Money: The Most Liquid Good

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"The false definitions of money divide up into two main groups : those that consider it to be something more, and those that consider it to be something less, than the most saleable commodity"

—-Wilhelm Roscher (1880) [1]

"important and difficult question ... not answered by the approach taken here : the integration of money in the theory of value"

—— Gerard Debreu, *Theory of Value* (1959)

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[1] Die Grundlagen der National ökonomie, (15th edition, 1880); (Principles of Political Economy). "Die falschen Definitionen von Geld lassen sich in zwei Hauptgruppen theilen : solche, die es für mehr, und solche, die es für weniger halten, als die currenteste Waare." Earlier editions omit the phrase 'die currenteste' (most saleable). Schumpeter (1954) translation. Thanks to Stefan Schmitz for confirming Schumpeter's translation.

Money, the common medium of exchange, is derived endogenously as the low-transaction cost good in a commodity-pairwise trading post exchange economy at general equilibrium. Budget constraints apply separately at each pairwise transaction. Transaction costs create a bid/ask spread. Examples compare monetary and barter equilibria of N commodities. Barter equilibrium has [N(N-1)/2] trading posts of low trading volume. Monetary equilibrium has fewer, N active trading posts of higher volume. Thick market externality reduces transaction costs in monetary equilibrium and leads to more efficient allocation. (JEL B13, C62, C68, D02, D40, D50, D51, D62, E40, E41, E42)

Prof. Hicks (1935) posed a challenge a century ago.

> This ... is really the central issue in the pure theory of money. Either we have to give an explanation of the fact that people do hold money when [its yield is inferior to other assets] .... or we have to evade the difficulty somehow.... the great evaders would not have denied that there must be some explanation of the fact. But they would have put it down to "frictions," and since there was no adequate place for frictions in the rest of their economic theory, a theory of money based on frictions did not seem to them a promising field for economic analysis. This is where I disagree. I think we have to look the frictions in the face, and see if they are really so refractory after all. This will, of course, mean that we cannot allow them to go to sleep under so vague a title....The most obvious sort of friction, and undoubtedly one of the most important, is the cost of transferring assets from one form to another.

This essay is a response to the challenge, by introducing a detailed model of frictions. Money, the common medium of exchange, is derived here endogenously as a result of general equilibrium. The common medium of exchange is not a distinct assumption; it is an outcome of the equilibrium. Professor Hahn (1982), echoing Debreu

(1959), reminds us that the theory of money is unfinished business in formal general equilibrium theory.

> The most serious challenge that the existence of money poses to the theorist is this: the best developed model of the economy cannot find room for it ... A first, and . . . difficult . . . task is to find an alternative construction without . . . sacrificing the clarity and logical coherence . . . of Arrow-Debreu.

Jevons(1875) provides guidance.

> The earliest form of exchange must have consisted in giving what was not wanted directly for that which was wanted. This simple traffic we call barter or truck, the French *troc*, and distinguish it from sale and purchase in which one of the articles exchanged is intended to be held only for a short time, until it is parted with in a second act of exchange. The object which thus temporarily intervenes in sale and purchase is money. At first sight, it might seem that money only doubles the trouble, by making two exchanges necessary where one was sufficient; but a slight analysis of the difficulties inherent in simple barter shows that the balance of trouble lies quite in the opposite direction.

The model below posits transactions to take place at commodity-pair trading posts with transaction costs, Shapley & Shubik (1972), resulting in distinct bid and ask prices. A commodity whose transactions are characterized by consistently narrow bid/ask spreads displays peculiarly low transaction cost; it is a candidate common medium of exchange.

Money is not here defined by government issue. This essay follows Walker (1891), "Money is that money does," (typically rephrased in the 20th century as 'money is what money does'). General use of the common medium of exchange (and its market-clearing price) is derived endogenously as a property of the market equilibrium from elementary conditions. Its market-clearing value is endogenously determined like all other prices. A commodity showing endogenously determined generally lowest transaction cost becomes the common medium of exchange. Money is not an assumption; it is a conclusion.

It is possible that 'money', be any low transaction cost commodity: cigarettes, gold, silver, or cattle. Or it may be government-issued non-interest bearing debt instruments or fiduciary liabilities denominated therein. That is to be market-determined.

The distinctive property of the model here is that transactions take place at commodity-pair trades. This is in contrast to an Arrow-Debreu Walrasian model where N commodities trade directly with a comprehensive market that accepts all goods in supply and renders all goods demanded. Trade here takes place at $\frac{1}{2}N(N-1)$ distinct commodity-pair trading posts. Then commodity (or fiduciary) money becomes useful to carry purchasing power from one transaction to the next. When all (or most) of $\frac{1}{2}N(N-1)$ trading posts are active in equilibrium, that is a barter equilibrium. A monetary equilibrium occurs when the trading posts are all priced, but active trade occurs at only N trading posts exchanging the N goods for a monetary instrument, or at $N-1$ trading posts where one of the N goods acts as a common medium of exchange, commodity money.

A class of examples demonstrates a foundation in price theory for several observations in monetary economics, Starr (2003):

(i) Trade is monetary. One side of almost all transactions is the economy's common medium of exchange. *Resolution:Money is derived as the lowest transaction cost commodity. Optimizing transactors choose to trade at trading posts that trade that instrument, implying high yield, low transaction cost, in satisfying the budget constraint.*

(ii) Money is (locally) unique. *Resolution: Scale economy in the transaction technology (a thick-market external effect treated parametrically at each transaction) leads to — and reinforces — concentration on a single instrument.*

(iii) Transactions displaying a double coincidence of wants — con-

ventionally thought to be suitable for barter exchange — are transacted with money rather than direct barter. *Resolution:The scale economy in transaction cost makes monetary exchange (on thick money-goods markets) for reciprocally desired goods less costly than direct exchange (on thin commodity-commodity markets).*

Several references explain that government-issued fiat money has equilibrium pricing at a positive value though fiat money conveys directly no utility or production, based on fiat money's acceptability in tax payments: Smith(1776), Knapp (1905), Lerner(1947), Tobin (1980), Goldberg (2009, 2018). Treating this issue is beyond the scope of this essay.

I. Liquidity

Carl Menger (1892) tells us

> Goods [are] ...more or less saleable [*absatzfahig* marketable], according to the greater or less facility with which they can be disposed of ... at current purchasing prices, or with less or more diminution.
>
> The theory of money necessarily presupposes a theory of the saleableness [*Absatzfahigkeit*, marketability] of goods.
>
> when any one has brought goods not highly saleable to market, the idea uppermost in his mind is to exchange them, not only for such as he happens to be in need of, but ... for other goods ... more saleable than his own .... By ... a mediate exchange, he gains the prospect of accomplishing his purpose more surely and economically than if he had confined himself to direct exchange .... Men have been led ... without convention, without legal compulsion,... to exchange ... their wares ... for other goods ... more saleable ... which ... become generally acceptable media of exchange.

This paper uses a general equilibrium model to formulate, as a result of price theory, the existence and universal use of a most liquid (saleable) good as a locally unique medium of exchange. In the Arrow-Debreu model, agents deal with a single budget constraint including all purchases and sales. In contrast, the present model uses

a trading post setting. For N commodities, there are $\frac{1}{2}N(N-1)$ commodity-pairwise trading posts, each with its own budget constraint and price ratios.

Walras (1874) puts it this way:

> we shall imagine that the place which serves as a market for the exchange of all the commodities (A), (B), (C), (D)for one another is divided into as many sectors as there are pairs of commodities exchanged. We should then have $\frac{m(m-1)}{2}$ special markets each identified by a signboard indicating the names of the two commodities exchanged there as well as their ... rates of exchange

A household trading in ξ transactions deals in ξ distinct payment requirements. That formalization creates the need for a medium(a) of exchange, a commodity carrying purchasing power from one trading post to the next. Transaction costs mean that there is a significant choice as to the commodity(ies) that fulfill that function. A thick market externality, reflecting an underlying scale economy, leads to local uniqueness of the medium of exchange.

There is now a formal theory of frictions. They are transaction costs. The treatments of Foley (1970), Hahn (1971, 1973), Heller and Starr (1976), Kiyotaki and Wright(1989), Kurz(1974), Starr(2008), and Starrett (1973) make transaction costs explicit. Baumol (1952) and Tobin (1956) incorporate explicit transaction costs in a partial equilibrium two-asset model.

To derive a theory of money from price theory, it is needed to use a model of at least three commodities and at least three transactors. In the case of two commodities, there is only one relative price and direct exchange can fulfill all trading needs. With only two traders, once general equilibrium price ratios have been determined, there is double coincidence of wants. Traders can then satisfactorily make direct exchange of excess supplies for excess demands. It is with at least three commodities and three traders that the problem of absence of 'double coincidence of wants' arises, Jevons(1875), Wicksell

(1935).

Sufficient conditions for existence of general equilibrium in a trading post model are developed in Starr(2008). This essay follows the notation there. N commodities are priced for commodity-pairwise trade at $\frac{1}{2}N(N-1)$ trading posts. Starr(2008) includes a time structure. In the present essay, the time index is suppressed; a single trading period is considered. An equilibrium occurs where all trading posts are priced, and trading posts clear. Some trading posts may be priced but inactive at equilibrium. Goods acquired by a household at one trading post and disbursed by the household at another are acting as commodity money.

Goods i and j are traded pairwise at trading post $\{i,j\} \equiv \{j,i\}$, at the post's bid and ask price ratios without distinguishing a unit of account. Let Δ represent the unit 1-simplex. At trading post $\{i,j\}$, the (relative) ask price of good i and (relative) bid price of good j are represented as $a_i^{\{i,j\}}$, and $b_j^{\{i,j\}}$. Prices at $\{i,j\}$ are represented as $p^{\{i,j\}} = (a_i^{\{i,j\}}, b_j^{\{i,j\}}) \in \Delta$ and $p^{\{j,i\}} = (a_j^{\{i,j\}}, b_i^{\{i,j\}}) \in \Delta$. The price ratio is meaningful. The ask price of i in units of j at j's bid price is $\frac{a_i^{\{i,j\}}}{b_j^{\{i,j\}}}$. The bid price of i in units of j is $\frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}}$, and so forth.

There are N tradeable commodities, and $\frac{1}{2}N(N-1)$ trading posts. Thus the overall price space is the $N(N-1)$-fold product of the unit simplex with itself, $\Delta^{N(N-1)}$. The space of commodities will be denoted $\mathcal{N}$.

For the household h or firm f, transactions at ask prices in i are $y_i^{h\{i,j\}}$, $y_i^{f\{i,j\}}$, respectively. Transactions at bid prices in j are $x_j^{h\{i,j\}}$, $x_j^{f\{i,j\}}$.

$y_i^{h\{i,j\}}$, $y_i^{f\{i,j\}}, x_j^{h\{i,j\}}$, and $x_j^{f\{i,j\}} \in R$. They are one-dimensional, scalars. They are i^{th} and j^{th} co-ordinates of $y^{h\{i,j\}}$, $y^{f\{i,j\}}, x^{h\{i,j\}}$, and $x^{f\{i,j\}} \in R^N$. Most of the co-ordinates of these vectors are zero, the i^{th} and j^{th} terms representing actual transactions. Household h's array of transactions at bid prices is

$$x^h \equiv \sum_{i,j \in \mathcal{N}, i \neq j} x^{h\{i,j\}}; \qquad x^h \in R^N \tag{1}$$

h's array of transactions at ask prices is

$$y^h \equiv \sum_{i,j \in \mathcal{N}, i \neq j} y^{h\{i,j\}}; \qquad y^h \in R^N \tag{2}$$

Then the budget constraint at each trading post is

$$0 = (a_i^{\{i,j\}}, b_j^{\{i,j\}}) \cdot (y_i^{h\{i,j\}}, x_j^{h\{i,j\}}) \tag{3a}$$

$$0 = (a_j^{\{i,j\}}, b_i^{\{i,j\}}) \cdot (y_j^{h\{i,j\}}, x_i^{h\{i,j\}}) \tag{3b}$$

$$0 = (a_i^{\{i,j\}}, b_j^{\{i,j\}}) \cdot (y_i^{f\{i,j\}}, x_j^{f\{i,j\}}) \tag{3c}$$

$$0 = (a_j^{\{i,j\}}, b_i^{\{i,j\}}) \cdot (y_j^{f\{i,j\}}, x_i^{f\{i,j\}}) \tag{3d}$$

(3a-d) say that purchases of i at the bid price are repaid by sales of j at the ask price, purchases of i at the ask price are repaid by sales of j at the bid price.

By definition, a competitive equilibrium occurs when housheolds h and firms f are optimizing at prices $p^* \in \Delta^{N(N-1)}$, so that trading posts clear. That is, for each $\{i,j\}$ so that $i,j, \in \mathcal{N}; i \neq j$

$$0 = \sum_{f \in F} (y_i^{f\{i,j\}}, x_j^{f\{i,j\}}) + \sum_{h \in H} (y_i^{h\{i,j\}}, x_j^{h\{i,j\}}) \tag{4}$$

where 0 is the zero vector in R^2.

Household h has endowment $r^h \in R_+^N$. The ℓ^{th} co-ordinate of r^h

is r_ℓ^h.

Household h's consumption vector is $c^h \in R^N$.

$$c^h \equiv r^h + \sum_{i,j \in \mathcal{N}, i \neq j} (y^{h\{i,j\}} + x^{h\{i,j\}}) \tag{5}$$

A household may acquire a commodity at one trading post and disburse it at another. That commodity is performing a monetary function, acting as a (commodity) money. Bid/ask price spreads are determined competitively as a result of transaction technology. Households make trading decisions considering prevailing prices that incorporate transaction costs. A transaction decision considering the tranaction costs embodied in the bid/ask spread does, as Hicks (1935) recommends, "look the frictions in the face."

A. Media of Exchange

Let (y^h, x^h) represent household h's $2N(N-1)$-dimensional transaction vector. Some transactions will be selling endowed goods to purchase acquisitions to be retained. Other purchases may be re-traded. The re-traded items are acting as commodity money. The x^h co-ordinates are at bid prices; they are typically sales (negative sign). The y^h co-ordinates are transactions at ask prices; they are typically purchases (positive sign). Then we can characterize h's gross transactions in good i as $\sum_j |y_i^{h\{i,j\}}| + \sum_j |x_i^{h\{i,j\}}| = \sum_j y_i^{h\{i,j\}} - \sum_j x_i^{h\{i,j\}} \equiv \gamma_i^h$. Further, the absolute value of h's net transactions in good i is $|\sum_j y_i^{h\{i,j\}} + \sum_j x_i^{h\{i,j\}}| \equiv \nu_i^h$.

The N-dimensional vector γ^h with typical element γ_i^h is h's gross trade. The N - dimensional vector ν^h with typical element ν_i^h is h's net trade vector (in absolute value). $\mu^h = \gamma^h - \nu^h$ is h's flow of goods as media of exchange, gross trades minus net trades.

The total (N-dimensional vector) flow of media of exchange among households is then $\sum_{h \in H} \mu^h$. This expression, is the flow of (com-

modity) monies. Thus, the trading post equilibrium establishes a well-defined demand for media of exchange as an outcome of the market equilibrium. Media of exchange (commodity monies) are commodities acquired to be re-traded. They are goods flows acting as the carrier of purchasing power between transactions, the difference between gross and net trades.

This paper assumes, without proof, that external effects in the trading post model are consistent with existence of competitive equilibrium. The issue is treated for an Arrow-Debreu model in Arrow and Hahn (1971). It is treated particularly for a trading post model in Starr and Spini (2021).

II. The Most Liquid Good

In a general equilibrium of the trading post economy, we can describe pricing that would characterize commodity m as a 'most liquid good.' Liquidity of m refers to the facility with which m can acquire and be acquired by other goods in trade. The definition below characterizes m's ability to deal with other goods with which it trades. There is no guarantee that a 'most liquid good' exist, nor that it be unique. The set of most liquid goods will be designated $\mathcal{M}$. Existence of such $m \in \mathcal{M}$ is a property of the equilibrium to be determined, for example as a cigarette currency, Radford (1945). The most liquid good may be endogenously determined as the non-interest-bearing debt of the government, Knapp (1905).

Let $1 \leq m, i, j, k \leq N$. There will be a net transaction cost on disbursing one good for m and acquiring another good in exchange for m. If m is a most liquid good, that loss should be narrower than for any other similar successive exchange. Then m is a most liquid good if commodity m has the following property:

$$(6) \qquad \frac{b_i^{\{m,i\}}}{a_m^{\{m,i\}}} \cdot \frac{b_m^{\{m,j\}}}{a_j^{\{m,j\}}} \cdot \frac{b_j^{\{m,j\}}}{a_m^{\{m,j\}}} \cdot \frac{b_m^{\{m,i\}}}{a_i^{\{m,i\}}} \geq \frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} \cdot \frac{b_j^{\{i,j\}}}{a_i^{\{i,j\}}}$$

with a strict inequality for all $j, i \notin \mathcal{M}$. Conversely, if there is m fulfilling (6) and i, j so that (6) is fullfilled with a strict inequality, then i, j are said to fulfill (*not* 6).

Property (6) says that a round trip from i to j back to i by way of m incurs lower transaction cost than direct transit i to j back to i.

(6) says that trade of i for j for i through m as an intermediary has a higher return than direct trade. And that observation is true for all i and j. That result says that m is 'most liquid'.

Of course, no individual trader sets out to make a round trip trade. But every trade of i for j includes a counterpart trading j for i. The net transaction cost of their trade enters into its shared desirability.

III. The Most Liquid Good is the Universal Medium of Exchange

In the trading post model, it is the price system, **including transaction costs**, that guides exchange. At an equilibrium allocation, the theorem below demonstrates that the most liquid good(s) is the medium of exchange, acquired at one trading post to be disbursed at another. That is not automatic. An autarkic equilibrium may occur if transaction costs are prohibitively high. The notion of transaction costs sufficiently moderate that buying and selling is desirable is summarized here in property (**D**).

Definition: $h \in H$ is said to have property (**D**) if there is $\mathcal{I}^h \subseteq \mathcal{N}, \mathcal{I}^h \neq \phi$, $m \in \mathcal{M}$,

where $y'^h_j \equiv y^h_j + \frac{b^{\{m,j\}}_m}{a^{\{m,j\}}_j}$ for all $j \in \mathcal{I}$ so that
$u^h(x^h + y'^h) > u^h(x^h + y^h)$.

Property (**D**) includes two qualities of the equilibrium allocation. It says that h finds goods in $\mathcal{I}^h$ desirable, and that – at prevailing prices, taking account of transaction costs — acquiring more by trading through m would be utility-increasing. In a microeconomic model without transaction costs (e.g. in an Arrow-Debreu model), it would be sufficient to assume that some goods in $\mathcal{I}^h$ are desirable. Taking account of transaction costs here makes that a more detailed evaluation. **Theorem** Let $\mathcal{M} \neq \phi$. Let $N \geq 4$, and

$[p, y^h, x^h, y^f, x^f]$,
$p \in \Delta^{N(N-1)}, h \in H, f \in F$, be a general equilibrium. For all $h \in H$, let h have property (**D**). For all $i, j, k \notin \mathcal{M}$ assume that property (not 7) holds.

Then there is no $h \in H$ where $i, j, k \notin \mathcal{M}$, so that both $x^{h\{i,j\}}_i < 0$ and $y^{h\{i,k\}}_i > 0$. Conversely, commodity(ies) $m, \ell \in \mathcal{M}$ is (are) the only commodity(ies) in which $x^{h\{\ell,j\}}_\ell < 0$ and $y^{h\{\ell,k\}}_\ell > 0$ or $x^{h\{m,j\}}_m < 0$ and $y^{h\{m,k\}}_m > 0$ are both non-zero in equilibrium for some $h \in H$.

The theorem hypthesizes that households can find attractive acquisitions through expenditure of ℓ, m. Buying and selling the same commodity m, ℓ is treating that commodity as a medium of exchange. Of course that is just what $m, \ell \in \mathcal{M}$ are well positioned to do. And other commodities have prohibitive transaction costs for doing so. So it is only m, ℓ that act as media of exchange.

Proof: Let m denote the typical element of $\mathcal{M}$. Here is a proof by contradiction. Suppose contrary to the theorem that there is household h, and commodities $i, j, k \notin \mathcal{M}$ so that $x^{h\{i,j\}}_i < 0$ and $y^{h\{i,k\}}_i > 0$. Then we have $(a^{\{i,j\}}_i, b^{\{i,j\}}_j) \in \Delta$, $(a^{\{i,j\}}_j, b^{\{i,j\}}_i) \in \Delta$,

$(a_i^{\{i,k\}}, b_k^{\{i,k\}}) \in \Delta$, $(a_k^{\{i,k\}}, b_i^{\{i,k\}}) \in \Delta$, so that
$b_i^{\{i,j\}} \cdot x_i^{h\{i,j\}} + a_j^{\{i,j\}} \cdot y_j^{h\{i,j\}} \leq 0$; equivalently

$$\frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} \cdot x_i^{h\{i,j\}} + y_j^{h\{i,j\}} \leq 0 \tag{7}$$

$$b_k^{\{i,k\}} \cdot x_k^{h\{i,k\}} + a_i^{\{i,k\}} \cdot y_i^{h\{i,k\}} \leq 0 \tag{8}$$

equivalently

$$\frac{b_k^{\{i,k\}}}{a_i^{\{i,k\}}} \cdot x_k^{h\{i,k\}} + y_i^{h\{i,k\}} \leq 0 \tag{9}$$

Selling k to acquire j with i as the intermediary good:

$$\frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} \cdot x_i^{h\{i,j\}} + y_j^{h\{i,j\}} \leq 0; \qquad \frac{b_k^{\{i,k\}}}{a_i^{\{i,k\}}} \cdot x_k^{h\{i,k\}} + y_i^{h\{i,k\}} \leq 0. \tag{10}$$

Household h is selling k for i and selling i for j. Net, h is trading k for j with i as intermediary. h is both buying and selling commodity i (at different trading posts). Commodity i is acting as a commodity money.

We claim that commodity m is priced to perform this function more rewardingly. Consider re-pricing these transactions in commodity m instead of i.

Selling k to acquire j with m as the intermediary good:

$$\frac{b_m^{\{m,j\}}}{a_j^{\{m,j\}}} \cdot x_m^{h\{m,j\}} + y_j^{h\{m,j\}} \leq 0; \qquad \frac{b_k^{\{m,k\}}}{a_m^{\{m,k\}}} \cdot x_k^{h\{m,k\}} + y_m^{h\{m,k\}} \leq 0. \tag{11}$$

Then, using m as the intermediary good the result is

$$|y_j^{h\{m,j\}}| = \frac{b_m^{\{m,j\}}}{a_j^{\{m,j\}}} \cdot \frac{b_k^{\{m,k\}}}{a_m^{\{m,k\}}} \cdot x_k^{h\{m,k\}}. \tag{12}$$

Alternatively, using i as the intermediary good, the result is

$$|y_j^{h\{i,j\}}| = \frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} \cdot \frac{b_k^{\{i,k\}}}{a_i^{\{i,k\}}} \cdot x_k^{h\{i,k\}}. \tag{13}$$

Then by hypothesis, for any value of $x_k^{h\{m,k\}}$ set equal to $x_k^{h\{i,k\}}$, it follows that $|y_j^{h\{m,j\}}| > |y_j^{h\{i,j\}}|$. Trading in m is more rewarding. Assuming h's fulfillment of property (D), m is the superior intermediary good. The contradiction is confirmed.

Monotonicity of f's maximand gives the same result. **QED**

Condition (D) formalizes the notion that because m is a most liquid good, it incurs lower transaction costs — delivers more — in buying and selling than goods not in $\mathcal{M}$. Then it is goods in $\mathcal{M}$ that are used as media of exchange, because they return to the transacting household more of the acquisitions.

IV. Endogenous Determination of Liquidity

Roth(2008) reminds us that liquidity requires a thick market. Diamond(1984) presents a model of endogenous determination of liquidity through market thickness, summarized below, as a model of trading frequency (*N.B. Notation*: There is a difference in notation: Diamond(1984) denotes by "m" a number of transactors and "f" denotes a function):

> Assume there are m individuals ... trying to buy the commodity and e individuals ... trying to sell. ... Then, the rate of completion

> of transactions will be written as $f(e,m)$, which is assumed to be twice continuously differentiable ....
>
> $f_e(e,m) > O, f_m(e,m) > O$ when $m > O, e > O$. ... we make the assumption of increasing returns to scale: ... $ef_e + mf_m > f$.

The notion of scale economy, for the economy as a whole, implies an externality for the typical transactor. Assume transaction costs to be inversely proportional to the rate of transaction completion in Diamond(1984). Then as trading volume increases, average and marginal transaction costs decline. Following the Diamond(1984) specification, we'll state this notion as a marginal transaction cost varying inversely with trading volume, $f(e,m)$.

This constitutes a thick market external economy, Coppola et al (2023), Rey(2001), Roth (2008).

V. Thick Market Externality and the Monetization of Trade

A. Monetary Trade in the Trading Post Economy

In the trading post model with N commodities, there are $\frac{1}{2}N(N-1)$ commodity pairwise trading posts. Each good can trade directly against any other at their joint trading post. If some single commodity becomes the common medium of exchange, then most of the trading posts are priced but inactive. There will then be N (or $N-1$) active posts. In general equilibrium, that's not about 'legal tender' or the power of gold. The result comes from the evolution of transaction costs and the resulting market equilibrium.

Prof. Hahn (1971) notes the scale economy.

> the pure transaction theory of money seems to require in an essential way that the set of marketing activities exhibits a certain kind of increasing returns

Menger (1892), focusing on the external economy generating consensus, tells us

> Money has not been generated by law. In its origin it is a social, and not a state-institution.

In the present paper assume that the scale economy is at the level of the trading post as a whole, where there are many active marketing firms, not on the part of any single firm. Scale then affects each market-maker as an external effect. Tobin (1980) describes this issue as a public good, analogous to language.

> Social institutions like money are public goods. ... Another time-honored observation ... is the analogy of money and language. Both are means of communication. The use of a particular language or money by one individual increases its value to other actual or potential users. Increasing returns to scale in this sense limits the number of languages or moneys in a society and indeed explains the tendency of one language or money to monopolize the field.

The treatment in this paper is to model unique money as the market equilibrium outcome of an external economy, rather than as a public good. General equilibrium with external effects is treated in Arrow and Hahn (1971) for an Arrow-Debreu model. Extension to a trading post model is in Starr and Spini (2021).

VI. A Class of Examples

The balance of the paper develops in detail a class of examples. There are several principal constructs: a tractable population of transactors; specification of a transaction cost structure that emphasizes the scale economy of a thick market externality; general equilibrium pricing.

There are assumed to be $N \geq 4$ distinct commodities, implying $\frac{1}{2}N(N-1)$ commodity pair trading posts. The trading posts are denoted $\{i,j\} \equiv \{j,i\}, i \neq j, 1 \leq i,j, \leq N$. They will all be priced in

equilibrium with bid and ask prices for each commodity pair. Each trading post may be active in equilibrium or inactive, depending on endowments, demands, and transaction costs. Barter and monetary trade are outcomes of the equilibrium. A barter equilibrium may have an abundance of trading posts active in equilibrium or only a few. A monetary equilibrium occurs if activity is concentrated on N or $N-1$ trading posts $\{m,j\},\{j,m\}, 1 \leq j \leq N$, where m is the endogenously determined locally unique monetary instrument.

VII. Transaction Costs with Thick Market Externality

The expression $\left[\frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} \cdot \frac{b_j^{\{i,j\}}}{a_i^{\{i,j\}}}\right]$ represents the yield of a a round trip transaction from i to j and back to i. Selling i at the bid price, buying j at the ask price, selling j at the bid price to purchase i at the ask price. The expression $1 - \frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} \cdot \frac{b_j^{\{i,j\}}}{a_i^{\{i,j\}}}$ is the proportion lost in the round trip, that is the bid/ask spread for i and j. In the case of perfect liquidity — the Arrow-Debreu model — the yield equals 1, unity. Alternatively if the bid price of i, $\frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}}$ were zero, 0, reflecting complete illiquidity, then the expression $1 - \frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} \cdot \frac{b_j^{\{i,j\}}}{a_i^{\{i,j\}}}$ would be unity. In that case, the bid/ask spread would be 100%, meaning that all the value in the transaction is expended in the transactions costs.

To develop the example in more detail we need to make transaction cost more precise. Let $V^{\{i,j\}}$ denote trading volume at trading post $\{i,j\}$ normalized so that volumes of i and j are commensurable. In a market equilibrium, buying and selling volume will be equivalent. Let $\alpha = 0.001; \beta = 0.999$. We'll use the following specification:

$$\frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} \cdot \frac{b_j^{\{i,j\}}}{a_i^{\{i,j\}}} \equiv 1 - [\alpha + \beta \times \frac{1}{1 + V^{\{i,j\}}}] \tag{14}$$

In this expression, at very low volume, transactions costs absorb nearly 100% of a trade. At very high volume they approach α, virtually negligible .

The specification reflects a thick markets externality. High trading volume — for the whole trading post (not only a single trading firm) — generates low marginal and average transaction cost. Transaction costs at $\{i, j\}$ for each of many price-taking firms will be treated as linear in the transactions executed at the firm but inversely proportional to the total volume of trade at the post $\{i, j\}$. The trading post's volume , $V^{\{i,j\}}$ is treated parametrically by each marketing firm.

VIII. How should we interpret the bid/ask spread in this model?

Transaction costs are characterized here by the bid/ask spread $1 - \frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} \cdot \frac{b_j^{\{i,j\}}}{a_i^{\{i,j\}}}$. That is by the resources used up in transaction costs stated as a proportion of the goods transacted. In well-organized financial markets the spread is recognized explicitly as the difference between bid and ask prices stated in terms of currency. But how should we think about it in the more general context of this model, where the terms of trade are stated in terms of commodity exchange ratios?

Transaction costs are formalized in the trading post general equilibrium model as incurred by marketing firms. The firms are peculiarly capable of buying at bid prices and selling at ask prices. Marketing firms, denoted $f \in F$, behave as competitive price tak-

ers, treating prevailing aggregate trading volumes and the resulting transaction costs parametrically. There is a distinct external effect so that trading volumes at trading post $\{i, j\}$ significantly affect firm f's trading technology, following the specification above. Competition among firms and eventual market clearing establishes the equilibrium bid and ask prices and the bid/ask spread. That formalization may be regarded as a metaphor for the calculation of a typical household transactor considering barter: "trading i for j directly, it takes rather a long time to find a counter-party. And there's transport cost, use of gasoline and wear on the car."

In a monetary trade, we can view the bid/ask spread as the difference between wholesale and retail prices. But barter emphasizes commodity to commodity trades. So the bid/ask spread is interpreted as the value, denominated in the goods transacted at the trading post, of the resources expended in the transaction process. When barter is not as efficient as monetary trade, that is demonstrated by the resources used up as transaction costs in barter being more valuable than those expended similarly in monetary trade.

IX. The Clock-Face Economy

This section specifies a class of examples suitable for demonstrating the transaction costs of monetary and barter trade. The specification allows: computation of results; variation in the number of commodities and the number of traders; variation from absence of double coincidence of wants to complete double coincidence; scale economy of transaction cost exogenous to each market-making firm based on a thick market externality.

Denote the following specification as the Clock-Face economy. Household endowments are distributed at numbered positions round a circle, like a clock-face. $h \in H$ is endowed with one unit of

$i = h(modN)$. Each household desires L goods clockwise from his endowed position.

Let the population be $H \equiv \{h|h = 1, 2, \ldots, N, N+1, \ldots, (K-1)N, \ldots KN\}$. $N \geq 3$. Each household h is identified with an endowed commodity $i = h(modN)$. The population consists of K replicas of a subpopulation of N households, totalling KN households in all. . Let the notation $\ell \oplus n$ denote $(\ell + n)(modN)$.

N is the number of distinct commodities; K is the number of replications of the subpopulation of N households in the grand population $H = \{h|h = 1, 2, \ldots, N, N+1, \ldots, 2N, 2N+1, \ldots, 3N, \ldots, KN-1, KN\}$. K may be termed the "size of the economy." A typical household h is endowed with one (divisible) unit of commodity $i = h(modN)$. $r_i^h = 1$, $x^h \equiv (x_1^h, x_2^h, \ldots, x_N^h)$. Household h's consumption vector is $c^h \equiv r^h + \sum_{i,j \in \mathcal{N}, i \neq j} (y^{h\{i,j\}} + x^{h\{i,j\}})$

A parameter of each specification will be $L \in \{1, 2, \ldots, (N-1))\}$, described as "scope of purchases."

$h \in H$ has utility function

(15) $u^h(x^h) = \min\{x_n^h | n = i \oplus 1, \ldots, i \oplus L\}$ where $i = h(modN)$.

u^h represents h's Leontieff preferences over L of the N commodities. Welfare will be represented as the sum over households $h \in H$ of $u^h(x^h)$.

Each $h \in H$ is endowed with 1 (divisible) unit of commodity $h(modN) \equiv i$ and desires consumption of commodities $i \oplus 1, i \oplus 2, \ldots, i \oplus L$. Think of the N commodities arrayed over a clockface. Each household desires the L commodities clockwise from his endowment.

Depending on the specification of L there are some notable cases of demand. For $L \leq (\frac{N-1}{2})$ there is complete absence of double coincidence of wants. In this case each household desires the L goods clockwise from his own; none of the owners of those goods

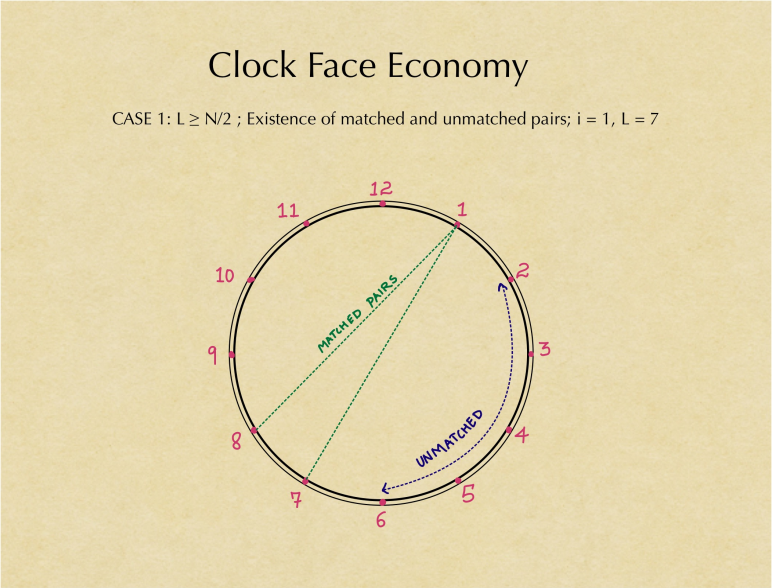


Figure 1: Clock Face Economy

wants his/her good $i = h(modN)$.

For $L = N - 1$ there is complete double coincidence of wants. Each household $h(modN) = i$ is endowed with good i and desires to exchange i for all other goods. And the owners of all those other goods desire good i. In that case, at each of $\frac{1}{2}N(N-1)$ pairwise trading posts $\{i, j\}, j \neq i; i, j \in \{1, 2, \ldots, N\}$ there are reciprocally satisfactory trades available, double coincidence of wants.

There may be multiple equilibria of the clock-face economy including barter and monetary equilibrium. A barter equilibrium will typically have active trade at each of the $\frac{1}{2}N(N-1)$ trading posts. A monetary equilibrium will include designation of a single common medium of exchange m with active trade at N or $N-1$ trading posts $\{i, m\}; i = 1, 2, \ldots, N$.

X. Transaction Yield

A. A single transaction

At a typical active trading post in a barter equilibrium, trading volume is K/L. For a single trade, the transaction yield (net of transaction cost) in the barter economy will be denoted $\mathcal{BAR}(K, L)$.

So

$$\mathcal{BAR}(K,L) = \frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} \cdot \frac{b_j^{\{i,j\}}}{a_i^{\{i,j\}}} = 1 - [\alpha + \beta \times [\frac{1}{1+(K/L)}]] \tag{16}$$

In the monetary equilibrium, the trading posts $\{m, j\}, j \in \mathcal{N}$, are the active trading posts. The transaction yield in the monetary economy will be denoted $\mathcal{MON}(K, L)$

So

$$\begin{aligned} \mathcal{MON}(K,L) &= \frac{b_i^{\{i,m\}}}{a_m^{\{i,m\}}} \cdot \frac{b_m^{\{i,m\}}}{a_i^{\{i,m\}}} \times \frac{b_m^{\{m,j\}}}{a_j^{\{m,j\}}} \cdot \frac{b_j^{\{m,j\}}}{a_m^{\{m,j\}}} \\ &= \left\{1 - \left[\alpha + \beta \times \left[\frac{1}{1+K}\right]\right]\right\}^2 \end{aligned} \tag{17}$$

For a single transaction the yield (net of transaction cost) reflects the differing costs depending on trading volume. Markets in the barter economy are thin, with typical volume (K/L). Monetary markets are thicker with typical volume K. Those differences appear in the denominators of the cost expressions. But monetary exchange requires two distinct transactions for each commodity pair exchanged. That shows up in the squared term of the monetary yield.

XI. No Coincidence Economy, L $< \frac{N}{2}$

A. Structure of Demand

When L $< \frac{N}{2}$, in a barter equilibrium, each $h \in H$ supplies his $i = h(modN)$ to trading posts $\{i, i \oplus 1\}, \{i, i \oplus 2\}, \ldots, \{i, i \oplus L\}$. None of the households at the receiving trading posts has use for i, so i will be retraded.

B. Notation

N commodities; N even, $N \geq 4$; K replications of N households; L breadth of household purchases;

$L \leq [\frac{N}{2} - 1]$

The transaction cost of a single purchase and sale transaction differs between barter and monetary transactions reflecting the markets respective trading volumes. Active barter markets are thin, typically with volume K/L.

C. Tracking Trade in a Barter Equilibrium of the No Coincidence Economy: L $< \frac{N}{2}$

In a clock face economy, a household of type n is endowed with one unit of good n and seeks to acquire goods type $n \oplus 1, n \oplus 2, \ldots, n \oplus L$. Trades at bid price, in quantity $(1/L)$ of good n, at posts $\{n, n \oplus 1\}, \{n, \oplus 2\}, \ldots, \{n, n \oplus L\}$ accomplish those trades. Eventually good n arrives at $\{n, n \oplus (-L)\}, \{n, n \oplus (-L+1)\}, \{n, n \oplus (-L+2)\}, \ldots, \{n, n \oplus (-1)\}$, where it is traded to a final demander. There are NL active trading posts, with sales at bid price, in quantity $\frac{K}{L}$ at each active trading post.

D. Yield per transaction

The yield at a single trade in the barter economy will be denoted $\mathcal{BAR}(K,L)$.

So

$$\mathcal{BAR}(K,L) = \frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} \cdot \frac{b_j^{\{i,j\}}}{a_i^{\{i,j\}}} = 1 - \{\alpha + \beta \times [\frac{1}{1+(K/L)}]\} \tag{18}$$

E. Total equilibrium Yield in the Barter Economy with $\mathbf{L < \frac{N}{2}}$

$$\mathcal{BBBALT}(N,K,L) \equiv NL\frac{K}{L}\mathcal{BAR}(K,L) \tag{19}$$

So

$$\mathcal{BBBALT}(N,K,L) = NL\frac{K}{L}\Big[1 - \{[[\alpha + \beta \times [\frac{1}{1+(K/L)}]\}\Big] \tag{20}$$

$$\mathcal{BBBALT}(N,K,L) = NK\Big[1 - \{[[\alpha + \beta \times [\frac{1}{1+(K/L)}]\}\Big] \tag{21}$$

For each $h \in H, h(modN) = i, i \neq j, j = i \oplus 1, j = i \oplus 2, \ldots, j = i \oplus L$,

$$\frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} = 1 - \{[[\alpha + \beta \times [[\frac{1}{1+(K/L)}]]\} \tag{22}$$

The pricing here, for $i = h(modN)$, and j at 1 to L clockwise positions from i, reflects that $h(modN) = i$ is eager to acquire j, so that i sells for j at a discounted price to cover <u>all</u> of the transaction cost. $\frac{b_j^{\{i,j\}}}{a_i^{\{i,j\}}} = 1$

$$\frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} \cdot \frac{b_j^{\{i,j\}}}{a_i^{\{i,j\}}} = 1 - \{[[\alpha + \beta \times [[\frac{1}{1 + (K/L)}]]\} \tag{23}$$

let $x_i^{h\{i,j\}} = -\frac{1}{L}$, $y_j^{h\{i,j\}} = \frac{1}{L} \times \Big[1 - \{[[\alpha + \beta \times [[\frac{1}{1+(K/L)}]]\}\Big]$. Let firm f be an active market maker at the $\{i, j\}$ trading post. f is assumed to act as a competitive price taker treating trading volume at trading post $\{i, j\}$ parametrically.

$$\sum_{h \in H} x_i^{h\{i,j\}} = -K\frac{1}{L} \tag{24}$$

$$\sum_{h \in H} y_j^{h\{i,j\}} = K\frac{1}{L} \times \Big[1 - \{[[\alpha + \beta \times [[\frac{1}{1 + (K/L)}]]\}\Big] \tag{25}$$

$$\sum_{f \in F} x_i^{f\{i,j\}} = K\frac{1}{L}; \tag{26}$$

$$\sum_{f \in F} y_i^{f\{i,j\}} = -K\frac{1}{L}\Big[1 - \{[[\alpha + \beta \times [[\frac{1}{1 + (K/L)}]]\}\Big] \tag{27}$$

all $i = 1, 2, \ldots, N$.

XII. Monetary Equilibrium of the Clock-Face Economy with $L < \frac{N}{2}$

The character m will typically denote the monetary instrument in the economy. In the case of commodity money, $m \in \mathcal{N}$. For fiat money, $m \notin \mathcal{N}$. In the case of fiat money, m is not one of the N com-

modities — no one actually desires m. Then in a fiat monetary equilibrium, there will be N active trading posts $\{m, i\}; i = 1, 2, \ldots, N$.

In the case of commodity money m is one of the N commodities, and there will be households desiring m. Then there will be $N - 1$ active trading posts $\{m, i\}; i = 1, 2, \ldots, m-1, m+1, \ldots, N$. There is no $\{m, m\}$ post.

To avoid repetitive notation, the examples will assume fiat money so that there are N non-monetary commodities. All trade in the monetary equilibrium is at the N trading posts $\{m, i\}; i = 1, 2, \ldots, N$.

XIII. Monetary Equilibrium with $L < \frac{N}{2}$

A. Yield per transaction

The transaction yield in the monetary economy will be denoted $\mathcal{MON}$(K, L) .

So

$$\begin{aligned} \mathcal{MON}(K,L) &= \frac{b_i^{\{i,m\}}}{a_m^{\{i,m\}}} \cdot \frac{b_m^{\{i,m\}}}{a_i^{\{i,m\}}} \times \frac{b_m^{\{m,j\}}}{a_j^{\{m,j\}}} \cdot \frac{b_j^{\{m,j\}}}{a_m^{\{m,j\}}} \\ &= \left\{1 - \left[\alpha + \beta \times \left[\frac{1}{1+K}\right]\right]\right\}^2 \end{aligned} \tag{28}$$

B. Total equilibrium Yield in monetary equilibrium

There are N active trading posts hosting K trades of m for commodities $i, j, \ldots$. Total yield then is

$$\mathcal{MMM}(N,K,L) \equiv (N)K \times \mathcal{MON}(K,L) \tag{29}$$

. Equivalently

$$\mathcal{MMM}(N,K,L) \equiv (N)K\Big[\{1 - \ [\alpha + \beta \times [\frac{1}{1+K}]]\}^2\Big] \tag{30}$$

C. Equilibrium Prices

The array of prices is $p^{\{i,m\}} = (a_i^{\{i,m\}}, b_m^{\{i,m\}}) \in \Delta$; $p^{\{m,i\}} = (a_m^{\{i,m\}}, b_i^{\{i,m\}}) \in \Delta$

$$\frac{b_i^{\{i,m\}}}{a_m^{\{i,m\}}} \cdot \frac{b_m^{\{i,m\}}}{a_i^{\{i,m\}}} = 1 - [\alpha + \beta \times [\frac{1}{1+K}]] \tag{31}$$

D. Equilibrium Allocation

For $i = h(modN)$, let

$$x_i^{h\{i,m\}} = -1 \tag{32}$$

$$y_m^{h\{i,m\}} = 1 \tag{33}$$

For $j = h \oplus 1, h \oplus 2, \ldots, h \oplus L$, let

$$x_m^{h\{j,m\}} = -\frac{1}{L} \tag{34}$$

$$y_j^{h\{j,m\}} = \frac{1}{L}\Big[1 - \{[\alpha + \beta \times [\frac{1}{1+K}]]\}\Big] \tag{35}$$

Let firm f be an active market maker at the $\{i,m\},$ and $\{j,m\}$ trading posts.

$$V^{\{i,m\}} = K \tag{36}$$

According to the specification

$$\frac{b_i^{\{i,m\}}}{a_m^{\{i,m\}}} \cdot \frac{b_m^{\{i,m\}}}{a_i^{\{i,m\}}} = 1 - [\alpha + \beta \times [\frac{1}{1 + V^{\{i,m\}}}]] \tag{37}$$

$$\frac{b_m^{\{m,j\}}}{a_j^{\{m,j\}}} \cdot \frac{b_j^{\{m,j\}}}{a_m^{\{m,j\}}} = 1 - [\alpha + \beta \times [\frac{1}{1 + V^{\{i,m\}}}]] \tag{38}$$

$$\frac{b_i^{\{i,m\}}}{a_m^{\{i,m\}}} \cdot \frac{b_m^{\{i,m\}}}{a_i^{\{i,m\}}} = \frac{b_m^{\{m,j\}}}{a_j^{\{m,j\}}} \cdot \frac{b_j^{\{m,j\}}}{a_m^{\{m,j\}}} = 1 - [\alpha + \beta \times [\frac{1}{1 + K}] \tag{39}$$

$$\frac{b_i^{\{i,m\}}}{a_m^{\{i,m\}}} = 1 \tag{40}$$

$$\frac{b_m^{\{i,m\}}}{a_i^{\{i,m\}}} = \Big[1 - \{[\alpha + \beta \times [\frac{1}{1 + K}]]\}\Big] \tag{41}$$

$$\frac{b_m^{\{m,j\}}}{a_j^{\{m,j\}}} = \Big[1 - \{[\alpha + \beta \times [\frac{1}{1 + K}]]\}\Big] \tag{42}$$

$$\frac{b_j^{\{m,j\}}}{a_m^{\{m,j\}}} = 1 \tag{43}$$

Then the net yield on the trade of i for j (and j for i) through m is

$$(44)\quad \frac{b_i^{\{i,m\}}}{a_m^{\{i,m\}}} \cdot \frac{b_m^{\{i,m\}}}{a_i^{\{i,m\}}} \times \frac{b_m^{\{m,j\}}}{a_j^{\{m,j\}}} \cdot \frac{b_j^{\{m,j\}}}{a_m^{\{m,j\}}} = [1 - \{[\alpha + \beta \times [\frac{1}{1+K}]]\}]^2$$

For $i, j, \neq m$, trading volume is zero so that prevailing prices for non-monetary trade are

$$(45)\quad \frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} \cdot \frac{b_j^{\{i,j\}}}{a_i^{\{i,j\}}} = \frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} \cdot \frac{b_j^{\{i,j\}}}{a_i^{\{i,j\}}} = \Big[1 - \{[\alpha + \frac{1}{1+0}\beta]\}\Big] = 0$$

At these nonmonetary prices, the return to barter trade is nil. Rational transactors will avoid barter trade finding monetary trade more rewarding. Transactions volume at trading posts $\{i, j\}$ for $i, j \neq m$ is zero.

The resulting equilibrium trades for trading posts $\{m, j\}, \{m, i\}$ are

$$(46)\quad \sum_f x_i^{f\{i,m\}} = K$$

$$(47)\quad \sum_f y_i^{f\{i,m\}} = -K \times \Big[1 - \{[\alpha + \beta \times [\frac{1}{1+K}]]\}\Big]$$

$$(48)\quad \sum_f y_m^{f\{i,m\}} = K$$

$$(49)\quad \sum_f y_m^{f\{i,m\}} = -K$$

$$\sum_f x_j^{f\{j,m\}} = K \tag{50}$$

$$\sum_f y_j^{f\{j,m\}} = -K \times \left[1 - \{[\alpha + \beta \times [\frac{1}{1+K}]]\}\right] \tag{51}$$

$$\sum_f y_m^{f\{j,m\}} = K \tag{52}$$

$$\sum_f y_m^{f\{j,m\}} = -K \tag{53}$$

Corresponding values for the households, summing over $h \in H$, are equal and opposite. Note that breadth of purchases does not enter into the cost situation for a monetary economy.

XIV. Barter with Double coincidence of wants : $N - 1 \geq L \geq \frac{N}{2}$

A clock-face model with values of $L \geq (N/2)$ include one or more double coincidence(s) of wants. A single barter trade at $\{i, j\}$ where $j = [i \oplus (N/2) + k]$, where $k = 0, 1, \ldots, (N/2) - 1$ can fulfill two outstanding demands with one transaction. Each of the pairs $\{i, j = i \oplus \frac{N}{2}\}, \{i \oplus 1, j = i \oplus (\frac{N}{2} + 1)\}, \ldots \{i \oplus [L - \frac{N}{2}], j = i \oplus L\}$, displays double coincidence of wants. Jevons (1875), Tobin (1968), Kiyotaki and Wright (1959) suggest that this is the setting where barter may be at least as efficient as monetary exchange.

A. Barter Matched Pairs; $N-1 \geq L \geq \frac{N}{2}$

For i, j with $j = i \oplus [(N/2)], i \oplus [(N/2)+1], ..., i \oplus [L], i$ and j are a matched pair, with reciprocally desired holdings to exchange. For each i, there are $2K(L-(\frac{N-1}{2}))$ fully matched commodity-pair holdings. Their exchange requires only one transaction. They are matched on thin markets. Equivalently, there are $KN(L-(\frac{N-1}{2}))$ fully matched pairs. Each pair accounts for two commodities, (diametrically opposite or one commodity of the pair clockwise from diametrically opposite) , with trading volume $\frac{1}{L}$ each. For each such pair the yield is

$$\mathcal{BLLR}(K,L) = \frac{1}{L} \cdot \{2 - [\alpha + \beta \times [\frac{1}{1+(K/L)}]]\} \tag{54}$$

B. Barter Unmatched commodity holdings; $N-1 \geq L \geq \frac{N}{2}$

For $L \geq [\frac{N-1}{2}]$, there are $KN\big[L - 2[L-(\frac{N-1}{2})]\big]$ unmatched commodity holdings, i, to be traded for desired (but unmatching) j. They are exchanged on thin meetings at volume $\frac{1}{L}$ each with j, necessitating an eventual re-trade, accounted separately as a trade of i. For each commodity i the yield net of transaction cost is

(55)

$$\mathcal{BUMBR}(K,L) = \frac{1}{L} \cdot \frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}} \frac{b_j^{\{i,j\}}}{a_i^{\{i,j\}}} = \frac{1}{L} \cdot [1 - [\alpha + \beta \times [\frac{1}{1+(K/L)}]]]$$

.

The transaction represented here is that unmatched i is traded for i's desired j, leaving the recipient of i to re-trade. There is no implied increase in the number of transactions; the recipient of i had intended to trade its previous holding j .

C. Total Yield in the barter economy with $N-1 \geq L \geq \frac{N}{2}$

$$\mathcal{BRRRB}(K,L) \equiv \Big[KN(L-(\frac{N-1}{2})) \times \mathcal{BLLR}\Big] \tag{56}$$

$$+\Big[KN\big[L-2[L-(\frac{N-1}{2})]\big] \times \mathcal{BUMBR}\Big] \tag{57}$$

(58)

$$\mathcal{BRRRB}(K,L) \equiv \Big[KN\left(L-\frac{N-1}{2}\right) \times \frac{1}{L} \cdot \Big[2-\Big[\alpha+\beta \times \Big[\frac{1}{1+(K/L)}\Big]\Big]\Big] + KN\left[L-2\left(L-\frac{N-1}{2}\right)\right] \times \frac{1}{L} \cdot \Big[1-\Big[\alpha+\beta \times \Big[\frac{1}{1+(K/L)}\Big]\Big]\Big]\Big]$$

XV. Yield comparison

The comparative effectiveness of barter and monetary exchange is characterized as the ratio of the yield in the monetary equilibrium $\mathcal{MMM}(N,K,L)$ to the yield in the barter equilibrium.

A. No coincidence case, $L < \frac{N}{2}$

The comparison ratio in the case where there is no double coincidence of wants is

$$\mathcal{WWWALT}(N,K,L) \equiv \frac{\mathcal{MMM}(N,K,L)}{\mathcal{BBBALT}(N,K,L)} \tag{59}$$

B. Coincidence of wants case, $N-1 \geq L \geq \frac{N}{2}$

The comparison ratio in the case where there is partial or complete double coincidence of wants is

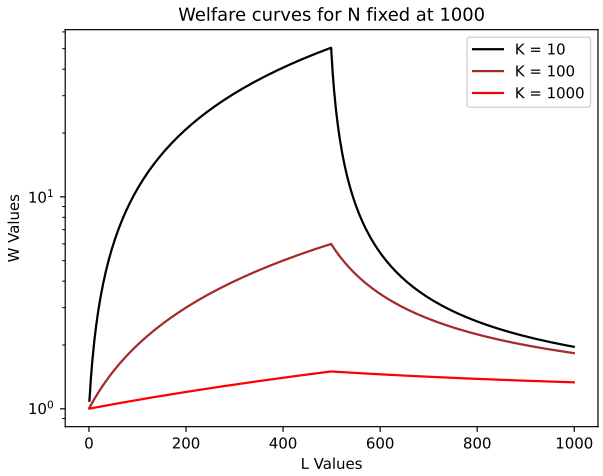


Figure 2: Welfare Curves for N = 1000

$$\mathcal{WCWLW}(K,L) \equiv \frac{\mathcal{MMM}(K,L)}{\mathcal{BRRRB}(K,L)} \tag{60}$$

Let welfare be evaluated as the sum of household utilities. Figure 2 represents (abbreviated as **W**) , $\mathcal{WWWALT}(N,K,L)$ and $\mathcal{WCWLW}(K,L)$, for the cases of no coincidence and double coincidence respectively. The transition occurs as L moves from less than $\frac{N}{2}$ to greater than $\frac{N}{2}$. The value **W** represents the factor by which monetary trade is superior to barter. But the increase in L represents broadening of the double coincidence of wants. Monetary trade remains superior even as the scope for successful barter grows.

The value of W, the measure of the superiority of monetary exchange, in Figure 2 peaks at $L = \frac{N-1}{2}$. As L increases from $\frac{N}{2}$ through $N-1$, W declines but remains greater than 1. That means that monetary trade remains more efficient than barter, but for $L \geq \frac{N}{2}$ barter becomes increasingly efficient. The increase in L be-

yond $\frac{N-1}{2}$ is an increase in the double coincidence of wants, which, as Jevons(1875) tells us, increases the scope for successful barter. Barter exchange however is necessarily restricted to comparatively small scale compared to monetary trade, hence incompletely benefiting from scale economy in the transactions technology.

XVI. Uniqueness of Money

Designation of m as money is a natural monopoly. Two moneys is less efficient than one.

The economy's yield with one money is

$$\mathcal{MMM}(N,K,L) \equiv (N)K\left[1-[\alpha+\beta\times[\frac{1}{1+K}]]\right]^2 \tag{61}$$

Total equilibrium yield of the monetary economy with Q equivalent moneys is

(62)

$$\begin{aligned}\mathcal{MTWOM}(N,K,L,Q) &\equiv Q\times\mathcal{MMM}\Big(N,\frac{K}{Q},L\Big)\\ &\equiv Q\times N\times\frac{K}{Q}\left[1-\left[\alpha+\beta\times\left[\frac{1}{1+\frac{K}{Q}}\right]\right]\right]^2\end{aligned}$$

Total yield in the single money equilibrium versus Q moneys is represented as the Efficiency Ratio $\mathcal{ER}$.

$$\mathcal{ER} \equiv \frac{\mathcal{MMM}(N,K,L)}{\mathcal{MTWOM}(N,K,L,Q)} \tag{63}$$

The table, Figure 3, represents Efficiency Ratios for the case $N = 1000$, for a variety of economy sizes, K, and number of alternative

K/Q	K = 1	K = 6	K = 11	K = 16	K = 21	K = 26	K = 31	K = 36	K = 41	K = 46
Q = 2	1.35	1.045	1.017	1.009	1.006	1.004	1.003	1.002	1.002	1.001
Q = 5	2.455	1.235	1.101	1.057	1.037	1.026	1.019	1.015	1.012	1.009
Q = 10	4.323	1.608	1.285	1.17	1.114	1.083	1.063	1.049	1.04	1.033
Q = 100	38.076	8.912	5.276	3.883	3.151	2.7	2.396	2.177	2.012	1.884
Q = 1000	375.688	82.43	45.924	31.919	24.533	19.975	16.882	14.646	12.955	11.631

Figure 3: Efficiency Ratios

moneys, Q. An Efficiency Ratio represents the proportion by which a single money increases efficiency of exchange versus Q multiple moneys. The Efficiency Ratios range from 1, unity, to dozens, to several hundred. An abundance of moneys significantly degrades the efficiency of monetary trade.

XVII. Conclusion

The trading post model in general equilibrium endogenously generates the medium(a) of exchange as the most liquid good(s) . There are, of course, multiple equilibria. Active trade of a medium of exchange creates a thick market externality ensuring distinctive low transaction cost for the prevailing medium. The thick market externality generates money's uniquely highest liquidity. Money's asymmetric function, as the medium of exchange, need not be a distinct assumption of the economic analysis. In the trading post model with transaction costs and thick market externality, the existence and local uniqueness of money is a consequence of the equilibrium allocation.

References

Arrow, K. J. and F. H. Hahn (1971) *General Competitive Analysis*, San Francisco: Holden Day

Baumol, W.J. (1952). "The Transactions Demand for Cash: An Inventory Theoretic Approach". *Quarterly Journal of Economics*. 66 (4): 545556.

Coppola, A., Krishnamurthy, A., Xu, C (2023), "Liquidity, Debt Denomination, and Currency Dominance," https://www.nber.org/papers/w30984

Diamond, P. (1982) "Aggregate Demand Management in Search Equilibrium," *Journal of Political Economy*, 1982, vol. 90, no. 5; pp. 881 - 894.

Diamond, P. (1984) "Money in Search Equilibrium" *Econometrica*, January, Vol. 52, No. 1 , pp. 1-20

Foley, D. K. (1970), "Economic equilibrium with costly marketing," *Journal of Economic Theory* Volume 2, Issue 3, September 1970, Pages 276-291

Goldberg, D. (2009), "The Massachusetts Paper Money of 1690," *Journal of Economic History*, 69, n. 4 (December), pp. 1091 - 1106.

Goldberg, D. (2023), *Easy Money: American Puritans and the Invention of Modern Currency*, Chicago: University of Chicago Press

Hahn, Frank (1965). "On Some Problems of Proving the Existence of an Equilibrium in a Monetary Economy". In Hahn; Brechling (eds.). *Theory of Interest Rates*. London: Macmillan.

Hahn, F. H. (1971) "Equilibrium with Transaction Costs ", *Econometrica* 39, No. (May 1971).

Hahn, F. H. (1973), "On Transaction Costs, Inessential Sequence Economies and Money" , *The Review of Economic Studies*, Vol. 40, No. 4 (Oct., 1973), pp. 449-461

Heller, W. P. and Starr, R. M, (1976), "Equilibrium with Non-convex Transactions Costs: Monetary and Non-monetary Economies," *The Review of Economic Studies*, Volume 43, Issue 2, June , pp. 195-215.

Hellwig, M. (1993), "The Challenge of Monetary Theory," *European Economic Review*, v. 37, pp. 215-242. North Holland.

Hicks, J. R. (1935), "A Suggestion for Simplifying the Theory of Money," *Economica*, v. 2 (5), pp. 1 - 19.

Jevons, W. S. (1875), *Money and the Mechanism of Exchange*. New York: D. Appleton and Co.

Kareken, J. H, and N. Wallace (1980), eds., *Models of Monetary Economies*, Federal Reserve Bank of Minneapolis

Kiyotaki, N. and R. Wright (1989) , "On Money as a Medium of Exchange, *Journal of Political Economy*, v. 97, pp. 927-54.

Knapp, G. F. (1905). *Staatliche Theorie des Geldes.* English Translation: Knapp, G. F. (1924). *The State Theory of Money.* London: Macmillan & Company Limited.

Kurz, M. (1974), "Equilibrium in a finite sequence of markets with transaction cost,", *Econometrica*, v. 42(1), pp. 1 - 20.

Lerner, A. (1947), "Money as a Creature of the State," *Proceedings of the American Economic Association* v. 37. pp.312-317.

Menger, K (1892), "On the Origin of Money" *The Economic Journal*, Vol. 2, No. 6 (Jun., 1892), pp. 239-255

Radford, R. A. (1945) "The Economic Organisation of a P.O.W. Camp," *Economica*, New Series, Vol. 12, No. 48 (Nov., 1945), pp. 189-201.

Rey, H. (2001) "International trade and currency exchange." *The Review of Economic Studies* 68.2 (2001): 443 - 464.

Roscher, William (1880), *Die Grundlagen der National ökonomie*, 15th edition; Stuttgart: J. G. Cotta, (Principles of Political Economy, New York: Henry Holt (1878))

Roth, A. (2008), " What Have We Learned from Market Design?" *The Economic Journal*, Volume 118, Issue 527, March 2008, Pages 285 - 310.

Schumpeter, J. A. (1954), *History of Economic Analysis*, New York: Oxford University Press.

Shapley, L. S. and M. Shubik (1977), "Trade Using One Commodity as a Means of Payment," *Journal of Political Economy* (Vol. 85, No. 5, pp. 937968),

Smith, Adam (1776), *An Inquiry into the Nature and Causes of the Wealth of Nations* (London: W. Strahan and T. Cadell),

Starr, R. M. (2003), 'Why is there money? Endogenous derivation of "money" as the most liquid asset: A class of examples,' *Economic Theory*, v. 21 (2 -3) March, pp. 455-474.

Starr, R. M. (2008) "Commodity money equilibrium in a convex trading post economy with transaction costs" *Journal of Mathematical Economics*, 44 (2008) , pp. 1413 - 1427

Starr, R. M. (2017) "Why Is There Money? Endogenous Monetization of an Arrow-Debreu Economy," The Rolf Mantel Lecture, Argentine Association of Political Economy,: econweb.ucsd.edu/ rstarr/RolfMantelLecture2017.pdf

Starr, R. M. and P. E. Spini (2021), "Making Money: Existence and Determination of Commodity Money in General Equilibrium," https://escholarship.org/uc/item/8qr0r0n7

Starrett, D.A. (1973). "Inefficiency and the demand for money in a sequence economy," *Review of Economic Studies* 40, pp. 437-48.

Tobin, J. (1956). "The Interest Elasticity of the Transactions Demand for Cash".*Review of Economics and Statistics.* 38 (3): pp. 241-247

Tobin, J. (1980) "Discussion," in Kareken, J. H. and N. Wallace (eds.), *Models of Monetary Economies*, Federal Reserve Bank of Minneapolis.

Walker, F. A. (1891), *Money*, New York: Henry Holt.

Walras, L. (1874), *Elements of Pure Economics* , Jaffe translation (1954), Homewood, Illinois: Irwin.

Wicksell, K. (1935) *Lectures on Political Economy, Vol. II: Money* Translated by E. Classen; edited with an introduction by Lionel Robbins. London: Routledge Kegan Paul, 1935