Where Bubbles Hide: Overvaluation and the Term Structure of Disagreement

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Abstract

We develop model-motivated, closed form empirical predictors of overvaluation persistence and level, above and beyond canonical shorting fees, idiosyncratic volatility, and one-period-ahead analyst dispersion. First, our model predicts that overvaluation correction over any horizon is proportional to the resolution share, the fraction of disputed value resolved within that horizon. Second, our model predicts that overvaluation levels equal disagreement times the present value of disputed cash flows. Bubbles therefore form where disagreement attaches to a large share of firm value, and rationally persist where those disputed cash flows are distant and slow to resolve. Taking these closed-form empirical predictors to the data using a new machine-learning measure of long-horizon disagreement, we find overvaluation persistence is predicted by the resolution share, and that overvaluation levels, as measured by existing mispricing proxies, have 8-12% of their cross-sectional variation explained by the present value of disputed cash flows. Together, our empirical and theoretical results suggest that it is not only disagreement, nor only duration, that affect overvaluation, but the term structure of disagreement.

1 Introduction

A canonical source of overvaluation is difficulty selling short, whether because of regulation, fees, or high idiosyncratic volatility creating shorting risk (Miller, 1977; Harrison and Kreps, 1978). For example, the collapse of the dot-com bubble has been traced in the literature to the expiration of lock-ups, effectively a form of short-selling restrictions expiring (Ofek and Richardson, 2003).

Yet many contemporary overvaluation debates seem to go beyond difficulty shorting. For example, in December 2024, Tesla and General Motors both have low shorting fees of 0.32% and 0.28%, respectively, and similar daily historic idiosyncratic volatility in 2023 of 2.8% and 1.8%, respectively.[1] Yet Tesla was in conversations regarding overvaluation more than General Motors was (Wall Street Journal, 2026). Why?

This paper seeks to understand sources of overvaluation in stocks beyond difficulty shorting (e.g, low shorting fees and low idiosyncratic volatility). We find, both theoretically and empirically, that overvaluation appears even in easy-to-short stocks. We rationalize this with a model demonstrating how disagreement leads to overvaluation even in easy-to-short, low-volatility stocks with arbitrageurs present. The primary novel contribution of our model, for which we also find evidence in the data, is the prediction that it is not just disagreement, but the term structure of disagreement, that predicts overvaluation levels and persistence. Intuitively, the harder a company's narrative is to falsify, the longer the impact of any initial disagreement can rationally endure. As a result, our results suggest that bubbles can rationally persist in easy-to-short stocks where the bulk of valuation comes from cash flows that near-term information cannot prove true or false.

We start with a motivating empirical exercise that extends Diether et al. (2002). Whereas this influential paper attributes their finding of disagreement predicting negative returns to pessimists not being able to express their view in the market, we find that their results also hold in low short fee, low idiosyncratic volatility stocks. Because canonical models of disagreement affecting valuation tend to rely on short sale constraints (Miller, 1977; Harrison and Kreps, 1978; Chen et al., 2002; Scheinkman and Xiong, 2003), this begs the question of what determines which easy-to-short stocks carry the overvaluation, and when it corrects.

We introduce a parsimonious model to find such closed-form empirical measures. The model gives a novel prediction for bubble persistence. It says that the quantitative persistence of a bubble from time t to some later time $t + h$ is directly proportional to how much disagreement gets resolved during that period. Intuitively, the greater this resolution share, the less of a bubble remains at time $t + h$. One can think of this statistic as similar to the duration of disagreement. This novel model-based prediction connects a large literature on duration and its impact on valuation (Van Binsbergen et al., 2012; Gormsen and Lazarus, 2023; Giglio et al., 2024) to a large literature on disagreement and its impact (Hong and Stein, 2007). Our contribution is that it is not simply duration, nor simply disagreement, that drive overvaluation, but, approximately speaking, the duration of disagreement.

In addition, our model predicts that the level of overvaluation at any point in time should

[1]We obtain short fee data from the Markit securities-lending database. Idiosyncratic volatility is the residual volatility from Fama-French three-factor model estimated on daily returns over the previous year.

be directly proportional to the present value of disputed cash flows even when shorting is costless. This result extends the classical empirical measure of disagreement – one-period-ahead dispersion (Diether et al., 2002) – to summarize the entire term structure of disputed cash flows.

Together, our model indicates that it is possible to predict not just which stocks are overvalued, but when that overvaluation should unwind, from the maturity of the cash flows investors disagree about.

We find empirical support for both predictions. Start with persistence. For each stock and each return window, we measure the resolution share: the fraction of the present value of disputed cash flows — which we call "present value disagreement" — that resolves within that window. We find that overvaluation predicts more negative returns when the resolution share is higher. For example, a one standard deviation higher present value disagreement predicts a 9% lower return over the next year if all of the stock's disputed cash flows realize *in that year* versus if all of the stock's disputed cash flows realize *after that year*. In other words, bubbles correct quickly when the disputed cash flows arrive quickly. When the disputed cash flows sit far in the future, little disagreement resolves, learning is slow, and the bubble persists. This finding holds even after controlling for literature-based measures of duration and disagreement.

Then consider levels. We find that the disputed present value of cash flows for a stock is a better predictor of overvaluation (as measured by the Cho and Polk (2024) and Van Binsbergen et al. (2023) measures) than canonical disagreement or short fees. Disputed present value remains a statistically significant predictor even after controlling for duration and idiosyncratic volatility. Precisely, model-predicted disputed present value explains 8% to 12% of overvaluation levels cross-sectionally, depending on the measure of overvaluation used. For comparison, the canonical Diether et al. (2002) disagreement measure explains less than 0.001% of variation in overvaluation.

To implement these model-motivated empirical measures, we develop a novel machine-learning based methodology that may be of interest in its own right. Measuring the term structure of disagreement is difficult in general due to the lack of subjective cash flow expectations at medium and long horizons. For example, most firms have little analyst coverage in I/B/E/S for both earnings per share (EPS) forecasts at horizons beyond two years long-term growth (LTG) expectations. Thus, we develop a new methodology based on partial least squares that leverages both subjective cash flow expectations and realized future cash flows to measure disagreement at long horizons, enabling us to compute present value disagreement for a wide cross section of firms. Intuitively, the method identifies "disputed states": latent factors that both predict future realized cash flows and explain

disagreement across analysts. For example, one disputed state may represent the current state of AI technology, which affects future cash flows and about which analysts disagree. The component of future cash flows driven by these disputed states is exactly the disputed cash flows in the model. For robustness, we also report results for measures constructed from more standard I/B/E/S data, and find results that are qualitatively similar, albeit with less statistical power since those measures are populated for a far smaller set of stocks, motivating our novel empirical measure.

The motivating exercise, model, and empirical work contribute to several literatures. First, we contribute to the empirical literature on overvaluation (Diether et al., 2002; Brunnermeier and Nagel, 2004; Nagel, 2005; Van Binsbergen et al., 2023; Cho and Polk, 2024) by finding a model-motivated first-order driver (e.g. the duration of disagreement, and more precisely, the resolution share) for bubble persistence, and by finding a model-motivated sufficient statistic (i.e., present value disagreement) for overvaluation even with costless shorting and low idiosyncratic volatility. Our prediction that resolution share predicts persistence also differentiates the paper from empirical work on disagreement affecting valuation through betas (Hong and Sraer, 2013).

Second, we contribute to a theoretical literature on disagreement and overvaluation. Canonical models rely on an absence of short sales (Miller, 1977; Harrison and Kreps, 1978; Chen et al., 2002; Scheinkman and Xiong, 2003; Duffie et al., 2002), and newer models remove the short sale constraint but rely on limits to arbitrage through volatility (Hong and Sraer, 2016; Atmaz and Basak, 2018). We contribute primarily by using our model to find new empirical drivers of bubble levels and persistence.

The paper proceeds as follows. Section 2 describes the data. Section 3 performs the empirical motivating exercise, finding that canonical overvaluation measures predict returns even in low idiosyncratic volatility, easy-to-short stocks. Section 4 introduces the model and derives its results, particularly its finding that the term structure of disagreement (rather than one-period ahead disagreement) affects valuation; the model also provides closed-form empirical sufficient statistics. Section 5 describes the empirical implementation of these sufficient statistics, via a novel machine learning method to estimate long-run disagreement in cash flows. Section 6 presents the empirical results. Section 7 concludes.

2 Data

We use five sets of data in this paper: analyst earnings forecasts, realized earnings, stock-level valuation and mispricing measures, stock returns, and stock characteristics (short fees, idiosyncratic volatility, and cash-flow duration). The time period for our analysis is May

2002 through December 2024, which is constrained by the availability of the short fee data from Markit.

2.1 Analyst Earnings Forecasts

We use analyst-level and consensus-level earnings-per-share (EPS) forecasts from the I/B/E/S Summary and Detail History databases, obtained through WRDS.

Fiscal-year-one forecasts. To construct the disagreement measure from Diether et al. (2002), we use current-fiscal-year EPS forecasts. We use both the consensus (mean) forecast and standard deviation across analysts from the I/B/E/S Summary History statistical period snapshots.

Fixed-horizon forecasts. To construct our novel measure of long-horizon disagreement, we use fixed-horizon EPS forecasts. Within each quarter, we aggregate all forecasts issued by analysts at institution a (e.g., Goldman Sachs) for stock n for fiscal year y by taking the median. We then interpolate between analyst institution-stock-fiscal year forecasts to obtain fixed h-year ahead forecasts for $h = 1, \ldots, 5$ years. We use forecasts only up to five years ahead because coverage at longer horizons is extremely sparse.

2.2 Realized Cash Flows

We use realized EPS from the I/B/E/S Actuals database. For each stock-quarter we collect realized EPS at horizons of one through ten fiscal years ahead. We then interpolate between end-of-fiscal year EPS realizations to obtain fixed h-year ahead EPS realizations for $h = 1, \ldots, 10$ years.

2.3 Valuation and Mispricing Measures

We use three different valuation metrics.

Market-to-book ratio. We compute the market-to-book ratio as the ratio of the market equity (calculated from CRSP prices) to the book equity (from the Compustat Fundamental database).

Adjusted value. We use the adjusted value measure from Cho and Polk (2024) provided by the authors. Adjusted-value is $z(B/M) + z(\text{Profitability}) - z(\text{Beta})$, where $z(\cdot)$ denotes the z-score of a characteristic's cross-sectional rank. The logic is that high book-to-market stocks are more likely to be genuinely underpriced when they are also profitable and safe (low beta).

Since the raw adjusted value is high for underpriced stocks, we multiply it by -1 so that, like the other outcomes, larger values indicate higher valuations. Adjusted value is observed once a year in June through 2022.

Price wedges. We use the Fama-French five-factor price wedge from Van Binsbergen et al. (2023) provided by the authors. The price wedge is defined as the log deviation of price from fundamental value: $PW = -\log(\text{fundamental value/market price})$. Positive values indicate overpricing. Van Binsbergen et al. (2023) estimate portfolio-level wedges from long-horizon cash flows and a CAPM stochastic discount factor. They then map these portfolio-level wedges to firm-level wedges by projecting onto five characteristics motivated by the Fama-French five-factor model: size, book-to-market, profitability, investment, and momentum. The wedges are available monthly through December 2017.

2.4 Stock Returns

We obtain stock prices, returns, shares outstanding, and market equity from the CRSP monthly stock database.

2.5 Stock Characteristics

The three main stock-level characteristics we use are short fees, idiosyncratic volatility, and cash flow duration.

Short-sale costs. We obtain short-sale costs from the Markit securities-lending database beginning in 2002. In particular, we use the annualized indicative stock-borrow short fee.[2]

Volatility. We use Fama-French three-factor idiosyncratic volatility from the Chen and Zimmermann (2022) database.[3]

Cash-flow duration. We use the cash-flow duration measure from Gormsen and Lazarus (2023), which we reconstruct using the authors' replication code. Specifically, Gormsen and Lazarus (2023) estimate cross-sectional regressions of median I/B/E/S long-run EPS growth (LTG) on the percentile ranks of four characteristics (operating profitability, investment, payout, and market beta). Then they define the duration characteristic as the fitted value.[4]

[2]We winsorize the short fee at the 1st and 99th percentiles within each quarter.
[3]We winsorize the idiosyncratic volatility at the 1st and 99th percentiles within each quarter.
[4]We winsorize duration at the 1st and 99th percentiles within each quarter.

3 Motivating Evidence

To motivate our investigation of channels outside traditional limits to arbitrage through which disagreement can lead to overvaluation, we revisit the empirical result that equity research analyst disagreement negatively predicts returns from Diether et al. (2002) (henceforth DMS).

We construct the DMS disagreement measure using I/B/E/S analyst earnings per share (EPS) forecasts for the end of the current fiscal year. Specifically, we measure for stock n in month t the standard deviation of EPS forecasts across analysts divided by the absolute value of the mean EPS forecast[5]:

$$D_{n,t} \equiv \frac{\text{std}(EPS_{a,n,t})}{|\text{mean}(EPS_{n,t})|}, \tag{1}$$

We then regress the stock return in month $t+1$ on the DMS disagreement measure in month t:

$$r_{n,t+1} = \beta D_{n,t} + \boldsymbol{\gamma} \boldsymbol{X}_{n,t} + FE_t + \epsilon_{n,t+1}, \tag{2}$$

where $\boldsymbol{X}_{n,t}$ is a vector of contemporaneous control variables and FE_t are month fixed effects. Table 1 displays the summary statistics for all the variables used in this analysis.

				Percentiles				
	Obs	Mean	StDev	1%	25%	50%	75%	99%
$D_{n,t}$	609,566	0.183	1.507	0.000	0.016	0.036	0.098	2.333
Analyst Count	609,566	9.42	7.09	2.00	4.00	7.00	13.00	32.00
Short Fee (%)	609,566	0.912	3.122	0.218	0.310	0.375	0.500	13.233
Idiosyncratic Vol. (%)	609,566	1.840	1.243	0.421	1.007	1.499	2.278	6.505
Monthly Return (%)	609,566	0.92	13.21	-33.17	-5.46	0.74	6.87	37.93

Table 1: **Summary statistics for DMS disagreement regression.** This table presents summary statistics for the variables used in the DMS disagreement regression (2). $D_{n,t}$ is the DMS disagreement measure defined in (1). Analyst count is analyst coverage per stock-month pair in the included sample. Short fee is the Markit indicative stock-borrow fee (displayed as a percentage). Idiosyncratic volatility is the Fama-French three-factor idiosyncratic volatility from the Chen-Zimmerman database (displayed as a percentage). Monthly returns are from CRSP (displayed as a percentage). The time period for the sample is 2002Q2 to 2024Q4.

Table 2 reports the regressions results. Column (1) displays the results of the univariate regression with only the DMS disagreement measure $D_{n,t}$. As in DMS, we find a significant negative association between disagreement and next-month returns: a one percentage point

[5]Following DMS, we include only stock-month pairs with at least two analysts, since we need at least two analysts to compute the standard deviation.

increase in disagreement (i.e., $D_{n,t}$ rises from 0.01 to 0.02) predicts a 4.2 basis point lower return in the next month. DMS attribute this predictability to disagreement leading to overvaluation in the presence of short-sales constraints that subsequently corrects.

However, in Column (2), we find quantitatively similar negative return predictability after adding short fees and Fama-French three-factor idiosyncratic volatility as controls. Moreover, in Column (3), we continue to find quantitatively similar negative return predictability even after restricting the sample to stock-month pairs for which the short fee and idiosyncratic volatility are below their median values.

	(1)	(2)	(3)
$D_{n,t}$	-0.042^{**}	-0.033^{*}	-0.057^{**}
	(-2.26)	(-1.88)	(-2.25)
Short Fee		-12.185^{***}	
		(-6.44)	
Idiosyncratic Volatility		-6.135	
		(-0.68)	
Month Fixed Effects	Yes	Yes	Yes
Stock and month clustering	Yes	Yes	Yes
Obs.	609,566	609,566	228,142
Within R^2	0.0000	0.0012	0.0000

Table 2: **Disagreement predicts returns even after controlling for short fees and idiosyncratic volatility.** This table presents the results of regression (2) of month $t+1$ returns on the DMS disagreement measure defined in (1). Columns (1) displays the results of the univariate regression with only the DMS disagreement measure $D_{n,t}$. Column (2) adds short fees and Fama-French three-factor idiosyncratic volatility as controls. Column (3) restricts the sample to stock-month pairs for which the unstandardized short fee is below median (37.5 basis points) and unstandardized idiosyncratic volatility is below median (1.41 percent). All columns include month fixed effects. All numbers are presented in absolute terms (i.e., 0.01 is 1%). Parentheses contain t statistics from stock and month two-way clustered standard errors. Levels of significance are presented as follows: *p<0.1; **p<0.05; ***p<0.01.

These results suggest that disagreement can lead to overvaluation through channels outside traditional limits to arbitrage, and so motivates our theoretical analysis in the next section.

4 A New Model: The Term Structure of Disagreement and Bubbles

4.1 Setup

Time is finite: $0, 1, .., T$. There is a single risk-free asset with known risk-free rate R. There is a single firm with fixed equity supply S. Its cash flows at time t are given by:

$$Cash_t = Agreed_t + Load_t\theta + u_t,$$

where u_t is i.i.d normally distributed noise: $u_t \sim N(0, \sigma_u^2)$. $Agreed_t$ may be an arbitrary sequence of real-numbers. $Load_t$ may be an arbitrary sequence of non-negative real numbers.

Investors agree on the agreed component of cash flows, and on the load at time t on disputed cash flows θ.

Investors come in two groups. Both describe their belief about θ with a normal distribution. They share the same initial variance $\sigma_{\theta,0}^2 > 0$ over θ but they center it differently.

Arbitrageurs (there are $M_{arb} > 0$ of them) have a prior on θ of $\hat{\theta}_0^{arb}$. Long-only investors (there are $M_{opt} > 0$ of them) estimate it higher, at $\hat{\theta}_0^{\text{long-only}} + Gap_0$, where $Gap_0 \geq 0$. Both groups know each others' priors. As the name suggests, long-only investors cannot short. Arbitrageurs can short at cost $c \geq 0$. We will be particularly interested in the case where arbitrageurs can short costlessly, $c = 0$.

We note that the assumption that $Gap_0 \geq 0$ can be microfounded through mandates. In particular, the above setup comes out of a model where the population-level beliefs of long-only investors are on average identical in the population to those of the arbitrageurs, but the long-only investors are censored in holdings. Appendix B.3 derives this microfoundation, which allows us to interpret arbitrageur beliefs as the average belief in the population.

Each investor type has the same utility function, taking standard CARA form $U(W) = -exp(-\gamma W)$ where $\gamma > 0$ is the degree of risk aversion.

All investors learn about θ through observing cash flows each period. Each investor group updates in a Bayesian manner.

For notational convenience, define:

$$AgreedPV_t \coloneqq \sum_{n>t} \frac{Agreed_n}{R^{n-t}} \text{ and } DisputedPV_t \coloneqq \sum_{n>t} \frac{Load_n}{R^{n-t}}.$$

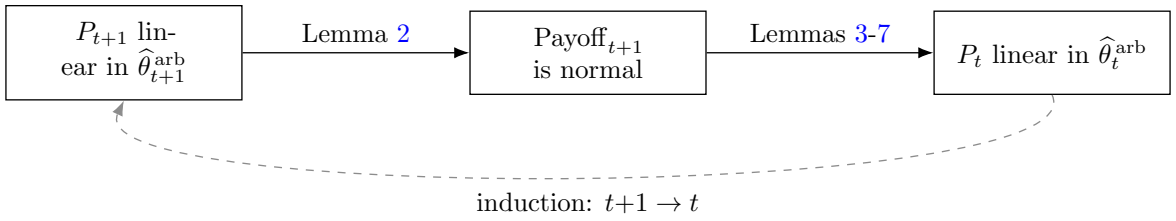


Figure 1: Proof strategy. The equilibrium price is found by backward induction from $P_T = 0$: if the resale price P_{t+1} is linear in beliefs, the one-period payoff is normal (Lemma 2), so CARA investors' demands clear the market (Lemmas 3-6) and return a price P_t that is again linear in beliefs (Lemma 7); and iterating downward covers every date. Lemma 1 supplies the common payoff variance and the deterministic valuation gap.

4.2 Results

Figure 1 illustrates the proof strategy to obtain a clean, closed form result for the price of the asset, from which we later obtain its bubble component. Equilibrium is built backwards by induction, and each step of that induction relies on three facts about a *single* date, proven in Lemmas 2 - 6. For expositional clarity, we state here the Lemmas but defer their proofs to Appendix B.1.

Lemma 1. *Consider any time t. The beliefs of investor type j at time t over θ are normally distributed. Further:*

1. *All investor types share the same posterior variance at time t. This posterior variance at time t is given by $\sigma^2_{\theta,t+1} = (1 - Resolve_t)\sigma^2_{\theta,t}$, where $Resolve_t = \frac{\sigma^2_{\theta,t} Load^2_{t+1}}{\sigma^2_{\theta,t} Load^2_{t+1} + \sigma^2_u}$.*
2. *The difference in mean beliefs of between the long-only investors and the arbitrageurs is given by Gap_t, which evolves deterministically according to $Gap_{t+1} = (1 - Resolve_t)Gap_t$, where $Resolve_t$ is defined as in (1).*

Lemma 2. *Fix a date t and suppose that next period's price is linear in the arbitrageurs' estimate:*

$$P_{t+1} = AgreedPV_{t+1} + DisputedPV_{t+1}\hat{\theta}^{arb}_{t+1} + Wedge_{t+1},$$

where $AgreedPV_{t+1}, DisputedPV_{t+1}$, and $Wedge_{t+1}$ are fixed numbers at time t.

Then in each group's eyes, the one-period payoff $Payoff_{t+1} = Div_{t+1} + P_{t+1}$ is normally distributed.

In particular, letting $Signal_{t+1} = Load_{t+1}\theta + u_{t+1}$,

$$Payoff_{t+1} = Known_t + Passthrough_t Signal_{t+1},$$

where $Known_t$ *is some constant known at time* t*, and* $Passthrough_t = 1 + DisputedPV_{t+1}Gain_t$ *is also known at time* t*.*

The expected value of $Payoff_{t+1}$ *in the eyes of investor type* j *is given by* $E_j[Payoff_{t+1}] = Known_t + Passthrough_t Load_{t+1}\hat{\theta}_t^j$*, and the variance of* $Payoff_{t+1}$ *is common across investor types and given by* $Var(Payoff_{t+1}) = Passthrough_t^2(Load_{t+1}^2\sigma_{\theta,t}^2 + \sigma_u^2)$*.*

As a result, the two investor types have valuations that differ by $ValGap_t = E_{long\text{-}only}[Payoff_{t+1}] - E_{arb}[Payoff_{t+1}] = Gap_t Exposure_t$ *where* $Exposure_t = Load_{t+1} + DisputedPV_{t+1}Resolve_t$*.*

Lemma 3. *Fix a date* t*. Suppose that each investor type perceives* $Payoff_{t+1}$ *(as defined in Lemma 2) to be normal, with common variance* $Var(Payoff_{t+1})$*.*

The utility-maximizing holdings for the long-only investors are given by:

$$q_{long\text{-}only} = \frac{max(E_{long\text{-}only}(Payoff_{t+1}) - RP_t, 0)}{\gamma Var(Payoff_{t+1})}$$

The utility-maximizing holdings for the arbitrageurs are given by:

$$q_{arb} = \frac{1}{\gamma Var(Payoff_{t+1})} \times \begin{cases} E_{arb}[Payoff_{t+1}] - RP_t, & RP_t \le E_{arb}[Payoff_{t+1}], \\ 0, & RP_t \in [E_{arb}[Payoff_{t+1}], E_{arb}[Payoff_{t+1}] + c], \\ E_{arb}[Payoff_{t+1}] + c - RP_t, & RP_t \ge E_{arb}[Payoff_{t+1}] + c. \end{cases}$$

Lemma 4. *At every date* t *the valuation gap is nonnegative:* $\mathrm{ValGap}_t \ge 0$*.*

Lemma 5. *Consider a trading date* t *with net supply* $S > 0$ *at which each investor type perceives the one-period payoff as normal with common variance* $Var(Payoff_{t+1})$*. Then at every market-clearing price, long-only investors hold a strictly positive position.*

Lemma 6. *Fix a date* t*. Define the per-period premium as* $Prem_t = RP_t - E_{arb}[Payoff_{t+1}]$*.*

Suppose that each investor type views $Payoff_{t+1}$ *as normally distributed with common variance* $Var(Payoff_{t+1})$*.*

Then the per-period premium has a closed form solution, given by three regions.

When $q_{arb} > 0$ *(equiv.* $Prem_t < 0$*),* $Prem_t = \frac{M_{long\text{-}only} ValGap_t - \gamma Var(Payoff_{t+1})S}{M_{long\text{-}only} + M_{arb}}$*.*

When $q_{arb} = 0$ *(equiv.* $Prem_t \in [0, c]$*),* $Prem_t = ValGap_t - \frac{\gamma Var(Payoff_{t+1})S}{M_{long\text{-}only}}$*.*

When $q_{arb} < 0$ *(equiv.* $Prem_t > c$*),* $Prem_t = \frac{M_{long\text{-}only} ValGap_t + M_{arb} c - \gamma Var(Payoff_{t+1})S}{M_{long\text{-}only} + M_{arb}}$*.*

Further $Prem_t$ *is continuous, increasing in* $ValGap_t$*, increasing in the short fee* c*, and decreasing in the supply* S*.*

Lemma 7. *At every date* t*, the price is affine in arbitrageurs' estimate of* θ*:* $P_t = Fund_t + Wedge_t$*, where* $Fund_t = AgreedPV_t + DisputedPV_t\hat{\theta}_t^{arb}$ *is the fundamental value of the firm*

(e.g. the present value of future cash flows) under arbitrageurs' beliefs, and $Wedge_t$ is a deterministic add-on.

In particular, the assumed condition in Lemma 2 holds at every date, and therefore the market clears at every date through Lemma 6.

Theorem 1. *In equilibrium, the time-t price is a linear function of the arbitrageurs' estimate of fundamental value, plus a deterministic wedge:*

$$P_t = Fund_t + Wedge_t,$$

where $Fund_t$ is as in Lemma 7.

The term $Wedge_t$ can be decomposed as:

$$Wedge_t = \underbrace{\sum_{t'=t}^{T-1} \frac{1}{R^{t'-t+1}} \frac{-\gamma Var(Payoff_{t'+1})S}{M_{long\text{-}only} + M_{arb}}}_{Risk\ Compensation} + Bubble_t,$$

where

$$Bubble_t = \sum_{t'=t}^{T-1} \frac{1}{R^{t'-t+1}} \left(Prem_{t'} + \frac{\gamma Var(Payoff_{t'+1})S}{M_{long\text{-}only} + M_{arb}} \right).$$

Further, $Bubble_t \geq 0$, and $Bubble_t = 0$ if and only if $ValGap_{t'} = 0$ at all dates $t' \geq t$.

Proof. By Lemma 7, the price at time t is $P_t = Fund_t + Wedge_t$, where $Fund_t = AgreedPV_t + DisputedPV_t \hat{\theta}_t^{arb}$ is the fundamental value of the firm (e.g. the present value of future cash flows) under arbitrageurs' beliefs, and $Wedge_t$ is a deterministic add-on.

We can use the last equation in the proof of Lemma 7 to iterate backwards and obtain $Wedge_t = \sum_{t'=t}^{T-1} \frac{Prem_{t'}}{R^{t'-t+1}}$.

We add and subtract $\frac{\gamma Var(Payoff_{t'+1})S}{M_{\text{long-only}}+M_{\text{arb}}}$ from each time within the sum to obtain the decomposition of $Wedge_t$ into risk compensation and bubble components.

We now justify why we call the first term risk compensation. Our interest in overvaluation is overvaluation arising from disagreement and resale value. When $ValGap_t = 0$, there is no disagreement over next period's payoff. Here, both arbitrageurs and long-only investors agree on the expected next period payoff. By Lemmas 3 and 6 ($ValGap_t = 0$ implies $Prem_t < 0$, so $q_{arb} > 0$), arbitrageurs hold a positive position. From Lemma 6, $Prem_t$ is therefore equal to $\frac{-\gamma Var(Payoff_{t+1})S}{M_{\text{long-only}}+M_{\text{arb}}}$. Thus, even in the absence of a valuation gap at time t, the wedge term would include the term $\frac{-\gamma Var(Payoff_{t+1})S}{M_{\text{long-only}}+M_{\text{arb}}}$. This is precisely the component that we are calling risk compensation and not including in our definition of the bubble. Intuitively, it

is downward pressure on the price arising from payoff uncertainty and risk-averse investors. Precisely, it is the part of the wedge that is not due to valuation disagreement.

The prior paragraph also illustrates that when $ValGap_{t'} = 0$, $Prem_{t'} = \frac{-\gamma Var(Payoff_{t'+1})S}{M_{\text{long-only}}+M_{\text{arb}}}$ and therefore $Bubble_t = 0$ when $ValGap_{t'} = 0$ for all $t' \geq t$. To prove the converse, from Lemma 4, $ValGap_{t'} \geq 0$ and therefore $Prem_{t'} \geq \frac{-\gamma Var(Payoff_{t'+1})S}{M_{\text{long-only}}+M_{\text{arb}}}$. If $Bubble_t = 0$, it therefore follows that $Prem_{t'} = \frac{-\gamma Var(Payoff_{t'+1})S}{M_{\text{long-only}}+M_{\text{arb}}}$, which from Lemma 6 can occur only if $ValGap_{t'} = 0$. □

For the remaining theoretical results, we focus on the costless shorting case ($c = 0$), since overvaluation with costless shorting forms the primary contribution of the paper. For completeness, Appendix B.2 derives the overvaluation level result for the case of costly shorting.

Theorem 2. *Suppose shorting is free: $c = 0$. Then the bubble has the closed form solution:*

$$Bubble_t = Gap_t DisputedPV_t \frac{M_{long\text{-}only}}{M_{long\text{-}only} + M_{arb}}.$$

And satisfies the following comparative statics:

1. *The bubble increases in the amount of disagreement about θ at time t, Gap_t.*
2. *The bubble increases in the present value of disputed cash flows, $DisputedPV_t$.*
3. *The bubble increases in the fraction of long-only investors.*

*The bubble does **not** depend on:*

- *The timing of disputed cash flows. Holding $DisputedPV_t$ fixed, duration should not matter.*
- *Risk aversion γ, float size S, or the payoff variance $Var(Payoff_{t+1})$. These move the price level through the risk-compensation term of Theorem 1, but net out of the bubble.*

Proof. At $c = 0$, from Lemma 3, arbitrageur demand is everywhere $q = \frac{E_j[Payoff_{t+1}]-RP_t}{\gamma Var(Payoff_{t+1})}$. As a result Lemma 6 tells us that $Prem_t = \frac{M_{\text{long-only}} ValGap_t - \gamma Var(Payoff_{t+1})S}{M_{\text{long-only}}+M_{\text{arb}}}$.

As a result, $Prem_t + \frac{\gamma Var(Payoff_{t'+1})S}{M_{\text{long-only}}+M_{\text{arb}}} = \frac{M_{\text{long-only}} ValGap_t}{M_{\text{long-only}}+M_{\text{arb}}}$.

We therefore obtain

$$Bubble_t = \frac{M_{\text{long-only}}}{M_{\text{long-only}} + M_{\text{arb}}} \sum_{t'=t}^{T-1} \frac{1}{R^{t'-t+1}} \left(ValGap_{t'}\right).$$

We now apply Lemma 2, in particular that $ValGap_t = Gap_t Exposure_t$. Plugging in and using that $Exposure_t = Load_{t+1} + DisputedPV_{t+1} Resolve_t$, we obtain:

$$
\begin{aligned}
Bubble_t &= \frac{M_{\text{long-only}}}{M_{\text{long-only}} + M_{\text{arb}}} \sum_{t'=t}^{T-1} \frac{1}{R^{t'-t+1}} \left(Gap_{t'}(Load_{t'+1} + DisputedPV_{t'+1} Resolve_{t'})\right) \\
&= \frac{M_{\text{long-only}}}{M_{\text{long-only}} + M_{\text{arb}}} \sum_{t'=t}^{T-1} \frac{1}{R^{t'-t+1}} \left(Gap_{t'} Load_{t'+1} + DisputedPV_{t'+1}(Gap_{t'} - Gap_{t'+1})\right) \\
&= \frac{M_{\text{long-only}}}{M_{\text{long-only}} + M_{\text{arb}}} \sum_{t'=t}^{T-1} \frac{1}{R^{t'-t+1}} \left(Gap_{t'}(Load_{t'+1} + DisputedPV_{t'+1}) - Gap_{t'+1} DisputedPV_{t'+1}\right) \\
&= \frac{M_{\text{long-only}}}{M_{\text{long-only}} + M_{\text{arb}}} \sum_{t'=t}^{T-1} \frac{1}{R^{t'-t+1}} \left(Gap_{t'}(RDisputedPV_{t'}) - Gap_{t'+1} DisputedPV_{t'+1}\right) \\
&= \frac{M_{\text{long-only}}}{M_{\text{long-only}} + M_{\text{arb}}} \sum_{t'=t}^{T-1} \left(\frac{Gap_{t'} DisputedPV_{t'}}{R^{t'-t}} - \frac{Gap_{t'+1} DisputedPV_{t'+1}}{R^{t'-t+1}}\right) \\
&= \frac{M_{\text{long-only}}}{M_{\text{long-only}} + M_{\text{arb}}} (Gap_t DisputedPV_t - \frac{Gap_{t+1} DisputedPV_{t+1}}{R} \\
&+ \frac{Gap_{t+1} DisputedPV_{t+1}}{R} - \frac{Gap_{t+2} DisputedPV_{t+2}}{R^2} + ...)
\end{aligned}
$$

The second line uses, from Lemma 1, that $Gap_{t'+1} = Gap_{t'} - Gap_{t'} Resolve_{t'}$. The fourth line uses the definition of $DisputedPV_{t'}$. We now observe that this is a telescoping sum, and therefore simplifies to $\frac{M_{\text{long-only}}}{M_{\text{long-only}} + M_{\text{arb}}} \left(Gap_t DisputedPV_t - \frac{Gap_T DisputedPV_T}{R^{T-t}}\right)$. Since $DisputedPV_T = 0$ (as there are no cash flows after time T), the overall expression simplifies to:

$$Bubble_t = \frac{M_{\text{long-only}}}{M_{\text{long-only}} + M_{\text{arb}}} Gap_t DisputedPV_t.$$

The comparative statics now read off the result. The bubble increases in all three of its terms. Holding these terms fixed, other terms do not affect its value. □

We now turn to bubble dynamics, and in particular abnormal payoffs. We define abnormal payoffs the usual way, as expected payoff above and beyond carry and risk compensation. We assume $c = 0$ to focus on the costless shorting case.

Definition 1. Fix a date t with $t \leq T - 1$. The expected abnormal payoff at date t is the expected one-period payoff in excess of the cost of carry and the risk premium:

$$A_t = E_{arb,t}[Payoff_{t+1}] - RP_t - \frac{\gamma Var(Payoff_{t+1}) S}{M_{long-only} + M_{arb}},$$

where the closed form for the risk premium (the last term) comes from Theorem 1.

Lemma 8. *At any date t, $A_t = -\frac{M_{long-only}}{M_{long-only}+M_{arb}} ValGap_t$.*

Theorem 3. *1. The abnormal payoff A_t is weakly negative at all times t, and is strictly negative at time t if and only if a disputed cash flow occurs the following period: $A_t < 0 \iff Load_{t+1} \neq 0$.*

2. Fix a horizon $H > 0$ and a time t. Define the present disputed value between times $t+1$ and $t+H+1$, the present disputed value beyond time $t+H+1$, and the news about θ delivered inside the window, as:

$$InsidePV_{t,H} = \sum_{j=1}^{H+1} \frac{Load_{t+j}}{R^j},$$
$$OutsidePV_{t,H} = \sum_{n>t+H+1} \frac{Load_n}{R^{n-t}},$$
$$\text{and } InsideNews_{t,H} = \sum_{j=1}^{H+1} \frac{Load_{t+j}^2}{\sigma_u^2},$$

respectively.

The cumulative expected underperformance over the window, per unit of the current bubble, equals the share of the disputed value paid inside the window, plus the share left unpaid times the fraction of disagreement resolved by the news delivered inside the window:

$$\frac{1}{Bubble_t} \sum_{h=0}^{H} \frac{-A_{t+h}}{R^{h+1}} = \frac{InsidePV_{t,H}}{InsidePV_{t,H} + OutsidePV_{t,H}} + \frac{OutsidePV_{t,H}}{InsidePV_{t,H} + OutsidePV_{t,H}} \left(1 - \frac{Gap_{t+H+1}}{Gap_t}\right).$$

In addition,

$$1 - \frac{Gap_{t+H+1}}{Gap_t} = \frac{\sigma_{\theta,t}^2 InsideNews_{t,H}}{1 + \sigma_{\theta,t}^2 InsideNews_{t,H}}.$$

Proof. By Lemma 8, $A_t = -\frac{M_{long-only}}{M_{long-only}+M_{arb}} ValGap_t$. Since each term is weakly greater than zero (Lemma 4), the entire expression must be weakly negative. Next, by Lemma 2, $ValGap_{t+1} = Gap_{t+1}\left(Load_{t+1} + DisputedPV_{t+1} Resolve_t\right)$. By Lemma 1 iterated, $Gap_{t+1} > 0$. Therefore $ValGap_{t+1} = 0$ if and only if the bracket is zero.

Recall that $Load_{t+1} \geq 0$ by assumption for all times t. If $Load_{t+1} = 0$, then $Resolve_t = 0$ by definition of $Resolve_t$ (Lemma 1), and therefore the term in brackets equals zero.

If $Load_{t+1} > 0$, then the first term of the bracket is positive; and the second term is non-negative, because $Load_t \geq 0$ for all t implies $DisputedPV_{t+1} \geq 0$ and $Resolve_t \in [0, 1]$.

Therefore the term in brackets is zero if and only if $Load_{t+1} = 0$; equivalently, the term in brackets is nonzero if and only if $Load_{t+1} \neq 0$. Combining this with the derivation from the top of the proof that $A_t \leq 0$ gives the result for (i).

To prove (ii), we take from the proof of Theorem 2 that $Bubble_t = \frac{M_{\text{long-only}}}{M_{\text{long-only}}+M_{\text{arb}}} \sum_{t'=t}^{T-1} \frac{1}{R^{t'-t+1}} (ValGap_{t'})$. From Lemma 8, $A_t = -\frac{M_{long-only}}{M_{long-only}+M_{arb}} ValGap_t$. Therefore $Bubble_t = \sum_{t'=t}^{T-1} \frac{A_{t'}}{R^{t'-t+1}}$. Rewriting as increments, $Bubble_t = \sum_{h=0}^{T-1-t} \frac{A_{t+h}}{R^{h+1}}$.

Split the sum at $h = H + 1$. Observe that in the split, for the sum after time $H + 1$, $\sum_{h=H+1}^{T-1-t} \frac{-A_{t+h}}{R^{h+1}} = \frac{1}{R^{H+1}} \sum_{j=0}^{T-1-(t+H+1)} \frac{-A_{t+H+1+j}}{R^{j+1}} = \frac{1}{R^{H+1}} Bubble_{t+H+1}$.

We therefore obtain $Bubble_t = \sum_{h=0}^{H+1} \frac{-A_{t+h}}{R^{h+1}} + \sum_{h=H+2}^{T-1-(t+H+1)} \frac{-A_{t+h}}{R^{h+1}}$, and, substituting the expression just derived in, $Bubble_t - \frac{1}{R^{H+1}} Bubble_{t+H+1} = \sum_{h=0}^{H+1} \frac{-A_{t+h}}{R^{h+1}}$. Dividing both sides by $Bubble_t$ gives

$$1 - \frac{1}{R^{H+1}} \frac{Bubble_{t+H+1}}{Bubble_t} = \frac{1}{Bubble_t} \sum_{h=0}^{H+1} \frac{-A_{t+h}}{R^{h+1}}.$$

We now consider the term $\frac{1}{R^{H+1}} \frac{Bubble_{t+H+1}}{Bubble_t}$. By the closed form expressed for the bubble at each time in Theorem 2, this expression is equal to $\frac{Gap_{t+H+1}}{Gap_t} \frac{DisputedPV_{t+H+1}}{DisputedPV_t R^{H+1}}$. Observe further that by definition of DisputedPV, $\frac{DisputedPV_{t+H+1}}{R^{H+1}} = \sum_{n>t+H+1} \frac{Load_n}{R^{n-t}} = OutsidePV_{t,H}$.

Finally, $DisputedPV_t = OutsidePV_{t,H} + \sum_{j=1}^{H+1} \frac{Load_{t+j}}{R^j} = OutsidePV_{t,H} + InsidePV_{t,H}$. Substituting into the expression for $\frac{1}{Bubble_t} \sum_{h=0}^{H+1} \frac{-A_{t+h}}{R^{h+1}}$ derived above, we obtain:

$$\begin{aligned}
\frac{1}{Bubble_t} \sum_{h=0}^{H+1} \frac{-A_{t+h}}{R^{h+1}} &= 1 - \frac{1}{R^{H+1}} \frac{Bubble_{t+H+1}}{Bubble_t} \\
&= 1 - \frac{Gap_{t+H+1}}{Gap_t} \frac{DisputedPV_{t+H+1}}{DisputedPV_t R^{H+1}} \\
&= 1 - \frac{Gap_{t+H+1}}{Gap_t} \frac{OutsidePV_{t,H}}{InsidePV_{t,H} + OutsidePV_{t,H}} \\
&= \frac{InsidePV_{t,H}}{InsidePV_{t,H} + OutsidePV_{t,H}} + \frac{OutsidePV_{t,H}}{InsidePV_{t,H} + OutsidePV_{t,H}} \left(1 - \frac{Gap_{t+H+1}}{Gap_t}\right),
\end{aligned}$$

where the last term comes from writing, in the third line, $\frac{Gap_{t+H+1}}{Gap_t} = 1 - \left(1 - \frac{Gap_{t+H+1}}{Gap_t}\right)$ and expanding.

Finally, to obtain the expression for $1 - \frac{Gap_{t+H+1}}{Gap_t}$, we appeal to Lemma 1, which says that

$Gap_{s+1} = (1 - Resolve_s)Gap_s$ and $\sigma_{\theta,s+1} = (1 - Resolve_s)\sigma^2_{\theta,s}$. Since both sides are positive, we obtain $\frac{Gap_{s+1}}{\sigma^2_{\theta,s+1}} = \frac{(1-Resolve_s)Gap_s}{(1-Resolve_s)\sigma^2_{\theta,s}} = \frac{Gap_s}{\sigma^2_{\theta,s}}$.

By induction, therefore, $\frac{Gap_{t+H+1}}{Gap_t} = \frac{\sigma^2_{\theta,t+H+1}}{\sigma^2_{\theta,t}}$.

From the conjugate normal update in the proof of Lemma 1, $\frac{1}{\sigma^2_{\theta,t+1}} = \frac{1}{\sigma^2_{\theta,t}} + \frac{Load^2_{t+1}}{\sigma^2_u}$.

We therefore obtain $\frac{1}{\sigma^2_{\theta,t+H+1}} - \frac{1}{\sigma^2_{\theta,t}} = \sum_{j=1}^{H+1} \frac{Load^2_{t+j}}{\sigma^2_u} = InsideNews_{t,H}$.

Moving the second term to the right and multiplying both sides by $\sigma^2_{\theta,t}$ gives $\frac{\sigma^2_{\theta,t}}{\sigma^2_{\theta,t+H+1}} = 1 + \sigma^2_{\theta,t} InsideNews_{t,H}$. Therefore $\frac{Gap_{t+H+1}}{Gap_t} = \frac{\sigma^2_{\theta,t+H+1}}{\sigma^2_{\theta,t}} = \frac{1}{1+\sigma^2_{\theta,t} InsideNews_{t,H}}$, completing the proof.

□

5 A New Measure of Long-Horizon Disagreement

In this section, motivated by the model in Section 4, we propose a novel machine-learning based measure of present value disagreement.

5.1 Setup

Theorem 2 decomposes the bubble term in valuations into disagreement about some latent state and the present value of the loading of the asset's future cash flows on that state. Taking this prediction to the data requires cash flow subjective forecasts over many future horizons. However, I/B/E/S EPS forecasts are not well-populated beyond the five-year horizon. Even longer-term forecasts such as the long-term earnings growth (LTG) forecasts are often reported by only a couple of analysts per stock-quarter.

Thus, we propose a new methodology for measuring long-horizon disagreement by combining analyst cash-flow forecasts with the maturity structure of subsequently realized cash flows.

Fix a quarter (and so drop quarter t subscripts). Let $F_{a,n,h}$ be analyst a's forecast at time t of stock n's cash flow at horizon h (i.e., at time $t + h$). Let $Y_{n,h}$ be the corresponding subsequently realized cash flow for stock n at horizon h.

Following the theory in Section 4, we model the realized future cash flow $Y_{n,h}$ as the sum

of an agreed cash flow term, a disputed cash flow term, and a noise term:

$$Y_{n,h} = \underbrace{\boldsymbol{\vartheta}_n' \boldsymbol{\beta}_h}_{\text{Agreed Cash Flow}} + \underbrace{\sum_{k=1}^{K} \underbrace{\theta_k}_{\text{Latent State } k} \cdot \underbrace{x_{n,k}\phi_k(h)}_{\text{Loading on State } k}}_{\text{Disputed Cash Flow}} + FE_h + \underbrace{u_{n,h}}_{\text{Noise}} \tag{3}$$

The agreed cash flow term is a linear function of the latent stock-specific state $\boldsymbol{\vartheta}_n \in \mathbb{R}^J$ and horizon-specific loadings $\boldsymbol{\beta}_h \in \mathbb{R}^J$ that are common to all stocks. The disputed cash flow depends on K latent states θ_k that are common to all stocks, thereby generalizing the single disputed state in the model in Section 4. For example, θ_k may represent the current (time-t) state of AI technology. Each stock has its own horizon-specific loading on θ_k: $x_{n,k}\phi_k(h)$. $\phi_k(h)$ captures how much state k impacts cash flows at horizon h for the average stock, and $x_{n,k}$ captures heterogeneity in loadings on state k across stocks. For example, if θ_k represents the current state of AI technology, then plausibly its cash flow impact will mostly occur at longer horizons: $\phi_k(h)$ is larger for higher h. Moreover, plausibly Nvidia is more exposed to θ_k than General Motors is: $x_{n,k}$ is larger for Nvidia than for General Motors.

We model analyst forecasts similarly, but with the crucial distinction that analysts can disagree about the value of each latent state θ_k:

$$F_{a,n,h} = \sum_{k=1}^{K} \theta_{a,k} \cdot x_{n,k}\phi_k(h) + FE_{n,h} + \varepsilon_{a,n,h}. \tag{4}$$

$\theta_{a,k}$ captures analyst a's belief about the value of latent state k. For example, analysts at J.P. Morgan versus Goldman Sachs may disagree about the current state of AI technology. Agreed cash flows are absorbed into the stock-horizon fixed effect $FE_{n,h}$. As in the model in Section 4, we assume that the state loadings $x_{n,k}\phi_k(h)$ are known and are common to all analysts.

Equations (3) and (4) set up a latent factor model that can be estimated in the cross-section of stocks within a quarter given a stock $\times$ horizon panel of realized cash flows and an analyst $\times$ stock $\times$ horizon panel of analyst forecasts. Crucially, these factor models can be estimated even if forecasts are not observed at all horizons for a given stock, as explained in Section 5.2. This methodology resembles partial least squares from machine learning: we identify latent factors that explain variation in forecasts across analysts that also forecast future realized cash flows.

Given these factor models in realized cash flows and analyst forecasts, we can estimate the empirical analogue of the present value disagreement term in the bubble term in Theorem 2.

Let $PV_{a,n}$ be analyst a's belief about the present value of the disputed cash flows for stock n:

$$PV_{a,n} = \sum_{h=1}^{H} \frac{1}{(1+r_f)^h} \sum_{k=1}^{K} \theta_{a,k} x_{n,k} \phi_k(h). \tag{5}$$

Then we can define present value disagreement — the analogue of $Gap_t \cdot DisputedPV_t$ in Theorem 2 — as the cross-analyst disagreement in $PV_{a,n,h}$:

$$PVD_n = \text{std}_a(PV_{a,n}). \tag{6}$$

For parsimony we deliberately focus only on the present value disagreement term in Theorem 2, as opposed to also allowing for heterogeneity across stocks in the ratio of long-only investor to arbitrageur mass.

We can also compute the empirical analogue of the inside present-value ratio from Theorem 3. Let analyst a's demeaned-versus-consensus expectation about the present value of the disputed cash flows up to horizon $\bar{h}$ be:

$$I_{a,n}\left(\bar{h}\right) = \sum_{h=1}^{\bar{h}} \frac{1}{(1+r_f)^h} \sum_{k=1}^{K} \left(\theta_{a,k} - \bar{\theta}_k\right) x_{n,k} \phi_k(h), \tag{7}$$

where $\bar{\theta}_k$ is the consensus value of latent state k (which may or may not equal θ_k). Similarly, we can define the empirical analogue of the demeaned-versus-consensus outside present-value ratio as

$$O_{a,n}\left(\bar{h}\right) = \sum_{h=\bar{h}+1}^{H} \frac{1}{(1+r_f)^h} \sum_{k=1}^{K} \left(\theta_{a,k} - \bar{\theta}_k\right) x_{n,k} \phi_k(h), \tag{8}$$

and the demeaned-versus-consensus total present-value ratio as

$$T_{a,n}\left(\bar{h}\right) = I_{a,n}\left(\bar{h}\right) + O_{a,n}\left(\bar{h}\right), \tag{9}$$

Then the squared disagreement across analysts about the total present value of the disputed cash flows is:

$$\mathbb{V}_a\left[T_{a,n}\left(\bar{h}\right)\right] = \mathbb{V}_a\left[I_{a,n}\left(\bar{h}\right)\right] + \mathbb{V}_a\left[O_{a,n}\left(\bar{h}\right)\right] + 2\,\text{Cov}_a\left(I_{a,n}\left(\bar{h}\right), O_{a,n}\left(\bar{h}\right)\right).$$

Thus, we can compute the empirical analogue of the inside present-value ratio as:

$$InsideRatio_n(h) = \frac{\mathbb{V}_a\left[I_{a,n}\left(\bar{h}\right)\right] + Cov_a\left(I_{a,n}\left(\bar{h}\right), O_{a,n}\left(\bar{h}\right)\right)}{\mathbb{V}_a\left[T_{a,n}\left(\bar{h}\right)\right]}. \tag{10}$$

For $K = 1$ latent state and $Cov_a\left(I_{a,n}\left(\bar{h}\right), O_{a,n}\left(\bar{h}\right)\right) = 0$, (10) simplifies exactly to the inside present-value ratio defined in Theorem 3. Note that depending on the sign of $Cov_a\left(I_{a,n}\left(\bar{h}\right), O_{a,n}\left(\bar{h}\right)\right)$, it is possible for $InsideRatio_n(h)$ to be greater than one or less than zero.

5.2 Identification and Estimation

Identification. The key objects (3) and (4) ($\theta_k, \theta_{a,k}, x_{n,k}$, and $\phi_k(h)$) can be identified in the cross section of stocks in a single quarter given a stock $\times$ horizon panel of realized cash flows and an analyst $\times$ stock $\times$ horizon panel of analyst forecasts.

Within the analyst-stock-horizon forecast panel, identification of $\theta_{a,k}, x_{n,k}$, and $\phi_k(h)$ arises from each object creating similar patterns across two of the three dimensions. Within a stock-horizon, analysts' deviations from consensus reveal who is relatively optimistic or pessimistic, thereby identifying $\theta_{a,k}$. Across stocks for a given horizon, certain groups of analysts repeatedly appearing as optimistic or pessimistic reveals which firms load on the same state, thereby identifying $x_{n,k}$. Across horizons for a given stock and analyst, the recurrence of the same forecasted cash flow pattern reveals when the cash-flow effects associated with a given state arrive, thereby identifying $\phi_k(h)$.

The realized-cash-flow panel both extends and disciplines this identification strategy. Analyst forecasts can identify $x_{n,k}\phi_k(h)$ for stocks and horizons for which forecasts are observed. The realized cash flow panel disiplines these loadings by ensuring they explain realized future cash flows. Moreover, the realized cash flow panel can help fill in these loadings for stocks and horizons for which forecasts are not observed. Additionally, the agreed component $\boldsymbol{\vartheta}_n'\boldsymbol{\beta}_h$ absorbs systematic realized-cash-flow variation across stocks and horizons that is not associated with disagreement across analysts.

Thus, this method essentially identifies which time-t disagreements across analysts are associated with realized cash flows that subsequently arrived early versus late.

Estimation. We estimate the factor models (3) and (4) jointly in each quarter t using alternating least squares. To circumvent arbitrary variation in EPS across stocks due to variation in the number of shares outstanding, we scale all forecasts and realized cash flows by book equity per share. That is, we essentially use realized and forecasted return on equity

over different horizons.[6]

For the baseline analysis we use $K = 7$ disputed states and $J = 1$ agreed state. We use only stock-quarter pairs for which we observe forecasts by ten different analysts (pooled across all horizons), and only analyst-quarter pairs for which we observe at least twenty forecasts (pooled across all horizons). We use a total of $H = 10$ annual horizons in the realized cash flow panel. [7] The analyst forecast panel only extends to five annual horizons due to limited coverage thereafter in I/B/E/S. We use a fixed risk-free rate of $r_f = 4\%$ when computing present values (5). Appendix A.1 provides further estimation details.

6 Empirical Results

In this section, we use the measures constructed using the methodology in Section 5 to test the implications of the model in Section 4.

6.1 Higher Present Value Disagreement is Positively Associated with Higher Mispricing

To test if the empirical present value disagreement term from (6) is positively associated with high valuations, as implied by Theorem 2 in Section 4, we regress valuation metrics $V_{n,t}$ on present value disagreement, which we standardize within each quarter to have zero mean and unit standard deviation.

$$V_{n,t} = \beta PVD_{n,t} + \boldsymbol{\gamma}' \boldsymbol{X}_{n,t} + FE_t + \epsilon_{n,t}, \tag{11}$$

where $V_{n,t}$ is market-to-book ratio or one of two mispricing metrics: the Cho and Polk (2024) adjusted value or Van Binsbergen et al. (2023) Fama-French five-factor price wedge. All valuation metrics are signed so that higher values indicate higher valuation. FE_t is a quarter fixed effect. Table 3 displays the summary statistics for all variables used in this regression.

Table 4 displays the results of regressions of (11). In Panel A, we find that present value disagreement is strongly associated with higher valuation across all three valuation metrics. Given that these are univariate regressions, the cross-sectional (i.e., within) R^2s of around 7% to 12% of variation explained are high for the two mispricing metrics. In Panels B and C, we find quantitatively similar results when controlling for short fees, idiosyncratic volatility,

[6] To avoid extremely small denominators, we retain only stock-quarter pairs with at least \$0.50 in book equity per share.

[7] The realized cash-flow panel is unbalanced: for each stock we use only the horizons at which a future realized EPS is observed, and a stock need not be observed at all ten horizons to enter estimation.

				Percentiles				
	Obs	Mean	StDev	1%	25%	50%	75%	99%
$PVD_{n,t}$	75,079	0.271	0.617	0.000	0.045	0.118	0.289	2.007
$InsideRatio_{n,t}\left(\bar{h}=1\right)$	75,079	0.099	0.142	-0.191	0.029	0.080	0.139	0.647
$InsideRatio_{n,t}\left(\bar{h}=2\right)$	75,079	0.222	0.294	-0.458	0.075	0.182	0.325	1.306
$InsideRatio_{n,t}\left(\bar{h}=3\right)$	75,079	0.342	0.427	-0.696	0.134	0.297	0.498	1.887
$InsideRatio_{n,t}\left(\bar{h}=4\right)$	75,079	0.428	0.470	-0.806	0.204	0.411	0.617	2.054
$InsideRatio_{n,t}\left(\bar{h}=5\right)$	75,079	0.504	0.463	-0.862	0.294	0.522	0.717	1.944
$InsideRatio_{n,t}\left(\bar{h}=6\right)$	75,079	0.596	0.434	-0.746	0.408	0.624	0.797	1.955
$InsideRatio_{n,t}\left(\bar{h}=7\right)$	75,079	0.694	0.370	-0.482	0.530	0.724	0.879	1.838
$InsideRatio_{n,t}\left(\bar{h}=8\right)$	75,079	0.799	0.283	-0.044	0.673	0.822	0.937	1.649
$InsideRatio_{n,t}\left(\bar{h}=9\right)$	75,079	0.900	0.165	0.421	0.835	0.914	0.974	1.399
Duration	75,079	0.090	1.023	-2.089	-0.628	0.104	0.791	2.356
DMS Disagreement	75,079	0.403	6.222	0.007	0.040	0.073	0.159	3.807
Short Fee (%)	75,079	0.726	1.584	0.250	0.375	0.375	0.500	10.000
Idiosyncratic Vol. (%)	75,079	1.697	1.010	0.462	0.998	1.452	2.107	5.250
Market-to-Book	72,426	3.742	16.694	0.445	1.408	2.258	3.803	21.260
Adjusted Value	17,278	0.415	1.292	-2.454	-0.470	0.375	1.243	3.793
Price Wedge	71,121	0.057	0.203	-0.372	-0.100	0.073	0.224	0.425
Annual Return (%)	75,079	12.16	46.53	-79.44	-14.58	8.83	32.90	161.10

Table 3: **Summary statistics valuation and return-prediction regressions.** This table presents summary statistics for the variables used in the return-predictability and valuation regressions. $PVD_{n,t}$ is the empirical present value disagreement measure from (6), reported before standardization. $InsideRatio_{n,t}\left(\bar{h}\right)$ is the empirical inside present value ratio measure from (10) for horizon $\bar{h}=1,\ldots,9$. Duration is the Gormsen and Lazarus (2023) cash flow duration. DMS disagreement is the Diether et al. (2002) disagreement measure used in Section 3. Short fee is the Markit indicative stock-borrow fee. Idiosyncratic volatility is the Fama-French three-factor idiosyncratic volatility from the Chen-Zimmerman database. Market-to-book is the market-to-book ratio. Adjusted value is the Cho and Polk (2024) adjusted value measure and is observed once a year in June. Price wedge is the Van Binsbergen et al. (2023) Fama-French five-factor price wedge and is observed quarterly. Annual returns are from CRSP and are displayed as percentages. The sample period is 2002Q2 to 2012Q4, where the end date is constrained because ten years of future realized cash flows are required to construct $PVD_{n,t}$.

the Diether et al. (2002) disagreement measure from Section 3, and Gormsen and Lazarus (2023) cash-flow duration, as well as when restricting the sample to stock-quarter pairs for which the short fee and idiosyncratic volatility are below their medians, respectively.

These results provide empirical support for Theorem 2: higher disagreement about disputed cash flows is associated with higher valuations.

6.2 Overvaluation Corrects More Quickly When Resolution Share is Larger

To test if overvaluation corrects more quickly when the resolution share is higher (i.e., when the inside present value ratio is higher), as implied by Theorem 3 in the model in Section 4, we regress future returns on present value disagreement interacted with the inside present value ratio. Specifically, we regress the annual geometric mean return from years $t+1$ to $t+\bar{h}$ ($\bar{r}_{n,t+1:t+\bar{h}}$) on present value disagreement interacted with the inside present value ratio:

$$\bar{r}_{n,t+1:t+\bar{h}} = \beta_1 PVD_{n,t} + \beta_2 PVD_{n,t} \cdot InsideRatio_{n,t} + \boldsymbol{\gamma}' \boldsymbol{X}_{n,t} + FE_t + \epsilon_{n,t+1:t+\bar{h}}, \tag{12}$$

where $\boldsymbol{X}_{n,t}$ is a vector of control variables, and FE_t is a quarter fixed effect. Theorem 3 predicts that $\beta_2 < 0$. Table 3 displays the summary statistics for all variables used in this regression.

Table 5 displays the results of regressions of (12). Panel A displays the results of the regression with no controls. Across all horizons $\bar{h} = 1, 5$, and 9, we find that β_2 is significantly negative: present value disagreement predicts returns more negatively when the inside present value ratio up to horizon $\bar{h}$ is higher. This pattern continues to hold when we control for short fees, idiosyncratic volatility, the Diether et al. (2002) disagreement measure from Section 3, and Gormsen and Lazarus (2023) cash-flow duration in Panel B, as well as when we subset to stock-quarter pairs or which short fee and idiosyncratic volatility are below their medians in Panel C.

These results provide empirical support for Theorem 3: overvaluation corrects more quickly when the disagreed-upon cash flows are realized more quickly.

6.3 Robustness

We repeat the empirical analyses from Sections 6.1 and 6.2 using alternate disagreement measures.

	(1)	(2)	(3)
Valuation Metric	Price-to-book	Adjusted value	FF5 price wedge
Panel A: Baseline			
$PVD_{n,t}$	2.266***	0.367***	0.0706***
	(8.06)	(11.56)	(19.18)
Quarter fixed effects	Yes	Yes	Yes
Obs.	76,883	18,378	75,541
Within-quarter R^2	0.0184	0.0748	0.1149
Panel B: Add controls			
$PVD_{n,t}$	2.199***	0.374***	0.0705***
	(7.70)	(12.45)	(20.37)
Gormsen–Lazarus Duration	−0.392**	0.344***	−0.0244***
	(−2.49)	(16.97)	(−7.58)
DMS Disagreement	0.011	−0.026**	−0.0082***
	(0.11)	(−3.05)	(−5.55)
Short fee	0.177*	0.037	0.0061**
	(1.70)	(1.66)	(2.53)
Three-factor idiosyncratic volatility	0.361***	0.015	−0.0088**
	(3.18)	(0.56)	(−2.26)
Quarter fixed effects	Yes	Yes	Yes
Obs.	76,883	18,378	75,541
Within-quarter R^2	0.0193	0.1537	0.1363
Panel C: Short fee and idiosyncratic volatility below medians			
$PVD_{n,t}$	2.240***	0.185***	0.0889***
	(9.25)	(6.46)	(19.33)
Quarter fixed effects	Yes	Yes	Yes
Obs.	34,219	7,757	33,318
Within-quarter R^2	0.0301	0.0194	0.1532

Table 4: **Valuation metrics are positively associated with higher valuations.** This table presents the results of regression (11) of valuation metrics $V_{n,t}$ (one of the market-to-book ratio, Cho and Polk (2024) adjusted value, or Van Binsbergen et al. (2023) Fama-French five-factor price wedge) on the empirical present value disagreement term defined in (6), standardized in each quarter to have mean 0 and standard deviation 1. All columns include quarter fixed effects. Then within R^2 is the proportion of variation explained after partialling out quarter fixed effects. Panel A runs the regression in the full sample with no controls. Panel B controls for short fees and Fama-French three-factor idiosyncratic volatility. Panel C restricts the sample to stock-quarter pairs for which the short fee and idiosyncratic volatility are below their medians. Parentheses contain t statistics from stock and quarter two-way clustered standard errors. Levels of significance are presented as follows: *p<0.1; **p<0.05; ***p<0.01.

	(1)	(2)	(3)
PV / Annualized Geometric Return Horizon	$\bar{h}=1$	$\bar{h}=5$	$\bar{h}=9$
Panel A: Baseline			
$PVD_{n,t} \times InsideRatio_{n,t}(\bar{h})$	-9.308^{***}	-2.476^{***}	-2.589^{***}
	(-4.17)	(-5.06)	(-2.96)
$PVD_{n,t}$	1.905^{***}	0.470	1.256
	(3.85)	(1.23)	(1.49)
Quarter Fixed Effects	Yes	Yes	Yes
Obs.	80,518	77,334	75,079
Within R^2	0.0016	0.0060	0.0067
Panel B: Add controls			
$PVD_{n,t} \times InsideRatio_{n,t}(\bar{h})$	-8.984^{***}	-2.370^{***}	-2.416^{***}
	(-4.16)	(-4.99)	(-2.94)
$PVD_{n,t}$	1.999^{***}	0.723^{*}	1.395^{*}
	(4.10)	(1.95)	(1.75)
Gormsen–Lazarus Duration	-0.260	-0.929^{***}	-0.996^{***}
	(-0.58)	(-4.03)	(-5.14)
DMS Disagreement	-0.476^{***}	-0.171^{*}	-0.243^{***}
	(-3.14)	(-1.96)	(-3.45)
Short Fee	-2.023^{***}	-1.637^{***}	-1.166^{***}
	(-4.41)	(-6.67)	(-5.72)
Idiosyncratic Volatility	-0.047	-1.987^{***}	-2.127^{***}
	(-0.07)	(-9.05)	(-11.85)
Quarter Fixed Effects	Yes	Yes	Yes
Obs.	80,518	77,334	75,079
Within R^2	0.0037	0.0339	0.0489
Panel C: Short fee and idiosyncratic volatility below medians			
$PVD_{n,t} \times InsideRatio_{n,t}(\bar{h})$	-8.542^{**}	-2.165^{**}	-1.848
	(-2.23)	(-2.45)	(-1.13)
$PVD_{n,t}$	1.436^{**}	0.774	0.819
	(2.32)	(1.28)	(0.57)
Quarter Fixed Effects	Yes	Yes	Yes
Obs.	8,854	8,444	8,215
Within R^2	0.0019	0.0032	0.0044

Table 5: **Overvaluation corrects more quickly when resolution share is larger.** This table presents the results of regression (12) of annual geometric mean returns from $t+1$ to $t+\bar{h}$ for $\bar{h}=1,5$, and 9 (indicated in the column names at the top of the table) on the empirical present value disagreement term defined in (6), standardized in each quarter to have mean 0 and standard deviation 1, and its interaction with the inside present value ratio defined in (10). Panel A displays the results of the regression without controls. Panel B adds short fees, Fama-French three-factor idiosyncratic volatility, Gormsen and Lazarus (2023) cash flow duration, and Diether et al. (2002) disagreement (all standardized within each quarter to have mean 0 and standard deviation of 1) as controls. Panel C restricts the sample to stock-quarter pairs for which the short fee and idiosyncratic volatility are below their medians. All columns include quarter fixed effects. Parentheses contain t statistics from stock and quarter two-way clustered standard errors. Levels of significance are presented as follows: *p<0.1; **p<0.05; ***p<0.01.

6.3.1 Valuation Regressions with Alternate Disagreement Measures

We repeat the valuation regressions from (11) using two alternate disagreement measures.

First, we use the Diether et al. (2002) current-fiscal-year disagreement measure used in Section 3.

Second, we compute a measure based only on I/B/E/S analyst forecasts: the present value disagreement about h-year ahead EPS for $h = 1, \ldots, 5$ years:

$$PVD_n^{IBES} = \sum_{h=1}^{5} \frac{1}{(1+r_f)^h} \operatorname{std}_a(F_{a,n,h}), \tag{13}$$

where $F_{a,n,h}$ is analyst a's forecast of the stock n's EPS for horizon h. PVD_n^{IBES} is an empirical proxy for the $Gap_t \cdot DisputedPV_t$ term in Theorem 2.[8] We then scale PVD_n^{IBES} by book equity per share at time t.

We scale both of these disagreement measures by book equity per share at time t and standardize them to have mean 0 and standard deviation 1.

Table 7 displays the results of the univariate valuation regression (11) on our empirical present value disagreement measure versus each of these alternate disagreement measures in matched samples. Panel A displays the results of the regression on present value disagreement versus the Diether et al. (2002) disagreement measure in a matched sample. While present value disagreement term is strongly associated with higher valuations across all three metrics, the Diether et al. (2002) disagreement measure is not. Panel B displays the results of the regression on present value disagreement term versus the present value of disagreement about h-year ahead EPS over the next five years PVD_n^{IBES} in a matched sample. PVD_n^{IBES} has a slightly higher R^2 for explaining variation in the market-to-book ratio, but $PVD_{n,t}$ has far higher R^2s for explaining variation in the two mispricing metrics.

These results suggest that our novel machine learning methodology allows us to construct a stronger empirical proxy for present value disagreement in Theorem 2 than is possible using traditional methods based solely on analyst forecasts.

6.3.2 Return-Prediction Regressions with Alternate Disagreement Measures

We repeat the return-prediction regressions from (12) using the alternate disagreement measure based solely on analyst forecasts from (13). We calculate the inside present value

[8] We use the present value of disagreement across analysts at each horizon as opposed to disagreement across analysts about the present value. Doing so maximizes the set of stocks for which we can construct this measure because few analysts report EPS expectations for all horizons $h = 1, \ldots, 5$ for a single stock.

	Market-to-Book		Adjusted Value		Price Wedge	
	(1)	(2)	(3)	(4)	(5)	(6)
Panel A: Present value disagreement versus DMS disagreement						
$PVD_{n,t}$	2.266***		0.367***		0.0706***	
	(8.06)		(11.56)		(19.18)	
DMS Disagreement		0.082		0.011		−0.0078***
		(0.74)		(1.26)		(−4.50)
Quarter Fixed Effects	Yes	Yes	Yes	Yes	Yes	Yes
Obs.	76,883	76,883	18,378	18,378	75,541	75,541
Within R^2	0.0184	0.0000	0.0748	0.0000	0.1149	0.0010
Panel B: Present value disagreement versus five-year IBES PV disagreement						
$PVD_{n,t}$	3.214***		0.468***		0.0441***	
	(3.22)		(9.18)		(10.89)	
PVD_n^{IBES}		9.084		0.473***		0.0200***
		(1.67)		(3.68)		(2.83)
Quarter Fixed Effects	Yes	Yes	Yes	Yes	Yes	Yes
Obs.	4,891	4,891	1,118	1,118	4,799	4,799
Within R^2	0.0095	0.0219	0.1779	0.0403	0.1322	0.0073

Table 6: **Valuation regression results with alternate disagreement measures.** This table presents the results of regression (11) of valuation metrics $V_{n,t}$ (one of the market-to-book ratio, Cho and Polk (2024) adjusted value, or Van Binsbergen et al. (2023) Fama-French five-factor price wedge) on the empirical present value disagreement term defined in (6), the Diether et al. (2002) disagreement measure from Section 3, and the present value of disagreement about h-year ahead EPS over the next five years defined in (13). All right-hand side regressors are standardized in each quarter to have mean 0 and standard deviation 1. All columns include quarter fixed effects. Then within R^2 is the proportion of variation explained after partialling out quarter fixed effects. Panel A runs the regressions on the empirical present value disagreement term versus the Diether et al. (2002) disagreement measure in a matched sample. Panel B runs the regression on the empirical present value disagreement term versus the present value of disagreement about h-year ahead EPS over the next five years in a matched sample. Parentheses contain t statistics from stock and quarter two-way clustered standard errors. Levels of significance are presented as follows: *p<0.1; **p<0.05; ***p<0.01.

ratio of this measure for horizon $\bar{h}$ as:

$$InsideRatio_{n,t}^{IBES}\left(\bar{h}\right) = \frac{\sum_{h=1}^{\bar{h}} \frac{1}{\left(1+r_f\right)^h} std_a(F_{a,n,h})}{\sum_{h=1}^{5} \frac{1}{\left(1+r_f\right)^h} std_a(F_{a,n,h})}, \tag{14}$$

Table 7 displays the results of the return-prediction regression (12) using this alternate disagreement measure. We find qualitatively similar results to those in Table 5. Panel A displays the results of the regression with no controls. Across all horizons $\bar{h} = 1, \ldots, 5$, we find that β_2 is negative: present value disagreement predicts returns more negatively when the inside present value ratio up to horizon $\bar{h}$ is higher. However, statistical power is much lower than in Table 5 because the sample size is an order of magnitude smaller, due to limited analyst coverage of EPS forecasts beyond the first two years. This pattern continues to hold when we control for short fees, idiosyncratic volatility, the Diether et al. (2002) disagreement measure from Section 3, and Gormsen and Lazarus (2023) cash-flow duration in Panel B. This pattern fails to hold in Panel C where we subset to stock-quarter pairs for which short fee and idiosyncratic volatility are below their medians. However, note that the sample size is extremely small in this subset.

These results are broadly consistent with overvaluation correcting more quickly when disagreement duration is shorter (e.g., resolution share is larger), as we find in our baseline analysis in Table 5. These results also highlight a major benefit of our novel machine learning methodology: it allows us to construct disagreement measures for stocks that we do not observe analyst forecasts for at long horizons. As a result, the panel used in the baseline Table 5 has far more observations than that used in Table 7, thereby creating more statistical power.

7 Conclusion

This paper develops model-based closed-form empirical predictors of the persistence and level of overvaluation, based on the term structure of disagreement. We make the argument in three parts. Empirically, we find that the canonical result that disagreement predicts negative returns survives among stocks with low short fees and low idiosyncratic volatility, where standard limits-to-arbitrage explanations are less compelling. Theoretically, we rationalize this in a model whose payoff is two closed-form empirical predictors: the bubble is proportional to disagreement times the present value of disputed cash flows, even absent shorting fees, and corrects in proportion to the resolution share, the fraction of disputed value resolved within a window. We find support for both predictions in the data: overvaluation predicts

PV / Annualized Geometric Return Horizon	(1) $\bar{h}=1$	(2) $\bar{h}=2$	(3) $\bar{h}=3$	(4) $\bar{h}=4$
Panel A: Four-control-complete common sample (univariate)				
$PVD_n^{IBES} \times InsideRatio_{n,t}^{IBES}(\bar{h})$	-27.879^{**}	-5.454	-2.741	-2.062
	(-2.42)	(-0.89)	(-0.40)	(-0.34)
PVD_n^{IBES}	-0.947	-2.386	-3.600	-3.998
	(-0.41)	(-0.92)	(-0.90)	(-0.84)
Quarter Fixed Effects	Yes	Yes	Yes	Yes
Obs.	4,768	4,768	4,768	4,768
Within R^2	0.0085	0.0123	0.0305	0.0494
Panel B: Four-control-complete common sample (with controls)				
$PVD_n^{IBES} \times InsideRatio_{n,t}^{IBES}(\bar{h})$	-26.641^{**}	-5.241	-3.262	-3.042
	(-2.40)	(-0.88)	(-0.49)	(-0.52)
PVD_n^{IBES}	-1.271	-1.938	-2.645	-2.530
	(-0.55)	(-0.78)	(-0.67)	(-0.55)
Gormsen–Lazarus Duration	0.872	0.703	0.117	0.141
	(1.00)	(0.96)	(0.18)	(0.22)
DMS Disagreement	-1.170^{*}	-0.325	-0.146	-0.060
	(-1.86)	(-0.65)	(-0.49)	(-0.20)
Short Fee	-0.643	-1.890	-1.942^{*}	-1.021
	(-0.24)	(-1.38)	(-1.86)	(-1.31)
Idiosyncratic Volatility	0.466	-1.656^{*}	-1.946^{**}	-2.452^{***}
	(0.31)	(-1.90)	(-2.57)	(-3.38)
Quarter Fixed Effects	Yes	Yes	Yes	Yes
Obs.	4,768	4,768	4,768	4,768
Within R^2	0.0097	0.0189	0.0441	0.0670
Panel C: Short fee and idiosyncratic volatility strictly below medians				
$PVD_n^{IBES} \times InsideRatio_{n,t}^{IBES}(\bar{h})$	-9.188	18.913	16.101	21.572
	(-0.22)	(1.30)	(1.41)	(1.48)
PVD_n^{IBES}	4.833	-9.597	-12.682	-19.420
	(0.46)	(-1.28)	(-1.35)	(-1.49)
Quarter Fixed Effects	Yes	Yes	Yes	Yes
Obs.	555	555	555	555
Within R^2	0.0039	0.0101	0.0159	0.0157

Table 7: **Overvaluation corrects more quickly when resolution share is larger.** This table presents the results of regression (12) of annual geometric mean returns from $t+1$ on the present value of disagreement about h-year ahead disagreement, defined in (13), standardized in each quarter to have mean 0 and standard deviation 1, and its interaction with the inside present value ratio defined in (10). Panel A displays the results of the regression without controls. Panel B adds short fees, Fama-French three-factor idiosyncratic volatility, Gormsen and Lazarus (2023) cash flow duration, and Diether et al. (2002) disagreement (all standardized within each quarter to have mean 0 and standard deviation of 1) as controls. Panel C restricts the sample to stock-quarter pairs for which the short fee and idiosyncratic volatility are below their medians. All columns include quarter fixed effects. Parentheses contain t statistics from stock and quarter two-way clustered standard errors. Levels of significance are presented as follows: *p<0.1; **p<0.05; ***p<0.01.

more negative returns precisely when the resolution share is higher, even after controlling for standard measures of duration and disagreement. Methodologically, these tests require measuring disagreement at horizons where analyst forecasts are sparse; our machine-learning measure does so, and explains an order of magnitude more variation in existing mispricing proxies than canonical one-period-ahead dispersion.

Our central conceptual contribution is thus that one-period-ahead disagreement is insufficient: the term structure of disagreement contains distinct information about both the level and persistence of overvaluation. When disputed value lies far in the future and little information arrives to resolve it, learning is slow and overvaluation rationally persists. This offers a discipline for contemporary valuation debates: the question is not only whether a company can be shorted, but whether its narrative can be falsified soon. It also connects two literatures – duration and disagreement – that have largely developed separately, but whose interaction we argue matters for overvaluation.

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A Empirical Appendix

A.1 Estimation Details

Let $\mathcal{O}_t^F$ and $\mathcal{O}_t^Y$ denote the observed forecast and cash-flow cells in quarter t. Suppressing the quarter subscript on the parameters, we minimize

$$\begin{aligned}\mathcal{L} = & \sum_{(a,n,h)\in\mathcal{O}_t^F} w_{a,n,h}^F \left[F_{a,n,h} - \sum_{k=1}^{K} \widetilde{\theta}_{a,k} x_{n,k} \phi_k(h)\right]^2 \\ & + \rho \sum_{(n,h)\in\mathcal{O}_t^Y} w_{n,h}^Y \left[Y_{n,h} - \ell_n^A \beta_h^A - \sum_{k=1}^{K} \theta_k x_{n,k} \phi_k(h)\right]^2 \\ & + \eta_{\widetilde{\theta}} \|\widetilde{\theta}\|_F^2 + \eta_x \big(\|x_{n,k}\|_F^2 + \|\ell_n^A\|_2^2\big) + \eta_\beta \|\beta^A\|_2^2 \end{aligned} \tag{15}$$

Here $w_{a,n,h}^F$ implements equal weighting of all analyst forecast observations. $w_{n,h}^Y$ implements equal weighting of all realized cash flow observations and rescales the cash block by its weighted variance. We set $\rho = 1$. The $\eta_\cdot$ are regularization parameters that penalized extreme magnitudes of the estimated objects.

The loss function is nonlinear jointly but linear in each parameter block conditional on the others. We therefore use alternating least squares with closed-form ridge updates: analyst by analyst for $\widetilde{\theta}_a$, globally for the true-state vector θ, horizon by horizon for $\phi(h)$ and β_h^A, and stock by stock for (x_n, ℓ_n^A). We initialize the horizon profiles with orthogonal polynomials on the ten-year horizon grid, obtain initial stock exposures from the cash-flow block, and initialize the agreed component from a rank-one decomposition of the remaining cash-flow matrix. These polynomial profiles only initialize the search; the final horizon profiles are unrestricted at the ten annual points.

We calculate the regularization parameters directly from the regressions used in each alternating update. For each analyst, stock, or horizon, we sum the weighted squares of the right-hand-side terms and divide by the number of coefficients being estimated. We then take the median of this quantity across analysts for $\eta_{\widetilde{\theta}}$, across stocks for η_x, and across horizons for η_β. We multiply these medians by the tuning values selected through cross-validation: a value from $\{1, 4, 16\}$ for $\eta_{\widetilde{\theta}}$, a value from $\{0.2, 0.5, 2\}$ for η_x, and the fixed value 0.75 for η_β. Finally, we add 10^{-8} to each penalty.

For example, the penalty on the analyst belief coefficients is

$$\eta_{\widetilde{\theta}} = 10^{-8} + [\text{selected value from } \{1, 4, 16\}] \, \text{median}_a \left\{ \frac{1}{K} \sum_{(n,h):(a,n,h)\in\mathcal{O}_t^F} w_{a,n,h}^F \sum_{k=1}^{K} [x_{n,k}\phi_k(h)]^2 \right\}. \tag{16}$$

Thus, the penalty is larger when the right-hand-side terms in the corresponding regressions are larger. We recalculate these penalties during the first 100 iterations and then hold them fixed for the remainder of the estimation.

We select regularization separately in every quarter using five deterministic row-level folds for both measurement blocks. The grid is $\kappa_{\widetilde{\theta}} \in \{1, 4, 16\}$ and $\kappa_x \in \{0.2, 0.5, 2\}$, with $\kappa_\beta = 0.75$. For each of the nine candidate pairs, we fit the model on four folds and evaluate weighted R^2 on the held-out forecast deviations and cash-flow cells. We select the pair that maximizes the sum of mean held-out R^2 across the two blocks and refit that specification on the full quarterly sample. The focal empirical results use $K = 7$ latent states and the $s = 10$ analyst–stock support rule.

The final fit uses a monotone, damped alternating procedure. After 100 iterations, we freeze the ridge scales and evaluate all subsequent updates against that fixed penalized objective. At each iteration we choose among a full update and blends of 0.5, 0.25, 0.125, or zero with the previous iterate; we also refresh the analyst loadings after updating the horizon profiles. We require at least 150 iterations and stop after 25 consecutive relative improvements no larger than 10^{-8}, subject to a maximum of 1,500 iterations. A stock with no training observations in a cross-validation fold is assigned its ridge-limit exposure of zero rather than carrying a stale loading into held-out prediction.

Normalization conventions. The trilinear representation is unique only up to state-specific scale, sign, and permutation. During estimation we impose $\|\phi_k\|_2 = 1$, choose the sign so that $\sum_h (1 + r_f)^{-h} \phi_k(h) \geq 0$.

We balance the norms of $\widetilde{\theta}_{\cdot k}$ and $x_{\cdot k}$. That is, because the analyst loading and stock exposure are identified only through their product, we remove their arbitrary relative scale by rescaling them to have equal Euclidean norms. This normalization leaves all fitted values and empirical measures unchanged.

We similarly normalize $\|\beta^A\|_2 = 1$ and orient its discounted sum positively. For reporting, we rescale each state so that $\text{std}_a(\widetilde{\theta}_{ak}) = 1$, orient $x_{\cdot k}$ to have a nonnegative cross-sectional mean, and order states by profile duration. The fitted values, present value disagreement, and disagreement duration are invariant to these normalizations.

B Theoretical Appendix

B.1 Lemmas

B.1.1 Proof of Lemma 1

Proof. We prove by induction. At date $t+1$, the investor sees the cash flow $Cash_{t+1} = Agreed_{t+1} + Load_{t+1}\theta + u_{t+1}$.

Since, by assumption, $Agreed_{t+1}$ is a known number, define the signal $Signal_{t+1} = Cash_{t+1} - Agreed_{t+1} = Load_{t+1}\theta + u_{t+1}$.

To prove (1), divide the signal by $Load_{t+1}$ to obtain $\frac{Signal_{t+1}}{Load_{t+1}} \sim N(\theta, \sigma_u^2/Load_{t+1}^2)$. Using the canonical conjugate priors result (Bishop (2006), p. 98), we obtain that the posterior variance satisfies $\frac{1}{\sigma^2_{\theta,t+1}} = \frac{1}{\sigma^2_{\theta,t}} + \frac{Load_{t+1}^2}{\sigma_u^2}$. Inverting and solving gives the equation displayed in (1). For the induction base case, observe that by assumption from Section 4, all investors share at time 0 the same belief for the variance of θ, $\sigma_{\theta,0}$.

To prove (2), another appeal to conjugate priors yields that the belief over the mean updates to $\hat{\theta}_{t+1} = \hat{\theta}_t + Gain_t\left(Signal_{t+1} - Load_{t+1}\hat{\theta}_t\right)$, where $Gain_t = \frac{\sigma^2_{\theta,t}Load_{t+1}}{\sigma_u^2 + Load_{t+1}^2\sigma^2_{\theta,t}}$. Observe that $Gain_t Load_{t+1} = Resolve_t$.

We now calculate

$$\begin{aligned}
Gap_{t+1} =& \hat{\theta}^{long}_{t+1} - \hat{\theta}^{arb}_t \\
=& \hat{\theta}^{long}_t - Gain_t Load_{t+1}\hat{\theta}^{long}_t \\
& - \hat{\theta}^{arb}_t + Gain_t Load_{t+1}\hat{\theta}^{arb}_t \\
=& (\hat{\theta}^{long}_t - \hat{\theta}^{arb}_t)(1 - Gain_t Load_{t+1}) \\
=& Gap_t(1 - Resolve_t).
\end{aligned}$$

Since Gap_t is known at time t, and $Resolve_t$ is a function of known values, the gap narrows deterministically over time. □

B.1.2 Proof of Lemma 2

Proof. We first prove normality given the assumption. The cash flow at time $t+1$ is given by $Cash_{t+1} = Agreed_{t+1} + Signal_{t+1}$ by substitution.

From the proof of Lemma 1, arbitrageurs' posterior after observing the signal is $\hat{\theta}^{arb}_{t+1} = \hat{\theta}^{arb}_t(1 - Gain_t Load_{t+1}) + Gain_t Signal_{t+1}$.

By assumption in the statement of the lemma, $P_{t+1} = AgreedPV_{t+1} + DisputedPV_{t+1}\hat{\theta}^{arb}_{t+1} +$

$Wedge_{t+1}$. Substituting in the previous equation for $\hat{\theta}^{arb}_{t+1}$, we obtain

$$\begin{aligned} P_{t+1} =& AgreedPV_{t+1} + Wedge_{t+1} \\ &+ DisputedPV_{t+1}(1 - Gain_t Load_{t+1})\hat{\theta}^{arb}_t \\ &+ DisputedPV_{t+1} Gain_t Signal_{t+1}. \end{aligned}$$

The payoff at time $t+1$ is therefore:

$$\begin{aligned} Payoff_{t+1} =& P_{t+1} + Cash_{t+1} \\ =& Agreed_{t+1} + Signal_{t+1} + P_{t+1}. \end{aligned}$$

Now in the terms for $Payoff_{t+1}$, observe that the only value not known at time t is $Signal_{t+1}$ (and therefore also $DisputedPV_{t+1}Gain_t Signal_{t+1}$).

As a result, we may write $Payoff_{t+1} = Known_t + Signal_{t+1}(1 + DisputedPV_{t+1}Gain_t)$, which is exactly the second statement of the Lemma. For the first statement, observe that $Signal_{t+1}$ is normally distributed by assumption in Section 4, and since $Known_t$, $Gain_t$, and $DisputedPV_{t+1}$ are known at time t (above derivation, Lemma 1, and Section 4) and an affine transformation of a normal is normal, it follows that $Payoff_{t+1}$ is normally distributed.

To obtain the third statement of the lemma (moments), observe that under investor type j's beliefs,

$$Signal_{t+1} \sim N(Load_{t+1}\hat{\theta}^j_t, Load^2_{t+1}\sigma^2_{\theta,t} + \sigma^2_u).$$

Since $Payoff_{t+1} = Known_t + Signal_{t+1}(1 + DisputedPV_{t+1}Gain_t)$, we obtain that $E_j[Payoff_{t+1}] = Known_t + Load_{t+1}\hat{\theta}^j_t(1 + DisputedPV_{t+1}Gain_t)$, and $Var_j[Payoff_{t+1}] = (1 + DisputedPV_{t+1}Gain_t)^2(Load^2_{t+1}\sigma^2_{\theta,t} + \sigma^2_u)$, as desired.

Finally, subtracting the two means gives $ValGap_t = Load_{t+1}(\hat{\theta}^{long-only}_t - \hat{\theta}^{arb}_t)(1 + DisputedPV_{t+1}Gain_t)$. Since $Load_{t+1}Gain_t = Resolve_t$ (from the proof of Lemma 1) and $Gap_t = \hat{\theta}^{long-only}_t - \hat{\theta}^{arb}_t$, we obtain $ValGap_t = (Load_{t+1} + DisputedPV_{t+1}Resolve_t)Gap_t$, as desired. □

B.1.3 Proof of Lemma 3

Proof. An investor of type j with wealth W_t who holds q shares pays qP_t now and puts the rest in the safe asset. Their next period wealth is therefore $W_{t+1} = R(W_t - qP_t) + qPayoff_{t+1} - cmax(-q, 0)$.

By assumption, $Payoff_{t+1}$ is normally distributed, and therefore W_{t+1} is normally dis-

tributed. Since utility takes the CARA form $-exp(-\gamma W)$ by assumption, an application of the CARA-normal expectation gives $E[-exp(-\gamma W)] = -exp\left(-\gamma\left(E[W] - \frac{\gamma}{2}Var[W]\right)\right)$.

Maximizing utility is therefore equivalent to maximizing $E(W_{t+1}) - \frac{\gamma}{2}Var(W_{t+1})$.

Computing, this expression is

$$R(W_t - qP_t) + qE_j[Payoff_{t+1}] - cmax(-q, 0) - q^2\frac{\gamma}{2}Var(Payoff_{t+1}).$$

This is a negative quadratic in q, therefore concave. Away from the kink at $q = 0$, it is therefore permissible to take a first order condition to maximize. Taking this first order condition, we obtain $E_j[Payoff_{t+1}] - RP_t - \gamma qVar(Payoff_{t+1})$ when $q > 0$, and the same expression minus c when $q < 0$.

Setting the slope to zero in each region gives:

- $q > 0$: $q = \frac{E_j[Payoff_{t+1}] - RP_t}{\gamma Var(Payoff_{t+1})}$.
- $q < 0$: $q = \frac{E_j[Payoff_{t+1}] - RP_t + c}{\gamma Var(Payoff_{t+1})}$.

Consistency requires $E_j[Payoff_{t+1}] > RP_t$ in the first region and $E_j[Payoff_{t+1}] + c < RP_t$ in the second region.

When $E_j[Payoff_{t+1}] < RP_t < E_j[Payoff_{t+1}] + c$, we obtain a negative slope in the $q > 0$ region and a positive slope in the $q < 0$ region, indicating that the optimum is at $q = 0$. This gives the three-part demand for arbitrageurs.

For long-only investors, observe that they are constrained to have $q \geq 0$. Therefore only the first region applies.

□

B.1.4 Proof of Lemma 4

Proof. By the definition of the valuation gap identity, $\text{ValGap}_t = \text{Gap}_t \text{Exposure}_t$, where $\text{Gap}_t = \widehat{\theta}_t^{\text{opt}} - \widehat{\theta}_t^{\text{arb}}$ and $\text{Exposure}_t = \text{Load}_{t+1} + \text{DisputedPV}_{t+1}\,\text{Resolve}_t$. We show each term is nonnegative.

First we show that exposure is non-negative. Because the dividend loadings are non-negative, $\text{Load}_n \geq 0$ for all n (a higher θ never lowers a dividend), the present value of future loadings DisputedPV_{t+1} is nonnegative. By definition of Resolve, $\text{Resolve}_t = \dfrac{\sigma_{\theta,t}^2 \text{Load}_{t+1}^2}{\sigma_{\theta,t}^2 \text{Load}_{t+1}^2 + \sigma_u^2} \in [0, 1]$. Therefore Exposure_t is non-negative.

Next, by assumption (Section 4) the long-only investors have the weakly higher initial estimate, so $\text{Gap}_0 = \widehat{\theta}_0^{\text{long-only}} - \widehat{\theta}_0^{\text{arb}} \geq 0$. By Lemma 1, $\text{Gap}_t = \text{Gap}_0 \prod_{u=0}^{t-1}\left(1 - \text{Resolve}_u\right)$, and since each $\text{Resolve}_u \in [0, 1]$, it follows that Gap_t is also non-negative. □

B.1.5 Proof of Lemma 5

Proof. Suppose towards a contradiction that at some market-clearing price the carrying cost weakly exceeds the long-only investors' valuation, $RP_t \geq E_{\text{long-only}}[\text{Payoff}_{t+1}]$. Then by the demand schedules (Lemma 3) the long-only investors then hold zero, $q_{\text{long-only}} = 0$.

By Lemma 4 the long-only investors value the payoff at least as much as the arbitrageurs: $E_{\text{long-only}}[\text{Payoff}_{t+1}] \geq E_{\text{arb}}[\text{Payoff}_{t+1}]$, so the supposition gives $RP_t \geq E_{\text{arb}}[\text{Payoff}_{t+1}]$. The arbitrageurs are then out of the market or short: by Lemma 3 their demand is zero on $\left[E_{\text{arb}}[\text{Payoff}_{t+1}],\ E_{\text{arb}}[\text{Payoff}_{t+1}]+c\right]$ and strictly negative above it, hence $q_{\text{arb}} \leq 0$. Aggregate demand is therefore $M_{\text{long-only}}\, q_{\text{long-only}} + M_{\text{arb}}\, q_{\text{arb}} \leq 0 < S$, contradicting market clearing $M_{\text{long-only}}\, q_{\text{long-only}} + M_{\text{arb}}\, q_{\text{arb}} = S$.

Hence it must be the case that $RP_t < E_{\text{long-only}}[\text{Payoff}_{t+1}]$, and by the demand schedule the long-only investors hold the strictly positive position $q_{\text{long-only}} = \frac{E_{\text{long-only}}[\text{Payoff}_{t+1}] - RP_t}{\gamma\, \text{Var}(\text{Payoff}_{t+1})} > 0$. □

B.1.6 Proof of Lemma 6

Proof. Market clearing requires $M_{\text{long-only}} q_{\text{long-only}} + M_{arb} q_{arb} = S$.

Substituting the expression for $Prem_t$ into the demand functions of Lemma 3 yields $q_{\text{long-only}} = \frac{max(0, ValGap_t - Prem_t)}{\gamma Var(Payoff_{t+1})}$, and $q_{arb} = \frac{-Prem_t}{\gamma Var(Payoff_{t+1})}$ when $q_{arb} \geq 0$ and $q_{arb} = \frac{-Prem_t + c}{\gamma Var(Payoff_{t+1})}$ when $q_{arb} < 0$.

Combining market clearing and the demand functions, we obtain that when $q_{arb} > 0$,

$$M_{\text{long-only}} \frac{max(0, ValGap_t - Prem_t)}{\gamma Var(Payoff_{t+1})} + M_{arb} \frac{-Prem_t}{\gamma Var(Payoff_{t+1})} = S$$

Using that $q_{\text{long-only}} > 0$ from Lemma 5, we obtain $Prem_t = \frac{ValGap_t M_{\text{long-only}} - S\gamma Var(Payoff_{t+1})}{M_{arb} + M_{\text{long-only}}}$.

Similarly when $q_{arb} < 0$, we obtain $Prem_t = \frac{(ValGap_t M_{\text{long-only}} + M_{arb} c - S\gamma Var(Payoff_{t+1})}{M_{arb} + M_{\text{long-only}}}$.

When $q_{arb} = 0$, long-only investors clear the market, so we obtain $Prem_t = ValGap_t - \frac{S\gamma Var(Payoff_{t+1})}{M_{\text{long-only}}}$.

Every input is known at time t, so $Prem_t$ is deterministic. Further, at $Prem_t = 0$, the formulas for $Prem_t$ when $q_{arb} > 0$ and $q_{arb} = 0$ agree; similarly when $Prem_t = c$, the formulas for $q_{arb} = 0$ and $q_{arb} < 0$ agree. Therefore, $Prem_t$ is continuous. Finally, $Prem_t$ is decreasing in S in each branch, and has zero slope for c in in the first two branches, increasing in c in the third branch. □

B.1.7 Proof of Lemma 7

Proof. At the last date, the firm ceases to exist, so $P_T = 0$, $AgreedPV_T = 0$, $DisputedPV_T = 0$, and we may set $Wedge_T = 0$, satisfying the hypothesis.

For the inductive step, assume that $P_{t+1} = Fund_{t+1} + Wedge_{t+1}$. We wish to prove the statement for P_t. We may apply Lemmas 2 - 6 to obtain that the market clears at a price P_t satisfying $RP_t = Prem_t + E_{arb}[Payoff_{t+1}]$, with $Prem_t$ deterministic as given in closed form in Lemma 6.

We derive:

$$\begin{aligned}
E_{arb}[Payoff_{t+1}] &= E_{arb}[Cash_{t+1} + P_{t+1}] \\
&= E_{arb}[Agreed_{t+1} + Load_{t+1}\theta + u_{t+1} \\
&+ AgreedPV_{t+1} + DisputedPV_{t+1}\theta + Wedge_{t+1}] \\
&= \hat{\theta}_t^{arb}(Load_{t+1} + DisputedPV_{t+1}) \\
&+ Agreed_{t+1} + AgreedPV_{t+1} \\
&+ Wedge_{t+1} \\
&= R\hat{\theta}_t^{arb} DisputedPV_t + RAgreedPV_t + Wedge_{t+1} \\
&= RFund_t + Wedge_{t+1}.
\end{aligned}$$

Substituting back in to $RP_t = Prem_t + E_{arb}[Payoff_{t+1}]$, we obtain:

$$P_t = \frac{Prem_t}{R} + Fund_t + \frac{Wedge_{t+1}}{R}.$$

Since $Prem_t$ is known at time t by Lemma 6 and $Wedge_{t+1}$ is known at time t because it is deterministic by the inductive step, we define $Wedge_t = \frac{Prem_t}{R} + \frac{Wedge_{t+1}}{R}$. This completes the inductive step. □

B.1.8 Proof of Lemma 8

Proof. By Lemma 6, $E_{arb,t}[Payoff_{t+1}] - RP_t = -Prem_t$. In addition, notice that the three cases of the Lemma become identical when $c = 0$. Substituting this expression into the definition of abnormal payoff yields

$$\begin{aligned}
A_t &= -Prem_t - \frac{\gamma Var(Payoff_{t+1})S}{M_{long-only} + M_{arb}} \\
&= -\frac{M_{long-only} ValGap_t - \gamma Var(Payoff_{t+1})S}{M_{long-only} + M_{arb}} - \frac{\gamma Var(Payoff_{t+1})S}{M_{long-only} + M_{arb}} \\
&= -\frac{M_{long-only} ValGap_t}{M_{long-only} + M_{arb}},
\end{aligned}$$

as desired. □

B.2 Costly Shorting Results

Theorem 4. *When the shorting fee is positive ($c > 0$), the size of the bubble is given by:*

$$Bubble_t = \underbrace{\frac{M_{long\text{-}only}}{M_{long\text{-}only} + M_{arb}} Gap_t DisputedPV_t}_{Free\text{-}Shorting\ Base} + \underbrace{\frac{M_{arb}}{M_{long\text{-}only} + M_{arb}} \sum_{t'=t}^{T-1} \frac{1}{R^{t'-t+1}} min\left(c, max(0, ValGap_{t'} - \frac{\gamma Var(Payoff_{t'+1})S}{M_{long\text{-}only}})\right)}_{Shorting\ Fee\ Premium}.$$

The Free-Shorting Base remains increasing in $M_{long\text{-}only}$, Gap_t, and $DisputedPV_t$.

The Shorting Fee Premium is increasing in the shorting fee c, and the timing of disputed cash flows above risk compensation: $\sum_{t'=t}^{T-1} \frac{1}{R^{t'-t+1}} min\left(c, max(0, ValGap_{t'} - \frac{\gamma Var(Payoff_{t'+1})S}{M_{long\text{-}only}})\right)$.

Proof. Consider the three cases of Lemmas 3 and 6. They give:

$$Prem_t + \frac{\gamma Var(Payoff_{t+1})S}{M_{\text{long-only}} + M_{\text{arb}}} = \begin{cases} \frac{M_{\text{long-only}} ValGap_t}{M_{\text{long-only}} + M_{\text{arb}}}, & q_{arb} > 0 \\ \frac{M_{\text{long-only}} ValGap_t}{M_{\text{long-only}} + M_{\text{arb}}} + \frac{M_{arb}}{M_{\text{long-only}} + M_{\text{arb}}}(ValGap_t - \frac{\gamma Var(Payoff_{t+1})S}{M_{\text{long-only}}}) & q_{arb} = 0 \\ \frac{M_{\text{long-only}} ValGap_t + M_{arb} c}{M_{\text{long-only}} + M_{\text{arb}}}, & q_{arb} < 0. \end{cases}$$

We now appeal to Lemma 6, which says that $Prem_t$ is a strictly increasing function of $ValGap_t$, to find the cutoffs for $Prem_t = 0$ and $Prem_t = c$ (e.g. the points at which the demand for arbitrageurs changes). At $Prem_t = 0$, from Lemma 3, $q_{arb} = 0$, so $ValGap_t = \frac{\gamma Var(Payoff_{t+1})S}{M_{\text{long-only}}}$. At $Prem_t = c$, it follows that $q_{arb} = 0$ and therefore that $ValGap_t = c + \frac{\gamma Var(Payoff_{t+1})S}{M_{\text{long-only}}}$.

As a result, we can therefore re-write the cases as:

$$Prem_t + \frac{\gamma Var(Payoff_{t+1})S}{M_{\text{long-only}} + M_{\text{arb}}}$$
$$= \begin{cases} \frac{M_{\text{long-only}} ValGap_t}{M_{\text{long-only}} + M_{\text{arb}}}, & ValGap_t < \frac{\gamma Var(Payoff_{t+1})S}{M_{\text{long-only}}} \\ \frac{M_{\text{long-only}} ValGap_t}{M_{\text{long-only}} + M_{\text{arb}}} + \frac{M_{arb}(ValGap_t - \frac{\gamma Var(Payoff_{t+1})S}{M_{\text{long-only}}})}{M_{\text{long-only}} + M_{\text{arb}}} & ValGap_t \in \left[\frac{\gamma Var(Payoff_{t+1})S}{M_{\text{long-only}}}, \frac{\gamma Var(Payoff_{t+1})S}{M_{\text{long-only}}} + c\right] \\ \frac{M_{\text{long-only}} ValGap_t + M_{arb} c}{M_{\text{long-only}} + M_{\text{arb}}}, & ValGap_t > \frac{\gamma Var(Payoff_{t+1})S}{M_{\text{long-only}}} + c. \end{cases}$$

Noticing that $\frac{M_{\text{long-only}} ValGap_t}{M_{\text{long-only}} + M_{\text{arb}}}$ appears in each term, we may write the above expression

compactly as:

$$Prem_t + \frac{\gamma Var(Payoff_{t+1})S}{M_{\text{long-only}} + M_{\text{arb}}}$$
$$= \frac{M_{\text{long-only}} ValGap_t}{M_{\text{long-only}} + M_{\text{arb}}} + \frac{M_{\text{arb}}}{M_{\text{long-only}} + M_{\text{arb}}} min\left(max\left(0, ValGap_t - \frac{\gamma Var(Payoff_{t+1})S}{M_{\text{long-only}}}\right), c\right)$$

Substituting into the equation for $Bubble_t$ (Theorem 1) and separating the two terms, with an appeal to Theorem 2 to simplify the first expression, gives the rest.

The claimed comparative statics are then immediate from the expression. □

B.3 Mandate-Based Microfoundation

The main text assumed that the long-only investors were optimists ($Gap_0 \geq 0$). This appendix derives that optimism from the mandate: long-only investors and arbitrageurs hold identical belief distributions and differ only in the ability to short; in equilibrium, the mandate censors the long-only pessimists, and the model of the main text, interpreted as the beliefs of the investors who choose to hold, emerges as the reduced form, with its main results intact.

B.3.1 Model

The model is identical to that in the main text, with two changes.

First, the long-only sector consists of k types, $k = 1, .., K$, with respective masses $m_k > 0$, and priors $\hat{\theta}_0^k = \hat{\theta}_0^{arb} + Gap_0^k$, where $Gap_0^1 \leq ... \leq Gap_0^K, \sum_k m_k Gap_0^k = 0$, and $Gap_0^1 < 0$: that is, the long-only sector share the same time 0 belief mean as the arbitrageurs, with genuine dispersion. All types share the prior variance $\sigma^2_{\theta,0}$, observe common cash flows, and update in a Bayesian manner: the only difference is that long-only investors cannot short.

Second, the asset is in zero net supply, $S = 0$; since Theorem 2 shows the bubble term does not depend on S, this normalization removes risk compensation while leaving the object of interest, the bubble term, intact. We make this assumption for tractability; Remark 1 states what changes when $S > 0$.

B.3.2 Re-Derived Results

Lemma 9. *At every date t, all $K + 1$ investor types hold normal posteriors over θ with common variance $\sigma^2_{\theta,t}$ of Lemma 1, and each type's gap decays deterministically by the common factor: $Gap_t^k = Gap_0^k \prod_{u=0}^{t-1}(1 - Resolve_u)$, where the common factor is strictly positive since each $Resolve_u \in [0, 1]$. In particular, signs, the ordering, and the zero cross-sectional mean $sum_k m_k Gap_t^k = 0$ are preserved at every date.*

Proof. Since the proof of Lemma 1 uses only that an investor holds a normal prior over θ with variance $\sigma^2_{\theta,t}$ and observes the common signal, it therefore applies to each of the $K+1$ types. Part (1) of Lemma 1 gives the common posterior variance, since all types start from $\sigma^2_{\theta,0}$. For the gap, the mean-update equation in the proof of Lemma 1 gives, for any type j, $\hat{theta}^j_{t+1} = \hat{\theta}^j_t + Gain_t(Signal_{t+1} - Load_{t+1}\hat{\theta}^j_t)$, with $Gain_t Load_{t+1} = Resolve_t$. Differencing type k against the arbitrageurs, the terms with $Signal_{t+1}$ cancel:

$$Gap^k_{t+1} = Gap^k_t(1 - Gain_t Load_{t+1}) = (1 - Resolve_t)Gap^k_t,$$

and iterating gives the desired equation. Since $\sigma^2_u > 0$ implies $Resolve_u \in [0,1)$, each factor lies in $(0,1]$, so multiplying every Gap^t_0 by the same positive number preserves signs, the ordering, and the zero weighted mean. □

Lemma 10. *Suppose, as in Lemma 2, that P_{t+1} is affine in $\hat{\theta}^{arb}_{t+1}$ with coefficients known at t. Then:*

1. *Every investor perceives $Payoff_{t+1}$ as normally distributed, with a variance $Var(Payoff_{t+1})$ common across types.*
2. *Type k's valuation exceeds the arbitrageurs' by $ValGap^k_t := E_k[Payoff_{t+1}] - E_{arb}[Payoff_{t+1}] = Gap^k_t Exposure_t$, with $Exposure_t = Load_{t+1} + DisputedPV_{t+1}Resolve_t \geq 0$ and $Exposure_t = 0$ if and only if $Load_{t+1} = 0$.*
3. *The demands are those of Lemma 3 for each type: $q_k = \frac{max(E_k[Payoff_{t+1} - RP_t, 0)}{\gamma Var(Payoff_{t+1})}$ for long-only type k, and $q_{arb} = \frac{E_{arb}[Payoff_{t+1} - RP_t}{\gamma Var(Payoff_{t+1})}$ at $c = 0$.*

Proof. By Lemma 9, every type holds a normal posterior over θ with a common variance. Lemma 2 is stated for an arbitrary investor type j with exactly such a posterior; it therefore applies to each of the $K+1$ types: every type perceives $Payoff_{t+1}$ as normal, with mean $E_j[Payoff_{t+1}] = Known_t + Passthrough_t Load_{t+1}\hat{\theta}^j_t$, where $Passthrough_t = 1 + DisputedPV_{t+1}Gain_t$. Subtracting the arbitrageurs' mean from type k's, exactly as in the last step of the proof of Lemma 2,

$$\begin{aligned} ValGap^k_t &= Passthrough_t Load_{t+1} Gap^k_t \\ &= (Load_{t+1} + DisputedPV_{t+1}Gain_t Load_{t+1})Gap^k_t \\ &= Gap^k_t Exposure_t. \end{aligned}$$

By the argument of Lemma 4, $Load_n \geq 0$ for all n gives $DisputedPV_{t+1} \geq 0$ and $Resolve_t \in [0,1)$, so $Exposure_t \geq 0$; and $Exposure_t = 0$ if and only if $Load_{t+1} = 0$ since

$Load_{t+1} = 0$ forces $Resolve_t = 0$. Finally, normality with common variance is the hypothesis of Lemma 3, whose demand schedules therefore apply to each type.

□

Lemma 11. *The equation* $\sum_{k=1}^{K} m_k max(Gap_0^k - b, 0) = M_{arb}p$ *has a unique solution* $b^* \in (0, Gap_0^K)$. *Let the participating types be defined as those with* $Gap_0^k > b^*$; *they include type* K, *exclude type 1, and their mass and holdings-weighted gap,* $M_{long-only} := \sum_{k:Gap_{0^k} > b^*} m_k$ *and* $Gap_0 = \frac{1}{M_{long-only}} \sum_{k:Gap_{0^k} > b^*} m_k Gap_0^k = \frac{M_{long-only} + M_a rb}{M_{long-only}} b^* > 0$, *are determined at time 0.*

Proof. The left side of the equation is continuous and weakly decreasing in p and the right side is continuous and strictly increasing, so a solution is unique if it exists. At $p = 0$, the left side is $\sum_{k=1}^{K} m_k max(Gap_0^k - p, 0) \geq m_K Gap_0^K > 0$ and the right side is 0; at $p = Gap_0^K$, the left side is 0 and the right side is $M_{arb} Gap_0^K > 0$; by the intermediate value theorem, a solution $b^* \in (0, Gap_0^K)$ exists. Type K participates since $b^* < Gap_0^K$; type 1 does not, since $Gap_0^1 < 0 < b^*$. Finally, at $p = b^*$, only participating types contribute to the left side of the equation, so $\sum_{k:Gap_{0^k} > b^*} m_k (Gap_0^k - b^*) = M_{arb} b^*$ if and only if $\sum_{k:Gap_{0^k} > b^*} m_k Gap_0^k = (M_{arb} + M_{long-only}) b^*$, by definition of $M_{long-only}$; the expression is positive because $b^* > 0$. □

Lemma 12. *The mandate economy has a unique equilibrium which coincides, date by date, with the equilibrium of the economy in the main text with* $S = 0$, *long-only mass* $M_{long-only}$, *and initial gap* Gap_0 *of Lemma 11.*

In particular, at every date, the long-only investors holding the asset are exactly the types with $Gap_0^k > b^*$ *(e.g., the same set forever); the price is affine in arbitrageur beliefs;* $P_t = Fund_t + Wedge_t$, *with* $Fund_t$ *and* $Wedge_t$ *as defined in the main text; and the per-period premium, defined exactly as in Lemma 6 (e.g.* $Prem_t = RP_t - E_{arb}[Payoff_{t+1}]$, *satisfies* $Prem_t = b^* \prod_{u=0}^{t-1} \frac{M_{long-only}}{M_{long-only} + M_{arb}} ValGap_t \geq 0$.

Proof. The proof is by backward induction on t, following Figure 1 and the proof of Lemma 7. We show that at every date the market has a unique clearing price, which is affine, $P_t = Fund_t + Wedge_t$, with $Wedge_t = (Prem_t + Wedge_{t+1})/R$, and $Prem_t$ given by the Lemma statement.

First, for the base case, take $t = T$. The firm ceases to exist at T, so $P_T = 0$, which is trivially affine, with $AgreedPV_T = DisputedPV_T = Wedged_T = 0$.

For the inductive step, fix $t \leq T - 1$ and suppose the claim holds at $t + 1$, so that P_{t+1} is affine with deterministic coefficients – the hypothesis of Lemma 10.

For any type k, at the proposed premium, $E_k[Payoff_{t+1}] - RP_t = ValGap_t^k - Prem_t$ by adding and subtracting $E_{arb}[Payoff_{t+1}]$. In turn, applying Lemmas 9 and 10 and the

expression for the $Prem_t$ candidate given by the statement of the Lemma, $ValGap_t^k - Prem_t = (Gap_0^k - b^*)\prod_{u<t}(1 - Resolve_u)Exposure_t$.

For a type with $Gap_0^k > b^*$, this expression is positive. Therefore the max in $max(E_k[Payoff_{t+1} - RP_t, 0)/(\gamma Var(Payoff_{t+1}))$ is slack, and by Lemma 10, type k's demand is $q_k = (E_k[Payoff_{t+1} - RP_t)/(\gamma Var(Payoff_{t+1}))$. For a type with $Gap_0^k \leq b^*$, the max binds and $q_k = 0$. Applying also the function for arbitrageur demand from Lemmas 10, aggregate demand is:

$$\frac{\prod_{u<t}(1 - Resolve_u)Exposure_t}{\gamma Var(Payoff_{t+1})}\left[\sum_{k:Gap_0^k > b^*} m_k max(Gap_0^k - b, 0) - M_{arb}b^*\right] = 0 = S,$$

by the definition of b^*. Therefore the market clears with the stated holders.

Next, observe that all demands fall in $Prem_t$ because $E_k[Payoff_{t+1}] - RP_t = ValGap_t^k - Prem_t$ as derived above. Therefore, there must be a unique $Prem_t$ that clears the market. Since we guessed and verified that the $Prem_t$ in the Lemma statement clears the market, that must be the unique market-clearing value for $Prem_t$.

Finally, to prove that the date t price is again affine, the computation in the proof of Lemma 7, of $E_{arb}[Cash_{t+1} + P_{t+1}]$, uses only the arbitrageurs, whose beliefs and updating here are identical to the main text; this part of Lemma 7's proof therefore applies unchanged. Therefore $P_t = Fund_t + Wedge_t$. This completes the proof.

We note that, having derived Gap_0 and $M_{long-only}$ in Lemma 11, the only remaining difference from the text is in $S = 0$; this is used in the main text only in Lemma 5. This proof has re-derived the later Lemmas without this result, instead using the market-clearing demands. □

Theorem 5. *Theorems 2 and 3 of the main text hold verbatim with $M_{long-only}$ and Gap_0 as derived in Lemma 11.*

Proof. By Lemma 12, the equilibrium price path is that of the main text with initial gap Gap_0. Theorems 1-3 apply to that economy verbatim, with the single Lemma 5 substitution recorded in the proof of Lemma 12 whenever their proofs route through it. At $S = 0$, the risk compensation terms vanish, so Theorem 1 gives $Wedge_t = Bubble_t$; Theorem 2 continues to give $Bubble_t = \frac{M_{long-only}}{M_{long-only}+M_{arb}} Gap_t DisputedPV_t$; Theorem 3(i) gives $A_t < 0$ if and only if $Load_{t+1} \neq 0$, as earlier; and Theorem 3(ii) is unchanged. □

Remark 1. Lemma 12 demonstrated why the participating share $M_{long-only}$ is unchanged over time: learning shrinks each type's price pressure at the same rate, so whoever stays out remains out. With $S > 0$, the risk premium becomes positive and does not shrink with

learning. As a result, the share of long-only investors participating can change over time, and the clean results of the main text are no longer feasible, though their intuition remains.