in school-choice environments with known responsive priorities, the schools relevant for students' worst-case reasoning. Decerf and Van der Linden (2021) develops a related safe-set formulation. In the special case of known responsive priorities, our interim safety-set coincides with their safe-set. The difference is that this literature uses the object to study incentives, whereas we use it to characterize legitimate complaints and incontestability.

The paper is organized as follows. We Section 2 presents the school-choice model together with a definition of students' information. Section 3 defines and characterizes legitimate complaints. Section 4 consider incontestable mechanisms. Section 5 studies incontestable mechanisms for school-side information rules. We discuss other classes of information rules in Section 6 and conclude in Section 7. Proofs are in the Appendix, with additional results in the online Appendix.

2 The model

2.1 School choice problems

We consider school choice problems with a finite set I of students and a finite set S of schools. The set S contains a school denoted by s_0, which we call the null school. Each student i's preferences over schools are given by a strict linear order $\succ_i$ over the set of schools S, and $s \succ_i s'$ means that student i prefers s to s', with $s, s' \in S$. A school s is unacceptable if $s_0 \succ_i s$ (and acceptable otherwise).[10] Given a set of students $J \subseteq I$, let $\succ_J = (\succ_i)_{i \in J}$. If $J = I$, the set of all students, we will generally omit the subscript and write $\succ$ instead of $\succ_I$. We denote by $\mathcal{P}_i$ the set of all preference relations for a student i. We assume that for

[9] See also Ehlers and Massó (2015).

[10] So, s_0 captures students' outside option (i.e., not being assigned, which could be, for instance, home schooling or being assigned to a school outside the problem).

each $i \in I$, $\mathcal{P}_i$ consists of all strict preference orderings over S. Let $\mathcal{P}$ denote the set of all preference profiles and $\mathcal{P}_{-i} = \Pi_{j \neq i} \mathcal{P}_j$.

Each school s has a capacity constraint $q_s > 0$, which denotes the maximum number of students that school s can enroll. For the null school s_0, we assume that $q_{s_0} \geq |I|$. Given its capacity constraint, each school s is endowed with a choice function $C_s : 2^I \to 2^I$ where for any set of students $J \subseteq I$, $C_s(J)$ denotes the set of students chosen by s from the set J. We assume that $C_s(J) \subseteq J$ and $|C_s(J)| \leq q_s$ for every $J \subseteq I$, and that $C_s(\varnothing) = \varnothing$. For any set $J \subseteq I$, let $R_s(J) = J \backslash C_s(J)$ denote the set of students rejected from J. Following the standard school-choice convention that schools do not deem students unacceptable, we assume that no school rejects students from a set whose size is weakly below its capacity: for each school s and each set $J \subseteq I$, if $|J| \leq q_s$, then $C_s(J) = J$. For the null school, we assume that for every set of students $J \subseteq I$, $C_{s_0}(J) = J$. We denote by $\mathcal{C}_s$ the set of all choice functions of school s, and let $\mathcal{C} = \Pi_{s \in S} \mathcal{C}_s$ denote the set of all choice profiles. For a subset of schools $\widehat{S} \subseteq S$, let $\mathcal{C}_{\widehat{S}} = \Pi_{s \in \widehat{S}} \mathcal{C}_s$. Note that, until Section 4 we de not make any assumption on $\mathcal{C}$.

Unless needed, we will not refer explicitly to schools' capacity constraints, as they are implicitly captured by their choice functions. Similarly, the preference profile $\succ$ and the choice profile C implicitly determine the sets of students and schools. Hence, for simplicity, we denote by $(\succ, C)$ a school choice problem.

Our intuitions can be easily captured using *responsive priorities*. We say that a school s has responsive priorities if its choice function C_s can be obtained from a strict priority ordering of students r_s (with the convention that $r_s(i) < r_s(j)$ means that i has higher priority than j) as follows. For any set of students $J \subseteq I$, $C_s(J)$ is obtained by ranking students in J according to r_s and admitting the highest-priority students up to capacity q_s.

Assignments: An assignment is a mapping $\mu : I \cup S \to 2^I \cup S$ such that for each student $i \in I$, $\mu(i) \in S$, for each school $s \in S$, $\mu(s) \in 2^I$ and $|\mu(s)| \leq q_s$, and for each student $i \in I$, $\mu(i) = s$ if, and only if, $i \in \mu(s)$. Let $\mathcal{M}$ denote the set of all assignments. For a given set of students $J \subseteq I$, we will abuse notation and denote by $\mu(J)$ the set of schools to which students in J are assigned, that is, $\mu(J) = \cup_{i \in J} \{\mu(i)\}$. Similarly, for any set $T \subseteq S$, we write $\mu(T)$ to denote the set of students that are assigned to the set of schools T under assignment μ, that is, $\mu(T) = \cup_{s \in T} \mu(s)$. A student i's preferences $\succ_i$ over schools implicitly define a weak preference relation over assignments, denoted, with some abuse of notation, by $\succeq_i$. So, $\mu \succeq_i \mu'$ if, and only if, $\mu(i) \succ_i \mu'(i)$ or $\mu(i) = \mu'(i)$.

Stability: Stability is a standard concept in the school choice literature and is often presented as the conjunction of three requirements: individual rationality, non-wastefulness, and absence of blocking pairs (or "justified envy"). In our setting, these requirements can be written compactly as the absence of a blocking *student–school pair* defined through the schools' choice functions. Fix a problem $(\succ, C)$ and an assignment μ. We say that a student–school pair (i, s) **blocks** μ if

$$s \succ_i \mu(i) \quad \text{and} \quad i \in C_s\big(\mu(s) \cup \{i\}\big). \tag{1}$$

An assignment μ is **stable** for a problem $(\succ, C)$ if it admits no blocking pair.[11] Stability rules out those situations in which a student prefers a school and the school would choose that student if she were added to its current assignment. Some immediate implications are: 1) Since all students can be assigned to the null school, this implies individual rationality of stable assignments. 2) stable assignments cannot exhibit justified envy. 3) If all school find all students acceptable, then stable allocations must be non-wasteful.

2.2 Student Information

We now define the realized information in a particular situation, which models the information available to students regarding the realization of school choice functions and assignments. In particular, this constitutes the information that students officially have access to, and which is admissible in an appeals procedure to contest their assignment. Therefore, we interpret this information as being disclosed to the students from the school system. We take this information as an exogenous primitive of the model, based on institutional requirements.[12,13] We will call information about school choice functions *school-side information*,

[11] In many-to-one matching problems, stability is often also required to satisfy school-side individual rationality: $C_s(\mu(s)) = \mu(s)$ for every $s \in S$. Under our maintained assumption that no student is unacceptable to any school, this condition is automatic. Indeed, feasibility of an assignment implies $|\mu(s)| \leq q_s$ for every school $s \in S$; hence, by the no-unacceptability assumption, $C_s(\mu(s)) = \mu(s)$ for every $s \in S$.

[12] For instance, England's School Admissions Code impose admission authorities to publish a certain number of information, such as capacities, catchment areas, or selection requirements (https://assets.publishing.service.gov.uk/government/uploads/system/uploads/attachment_data/file/1001050/School_admissions_code_2021.pdf). Similarly, California Education Code §35160.5 publicly restrict schools' choice functions (https://leginfo.legislature.ca.gov/faces/codes_displaySection.xhtml?lawCode=EDC&sectionNum=35160.5.).

[13] In the body of the paper, we consider information environments that only disclose information about schools' choice functions and/or the realized assignment. This is most natural case from an application

and information about the realized assignment *assignment-side information*. We will assume that school-side information is disclosed to students first, and refer to it as *interim information*, and assignment-side information is disclosed after, called *ex-post information*. This distinction will be relevant when we analyze incontestability of assignment mechanisms in section 4, as we will show that under certain conditions, incontestability can be characterized in terms of interim information, which significantly reduces the problem of verifying incontestability.

Each student $i \in I$ has an interim information set $\mathcal{K}_i \subset \mathcal{C}$, describing her available school-side information and the set of school choice profiles the student considers possible. A student's ex-post information set given an assignment μ is $\mathcal{H}_i = \mathcal{K}_i \times \mathcal{A}_i$, where $\mathcal{A}_i \subseteq \mathcal{M}_i(\mu) \equiv \{\mu' \in \mathcal{M} : \mu'(i) = \mu(i)\}$, which captures that ex-post, i always knows her own assignment. The set $\mathcal{H}_i$ describes the set of school choice profiles and assignments the student considers possible given her interim information set $\mathcal{K}_i$ and assignment-side information $\mathcal{A}_i$. We assume that the information is always compatible with the truth: if C is the true school choice profile and μ the realized assignment, then $C \in \mathcal{K}_i$, and $(C, \mu) \in \mathcal{H}_i$.

We do not make any additional assumption regarding the structure of student's information. For instance, if two choice profiles C and C' are deemed possible given a student's information, we do not make any assumption about how C compares to C'. Our methodology works for any type of partial information.

A key concept in the paper is the set of problems and ex-post realized assignments that are plausible given a student's information. With our set-based definition, compatibility is simply set membership (or membership in a projection when we suppress some components). We say that a choice profile C or a pair (C, μ) is *compatible* with student i's information set $\mathcal{K}_i$ or $\mathcal{H}_i$ if it is not ruled out by that information: that is, the choice profile or the choice together with the assignment is in the relevant information set.

perspective, since school systems typically will reveal information about students' rankings at schools, and assignment of students, but for privacy reasons, not other students' preferences over schools. In the online appendix, we extend the model to allow for information environments to include information about other students' preferences.

3 Legitimate complaints under partial information

The standard notion of a blocking pair is defined relative to a complete-information problem: given the realized assignment and schools' choice functions, one asks whether a student prefers some school to her assignment and that school would choose her over its current assignees. Under partial information, however, a student may be unable to identify the blocking pairs that she could form. We therefore require a complaint to be justified by the information available to the student. Specifically, a student has a legitimate complaint only if her information establishes that she can block with some more-preferred school in every problem and assignment compatible with what she knows.

Definition 1 *Let μ be an assignment, and let i be a student with ex-post information set $\mathcal{H}_i$. Student i has a* ***legitimate complaint at*** *μ if, for every $(\widehat{C}, \widehat{\mu})$ compatible with $\mathcal{H}_i$, there exists a school s such that $s \succ_i \mu(i)$ and*

$$i \in \widehat{C}_s(\widehat{\mu}(s) \cup \{i\}). \tag{2}$$

Given a problem $(\succ, C)$ and information profile $\mathcal{H}$, an assignment μ is **incontestable** if no student has a legitimate complaint.

Under full information, the statement that i has a legitimate complaint is equivalent to the statement that i blocks with some school. An important feature of the definition of a legitimate complaint is the order of quantifiers: *for each* compatible problem *there exists* a school such that .... That is, the school with which the student can block —s in Eq. (2)— need not be the same across compatible choice profile and assignments. A student may thus be able to conclude that she can block, while being unable to determine with which school she can block and, in the case of justified envy, against which student.[14]

A legitimate complaint only concerns the student's *own* assignment. Under complete information this restriction is immaterial: whenever an assignment has a blocking pair, the student in that pair can identify it. Under partial information, however, a student may be able to infer that some blocking pair exists without being able to establish that she belongs to one (for instance based on some private or inadmissible information outside the

[14] We discuss in the online appendix the case when the school a student can block with must be the same for all compatible choice profile and assignments. This definition turns out to be very restrictive. For instance, when schools have responsive priorities and students only know their assignment a legitimate complaint is possible only if the student is among the top q_s students.

official record about other students' preferences or assignments). Our definition preserves the individual-standing structure of ordinary blocking pairs, rather than allowing students to challenge assignments on the basis of global instability or third-party claims. In our setting, this is a restriction, but one which is strongly motivated by institutional realities. School systems do not typically allow students to make complaints based on information about other students—for instance, doing so might violate the privacy of other students, as even verifying or rebutting such a complain could reveal protected information about others. Our definition therefore treats a complaint as legitimate only when the student's admissible information establishes that her own assignment cannot be justified.

3.1 High-priority sets and safety cutoffs

The definition of a legitimate complaint uses the idea that a student may know that some set of schools is safe for her: for any choice profile and assignment compatible with her information, she cannot be rejected by all schools in that set. In other words, at least one of these schools must accept her.

We call such a set an (ex-post) $\mathcal{H}_i$-high-priority set. Formally, a non-empty set $\widehat{S} \subseteq S$ is an **ex-post $\mathcal{H}_i$-high-priority set** for student i if, for every $(\widehat{C}, \widehat{\mu})$ compatible with $\mathcal{H}_i$,

$$i \in \widehat{C}_s\big(\widehat{\mu}(s) \cup \{i\}\big) \quad \text{for some } s \in \widehat{S}. \tag{3}$$

The property is monotone: if $\widehat{S}$ is an $\mathcal{H}_i$-high-priority set for i, then any superset of $\widehat{S}$ is also an $\mathcal{H}_i$-high-priority set. In particular, since $q_{s_0} \geq |I|$ and $i \in C_{s_0}(J \cup \{i\})$ for every $J \subseteq I$, any set containing the null school is high-priority. Also, since any assignment $\widehat{\mu}$ compatible with $\mathcal{H}_i$ maintains student i's assignment, so that $\widehat{\mu}(i) = \mu(i)$, $\{\mu(i)\}$ is always an $\mathcal{H}_i$-high-priority set. Hence, the collection of $\mathcal{H}_i$-high-priority sets is nonempty.[15]

However, a high-priority set by itself does not tell us which assignments leave i without a legitimate complaint. For that, we need a notion of "best" high priority set for i, such that i cannot be assigned something worse than everything in that set. While in general,

[15]Under known responsive priorities, a set T is high-priority precisely when no assignment compatible with $\mathcal{H}_i$ can fill every school $s \in T$ with q_s students who have higher priority than i. When those responsive priorities are known by student i, it is therefore relatively easy to check whether a given set $\widehat{S}$ is a high-priority set. One only has to find a maximum assignment restricting attention to the students ranked above i, under the constraint that the assignment is compatible with $\mathcal{H}_i$. If it is not possible to fill the capacity of every school in $\widehat{S}$, then $\widehat{S}$ is a high-priority set.

high-priority sets need not be nested, for each s, the upper contour sets of schools i weakly prefers to s,

$$U_i(s) := \{t \in S : t \succeq_i s\},$$

are nested. Moreover, the smallest upper contour set (in the sense of set-inclusion) that is also a high-priority set gives us precisely the notion of a best set. Student i's *ex-post safety-set* of schools is therefore

$$\text{Safe}^E(\succ_i, \mathcal{H}_i) := \min_{\subseteq}\{U_i(s) : s \in S, U_i(s) \text{ is an } \mathcal{H}_i\text{-high-priority set}\}. \tag{4}$$

The safety-set is the best set of schools, from i's perspective, within which she can guarantee admission to at least one school. This property refers to the set as a whole—the student is guaranteed acceptance by some school in the set, not by each of its members. Moreover, membership in the safety-set does not a priori imply that a school is attainable under a particular assignment mechanism.[16] The set is well-defined, since upper contour sets are nested and the upper contour set $U_i(s_0)$ is a high-priority set. Since $\mathcal{H}_i$ is ex-post information, i knows her own assignment, so any school in the safety-set must be at least as good as $\mu(i)$. The safety-set therefore suggests a simple test for a legitimate complaint: whether i was assigned a school in her safety-set. If not, she must have a legitimate complaint since the safety-set is an upper contour set and all schools better than the worst school in her safety-set are included.

Finally, notice that we can represent this requirement with a single safety-cutoff: since the safety-set is an upper contour set, this test only requires checking if i's assignment is worse than the least preferred school in her safety-set. We define

$$\underline{s}^E(\succ_i, \mathcal{H}_i) := \min_{\succ_i} \text{Safe}^E(\succ_i, \mathcal{H}_i)$$

as i's *ex-post safety cutoff*. This gives us the following result:

Theorem 1 *Let μ be an assignment, and let i be a student with ex-post information $\mathcal{H}_i$. Then i has a legitimate complaint at μ given $\mathcal{H}_i$ if and only if*

$$\mu(i) \notin \text{Safe}^E(\succ_i, \mathcal{H}_i),$$

or equivalently, if

$$\underline{s}^E(\succ_i, \mathcal{H}_i) \succ_i \mu(i).$$

[16] Theorem 3 will establish this under some conditions.

The safety cutoff is thus the best threshold such that, given her information, student i can establish that she must be guaranteed to be accepted by *some* school weakly better than that threshold. Importantly, this does not imply that the cutoff school itself must accept her: the school that sustains the guarantee may differ across compatible choice profiles and assignments. Theorem 1 says that i has a legitimate complaint precisely when this threshold is strictly better than her realized assignment.

The result nests the standard blocking logic under full information. If students have full information, the safety-set characterization reduces to the standard blocking condition. If i has a blocking pair with some school $s \succ_i \mu(i)$, then $\{s\}$ is a high-priority set, so the safety-set lies entirely above i's assignment and i has a legitimate complaint. Conversely, if the assignment is stable, no school preferred to $\mu(i)$ would accept i, so no upper contour set lying strictly above $\mu(i)$ can be high-priority. Hence i's safety-set contains her assignment and she has no legitimate complaint.

3.2 Example

The example first illustrates how a publicly known admissions rule can induce incomplete admissible information about schools' choice functions. It then illustrates that the efficiency gains permitted by incontestability do not depend on this incompleteness.

Consider a large urban district with ten schools s_h, $h = 1, \ldots, 10$, each with 100 seats, and 1,000 students. Each school gives priority to students who live in its walk zone. Within the walk-zone and non-walk-zone groups, students are ranked according to a common district examination score. Thus, school s_h first ranks the students eligible for its walk-zone priority, in examination-score order, and then ranks all remaining students in examination-score order. All students are acceptable.

There are exactly 30 students in the walk zone of each school, and the ten walk zones are disjoint. The admissions rule, school capacities, aggregate walk-zone counts, and disjointness of the zones are public. The district observes and verifies every student's address and examination score. In an individual appeal, however, a student may use her own records and the published aggregates but may not introduce the address or score records of specific students.

Consider a student i who is not eligible for walk-zone priority at any school and who ranks 300th in the district examination, so that exactly 299 students have a higher score.

Her preferences are

$$s_1 \succ_i s_2 \succ_i \cdots \succ_i s_{10}.$$

Student i does not know which named students are walk-zone eligible at each school. Hence, although she knows the admissions rule, her admissible information does not identify the responsive priority ordering—and therefore the choice function—over named students. For instance, whether a lower-scoring student j ranks above i at s_h depends on whether j is eligible for s_h's walk-zone priority.

We now calculate i's interim safety-set. If a school rejects i, it must admit 100 students. At most 30 of these students can rank above i solely because of walk-zone priority. Every other admitted student must have a higher examination score than i. Thus, each school rejecting i requires at least 70 students with higher examination scores.

It is possible for the first four schools to reject i. Consider a compatible classification in which all 299 higher-scoring students live outside the ten walk zones. Each of $s_1, \ldots, s_4$ can then admit its 30 walk-zone students together with 70 distinct higher-scoring students. This uses $4 \times 70 = 280 \leq 299$ higher-scoring students. By contrast, if the first five schools all rejected i, their disjoint rosters would have to contain at least $5 \times 70 = 350 > 299$ distinct students with higher examination scores. This is impossible in every choice profile and assignment compatible with i's information. Hence,

$$\text{Safe}^I(\succ_i, \mathcal{K}_i) = \{s_1, s_2, s_3, s_4, s_5\}.$$

The guarantee applies to the set as a whole: for every compatible residential distribution, at least one of these five schools chooses i. It does not imply that every school in the set chooses her. Consequently, if i is assigned to s_5, she has no legitimate complaint. If she is assigned to s_6, she does have a legitimate complaint: no compatible choice profile and assignment can make all five schools she prefers reject her.

4 Incontestable Assignment Mechanisms

The previous section characterized legitimate complaints for a fixed assignment, taking as given students' preferences, school choice functions, and the admissible information available to students. Theorem 1 shows that, given student i's admissible information $\mathcal{H}_i$, the absence of a legitimate complaint for an assignment is equivalent to assigning i a school in her safety-set. In this section, we move from specific assignments to matching mechanisms and their

incontestability. A matching mechanism specifies the assignment that will be selected for each school choice problem $(\succ, C)$. Formally, a mechanism is a map $\varphi : \mathcal{P} \times \mathcal{C} \to \mathcal{M}$.

4.1 Information Rules and Incontestability

For each realized problem and assignment, the information available to students can generally differ, and this is captured by an *information rule*, which specifies the admissible information available to students for each school choice problem (and assignment in the case of ex-post information). We ask when a mechanism is incontestable not only under a single information rule, but under a class of information rules. Such a class can represent the disclosure and appeals rules that arise across different districts or implementations using the same mechanism. An information rule specifies, for each student i, how the admissible information in Section 3 is generated from the realized problem and assignment.

Definition 2 *An **information rule** π specifies, for each student i, an interim information set $\pi_i^I(\succ_i, C) \subseteq \mathcal{C}$ for each $(\succ_i, C) \in \mathcal{P}_i \times \mathcal{C}$, and an ex-post information set $\pi_i^E(\succ_i, C, \mu) \subseteq \mathcal{C} \times \mathcal{M}$ for each $(\succ_i, C, \mu) \in \mathcal{P}_i \times \mathcal{C} \times \mathcal{M}$. The information sets induced by π are assumed to satisfy the compatibility requirements described in Section 4.*

Note that as defined, a mechanism does not depend directly on the admissible-information rule.[17] This is a deliberate restriction: contestability depends on what information students can use in appeals, and different districts may differ in their disclosure rules, privacy constraints, or appeal procedures. Our interpretation is that the same assignment mechanism is used across a number of districts or implementations, while the admissible-information rule may vary across them. We therefore ask whether a single mechanism remains incontestable for every information rule in a class Π, which we define as a set of information rules.

Definition 3 (Incontestability) *For any information rule π, a mechanism φ is π-**incontestable** if, for every problem $(\succ, C)$ and every student i, student i has no legitimate complaint at $\varphi(\succ, C)$ given the ex-post information set $\pi_i^E(\succ_i, C, \varphi(\succ, C))$.*

*For any class of information rules Π, a mechanism φ is Π-**incontestable** if φ is π-incontestable for all $\pi \in \Pi$.*

[17] Although, as we explain below, this case is covered in our analysis by taking $\Pi = \{\pi\}$ for some information rule π and applying Theorem 1 to each problem $(\succ, C)$.

Depending on the institutional environments across which a mechanism will be applied, the class of information rules with respect to which the mechanism needs to be incontestable can vary. For instance, if the mechanism is being designed for a particular district with one specific information rule, or a number of districts which all use the same information rule π, we would consider the singleton case $\Pi = \{\pi\}$. On the other hand, if the mechanism will be applied to many districts with differing information rules, the mechanism must be incontestable with respect to all of these information rules. More generally, the mechanism designer may not know the specifics of every information rule used by different districts, and therefore look for a mechanism that is incontestable with respect to a large set of possible information rules, to ensure that the mechanism is robustly incontestable.

We now provide a general characterization of incontestability under any class of information rules. For a fixed information rule π, π-incontestability requires the assignment-level criterion in Theorem 1 to hold for every assignment generated by the mechanism, using the ex-post information induced by π at that problem and assignment. For a class Π, the same no-complaint condition must hold for every information rule in the class. The following results express these requirements as safety-set conditions and are key to analyze incontestability for various classes of information rules.[18]

Corollary 1 *Let Π be a non-empty class of information rules. A mechanism φ is Π-incontestable if and only if, for every $(\succ, C)$, every student i, and every information rule $\pi \in \Pi$,*

$$\varphi_i(\succ, C) \in \text{Safe}^E(\succ_i, \pi_i^E(\succ_i, C, \varphi(\succ, C)))\,.$$

This is an immediate corollary of Theorem 1, but we state it on its own as it clarifies the role of incontestability with respect to classes of information rules. The characterization states that for a mechanism to be Π-incontestable, in every realized problem, the assignment to each student must belong to her ex-post safety-set under every admissible information rule in the class.

The ordering of information rules induces a corresponding ordering of safety-sets. If an ex-post information set $\mathcal{H}_i$ is more informative than $\mathcal{H}_i'$, so that $\mathcal{H}_i \subseteq \mathcal{H}_i'$, then

$$\text{Safe}^E(\succ_i, \mathcal{H}_i) \subseteq \text{Safe}^E(\succ_i, \mathcal{H}_i')\,.$$

[18] We omit the proofs of Corollary 1 and Proposition 1 below because they are straightforward applications of Theorem 1.

Indeed, every high-priority set under the coarser information set $\mathcal{H}_i'$ is also high-priority under the finer information set $\mathcal{H}_i$. Thus, if a class of information rules contains a most informative rule, the safety-set condition generated by that rule is the most demanding one.

We can compare information rules by the informativeness of the information sets they generate. We say that an information rule π is weakly more informative than an information rule π' ex post if, for every student i, every $(\succ_i, C, \mu)$,

$$\pi_i^E(\succ_i, C, \mu) \subseteq \pi_i'^E(\succ_i, C, \mu)\,.$$

Thus, every choice profile and assignment pair compatible with π is also compatible with π'. This immediately gives us the following result, which says that if a class Π of information rules contains a "most informative" information rule, Π-incontestability reduces to checking that the mechanism assigns each student in the ex-post safety-set generated by that information rule.

Proposition 1 *Let Π be a non-empty class of information rules. Suppose there exists $\overline{\pi} \in \Pi$ such that $\overline{\pi}$ is weakly more informative than every $\pi \in \Pi$. Then a mechanism φ is Π-incontestable if and only if, for every $(\succ, C)$ and every student i,*

$$\varphi_i(\succ, C) \in \text{Safe}^E\left(\succ_i, \overline{\pi}_i^E(\succ_i, C, \varphi(\succ, C))\right).$$

4.2 School-Side Information Rules

We now focus on information rules that disclose information about school choice functions, but ex-post disclose nothing about the realized assignment other than the student's own assignment: school-side information rules. An information rule π is a **school-side information rule** if, for every student i and every $(\succ_i, C, \mu)$,

$$\pi_i^E(\succ_i, C, \mu) = \pi_i^I(\succ_i, C) \times \mathcal{M}_i(\mu)\,.$$

These rules are natural in most applications and correspond to the kind of information that is typically disclosed by school districts—students may have access to information about school priorities and capacities, but generally are not given information about other students' assignments.

Under such rules, we provide a sharper characterization of incontestability. Incontestability as defined is inherently an ex-post notion, because students appeal based on knowing

their own assignment. By Theorem 1, to ensure π-incontestability for an information rule, we need to check that each student's assignment belongs to her ex-post safety-set given her school-side and assignment-side information. As students know their own assignment, this requires computing each ex-post safety-set for every possible mechanism outcome. We will show that this can be reduced to checking a single interim set for each student. We define analogous notions of *interim safety-sets* and *interim safety-cutoffs*, which depend only on a student's school-side information.

Definition 4 *Let $\mathcal{K}_i$ be an interim information set for student i. A nonempty set of schools $\hat{S} \subseteq S$ is an **interim $\mathcal{K}_i$-high-priority set** for student i if for every $(C, \mu) \in \mathcal{K}_i \times \mathcal{M}$,*

$$i \in C_s(\mu(s) \cup \{i\}) \text{ for some } s \in \hat{S}.$$

*Student i's **interim safety-set** is*

$$\text{Safe}^I(\succ_i, \mathcal{K}_i) := \min_{\subseteq}\{U_i(s) : s \in S, U_i(s) \text{ is an interim } \mathcal{K}_i\text{-high-priority set}\}.$$

*Student i's **interim safety-cutoff** is*

$$\underline{s}^I(\succ_i, \mathcal{K}_i) := \min_{\succ_i} \text{Safe}^I(\succ_i, \mathcal{K}_i).$$

Interim safety-sets are conceptually similar to ex-post safety-sets, but are computed before an assignment is realized and therefore range over all possible assignments. The following example illustrates how those two sets differ.

Example 2 *Consider two students, i and j, and two schools, s and s', each with one seat. Suppose that school s ranks j above i, whereas school s' ranks i above j, and that student i knows this. Let i prefer s to s'. Clearly,* $\text{Safe}^I(\succ_i, \{C\}) = \{s, s'\}$.

Suppose now that $\mu(i) = s$, and that $\mathcal{H}_i$ is such that i does not know j's assignment. In the ex-post problem, the high-priority condition is imposed only on assignments compatible with what i knows about the realized assignment. Since i already knows that she is assigned to s, the singleton $\{s\}$ is an ex-post high-priority set for i. Hence, $\text{Safe}^E(\succ_i, \mathcal{H}_i) = \{s\}$. □

Although the interim and ex-post safety-sets may differ, the following result shows that for school-side information rules, they agree on the relevant membership question: checking that the mechanism assigns students to their ex-post safety-sets is equivalent to checking that it assigns students to their interim safety-sets.

[18] It is easy to check that interim safety-sets satisfy the same monotonicity property as ex-post safety-sets: if $\mathcal{K}_i \subseteq \mathcal{K}_i'$, then $\text{Safe}^I(\succ_i, \mathcal{K}_i) \subseteq \text{Safe}^I(\succ_i, \mathcal{K}_i')$

Theorem 2 *Let Π be a class of school-side information rules. For any $\pi \in \Pi$, problem $(\succ, C)$, assignment μ and student i,*

$$\mu(i) \in \text{Safe}^E(\succ_i, \pi_i^E(\succ_i, C, \mu)) \iff \mu(i) \in \text{Safe}^I(\succ_i, \pi_i^I(\succ_i, C)) .$$

Consequently, a mechanism φ is Π-incontestable if and only if for every $\pi \in \Pi$, every problem $(\succ, C)$, and every student i,

$$\varphi_i(\succ, C) \in \text{Safe}^I(\succ_i, \pi_i^I(\succ_i, C)) ,$$

or, equivalently,

$$\varphi_i(\succ, C) \succeq_i \underline{s}^I(\succ_i, \pi_i^I(\succ_i, C)).$$

The equivalence concerns whether the realized assignment belongs to the two safety-sets, not whether the safety-sets themselves are equal. One direction is immediate. Under a school-side information rule, the ex-post state space is $\mathcal{K}_i \times \mathcal{M}_i(\mu)$, which is contained in the interim state space $\mathcal{K}_i \times \mathcal{M}$. Hence, ex-post information can only shrink the student's safety-set, so membership in the ex-post safety-set implies membership in the interim safety-set.

The intuition for the converse is that the only new information the student receives ex post is her own assignment. Learning that she is assigned to $\mu(i)$ may therefore shrink the interim safety-set by effectively truncating it at $\mu(i)$: the student now knows that she has obtained a school at least as good as $\mu(i)$. But this information cannot make the safety-set shrink past $\mu(i)$, since the student learns nothing about the assignments at schools she prefers to $\mu(i)$. Thus, although the interim and ex-post safety-sets may differ, learning one's own assignment cannot change the membership verdict: the realized assignment belongs to the ex-post safety-set if and only if it belongs to the interim safety-set.

As with the ex-post safety-set characterization, if a class has a most informative rule (weakly more informative than any other rule in the class), incontestability can be checked with respect to only that rule:

Corollary 2 *Let Π be a class of school-side information rules that contains a most informative information rule $\bar{\pi}$. Then a mechanism φ is Π-incontestable if and only if for every problem $(\succ, C)$ and every student i,*

$$\varphi_i(\succ, C) \in \text{Safe}^I(\succ_i, \bar{\pi}_i^I(\succ_i, C)) .$$

5 A Robust School-Side Information Class: Π^C

We now consider a focal class of school-side information rules, Π^C, which contains *all* information rules that disclose only school-side information. Π^C-incontestability takes a robust approach whereby the designer of the mechanism does not know anything about which school-side information rules will be used, and requires a mechanism to be incontestable under *every* possible school-side information rule. By the preceding theorem, this is equivalent to requiring that the mechanism always assigns students in the interim safety-set generated by every school-side rule. Π^C contains a most informative rule—namely the one that fully discloses the realized school choice profile C. Corollary 2 therefore implies that Π^C-incontestability can be checked using only the interim safety-sets generated under full knowledge of C, which we denote $\text{Safe}^I(\succ_i, \{C\})$. To simplify terminology, we will say that a mechanism **respects safety-sets** if, for every problem $(\succ, C)$ and every student i,

$$\varphi_i(\succ, C) \in \text{Safe}^I(\succ_i, \{C\}).^{19}$$

When C is induced by responsive priorities, the interim safety-set $\text{Safe}^I(\succ_i, \{C\})$ is precisely the safe set of Decerf and Van der Linden (2021), and the least preferred school in that set is the safety school of Troyan and Morrill (2020), but generalized to any school choice profile. Since Π^C admits a most informative rule, a direct consequence of Corollary 2 is the following,

Corollary 3 *A mechanism is Π^C-incontestable if and only if it respects safety-sets.*

5.1 Incontestability and Efficiency

A priori, requiring incontestability under every school-side information rule could be highly restrictive. The class Π^C includes rules ranging from no disclosure about school choice functions to full disclosure of the entire choice profile, so a Π^C-incontestable mechanism must remain appeal-proof regardless of the precise disclosure regime used. The following proposition gives the main applied implication of our characterization: several prominent mecha-

[19] Strictly speaking, this property is respect for interim safety sets; however, because it is evaluated under the most informative rule in Π^C, which fully discloses the choice profile, the relevant safety set is determined entirely by the standard school choice problem $(\succ, C)$. We therefore omit "interim" and treat respect for safety sets as an ordinary property of an assignment at a school choice problem, and hence as a property of a mechanism.

nisms that may produce unstable assignments (including mechanisms designed to improve efficiency relative to stable assignments) remain incontestable with respect to Π^C.

Proposition 2 *The following mechanisms are Π^C-incontestable:*

(a) Any stable mechanism.

(b) Under responsive priorities, TTC.

(c) Under responsive priorities, Efficiency Adjusted Deferred Acceptance (Kesten, 2010).

(d) Any mechanism that Pareto-dominates a Π^C-incontestable mechanism.

The proof of Proposition 2 can be found in Appendix E.[20] Statement *(a)* is obvious (stable mechanisms are trivially incontestable and thus by Corollary 2 respect safety-sets). Since the relevant interim safety-sets are upper contour sets, a Pareto improvement of an assignment that respects safety-sets must itself respect safety-sets. Hence *(d)* follows. Since the Efficiency Adjusted Deferred Acceptance Pareto dominates any stable assignment (Kesten, 2010, Theorem 1), *(a)* and *(d)* together imply *(c)*.

As stability is a stronger notion than incontestability, of course any stable mechanism is Π^C-incontestable. The main insight of the result is that mechanisms that are prominent and may produce efficiency improvements are incontestable, despite violating stability. In particular, the fourth point implies that any mechanism that Pareto dominates a stable mechanism is Π^C-incontestable. Therefore, if we think that students in practice may not have access to or be permitted to use assignment-side information in appeals, our result shows that these mechanisms are still be appeals-proof, and provides a rationale for their use.

The proposition establishes robustness across the entire range of school-side disclosure regimes. Since Π^C contains every school-side information rule, any Π^C-incontestable mechanism is also incontestable with respect to every narrower class of such rules. The positive conclusions therefore do not depend on a district's precise disclosure policy: the mechanisms remain incontestable whether students receive very little information about schools' choice functions, full information, or anything in between. In this sense, Π^C-incontestability is

[20] In the online appendix we show that two variations of the TTC mechanism proposed by (Morrill, 2015), *Clinch and Trade*, and *First Clinch and Trade*, are also Π^C-incontestable.

the strongest test of appeal-proofness within the class of school-side information environments. By contrast, failure of Π^C-incontestability need not imply contestability under every narrower class, because coarser information may enlarge students' interim safety-sets.

At the same time, the proposition is not an "anything goes" result. Efficiency alone is not sufficient for Π^C-incontestability. As the following example shows, the Boston mechanism may assign a student ouside her interim safety-set, and is therefore contestable under the most informative school-side information rule, even though the resulting assignment is efficient. Thus, while Π^C-incontestability permits important departures from stability, it continues to impose a substantive restriction on the assignments a mechanism may select.[21]

Example 3 *Consider three students i_h, $h \in \{1,2,3\}$, and three schools s_h, $h \in \{1,2,3\}$. Each school has one seat and schools have responsive priorities. Preferences and priorities are depicted in Table 1.*

i_1	i_2	i_3
s_1	s_1	s_2
s_2	s_2	s_1
s_3	s_3	s_3

s_1	s_2	s_3
i_1	i_1	i_1
i_2	i_2	i_2
i_3	i_3	i_3

Table 1: The Boston mechanism is not Π^C-incontestable

For student i_2, the set $\{s_1, s_2\}$ is a high-priority set at C. Hence, $\text{Safe}^I(\succ_{i_2}, \{C\}) = \{s_1, s_2\}$. Under the Boston mechanism, however, given the preferences of students i_1 and i_3, student i_2 is assigned to s_3. The assignment therefore does not respect safety-sets, and thus the Boston mechanism is not Π^C-incontestable. □

5.2 Attainability

For the remainder of this section, we assume that school choice functions are substitutable and consistent. These assumptions are common in assignment problems with general choice functions. They also allow us to focus on the informational content of complaints rather than on failures of stability generated by pathologies of the choice rules.

- C_s satisfies **consistency** if for any $J, J' \subseteq I$, $C_s(J) \subseteq J' \subseteq J$ implies $C_s(J) = C_s(J')$;

[21] We show in the online Appendix that the Equitable TTC (Hakimov and Kesten, 2018) and the Application-Rejection mechanism (Chen and Kesten, 2017) do not respect safety-sets.

- C_s satisfies **substitutability** if for any $J, J' \subseteq I$ with $J \subseteq J'$, $R_s(J) \subseteq R_s(J')$.

It is well known that a choice function satisfies substitutability and consistency if and only if it satisfies path independence (Aizerman and Malishevski, 1981; Moulin, 1985), that is, for every $J, J' \subseteq I$,

$$C_s(J \cup J') = C_s(C_s(J) \cup C_s(J')).$$

The preceding results show that a Π^C-incontestable mechanism must always assign each student within her interim safety-set. We now ask whether this safety-set merely places an upper bound on the assignments a student may receive, or whether it describes exactly the schools that she can receive across problems compatible with her interim information.

Fix a school-side information rule π, a problem $(\succ, C)$, and a student i with interim information set $\mathcal{K}_i = \pi_i^I(\succ_i, C)$. We say that a school s is **attainable for student** i **under** φ **given** $(\succ_i, \mathcal{K}_i)$ if there exist a choice profile $\widehat{C} \in \mathcal{K}_i$ and a preference profile $\widehat{\succ}_{-i} \in \mathcal{P}_{-i}$ such that

$$\varphi_i\big((\succ_i, \widehat{\succ}_{-i}), \widehat{C}\big) = s.$$

Thus, attainability is evaluated from the student's interim perspective: her own preferences are held fixed, while the choice profile and the preferences of other students may vary over the problems compatible with her information.

To obtain an exact characterization of attainable schools, we impose non-wastefulness and a weak consistency requirement. A mechanism φ is **non-wasteful** if, for every problem $(\succ, C)$,

$$s \succ_i \varphi_i(\succ, C) \quad \Rightarrow \quad |\varphi_s(\succ, C)| = q_s.$$

We next define the relevant consistency requirement. A pair (i, s) is a **top–top pair** at a problem $(\succ, C)$ if s is student i's most-preferred school and

$$iinC_s(T \cup \{i\}) \qquad \text{for every } T \subseteq I \setminus \{j\}.$$

Under responsive priorities, (i, s) is a top-top pair if i has the highest priority at s (and s is i's most preferred school).

For a top–top pair (j, s), let $(\succ^{j,s}, C^{j,s})$ denote the reduced problem obtained by removing student j and one seat at school s.[22] A mechanism φ is **top–top consistent** if, for every

[22]The formal construction of the reduced problem, including the contraction of the choice function at school s, is provided in Appendix C.

problem $(\succ, C)$ and every top–top pair (j, s),

$$\varphi_j(\succ, C) = s \quad \Longrightarrow \quad \varphi_h(\succ, C) = \varphi_h(\succ^{j,s}, C^{j,s}) \quad \text{for every } h \neq j.$$

Thus, once a top–top student is assigned to her top–top school, removing that student and one seat at the school does not change the assignments of the remaining students. We show in the online Appendix that the student-optimal stable mechanism, TTC, and Kesten's (2010) mechanism satisfy top-top consistency.

The following theorem shows that, under these conditions, the interim safety-set is exactly the student's set of attainable schools.

Theorem 3 *Let φ be a Π^C-incontestable, non-wasteful, and top–top consistent mechanism. Fix an information rule $\pi \in \Pi^C$, a problem $(\succ, C)$, and a student i, and let $\mathcal{K}_i = \pi_i^I(\succ_i, C)$. Then, for every school $s \in S$,*

$$s \in \text{Safe}^I(\succ_i, \mathcal{K}_i) \quad \Longleftrightarrow \quad \varphi_i\big((\succ_i, \widehat{\succ}_{-i}), \widehat{C}\big) = s$$

for some $\widehat{C} \in \mathcal{K}_i$ and some $\widehat{\succ}_{-i} \in \mathcal{P}_{-i}$.

One direction follows from Π^C-incontestability. At every problem compatible with $\mathcal{K}_i$, the mechanism must assign student i within the interim safety-set generated by the realized choice profile; by monotonicity of safety-sets, this set is contained in $\text{Safe}^I(\succ_i, \mathcal{K}_i)$. Hence, every attainable school belongs to the student's interim safety-set.

The converse is the substantive part of the theorem. It shows that every school in the interim safety-set can actually arise as the student's assignment at some problem compatible with her information. The proof uses non-wastefulness and top–top consistency to construct such a problem recursively.

An immediate implication is an attainable-set equivalence across mechanisms. Any two Π^C-incontestable, non-wasteful, and top–top consistent mechanisms generate exactly the same set of attainable schools for every student and every interim information set: that set is $\text{Safe}^I(\succ_i, \mathcal{K}_i)$. The mechanisms may select different assignments at a particular problem, but from the student's interim perspective they offer the same range of possible schools.

This equivalence concerns attainable outcomes rather than the pointwise behavior of the mechanisms. The mechanisms need not select the same assignment at a given problem, nor need they have the same incentive or efficiency properties. Rather, conditional on a student's preferences and interim information, the identity of the mechanism does not affect the set

of schools that she may receive. In this specific sense, the mechanisms are indistinguishable from the student's interim perspective.[23]

Theorem 3 is also related to Lemma 1 and Corollary 1 of Troyan and Morrill (2020), who obtain a similar attainability result for stable-dominating mechanisms when schools have responsive priorities that are perfectly known to students. The two results are not formally nested. Their result does not impose top–top consistency, whereas our theorem applies to mechanisms, such as TTC, that need not Pareto dominate a stable mechanism. Our result also allows for general school-side information and consistent and substitutable choice functions.

5.3 Maxmin Incentives

We now use the characterization of attainable schools to derive a maxmin incentive property for mechanisms satisfying the conditions of Theorem 3. Reny (2022) studies the incentive properties of the Efficiency Adjusted Deferred Acceptance Mechanism (Kesten, 2010).[24] That mechanism is not strategy-proof, but Reny shows that it still satisfies a robust incentive property: if students compare strategies by their worst-case outcome, according to their true preferences, then truth-telling is optimal. This belief-free criterion is close in spirit to our use of safety-sets. In our framework, a student may also be uncertain about schools' choice functions, so the relevant worst case ranges over all problems compatible with the student's interim information.

Fix a mechanism φ and a student i with true preferences $\succ_i$ and interim information set $\mathcal{K}_i \subseteq \mathcal{C}$. When student i submits a preference ordering $\succ_i' \in \mathcal{P}_i$, let $w_i(\succ_i' | \succ_i, \varphi, \mathcal{K}_i)$ denote the worst school, according to $\succ_i$, that i can receive as the choice profile ranges over $\mathcal{K}_i$ and the preferences of the other students range over $\mathcal{P}_{-i}$. Formally,

$$w_i(\succ_i' | \succ_i, \varphi, \mathcal{K}_i) \equiv \min_{\substack{C \in \mathcal{K}_i \\ \succ_{-i} \in \mathcal{P}_{-i}}} \varphi_i\big((\succ_i', \succ_{-i}), C\big), \tag{5}$$

where the minimum is taken with respect to $\succ_i$.

[23] This conclusion is related in spirit to the notion of i-indistinguishability in Li (2017), but is distinct from it. Li's notion concerns an agent's experience within an extensive-form mechanism, whereas our result concerns only the set of final schools attainable by the student.

[24] Reny (2022) proposes a different mechanism than EADAM, but the two are outcome equivalent when all students waive their priorities.

Theorem 3 gives the following extension of Reny's maxmin result.[25]

Proposition 3 *Fix a student i with interim information set $\mathcal{K}_i$. Let φ be a Π^C-incontestable, non-wasteful, and top-top consistent mechanism. Then, for every preference ordering $\succ_i' \in \mathcal{P}_i$,*

$$w_i(\succ_i \mid \succ_i, \varphi, \mathcal{K}_i) \;\succeq_i\; w_i(\succ_i' \mid \succ_i, \varphi, \mathcal{K}_i). \tag{6}$$

The intuition follows directly from the attainability result. Under truthful reporting, the set of schools that student i can receive is exactly $\text{Safe}^I(\succ_i, \mathcal{K}_i)$ with safety-cutoff $\underline{s}^I(\succ_i, \mathcal{K}_i)$, which is therefore the student's worst-case outcome under truthful reporting. Under any alternative report $\succ_i'$, the attainable set is $\text{Safe}^I(\succ_i', \mathcal{K}_i)$. This set is itself an interim $\mathcal{K}_i$-high-priority set, but it is selected using the reported rather than the true preferences. Let s be the least-preferred school in $\text{Safe}^I(\succ_i', \mathcal{K}_i)$ according to the true preference $\succ_i$, so this is the worst-case outcome under the misreport. Then $U_i(s)$ contains $\text{Safe}^I(\succ_i', \mathcal{K}_i)$, and is therefore also high-priority. Since $\text{Safe}^I(\succ_i, \mathcal{K}_i)$ is the smallest high-priority upper contour set according to the true preferences, its safety-cutoff must be weakly preferred to s. A misreport may therefore improve the student's assignment at some problems, but it cannot improve her worst-case outcome.

6 Other classes of information rules

To place our analysis of school-side information rules and Π^C in context, we briefly consider two boundary classes of information rules. The first, Π^μ, allows students to receive information about the realized assignment but no information about schools' choice functions. The second, $\Pi^{C,\mu}$, allows students to receive information about both objects.

Because capacities are embedded in choice functions, students under rules in Π^μ do not know capacities either. In this environment, Π^μ-incontestability is equivalent to individual rationality. A student assigned to an unacceptable school can always complain by claiming the null school. Conversely, an individually rational student cannot establish that any preferred school would accept her: without information about choice functions, it remains possible that every preferred school would choose its current assignees over her. Thus, even extensive information about the assignment has little force when students do not know how schools choose.

[25] The proof can be found in Appendix F.

A slightly richer benchmark is $\Pi^{\mu,q}$, under which students also know schools' capacities. In this case, incontestability is equivalent to individual rationality and non-wastefulness. A vacant seat at a preferred school supports a complaint independently of the school's priorities. If no such vacancy exists, however, the unknown choice functions can again be completed so that every preferred school rejects the student. These two cases illustrate why school-side information is essential for appeals to impose more than minimal restrictions on assignments.

At the other boundary, rules in $\Pi^{C,\mu}$ may disclose information about both the choice profile and the realized assignment. This class contains the full-revelation rule, under which a legitimate complaint is simply an ordinary blocking complaint. Consequently, $\Pi^{C,\mu}$-incontestability is equivalent to stability. Necessity follows from the presence of full revelation. Conversely, if the realized assignment is stable, no student can have a legitimate complaint under any information rule, because the true choice profile and assignment are compatible with her information and contain no blocking pair.

These boundary cases clarify why Π^{C} is the substantively interesting environment. Without school-side information, incontestability imposes little beyond individual rationality and non-wastefulness; once the full assignment may also be revealed, incontestability collapses to stability. Under Π^{C}, students may substantiate claims using information about how schools choose, while their inability to observe other students' assignments still permits robust departures from stability. Formal proofs are provided in the Online Appendix.

7 Conclusion

We propose in this paper an epistemic definition of stability for assignment problems. Our results highlight that the information available to students (about participants, preferences, priorities or choice rules, and the realized assignment) should be treated as part of the design problem rather than as a background feature of the market. Theorem 1 shows that a student's ability to justify a complaint is characterized by a simple test: whether her assignment belongs to her ex-post safety-set. A second key result, Corollary 3 , characterizes incontestable institutions whenever students' ex-post information regarding the assignments contains only their own assignments, which is the standard information rule in most school districts. The class of mechanisms is large. It includes not only the student-optimal stable mechanism, but also TTC and any mechanism that weakly Pareto dominates a stable mechanism (and is non-wasteful and top-top consistent). Hence, there is no inherent trade-

off between incontestability and efficiency. At the same time, incontestability is not an "anything goes" requirement, as it excludes efficient mechanisms such as the Boston mechanism. Figure 1 gives an overview of various assignment mechanisms. Our third main result, Theorem 3, establishes that mechanisms that are Π^C-incontestable, top–top consistent, and non-wasteful are observationally indistinguishable from the students' point of view.

Our approach to epistemic stability opens several directions for future research. First, we have focused on one-sided appeals. A natural extension would study appeal procedures in which both sides possess private information and can challenge an assignment, building on the literature on two-sided matching under incomplete information (Roth, 1989; Liu, 2020; Chen and Hu, 2023). Second, legitimate complaints are individual, while information may be shared. An assignment may be informationally unstable without generating any individual legitimate complaint, yet become vulnerable once students pool their information and submit a joint complaint. Characterizing information rules that are joint-complaint-proof would clarify when information sharing restores the force of stability. Third, the paper studies three broad classes of information rules. A finer taxonomy could identify the minimal disclosure needed for incontestability, or reveal additional mechanisms that are incontestable under more detailed information structures. Such research would tighten the relation between epistemic stability and the institutional design of appeals.

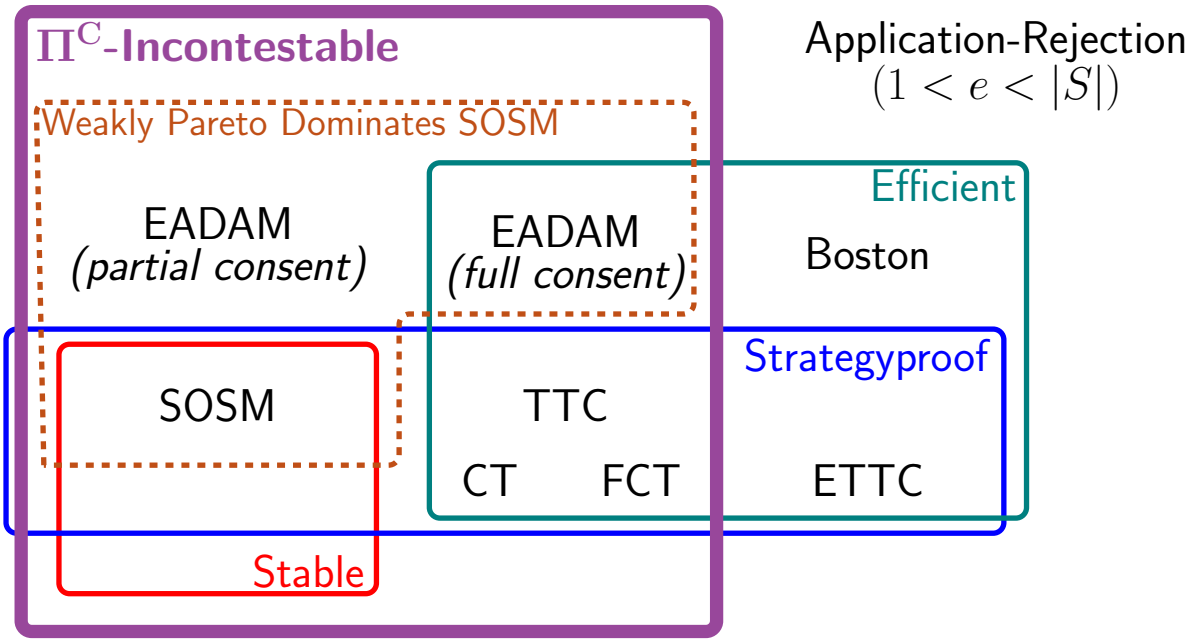


Figure 1: Comparison of mechanisms

Note: CT denotes "Clinch and Trade", FCT denotes "First Clinch and Trade", ETTC denotes "Equitable TTC". We show in the Online Appendix that CT and FCT respect safety-sets and are top-top consistent, while ETTC violates respect for safety-sets. A description of CT, FCT, Application-Rejection, and ETTC can be found in the online Appendix.

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A Proof of Theorem 1

Suppose that $\mu(i) \notin \text{Safe}^E(\succ_i, \mathcal{H}_i)$. Let $\widehat{S} = \text{Safe}^E(\succ_i, \mathcal{H}_i)$. Then $\underline{s}^E(\succ_i, \mathcal{H}_i) \succ_i \mu(i)$, and thus $s \succ_i \mu(i)$ for every school $s \in \widehat{S}$. Fix any $(\widehat{C}, \widehat{\mu})$ compatible with $\mathcal{H}_i$ (so $\widehat{\mu}(i) = \mu(i)$). Since $\widehat{S}$ is an $\mathcal{H}_i$-high-priority set, there exists $s \in \widehat{S}$ such that $i \in \widehat{C}_s(\widehat{\mu}(s) \cup \{i\})$. Since $s \succ_i \mu(i)$ and $\widehat{\mu}(i) = \mu(i)$, student i has a legitimate complaint at μ given $\mathcal{H}_i$.

Suppose that i has a legitimate complaint at μ given $\mathcal{H}_i$. Let $T = \{s \in S : \ s \succ_i \mu(i)\}$. Then $T \neq \varnothing$; otherwise no school would be strictly preferred to $\mu(i)$, contradicting the existence of a legitimate complaint. Since i has a legitimate complaint, for every compatible pair $(\widehat{C}, \widehat{\mu})$, there exists a school $s \in T$ such that $i \in \widehat{C}_s(\widehat{\mu}(s) \cup \{i\})$. Hence T is an $\mathcal{H}_i$-high-priority set for i. Since $\text{Safe}^E(\succ_i, \mathcal{H}_i)$ maximizes the least-preferred school among all $\mathcal{H}_i$-high-priority sets, we have $\underline{s}(\succ_i, \text{Safe}^E(\succ_i, \mathcal{H}_i)) \succeq_i \underline{s}(\succ_i, T)$. But $\underline{s}(\succ_i, T) \succ_i \mu(i)$ by construction of T. Therefore $\underline{s}(\succ_i, \text{Safe}^E(\succ_i, \mathcal{H}_i)) \succ_i \mu(i)$. Hence $\mu(i) \notin \{s \in S : \ s \succeq_i \underline{s}_i(\text{Safe}^E(\succ_i, \mathcal{H}_i))\}$, which means that $\mu(i) \notin \text{Safe}^E(\succ_i, \mathcal{H}_i)$.

B Proof of Theorem 2

Fix $\pi \in \Pi$, a problem $(\succ, C)$, an assignment μ, and a student i. Let $\mathcal{K}_i = \pi_i^I(\succ_i, C)$, and $\mathcal{H}_i = \pi_i^E(\succ_i, C, \mu) = \mathcal{K}_i \times \mathcal{M}_i(\mu)$.

(If) Suppose that $\mu(i) \in \text{Safe}^I(\succ_i, \mathcal{K}_i)$. Assume to the contrary that $\mu(i) \notin \text{Safe}^E(\succ_i, \mathcal{H}_i)$, and let $T = \text{Safe}^E(\succ_i, \mathcal{H}_i)$. By definition, T is an ex-post $\mathcal{H}_i$-high-priority set. We show that it is also an interim $\mathcal{K}_i$-high-priority set.

Fix any $(C', \mu') \in \mathcal{K}_i \times \mathcal{M}$. If $\mu'(i) \in T$, then, for $s = \mu'(i)$, since $|\mu'(s)| \leq q_s$ and since all students are acceptable for all schools, we have $i \in C'_s(\mu'(s) \cup \{i\})$. Suppose instead that $\mu'(i) \notin T$. Since also $\mu(i) \notin T$, we can construct an assignment $\widetilde{\mu} \in \mathcal{M}_i(\mu)$ such that $\widetilde{\mu}(s) = \mu'(s)$ for every $s \in T$. Indeed, if $\mu'(i) = \mu(i)$, take $\widetilde{\mu} = \mu'$. Otherwise, move i from $\mu'(i)$ to $\mu(i)$ and, if $\mu(i)$ is full under μ', move one of its assignees to the seat vacated by i at $\mu'(i)$. All these adjustments concern schools outside T.

Thus, $(C', \widetilde{\mu}) \in \mathcal{H}_i$. Since T is an ex-post $\mathcal{H}_i$-high-priority set, there exists $s \in T$ such that

$$i \in C'_s(\widetilde{\mu}(s) \cup \{i\}) = C'_s(\mu'(s) \cup \{i\}).$$

Hence, T is an interim $\mathcal{K}_i$-high-priority set. Since T is an upper contour set, the definition

of the interim safety-set implies

$$\text{Safe}^I(\succ_i, \mathcal{K}_i) \subseteq T = \text{Safe}^E(\succ_i, \mathcal{H}_i).$$

This contradicts $\mu(i) \in \text{Safe}^I(\succ_i, \mathcal{K}_i)$ and $\mu(i) \notin T$.

(Only if) Since $\mathcal{H}_i \subseteq \mathcal{K}_i \times \mathcal{M}$, every interim $\mathcal{K}_i$-high-priority set is also an ex-post $\mathcal{H}_i$-high-priority set. Therefore,

$$\text{Safe}^E(\succ_i, \mathcal{H}_i) \subseteq \text{Safe}^I(\succ_i, \mathcal{K}_i).$$

Thus, $\mu(i) \in \text{Safe}^E(\succ_i, \mathcal{H}_i)$ implies that $\mu(i) \in \text{Safe}^I(\succ_i, \mathcal{K}_i)$.

Finally, by Theorem 1 and the definition of Π-incontestability, a mechanism is Π-incontestable if and only if its assignment belongs to the corresponding ex-post safety-set for every information rule, problem, and student. Applying the equivalence above to $\mu = \varphi(\succ, C)$ gives the stated interim safety-set condition.

C Reduced problem with top-top pairs

Definition 5 (Top-top pair) *Fix a problem $(\succ, C)$. A pair $(i, s) \in I \times S \backslash \{s_0\}$ is a **top-top pair** in $(\succ, C)$ if*

(a) $s \succ_i s'$ for all $s' \in S \backslash \{s\}$; and

(b) i has top priority at s in the following sense: if C_s is induced by responsive priorities r_s, then $r_s(i) < r_s(j)$ for every $j \in I \backslash \{i\}$; otherwise, $i \in C_s(T \cup \{i\})$ for every $T \subseteq I \setminus \{i\}$.

The reduced problem removes the top-top student and one seat at the corresponding school.[26]

Definition 6 *Fix a problem $(I, S, \succ, C, q)$ and a pair $(i, s) \in I \times S \backslash \{s_0\}$ such that $q_s \geq 1$. Assume that i has* top priority *at s, that is, $i \in C_s(T \cup \{i\})$ for each $T \subseteq I \backslash \{i\}$.*

*Define the **reduced problem** $(I^{i,s}, S^{i,s}, \succ^{i,s}, C^{i,s}, q^{i,s})$ obtained from $(I, S, \succ, C, q)$ by withdrawing student i and one seat at s as follows:*

[26] The definition is somewhat heavy because one has to specify how choice functions are projected onto the reduced set of students. This is immediate with responsive priorities (or in the one-to-one case). With general substitutable choice functions, the reduction (condition *(e)* in Definition 6) at the school whose capacity is reduced must be defined as a contracted choice rule, that is, the domain of a school choice function in the reduced problem is no longer I but $I \backslash \{i\}$, where i is the student removed from the problem.

(a) $I^{i,s} \equiv I \setminus \{i\}$.

(b) *If* $q_s > 1$, *then* $S^{i,s} \equiv S$ *and* $q_s^{i,s} \equiv q_s - 1$. *If* $q_s = 1$, *then* $S^{i,s} \equiv S \setminus \{s\}$. *For every* $s' \in S^{i,s} \setminus \{s\}$, *set* $q_{s'}^{i,s} \equiv q_{s'}$.

(c) *For each student* $j \in I^{i,s}$, *let* $\succ_j^{i,s}$ *be the restriction of* $\succ_j$ *to* $S^{i,s}$.

(d) *For each school* $r \in S^{i,s}$ *with* $r \neq s$, *define*

$$C_r^{i,s}(J) \equiv C_r(J) \qquad \text{for all } J \subseteq I^{i,s}.$$

(e) *If* $q_s > 1$, *define the reduced choice function at* s *by*

$$C_s^{i,s}(J) \equiv C_s(J \cup \{i\}) \setminus \{i\} \qquad \text{for all } J \subseteq I^{i,s}.^{27} \tag{7}$$

D Proof of Theorem 3

The proof of Theorem 3 relies on the following lemmas.

Lemma 1 (Aizerman and Malishevski, 1981) *Let* $C_s : 2^I \to 2^I$ *be a choice function satisfying path independence. Let* $\mathcal{R}_s = \{C_s(X) : X \subseteq I\}$ *be the range of* C_s. *Define a binary relation* $\trianglerighteq_s$ *on* $\mathcal{R}_s$ *by*

$$T \trianglerighteq_s T' \quad \Leftrightarrow \quad C_s(T \cup T') = T. \tag{8}$$

Then $\trianglerighteq_s$ *is a partial order on* $\mathcal{R}_s$ *(i.e., it is reflexive, antisymmetric, and transitive).*

Lemma 2 *Let* $\mathcal{C}$ *be such that for each* $C \in \mathcal{C}$, C_s *is path independent for each* $s \in S$. *Let* $\pi \in \Pi^C$, *a problem* $(\widehat{\succ}, \widehat{C})$, *and a student* i *and let* $\mathcal{K}_i = \pi_i^I(\widehat{\succ}_i, \widehat{C})$. *Let* C *be a choice profile compatible with* $\mathcal{K}_i$. *Suppose there exist a non-empty set of schools* $\{s_1, \ldots, s_k\} \subseteq S \setminus \{s_0\}$ *and pairwise disjoint sets* $J_1, \ldots, J_k \subseteq I \setminus \{i\}$ *such that for each* $h = 1, \ldots, k$,

$$|J_h| = q_{s_h} \qquad \text{and} \qquad J_h = C_{s_h}(J_h \cup \{i\}). \tag{9}$$

[27] Top priority implies $C_s^{i,s}(J) \subseteq J$ and $|C_s^{i,s}(J)| \leq q_s - 1$. If C_s is substitutable and consistent (i.e., path independent), then the contracted rule $C_s^{i,s}$ is also path independent, hence substitutable and consistent.

Then there exists an assignment μ such that $\mu(i) = s_0$ and

$$\mu(s) = \begin{cases} C_s(\mu(s) \cup \mu(s_0)) & \text{if } s \in \{s_1, \ldots, s_k\}, \\ \varnothing & \text{if } s \notin \{s_0, s_1, \ldots, s_k\}, \\ I \setminus \cup_{h=1}^{k} \mu(s_h) & \text{if } s = s_0\,. \end{cases} \tag{10}$$

Proof Fix i, $\mathcal{K}_i$, a choice profile C compatible with $\mathcal{K}_i$, a non-empty set $\widehat{S} = \{s_1, \ldots, s_k\} \subseteq S$, and pairwise disjoint sets $J_1, \ldots, J_k \subseteq I \setminus \{i\}$ satisfying Eq. (9).

Let μ^0 be the assignment such that $\mu^0(s_h) = J_h$ for each $h = 1, \ldots, k$ and $\mu^0(s) = \varnothing$ for $s \notin \widehat{S} \cup \{s_0\}$, and $\mu^0(s_0) = I \setminus \cup_h J_h$. Let $Z^0 = \mu^0(s_0)$. Since $i \notin J_h$ for all h, we have $\mu^0(i) = s_0$, hence $i \in Z^0$. We consider the following multi-step procedure

Step $\mathbf{t \geq 1}$ Let $\mathcal{S}^t = \{s \in \widehat{S} : C_s\big(\mu^{t-1}(s) \cup Z^{t-1}\big) \neq \mu^{t-1}(s)\}$. If $\mathcal{S}^t = \varnothing$, stop and set $\mu = \mu^{t-1}$ and $Z = Z^{t-1}$. Otherwise, pick any $s^t \in \mathcal{S}^t$ and let μ^t be defined as

$$\mu^t(s) = \begin{cases} C_s\big(\mu^{t-1}(s^t) \cup Z^{t-1}\big) & \text{if } s = s^t, \\ \mu^{t-1}(s) & \text{if } s \neq s^t, s_0\,, \end{cases}$$

and $\mu^t(s_0) = I \setminus \cup_{s \neq s_0} \mu^t(s)$. Let $Z^t = \mu^t(s_0)$.

We first show that the procedure must eventually stop after a finite number of steps. To that end, consider any step t, and let $s = s^t$, $B = \mu^{t-1}(s)$ and $A = \mu^t(s) = C_s(B \cup Z^{t-1})$. Since $A \subseteq A \cup B \subseteq B \cup Z^{t-1}$, consistency yields $C_s(A \cup B) = C_s(B \cup Z^{t-1}) = A$. Therefore, $A \trianglerighteq_s B$ (where $\trianglerighteq_s$ is defined in Lemma 1). Since, $A \neq B$, we have $A \rhd_s B$, where $\rhd_s$ denotes the irreflexive part of $\trianglerighteq_s$. Since $\mathcal{R}_s$ is finite for each s and $\rhd_s$ is a partial order by Lemma 1, each school can update its assignment only finitely many times. Hence the procedure terminates.

Denote by T the last step at which a school updates its assignment. That is, $\mathcal{S}^{T+1} = \varnothing$. So, for every $s \in \widehat{S}$, $\mu(s) = \mu^T(s)$ and by construction, $C_s(\mu(s) \cup Z) = \mu(s)$. Also by construction, $\mu(s) = \varnothing$ for each $s \notin \widehat{S} \cup \{s_0\}$, and $\mu(s_0) = I \setminus \mu(\widehat{S})$. This is exactly Eq. (10).

It remains to show that $i \in Z^T$. We prove a stronger claim by induction on t: for every $t \geq 0$,

$$i \in Z^t \qquad \text{and} \qquad i \notin C_s\big(\mu^t(s) \cup \{i\}\big) \ \text{ for all } s \in \widehat{S}. \tag{11}$$

If this claim holds, then since $Z^t = \mu^t(s_0)$, the first part of Eq. (11) implies that i is never assigned to a school in $\widehat{S}$ at any step. In particular, $i \in Z^T$, as desired.

We first show that Eq. (11) holds for $t = 0$. By construction, $\mu^0(s_h) = J_h$ for each $h = 1, \ldots, k$, and $i \notin J_h$ since $J_h \subseteq I \setminus \{i\}$. Hence $i \notin \mu^0(s)$ for every $s \neq s_0$, so $i \in Z^0 = \mu^0(s_0)$. Moreover, for each $s_h \in \widehat{S}$, Eq. (9) gives $C_{s_h}\big(\mu^0(s_h) \cup \{i\}\big) = C_{s_h}(J_h \cup \{i\}) = J_h = \mu^0(s_h)$. So, $i \notin C_{s_h}(\mu^0(s_h) \cup \{i\})$. For schools $s \notin \widehat{S} \cup \{s_0\}$, $\mu^0(s) = \varnothing$ and they never update. Also, $i \in \mu(s_0) = Z^0$, so the second part of Eq. (11) is trivially satisfied on $\widehat{S}$ for schools not in $\widehat{S}$. Thus Eq. (11) holds at $t = 0$.

Fix $t \geq 1$ and assume Eq. (11) holds at $t-1$. If $\mathcal{S}^t = \varnothing$, then $\mu^t = \mu^{t-1}$ and $Z^t = Z^{t-1}$, so Eq. (11) also holds at t. Suppose instead that $\mathcal{S}^t \neq \varnothing$ and the procedure updates school $s^t \in \mathcal{S}^t$. Let $M \equiv \mu^{t-1}(s^t) \cup Z^{t-1}$. By the induction hypothesis, $i \in Z^{t-1}$, hence $\mu^{t-1}(s^t) \cup \{i\} \subseteq M$. Also by the induction hypothesis, $i \notin C_{s^t}(\mu^{t-1}(s^t) \cup \{i\})$, i.e., $i \in R_{s^t}(\mu^{t-1}(s^t) \cup \{i\})$. By substitutability, $R_{s^t}(\mu^{t-1}(s^t) \cup \{i\}) \subseteq R_{s^t}(M)$, so $i \in R_{s^t}(M)$ and therefore $i \notin C_{s^t}(M) = \mu^t(s^t)$. No other school $s \neq s^t, s_0$ changes its assigned set at step t, so $i \notin \mu^t(s)$ for every $s \neq s_0$. Hence, by definition of $Z^t = \mu^t(s_0) = I \setminus \bigcup_{s \neq s_0} \mu^t(s)$, we obtain $i \in Z^t$.

It remains to verify the second part of Eq. (11) at step t. For any school $s \in \widehat{S}$ with $s \neq s^t$ we have $\mu^t(s) = \mu^{t-1}(s)$, so $i \notin C_s(\mu^t(s) \cup \{i\})$ follows from the induction hypothesis. For $s = s^t$, note that $\mu^t(s^t) = C_{s^t}(M) \subseteq M$ and $i \in M$, so $\mu^t(s^t) \subseteq \mu^t(s^t) \cup \{i\} \subseteq M$. By consistency, since $C_{s^t}(M) = \mu^t(s^t)$ and $\mu^t(s^t) \subseteq \mu^t(s^t) \cup \{i\} \subseteq M$, we have $C_{s^t}\big(\mu^t(s^t) \cup \{i\}\big) = C_{s^t}(M) = \mu^t(s^t)$, and thus $i \notin C_{s^t}(\mu^t(s^t) \cup \{i\})$. This completes the induction step and proves Eq. (11) for all t. Taking $t = T$ yields $i \in Z^T$, as claimed. ■

Lemma 3 (Aizerman and Malishevski (1981)) *A choice function C_s is path-independent if, and only if there exists a finite sequence of linear ordering $\{\succ^h\}_{h=1,\ldots k}$ over I such that, for all $J \subseteq I$,*

$$C_s(J) = \cup_h \max_J(\succ^h)$$

where $\max_J(\succ^h) = \{j \in J \ : \ \nexists j' \in J \text{ such that } j' \succ^h j\}$.

Lemma 4 *Let $\widehat{S} = \{s_1, \ldots, s_k\} \subseteq S \backslash \{s_0\}$ and μ satisfy Eq. (10). Let s be any school in $\widehat{S}$ and $J \subseteq I$ such that $J \cap \mu(s) \neq \varnothing$. Then there exists $\ell \in J \cap \mu(\widehat{S})$ such that $\ell \in C_s(J)$.*

Proof Let ℓ be a student in $J \cap \mu(s)$ (such a student exists by assumption). Let $\{\succ^h\}_{h=1,\ldots,m}$ be an Aizerman and Malishevski decomposition of C_s. Let $\succ^h$ be the ordering from the decomposition for which ℓ is maximal in $\mu(s) \cup \mu(s_0)$. That ordering exists because

$\ell \in \mu(s) = C_s(\mu(s) \cup \mu(s_0))$. If $\ell \in \max_J(\succ^h)$ then we are done (it will imply that $\ell \in C_s(J)$). Otherwise, let $\ell' \in \max_J(\succ^h)$. So, $\ell' \in C_s(J)$ and $\ell' \succ^h \ell$. Hence, $\ell' \succ^h j$ for each $j \in \mu(s_0)$. So, $\ell' \notin \mu(s_0)$ and thus $\ell' \in \mu(\widehat{S})$. ■

Lemma 5 *Let i be a student with interim information set $\mathcal{K}_i$ and preference list $\succ_i$. Let C be a choice profile compatible with $\mathcal{K}_i$ such that for a non-empty set of schools $\widehat{S} = \{s_1, \ldots, s_k\} \subseteq S \backslash \{s_0\}$ there are pairwise disjoint sets $J_1, \ldots, J_k \subseteq I \setminus \{i\}$ such that for each $h = 1, \ldots, k$, $|J_h| = q_{s_h}$ and $J_h = C_{s_h}(J_h \cup \{i\})$. Let φ be any mechanism that respects safety-sets and is top-top consistent. Then there exists $\succ_{-i}$ such that $\varphi_i((\succ_i, \succ_{-i}), C) \notin \widehat{S}$, where $\succ_{-i}$ is such that for any student $j \notin \varphi_{\widehat{S}}((\succ_i, \succ_{-i}), C) \cup \{i\}$ and for any school $s \notin \widehat{S} \cup \{s_0\}$, $s_0 \succ_j s$.*

Proof Let $\mathcal{K}_i$, C, $\widehat{S} = \{s_1, \ldots, s_k\}$, and $J_1, \ldots, J_k$ satisfy the conditions of the lemma. By Lemma 2, there exists an assignment μ satisfying Eq. (10). In particular, $\mu(i) = s_0$ and therefore $i \notin \mu(\widehat{S})$. Moreover, for each $s \in \widehat{S}$, $|\mu(s)| = q_s$.

We first construct a sequence of top-top pairs. The construction is based on a TTC-like procedure, but this procedure is used only as a device to identify top-top pairs; it is not the assignment mechanism under consideration.

Let $U \equiv \mu(\widehat{S})$. We start from the original problem and an auxiliary assignment ν^0 given by $\nu^0(s) = \mu(s)$ for every $s \in \widehat{S}$ and $\nu^0(s_0) = \mu(s_0)$. At any step of the construction, we consider the current reduced problem, obtained from the original problem by removing the top-top pairs already identified. Let I^t be the set of remaining students, let C^t be the corresponding profile of reduced choice functions, and let q_s^t be the remaining capacity of school s. Let $U^t \equiv U \cap I^t$ be the students in U who have not yet been removed, and let $\widehat{S}^t \equiv \{s \in \widehat{S} : \ q_s^t > 0\}$.

For each $s \in \widehat{S}^t$, let $\nu^t(s)$ denote the students in $\mu(s)$ who have not yet been removed. The construction maintains the following for each step t,

$$|\nu^t(s)| = q_s^t \quad \text{and} \quad \nu^t(s) = C_s^t\big(\nu^t(s) \cup \nu^t(s_0)\big) \qquad \text{for every } s \in \widehat{S}^t. \tag{12}$$

Eq. (10) holds at $t = 0$ and will continue to hold after each step $t \geq 1$. To see this, not that whenever a school in $\widehat{S}^t$ loses one seat, exactly one student in its current set $\nu^t(s)$ is removed. Since reduced choice functions preserve path independence, the same argument as in Lemma 2 gives Eq. (12) in the reduced problem.

At each step t, each remaining student $j \in U^t$ points to the unique school $s \in \widehat{S}^t$ such that $j \in \nu^t(s)$. Each school $s \in \widehat{S}^t$ points to a student in $C_s^t(I^t) \cap U^t$. Since $q_s^t > 0$, we have $\nu^t(s) \neq \varnothing$, and applying Lemma 4 to the current reduced problem and to the set $J = I^t$ gives $C_s^t(I^t) \cap U^t \neq \varnothing$.

If C_s^t is induced by responsive priorities r_s^t, then school s points to the highest-priority student in $C_s^t(I^t) \cap U^t$. Otherwise, it points to any student in $C_s^t(I^t) \cap U^t$.

Since every student in U^t points to a school in $\widehat{S}^t$ and every school in $\widehat{S}^t$ points to a student in U^t, there is at least one cycle. Consider one such cycle. For each school s in the cycle, let j_s be the student to whom s points. We record the pair (j_s, s).

We claim that each recorded pair is a top-top pair in the current reduced problem. The student side will be imposed by the preference profile constructed below: s will be the favorite school of j_s. For the school side, first suppose that C_s^t is not induced by responsive priorities. Since $j_s \in C_s^t(I^t)$, substitutability implies $j_s \in C_s^t(T \cup \{j_s\})$ for every $T \subseteq I^t \setminus \{j_s\}$. So, (j_s, s) is a top-top pair.

Now suppose that C_s^t is induced by responsive priorities r_s^t. By Eq. (12), the students not in U^t (who are assigned to the null school in the auxiliary assignment ν^t) have a lower priority than the students in $\nu^t(s)$. Hence the highest priority student in $C_s^t(I^t) \cap U^t$ is the highest priority student among all remaining students in I^t. Since the pointing rule selects precisely that student, we have $r_s^t(j_s) < r_s^t(h)$ for every $h \in I^t \setminus \{j_s\}$. Thus, (j_s, s) is also a top-top pair in the responsive-priority case.

After recording all pairs in the cycle, we reduce the problem by these pairs. That is, for each recorded pair (j_s, s), we remove student j_s and one seat of school s. Note that for students within the same cycle, the order in which these reductions are performed has no impact: each student in a cycle and the school that point to her constitute a top-top pair. Once all students in cycles are removed (and schools' capacities adjusted accordingly), we proceed to the next step of this TTC-like algorithm. The procedure terminates after finitely many steps because at least one student in U^t is removed at each step.

Let $(j_1, \sigma_1), \ldots, (j_m, \sigma_m)$ be the sequence of recorded pairs, in the order in which they are removed. By construction, $\{j_1, \ldots, j_m\} = U = \mu(\widehat{S})$, and for each school $s \in \widehat{S}$, the number of indices ℓ such that $\sigma_\ell = s$ is exactly q_s. That is, the recorded pairs fill all seats at schools in $\widehat{S}$.

We now construct the preference profile. Let $\succ^*$ be any profile such that $\succ_i^* = \succ_i$, and for each $\ell = 1, \ldots, m$, $\sigma_\ell \succ_{j_\ell}^* s$ for every $s \in S \setminus \{\sigma_\ell\}$. For every student $j \notin U \cup \{i\}$, choose

preferences $\succ_j^*$ such that $s_0 \succ_j^* s$ for every $s \notin \widehat{S} \cup \{s_0\}$.

We next show that the mechanism φ assigns each student j_ℓ to σ_ℓ. Let Γ^0 denote the original problem $((\succ_i, \succ_{-i}^*), C)$. For each $\ell \geq 1$, let Γ^ℓ be the problem obtained from $\Gamma^{\ell-1}$ by reducing it at the top-top pair (j_ℓ, σ_ℓ).

We prove by induction on ℓ that $\varphi_{j_\ell}(\Gamma^0) = \sigma_\ell$. Consider first $\ell = 1$. By construction, σ_1 is j_1's favorite school, and (j_1, σ_1) is a top-top pair in Γ^0. Hence the singleton $\{\sigma_1\}$ is the safety-set of j_1 in Γ^0. Since φ respects safety-sets, $\varphi_{j_1}(\Gamma^0) = \sigma_1$.

By top-top consistency, the assignments of all remaining students in Γ^0 coincide with their assignments in the reduced problem Γ^1.

Now suppose the claim has been proved for $1, \ldots, \ell - 1$. By repeated applications of top-top consistency, the assignment of j_ℓ in the original problem Γ^0 coincides with her assignment in the reduced problem $\Gamma^{\ell-1}$. By construction, (j_ℓ, σ_ℓ) is a top-top pair in $\Gamma^{\ell-1}$, and σ_ℓ is j_ℓ's favorite school in that reduced problem. Hence $\text{Safe}^I(\succ_{j_\ell}^*, \{C(\Gamma^{\ell-1})\}) = \{\sigma_\ell\}$, where $C(\Gamma^{\ell-1})$ denotes the schools' choice profile in the problem $\Gamma^{\ell-1}$. Since φ respects safety-sets, $\varphi_{j_\ell}(\Gamma^{\ell-1}) = \sigma_\ell$. Therefore, $\varphi_{j_\ell}(\Gamma^0) = \sigma_\ell$. This completes the induction.

It follows that, under φ, all seats at schools in $\widehat{S}$ are filled by the students $j_1, \ldots, j_m$. Since $\{j_1, \ldots, j_m\} = U = \mu(\widehat{S})$ and $i \notin U$, feasibility of assignments implies $\varphi_i((\succ_i, \succ_{-i}^*), C) \notin \widehat{S}$.

It remains only to check the last requirement in the statement of the lemma. We have just shown that $\varphi_{\widehat{S}}((\succ_i, \succ_{-i}^*), C) = U$. Thus, if $j \notin \varphi_{\widehat{S}}((\succ_i, \succ_{-i}^*), C) \cup \{i\}$, then $j \notin U \cup \{i\}$. By construction of $\succ^*$, this implies $s_0 \succ_j^* s$ for every $s \notin \widehat{S} \cup \{s_0\}$. Taking $\succ_{-i} = \succ_{-i}^*$ proves the lemma. ■

Proof of Theorem 3 Since φ is Π^C-incontestable, Corollary 2 implies that φ respects safety-sets. So, we only need to show that for any $s \in \text{Safe}^I(\succ_i, \mathcal{K}_i)$, there is some $\widehat{\succ}_{-i}$ and $\widehat{C} \in \mathcal{K}_i$ such that $\varphi_i((\succ_i, \widehat{\succ}_{-i}), \widehat{C}) = s$.

Recall that since $C_{s_0}(J) = J$ for every $J \subseteq I$, for any student j and any information set $\mathcal{K}_j$, we have $\text{Safe}^I(\succ_j, \mathcal{K}_j) \subseteq \{s \in S : \ s \succeq_j s_0\}$. Consider any student i with interim information $\mathcal{K}_i$ and let $\widehat{S} = \text{Safe}^I(\succ_i, \mathcal{K}_i)$. We first show that for any $s^* \in \widehat{S}$, there exists $\succ_{-i}$, and C compatible with $\mathcal{K}_i$ such that $\varphi_i(\succ, C) = s^*$.

Case 1: $s^* = s_0$. Let $\widetilde{S} = \{s \in S : \ s \succ_i s_0\}$. If $\widetilde{S} = \varnothing$, then, for any problem $(\succ, C)$ compatible with $\mathcal{K}_i$, respect for safety-sets and the fact that $\text{Safe}^I(\succ_i, \mathcal{K}_i) \subseteq \{s \in S : \ s \succeq_i s_0\}$, we have $\varphi_i(\succ, C) \in \text{Safe}^I(\succ_i, \{C\}) \subseteq \{s \in S : \ s \succeq_i s_0\} = \{s_0\}$, and thus $\varphi_i(\succ, C) = s_0$, as desired. Suppose now that $\widetilde{S} \neq \varnothing$. Since $s_0 \in \widehat{S}$, we then have $\widehat{S} = \{s \in S : \ s \succeq_i s_0\}$, and

$\widetilde{S}$ is not an $\mathcal{K}_i$-high-priority set. Therefore, there exist a choice profile C compatible with $\mathcal{K}_i$ and an assignment μ such that, for every $s \in \widetilde{S}$, $i \notin C_s(\mu(s) \cup \{i\})$. Since $i \in C_s(J \cup \{i\})$ whenever $|J| < q_s$, we have $|\mu(s)| = q_s$ for every $s \in \widetilde{S}$. By consistency, $\mu(s) = C_s(\mu(s) \cup \{i\})$ for every $s \in \widetilde{S}$. Thus the hypotheses of Lemma 5 are satisfied with $\widetilde{S}$. Hence there exists $\succ_{-i}$ such that $\varphi_i((\succ_i, \succ_{-i}), C) \notin \widetilde{S}$. Respect for safety-sets implies $\varphi_i((\succ_i, \succ_{-i}), C) \in \text{Safe}^I(\succ_i, \{C\}) \subseteq \{s \in S : s \succeq_i s_0\} = \widetilde{S} \cup \{s_0\}$. Together with $\varphi_i((\succ_i, \succ_{-i}), C) \notin \widetilde{S}$ we get $\varphi_i((\succ_i, \succ_{-i}), C) = s_0 = s^*$.

Case 2: $s^* \neq s_0$ and s^* is i's most preferred school. Choose any choice profile C compatible with $\mathcal{K}_i$. Let $\succ_{-i}$ be such that $s_0 \succ_j s^*$ for every $j \neq i$. For every student $j \neq i$, respect for safety-sets and $\text{Safe}^I(\succ_j, \{C\}) \subseteq \{s : s \succeq_j s_0\}$ therefore implies that $\varphi_j((\succ_i, \succ_{-i}), C) \neq s^*$. By non-wastefulness, we then have $\varphi_i((\succ_i, \succ_{-i}), C) = s^*$.

Case 3: $s^* \neq s_0$ and s^* is not i's most preferred school. Let $\widetilde{S} = \{s \in S : s \succ_i s^*\}$. Then $\widetilde{S} \neq \varnothing$. Moreover, since $s^* \in \widehat{S} \subseteq \{s \in S : s \succeq_i s_0\}$ and $s^* \neq s_0$, we have $s^* \succ_i s_0$, and therefore $s_0 \notin \widetilde{S}$. Note that $\widetilde{S}$ cannot be an $\mathcal{K}_i$-high-priority set. Therefore, there exist a choice profile C compatible with $\mathcal{K}_i$ and an assignment μ such that, for every $s \in \widetilde{S}$, $i \notin C_s(\mu(s) \cup \{i\})$. As in Case 1, we must have $|\mu(s)| = q_s$ and $\mu(s) = C_s(\mu(s) \cup \{i\})$ for every $s \in \widetilde{S}$. Thus, by Lemma 5, there exists $\succ_{-i}$ such that

$$\varphi_i((\succ_i, \succ_{-i}), C) \notin \widetilde{S}, \tag{13}$$

and such that, for every student $j \notin \varphi_{\widetilde{S}}((\succ_i, \succ_{-i}), C) \cup \{i\}$, and every school $s \notin \widetilde{S} \cup \{s_0\}$, $s_0 \succ_j s$. Since $s^* \notin \widetilde{S} \cup \{s_0\}$, this implies in particular that $s_0 \succ_j s^*$ for every $j \notin \varphi_{\widetilde{S}}((\succ_i, \succ_{-i}), C) \cup \{i\}$.

We now show that no student other than i can be assigned to s^*. Suppose, toward a contradiction, that some $j \neq i$ is assigned to s^*. Since $s^* \notin \widetilde{S}$, this student is not in $\varphi_{\widetilde{S}}((\succ_i, \succ_{-i}), C)$, and hence $s_0 \succ_j s^*$. Therefore, $s^* \notin \text{Safe}^I(\succ_j, \{C\})$, contradicting respect for safety-sets. By non-wastefulness, we then must have $\varphi_i((\succ_i, \succ_{-i}), C) = s^*$. ■

E Proof of Proposition 2

E.1 Any mechanism that Pareto-dominates a Π^C-incontestable mechanism

For that result we use the following generalization of the Rural Hospital Theorem (See Alva and Manjunath (2019) for a related result). The proof can be found in the online Appendix.

Theorem 4 (Generalized Rural Hospital Theorem) *Fix a school choice problem with a finite set of students I and schools S. Each school $s \in S$ has capacity $q_s \in \mathbb{N}$ and a choice function $C_s : 2^I \to 2^I$ satisfying the following acceptability condition: for every $J \subseteq I$ with $|J| < q_s$ and every $i \in I \setminus J$,*

$$i \in C_s(J \cup \{i\}).$$

(Equivalently, a school cannot reject a student when it has a vacant seat.) Let μ be an assignment that Pareto dominates an individually rational and non-wasteful assignment μ'. Then the following holds:

(i) for each school $s \in S$, $|\mu(s)| = |\mu'(s)|$.

(ii) If $|\mu'(s)| < q_s$ then $\mu(s) = \mu'(s)$.

(iii) for each student $i \in I$, $\mu(i) \in S \setminus \{s_0\}$ if, and only if $\mu'(i) \in S \setminus \{s_0\}$.

Lemma 6 *Fix a problem $(\succ, C)$ and let μ and μ' be assignments. If μ weakly Pareto dominates μ' and μ' respects safety-sets at $(\succ, C)$, then μ respects safety-sets at $(\succ, C)$. Moreover, if μ' is non-wasteful at $(\succ, C)$, then μ is non-wasteful at $(\succ, C)$.*

Proof Fix a problem $(\succ, C)$. Let $i \in I$ and let $\widehat{S}_i = \text{Safe}^I(\succ_i, \{C\})$. Since μ' respects safety-sets, $\mu'(i) \in \widehat{S}_i$. By definition, $\widehat{S}_i$ is an upper contour set: if $s \in \widehat{S}_i$ and $t \succeq_i s$, then $t \in \widehat{S}_i$. Since μ weakly Pareto dominates μ', we have $\mu(i) \succeq_i \mu'(i)$, and hence $\mu(i) \in \widehat{S}_i$. Since i was arbitrary, μ respects safety-sets at $(\succ, C)$.

Assume now that μ' is non-wasteful, and suppose, toward a contradiction, that μ is wasteful. Then there exist a student i and a school s such that $s \succ_i \mu(i)$ and $|\mu(s)| < q_s$. Since μ weakly Pareto dominates μ', we have $\mu(i) \succeq_i \mu'(i)$, and therefore $s \succ_i \mu'(i)$. Moreover, since μ weakly Pareto dominates μ' and μ' is non-wasteful, Theorem 4 implies $|\mu(s)| = |\mu'(s)|$. Hence $|\mu'(s)| < q_s$, contradicting the non-wastefulness of μ'. ■

Fix a problem $(\succ, C)$ and let $\mu = \varphi(\succ, C)$ and $\mu' = \psi(\succ, C)$. Since ψ is Π^C-incontestable, ψ respects safety-sets, and thus $\mu'(i) \in \text{Safe}^I(\succ_i, \{C\})$ for each $i \in I$. By assumption, μ weakly Pareto dominates μ'. Lemma 6 therefore implies that μ respects safety-sets at $(\succ, C)$. Since $(\succ, C)$ was arbitrary, Corollary 2 implies that φ is Π^C-incontestable. If ψ is non-wasteful, then μ' is non-wasteful at every problem. The non-wastefulness of φ follows from the second part of Lemma 6.

E.2 TTC is Π^C-incontestable

For the Top Trading Cycle mechanism we assume that schools have responsive priorities. So, throughout this subsection, a problem is denoted $(\succ, r)$, where $\succ$ is the profile of students' preferences and $r = (r_s)_{s \in S}$ is the profile of schools' strict priority rankings (with the convention that $r_s(i) < r_s(j)$ means that i has higher priority than j at s). We keep capacities $(q_s)_{s \in S}$ fixed. Hence, under responsive priorities and the assumption that for each school all students are acceptable, $C_s(J)$ is the set of q_s students with the highest priority at s if $|J| > q_s$, or J itself if $|J| \leq q_s$.

Lemma 7 *The Top-Trading Cycle (TTC) mechanism is Π^C-incontestable.*

Proof It suffices to show that TTC respects safety-sets. Fix a student i and let $\widehat{S} = \text{Safe}^I(\succ_i, r)$. Suppose, toward a contradiction, that $\mu(i) \notin \widehat{S}$. By definition of the safety-set, every school in $\widehat{S}$ is strictly preferred by i to every school in $S \setminus \widehat{S}$; hence $s \succ_i \mu(i)$ for all $s \in \widehat{S}$.

Let k_i be the step of the TTC algorithm at which student i is removed (and thus assigned). Let $k_{\widehat{S}}$ be the step at which the *last remaining seat* among schools in $\widehat{S}$ is allocated.[28] Since student i points at each step to her most preferred school among those that still have an available seat, and since every school in $\widehat{S}$ is preferred to every school in $S \setminus \widehat{S}$, student i must point to a school in $\widehat{S}$ at every step prior to $k_{\widehat{S}}$. Therefore, if $\mu(i) \notin \widehat{S}$, student i can only be assigned after all schools in $\widehat{S}$ have filled their capacities, that is,

$$k_i > k_{\widehat{S}}. \tag{14}$$

Next, let $\widetilde{S}$ be a high-priority set for i at $(\succ_i, r)$ whose least-preferred element is maximal (so that $\widehat{S} = \{s \in S : s \succeq_i \underline{s}(\succ_i, \widetilde{S})\}$). In particular, $\widetilde{S} \subseteq \widehat{S}$. Since $\widetilde{S}$ is a high-priority set,

[28] Equivalently: at the beginning of step $k_{\widehat{S}}$, at least one school in $\widehat{S}$ still has an available seat, and at the beginning of step $k_{\widehat{S}} + 1$ all schools in $\widehat{S}$ are full.

there exists a school $s \in \widetilde{S}$ such that $i \in C_s(\mu(s) \cup \{i\})$. Since $s \in \widehat{S}$ and $\mu(i) \notin \widehat{S}$, we have $s \succ_i \mu(i)$; and since μ is non-wasteful, it follows that $|\mu(s)| = q_s$. Under responsive priorities, $i \in C_s(\mu(s) \cup \{i\})$ and $|\mu(s)| = q_s$ imply that there exists a student $j \in \mu(s)$ such that $r_s(i) < r_s(j)$.

Consider the step of the TTC algorithm at which student j is assigned to school s (i.e., the step at which one of s's remaining seats is allocated to j). In TTC, at every step each school points to the highest-priority student among those who are still present. Since $r_s(i) < r_s(j)$, school s cannot point to j while student i is still present. Hence, i must have been removed strictly before j is assigned to s. Since $s \in \widehat{S}$, the assignment of j to s occurs no later than step $k_{\widehat{S}}$. Therefore, $k_i < k_{\widehat{S}}$, contradicting (14). This contradiction shows that $\mu(i) \in \widehat{S}$. Since i was arbitrary, μ respects safety-sets at $(\succ, r)$, and thus TTC is Π^C-incontestable. ■

F Proof of Proposition 3

Fix i, $\succ_i$, $\mathcal{K}_i$, φ, and a preference list $\succ_i' \in \mathcal{P}_i$. Since φ is Π^C-incontestable, φ respects safety-sets. Let $\widehat{S} = \text{Safe}^I(\succ_i, \mathcal{K}_i)$ and $\widetilde{S} = \text{Safe}^I(\succ_i', \mathcal{K}_i)$. By Theorem 3, when i reports $\succ_i$ truthfully, the set of schools that i can obtain across problems compatible with $\mathcal{K}_i$ is exactly $\widehat{S}$. Hence,

$$w_i(\succ_i \mid \succ_i, \varphi, \mathcal{H}_i) = \underline{s}_i(\widehat{S}), \tag{15}$$

where $\underline{s}_i(\cdot)$ denotes the least-preferred school according to $\succ_i$.

When i reports $\succ_i'$, the mechanism is evaluated at preference profiles of the form $(\succ_i', \succ_{-i})$. Thus the set of schools that i can obtain across problems compatible with $\mathcal{K}_i$ is the same as the set she could obtain by being truthful in the counterfactual environment in which her preference ordering is $\succ_i'$, while her information about C is $\mathcal{K}_i$. Applying Theorem 3 to that counterfactual environment yields that this set is exactly $\widetilde{S}$. Therefore,

$$w_i(\succ_i' \mid \succ_i, \varphi, \mathcal{K}_i) = \underline{s}_i(\widetilde{S}), \tag{16}$$

where the minimum is still evaluated according to the true preference $\succ_i$.

We claim that $\widetilde{S}$ is a $\mathcal{K}_i$-high-priority set for i. Indeed, the threshold defining $\widetilde{S}$ is obtained from some $\mathcal{K}_i$-high-priority set T' that maximizes the least-preferred element according to $\succ_i'$. Since $T' \subseteq \widetilde{S}$, and since the high-priority set property is monotone in set inclusion, $\widetilde{S}$ is itself an $\mathcal{K}_i$-high-priority set.

Now $\underline{s}(\succ_i, \mathcal{K}_i)$ is the most preferred worst school (according to $\succ_i$) across all $\mathcal{H}_i$-high-priority sets, and $\widehat{S} = \{s \in S : s \succeq_i \underline{s}(\succ_i, \mathcal{K}_i)\}$. Hence $\underline{s}(\succ_i, \widehat{S}) = \underline{s}(\succ_i, \mathcal{K}_i)$. Since $\widetilde{S}$ is a $\mathcal{K}_i$-high-priority set, $\underline{s}(\succ_i, \widetilde{S})$ is one of the feasible candidates in the maximization that defines $\underline{s}(\succ_i, \mathcal{K}_i)$. Therefore, $\underline{s}(\succ_i, \widehat{S}) = \underline{s}(\succ_i, \mathcal{K}_i) \succeq_i \underline{s}(\succ_i, \widetilde{S})$. Combining this with Eq. (15) and Eq. (16) gives $w_i(\succ_i \mid \succ_i, \varphi, \mathcal{H}_i) \succeq_i w_i(\succ_i' \mid \succ_i, \varphi, \mathcal{H}_i)$, the desired result.