Two-Sided Matching with Partial Participation

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Abstract

We introduce the concept of safe partial matchings to address the problem of two-sided matching with partial participation. Safety provides a strong guarantee that assigned agents will not be part of blocking coalitions. We prove existence, study the structure of the set of safe matchings, derive comparative statics with respect to participation and entry, and show how to compute maximal safe matchings.

1 Introduction

Marketplaces do not operate in a vacuum. Rather, they are "small parts of large markets" (Roth, 2018). For a marketplace to succeed, thickness is often required. Thickness is achieved when many agents in the market *participate* in the marketplace. For this reason, participation is a key consideration in practical market design. For example, the National Residency Match Program (NRMP) has an "all-in policy" that forbids participants from operating outside the marketplace, and it may cancel fellowship matches if participation rates fall below 75 percent. Similarly, the Canadian Resident Matching Service (CaRMS) has rules to prevent matched participants from matching elsewhere (and, in particular, through the NRMP).[1]

Although partial participation is common in practice (see, e.g., Roth and Xing, 1994), little is known about how to operate a marketplace with partial participation. Prior studies have mostly assumed full participation (for a textbook treatment, see Haeringer, 2018), and when the issue of participation was considered, the questions focused on incentivizing full participation (e.g., Kagel and Roth, 2000; Roth and Shorrer, 2021),[2] showing that it can be difficult to achieve in practice (Ekmekci and Yenmez, 2019; Peivandi and Vohra, 2021), and highlighting its benefits (Ashlagi and Roth, 2014; Ekmekci and Yenmez, 2019). This paper asks a different question: *What can a market designer do when participation is partial?*

Our answer is in the form of a *partial* matching. To see why we rely on partial matchings, consider a medical residency matching marketplace, where participation of doctors and residency programs

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[1]See Niederle et al. (2006), the NRMP main residency match's all-in policy and match agreement, the Specialties Matching Service's participation agreement for programs, and CaRMS's applicant contract.

[2]Other examples, outside the domain of two-sided matching, include Abdulkadiroğlu and Sönmez (1999), Chen and Sönmez (2002), Roth et al. (2005), Ashlagi and Roth (2014), and Sönmez et al. (2020).

is strictly partial. One of the participants is Alice, a doctor whose most desired residency program, MGH, does not participate in the marketplace. Clearly, MGH cannot be matched by the marketplace, which prevents the marketplace from matching Alice to her first choice. As a result, if Alice is matched by the marketplace (to another residency program), she might *potentially* deviate and match with MGH off the marketplace. A cautious market designer might therefore prefer not to match Alice to any participating residency program. Such caution is particularly important in nascent marketplaces, where the marketplace's reputation for being reliable is crucial, and a matched resident not showing up might tarnish it. Moreover, when the marketplace does not replace an existing centralized matching system, leaving some participants with no recommendation is the status quo.

We propose *safety*, a generalization of stability, the standard solution concept for two-sided matching markets with full participation. In addition to the requirements of stability (individual rationality and the absence of blocking pairs), safety requires that only participants be assigned, and precludes the possibility of blocking pairs that involve one assigned agent (not making any assumptions on how such blocks might arise). Safety can be interpreted as a combination of a requirement of *internal stability* (on the support of the partial matching) and a requirement of *robustness* against off-marketplace deviations, where the robustness requirement takes a worst-case information stance. The robustness requirement reflects an assumption that the market designer has no information or control over nonparticipants.

This worst-case approach can be extremely restrictive and even lead to the empty matching (in which no agent is assigned) being the unique safe matching (see examples in Section 2.1). However, the tools we develop are flexible enough to accommodate practical considerations. Returning to our example, the market designer can ignore the risk of potential off-marketplace blocking (involving Alice and MGH) if they know that MGH will never consider Alice. Section 6 generalizes our analysis to cover a range of informational assumptions from the restrictive (which corresponds to the definition of safety) to the most permissive (which rules out all matches with nonparticipants, and considers only "standard" blocking). We defer this general case for didactical reasons, as our model already introduces several novelties relative to the standard model of two-sided matching markets.

To further motivate the problem, we note that our analysis formalizes a main feature in the design of the Israeli Law Internship Match (ILIM)—a matching marketplace that we designed. Prior to 2018, the law-internships market suffered from severe unraveling, which led the Israel Bar Association to enforce a uniform offer date (i.e., the market was in Stage 2 using the taxonomy of Roth and Xing, 1994). As is often the case in Stage 2, there was a great deal of congestion and offers came with extremely short deadlines. For example, a major daily business newspaper reports that

> *"... at 8 am, students start receiving phone calls from firms in which they interviewed. If a student gets a call from a firm that is not his favorite, then he is in a difficult situation—on the one hand, he hopes that he will still get a call from the firm he wanted, and on the other hand he knows that if they did not call already in the morning, they will likely never call. On another hand, if he refuses the firm that called, he might be left with no good internship slot, and on still another hand, there may be other students in his position—so maybe a spot in his favorite firm will nevertheless become available."* (Neuman, E., February 7, 2018, TheMarker, Hebrew)

In its first year, participation in the ILIM was very partial (most applicants were seniors in a single elite law school, and many internships did not participate). Since the rules of the Israel Bar Association prevented the marketplace from forming committing agreements, the ILIM could only *recommend* matches. Robustness was therefore crucial to reassure students that they can reject off-platform offers without worrying about the strategic considerations highlighted in the quote above. It was especially important due to the "startup" nature of the ILIM: growing participation in future years would be hindered by stories of recommended matches not materializing.

Our startup approach proved useful. Over a few years, participation rates in the ILIM rapidly increased. In 2020, the annual report of the State Comptroller of Israel mentioned the ILIM positively. (Along with other measures to stop unraveling, this was the only positive point in the 96-page chapter dedicated to the Israel Bar Association.) The report specifically recognized that

> *"private entities currently operate the system on a voluntary basis, but naturally only a limited number of supervisors and internship candidates participate in it."* (State Comptroller of Israel, Annual Report 70B (2020) – Israel Bar Association, p. 2116, Hebrew)

Shortly thereafter, participation was mandated by the Israel Bar Association.[3]

Summary of the results. We study one-to-one two-sided matching markets with partial participation. After introducing the model in Section 2, Section 3 presents results on the structure of the set of safe matchings. We show that there exists a *men-optimal maximal safe matching (MOMS)*, which assigns each agent that can be assigned in some safe matching and is weakly preferred by each assigned man over any safe matching in which he is assigned, and a *women-optimal maximal safe matching (WOMS)*, which achieves the same for women. We note that the empty matching (which assigns no one) is always safe, so classic results like the Lone Wolf Theorem (McVitie and Wilson, 1970) and the conflict of interests between men and women (Knuth, 1976) do not directly extend. However, we present modified versions that cover our setting, as well as additional structural results.

Section 4 considers the effects of entry and exit. Our framework allows us to distinguish entry to the *market* from participation in the *marketplace*. Namely, an incumbent nonparticipant may join the set of participants; an agent may join the market as a nonparticipant; and an agent may join the market as well as the set of participants. Matching theory with full participation only allows for the last option and documents the *competition effect*: A set of results that show that the entry of a new man (woman) into the market leads all incumbent men to weakly worse (better) outcomes under both the men-optimal and women-optimal stable matchings (See Demange and Gale, 1985, Roth and Sotomayor, 1990, Theorems 2.25 and 2.26, and Crawford, 1991). By contrast, we find that under MOMS when an agent (of any gender) joins the set of participants, the set of assigned agents weakly grows and all men (women) that were originally assigned are weakly better (worse) off (and symmetrically for WOMS). Specifically, the effect *does not* depend on the gender of the new participant and *does* depend on the choice between MOMS and WOMS. When an agent enters the market but does not participate, the set of agents assigned under MOMS weakly shrinks and all men (women) that continue to be assigned are weakly worse (better) off (and symmetrically for WOMS). Here, too, the

[3]See Globes (Greenzeig, A., February 23, 2020, Hebrew) and Globes (Shafir, N., February 16, 2021, Hebrew).

effect does not depend on the gender of the entrant and does depend on the choice between MOMS and WOMS. Finally, we show that "anything goes" when agents simultaneously enter the market and join the set of participants.

Section 5 shows that MOMS can be calculated in time polynomial in the number of agents in the market. Our *Iterative Market-Driven Elimination (IMDE)* algorithm *endogenously determines* the set of assigned agents, taking into account the challenging fact that the more agents are unassigned, the more opportunities for potential blocking arise.

Section 6 generalizes our analysis to accommodate situations where some potential blocks with unassigned agents can be ruled out. We show that information about nonparticipants relaxes the constraints posed by potential blocking, making more partial matchings safe. As a result, MOMS assigns more agents, improves the assignments of originally assigned men, and hurts originally assigned women (the symmetric results hold with respect to WOMS). As we will explain, our findings also clarify some of the reasons for the difference between our results on entry (Section 4) and those of matching theory with full participation (where information about nonparticipants is not an issue).

Our analysis covers most of the central themes in the full-participation matching literature, but leaves one canonical issue unaddressed: incentives. We do so since analyzing incentives requires taking a stance on the outcomes of unassigned agents, which we were careful to avoid throughout the paper. For instance, if a resident is extremely pessimistic about her prospects outside the marketplace, she may prefer to report that she is not interested in a non-participating program to avoid being unassigned. The opposite holds when she is optimistic (she may prefer to remain unassigned, which she can achieve by reporting a non-participating program as her first choice). Additionally, even with full participation, it is well known that achieving strategy-proofness for all agents is not possible (Roth, 1982).[4] Section 7 proposes some paths forward. It also discusses our contributions and outlines several directions for future research.

Related literature. We have discussed several related papers, and we discuss additional related studies throughout the paper. Here, we highlight a few closely related papers.

The most closely related paper is Haeringer and Iehlé (2019), despite the different motivation and focus. Haeringer and Iehlé use partial data (applicants' preferences and programs' acceptable applicants) to predict parts of a stable matching. Although our motivation is different, our result that any safe matching is a submatching of a stable matching (Lemma 3 in Section 3) can also be used to predict parts of a stable matching using partial data (participants' preferences). In the same vein, our paper is related to Rastegari et al. (2013) and Ashlagi et al. (2020), and—considering other matching problems—to Saban and Sethuraman (2015) and Decerf et al. (2024).[5]

A close read of Sotomayor (1996) and Wu and Roth (2018) also reveals some connections. In these papers, single agents are treated—in spirit—like our unassigned agents. Furthermore, their

[4] Applicant-proposing deferred acceptance is strategy-proof for applicants under standard assumptions (Dubins and Freedman, 1981; Roth, 1982). A recent literature shows that many agents fail to follow the truthful dominant strategy, and attributes this failure to cognitive limitations and non-standard preferences (e.g. Li, 2017; Dreyfuss et al., 2022; Meisner and Von Wangenheim, 2023; Rees-Jones, 2018).

[5] Even more broadly, as empirical studies use stability to predict matching patterns (e.g., Hitsch et al., 2010; Banerjee et al., 2013), our study is related to studies of partial identification (Manski, 1990).

restriction on the set of blocking pairs that constitute a "problem" is akin to our analysis in Section 6.

Roth and Shorrer (2021) also consider a model of marketplace design within a broader economic environment, and also recognize that the individual rationality of participation is endogenous to the set of participants.[6] However, they focus on incentivizing participation. Furthermore, they assume that agents can commit (i.e., delegate their action to the marketplace) and that the designer understands how the broader market operates. By contrast, we assume that agents are free to form blocking coalitions and make no assumptions on how the broader market operates (beyond ruling out violations of individual rationality).

2 Model

A *matching market* is a tuple $\langle M, W, \succ \rangle$, where M and W are disjoint sets of men and women and $\succ$ is a profile of strict preferences for each agent over potential partners on the other side and the option of remaining unmatched (which we identify with being self-matched). A *two-sided matching problem with partial participation* is a tuple $\langle M, W, \succ, P \rangle$, where $\langle M, W, \succ \rangle$ is a matching market and $P \subseteq M \cup W$ is the set of participants. In the following definitions, we fix a specific two-sided matching problem with partial participation. As we formalize below, we consider *partial matchings*—in which some agents are not assigned—and use the symbol $\boxdot$ to indicate the status of being "unassigned."

Definition 1. A *partial matching* is a function $\mu : M \cup W \to M \cup W \cup \{\boxdot\}$ such that

1. $\mu(m) \in W \cup \{m, \boxdot\}$ for each $m \in M$,
2. $\mu(w) \in M \cup \{w, \boxdot\}$ for each $w \in W$, and
3. $\mu^2(i) = i$ whenever $\mu(i) \neq \boxdot$.

We say that an agent $i \in M \cup W$ is *assigned* under μ if $\mu(i) \neq \boxdot$. We denote the *support* of μ—the set of all assigned agents—by $\operatorname{supp} \mu := \{i \mid \mu(i) \neq \boxdot\}$. We emphasize that, in contrast to much of the previous literature, if $\mu(i) = i$ we say that i is assigned (to be single). We reserve the term *matched* to indicate assignment to another partner, and denote, for a partial matching μ, $\mathcal{M}_M(\mu) = \{m \mid m \in M,\ \mu(m) \in W\}$ and $\mathcal{M}_W(\mu) = \{w \mid w \in W,\ \mu(w) \in M\}$, the sets of matched men and women, respectively.

It is illustrative to contrast Definition 1 with the standard definition of (non-partial) matching. The standard definition is the special case of a partial matching such that no agent is unassigned (i.e., $\operatorname{supp} \mu = M \cup W$).

We say that i is *acceptable* to j if $i \succcurlyeq_j j$, and it is *unacceptable* otherwise.

Definition 2. A partial matching μ is *individually rational* if no agent is assigned to an unacceptable partner. Formally, $\mu(i) \succcurlyeq_i i$ for each $i \in \operatorname{supp} \mu$.

[6] Roth and Shorrer's (2021) individual rationality (of participation) requires that agents are not hurt by participating relative to operating in the broader market. It should not be confused with our individual rationality (of an assignment), which requires that any assigned agent prefers her assignment to being single.

Definition 3. A *P-matching* is a partial matching that assigns only members of P. Formally, μ is a P-matching if $\operatorname{supp}\mu \subseteq P$.

We sometimes attach the subscript P to P-matchings to emphasize that only members of this set are assigned.

Definition 4. A partial matching μ is *(surely) blocked* by a pair $(m,w) \in M \times W$ if both m and w are assigned under μ, and both prefer to match with each other over their assignment. Formally, $\{m,w\} \subseteq \operatorname{supp}\mu$, $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$.

We say that j is a *potential match* for i if j finds i acceptable, or $j \notin P$. The assumption that j is a potential match for i whenever $j \notin P$ can be interpreted as a conservative presumption that each nonparticipant finds all participants acceptable. (In Section 6, we extend the analysis to cover less conservative assumptions.)

Definition 5. A partial matching μ is *potentially blocked* by a pair $(m,w) \in (M \times W)$ if $\{m,w\} = \{i,j\}$, i is assigned, j is unassigned, $j \succ_i \mu(i)$, and j is a potential match for i.

Definition 6. A partial matching for $\langle M, W, \succ, P\rangle$ is *safe* if it is an individually rational P-matching that is neither blocked nor potentially blocked by any pair.[7]

A few points deserve attention. First, safety is defined with respect to a two-sided matching problem with partial participation $\langle M, W, \succ, P\rangle$. Hence, when the set of participants changes, the set of safe matchings may change. Second, safety is independent of the preferences of non-participants. We include these preferences in the description of two-sided matching problems with partial participation for convenience (this notation is especially useful when we consider the implications of changing the set of participants, P, holding the rest of the market structure fixed). Finally, while participants' preferences consist of full rankings (including preferences over all non-participants), our analysis only requires knowing each participant's ranking of their most preferred non-participant, which may be more convenient in certain applications.[8]

Safety is a generalization of *stability*, a standard solution concept in the theory of two-sided matching markets with full participation. That theory considers only full-support matchings (μ is a full-support matching if $\operatorname{supp}\mu = M \cup W$), and stability requires that the matching is individually rational and that there are no blocking pairs. We note that under full participation, if μ is a full-support matching, then safety coincides with *stability*.

Lemma 1. *Any $\langle M, W, \succ, P\rangle$ has a safe partial matching.*

Proof. The empty partial matching $\mu^{\boxdot}(\cdot) \equiv \boxdot$ is a P-matching, as its (empty) support is a subset of P. Additionally, a violation of any of the other conditions in Definition 6 requires that at least one agent be assigned. Hence, the empty matching is safe. □

[7] The requirement that μ is a P-matching can be interpreted as a limitation on the market designer or as eliminating "potential violations of individual rationality."

[8] We thank Aislinn Bohren for this observation.

2.1 Examples

Before we continue to the general analysis, it is instructive to consider a few concrete examples. Our examples will require some additional definitions and notation.

We say that the partial matchings μ and μ' are *disjoint* if $\operatorname{supp}\mu \cap \operatorname{supp}\mu' = \varnothing$.

Definition 7. If the partial matchings μ and μ' are disjoint then their *pasting* is the matching $\mu \oplus \mu'$, where

$$(\mu \oplus \mu')(i) := \begin{cases} \mu(i) & \text{if } i \in \operatorname{supp}\mu \\ \mu'(i), & \text{otherwise.} \end{cases} \tag{1}$$

Lemma 2. *If μ and μ' are safe and disjoint, then $\mu \oplus \mu'$ is safe.*

Proof. The proof is simple and we omit it as the lemma is an immediate corollary of a stronger result that we prove below.[9] □

Definition 8. The partial matching μ is a *submatching* of ν if $\mu(i) \in \{\nu(i), \boxdot\}$ for all $i \in M \cup W$.

Example 1 (Disconnected markets). Table 1 describes a market with 4 men and 4 women. We follow the convention in the literature and truncate each agent i's preferences above their ranking of i (the option of being assigned to be single).

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{m_3}$	$\succ_{m_4}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$	$\succ_{w_4}$
w_2	w_1	w_4	w_3	m_1	m_2	m_3	m_4
w_1	w_2	w_3	w_4	m_2	m_1	m_4	m_3

Table 1: Disconnected markets

Agents in $A_1 := \{m_1, m_2, w_1, w_2\}$ are only interested in partners in A_1 (formally, any other partner is not acceptable for them) and agents in $A_2 := \{m_3, m_4, w_3, w_4\}$ are only interested in partners in A_2. As a result, in any safe partial matching μ, if m_1 is assigned, then w_2 must be matched (either to m_1 or to m_2). To see this, note that if w_2 is not assigned, (m_1, w_2) are a potentially blocking pair, if w_2 is assigned to be single, (m_1, w_2) are a blocking pair, and if w_2 is matched to a man in A_2, individual rationality is violated. Symmetrically, if w_2 is assigned, then m_2 must be matched; if m_2 is assigned, then w_1 must be matched; and if w_1 is assigned, then m_1 must be matched. And a symmetric argument applies to agents in A_2.

Furthermore, if $A_i \subseteq P$, any stable matching of the restriction of the market to A_i can be extended to a safe partial matching (by setting the assignment of all agents in A_{3-i} to $\boxdot$).[10] Thus, all matchings such that all agents in A_i are matched to a partner in A_i and other agents are unassigned are safe. Finally, by Lemma 2 the pasting of such matchings is also safe. Altogether, a partial matching is safe

[9] Specifically, using the notation we will introduce later, the lemma follows from Lemma 4 as $\mu \vee \mu' = \mu \oplus \mu'$ whenever $\operatorname{supp}\mu$ and $\operatorname{supp}\mu'$ are disjoint.

[10] Formally, the restriction of $\langle M, W, \succ \rangle$ to $R \subset M \cup W$ is the market $\langle M \cap R, W \cap R, \{\succ_i|_R\}_{i \in R} \rangle$.

if and only if for each $i \in \{1,2\}$ all agents in A_i are matched or none of them is. This identifies the set of all safe matchings. For a graphical summary, see Figure A.1 in Appendix B.

Example 2 (Connected local markets). Table 2 modifies Example 1 so that members of A_1 find members of A_2 acceptable and vice versa, but members of each group rank partners from their group higher than partners from the other group.

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{m_3}$	$\succ_{m_4}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$	$\succ_{w_4}$
w_2	w_1	w_4	w_3	m_1	m_2	m_3	m_4
w_1	w_2	w_3	w_4	m_2	m_1	m_4	m_3
w_4	w_3	w_2	w_1	m_3	m_4	m_1	m_2
w_3	w_4	w_1	w_2	m_4	m_3	m_2	m_1

Table 2: Connected local markets

It is not difficult to verify the safety of all partial matchings identified as safe in Example 1. Additionally, no safe matching can match a member of A_1 to a member of A_2. To see this, note that if a man $m \in A_i$ is matched to a woman in A_{3-i} then simple counting shows that at least one woman $w \in A_i$ is not matched to a man in A_i, and so w and m form a blocking pair (if w is assigned) or a potentially blocking pair (if w is unassigned). And, as in Example 1, no agent can be assigned to be single. Altogether, the set of all safe matchings is identical to Example 1.

Examples 1 and 2 show that when markets are sufficiently fragmented, it may still be possible to safely assign members of one fragment even when members of other fragments do not fully participate.

Example 3 (Tiered market). Table 3 describes a market in which all agents prefer partners in $A_1 := \{m_1, m_2, w_1, w_2\}$ over partners in $A_2 := \{m_3, m_4, w_3, w_4\}$.

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{m_3}$	$\succ_{m_4}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$	$\succ_{w_4}$
w_2	w_1	w_2	w_1	m_1	m_2	m_1	m_2
w_1	w_2	w_1	w_2	m_2	m_1	m_2	m_1
w_4	w_3	w_4	w_3	m_3	m_4	m_3	m_4
w_3	w_4	w_3	w_4	m_4	m_3	m_4	m_3

Table 3: Tiered market

As in Example 1, safety requires that if any agent in A_1 is assigned, then all agents in A_1 are matched to one another (otherwise, they form blocking pairs, or potentially blocking pairs). Furthermore, agents in A_2 can only be assigned if all agents in A_1 are matched. As a result, if $A_1 \nsubseteq P$ only the empty matching is safe. Additionally, agents in A_2 can only be assigned to agents in A_2 in any safe matching. And, as in Example 1, if one agent in A_2 is assigned, then all of them must be matched. Therefore, they can only be matched in a safe partial matching if $P = A_1 \cup A_2 = M \cup W$. Finally, we note that any matching of all men in A_1 to all women in A_1 does not introduce challenges to safety, and—whenever all agents in A_1 are matched to each other—any matching of all men in A_2 to all

women in A_2 does not introduce challenges to safety. This identifies the set of all safe matchings. For a graphical summary, see Figure A.2 in Appendix B.

Example 3 shows that when markets are sufficiently hierarchical, it may still be possible to safely assign members of the highest tiers members of lower tiers do not fully participate. This observation echos an observation made by the leadership of the Israeli Psychology Masters Match (IPMM), a market we designed over a decade ago (Hassidim et al., 2017). In Hassidim et al. (2021) we write

> *"... one college that offers graduate degrees in psychology did not participate. This college was not part of the pre-existing decentralized protocol, and was not considered a competitor by the participating institutions."*

Using the terminology developed in Section 2, the quote dismisses the possibility of (potential) blocks with a program from an unquestionably lower tier.

3 The Structure of the Set of Safe Partial Matchings

Our model generalizes the standard model of two-sided one-to-one matching markets, which corresponds to the case of full participation and full-support matching. In that case, our solution concept generalizes the standard solution concept of stability. In this section, we generalize some key structural results on the set of stable matchings (Roth and Sotomayor, 1990). We begin by stating our generalized theorems and comparing them with their versions for stable matchings. Then, in Section 3.1, we present the proofs and additional results of independent interest. Throughout, we fix a specific two-sided matching problem with partial participation, $\langle M, W, \succ, P\rangle$.

Theorem S.1. *An agent cannot be matched in one safe matching and assigned to be single in another. Formally, if μ and ν are safe and $\mu(i) = i$, then $\nu(i) \in \{i, \boxdot\}$.*

When considering the case of full participation and full-support matchings, Theorem S.1 implies the Lone Wolf Theorem (McVitie and Wilson, 1970). We note that, in contrast to that special case, the theorem does not state that the set of assigned or matched individuals is identical across all safe matchings. Instead, the theorem states that an agent may either have a partner in some safe partial matching, or may be assigned to be single in some safe partial matching, but not both.

Theorem S.2. *There exists a safe matching μ_P^{MOMS}, that*

1. *assigns each agent that can be assigned in some safe matching,*
2. *matches (to a partner) any agent that can be matched in some safe matching, and*
3. *is weakly preferred by each assigned man over any safe matching in which he is assigned.*

We call μ_P^{MOMS} the *men-optimal maximal safe matching*. By symmetry, there exists a *women-optimal maximal safe matching*, μ_P^{WOMS}, with symmetric properties with respect to women. Since both matchings assign each agent that can be assigned in a safe matching, we have supp μ_P^{WOMS} = supp μ_P^{MOMS}.

We note that with full participation (i.e., when $P = M \cup W$) $\mu^{MOMS}_{M\cup W}$ is the men-optimal stable matching, μ^M, and $\mu^{WOMS}_{M\cup W}$ is the women-optimal stable matching, μ^W (Gale and Shapley, 1962), and Theorem S.2 implies the consensus property (Roth, 1985b).

Theorem S.3. *Let μ, μ' be safe matchings. Then, if all men in* $\operatorname{supp}\mu \cap \operatorname{supp}\mu'$ *weakly prefer μ to μ', then all women in* $\operatorname{supp}\mu \cap \operatorname{supp}\mu'$ *weakly prefer μ' to μ.*

Corollary S.1. *The sets of assigned agents under μ^{WOMS}_P and μ^{MOMS}_P are identical. All assigned men weakly prefer μ^{MOMS}_P and all assigned women weakly prefer μ^{WOMS}_P.*

In the special case of full-support matchings, Theorem S.3 implies the conflict of interests between men and women over the set of stable matchings (Knuth, 1976). We note that, in the general case, Theorem S.3 does not make any assertion with respect to agents that are unassigned under one of the matchings. This is conceptually consistent with us not taking a stance on the outcomes of unassigned agents. Furthermore, even if we assumed that being unassigned yields the same utility as being assigned to be single, the statement would not hold. For example, if we compare the empty matching to another safe matching that matches man m to woman w, then both m and w would prefer the latter.

Theorem S.4. *If μ and μ' are safe, agents in* $\operatorname{supp}\mu \cap \operatorname{supp}\mu'$ *are matched among themselves under both μ and μ'. Formally,* $\mu'(\operatorname{supp}\mu \cap \operatorname{supp}\mu') = \operatorname{supp}\mu \cap \operatorname{supp}\mu'$.

We note that the conclusion of Theorem S.4 holds trivially with respect to any pair of full matchings (not necessarily stable). It is also worth commenting that Theorem S.4 can be used to construct a tier structure in the spirit of Aryal et al. (2025).[11]

3.1 Proofs

Our proofs use Lemma 3, which shows that safe matching can be thought of as a stepping stone toward stability: each safe matching is a submatching of a stable matching of the entire market.

Lemma 3. *If μ is a safe partial matching for $\langle M, W, \succ, P\rangle$, then it is a submatching of a stable matching of $\langle M, W, \succ\rangle$.*

Proof. Denote $R := (M \cup W) \setminus \operatorname{supp}\mu$, the set of unassigned agents under μ. Let μ_R denote some stable matching in the restriction of $\langle M, W, \succ\rangle$ to R. Existence of such matching is guaranteed by Gale and Shapley (1962). Let ν denote the matching such that

$$\nu(i) = \begin{cases} \mu(i), & \text{if } i \notin R \\ \mu_R(i), & \text{if } i \in R. \end{cases}$$

By the properties of μ_R and μ, it is clear that ν is a matching and that μ is a submatching of ν. To see that ν is stable, first note that individual rationality follows from the individual rationality of μ and μ_R. Furthermore, if a pair (m, w) blocks ν and both agents are in R, then the pair blocks μ_R in the

[11] To wit, we can say a set of agents forms a tier higher than another set of agents if agents in the former set can be matched in a safe matching while agents in the latter do not participate, but the converse does not hold.

restriction of $\langle M, W, \succ \rangle$ to R. Similarly, if a pair (m, w) blocks ν and both agents are not in R, then the pair blocks μ. Finally, if only one of (m, w) is in R, then the pair forms a potential block for μ. □

In light of Theorem S.2 and Lemma 3, it is tempting to think that μ_P^{MOMS} is a submatching of the men-optimal stable matching, μ^M. The following example shows that this is not necessarily the case.

Example 4. Consider the market described in Table 4.

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{m_3}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$
w_2	w_1	w_3	m_1	m_2	m_3
w_1	w_2		m_3	m_1	
			m_2		

Table 4: Example 4 preference profile

The men-optimal stable matching, μ^M, has all men matched with their top choices. However, when the set of participants contains only agents with index 1 or 2, $P = \{m_1, m_2, w_1, w_2\}$, $\mu_P^{MOMS}(m_1) = w_1$ and $\mu_P^{MOMS}(m_2) = w_2$ (namely, men are matched with their least preferred acceptable partner). To see why, note first that the matching is safe as there are no blocking pairs or violations of individual rationality, and the only unassigned potential match for any agent in P is m_3 (for w_1), but w_1 prefers her assignment under μ_P^{MOMS} over m_3. To see that men cannot be better off, note that since $m_3 \notin P$, if w_1 is assigned in a safe matching, her partner must be m_1. Since m_1 prefers w_2 to w_1, woman w_2 must be assigned to a man she likes more than m_1. The only possibility is m_2. A nearly identical argument shows that if any agent in P is assigned, all agents in P must be matched.

It is interesting to note that when the set of participants is $P' = P \cup \{m_3\}$, members of P continue to be matched with each other in any non-empty safe partial matching. Furthermore, m_3 is not assigned in any safe partial matching. However, the information that is revealed through the participation of m_3 (specifically, that w_1 is not acceptable for m_3) expands the set of safe partial matchings, making the restriction of μ^M to P safe. We elaborate on the effects of participation in Section 4 and on the effects of information in Section 6.

Proof of Theorem S.1. Towards contradiction, assume that μ and ν are safe in $\langle M, W, \succ, P \rangle$, $\mu(i) = i$ and $\nu(i) = j \notin \{i, \boxdot\}$. Then, by Lemma 3, there exist two stable matchings of $\langle M, W, \succ \rangle$such that in one of them i is matched to a partner from the other side of the market and in the other i is single (assigned to i). However, this contradicts the Lone Wolf Theorem (McVitie and Wilson, 1970) and its generalization, the Rural Hospitals Theorem (Roth, 1986). □

For the purpose of the following analysis, for each man m we extend $\succ_m$ to the order $\succ_m^*$ over $W \cup \{m\} \cup \{\boxdot\}$ as follows:

1. For all $i, j \in W \cup \{m\}$, $i \succ_m^* j$ if and only if $i \succ_m j$.
2. For all $i \in W \cup \{m\}$, $i \succ_m^* \boxdot$.

Definition 9. Given a pair of safe partial matchings μ and μ', we define the join $\mu \vee \mu'$ by

$$(\mu \vee \mu')(i) := \begin{cases} \max_{\succ_i^*}\{\mu(i), \mu'(i)\} & \text{if } i \in M \\ m & \text{if } i \in W \text{ and } i = \max_{\succ_m^*}\{\mu(m), \mu'(m)\} \text{ for a unique } m \in M \\ i & \text{otherwise, if } \mu(i) = i, \text{or } \mu'(i) = i \\ \boxdot & \text{otherwise} \end{cases}$$

Lemma 4. *Let μ and μ' be a pair of safe partial matchings, then*

1. *$\mu \vee \mu'$ is a safe partial matching, and*
2. *$\operatorname{supp} \mu \vee \mu' = \operatorname{supp} \mu \cup \operatorname{supp} \mu'$.*

The proof is deferred to Appendix A as it is relatively involved. It proceeds in three steps. In the first step, we show that $\mu \vee \mu'$ is a partial matching (specifically, if $w = \max_{\succ_m^*}\{\mu(m), \mu'(m)\}$ for some man $m \in M$, then m is the only man for whom this holds). In the second step, we show that the support of $\mu \vee \mu'$ is the union of the supports of μ and μ'. The delicate part involves showing that the assignment of women who were matched in μ or μ' must be described in the second line of Definition 9, rather than the fourth. Finally, in the third step, we show that the previous steps guarantee that $\mu \vee \mu'$ is safe.

Proof of Theorem S.2. Let $\{\mu_i\}_{i=1}^n$ be an enumeration of all safe matchings. We will show that

$$\mu_P^{MOMS} := (((\mu_1 \vee \mu_2) \vee \mu_3) \dots \vee \mu_n)$$

satisfies all requirements. This, in turn, proves that the resulting partial matching does not depend on the specific enumeration, justifying the independence of the notation μ_P^{MOMS} from the enumeration.

For (1), note that $\operatorname{supp} \mu_P^{MOMS} = \bigcup_{i=1}^n \operatorname{supp} \mu_i$ by Lemma 4. Additionally, (2) follows from (1) by Theorem S.1. Finally, for (3), for each man $m \in M$, let $j(m) \leqslant n$ be the first safe matching such that $\mu_{j(m)}(m) = \max_{\succ_m^*}\{\mu_i(m)\}_{i=1}^n$. Then, a simple induction shows that for each m, for each $k \geqslant j(m)$, we have $(((\mu_1 \vee \mu_2) \vee \mu_3) \dots \vee \mu_k)(m) = \mu_{j(m)}(m)$. □

Proof of Theorem S.3. Towards contradiction, assume that all men in $\operatorname{supp} \mu \cap \operatorname{supp} \mu'$ weakly prefer μ to μ' and some woman $w \in \operatorname{supp} \mu \cap \operatorname{supp} \mu'$ strictly prefers μ to μ'. Since $\mu(w) \succ_w \mu'(w)$ and both μ and μ' are individually rational, we have that $\mu(w)$ is a man (i.e., $\mu(w) = m \in M$). Since $\mu'(w) \neq m$, $\mu'(m) \neq w$.

Case 1. ($\mu'(m) \neq \boxdot$): Since m is a man in $\operatorname{supp} \mu \cap \operatorname{supp} \mu'$, m weakly prefers $\mu(m)$ to $\mu'(m)$. Thus $w = \mu(m) \succ_m \mu'(m)$, as preferences are strict and $\mu(m) \neq \mu'(m)$. Thus, (m, w) forms a blocking pair for μ', a contradiction.

Case 2. ($\mu'(m) = \boxdot$): Since $\mu(w) = m$ and μ is individually rational, m is a potential match for w. But this implies that μ' is potentially blocked by (m, w), as $m \succ_w \mu'(w)$ and $\mu'(m) = \boxdot$, a contradiction.[12] □

Proof of Theorem S.4. First, if $\mu(i) = i$ and $i \in \operatorname{supp}\mu \cap \operatorname{supp}\mu'$ then $\mu'(i) = i$ by Theorem S.1.

Next, recall that $\mu \vee \mu'$ is safe (Lemma 4). Assume that $\mu(m) \in \operatorname{supp}\mu \cap \operatorname{supp}\mu'$ but $\mu'(m) \in \operatorname{supp}\mu' \backslash \operatorname{supp}\mu$. Then, $(\mu \vee \mu')(m) = \mu(m)$ as otherwise $(m, \mu'(m))$ is a potential blocking pair for μ. Similarly, if $\mu'(m) \in \operatorname{supp}\mu \cap \operatorname{supp}\mu'$ but $\mu(m) \in \operatorname{supp}\mu \backslash \operatorname{supp}\mu'$, then, $(\mu \vee \mu')(m) = \mu'(m)$. Thus, any man who is matched with a woman that is matched in both μ and μ' must be matched with such a woman in $\mu \vee \mu'$:

$$\mu\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right) \cup \mu'\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right) \subseteq (\mu \vee \mu')\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right),$$

and so

$$\left|\mu\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right) \cup \mu'\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right)\right| \leqslant \left|(\mu \vee \mu')\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right)\right|. \tag{2}$$

However,

$$\left|\mu\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right)\right| = \left|\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right|, \tag{3}$$

$$\left|(\mu \vee \mu')\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right)\right| = \left|\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right|, \tag{4}$$

and

$$\begin{aligned}\left|\mu\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right) \cup \mu'\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right)\right| = \\ \left|\mu\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right)\right| + \left|\mu'\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right) \backslash \mu\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right)\right|.\end{aligned} \tag{5}$$

Substituting Equations (3) to (5) into Equation (2), we get

$$\left|\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right| + \left|\mu'\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right) \backslash \mu\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right)\right| \leqslant \left|\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right|,$$

which implies

$$\left|\mu'\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right) \backslash \mu\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right)\right| \leqslant 0.$$

By symmetry, we also have

$$\left|\mu\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right) \backslash \mu'\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right)\right| \leqslant 0.$$

Altogether, we conclude that

$$\mu'\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right) = \mu\left(\mathcal{M}_W(\mu) \cap \mathcal{M}_W(\mu')\right),$$

namely, women who are matched in both μ and μ' are matched to the same set of men. Hence, these

[12] In fact, Case 2 is ruled out by Theorem S.4.

men are members of supp $\mu \cap$ supp μ'. By symmetry, the same holds for men, which completes the proof. □

4 The Effects of Entry and Participation

Recall that the *competition effect* refers to a set of results in matching theory with full participation which establish that the entry of a new man (woman) to the market leads all incumbent men to weakly worse (better) outcomes under both the men-optimal and women-optimal stable matchings (See Roth and Sotomayor, 1990, Theorems 2.25 and 2.26).

Our framework allows for richer notions of entry. First, an incumbent nonparticipant may join the set of participants. Second, an agent may join the market as a nonparticipant. Third, an agent may join the market as well as the set of of participants. (We present our results in the context of entry, but they have an analogous interpretation in the context of exit).

Theorem P.1. *An agent cannot be matched in one safe matching and assigned to be single in another, even with a different set of participants. Formally, if μ is safe in $\langle M, W, \succ, P\rangle$ and $\mu(i) = i$, then for any $P' \subseteq M \cup W$ if ν is safe in $\langle M, W, \succ, P'\rangle$, then $\nu(i) \in \{i, \boxdot\}$.*

Theorem P.1 is a strengthening of Theorem S.1 that states that the status "lone wolf" cannot be overturned by changes to the set of participants. Next, we consider the effect of participation.

Theorem P.2. *Let $P \subset P' \subseteq M \cup W$. Then,*

1. *any agent that is assigned under μ_P^{MOMS} is assigned under $\mu_{P'}^{MOMS}$,*
2. *any man that is assigned under μ_P^{MOMS} weakly prefers $\mu_{P'}^{MOMS}$ to μ_P^{MOMS}, and*
3. *any woman that is assigned under μ_P^{MOMS} weakly prefers μ_P^{MOMS} to $\mu_{P'}^{MOMS}$.*

Theorem P.2 states that when an agent (of any gender) joins the set of participants, the set of agents assigned under MOMS weakly grows and all men (women) that were originally assigned are weakly better (worse) off. We prove it by showing that increased participation makes the set of safe matchings grow (i.e., a safe matching prior to participation is also safe post participation). The statement then follows from MOMS being the join over all safe matchings, as the post participation join includes the pre-entry MOMS.

Example 4 in Section 3 shows that an additional participant can lead to strict improvement (deterioration) for all assigned men (women). The example also illustrates that the effect may be the result of information (certain potential blocks being ruled out based on new participants' preferences) rather than through the possibility to assign the new participant. Indeed, in the example, while the new participant changes the assignment of all others, he himself remains unassigned (we discuss the role of additional information in Section 6).

It is interesting to note the asymmetry between our results and the competition effect from full-participation matching theory. In contrast to full-participation matching theory, the gender of the new participant *does not* affect the results, but the selection between MOMS and WOMS *does* matter.

For instance, in Example 4, the additional participant is a man, yet all men are better off after his entry. For completeness, Example A.1 shows an instance where the additional participant is a woman and all assigned men (women) are better (worse) off under MOMS after her entry. These examples also illustrate the importance of MOMS vs. WOMS (which can be seen by relabeling the genders).

Next we consider the (pure) effect of competition. Namely, the case of a nonparticipant entrant.

Theorem P.3. *Let $\langle M, W, \succ, P\rangle$ denote a matching problem with partial participation, and $\langle M', W', \succ', P\rangle$ be another problem that includes additional agents, keeping fixed the set of participants. Formally, $M \subseteq M'$, $W \subseteq W'$, and $\succ'|_{M\cup W} = \succ$. Let μ_P^{MOMS} and $\mu_P'^{MOMS}$ denote the corresponding MOMS matchings. Then,*

1. *any agent that is assigned under $\mu_P'^{MOMS}$ is also assigned under μ_P^{MOMS},*
2. *any man that is assigned under $\mu_P'^{MOMS}$ weakly prefers his assignment under μ_P^{MOMS}, and*
3. *any woman that is assigned under $\mu_P'^{MOMS}$ weakly prefers her assignment to her assignment under μ_P^{MOMS}.*

Theorem P.3 states that when an agent (of any gender) enters the market but does not participate, the set of agents assigned under MOMS weakly shrinks and all men (women) that continue to be assigned are weakly worse (better) off. As in Theorem P.2, the effect does not depend on the gender of the entrant and does depend on the choice between MOMS and WOMS. The proof is similar. First, we show that entry shrinks the set of safe matchings (i.e., a safe matching post entry is also safe pre-entry). The statement then follows from MOMS being the join over all safe matchings, since the pre-entry collection of safe matchings includes the post-entry MOMS.

Finally, Theorem P.4 states that when agents enter the market and simultaneously join the set of participants, either effect may dominate, yielding an "anything goes" result.

Theorem P.4. *A new entrant that joins the set of participants can improve or deteriorate the assignment of assigned men (women) under the men-optimal maximal safe matching.*

4.1 Proofs

Lemma 5. *If μ is safe in $\langle M, W, \succ, P\rangle$ and $P \subseteq P'$, then μ is safe in $\langle M, W, \succ, P'\rangle$.*

Proof. Since μ is a P-matching $\operatorname{supp}\mu \subseteq P \subseteq P'$, so μ is a P'-matching. Individual rationality and the absence of blocking pairs are inherited from $\langle M, W, \succ, P\rangle$. Furthermore, the set of potential matches for each $i \in \operatorname{supp}\mu$ under $\langle M, W, \succ, P'\rangle$ is a subset of the set of potential matches for each $i \in \operatorname{supp}\mu$ under $\langle M, W, \succ, P\rangle$, so the absence of potential blocking pairs for μ in $\langle M, W, \succ, P\rangle$ implies the absence of potential blocking pairs for μ in $\langle M, W, \succ, P'\rangle$. □

Proof of Theorem P.1. By Lemma 5 and Theorem S.1. □

Proof of Theorem P.2. By Lemma 5 μ_P^{MOMS} is safe in $\langle M, W, \succ, P'\rangle$. The first two parts therefore follow from Theorem S.2 and the third part follows by Theorem S.3. □

Lemma 6. *Using the definitions of Theorem P.3, if μ' is safe in $\langle M', W', \succ', P\rangle$ then $\mu := \mu'|_{M\cup W}$ is safe in $\langle M, W, \succ, P\rangle$.*

Proof of Lemma 6. Since μ' is a P-matching, $\operatorname{supp}\mu' \subseteq P \subseteq M \cup W$. Additionally, relative to μ' in $\langle M', W', \succ', P\rangle$, there are fewer (in the set inclusion sense) potentially blocking pairs for μ in $\langle M, W, \succ, P\rangle$, and the rest of the threats to safety are identical. □

Proof of Theorem P.3. By Lemma 6, $\mu'^{MOMS}_P|_{M\cup W}$ is safe in $\langle M, W, \succ, P\rangle$. Hence, the first two results follow from Theorem S.2 and the third part follows from Theorem S.3. □

Proof of Theorem P.4. Since the men-optimal stable matching is a special case of the men-optimal maximal safe matching (when participation is full), it is well known that a male entrant-participant can hurt all men and benefit all women under a MOMS matching, and that the opposite for a woman entrant (due to the competition effect in stable matchings). Examples 5 and 6 show that the opposite may also hold (when participation is not full), thus completing the proof. □

Example 5 (Entry of man participant). Table 5 describes a problem where m_5 is an entrant who joins the set of participants (the only non participant is w_4; changes after entry are in boldface font).

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{m_3}$	$\succ_{m_4}$	$\succ_{\mathbf{m_5}}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$	$\succ_{w_4}$	$\succ_{w_5}$
w_1	w_2	w_3	w_4	$\mathbf{w_3}$	m_2	m_1	$\mathbf{m_5}$	.	m_3
w_2	w_1	w_5	w_3		m_3	m_2	m_4	.	
		w_1			m_1		m_3	.	

Table 5: Entry of man participant

Prior to entry, MOMS assigns m_1 and m_2 to their second choices, and all other agents are unassigned (m_4 cannot be assigned since his first choice, w_4 does not participate. As a result, w_3, whose first choice prior to entry is m_4, cannot be assigned. And thus, m_3 cannot be assigned as well. Additionally, since m_3 is unassigned, w_1 can only be assigned to m_2).

We now consider the post-entry problem. It is easy to verify that MOMS matches m_1, m_2, and m_5 to their first choices, m_3 to w_5, while m_4 and w_4 remain unassigned. Specifically, both men who were assigned pre-entry are better off and both women who were assigned pre-entry are worse off.

Example 6 (Entry of woman participant). Table 6 describes the post-entry problem from Example 5 with w_6 being an entrant who joins the set of participants (the changes following entry are in boldface font).

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{m_3}$	$\succ_{m_4}$	$\succ_{m_5}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$	$\succ_{w_4}$	$\succ_{w_5}$	$\succ_{\mathbf{w_6}}$
w_1	w_2	w_3	w_4	$\mathbf{w_6}$	m_2	m_1	m_5	.	m_3	$\mathbf{m_4}$
w_2	w_1	w_5	w_3	w_3	m_3	m_2	m_4	.		$\mathbf{m_5}$
		w_1	$\mathbf{w_6}$		m_1		m_3	.		

Table 6: Entry of woman participant

After entry, MOMS assigns m_1 and m_2 with their second-ranked partners, leaving all others unassigned: w_4 who does not participate cannot be assigned in a safe matching, so m_4 cannot be assigned

(his first choice is w_4 who does not participate), so w_6 cannot be assigned, hence m_5 cannot be assigned, and so w_3 cannot be assigned, and therefore m_3 cannot be assigned, which means that w_5 cannot be assigned. Additionally, since m_3 is unassigned, w_1 can only be assigned to m_2. We note that prior to entry, MOMS assigned m_1 and m_2 to their first-ranked partners, and that w_1 and w_2 get their first-ranked partners post entry.

5 Computation of the Men-Optimal Maximal Safe Matching

As the empty matching is always safe, identifying some safe matching is extremely simple. However, in many applications, one may be interested in identifying a safe matching that maximizes the set of matched (assigned) individuals. Problems of this type are often complicated, and may be computationally infeasible (e.g., Manlove et al., 2002; Biró et al., 2022). Fortunately, as we show below, the men-optimal maximal safe matching can be computed efficiently.

To compute a maximal safe matching, one's first instinct may be to use the deferred acceptance algorithm in the market restricted to the set of participants. While this approach yields a P-matching that is individually rational and is not blocked, it is clear, upon reflection, that this approach is not satisfactory. For example, if the first-choice partners of all participants are nonparticipants, then the resulting matching will be potentially blocked.

It is tempting to think that this issue can be overcome by first truncating the preferences of each participant at their most preferred non-participating partner and then running deferred acceptance on the modified profile of preferences.[13] However, this approach also fails. To see this, consider the market from Example 1 with the set of participants $P = \{m_1, m_2, w_1\}$. Using the approach we outline above, we would get that m_2 is matched with w_1. However, we have shown in Example 1 that the empty matching is the only safe matching in this problem.

The failure of the truncation approach in this example arises due to the *endogeneity* of the restrictions posed by the requirement of no potential blocking with respect to the matching. Specifically, the more agents are unassigned, the more restrictive the requirement becomes. In general, problems where blocking is endogenous to the entire matching (and does not depend solely on the assignment of the members of the blocking coalition) are notoriously challenging, both in terms of existence and of computations. Specific examples include peer effects and matching with couples (see, e.g., Klaus and Klijn, 2005; Pycia, 2012; Kojima et al., 2013; Biró and Klijn, 2013).

Our approach builds on the idea of truncations, but we do so iteratively with the level of truncation becoming more conservative in each step. Intuitively, we first truncate and run deferred acceptance. If that does not result in a safe matching, we "eliminate" some (carefully chosen) problematic agents and determine that they would not be assigned. In the next iteration, we further truncate the preferences of the remaining agents (who were not eliminated), taking into account the possibility of potential blocks with newly eliminated agents. In practice, we implement this approach by constructing a sequence of auxiliary markets (and not through truncation).

Before turning to our approach, we address a recurring question about a simple alternative ap-

[13] Formally, for a man m, this truncation point would be $\max_{\succ_m}\{W \setminus P\}$, and analogously for women.

proach (which, unfortunately, fails). The question has several variants, which all boil down to asking if, after truncating participants' preferences at their most preferred non-participating partner, it is possible to identify the maximal support of a safe matching by looking at the identities of the partners that were not truncated (but not the specific order in which they are ranked). Example 7 rules out this possibility.

Example 7. Table 7 describes two problems with partial participation, both with three men and three women, and both have the same set of participants $P = \{m_1, m_2, w_1, w_2\}$. The only difference is that, in one case (labeled $\succ_{m_2}$) m_2 ranks w_1 first and w_2 second, and in the other case (labeled $\succ'_{m_2}$) m_2 ranks w_2 first and w_1 second. In both cases, the third ranked option is the nonparticipating woman, w_3.

$\succ_{m_1}$	$\succ_{m_2}$	$\succ'_{m_2}$	$\succ_{m_3}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$
w_1	w_1	w_2	.	m_2	m_2	.
w_3	w_2	w_1	.	m_1	m_1	.
w_2	w_3	w_3	.	m_3	m_3	.

Table 7: Example 7 preference profiles

In the first case, the men-optimal maximal safe matching assigns m_2 to w_1 and all other agents remain unassigned. In the second case, the men-optimal maximal safe matching assigns m_2 to w_2 and m_1 to w_1. Specifically, the support of the men-optimal maximal safe matching changes even though the identities of partners that are ranked higher than nonparticipants is identical in both cases.

5.1 The Iterative Market-Driven Elimination Algorithm

Related auxiliary matching market. Given a problem $\langle M, W, \succ, P\rangle$ and a set of agents E such that $(M \cup W)\backslash P \subseteq E$, the *related E-auxiliary two-sided matching market* (the related E-market, for short) consists of a set of men $\widehat{\mathrm{M}} := M$, a set of women $\widehat{\mathrm{W}} := (W\backslash E) \cup \bigcup_{m\in M,\, w\in W\cap E} \{w_m\}$, and preferences given by the profile $\succ'$ as follows:

1. For each non-eliminated woman $w \in W\backslash E$, $\succ'_w := \succ_w$.
2. For each eliminated woman $w \in W \cap E$ and $m \in M$, $\succ'_{w_m} := m$ if w is a potential match for m and otherwise $\succ'_{w_m} := \varnothing$.
3. For each non-eliminated participating man $m \in M\backslash E$, $\succ'_m$ is identical to $\succ_m$ with w_m replacing w for all $w \in E$.
4. For each eliminated participating man $m \in E \cap P$, $\succ'_m = \succ_m|_{W\backslash E}$.
5. For each non-participating man $m \in M\backslash P$, $\succ'_m$ is an arbitrary ranking such that all and only women in $W\backslash E$ are acceptable.

In the context of the Iterative Market-Driven Elimination algorithm (IMDE, Algorithm 1), we sometimes refer to members of E as *eliminated* and denote the corresponding set of agents in the

related E-market $E^R := (\widehat{M} \cup \widehat{W}) \setminus ((M \cup W) \setminus E)$. The set E^R includes an m-specific copy of each eliminated woman $w \in E$ for each man. This copy has m as its only acceptable partner if w is a potential match for m, and has no acceptable partner otherwise. Men in this set have their preferences modified to not include eliminated women as acceptable.

Given a problem $\langle M, W, \succ, P \rangle$ and a set of agents E such that $(M \cup W) \setminus P \subseteq E$, denote by $\mu^M[E]$ the matching outputted by the men-proposing deferred acceptance algorithm for the related E-market. We denote by π^E the operator that takes a matching for the related E-market and projects it onto the original problem:

$$\pi^E\left(\mu^M[E]\right)(i) := \begin{cases} \mu^M[E](i) & \text{if } i \in (M \cup W) \setminus E \\ \boxdot & \text{if } i \in E. \end{cases} \tag{6}$$

When E is such that $\mu^M[E]\left((M \cup W) \setminus E\right) = (M \cup W) \setminus E$, we have that $\pi^E\left(\mu^M[E]\right)$ is a partial matching.

Algorithm 1 Iterative Market-Driven Elimination (IMDE)

1: $E \leftarrow (M \cup W) \setminus P$
2: **while** $\Delta E := \left\{ i \in (M \cup W) \setminus E \mid \mu^M[E](i) \in E^R \right\} \neq \varnothing$ **do**
3: $\quad E \leftarrow E \cup \Delta E$
4: **return** $\mu_P^{\text{IMDE}} := \pi^E\left(\mu^M[E]\right)$

Theorem C.1. *IMDE terminates within* $O\left(|M||W|\left(|M| + |W|\right)\right)$ *steps.*

Proof. For a fixed E, the while loop requires running the men-proposing deferred acceptance algorithm in the related E-market. This takes $O(|M||W|)$ steps (which includes updating the related E-market representation).[14] Additionally, since in each step at least one agent is added to E, the loop is activated at most $|M| + |W|$ times. □

Theorem C.2. *IMDE outputs* μ_P^{MOMS}.

The proof of Theorem C.2 is deferred to Appendix A as it is involved. It proceeds in several steps. First, we show that for any E that includes all nonparticipants, if ΔE is empty (namely, non-eliminated agents are not matched to agents in E^R under $\mu^M[E]$), then $\pi^E\left(\mu^M[E]\right)$ is safe. Second, we show that if μ is safe, then $\pi^{(M \cup W) \setminus \operatorname{supp} \mu}\left(\mu^M\left[(M \cup W) \setminus \operatorname{supp} \mu\right]\right)$ is safe and men-optimal maximal among safe matchings with the same support. Third, we show that if, during the run of IMDE, a man is added to the eliminated set (i.e., $\Delta E^M := \Delta E \cap M \neq \varnothing$) there is no way to "save" this man from being eliminated by adding other agents to the set of eliminated agents. Fourth, we show that the same holds for women (using the fact that all men in $\Delta E^M = \varnothing$ cannot be "saved"). Fifth, we use the previous steps to show that any agent that is assigned under MOMS, is also assigned under

[14] A naive bound on the number of steps before the men-proposing deferred acceptance algorithm terminates is the number of men times the maximal number of acceptable women for a man (Gale and Shapley, 1962; Knuth, 1976). In any related E-market, the number of men is $|M|$ and no man finds more than $|W|$ women acceptable.

IMDE. Sixth, since IMDE outputs a safe matching, by Theorem S.2, any agent that is assigned under IMDE, is also assigned under MOMS. Thus, the supports of IMDE and MOMS coincide. Finally, since $\pi^{(M\cup W)\setminus \operatorname{supp}\mu_P^{\text{IMDE}}}\left(\mu^M\left[(M\cup W)\setminus \operatorname{supp}\mu_P^{\text{IMDE}}\right]\right)$ is optimal among safe matchings with the same support but also equal to μ_P^{IMDE}, we conclude that $\mu_P^{\text{IMDE}} = \mu_P^{MOMS}$.

Example 8. We consider the problem described in Table 8, where $P = (M \cup W)\setminus\{w_4\}$. We do not detail the preferences of w_4 as she is not in P (and therefore her preferences are irrelevant for our analysis).

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{m_3}$	$\succ_{m_4}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$	$\succ_{w_4}$
w_1	w_2	w_3	w_4	m_2	m_1	m_4	.
w_2	w_1	w_1	w_3	m_3	m_2	m_3	.
				m_1			.

Table 8: Example 8 preference profile

Figure 1 demonstrates the run of IMDE on the problem depicted in Table 8. According to Theorem C.2, IMDE should output MOMS. To confirm this prediction, we provide a direct argument that identifies MOMS.

Since w_4 is not a participant, she cannot be assigned in any safe matching. As a result, m_4, whose first choice is w_4, can also not be assigned in any safe matching (otherwise, m_4 and w_4 would form a potentially blocking pair). For the same reason, w_3—whose first choice is m_4—cannot be assigned in any safe matching. And so, m_3—whose first choice is w_3—cannot be assigned in any safe matching as well.

It is not hard to check that the matching in which w_1 and w_2 are matched with their first choices (and all others are unassigned) is safe. Furthermore, since m_3 is not assigned in any safe matching, w_1 must be assigned to m_2 in any safe matching that assigns her. Based on this fact, and since μ_P^{MOMS} assigns all agents that can be assigned in a safe matching (Theorem S.2), the matching in which w_1 and w_2 are matched with their first choices (and all others are unassigned) is μ_P^{MOMS}.

6 Additional Information about Nonparticipants

As we discuss above, the assumption that j is a potential match for i whenever $j \notin P$ can be interpreted as a conservative presumption that each nonparticipant is a potential match for all agents. In practice, certain potential blocks can be ruled out in some settings.[15] To accommodate such cases, let a *theory (of potential matches)* be a collection of sets $T = \{T_i\}_{i\in M\cup W}$ such that T_i is the set of acceptable partners for each $i \in P$, and a set of agents from the opposite gender otherwise.

Definition 10. Given a theory T, A partial matching μ is *potentially blocked according to* T by a pair $(m, w) \in (M \times W)$ if $\{m, w\} = \{i, j\}$, i is assigned, j is unassigned, $j \succ_i \mu(i)$, and $i \in T_j$.

[15] For example, the analyst or market designer may have information about the set of acceptable partners of some nonparticipants, or may be confident about the "reservation utility" they will achieve by matching with other non-participants. Alternatively, the analyst or market designer may know of some unmodeled sanctions (such as disbarment, license probation, or exclusion from future year matches) that make certain coalitions implausible.

Step 1. We initialize $E = (M \cup W) \backslash P = \{w_4\}$. In the related market, we create a copy of w_4 for each man in M. We use boldface font for members of E^R ("copies" of members of E).

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{m_3}$	$\succ_{m_4}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$	$\succ_{\mathbf{w_4^{m_1}}}$	$\succ_{\mathbf{w_4^{m_2}}}$	$\succ_{\mathbf{w_4^{m_3}}}$	$\succ_{\mathbf{w_4^{m_4}}}$
$\boxed{w_1}$	$\boxed{w_2}$	$\boxed{w_3}$	$\boxed{w_4^{m_4}}$	m_2	m_1	m_4	m_1	m_2	m_3	$\boxed{m_4}$
w_2	w_1	w_1	w_3	m_3	$\boxed{m_2}$	$\boxed{m_3}$				
				$\boxed{m_1}$						

Step 2. In Step 1, m_4 was assigned to a woman in E^R (the men-optimal stable matching in the related market, μ^M, is indicated by rectangles). As a result, in Step 2, m_4 is added to E. We remove women in E^R from his preferences.

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{m_3}$	$\succ_{\mathbf{m_4}}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$	$\succ_{\mathbf{w_4^{m_1}}}$	$\succ_{\mathbf{w_4^{m_2}}}$	$\succ_{\mathbf{w_4^{m_3}}}$	$\succ_{\mathbf{w_4^{m_4}}}$
w_1	w_2	w_3	$\boxed{w_3}$	$\boxed{m_2}$	$\boxed{m_1}$	$\boxed{m_4}$	m_1	m_2	m_3	m_4
$\boxed{w_2}$	$\boxed{w_1}$	w_1		m_3	m_2	m_3				
				m_1						

Step 3. In Step 2, w_3 was assigned to a man in E^R (the men-optimal stable matching in the related market, μ^M, is indicated by rectangles). As a result, in Step 3, w_3 is added to E. We create a copy of w_3 for each man (to facilitate reading, the table omits the copies designated for m_1 and m_2, who find no man acceptable), and omit copies of w_3 for the preference ranking of members of E^R (namely, m_4).

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{m_3}$	$\succ_{\mathbf{m_4}}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{\mathbf{w_3^{m_3}}}$	$\succ_{\mathbf{w_3^{m_4}}}$	$\succ_{\mathbf{w_4^{m_1}}}$	$\succ_{\mathbf{w_4^{m_2}}}$	$\succ_{\mathbf{w_4^{m_3}}}$	$\succ_{\mathbf{w_4^{m_4}}}$
$\boxed{w_1}$	$\boxed{w_2}$	$\boxed{w_3^{m_3}}$		m_2	m_1	$\boxed{m_3}$	m_4	m_1	m_2	m_3	m_4
w_2	w_1	w_1		m_3	$\boxed{m_2}$						
				$\boxed{m_1}$							

Step 4. In Step 3, m_3 was assigned to a woman in E^R. As a result, in Step 4, m_3 is added to E. We remove women in E^R from his preferences.

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{\mathbf{m_3}}$	$\succ_{\mathbf{m_4}}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{\mathbf{w_3^{m_3}}}$	$\succ_{\mathbf{w_3^{m_4}}}$	$\succ_{\mathbf{w_4^{m_1}}}$	$\succ_{\mathbf{w_4^{m_2}}}$	$\succ_{\mathbf{w_4^{m_3}}}$	$\succ_{\mathbf{w_4^{m_4}}}$
w_1	w_2	w_1		$\boxed{m_2}$	$\boxed{m_1}$	m_3	m_4	m_1	m_2	m_3	m_4
$\boxed{w_2}$	$\boxed{w_1}$			m_3	m_2						
				m_1							

Output. The men-optimal stable matching calculated in Step 4 (highlighted in rectangles) did not match any member of E^R. Therefore, the algorithm returns μ_P^{IMDE}, which matches m_1 with w_2 and m_2 with w_1, and leaves all other agents unassigned.

Figure 1: IMDE applied to the problem described in Example 8

Our conservative Definition 5 corresponds to the special case that T_j is maximal for each nonparticipant $j \notin P$ (i.e., T_j contains all agents of the opposite gender).

Definition 11. Given a theory T, a partial matching for $\langle M, W, \succ, P\rangle$, is *safe according to* T if it is an individually rational P-matching that is neither blocked nor potentially blocked by any pair according to T.

When T_j is maximal for each nonparticipant $j \notin P$, a matching is safe according to T if and only if it is safe. By contrast, when T_j is minimal (i.e., empty) for each nonparticipant $j \notin P$, any matching that is stable in the restriction of $\langle M, W, \succ\rangle$ to P is safe according to T.

Given a problem $\langle M, W, \succ, P\rangle$ and a theory T, let $\langle M^*, W^*, \succ^*, M \cup W\rangle$ denote the *related* T-*problem*, defined as follows:

1. $M^* := M \cup \{m^*\}$
2. $W^* := W \cup \{w^*\}$
3. $\succ_i^* := \succ_i$, if $i \in P$
4. $\succ_i^*$ ranks m^* or w^* (depending on i's gender) first, followed by a complete ranking of members of T_i (and only them), if $i \in (M \cup W) \backslash P$
5. $\succ_{m^*}^*$ and $\succ_{w^*}^*$ are arbitrary.

Let π^* be the operator that takes a partial matching μ of $\langle M, W, \succ\rangle$and projects it onto a corresponding partial matching of $\langle M^*, W^*, \succ^*\rangle$ such that $\pi^*(\mu)(i) = \mu(i)$ for all $i \in M \cup W$, and $\pi^*(\mu)(m^*) = \pi^*(\mu)(w^*) = \boxdot$.

Lemma 7. *In any safe matching in the related T-problem, m^* and w^* are unassigned. Furthermore, a matching μ is safe according to T if and only if $\pi^*(\mu)$ is safe in the related T-problem.*

Proof. Since m^* and w^* are nonparticipants in the related T-problem, they cannot be matched in any safe matching. Hence, nonparticipants (in the original problem) cannot be assigned in a safe matching of the related T-problem (since they rank m^* or w^* first). Hence, restricting attention to matchings that do not violate the above constraints, potential blocks in the related T-problem coincide with potential blocks according to T (by construction, no member of P finds m^* and w^* acceptable in the related T-problem). This completes the proof. □

6.1 Structure

Theorem T.1. *An agent cannot be matched in one matching that is safe according to T and assigned to be single in another. Formally, if μ and ν are safe in $\langle M, W, \succ, P\rangle$ according to T and $\mu(i) = i$, then $\nu(i) \in \{i, \boxdot\}$.*

Theorem T.2. *For any $\langle M, W, \succ, P\rangle$ and theory T, there exists a safe matching $\mu_{P,T}^{MOMS}$ that*

1. *assigns each agent that can be assigned in some matching that is safe according to T,*
2. *matches (to a partner) any agent that can be matched in some matching that is safe according to T, and*

3. *is weakly preferred by each assigned man over any matching that is safe according to T in which he is assigned.*

Theorem T.3. *Let μ, μ' be safe matchings for $\langle M, W, \succ, P\rangle$ according to T, then if all men in $\operatorname{supp}\mu \cap \operatorname{supp}\mu'$ weakly prefer μ to μ' then all women in $\operatorname{supp}\mu \cap \operatorname{supp}\mu'$ weakly prefer μ' to μ.*

Theorem T.4. *If μ and μ' are safe in $\langle M, W, \succ, P\rangle$ according to T, then agents in $\operatorname{supp}\mu \cap \operatorname{supp}\mu'$ are matched among themselves under both μ and μ'. Formally, $\mu'(\operatorname{supp}\mu \cap \operatorname{supp}\mu') = \operatorname{supp}\mu \cap \operatorname{supp}\mu'$.*

Theorem T.5. *$IMDE^* := IMDE(\langle M^*, W^*, \succ^*, M \cup W\rangle)$ outputs $\mu^{MOMS}_{P,T}$ in time polynomial in $|M| + |W|$.*

Using Lemma 7, Theorems T.1 to T.4 follow directly from Theorems S.1 to S.4, respectively, and Theorem T.5 follows from Theorems C.1 and C.2.

6.2 Additional Information

We now show that more informative theories expand the set of safe matchings in the set-inclusion sense.

Definition 12. Theory T' is *more informative* than theory T if $T'_i \subseteq T_i$ for each agent i.

Definition 13. We say that T is an *information-based* theory if, for each agent i, the set T_i includes (at least) all of i's acceptable partners.

Lemma 8. *If T' is more informative than T, then any matching that is safe according to T is also safe according to T'.*

Proof. Safety according to T' imposes a subset of the conditions required by safety according to T. □

Corollary T.1. *For any $\langle M, W, \succ, P\rangle$, if T' is more informative than T then*

1. *$\mu^{MOMS}_{P,T'}$ assigns each agent that is assigned under $\mu^{MOMS}_{P,T}$,*
2. *$\mu^{MOMS}_{P,T'}$ matches (to a partner) any agent that is matched under $\mu^{MOMS}_{P,T}$, and*
3. *$\mu^{MOMS}_{P,T'}$ is weakly preferred to $\mu^{MOMS}_{P,T}$ by each man that is assigned in both matchings.*

Corollary T.1 offers a new perspective for understanding the effects of entry (Theorem P.3) and participation (Theorem P.2). A new (non-participating) entrant i has the same effect as changing T_i in the post-entry problem from the empty theory to the full theory (holding fixed $\{T\}_{-i}$: maximal for nonparticipants, the set of acceptable partners for participants). For the effect of participation, consider an agent j joining the set of participants P. Even if we focus on P-matchings (which do not match the new participant j), the information about potential matches that is revealed by j's participation is equivalent to shrinking theory T_j from the maximal theory to j's acceptable partners. Corollary T.1 tells us that the men-optimal maximal safe matching in this domain already matches more agents and benefits (hurts) assigned men (women). And since $\mu^{MOMS}_{P\cup\{j\}}$ is maximal over a larger set of safe matchings, the results may be stronger.

Theorem T.6. *If T is information based and μ is safe according to T in $\langle M, W, \succ, P\rangle$, then μ is a submatching of a stable matching of $\langle M, W, \succ\rangle$.*

Proof of Theorem T.6. Consider the correctly specified theory T' (where T'_i consist of i's acceptable partners). Since T is information-based, T' is more informative than T, so any safe matching according to T is also safe according to T'.

Let μ denote a safe matching according to T'. Let $R := (M \cup W) \setminus \operatorname{supp} \mu$. Let μ_R denote a stable matching in the restriction of the market to R. Then, as in Lemma 3, the matching ν such that

$$\nu(i) = \begin{cases} \mu(i), & \text{if } i \notin R \\ \mu_R(i), & \text{if } i \in R. \end{cases}$$

is stable. The only difference to be mindful of is that partners ruled out by T' could not form blocking pairs of ν as T' is information based and ν is individually rational. □

If T is not an information-based theory, then the conclusion of the theorem does not necessarily hold. Example 9 shows that there may exist a matching that is safe according to T that all participating men strictly prefer to the men-optimal stable matching in the market (with full participation). Furthermore, these men may be matched even though they are unmatched in any stable matching of the market.

Example 9. Consider a market with two men $M = \{m_1, m_2\}$ and one woman $W = \{w_1\}$. Both men find w_1 acceptable, and w_1 prefers m_1 to m_2 to remaining single. The unique stable matching in this market has m_1 matched to w_1 and m_2 remaining unmatched. Now, consider $\langle M, W, \succ, P\rangle$ where $P = \{m_2, w_1\}$. The only safe matching in this market is the empty matching. Now, Let T be the theory such that $T_{m_1} = \varnothing$. Then the matching that assigns m_2 to w_1 is safe with respect to T.

We note that when T is information based, our results on the effects of entry and participation (Section 4) generalize naturally as above. The effects are more complex when certain potential blocks are ruled out on other grounds. (To see this, consider the full participation case of Example 9, recalling that under full participation MOMS is a stable matching.)

7 Discussion

Motivated by our experience designing the Israeli Law Internship Match, a marketplace that generates a partial matching due to partial participation, this paper makes three main contributions. First, conceptually, we propose partial matchings to address the question of partial participation in matching markets. Second, theoretically, we introduce *safety*—a solution concept for two-sided matching problems with partial participation—and study its implications. Third, concretely, we provide a computationally feasible algorithm for computing a maximal safe matching (which assigns each agent that can be safely assigned). Our work opens the way for future research on all three fronts.

On the conceptual front, although the current paper applies our idea to one-to-one two-sided matching problems with partial participation, there are many other important settings in which partial participation would yield new insights. Examples include housing markets (Shapley and Scarf,

1974), object allocation (Abdulkadiroğlu and Sönmez, 1998), many-to-one two-sided matching markets (Roth, 1985a), and matching with contracts (Crawford and Knoer, 1981; Hatfield and Milgrom, 2005; Kelso and Crawford, 1982; Roth, 1985a).

On the theoretical front, we note that safety, our solution concept, requires robustness against off-marketplace matches taking a worst-case informational standpoint. We have shown in Section 6 how to relax this robustness requirement using theories. But, theories are not the only possible relaxation. In particular, theories make assumptions on specific agents, but do not make any assumption about the broader market. (Roth and Shorrer, 2021, lies at the other extreme, as they fully specify how the broader market operates.) Making assumptions about the broader market may provide new insights and expand the set of possible matchings.

Additionally, we generalize many classic results on stable matchings (with full participation) to safe partial matchings in markets with partial participation. These include the existence of a side-optimal maximal safe matching, structural results in the spirit of the Lone Wolf Theorem, and, importantly, comparative statics on various forms of entry. As mentioned in the introduction, we left out the study of incentives, as it requires taking a stance on the outcomes of unassigned agents, which is something we avoided throughout the paper (for example, Theorem S.3, which discusses the conflict of interests between genders, makes no assertion with respect to agents that are assigned in one safe matching but not in the other). There are at least two possible routes towards a constructive result with respect to incentives. The first approach is to rely on sufficiently restrictive theories (an extreme example is a theory that rules out matching outside the marketplace). The second is a statistical approach that makes distributional assumptions (e.g., in the spirit of Roth and Rothblum, 1999).

Finally, on the concrete front, our approach breaks new ground for market designers confronting an all too common reality: reform typically begins with a coalition of allies—some stakeholders that are ready for centralization and trust that a new market design will make positive impact—while others remain reserved and refuse to join the marketplace, at least initially.[16] Rather than convincing all stakeholders to participate (or, often, failing to do so), we propose an alternative "startup approach" for market design. And, we back this proposal with IMDE—a computationally feasible algorithm that matches a maximal set of participants—and ways to modify it taking into account additional information that might be available to the designer in practice (Section 6). We hope that, like us, others will also find our startup approach useful as a stepping stone towards full participation.

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[16] For examples see Roth (2002, 2018) and Sönmez (2023).

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A Omitted Proofs

A.1 Proof of Lemma 4

We prove the lemma in steps.

Step 1. ($\mu \vee \mu'$ is a partial matching): We need to show that, for each woman w, the set of men such that $\max_{\succ_m^*}\{\mu(m), \mu'(m)\} = w$ contains at most one man (otherwise, two men are assigned to the same woman, in violation of Definition 1). Towards contradiction, assume that there exists two distinct men, m, m', such that $\max_{\succ_m^*}\{\mu(m), \mu'(m)\} = w$ and $\max_{\succ_{m'}^*}\{\mu(m'), \mu'(m')\} = w$. Since w can only match one man in each partial matching, we can assume without loss of generality, that $\mu(m) = w$ $\mu'(m') = w$. We furthermore assume, without loss of generality, that $m \succ_w m'$. Then, if m is assigned in μ', the pair (m, w) is a blocking pair (since $\max_{\succ_{m'}^*}\{\mu(m'), \mu'(m')\} = w$ implies $w = \mu(m) \succ_m \mu'(m)$). Otherwise, if m is unassigned in μ', (m, w) form a potentially blocking pair, contradicting safety of μ'.

Step 2. ($\text{supp}\, \mu \vee \mu' = \text{supp}\, \mu \cup \text{supp}\, \mu'$): By construction , any man m that is assigned under either μ or μ' is also assigned under $\mu \vee \mu'$ (as $\boxdot = \min_{\succ_m^*}$). We now show that if $w \in W$ is assigned under either matching, she is also assigned under $\mu \vee \mu'$.

First, if $\mu(w) = w$ or $\mu'(w) = w$, then by construction $(\mu \vee \mu')(w) \neq \boxdot$.[17] Next, assume that $\mu(w_0) = m_0 \in M$, $\mu'(w_0) = \boxdot$, and (towards contradiction) $(\mu \vee \mu')(w_0) = \boxdot$. Since $\mu'(w_0) = \boxdot$ and $\mu'(m_0) \succ_{m_0}^* \mu(m_0) = w_0$, we have $\mu'(m_0) = w_1 \in W$ with $w_1 \succ_{m_0} w_0$ (specifically, $w_1 \neq w_0$). Since μ is safe, and $w_1 \succ_{m_0} w_0$, it must be the case that $\mu(w_1) = m_1 \in M$ and $m_1 \succ_{w_1} m_0$ (specifically, $m_1 \neq m_0$). We can proceed this argument to get a sequence of men and women such that

$$\mu'(m_i) = w_{i+1} \succ_{m_i} w_i = \mu(m_i),$$

and

$$\mu(w_{i+1}) = m_{i+1} \succ_{w_{i+1}} m_i = \mu'(w_{i+1}).$$

The sequence $< w_0, m_0, w_1, m_1, \cdots >$ draws elements from the finite set $M \cup W$, and so the same element must repeat multiple times. Consider the first instance in the sequence in which some agent appears for the second time (we denote that agent by i^*). If $i^* = w_0$, then its predecessor (in the second occurrence) cannot be m_0 (as this would contradict w_0 being the first recurrence), so we find that w_0 is matched with two distinct men in μ and μ', a contradiction. Otherwise, i^* has distinct predecessor and successor in its first occurrence and its predecessor in the second occurrence must be different from both of them (as otherwise we get a contradiction to i^* being the first recurrence). Thus, we find that i^* has three distinct assignments under μ and μ', a contradiction.

Lastly, assume that $\mu(w_0) = m_0 \succ_{w_0} \hat{m} = \mu'(w_0)$ where $m_0, \hat{m} \in M$ and (towards contradiction) $(\mu \vee \mu')(w_0) = \boxdot$. Then, using the same construction,[18] if $i^* \neq w_0$ we have the same contradiction. Otherwise, $i^* = w_0$, but then the predecessor of w_0's second occurrence is a man $m^* \in M$ for whom

[17] In fact, by Theorem S.1 $\mu \vee \mu'(w) = w$.

[18] For the first step, we rely on $m_0 \succ_{w_0} \hat{m}$ rather than on $\mu'(w_0) = \boxdot$.

$w_0 = \max_{\succ_{m^*}} \{\mu(m^*), \mu'(m^*)\}$ and so $\mu \vee \mu'(m^*) = w_0$, hence $\mu \vee \mu'(w_0) = m^*$, a contradiction.

Step 3. ($(\mu \vee \mu')(\cdot)$ is safe): That μ is an individually rational P-matching is inherited from μ and μ' by construction. Next, assume (m, w) forms a blocking pair. Since w is assigned, she must be assigned in one of $\mu(w)$ or $\mu'(w)$. Assume, without loss, that $(\mu \vee \mu')(w) = \mu(w)$. Then if $\mu(m) = \boxdot$, then the pair (m, w) forms a potentially blocking pair of μ, and otherwise it forms a blocking pair (as m prefers w to $\max_{\succ_m^*}\{\mu(m), \mu'(m)\}$).

Next, assume (m, w) forms a potentially blocking pair and m is unassigned. Then $\mu(m) = \mu'(m) = \boxdot$, and w is assigned. Without loss of generality, assume $(\mu \vee \mu')(w) = \mu(w)$. Then (m, w) is a potentially blocking pair of μ, a contradiction to μ being safe.

Finally, assume (m, w) forms a potentially blocking pair and w is unassigned. By Step 2, $\mu(w) = \mu'(w) = \boxdot$. Without loss of generality, assume $(\mu \vee \mu')(m) = \mu(m)$. Then (m, w) is a potentially blocking pair of μ (that w is a potential match for m is implied by them being a potentially blocking pair for $\mu \vee \mu'$). □

A.2 Proof of Theorem C.2

Definition 14. The set E is *terminal* if $(M \cup W)\backslash P \subseteq E$ and $\Delta E := \left\{i \in (M \cup W)\backslash E \mid \mu^M[E](i) \in E^R\right\}$ is empty.

Lemma A.1. *If E is terminal, then $\mu' := \pi^E\left(\mu^M[E]\right)$ is safe.*

Proof. μ' is a partial matching since E is terminal. Furthermore, since E is terminal, $(M \cup W)\backslash P \subseteq E$, and so μ' is a P-matching. A blocking pair of μ' would also block $\mu^M[E]$ in the E-related market, but $\mu^M[E]$ is stable in that market. Similarly, any violation of individual rationality in μ' would also violate the individual rationality of $\mu^M[E]$ in the E-related market, contradicting its stability. Finally, if μ' is potentially blocked by a pair, then the assigned member of the pair is in $(M \cup W)\backslash E$ and the unassigned member is in E. If the assigned member is a man m and the unassigned member is a woman w, then (m, w_m) block $\mu^M[E]$. Alternatively, if the assigned member is a woman w and the unassigned member is a man m, then $\mu^M[E](m) = m$ (the only individually rational option in the related E-market that is compatible with E being terminal), and w is acceptable to m, so both m and w prefer each other to their assignment. In either case, we arrive at a contradiction. □

Lemma A.2. *For any safe matching μ:*

1. *$((M \cup W)\backslash \operatorname{supp}\mu)$ is terminal,*
2. *$\mu' := \pi^{(M\cup W)\backslash \operatorname{supp}\mu}\left(\mu^M\left[(M \cup W)\backslash \operatorname{supp}\mu\right]\right)$ is safe, and*
3. *$\mu' \vee \mu = \mu'$.*

Proof. To see that $(M \cup W)\backslash \operatorname{supp}\mu$ is terminal, consider the matching

$$\hat{\mu}(i) = \begin{cases} \mu(i), & \text{if } i \in \operatorname{supp}\mu \\ i, & \text{otherwise.} \end{cases} \tag{7}$$

We claim that $\hat{\mu}$ is stable in the $((M \cup W) \setminus \operatorname{supp} \mu)$-related market. To see this, note that violations of individual rationality can only arise from members of $\operatorname{supp} \mu$ (as only they are matched), but they are assigned to the same partner as in μ and their preferences are not changed.[19] Next, we show that no pair (m', w') can block $\hat{\mu}$. First, if both m' and w' do not belong to $\operatorname{supp} \mu$, then they do not find each other acceptable. If both belong to $\operatorname{supp} \mu$, then they would have blocked μ, contradicting its safety. If only m' is in $\operatorname{supp} \mu$, then w' is some $w_{m'}$ and (m', w) would be a potential blocking pair of μ, contradicting its safety. Alternatively, If only w' is in $\operatorname{supp} \mu$, then (m', w') would be a potential blocking pair of μ, contradicting its safety. In sum, all cases lead to a contradiction.

Next, we note that, by definition, $\mu^M [(M \cup W) \setminus \operatorname{supp} \mu]$ is the men-optimal stable matching in the $[(M \cup W) \setminus \operatorname{supp} \mu]$-related market. The Lone Wolf Theorem guarantees that agents that remained single under $\hat{\mu}$ are also single under $\mu^M [(M \cup W) \setminus \operatorname{supp} \mu]$. In particular, this includes all members of $(M \cup W) \setminus \operatorname{supp} \mu$, which implies that $(M \cup W) \setminus \operatorname{supp} \mu$ is terminal. Part 2 therefore follows from Lemma A.1 and Part 3 follows from the $\mu^M [(M \cup W) \setminus \operatorname{supp} \mu]$ being men-optimal and Lemma 4. □

Lemma A.3. *If during the run of IMDE, $m \in \Delta E$, E' is terminal, and $E \subset E'$, then $m \in E'$.*

Proof. We prove Lemma A.3 in steps.

Step 1. (The statement holds if $E' = E \cup \{m'\}$): Consider the case that $E' = E \cup \{m'\}$ for some man $m' \in M \setminus E$. We will show that $m' = m$. Toward contradiction, assume that $m' \neq m$, but E' is terminal. Then, the only difference between the related E-market and the related E'-market is that women in $\widehat{W} \cap E^R$ are unacceptable for m' in the latter.

Recall that $\mu^M[E]$ is the outcome of the men-proposing deferred-acceptance algorithm in the related E-market. Using the McVitie-Wilson algorithm (McVitie and Wilson, 1970), $\mu^M[E']$ can be calculated by first arriving to $\mu^M[E]$ and then, if m' is assigned to a member of E^R, erasing that woman and all other members of E^R from the preferences of m' and continuing to run DA with the modified preferences (see Kojima and Pathak, 2009, Appendix E).[20] Since, as the algorithm progresses, men can only be worse off, $\mu^M[E](m) \succcurlyeq_m \mu^M[E'](m)$. On the other hand, $\mu^M[E](m) = w_m$ for some $w \in E \cap W$. Since m is w_m's only acceptable partner, he cannot be assigned a partner he likes less under $\mu^M[E']$. Hence, $\mu^M[E'](m) = \mu^M[E](m)$, so $\mu^M[E'](m) \in E^R \subset E'^R$, a contradiction to E' being terminal.

Step 2. (The statement holds if $E' \setminus E \subset M$): As we have seen in Step 1, adding a man $m' \neq m$ to E keeps $\mu^M[E](m) = \mu^M[E \cup \{m'\}](m)$ so $m \in \Delta(E \cup \{m'\})$, and by induction, if $m \notin E' \setminus E$, we get $m \in \Delta(E \cup (E' \setminus E))$.

Step 3. (The statement holds for any E'): Let E' denote a superset of E. We have established in Step 2 that if $m \notin E' \setminus E \subseteq M$, then E' cannot be terminal. Assume, toward contradiction that $(E' \setminus E) \cap W \neq \varnothing$, E' is terminal, and $m \notin E'$.

[19] Other than renaming w as w_m in some men preferences, but in these instances the men are assigned to alternatives they prefer to all these women.

[20] Observe that in this case m' is assigned to his highest-ranked member of E^R.

Denote $\bar{E} := E \cup (E' \cap M)$. In words, $\bar{E}$ is consists of E and all *men* in E'. Then, $E' \backslash \bar{E} \subseteq W$. Furthermore, from Step 2, we know that $m \in \Delta\bar{E}$.

Now, we consider the market that results from removing all $w \in E' \backslash \bar{E}$ from the $\bar{E}$-related market. By the competition-effect result of Gale and Sotomayor (1985), all men are weakly worse off under the men-optimal stable matching then they are without the removal. But, since $m \in \Delta\bar{E}$, m is matched to some w_m prior to the removal, and since w_m is not removed and she ranks m first, m must be assigned to an option he likes at least as much as w_m after the removal. Altogether, the removal does not affect m.

On the other hand, since E' is terminal, for all $w \in E' \backslash \bar{E}$ and for any m, we have $\mu^M[E'](w_m) = w_m$. Thus, removing each w_m such that $w \in E' \backslash \bar{E}$ from the E'-related market does not change the men-optimal stable matching of other agents in the E'-related market. However, after the removal, the market is identical to the market from the previous paragraph,[21] which implies that

$$\mu^M[E'](m) = \mu^M[\bar{E}](m) \in \bar{E} \subset E',$$

a contradiction to E' being terminal. □

Definition 15. Denote $\Delta E^M := \Delta E \cap M$ and $\Delta E^W := \Delta E \cap W$.

Lemma A.4. *If during the run of IMDE, $w \in \Delta E$, E' is terminal, and $E \cup \Delta E^M \subset E'$, then $w \in E'$.*

Proof. Towards contradiction, assume that E' is terminal and $E \cup \Delta E^M \subset E'$, but $w \notin E'$. Since $w \in \Delta E$, there exists $m_0 \in E$ such that $\mu^M[E](m_0) = w$. Since $E \subseteq E'$, and E' is terminal (i.e., all members of E' are single in $\mu^M[E']$), we have $\mu^M[E'](m_0) = m_0$.

Since $w \in W \backslash E'$ is acceptable to m_0 in the E'-related market (as $\mu^M[E](m_0) = w$ and $m_0 \in E$) and $\mu^M[E'](m_0) = m_0$, by stability of $\mu^M[E']$ we have $\mu^M[E'](w) \succ_w m_0$. Specifically, $m_1 := \mu^M[E'](w)$ is a man in $M \backslash E'$.

Since $m_1 \succ_w m_0$ and $\mu^M[E](w) = m_0$, by stability of $\mu^M[E]$ we have $\mu^M[E](m_1) \succ_{m_1} w$. Specifically, $w_1 := \mu^M[E](m_1)$ is a woman. And, using the fact that $\Delta E^M \subset E'$, $w_1 \in W \backslash E$.

Since $w_1 \succ_{m_1} w$, $\mu^M[E'](m_1) = w \neq w_1$, and $w \notin E'$, by stability of $\mu^M[E']$ and E' being terminal we have that $w_1 \notin E'$ and $\mu^M[E'](w_1) \succ_{w_1} m_1$. Specifically, $m_2 := \mu^M[E'](w_1)$ is a man in $M \backslash E'$.

Repeating this argument, we get an alternating sequence of men and women in $M \cup W$

$$\langle m_0, w_0, m_1, w_1, m_2, \dots \rangle$$

such that $w_0 = w$, $\mu^M[E](m_i) = w_i$, and $\mu^M[E'](w_i) = m_{i+1}$. Since $M \cup W$ is finite, the sequence must recur. Assume that the first time that the sequence repeats itself is at w_j (which equals w_l for some $l < j$). Then $\mu^M[E](m_j) = w_j = w_l = \mu^M[E](m_l)$. This is a contradiction as, by minimality of w_j we have $m_l \neq m_j$ in contradiction to $\mu^M[E]$ being a matching. Assume therefore that the first time that the sequence repeats itself is at m_j (which equals m_l for some $l < j$). We will first address the case that

[21] In the first case women in $E' \backslash \bar{E}$ were not eliminated and they were removed directly. In the second case they were eliminated and split into many "copies" (w_m's), but then all of these w_m's were removed.

$l > 0$. Then, $\mu^M[E'](w_{l-1}) = m_l = m_j = \mu^M[E'](w_{j-1})$, a contradiction to $\mu^M[E']$ being a matching, as, by minimality of m_j, we have $w_{l-1} \neq w_{j-1}$. We now address the case that $l = 0$. In this case, $m_0 = m_j = \mu^M[E'](w_{j-1})$. But this is a contradiction to $\mu^M[E'](m_0) = m_0 \in M$ as $w_{j-1} \in W$. □

Proof of Theorem C.2. Toward contradiction assume $\mu_P^{\text{IMDE}} \neq \mu_P^{MOMS}$.

Case 1. We first consider the case that $\operatorname{supp} \mu_P^{MOMS} \setminus \operatorname{supp} \mu_P^{\text{IMDE}} \neq \varnothing$. Consider the first time during the run of IMDE in which an agent from $\operatorname{supp} \mu_P^{MOMS} \setminus \operatorname{supp} \mu_P^{\text{IMDE}}$ is added to E. Recall that since μ_P^{MOMS} is safe, $(M \cup W) \setminus \operatorname{supp} \mu_P^{MOMS}$ is terminal (Lemma A.2). Furthermore, since this is the first instance, prior to the addition we had $E \subseteq (M \cup W) \setminus \operatorname{supp} \mu_P^{MOMS}$. Therefore, any man that was added in this step of the algorithm belongs to any terminal history that includes E (Lemma A.3). Therefore, the agent must be a woman. However, since women are added to E alongside ΔE^M, the woman must belong to any terminal history that includes $E \cup \Delta E^M$ (Lemma A.4), and therefore to any terminal history that includes E (by Lemma A.3), a contradiction.

Case 2. Next, suppose that $\operatorname{supp} \mu_P^{MOMS} \setminus \operatorname{supp} \mu_P^{\text{IMDE}} = \varnothing$ but $\operatorname{supp} \mu_P^{\text{IMDE}} \setminus \operatorname{supp} \mu_P^{MOMS} \neq \varnothing$. In this case, $\operatorname{supp} \mu^{MOMS} \subsetneq \operatorname{supp} \mu_P^{\text{IMDE}}$, which contradicts Theorem S.2 (as μ_P^{IMDE} is safe by Lemma A.1).

Case 3. Finally, we consider the case that $\operatorname{supp} \mu_P^{MOMS} = \operatorname{supp} \mu_P^{\text{IMDE}}$. By definition,

$$\mu_P^{\text{IMDE}} = \pi^{(M \cup W) \setminus \operatorname{supp} \mu_P^{\text{IMDE}}} \left(\mu^M \left[(M \cup W) \setminus \operatorname{supp} \mu_P^{\text{IMDE}} \right] \right).$$

Since $\operatorname{supp} \mu_P^{MOMS} = \operatorname{supp} \mu_P^{\text{IMDE}}$, this implies

$$\mu_P^{\text{IMDE}} = \pi^{(M \cup W) \setminus \operatorname{supp} \mu_P^{MOMS}} \left(\mu^M \left[(M \cup W) \setminus \operatorname{supp} \mu_P^{MOMS} \right] \right).$$

However, denoting $E = (M \cup W) \setminus \operatorname{supp} \mu_P^{MOMS}$, by Lemma A.2,

$$\pi^E \left(\mu^M \left[(M \cup W) \setminus \operatorname{supp} \mu_P^{MOMS} \right] \right) \vee \mu_P^{MOMS} = \pi^E \left(\mu^M \left[(M \cup W) \setminus \operatorname{supp} \mu_P^{MOMS} \right] \right),$$

which, by Theorem S.2, can only hold if $\pi^E \left(\mu^M \left[(M \cup W) \setminus \operatorname{supp} \mu_P^{MOMS} \right] \right) = \mu_P^{MOMS}$. We therefore conclude that $\mu_P^{\text{IMDE}} = \mu_P^{MOMS}$ as required. □

B Additional Examples and Figures

Example A.1. Consider the market described in Table 9.

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{m_3}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$
w_2	w_1	w_3	m_1	m_2	m_3
w_1	w_2	w_1	m_3	m_1	
			m_2		

Table 9: Example 4 preference profile

The men-optimal stable matching, μ^M, has all men matched with their top choices. Recall that this matching coincides with MOMS under full participation. However, when the set of participants contains all agents but w_3, $P = \{m_1, m_2, m_3, w_1, w_2\}$, $\mu_P^{MOMS}(m_1) = w_1$ and $\mu_P^{MOMS}(m_2) = w_2$ (namely, assigned men are matched with their least preferred acceptable partner). To see why, note first that the matching is safe as there are no blocking pairs or violations of individual rationality, and the only unassigned potential match for any agent in P is m_3 (for w_1), but w_1 prefers her assignment under μ_P^{MOMS} over m_3. To see that men cannot be better off, note that since $m_3 \notin P$, if w_1 is assigned in a safe matching, her partner must be m_1. Since m_1 prefers w_2 to w_1, woman w_2 must be assigned to a man she likes more than m_1. The only possibility is m_2. A nearly identical argument shows that if any agent in P is assigned, all agents in P must be matched.

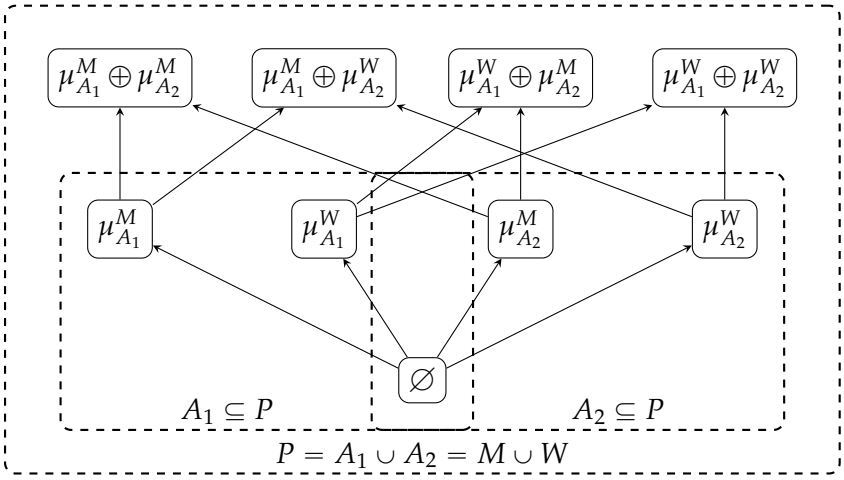


Figure A.1: Safe partial matchings for the market described in Example 1 under various sets of participants, P. When neither $A_1 \subseteq P$ nor $A_2 \subseteq P$, the only safe matching is the empty matching. Arrows indicate that one matching is a submatching of the other. $\mu^M_{A_i}$ and $\mu^W_{A_i}$ are the matchings that assign each man (resp. woman) in A_i to their most preferred partner.

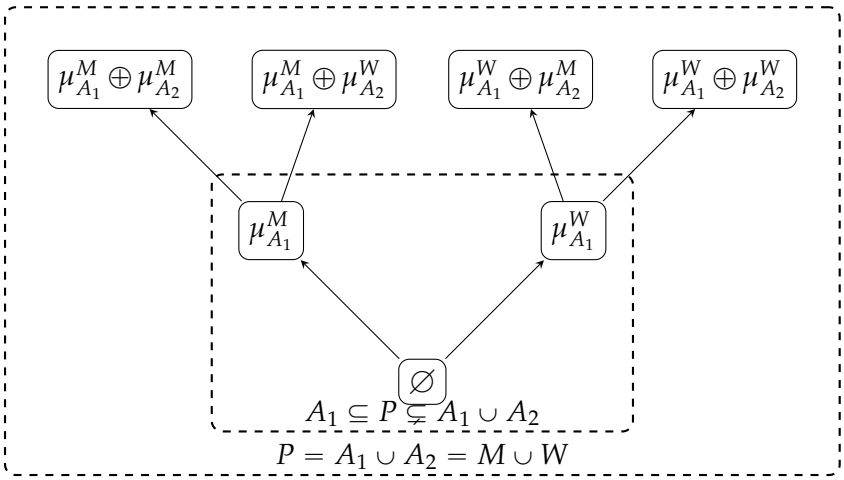


Figure A.2: Safe partial matchings for the market described in Example 3 under various sets of participants, P. When $A_1 \nsubseteq P$, the only safe matching is the empty matching. Arrows indicate that one matching is a submatching of the other. $\mu^M_{A_i}$ and $\mu^W_{A_i}$ are the matchings that assign each man (resp. woman) in A_i to their most preferred partner in A_i.