# The Variance of Revenue in Uniform and Discriminatory Auctions

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We study the stochastic properties of revenue in uniform and discriminatory multi-unit auctions with unit-demand bidders. The revenue equivalence theorem implies that the two formats yield the same expected revenue for risk-neutral bidders with independent private values. We show that there is no equivalence at the second moment: revenue variance is lower in the discriminatory auction than in the uniform auction. We further establish that the two formats are extremal in variance and, more generally, in any convex risk measure of revenue, within a large class of revenue-equivalent auctions satisfying ex-post individual rationality and competitive pricing.

*JEL codes:* D44, D81, D82.

## 1. Introduction

Uniform and discriminatory price auctions are two canonical formats for selling multiple identical units. Under independent private values with unit demand and risk-neutral bidders, the revenue equivalence theorem implies that these two formats yield the same expected revenue in their symmetric strictly increasing equilibria, which is the standard basis for comparison across formats (see, e.g., Mas-Colell et al., 1995). Since the revenue distributions differ, it is not a priori clear whether there is a ranking in revenue risk that holds across valuation distributions and across other revenue-equivalent formats. In this paper, we establish that within a broad class of such auctions, the discriminatory and uniform formats are in fact extremal in revenue risk for every valuation distribution and any risk-averse seller's preferences.

Underlying the extremality of these two formats is a basic choice for the seller: the uncertainty inherent in private valuations can be absorbed in revenue or pushed into bidders' strategic problem. In the uniform auction, bidder participation is strategically simple since truthful bidding is an

equilibrium and a winner's payment does not depend on her own bid (she pays the highest losing bid), so the seller's revenue inherits the full variability of the cutoff. In the discriminatory auction, equilibrium bidding is a nontrivial decision problem as a winner's payment depends fully on her own bid (she pays her bid in full), in turn reducing the uncertainty of the seller's revenue. Specifically, we establish that the discriminatory and uniform auctions are extremal among market-clearing auctions, formally defined in Section 2, a class of revenue-equivalent auctions characterized by ex-post individual rationality (every bidder has non-negative payoff on the equilibrium path) and competitive pricing (each winner's payment is nondecreasing in every bid and at least the highest losing bid). Given these defining properties, most auction formats used for market-clearing naturally belong to the class.

The idea that revenue-equivalent auction formats can differ in revenue risk goes back to Vickrey (1961, Appendix I) in the single-item case, sharpened by Krishna (2009, Proposition 2.4) who shows that the equilibrium price distribution in the second-price auction is a mean-preserving spread of the first-price distribution. Risk aversion in auctions has been studied from the bidder side (Maskin and Riley, 1984; Matthews, 1987) and from the seller side, with Waehrer et al. (1998) establishing that risk-averse sellers prefer first-price auctions to any standard single-item auction. McAfee et al. (2025) generalize this approach, showing that the discriminatory single-item auction minimizes revenue variance, and more generally any convex risk measure, among all interim individually rational payment rules. However, this minimization result does not extend to multi-item settings, where Bojko et al. (2026) construct a counterexample. Nor does the uniform auction maximize variance in general; this can fail even in the single-item case, as the example in Section 2.1 shows. Still, Pekeč and Smilgins (2025) and Bojko et al. (2026) recently established that, in the multi-item setting, the discriminatory auction has lower revenue variance than the uniform auction. We show that this pairwise comparison is a consequence of a broader extremality result: within the class of market-clearing auctions, the discriminatory and uniform formats sit at the two extremes of revenue risk.

Our analysis relies on a natural decomposition of the revenue variance in multi-unit market-clearing auctions into terms that are in one-to-one correspondence across formats: each winner's payment variance and each covariance between a pair of winner payments. Our proofs introduce two technical tools of independent interest. The first is a *lottery auction* construction, an auxiliary single-winner mechanism with the same equilibrium bidding function as the original multi-unit auction; it enables term-by-term comparison of payment variances and pairwise covariances across formats, yielding the argument that the discriminatory auction minimizes variance among all market-clearing auctions (Section 3). The second is a local, bidder-level differential inequality that controls how a pricing rule amplifies cutoff variability; this is the engine of the argument that

the uniform auction maximizes variance (Section 4). The convex-order strengthening (Section 5) builds on both tools and applied-probability machinery linking variance and convex order under equal means (Shaked and Shanthikumar, 2007).

## 2. Model

We consider auctions in which a seller offers $k$ identical indivisible items to $N > k \geq 1$ risk-neutral bidders, each with unit demand.[1]

Each bidder $i$ has a private valuation $v_i$, drawn independently and identically from the interval $[0, \omega]$, where $\omega \in (0, \infty]$. Let $X$ denote the common random variable, with cumulative distribution function $F$, continuous density $f$ with full support, and finite second moment $\mathbb{E}[X^2] < \infty$.

Let

$$X_1^{(N)} \geq X_2^{(N)} \geq \cdots \geq X_N^{(N)}$$

denote the order statistics of $(X_1, \ldots, X_N)$ in descending order (so, e.g., $X_1^{(N)} = \max_i X_i$).

An auction $A$ is characterized by its allocation and payment rules. We restrict our attention to auctions that allocate the $k$ units to the $k$ highest bidders. We further assume that payments are anonymous, i.e., the payment rule is invariant under permutations of bidders' identities, and that the payment of the $r$-th highest bidder, $p_r^A$, is a non-negative deterministic function of the submitted bids.

Without loss of generality, we index bidders by their bid rank, i.e.,

$$b_1^A \geq b_2^A \geq \cdots \geq b_N^A,$$

with ties broken uniformly at random. If there exists a symmetric increasing bidding function $b^A : [0, \omega] \to \mathbb{R}_+$ that transforms valuations into bids, the ranks of bids and valuations then coincide, so

$$b_r^A = b^A\left(X_r^{(N)}\right), \qquad r = 1, \ldots, N.$$

Further, with $\mathbf{b}^A := \left(b_1^A, \ldots, b_N^A\right)$, the payment of the $r$-th highest bidder is

$$p_r^A := p_r^A(\mathbf{b}^A),$$

and the auction revenue is

$$R^A := \sum_{r=1}^{N} p_r^A(\mathbf{b}^A). \tag{1}$$

[1] Hence, $N > k$ ensures an excess demand.

In this setting, the classic revenue equivalence theorem of Myerson (1981) applies: any two auction formats admitting a symmetric strictly increasing equilibrium with the same interim allocation rule $a(v)$ and zero rent at type zero, $\pi(0)=0$, generate the same interim expected payment

$$\pi(v) = v\,a(v) - \int_0^v a(u)\,du \qquad \text{for all } v\in[0,\omega], \tag{2}$$

and hence the same expected revenue. We refer to such auctions as *revenue-equivalent*.

There are two canonical auction formats among revenue equivalent auctions. In the *uniform-price auction*, each winner pays the highest losing bid, i.e. the $(k+1)$-st highest bid. Under unit demand, truthful bidding is weakly dominant (see, e.g., Krishna, 2009), so $b^U(v)=v$ and hence $b^U_r = X^{(N)}_r$ for all $r$. The payment function is

$$p^U_r(\mathbf{b}^U) = \begin{cases} b^U_{k+1} = X^{(N)}_{k+1}, & r=1,\dots,k,\\ 0, & r=k+1,\dots,N. \end{cases}$$

The seller revenue is

$$U := \sum_{r=1}^{N} p^U_r(\mathbf{b}^U) = k\,X^{(N)}_{k+1}. \tag{3}$$

In the *discriminatory auction*, each winner pays her own bid. The payment function is

$$p^D_r(\mathbf{b}^D) = \begin{cases} b^D_r = b^D\left(X^{(N)}_r\right), & r=1,\dots,k,\\ 0, & r=k+1,\dots,N. \end{cases}$$

The seller revenue is

$$D := \sum_{r=1}^{N} p^D_r(\mathbf{b}^D) = \sum_{r=1}^{k} b^D_r = \sum_{r=1}^{k} b^D\left(X^{(N)}_r\right). \tag{4}$$

Let $Y_k$ denote the $k$-th highest valuation among a bidder's $N-1$ rivals, and let $g$ denote the density of $Y_k$. Then the symmetric increasing equilibrium bidding function in the discriminatory auction satisfies

$$b^D(v) = \mathbb{E}[Y_k \mid Y_k < v] \tag{5}$$

(see, for example, Krishna, 2009).[2]

Discriminatory and uniform auctions are revenue equivalent, but their revenue distributions differ. In fact, as we establish, these two canonical formats are extremal in terms of variance (and, more generally, any convex risk measure of revenue) within a large class of auctions whose pricing rules respect basic market-clearing principles, which we now define.

**Market-clearing auctions.** Let $\mathcal{M}$ denote the class of revenue-equivalent auctions admitting a symmetric strictly increasing equilibrium, in which:

[2] We use separate notation, $U$ and $D$ (rather than $R^U$ and $R^D$), to denote revenue for these two canonical auction formats. Also, we will simplify notation by suppressing the superscript $A$, whenever auction $A$ is unambiguous. With this simplification, $p_r = p_r(\mathbf{b})$ denotes the payment of the bidder with the $r$-th highest bid, and the revenue is $R = \sum_{r=1}^{k} p_r$.

(IR) *Individual rationality.* Losers pay zero and winners pay no more than their bid: $p_r = 0$ for $r = k+1, \dots, N$, and $p_r \le b_r$ for $r = 1, \dots, k$.

(CP) *Competitive pricing.* Each winning bidder's payment is nondecreasing in every bid, and $p_r \ge b_{k+1}$ for $r = 1, \dots, k$.

Condition (IR) is ex post individual rationality on the equilibrium path: every bidder, winner or loser, receives non-negative payoff for any realization of values. On the other hand, Condition (CP) ensures that the seller's revenue from each winner tracks competitive market pressure: the payment is at the market-clearing level (at least the highest losing bid, so no rejected bidder is willing to match it), and is monotone in values (because $b$ is increasing), so revenue rises with demand. Together, (IR) and (CP) capture the basic logic of market-clearing so that participation is rational for both sides at the realized prices.

The discriminatory and uniform auctions belong to $\mathcal{M}$, as do many other auction formats used in practice. For example, a common alternative uniform-pricing rule is the *lowest-accepted-bid (LAB) auction*, in which each winner pays the $k$-th highest bid:[3]

$$p_r^{LAB}(\mathbf{b}^{LAB}) = \begin{cases} b_k^{LAB} = b^{LAB}\left(X_k^{(N)}\right), & r = 1, \dots, k, \\ 0, & r = k+1, \dots, N, \end{cases}$$

with revenue

$$R^{LAB} := \sum_{r=1}^{N} p_r^{LAB}(\mathbf{b}^{LAB}) = k\, b_k^{LAB} = k\, b^{LAB}\left(X_k^{(N)}\right).$$

While the discriminatory, uniform, and many other practical formats satisfy both (IR) and (CP), there are auction formats that fail one of these conditions. A canonical example of an (IR) failure is the *all-pay auction*, in which the $k$ highest bidders receive the items but every bidder, winner or loser, pays her own bid:[4]

$$p_r^{AP}(\mathbf{b}^{AP}) = b_r^{AP}, \quad r = 1, \dots, N, \qquad R^{AP} := \sum_{r=1}^{N} p_r^{AP}(\mathbf{b}^{AP}) = \sum_{r=1}^{N} b_r^{AP}.$$

The all-pay auction is revenue-equivalent to those in $\mathcal{M}$, but losers pay, and these loser payments contribute to revenue and can drive revenue variance above that in the uniform auction, as the illustrative examples in Section 2.1 show. Failures of (CP), on the other hand, require payment rules that respond to bids in artificial ways, e.g., by engineering negative correlation among winners' payments and thereby reducing revenue variance below that in the discriminatory auction (for a concrete multi-item construction, see Bojko et al., 2026). In this sense, $\mathcal{M}$ is a natural class within which our extremality results hold.

[3] Mezzetti and Tsetlin (2008) establish the symmetric increasing equilibrium for the LAB auction.

[4] All-pay auctions are typically used as models of R&D competition, contests with sunk costs, and related settings, rather than as market-clearing mechanisms (see, e.g., Krishna and Morgan, 1997).

### 2.1. Illustrative examples

We illustrate differences in the variance of revenues for uniform, discriminatory, all-pay and LAB auctions in the setting with i.i.d. private values $X \sim \text{Unif}[0,1]$, which in turn yields $X_{k+1}^{(N)} \sim \text{Beta}(N-k, k+1)$ (see, e.g., David and Nagaraja, 2003). So, using $\mathbb{E}[\text{Beta}(N-k,k+1)] = (N-k)/(N+1)$ to compute $\mathbb{E}[U]$ and utilizing revenue equivalence, we have

$$\mathbb{E}[U] = \mathbb{E}[D] = \mathbb{E}[R^{AP}] = \mathbb{E}[R^{LAB}] = k\,\mathbb{E}[X_{k+1}^{(N)}] = \frac{k(N-k)}{N+1}.$$

We relegate to Appendix S1 the derivation of the equilibrium bidding functions and revenues for auctions discussed in this section.

We first consider the case $N=3$ and $k=1$, where

$$U = X_2^{(3)}, \qquad D = R^{LAB} = \frac{2}{3}X_1^{(3)}, \qquad R^{AP} = \frac{2}{3}\sum_{i=1}^{3} X_i^3.$$

Figure 1 shows the revenue distributions for these auction formats. The expected revenue $\mathbb{E}[R] = 1/2$ is identical and we have

$$\text{Var}(D) = \text{Var}(R^{LAB}) = \frac{1}{60}, \qquad \text{Var}(U) = \frac{1}{20}, \qquad \text{Var}(R^{AP}) = \frac{3}{28}.$$

Thus,

$$\text{Var}(D) = \text{Var}(R^{LAB}) < \text{Var}(U) < \text{Var}(R^{AP}).$$

In this single-item case, the lower variance of the discriminatory auction follows directly from the fact that the revenue is equal to the highest bid, and that the equilibrium bidding function $b^D$ is a contraction of $X_1^{(3)}$. So, using the fact that in this setting we have $\text{Var}(X_1^{(3)}) < \text{Var}(X_2^{(3)})$, it follows immediately that $\text{Var}(D) = \text{Var}(b^D(X_1^{(3)}) < \text{Var}(X_1^{(3)}) < \text{Var}(X_2^{(3)}) = \text{Var}(U)$. In contrast, each of the three bids, as well as covariances among each pair of bids, contributes to $\text{Var}(R^{AP})$.

We next consider the case $N=3$ and $k=2$. Here, we have

$$U = 2X_3^{(3)}, \quad D = b^D\left(X_1^{(3)}\right) + b^D\left(X_2^{(3)}\right), \quad R^{LAB} = 2b^{LAB}(X_2^{(3)}), \quad R^{AP} = \sum_{i=1}^{3}\left(X_i^2 - \frac{2}{3}X_i^3\right).$$

Revenue distributions are depicted in Figure 1 and the common expected revenue is again $\mathbb{E}[R] = 1/2$, while

$$\text{Var}(D) \approx 0.0143, \qquad \text{Var}(R^{AP}) = \frac{17}{420} \approx 0.0405, \qquad \text{Var}(R^{LAB}) = \frac{1}{20}, \qquad \text{Var}(U) = \frac{3}{20}.$$

Thus,

$$\text{Var}(D) < \text{Var}(R^{AP}) < \text{Var}(R^{LAB}) < \text{Var}(U).$$

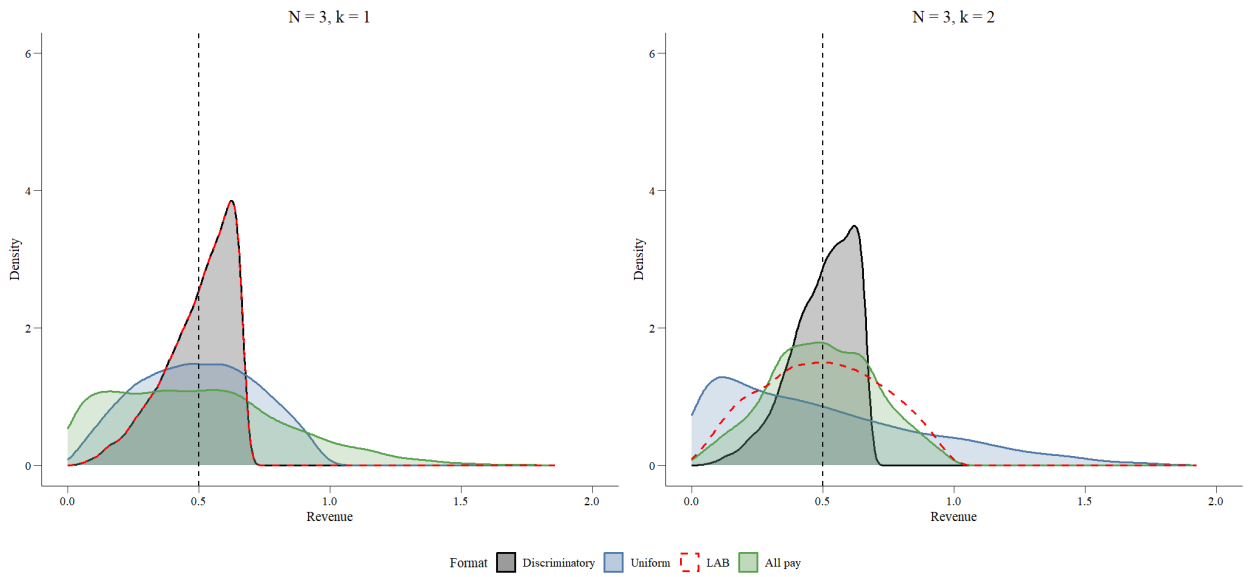



**Figure 1** **Revenue densities for $(N,k)=(3,1)$ and $(3,2)$ under i.i.d. $\mathrm{Unif}[0,1]$ values. The dashed vertical line marks the common expected revenue $1/2$. (Depicted using kernel estimates from 100000 simulations.)**

Unlike the single-item case, establishing a variance ranking when $k=2$ is more nuanced. Revenue is no longer driven by a single order statistic, but by the sum of at least two transformed order statistics. As a result, the comparison requires accounting not only for covariances, but also for counteracting effects. On the one hand, the symmetric equilibrium bidding function is a contraction, which tends to reduce variance. On the other hand, the third order statistic (winners' payments in the uniform auction) has the lowest variance among the order statistics (with transformations of the top two order statistics corresponding to winners' payments in the discriminatory auction). Hence, the variance ranking is determined by the interaction between bid compression, the variance ordering of the relevant order statistics, and dependence across multiple revenue components.

In both instances, we have $\mathrm{Var}(D)<\mathrm{Var}(U)$ and this variance ranking persists across a range of values of $N$ and $k$, as reported in Appendix S1.

## 3. Discriminatory Auctions and Variance Minimization

We show that the discriminatory auction minimizes revenue variance among market-clearing auctions: $\mathrm{Var}(D)\le\mathrm{Var}(R)$ for any $A\in\mathcal{M}$. Since market-clearing auctions are revenue equivalent, $\mathbb{E}[R]=\mathbb{E}[D]$, minimizing variance is equivalent to minimizing the second moment of the revenue, $\mathbb{E}[D^2]\le\mathbb{E}[R^2]$. By (IR), losers pay zero, so revenue comes from the $k$ winners:

$$R^2=\left(\sum_{r=1}^{k}p_r\right)^2=\sum_{r=1}^{k}p_r^2+\sum_{\substack{r,s=1\\ r\neq s}}^{k}p_rp_s. \tag{6}$$

Since in a strictly increasing symmetric equilibrium, bid ranking corresponds to valuation ranking, there is a natural matching of the terms of (6) across market-clearing auction formats.

The proof argument separately considers diagonal and off-diagonal terms of (6), where comparison is made via randomization. The key construction to facilitate this argument involves a single-item representation of a $k$-item auction. More precisely, for any $A \in \mathcal{M}$, let $A_L$ be the single-item *winner-lottery auction*. In $A_L$, the allocation and payment rule of $A$ are used to identify the $k$ eligible winners and the payments they would make in $A$. Then one of these $k$ eligible winning ranks is selected uniformly at random, i.e., $J \sim \mathrm{Unif}\{1,\dots,k\}$, independently of the valuation profile. Only the selected eligible winner receives an item and pays the payment she would have made in $A$. All other bidders receive no item and pay zero. The revenue of the winner-lottery auction is therefore $p_J$; in particular, for the discriminatory winner-lottery auction, we have $b^D(X_J^{(N)})$.

By construction, in a winner-lottery auction, each bidder expects to earn exactly $1/k$ of her payoff in $A$, hence $A_L$ has the same equilibrium as $A$, and $p_J$ is the payment of a random winner at $A$'s equilibrium bids. Consequently, we have $\mathbb{E}[p_J] = \mathbb{E}[R]/k = \mathbb{E}[D]/k = \mathbb{E}[b^D(X_J^{(N)})]$ (where revenue equivalence gives middle equality), and in fact the equality holds conditionally,

$$\mathbb{E}\Big[p_J \mid X_J^{(N)}\Big] = b^D\Big(X_J^{(N)}\Big).$$

So, the selected discriminatory payment removes the residual payment risk in the $A$-lottery payment once the selected winner's value is fixed. Thus the selected payment $p_J$ is a mean-preserving spread of the corresponding discriminatory payment $b^D(X_J^{(N)})$. We use this and Jensen's inequality to establish the following lemma.[5]

**Lemma 3.1.** *For every $A \in \mathcal{M}$,*

$$\sum_{r=1}^{k} \mathbb{E}[p_r^2] \;\geq\; \sum_{r=1}^{k} \mathbb{E}\Big[\big(b^D(X_r^{(N)})\big)^2\Big].$$

Lemma 3.1 compares the expected values of diagonal terms in the second moment of total revenue, i.e., terms under the first summation of the righthand side of (6).

To compare the off-diagonal terms, i.e., terms under the second summation of the righthand side of (6), we use a similar randomization construction.

**Lemma 3.2.** *If $k \geq 2$, then for every $A \in \mathcal{M}$,*

$$\sum_{\substack{r,s=1 \\ r\neq s}}^{k} \mathbb{E}[p_r p_s] \geq \sum_{\substack{r,s=1 \\ r\neq s}}^{k} \mathbb{E}\big[b^D\big(X_r^{(N)}\big)\, b^D\big(X_s^{(N)}\big)\big].$$

[5] Discussion of the properties of the winner-lottery auction and proofs of the lemmas in this section are in Appendix A.

Note that the expected value of off-diagonal terms in (6) represents the average product of two distinct winning payments. Thus, (CP) is the key condition for establishing the lemma, as it rules out payment rules that reduce revenue variance by engineering negative dependence among winners' payments.

Finally, we use Lemma 3.1 and Lemma 3.2 to establish the lower-extremality result.

**Theorem 3.3.** *Let $A \in \mathcal{M}$. Then*

$$\mathrm{Var}(D) \le \mathrm{Var}(R).$$

*Proof.* Taking expectations in (6) gives

$$\mathbb{E}[R^2] = \sum_{r=1}^{k} \mathbb{E}[p_r^2] + \sum_{\substack{r,s=1 \\ r \neq s}}^{k} \mathbb{E}[p_r p_s].$$

Likewise, for the discriminatory auction,

$$\mathbb{E}[D^2] = \sum_{r=1}^{k} \mathbb{E}\left[\left(b^D\left(X_r^{(N)}\right)\right)^2\right] + \sum_{\substack{r,s=1 \\ r \neq s}}^{k} \mathbb{E}\left[b^D\left(X_r^{(N)}\right) b^D\left(X_s^{(N)}\right)\right].$$

Since $J$ is uniform on $\{1,\dots,k\}$, the diagonal sums equal $k\,\mathbb{E}[p_J^2]$ and $k\,\mathbb{E}[(b^D(X_J^{(N)}))^2]$, respectively, and Lemma 3.1 yields

$$\sum_{r=1}^{k} \mathbb{E}[p_r^2] \;\ge\; \sum_{r=1}^{k} \mathbb{E}\left[\left(b^D(X_r^{(N)})\right)^2\right].$$

The off-diagonal sums (empty when $k=1$) are compared by Lemma 3.2. Combining the two yields $\mathbb{E}[R^2] \ge \mathbb{E}[D^2]$. Finally, since revenue equivalence gives $\mathbb{E}[R] = \mathbb{E}[D]$, we conclude that $\mathrm{Var}(R) \ge \mathrm{Var}(D)$.

□

## 4. Uniform Auctions and Variance Maximization

We now turn to the uniform auction, where truthful bidding is an equilibrium, so revenue inherits the variability of the valuations through the $(k+1)$-st order statistic. In contrast, a strictly increasing symmetric equilibrium function in any other market-clearing auction involves bid shading, that is, a contraction of valuations. Since variability of bids is then lower than that of valuations (which are bids in the uniform auction), revenue equivalence suggests the uniform auction is the most variable market-clearing format.

We again focus on second moments due to revenue equivalence, and the diagonal/off-diagonal decomposition of $R^2$ in (6). Applying the Cauchy–Schwarz inequality gives

$$R^2 \le k \sum_{r=1}^{k} p_r^2,$$

with equality only when the payments coincide, which is the case for the uniform auction. Thus, to show that revenue variance is maximized by the uniform auction, it suffices to show that the average second moment of the winners' payments is not larger than that of the uniform payment $X_{k+1}^{(N)}$.

To establish this, we again use the winner-lottery construction of Section 3: taking expectations above and using $\sum_{r=1}^{k} \mathbb{E}[p_r^2] = k\,\mathbb{E}[p_J^2]$ gives $\mathbb{E}[R^2] \le k^2\,\mathbb{E}[p_J^2]$. Consequently, with $\mathbb{E}[R] = k\,\mathbb{E}[p_J]$, we have $\mathrm{Var}(R) \le k^2 \mathrm{Var}(p_J)$, so it remains to show $\mathrm{Var}(p_J) \le \mathrm{Var}(X_{k+1}^{(N)})$.

This cannot be argued pointwise, since a winner's payment in $A$ need not lie below the cutoff in any given realization. We instead compare the two in expectation, type by type. Fix a bidder $i$ and recall that $Y_k$ denotes the $k$-th highest value among her $N-1$ opponents. Thus, bidding the equilibrium bid for a type $x$, she wins exactly when $x > Y_k$, in which case she faces cutoff $Y_k$. Writing $p(x; X_{-i})$ for her payment on winning, let

$$T_v := p(v; X_{-i})\,\mathbf{1}\{Y_k < v\}$$

be the payment of a type-$v$ bidder, extended by zero when she loses. Analogously, in the uniform auction she pays the cutoff, $Y_k\,\mathbf{1}\{Y_k < v\}$.

Establishing this comparison formally requires differentiating along the payment path, so we need a mild regularity condition on payments which ensures that a winner's payment path is sufficiently smooth as her own bid varies.

*Locally Lipschitz payments.* For almost every profile of opponents' values $\mathbf{v}_{-i}$, the map $x \mapsto p(x; \mathbf{v}_{-i})$ is locally Lipschitz on the winning region $(y(\mathbf{v}_{-i}), \omega]$. Moreover, for every compact interval $I \subset (0, \omega]$, the corresponding local Lipschitz constant admits an integrable envelope.

By (CP), the own-payment path is already weakly nondecreasing on the winning region: conditional on remaining a winner, a bidder cannot lower her payment by raising her own bid. Requiring that payments are locally Lipschitz bounds the rate of response, ruling out pathological pricing rules in which arbitrarily small bid changes generate unlimited payment changes. The condition is mild: a payment path that is locally 1-Lipschitz suffices, so the payments in all standard market-clearing formats are locally Lipschitz.

The next lemma quantifies the contraction intuition: the second-moment gap between a winner's payment in $A$ and in the uniform auction widens at a rate driven by her squared bid shading $(v - b(v))^2$. (The proof is in Appendix B.)

**Lemma 4.1.** *Let $A \in \mathcal{M}$ with locally Lipschitz payments. Then for almost every $v \in (0, \omega]$,*

$$\frac{d}{dv}\Big(\mathbb{E}[Y_k^2 \mathbf{1}\{Y_k < v\}] - \mathbb{E}[T_v^2]\Big) \;\ge\; (v - b(v))^2\, g(v),$$

*and hence*

$$\mathbb{E}[T_v^2] \le \mathbb{E}[Y_k^2 \mathbf{1}\{Y_k < v\}] \quad \textit{for every } v \in (0, \omega].$$

The type-by-type comparison of Lemma 4.1 combined with the lottery reduction is the key to establish the main result of this section: the extremality of uniform auction in terms of the variance of revenue.

**Theorem 4.2.** *Let $A \in \mathcal{M}$ with locally Lipschitz payments. Then* $\mathrm{Var}(R) \le \mathrm{Var}(U)$.

*Proof.* Since market-clearing auctions are revenue equivalent, by $\mathrm{Var}(R) = \mathbb{E}[R^2] - \mathbb{E}[R]^2$, it suffices to compare the second moments $\mathbb{E}[R^2]$ and $\mathbb{E}[U^2]$, which we first express as integrals over types. By symmetry, a uniformly selected winner is distributionally equivalent to a uniformly selected bidder conditional on winning, so

$$\frac{k}{N}\,\mathbb{E}[p_J^2] = \int_0^\omega \mathbb{E}[T_v^2]\, f(v)\, dv,$$

and, since a winner faces cutoff $X_{k+1}^{(N)} = Y_k$, the same identity for the uniform payment gives

$$\frac{k}{N}\,\mathbb{E}\Big[\big(X_{k+1}^{(N)}\big)^2\Big] = \int_0^\omega \mathbb{E}[Y_k^2 \mathbf{1}\{Y_k < v\}]\, f(v)\, dv.$$

Applying the pointwise bound $\mathbb{E}[T_v^2] \le \mathbb{E}[Y_k^2 \mathbf{1}\{Y_k < v\}]$ of Lemma 4.1 to compare the two integrals yields the comparison

$$\int_0^\omega \mathbb{E}[T_v^2]\, f(v)\, dv \;\le\; \int_0^\omega \mathbb{E}[Y_k^2 \mathbf{1}\{Y_k < v\}]\, f(v)\, dv,$$

so, we have

$$\mathbb{E}[p_J^2] \le \mathbb{E}\Big[\big(X_{k+1}^{(N)}\big)^2\Big]. \tag{7}$$

Finally,

$$\mathbb{E}[R^2] \;\le\; k \sum_{r=1}^{k} \mathbb{E}[p_r^2] \;=\; k^2\,\mathbb{E}[p_J^2] \;\le\; k^2\,\mathbb{E}\Big[\big(X_{k+1}^{(N)}\big)^2\Big] \;=\; \mathbb{E}[U^2],$$

where the first inequality is Cauchy–Schwarz, the equality uses $\sum_{r=1}^{k} \mathbb{E}[p_r^2] = k\,\mathbb{E}[p_J^2]$ from the lottery construction, and the last inequality is (7). Since revenue equivalence gives $\mathbb{E}[R] = \mathbb{E}[U]$, we conclude $\mathrm{Var}(R) \le \mathrm{Var}(U)$. □

## 5. Risk-Averse Sellers and Convex-Order Extremality

The preceding sections compare revenue risk using variance. We now generalize this utilizing convex order and show that, under the locally Lipschitz payment condition introduced in Section 4, the same extremal ranking holds for a much larger class of risk comparisons.

For integrable random variables $Z$ and $Z'$, we write $Z \preceq_{\mathrm{cx}} Z'$, and say $Z$ is smaller than $Z'$ in the *convex order*, if

$$\mathbb{E}[\varphi(Z)] \le \mathbb{E}[\varphi(Z')]$$

for every convex function $\varphi$ for which the corresponding expectations exist. Equivalently, $\mathbb{E}[u(Z)] \geq \mathbb{E}[u(Z')]$ for every concave $u$, so the convex order ranks $Z$ and $Z'$ as every risk-averse decision-maker would, risk aversion being captured by concavity of utility. When $\mathbb{E}[Z] = \mathbb{E}[Z']$, convex order coincides with the mean-preserving spread order of Rothschild and Stiglitz (1970); in particular, $Z \preceq_{\text{cx}} Z'$ implies that $Z$ second-order stochastically dominates $Z'$ (e.g., Hadar and Russell, 1969; Shaked and Shanthikumar, 2007).

The next theorem uses the convex order to generalize the extremality results from Sections 3 and 4. The lower comparison establishes that the discriminatory auction is the least risky market-clearing format. Its proof uses the same ideas as Section 3: revenue equivalence pins down conditional expected payments, while (CP) rules out negative dependence among winner payments. The upper comparison establishes that the uniform auction is the most risky format among auctions with locally Lipschitz payments. Its proof builds on the bidder-level comparison from Theorem 4.2. The proof is in Appendix C.

**Theorem 5.1.** *Let $A \in \mathcal{M}$ have locally Lipschitz payments, and let $R$ denote its revenue. Then*

$$D \preceq_{\text{cx}} R \preceq_{\text{cx}} U.$$

Theorem 5.1 extends the extremality of discriminatory and uniform auctions from variance to every convex risk measure of revenue.[6] Thus, within the class of market-clearing auctions with locally Lipschitz payments, the discriminatory auction is not only the low-variance benchmark: it is the least risky auction in convex order. Similarly, the uniform auction is the most risky auction in convex order. In particular, Theorem 5.1 readily implies that any risk-averse seller (i.e., with increasing concave utility) weakly prefers the discriminatory auction to any market clearing auction.

## 6. Concluding Remarks

We study the stochastic properties of revenues in uniform and discriminatory multi-unit auctions with unit-demand bidders. In the benchmark independent private values model with risk-neutral bidders, revenue equivalence implies that these canonical formats generate the same expected revenue. Our results show that this equivalence does not extend to the second moment: the discriminatory auction has weakly lower revenue variance than the uniform auction. Moreover, within a broad class of revenue-equivalent market-clearing mechanisms, the two formats serve as extremal benchmarks for revenue variance and, more generally, for any convex risk measure of revenue. [7]

[6] Using revenue equivalence and taking $\varphi(z) = z^2$ in $D \preceq_{\text{cx}} R \preceq_{\text{cx}} U$, yields $\text{Var}(D) \leq \text{Var}(R) \leq \text{Var}(U)$.

[7] In Appendix S2 we show that the revenue variance of single-item mixed-price auctions, a subclass of $\mathcal{M}$ whose payments combine discriminatory and uniform payments linearly, spans every value between those of the extremal discriminatory and uniform auctions.

The key driver behind the analysis is the different source of payment uncertainty across the two formats. In discriminatory auctions, conditional on winning, a bidder's payment is pinned down by her own bid, which reduces conditional uncertainty and dampens revenue fluctuations. In the uniform auction, by contrast, truthful bidding is the equilibrium, so there is no compression of values into bids and the full variability of the cutoff is transmitted into payments. These different intuitions suggest that the two extremal results should be proved by different techniques. The lower bound is driven by the random-winner method developed in Section 3, which allows us to compare the relevant diagonal and off-diagonal terms in the second moment of revenue. The upper bound combines the same reduction with a bidder-level differential argument that compares payment risk to cutoff risk along the bidder's own type. This distinction is conceptually useful: it isolates the point at which upper bounds on revenue variance become more delicate than lower bounds.

In closing, we briefly note that the assumptions of deterministic payments and unit demand also play an essential role. If payments were stochastic, revenue variance can be made arbitrarily large even when all conditions are satisfied in expectation. To see this, consider a single-item auction ($N=2$, $k=1$, $X\sim\mathrm{Unif}[0,1]$) in which the winner pays $Mb$ with probability $1/(M+1)$ and $b/M$ with probability $M/(M+1)$, for some $M>1$. Since the expected payment equals the bid, the equilibrium coincides with the first-price auction: $b(v)=v/2$. Yet, conditional on the winner's bid $b$, the payment variance $b^2(M-1)^2/M$ can be arbitrarily large as $M\to\infty$. Finally, when bidders may demand multiple units, equilibrium behavior in the uniform-price auction can involve demand reduction, in which case both revenue equivalence and the variance ranking can fail. For example, with $N=2$, $k=2$, and decreasing marginal values, $v_i\sim\mathrm{Unif}[0,1]$ for the first unit and $\alpha v_i$ for the second unit (where $0<\alpha<1$), the symmetric equilibrium of the uniform-price auction involves complete demand reduction on the second unit (see Ausubel et al., 2014; Krishna, 2009), yielding $\mathrm{Var}(D)>0=\mathrm{Var}(U)$. Thus, once bidders demand more than one unit, neither revenue equivalence nor the variance ranking is guaranteed.

# Appendices

## A. Proofs for Section 3

Before presenting proofs of Lemmas 3.1 and 3.2, we discuss the basic properties of the winner-lottery auction. We first note that every symmetric equilibrium bidding function $b$ of $A$ is also one of $A_L$. To see this, fix a bidder with value $v$, fix the opponents' bids, and compare the expected payoff from any bid $x$ in $A$ and in $A_L$. If the bid does not place the bidder among the $k$ eligible winners, her payoff is zero in both auctions. If the bid places her among the eligible winners, say at rank $r$, then her payoff in $A$ is $v - p_r(\mathbf{b})$. In $A_L$, she is selected with probability $1/k$; if selected, she receives the item and pays $p_r(\mathbf{b})$, and otherwise she receives no item and pays zero. Thus every bid yields exactly $1/k$ times the expected payoff it yields in auction $A$. Since multiplying payoffs by a positive constant does not change best responses, $b$ remains an equilibrium bidding function in $A_L$.

Second, since the lottery selects one of the $k$ winner payments uniformly at random,

$$\mathbb{E}[p_J] = \mathbb{E}\left[\frac{1}{k}\sum_{r=1}^{k} p_r\right] = \frac{1}{k}\mathbb{E}[R].$$

Applying the same argument to the discriminatory auction gives

$$\mathbb{E}\left[b^D(X_J^{(N)})\right] = \frac{1}{k}\mathbb{E}[D].$$

Hence, by revenue equivalence of the original auctions,

$$\mathbb{E}[p_J] = \mathbb{E}\left[b^D(X_J^{(N)})\right].$$

Lemma 3.1 provides the diagonal second-moment comparison needed for Theorem 3.3. The key observation is that, conditional on the selected winner's value, the discriminatory payment is the conditional expectation of the winner-lottery payment.

*Proof of Lemma 3.1.* We first establish that

$$\mathbb{E}[p_J^2] \geq \mathbb{E}\left[\left(b^D(X_J^{(N)})\right)^2\right],$$

for $J \sim \mathrm{Unif}\{1,\ldots,k\}$, and then we take the expectation with respect to $J$ to complete the proof.

For a bidder $i$, let $p_i$ denote her realized payment, let $W_i$ indicate that she is a winner, and let $a(v) := \Pr(W_i = 1 \mid X_i = v)$. Since all auctions in $\mathcal{M}$ allocate the $k$ units to the $k$ highest bidders and admit strictly increasing symmetric equilibria, $a(v)$ is common across auctions.

Let $\pi(v)$ denote bidder $i$'s interim expected payment in auction $A$. Because losers pay zero,

$$\pi(v) = a(v)\,\mathbb{E}[p_i \mid X_i = v, W_i = 1].$$

In the discriminatory auction, a type-$v$ bidder pays $b^D(v)$ if she wins and pays zero otherwise, so $\pi^D(v) = a(v)b^D(v)$. Revenue equivalence gives $\pi(v) = \pi^D(v)$. Thus, for all $v$ with $a(v) > 0$,

$$\mathbb{E}[p_i \mid X_i = v, W_i = 1] = b^D(v).$$

By anonymity and exchangeability, selecting one of the $k$ winning ranks uniformly at random is distributionally equivalent to selecting a labelled bidder conditional on winning. Therefore,

$$\mathbb{E}\left[p_J \mid X_J^{(N)}\right] = b^D\left(X_J^{(N)}\right).$$

Jensen's inequality then gives

$$\mathbb{E}[p_J^2] \geq \mathbb{E}\left[\left(\mathbb{E}[p_J \mid X_J^{(N)}]\right)^2\right] = \mathbb{E}\left[\left(b^D(X_J^{(N)})\right)^2\right].$$

Taking the expectation with respect to $J \sim \mathrm{Unif}\{1, \ldots, k\}$, we get

$$\frac{1}{k}\sum_{r=1}^{k}\mathbb{E}[p_r^2] \;\geq\; \frac{1}{k}\sum_{r=1}^{k}\mathbb{E}\left[\left(b^D(X_r^{(N)})\right)^2\right].$$

□

*Proof of Lemma 3.2.* Let $i$ and $j$ be bidder labels sampled uniformly without replacement from $\{1, \ldots, N\}$, independently of values. Let $p_i$ and $p_j$ denote their realized payments, and let $W_i$ and $W_j$ indicate that they are winners. Set $B := \{W_i = W_j = 1\}$.

Conditional on $B$, the ordered pair $(i, j)$ is uniformly distributed over ordered pairs of distinct winners. Hence

$$\mathbb{E}[p_i p_j \mid B] = \frac{1}{k(k-1)}\sum_{\substack{r,s=1 \\ r\neq s}}^{k}\mathbb{E}[p_r p_s],$$

and similarly,

$$\mathbb{E}[b^D(X_i)b^D(X_j) \mid B] = \frac{1}{k(k-1)}\sum_{\substack{r,s=1 \\ r\neq s}}^{k}\mathbb{E}\left[b^D\left(X_r^{(N)}\right)b^D\left(X_s^{(N)}\right)\right].$$

Thus, to prove the Lemma, it suffices

to show that

$$\mathbb{E}[p_i p_j \mid B] \geq \mathbb{E}[b^D(X_i)b^D(X_j) \mid B].$$

Define

$$\mu_i(v, w) := \mathbb{E}[p_i \mid X_i = v, X_j = w, B], \qquad \mu_j(v, w) := \mathbb{E}[p_j \mid X_i = v, X_j = w, B].$$

We use three consequences of revenue equivalence, exchangeability, and the labelled monotonicity clause of (CP). First,

$$\mathrm{Cov}(p_i, p_j \mid X_i = v, X_j = w, B) \geq 0.$$

To see this, let $t = \min\{v, w\}$, and let $S$ be the $(k-1)$-st highest value among the remaining $N-2$ bidders, so that $B = \{S \le t\}$. Conditional on $S = s$, apart from the fixed order statistic $s$, the remaining values split into independent draws above and below $s$. By (CP), both labelled payments are coordinatewise nondecreasing functions of these draws, so Harris's inequality gives nonnegative conditional covariance. The corresponding conditional means are nondecreasing in $s$, so Chebyshev's integral inequality and the law of total covariance give the displayed inequality.

Second,

$$\mathbb{E}[p_i \mid X_i = v, B] = b^D(v), \qquad \mathbb{E}[p_j \mid X_j = w, B] = b^D(w).$$

Indeed, revenue equivalence and losers-pay-zero give

$$\mathbb{E}[p_i \mid X_i = v, W_i = 1] = b^D(v).$$

Conditioning further on $W_j = 1$ does not change this conditional mean. To see this, condition on $(X_i = v, W_i = 1)$. Exactly $k-1$ of the remaining $N-1$ bidders are winners, and by exchangeability the expectation

$$\mathbb{E}[p_i \mid X_i = v, W_i = 1, W_\ell = 1]$$

is the same for every $\ell \neq i$. Averaging over the $k-1$ winning opponents therefore gives the same mean as conditioning only on $(X_i = v, W_i = 1)$. Hence

$$\mathbb{E}[p_i \mid X_i = v, W_i = 1, W_j = 1] = \mathbb{E}[p_i \mid X_i = v, W_i = 1] = b^D(v).$$

The argument for bidder $j$ is symmetric.

Third, for each fixed $w$, the maps $v \mapsto \mu_i(v, w)$ and $v \mapsto \mu_j(v, w)$ are nondecreasing. This follows by coupling the relevant order statistics as $v$ increases. The winning condition weakly relaxes, the remaining conditional distributions shift upward in first-order stochastic dominance, and labelled winner payments are coordinatewise nondecreasing by (CP).

Now use the law of total covariance:

$$\begin{aligned}\mathbb{E}[p_i p_j \mid B] = \mathbb{E}[\mathrm{Cov}(p_i, p_j \mid X_i, X_j, B) \mid B] \\ + \mathbb{E}[\mu_i(X_i, X_j)\mu_j(X_i, X_j) \mid B]\,.\end{aligned}$$

The first term is nonnegative. It remains to compare the second term with $\mathbb{E}[b^D(X_i)b^D(X_j) \mid B]$. Let

$$\Delta_i := \mu_i(X_i, X_j) - b^D(X_i), \qquad \Delta_j := \mu_j(X_i, X_j) - b^D(X_j).$$

Then

$$\mu_i\mu_j - b^D(X_i)b^D(X_j) = b^D(X_i)\Delta_j + b^D(X_j)\Delta_i + \Delta_i\Delta_j.$$

By exchangeability of $i$ and $j$ conditional on $B$,

$$\mathbb{E}[b^D(X_j)\Delta_i \mid B] = \mathbb{E}[b^D(X_i)\Delta_j \mid B],$$

and therefore

$$\mathbb{E}[\mu_i\mu_j - b^D(X_i)b^D(X_j) \mid B] = \mathbb{E}[(b^D(X_i) + \mu_i(X_i, X_j))\Delta_j \mid B].$$

Condition on $(X_j = w, B)$. The function $v \mapsto b^D(v) + \mu_i(v, w)$ is nondecreasing, and so is $v \mapsto \Delta_j(v, w) = \mu_j(v, w) - b^D(w)$. Moreover,

$$\mathbb{E}[\Delta_j \mid X_j = w, B] = 0.$$

Chebyshev's integral inequality gives

$$\mathbb{E}[(b^D(X_i) + \mu_i(X_i, w))\Delta_j(X_i, w) \mid X_j = w, B] \geq 0.$$

Averaging over $X_j$ proves

$$\mathbb{E}[\mu_i(X_i, X_j)\mu_j(X_i, X_j) \mid B] \geq \mathbb{E}[b^D(X_i)b^D(X_j) \mid B],$$

and the result follows. □

## B. Proofs for Section 4

*Proof of Lemma 4.1.* Fix bidder $i$ and recall that $Y_k$ is the $k$-th highest value among bidder $i$'s opponents and $T_v := p(v; X_{-i})\,\mathbf{1}\{Y_k < v\}$. Recall that $g$ denotes the density of $Y_k$. By revenue equivalence,

$$\mathbb{E}[T_v] = \int_0^v y\, g(y)\, dy = \mathbb{E}[Y_k \mathbf{1}\{Y_k < v\}]\,. \tag{8}$$

Fix a realization of $X_{-i}$. If $v \downarrow Y_k$ from the winning region, then bidder $i$'s own bid converges to $b(Y_k)$, and the highest losing bid also converges to $b(Y_k)$. The market-clearing bounds imply that the winning-side extension satisfies

$$p(Y_k; X_{-i}) = b(Y_k).$$

Moreover, the bid cap and the competitive lower bound imply $0 \leq T_v \leq b(v)$.

By (CP), the path $v \mapsto p(v; X_{-i})$ is weakly nondecreasing on the winning region. By locally Lipschitz payments, it is differentiable almost everywhere and $\partial_v T_v \geq 0$ for almost every $v$. In particular, for every compact interval $I \subset (0, \omega]$, there is an integrable $L_I(X_{-i})$ such that

$$|p(x; X_{-i}) - p(x'; X_{-i})| \leq L_I(X_{-i})\,|x - x'|$$

for all $x, x' \in I \cap (y(X_{-i}), \omega]$. This envelope justifies the use of Fubini's theorem below.

Using the boundary value $p(Y_k; X_{-i}) = b(Y_k)$ and absolute continuity on the winning region, we have

$$T_v = b(Y_k)\mathbf{1}\{Y_k < v\} + \int_0^v \partial_x T_x \, dx,$$

and

$$T_v^2 = b(Y_k)^2\mathbf{1}\{Y_k < v\} + 2\int_0^v T_x \, \partial_x T_x \, dx.$$

Taking expectations and applying Fubini's theorem gives

$$\mathbb{E}[T_v] = \int_0^v b(y)g(y) \, dy + \int_0^v \mathbb{E}[\partial_x T_x] \, dx.$$

Differentiating and using (8) yields

$$\mathbb{E}[\partial_v T_v] = (v - b(v))g(v). \tag{9}$$

Similarly,

$$\frac{d}{dv}\mathbb{E}[T_v^2] = b(v)^2 g(v) + 2\mathbb{E}[T_v \partial_v T_v].$$

Writing

$$\Phi(v) := \mathbb{E}[T_v^2] - \int_0^v y^2 g(y) \, dy = \mathbb{E}[T_v^2] - \mathbb{E}[Y_k^2\mathbf{1}\{Y_k < v\}]$$

for the negative of the gap in Lemma 4.1, differentiating gives

$$\Phi'(v) = (b(v)^2 - v^2)g(v) + 2\mathbb{E}[T_v \partial_v T_v]. \tag{10}$$

Finally, using $0 \le T_v \le b(v)$ and $\partial_v T_v \ge 0$,

$$\mathbb{E}[T_v \partial_v T_v] \le b(v)\mathbb{E}[\partial_v T_v].$$

Substituting (9) into (10) gives

$$\Phi'(v) \le -(v - b(v))^2 g(v) \le 0.$$

Equivalently,

$$\frac{d}{dv}\left(\mathbb{E}[Y_k^2\mathbf{1}\{Y_k < v\}] - \mathbb{E}[T_v^2]\right) \ge (v - b(v))^2 g(v).$$

Since $\Phi(0) = 0$, we obtain $\Phi(v) \le 0$ for every $v$, or

$$\mathbb{E}[T_v^2] \le \mathbb{E}\left[Y_k^2\mathbf{1}\{Y_k < v\}\right].$$

□

## C. Proofs for Section 5

We prove the lower and upper convex-order comparisons separately. It is enough to prove the convex-order inequalities for continuously differentiable convex functions with bounded derivative; the general case follows by standard approximation.

**Discriminatory revenue is minimal in convex order**

Fix $A \in \mathcal{M}$. Let $\mathcal{B}$ be the event that labelled bidders $1, \dots, k$ are the winners. In this proof, let $p_i$ denote the realized payment of labelled bidder $i$. By exchangeability, the unconditional distributions of $R$ and $D$ are the same as the conditional distributions of $\sum_{i=1}^{k} p_i$ and $\sum_{i=1}^{k} b^D(X_i)$, respectively, given $\mathcal{B}$.

For $i = 1, \dots, k$, define

$$\mu_i(x_1, \dots, x_k) := \mathbb{E}[p_i \mid X_1 = x_1, \dots, X_k = x_k, \mathcal{B}],$$

and set

$$\Delta_i := \mu_i(X_1, \dots, X_k) - b^D(X_i).$$

**Proposition C.1.** *For every $A \in \mathcal{M}$,*

$$D \preceq_{\mathrm{cx}} R.$$

*Proof.* We use three auxiliary facts. First,

$$\mathbb{E}[\Delta_i \mid X_i = x, \mathcal{B}] = 0.$$

Indeed,

$$\mathbb{E}[\mu_i(X_1, \dots, X_k) \mid X_i = x, \mathcal{B}] = \mathbb{E}[p_i \mid X_i = x, \mathcal{B}].$$

Conditional on bidder $i$ having value $x$ and being a winner, exactly $k-1$ opponent labels are also winners. By exchangeability, conditioning on the particular set of winning opponent labels does not change bidder $i$'s expected payment. Hence

$$\mathbb{E}[p_i \mid X_i = x, \mathcal{B}] = \mathbb{E}[p_i \mid X_i = x, \text{ bidder } i \text{ wins}] = b^D(x),$$

where the last equality follows from revenue equivalence and losers-pay-zero.

Second, each $\mu_i$ is coordinatewise nondecreasing. If $x \le x'$ coordinatewise, then the losing values under $x'$ can be coupled to be weakly larger than those under $x$, since the truncation point $\min_j x'_j$ is weakly higher than $\min_j x_j$. Along this coupled path, bidders $1, \dots, k$ remain winners, all relevant bids weakly increase, and the labelled monotonicity clause of (CP) implies that each winner's payment weakly increases. Thus $\mu_i(x') \ge \mu_i(x)$. Hence, for fixed $x_i$, $\Delta_i$ is coordinatewise nondecreasing in the other winner values.

Third, conditional on $(X_i = x, \mathcal{B})$, the vector $(X_j)_{j\neq i,\, j\leq k}$ is associated. Its conditional density is proportional to

$$\prod_{j\neq i,\, j\leq k} f(x_j)F(\min\{x, x_j : j \neq i,\ j \leq k\})^{N-k}.$$

This density is $\mathrm{MTP}_2$: the product term is $\mathrm{MTP}_2$, and the minimum term satisfies the lattice inequality because $F^{N-k}$ is nondecreasing. Hence association follows from the FKG/Harris inequality.

Let $\varphi$ be continuously differentiable and convex. Conditional on $(X_1, \ldots, X_k, \mathcal{B})$, Jensen's inequality gives

$$\mathbb{E}\left[\varphi\left(\sum_{i=1}^{k} p_i\right) \mid X_1, \ldots, X_k, \mathcal{B}\right] \geq \varphi\left(\sum_{i=1}^{k} \mu_i(X_1, \ldots, X_k)\right).$$

It remains to compare the right-hand side with the discriminatory sum. For $\lambda \in [0, 1]$, define

$$Z_\lambda := \sum_{i=1}^{k} b^D(X_i) + \lambda \sum_{i=1}^{k} \Delta_i, \qquad H(\lambda) := \mathbb{E}[\varphi(Z_\lambda) \mid \mathcal{B}].$$

Then

$$H'(\lambda) = \sum_{i=1}^{k} \mathbb{E}[\varphi'(Z_\lambda)\Delta_i \mid \mathcal{B}].$$

Fix $i$ and condition on $X_i = x$. By the facts above, $\mathbb{E}[\Delta_i \mid X_i = x, \mathcal{B}] = 0$, $\Delta_i$ is coordinatewise nondecreasing in the other winner values, and

$$Z_\lambda = (1-\lambda) \sum_{m=1}^{k} b^D(X_m) + \lambda \sum_{m=1}^{k} \mu_m(X_1, \ldots, X_k)$$

is coordinatewise nondecreasing in those values. Since $\varphi'$ is nondecreasing, $\varphi'(Z_\lambda)$ is also coordinatewise nondecreasing. Association therefore implies

$$\mathbb{E}[\varphi'(Z_\lambda)\Delta_i \mid X_i = x, \mathcal{B}] \geq 0.$$

Averaging over $X_i$ and summing over $i$ gives $H'(\lambda) \geq 0$. Hence $H(1) \geq H(0)$. Combining this with the conditional Jensen inequality yields

$$\mathbb{E}[\varphi(R)] \geq \mathbb{E}[\varphi(D)].$$

Thus $D \preceq_{\mathrm{cx}} R$. □

**Uniform auction revenue is maximal in convex order**

For the upper comparison, use the notation from Appendix B. Thus $Y_k$ is the $k$-th highest opponent value and

$$T_v = p(v; \mathbf{X}_{-i})\mathbf{1}\{Y_k < v\}.$$

**Proposition C.2.** *Suppose $A \in \mathcal{M}$ has locally Lipschitz payments. Then*

$$R \preceq_{\text{cx}} U.$$

*Proof.* Let $\varphi$ be continuously differentiable and convex, and subtract a constant so that $\varphi(0) = 0$. Define

$$\Psi_\varphi(v) := \mathbb{E}[\varphi(T_v)] - \mathbb{E}[\varphi(Y_k \mathbf{1}\{Y_k < v\})].$$

The same integral-representation argument as in Appendix B, with $\varphi(T_v)$ in place of $T_v^2$, gives, for almost every $v$,

$$\Psi'_\varphi(v) = \big(\varphi(b(v)) - \varphi(v)\big) g(v) + \mathbb{E}[\varphi'(T_v)\partial_v T_v].$$

Since $0 \le T_v \le b(v)$, $\partial_v T_v \ge 0$, and $\varphi'$ is nondecreasing,

$$\mathbb{E}[\varphi'(T_v)\partial_v T_v] \le \varphi'(b(v))\mathbb{E}[\partial_v T_v].$$

Using $\mathbb{E}[\partial_v T_v] = (v - b(v))g(v)$, we get

$$\Psi'_\varphi(v) \le \Big[\varphi(b(v)) + \varphi'(b(v))(v - b(v)) - \varphi(v)\Big] g(v) \le 0,$$

where the last inequality is the supporting-hyperplane inequality for convex $\varphi$. Since $\Psi_\varphi(0) = 0$,

$$\mathbb{E}[\varphi(T_v)] \le \mathbb{E}[\varphi(Y_k \mathbf{1}\{Y_k < v\})] \qquad \text{for every } v.$$

Integrating over bidder types gives the random-winner comparison. Since $\varphi(0) = 0$,

$$\mathbb{E}[\varphi(p_J)] = \frac{N}{k} \int_0^\omega \mathbb{E}[\varphi(T_v)] f(v)\, dv \le \frac{N}{k} \int_0^\omega \mathbb{E}[\varphi(Y_k \mathbf{1}\{Y_k < v\})] f(v)\, dv = \mathbb{E}[\varphi(X_{k+1}^{(N)})].$$

Thus $p_J \preceq_{\text{cx}} X_{k+1}^{(N)}$.

Finally, conditional on the realized winner payments $(p_1, \dots, p_k)$, $p_J$ is uniformly distributed over these $k$ payments, and

$$\frac{R}{k} = \mathbb{E}[p_J \mid p_1, \dots, p_k].$$

Jensen's inequality gives $R/k \preceq_{\text{cx}} p_J$. Combining this with $p_J \preceq_{\text{cx}} X_{k+1}^{(N)}$, and multiplying by $k > 0$, gives

$$R \preceq_{\text{cx}} k X_{k+1}^{(N)} = U.$$

□

*Proof of Theorem 5.1.* Proposition C.1 gives $D \preceq_{\text{cx}} R$, and Proposition C.2 gives $R \preceq_{\text{cx}} U$. Combining them gives $D \preceq_{\text{cx}} R \preceq_{\text{cx}} U$. □

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# Supplemental Appendices

## S1. Equilibrium Bidding and Revenue Moments in Discriminatory, Uniform, LAB, and All-Pay Auctions. Illustrative Examples

This appendix collects the calculations underlying the examples in Section 2.1, which assume that valuations are i.i.d. Unif$[0,1]$.

Using the order-statistics notation, $X_1^{(N)} \geq X_2^{(N)} \geq \cdots \geq X_N^{(N)}$, the $j$-th highest order statistic satisfies

$$X_j^{(N)} \sim \text{Beta}(N-j+1, j) \qquad \text{(see, e.g., David and Nagaraja (2003)).}$$

Hence

$$\mathbb{E}\Big[X_j^{(N)}\Big] = \frac{N-j+1}{N+1}, \qquad \text{Var}\Big(X_j^{(N)}\Big) = \frac{j(N-j+1)}{(N+1)^2(N+2)}.$$

In particular, by revenue equivalence and the expression for the uniform-price auction,

$$\mathbb{E}[R] = \mathbb{E}[U] = k\,\mathbb{E}\Big[X_{k+1}^{(N)}\Big] = \frac{k(N-k)}{N+1}.$$

This is the common expected revenue in all four auction formats considered in Section 2.1.

### Uniform auction

*Bidding function.* In the uniform-price auction, truthful bidding is weakly dominant, so

$$b^U(v) = v.$$

Revenue is therefore

$$U = kX_{k+1}^{(N)}.$$

*Case $k=1$.* When $k=1$, we have

$$U = X_2^{(N)}.$$

Hence

$$\mathbb{E}[U] = \frac{N-1}{N+1}, \qquad \text{Var}(U) = \frac{2(N-1)}{(N+1)^2(N+2)}.$$

*Case $k>1$.* When $k>1$, we have

$$U = kX_{k+1}^{(N)}.$$

Since $X_{k+1}^{(N)} \sim \text{Beta}(N-k, k+1)$,

$$\mathbb{E}[U] = \frac{k(N-k)}{N+1}, \qquad \text{Var}(U) = k^2\text{Var}\Big(X_{k+1}^{(N)}\Big) = \frac{k^2(N-k)(k+1)}{(N+1)^2(N+2)}.$$

*Examples from Section 2.1* . For $(N,k) = (3,1)$,

$$U = X_2^{(3)}, \qquad \mathbb{E}[U] = \frac{1}{2}, \qquad \text{Var}(U) = \frac{1}{20}.$$

For $(N,k) = (3,2)$,

$$U = 2X_3^{(3)}, \qquad \mathbb{E}[U] = \frac{1}{2}, \qquad \text{Var}(U) = \frac{3}{20}.$$

**Discriminatory auction**

*Bidding function.* Let $Y_k$ denote the $k$-th highest valuation among a bidder's $N-1$ rivals. By revenue equivalence, the interim expected payment of type $v$ is

$$\pi(v) = v\,a(v) - \int_0^v a(u)\,du.$$

Under i.i.d. Unif$[0,1]$ values, $Y_k \sim \text{Beta}(N-k,k)$, so

$$a(v) = \Pr(Y_k < v) = I_v(N-k,k),$$

where $I_v(a,b)$ denotes the regularized incomplete beta function, defined by

$$I_v(a,b) := \frac{B_v(a,b)}{B(a,b)} = \frac{\int_0^v t^{a-1}(1-t)^{b-1}\,dt}{\int_0^1 t^{a-1}(1-t)^{b-1}\,dt}, \qquad a,b>0,\ v\in[0,1],$$

where $B_v(a,b)$ is the (lower) incomplete beta function and $B(a,b)$ is the beta function. Equivalently, $I_v(a,b)$ is the cumulative distribution function of a Beta$(a,b)$ random variable evaluated at $v$.

The symmetric increasing equilibrium bid is

$$b^D(v) = \mathbb{E}[Y_k \mid Y_k < v] = \frac{\int_0^v y f_{Y_k}(y)\,dy}{\Pr(Y_k<v)} = \frac{N-k}{N}\,\frac{I_v(N-k+1,k)}{I_v(N-k,k)},$$

where $I_v(a,b)$ denotes the regularized incomplete beta function. Revenue is

$$D = \sum_{r=1}^{k} b^D\big(X_r^{(N)}\big).$$

*Case $k=1$.* When $k=1$, the bid function simplifies to

$$b^D(v) = \frac{N-1}{N}v,$$

so

$$D = \frac{N-1}{N}X_1^{(N)}.$$

Therefore

$$\mathbb{E}[D] = \frac{N-1}{N+1}, \qquad \text{Var}(D) = \frac{(N-1)^2}{N(N+1)^2(N+2)}.$$

*Case $k>1$.* When $k>1$, the equilibrium bid $b^D$ is generally nonlinear and revenue is the sum of the top $k$ transformed order statistics,

$$D = \sum_{r=1}^{k} b^D\big(X_r^{(N)}\big).$$

To compute its moments, let

$$f_r(x) = \frac{N!}{(N-r)!(r-1)!}\,x^{N-r}(1-x)^{r-1}, \qquad x\in[0,1],$$

denote the density of $X_r^{(N)}$, and for $1 \le r < s \le N$ let

$$f_{r,s}(x,y) = \frac{N!}{(N-s)!(s-r-1)!(r-1)!}\, y^{N-s}(x-y)^{s-r-1}(1-x)^{r-1}, \qquad 1 \ge x \ge y \ge 0,$$

denote the joint density of $\left(X_r^{(N)}, X_s^{(N)}\right)$. Then

$$\mathbb{E}[D] = \sum_{r=1}^{k} \int_0^1 b^D(x) f_r(x)\, dx,$$

and

$$\mathbb{E}[D^2] = \sum_{r=1}^{k} \int_0^1 \left(b^D(x)\right)^2 f_r(x)\, dx + 2 \sum_{1 \le r < s \le k} \int_0^1 \int_y^1 b^D(x) b^D(y) f_{r,s}(x,y)\, dx\, dy.$$

Hence

$$\mathrm{Var}(D) = \mathbb{E}[D^2] - \left(\mathbb{E}[D]\right)^2.$$

*Examples from Section 2.1* . For $(N,k) = (3,1)$, $Y_1$ is the highest of two rival values, so

$$\Pr(Y_1 < v) = v^2, \qquad \int_0^v y f_{Y_1}(y)\, dy = \int_0^v 2y^2\, dy = \frac{2}{3}v^3.$$

Thus

$$b^D(v) = \frac{(2/3)v^3}{v^2} = \frac{2}{3}v, \qquad D = \frac{2}{3}X_1^{(3)}.$$

Therefore

$$\mathbb{E}[D] = \frac{2}{3}\,\mathbb{E}\left[X_1^{(3)}\right] = \frac{1}{2}, \qquad \mathrm{Var}(D) = \frac{4}{9}\,\mathrm{Var}\left(X_1^{(3)}\right) = \frac{1}{60}.$$

For $(N,k) = (3,2)$, $Y_2$ is the lower of two rival values, so

$$\Pr(Y_2 < v) = 1 - (1-v)^2 = 2v - v^2,$$

and

$$\int_0^v y f_{Y_2}(y)\, dy = \int_0^v 2y(1-y)\, dy = v^2 - \frac{2}{3}v^3.$$

Therefore

$$b^D(v) = \frac{v^2 - \frac{2}{3}v^3}{2v - v^2} = \frac{v(3-2v)}{3(2-v)},$$

so that

$$D = b^D\left(X_1^{(3)}\right) + b^D\left(X_2^{(3)}\right).$$

Since the joint density of $\left(X_1^{(3)}, X_2^{(3)}\right)$ is

$$f_{1,2}(x_1,x_2) = 6x_2, \qquad 1 \ge x_1 \ge x_2 \ge 0,$$

we have

$$\mathbb{E}[D] = 6\int_0^1 \int_{x_2}^1 \left(\frac{x_1(3-2x_1)}{3(2-x_1)} + \frac{x_2(3-2x_2)}{3(2-x_2)}\right) x_2\, dx_1\, dx_2 = \frac{1}{2},$$

and

$$\mathbb{E}[D^2] = 6\int_0^1 \int_{x_2}^1 \left(\frac{x_1(3-2x_1)}{3(2-x_1)} + \frac{x_2(3-2x_2)}{3(2-x_2)}\right)^2 x_2\, dx_1\, dx_2 = \frac{272}{135} - \frac{56}{9}\log 2 + \frac{16}{3}(\log 2)^2.$$

Therefore

$$\mathrm{Var}(D) = \frac{953}{540} - \frac{56}{9}\log 2 + \frac{16}{3}(\log 2)^2 \approx 0.0143.$$

**All-pay auction**

*Bidding function.* In the all-pay auction, every bidder pays her bid. Revenue equivalence therefore implies

$$b^{AP}(v) = v\,\Pr(Y_k < v) - \int_0^v \Pr(Y_k < u)\,du = \frac{N-k}{N} I_v(N-k+1,k).$$

Revenue is

$$R^{AP} = \sum_{i=1}^{N} b^{AP}(X_i).$$

*Case $k=1$.* When $k=1$, the bid function simplifies to

$$b^{AP}(v) = \frac{N-1}{N} v^N,$$

so

$$R^{AP} = \frac{N-1}{N} \sum_{i=1}^{N} X_i^N.$$

Because $X_1, \ldots, X_N$ are i.i.d.,

$$\mathbb{E}[R^{AP}] = \frac{N-1}{N+1}, \qquad \mathrm{Var}(R^{AP}) = \frac{(N-1)^2}{N}\left(\frac{1}{2N+1} - \frac{1}{(N+1)^2}\right).$$

*Case $k>1$.* When $k>1$, the same bid formula applies,

$$b^{AP}(v) = \frac{N-k}{N} I_v(N-k+1,k), \qquad R^{AP} = \sum_{i=1}^{N} b^{AP}(X_i).$$

Because $X_1, \ldots, X_N$ are i.i.d. and $b^{AP}$ is deterministic,

$$\mathbb{E}[R^{AP}] = N \int_0^1 b^{AP}(v)\,dv,$$

and

$$\mathrm{Var}(R^{AP}) = N\left(\int_0^1 \left(b^{AP}(v)\right)^2 dv - \left(\int_0^1 b^{AP}(v)\,dv\right)^2\right).$$

Thus, even when $k>1$, the exact all-pay moments reduce to one-dimensional integrals.

*Examples from Section 2.1* . For $(N,k)=(3,1)$, we have

$$b^{AP}(v) = \frac{2}{3} v^3, \qquad R^{AP} = \frac{2}{3} \sum_{i=1}^{3} X_i^3.$$

Therefore

$$\mathbb{E}[R^{AP}] = 3 \cdot \frac{2}{3}\,\mathbb{E}[X^3] = \frac{1}{2},$$

and

$$\mathrm{Var}(R^{AP}) = 3\,\mathrm{Var}\left(\frac{2}{3} X^3\right) = \frac{3}{28}.$$

For $(N,k)=(3,2)$, we have

$$b^{AP}(v)=v^2-\frac{2}{3}v^3, \qquad R^{AP}=\sum_{i=1}^{3}\left(X_i^2-\frac{2}{3}X_i^3\right).$$

Therefore

$$\mathbb{E}[R^{AP}]=3\left(\mathbb{E}[X^2]-\frac{2}{3}\mathbb{E}[X^3]\right)=\frac{1}{2},$$

while

$$\mathbb{E}\left[\left(X^2-\frac{2}{3}X^3\right)^2\right]=\mathbb{E}[X^4]-\frac{4}{3}\mathbb{E}[X^5]+\frac{4}{9}\mathbb{E}[X^6]=\frac{13}{315}.$$

Hence

$$\mathrm{Var}(R^{AP})=3\left(\frac{13}{315}-\frac{1}{36}\right)=\frac{17}{420}\approx 0.0405.$$

**Lowest-accepted-bid (LAB) auction**

*Bidding function.* When $k=1$, the LAB auction coincides with the discriminatory auction, so

$$b^{LAB}(v)=\frac{N-1}{N}v.$$

For $k\geq 2$, the LAB auction with unit-demand bidders has been analyzed by Mezzetti and Tsetlin (2008). The next lemma records the specialization of their equilibrium characterization to the i.i.d. Unif$[0,1]$ case.

**Lemma S1.1.** *For $k\geq 2$, consider the auction in which the $k$ highest bids win one unit and each winner pays the lowest accepted bid, i.e. the $k$-th highest bid. Under i.i.d.* Unif$[0,1]$ *values, the unique increasing symmetric equilibrium bidding function is*

$$b^{LAB}(v)=\frac{N-k}{N-k+1}v.$$

*Proof.* Let $b^{LAB}$ be an increasing symmetric equilibrium and interpret a deviation to $b^{LAB}(x)$ as bidding as if the bidder's value were $x$. Let $Y_{k-1}\geq Y_k$ denote the $(k-1)$-st and $k$-th highest values among the $N-1$ rivals. If $x\leq Y_k$, the bidder loses and pays 0. If $Y_k<x<Y_{k-1}$, the bidder is the marginal winner and pays $b^{LAB}(x)$. If $x>Y_{k-1}$, the bidder wins but is not marginal and pays $b^{LAB}(Y_{k-1})$.

The expected utility of a type-$v$ bidder who chooses $x$ is therefore

$$u(v,x)=\int_0^x (v-b^{LAB}(y))f_{Y_{k-1}}(y)\,dy\;+(v-b^{LAB}(x))\,\Pr(Y_k<x<Y_{k-1}).$$

Write

$$q(x):=\Pr(Y_k<x<Y_{k-1})=F_{Y_k}(x)-F_{Y_{k-1}}(x),$$

so that

$$q'(x)=f_{Y_k}(x)-f_{Y_{k-1}}(x).$$

Differentiating $u(v,x)$ with respect to $x$ gives

$$\frac{\partial u(v,x)}{\partial x} = (v - b^{LAB}(x)) f_{Y_k}(x) - \big(b^{LAB}\big)'(x)\, q(x).$$

Imposing the first-order condition at $x = v$ yields

$$\big(b^{LAB}\big)'(v) = \frac{f_{Y_k}(v)}{\Pr(Y_k < v < Y_{k-1})}\big(v - b^{LAB}(v)\big).$$

Under Unif$[0,1]$ values,

$$f_{Y_k}(v) = \frac{(N-1)!}{(N-k-1)!\,(k-1)!}\, v^{N-k-1}(1-v)^{k-1},$$

and

$$\Pr(Y_k < v < Y_{k-1}) = \Pr(\text{exactly } k-1 \text{ rivals exceed } v) = \binom{N-1}{k-1}(1-v)^{k-1}v^{N-k}.$$

Hence

$$\big(b^{LAB}\big)'(v) = \frac{N-k}{v}\big(v - b^{LAB}(v)\big), \qquad b^{LAB}(0) = 0.$$

The unique solution is

$$b^{LAB}(v) = \frac{N-k}{N-k+1}v.$$

□

*Case* $k = 1$. When $k = 1$, LAB and discriminatory auctions coincide, so

$$R^{LAB} = \frac{N-1}{N} X_1^{(N)}.$$

Therefore

$$\mathbb{E}[R^{LAB}] = \frac{N-1}{N+1}, \qquad \mathrm{Var}(R^{LAB}) = \frac{(N-1)^2}{N(N+1)^2(N+2)}.$$

*Case* $k > 1$. By Lemma S1.1,

$$b^{LAB}(v) = \frac{N-k}{N-k+1}v.$$

Revenue is therefore

$$R^{LAB} = k\, b^{LAB}\Big(X_k^{(N)}\Big) = k\,\frac{N-k}{N-k+1} X_k^{(N)}.$$

Hence

$$\mathbb{E}[R^{LAB}] = \frac{k(N-k)}{N+1},$$

and

$$\mathrm{Var}(R^{LAB}) = k^2\left(\frac{N-k}{N-k+1}\right)^2 \mathrm{Var}\Big(X_k^{(N)}\Big) = \frac{k^3(N-k)^2}{(N-k+1)(N+1)^2(N+2)}.$$

*Examples from Section 2.1* . For $(N,k)=(3,1)$, LAB coincides with the discriminatory auction, so

$$R^{LAB}=\frac{2}{3}X_1^{(3)},\qquad \mathbb{E}[R^{LAB}]=\frac{1}{2},\qquad \mathrm{Var}(R^{LAB})=\frac{1}{60}.$$

For $(N,k)=(3,2)$, Lemma S1.1 gives

$$b^{LAB}(v)=\frac{1}{2}v,\qquad R^{LAB}=2b^{LAB}\left(X_2^{(3)}\right)=X_2^{(3)}.$$

Therefore

$$\mathbb{E}[R^{LAB}]=\frac{1}{2},\qquad \mathrm{Var}(R^{LAB})=\frac{1}{20}.$$

**Additional numerical examples**

Tables 1 and 2 report additional numerical illustrations under i.i.d. Unif$[0,1]$ values. All entries are rounded to four decimals.

We begin with the single-unit case, $k=1$, and vary the number of bidders $N$. In this case, all entries follow directly from the closed-form expressions derived above.

| $N$ | $k$ | $\mathbb{E}[R]$ | $\mathrm{Var}(D)$ | $\mathrm{Var}(U)$ | $\mathrm{Var}(R^{AP})$ | $\mathrm{Var}(R^{LAB})$ |
|---|---|---|---|---|---|---|
| 2 | 1 | 0.3333 | 0.0139 | **0.0556** | 0.0444 | 0.0139 |
| 3 | 1 | 0.5000 | **0.0167** | 0.0500 | 0.1071 | **0.0167** |
| 4 | 1 | 0.6000 | 0.0150 | 0.0400 | 0.1600 | 0.0150 |
| 5 | 1 | 0.6667 | 0.0127 | 0.0317 | 0.2020 | 0.0127 |
| 6 | 1 | 0.7143 | 0.0106 | 0.0255 | 0.2355 | 0.0106 |
| 7 | 1 | 0.7500 | 0.0089 | 0.0208 | 0.2625 | 0.0089 |
| 8 | 1 | 0.7778 | 0.0076 | 0.0173 | 0.2847 | 0.0076 |
| 9 | 1 | 0.8000 | 0.0065 | 0.0145 | 0.3032 | 0.0065 |
| 10 | 1 | 0.8182 | 0.0056 | 0.0124 | **0.3188** | 0.0056 |

**Table 1 Additional examples with $k=1$ and $N=2,\dots,10$ under i.i.d. Unif$[0,1]$ values. All values are rounded to four decimals. Within each variance column, the largest entry is marked in bold.**

When $k=1$, the discriminatory and LAB auctions coincide, so the corresponding variance columns are identical. As $N$ increases, expected revenue increases monotonically, reflecting stronger competition among bidders. Over the range reported in Table 1, the discriminatory auction has the smallest revenue variance among the four formats. The all-pay variance increases with $N$, whereas the market-clearing formats have much smaller variances: the uniform-auction variance is highest at $N=2$ and then decreases, while the discriminatory/LAB variance reaches its maximum at $N=3$ and then decreases. Accordingly, the ordering between the uniform and all-pay auctions depends on $N$: at $N=2$ the uniform auction has the larger variance, whereas for $N\geq 3$ in the table the all-pay auction does.

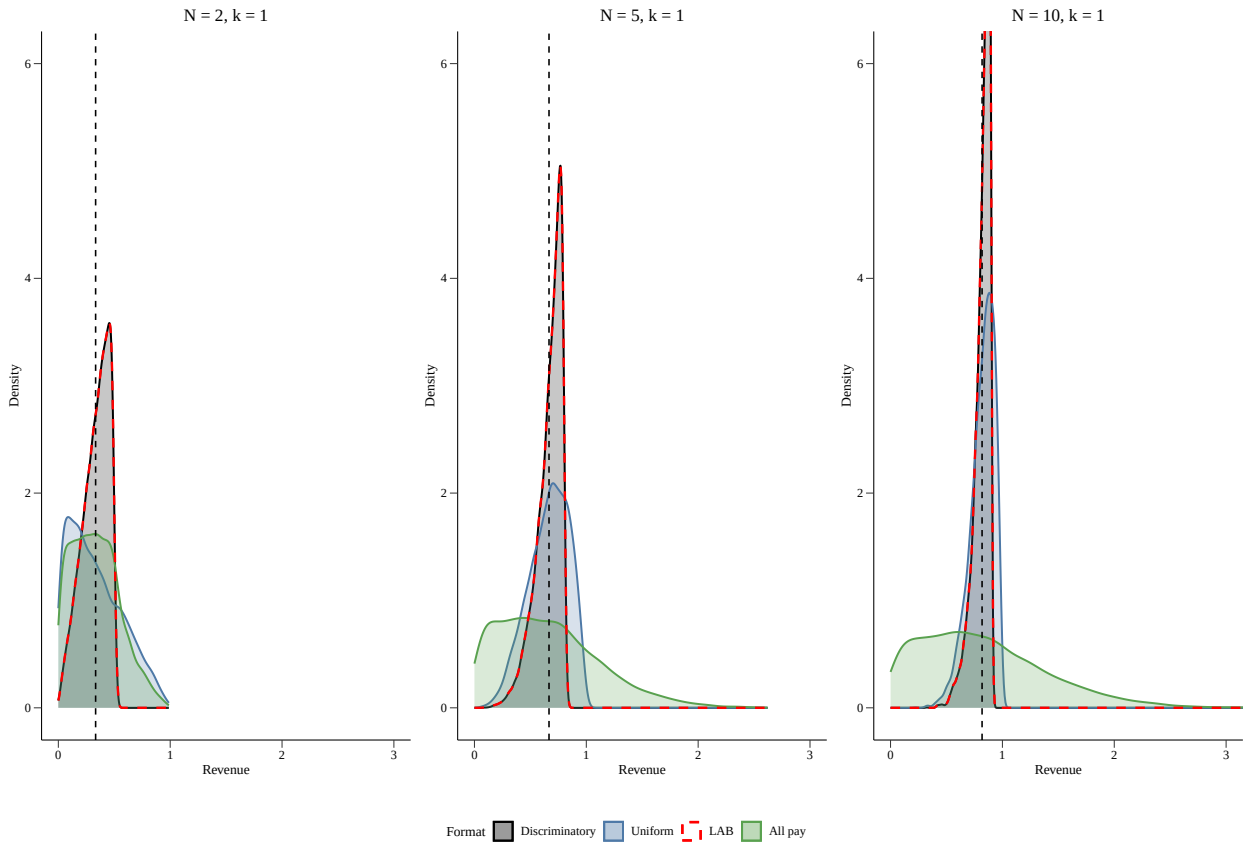



**Figure 2** **Simulation-based kernel density estimates of revenue for $k = 1$ and $N = 2, 5, 10$ under i.i.d.** Unif$[0, 1]$ **values. The dashed vertical line in each panel marks the common expected revenue. (Depicted using kernel estimates from 100000 simulations)**

Figure 2 plots simulation-based kernel density estimates of the revenue distributions for $N = 2, 5, 10$. Because the curves are based on simulation, they may smooth support boundaries and endpoint discontinuities.

Next, we fix $N = 10$ and vary the number of items $k$. The uniform and LAB columns again follow from the closed-form expressions above, while the discriminatory and all-pay columns are numerical evaluations of the exact moment formulas in the corresponding subsections.

With $N = 10$ fixed, expected revenue equals $k(10 - k)/11$, so it increases with $k$ up to $k = 5$ and then decreases. Across all values of $k$ reported in Table 2, the discriminatory auction has the smallest revenue variance. The comparison between the uniform and all-pay auctions again depends on the parameter values: for smaller values of $k$ (here $k \leq 4$), the all-pay auction has the larger variance, whereas for larger values of $k$ (here $k \geq 5$), the uniform auction does. The uniform auction has the larger variance than the LAB auction for all values of $k$. More generally, all four variance profiles are non-monotone in $k$ over this range, although their peaks occur at different values of $k$.

| $N$ | $k$ | $\mathbb{E}[R]$ | $\mathrm{Var}(D)$ | $\mathrm{Var}(U)$ | $\mathrm{Var}(R^{AP})$ | $\mathrm{Var}(R^{LAB})$ |
|---|---|---|---|---|---|---|
| 10 | 1 | 0.8182 | 0.0056 | 0.0124 | 0.3188 | 0.0056 |
| 10 | 2 | 1.4545 | 0.0161 | 0.0661 | 0.5584 | 0.0392 |
| 10 | 3 | 1.9091 | 0.0268 | 0.1736 | **0.6173** | 0.1139 |
| 10 | 4 | 2.1818 | 0.0341 | 0.3306 | 0.5494 | 0.2267 |
| 10 | 5 | 2.2727 | **0.0361** | 0.5165 | 0.4152 | 0.3587 |
| 10 | 6 | 2.1818 | 0.0323 | 0.6942 | 0.2658 | 0.4760 |
| 10 | 7 | 1.9091 | 0.0238 | **0.8099** | 0.1373 | **0.5315** |
| 10 | 8 | 1.4545 | 0.0131 | 0.7934 | 0.0504 | 0.4702 |
| 10 | 9 | 0.8182 | 0.0038 | 0.5579 | 0.0087 | 0.2510 |

**Table 2** **Additional examples with $N = 10$ and $k = 1, \ldots, 9$ under i.i.d. $\mathrm{Unif}[0, 1]$ values. All values are rounded to four decimals. Within each variance column, the largest entry is marked in bold.**

Figure 3 plots simulation-based kernel density estimates of the revenue distributions for $k = 2, 4, 7$.

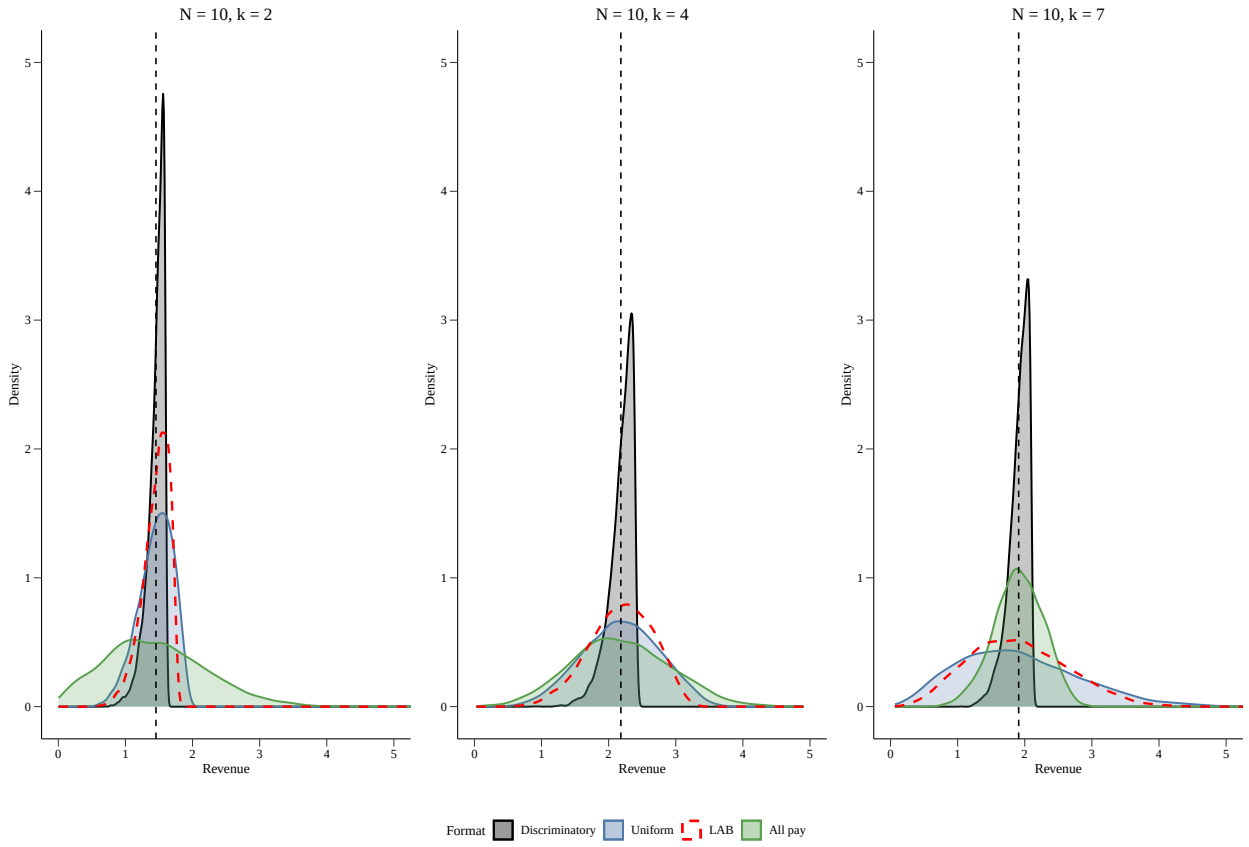



**Figure 3** **Simulation-based kernel density estimates of revenue for $N = 10$ and $k = 2, 4, 7$ under i.i.d. $\mathrm{Unif}[0, 1]$ values. The dashed vertical line in each panel marks the common expected revenue. (Depicted using kernel estimates from 100000 simulations).**

## S2. Filling the Variance Interval: Mixed-Price Auctions

The main result identifies the discriminatory and uniform auctions as the lower and upper benchmarks for revenue variance within the class of market-clearing auctions. For fixed $N$, $k$, and value distribution $F$ these two benchmarks define an interval of feasible revenue risk: $\big[\mathrm{Var}(D), \mathrm{Var}(U)\big]$.It is then natural to ask whether intermediate variance levels inside this interval can also be achieved by market-clearing formats.

A natural way to fill the interval is to consider pricing rules that interpolate between the two canonical formats. The mixed-price class does exactly this, as winners pay a convex combination of their own bid and the highest losing bid. For general $N > k \geq 1$, define the mixed-price auction with parameter $\alpha \in [0,1]$ by

$$p_r^\alpha(\mathbf{b}^\alpha) = \alpha b_r^\alpha + (1-\alpha) b_{k+1}^\alpha, \qquad r = 1, \ldots, k,$$

and

$$p_r^\alpha(\mathbf{b}^\alpha) = 0, \qquad r = k+1, \ldots, N.$$

Thus $\alpha = 0$ gives the uniform auction, while $\alpha = 1$ gives the discriminatory auction.

This appendix establishes variance attainability explicitly in the single-item case ($k = 1$) with $X \sim \mathrm{Unif}[0,1]$. In this setting, we show that as $\alpha$ varies from 0 to 1, revenue variance moves continuously and monotonically from the uniform-price benchmark to the discriminatory benchmark. Consequently, for fixed $N$ and $X \sim \mathrm{Unif}[0,1]$, every variance level between the two extremes can be achieved by a suitable mixed-price rule.

In this setting, only the highest bidder wins and pays $p_1^\alpha(\mathbf{b}^\alpha) = \alpha b_1^\alpha + (1-\alpha) b_2^\alpha$, while the remaining bidders pay 0, i.e. $p_r^\alpha(\mathbf{b}^\alpha) = 0$ for $r = 2, \ldots, N$. Suppose a bidder with value $v$ reports $x$, while all rivals use the increasing bidding function $b^\alpha$. Let $Y$ be the highest rival value, so $G(x) := \Pr(Y < x) = x^{N-1}$ and $g(x) := G'(x) = (N-1)x^{N-2}$. The expected utility from reporting $x$ is

$$u(v,x) = \big(v - \alpha b^\alpha(x)\big) G(x) - (1-\alpha) \int_0^x b^\alpha(t) g(t)\, dt.$$

For $\alpha > 0$, the first-order condition at the symmetric optimum $x = v$ is

$$\big(v - b^\alpha(v)\big) g(v) = \alpha (b^\alpha)'(v) G(v),$$

or equivalently,

$$(b^\alpha)'(v) = \frac{v - b^\alpha(v)}{\alpha} \frac{N-1}{v}.$$

Solving this differential equation with $b^\alpha(0) = 0$ gives

$$b^\alpha(v) = \frac{N-1}{N-1+\alpha} v.$$

As expected, for $\alpha = 0$, this formula gives truthful bidding, $b^0(v) = v$. This is the unique increasing symmetric equilibrium, as using the differential equation above, the derivative of utility can be written as

$$\frac{\partial u(v,x)}{\partial x} = (v-x)g(x).$$

Thus the derivative is positive for $x < v$ and negative for $x > v$, so the unique best response of type $v$ is to report $x = v$.

Equilibrium revenue is therefore

$$R^\alpha = \frac{N-1}{N-1+\alpha}\left(\alpha X_1^{(N)} + (1-\alpha)X_2^{(N)}\right).$$

Using the standard moments of uniform order statistics,

$$\mathrm{Var}(X_1^{(N)}) = \frac{N}{(N+1)^2(N+2)}, \quad \mathrm{Var}(X_2^{(N)}) = \frac{2(N-1)}{(N+1)^2(N+2)},$$

and

$$\mathrm{Cov}(X_1^{(N)}, X_2^{(N)}) = \frac{N-1}{(N+1)^2(N+2)},$$

we obtain

$$\mathrm{Var}(R^\alpha) = \frac{(N-1)^2}{(N-1+\alpha)^2} \cdot \frac{N\alpha^2 + 2(N-1)(1-\alpha)}{(N+1)^2(N+2)}.$$

Moreover,

$$\frac{d}{d\alpha}\mathrm{Var}(R^\alpha) = -\frac{2(N-1)^3(1-\alpha)}{(N+1)(N+2)(N-1+\alpha)^3} < 0 \qquad \text{for } \alpha \in [0,1).$$

Thus revenue variance is continuous and strictly decreasing in $\alpha$. The mixed-price class therefore attains every variance level between the uniform and discriminatory single-item benchmarks.

For the $N = 3$, $k = 1$, $X \sim \mathrm{Unif}[0,1]$ example in Section 2.1, $b^\alpha(v) = \frac{2}{2+\alpha}v$ and $\mathrm{Var}(R^\alpha) = \frac{3\alpha^2-4\alpha+4}{20(\alpha+2)^2}$. Thus, every variance level between $\mathrm{Var}(D) = 1/60$ and $\mathrm{Var}(U) = 1/20$ can be attained by a suitable value of $\alpha$. For example, solving for the midpoint variance $\mathrm{Var}(R^\alpha) = \frac{1/20+1/60}{2} = 1/30$ gives

$$\alpha^\star = \frac{10-6\sqrt{2}}{7} \approx 0.2164.$$

Thus, the mixed-price auction with $\alpha^\star$ attains exactly the midpoint variance between the uniform and discriminatory auctions.