Optimal Population Growth in the Diamond OLG Model

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Abstract

This paper studies the effects of population growth on welfare in an OLG model, in which lower population growth raises output per worker but also reduces the ratio of workers to retirees. Under fairly mild restrictions on the model's parameters, there exists an optimal population-growth rate—one that maximizes steady-state welfare. This growth rate may well be negative: it may be optimal for the population to shrink over time. A pay-as-you-go social security system has an ambiguous effect on optimal population growth.

1 Introduction

In the late 20th century, economists worried that high rates of population growth were a threat to prosperity. An intellectual foundation for this view was the neoclassical growth model (Solow, 1956), in which population growth reduces capital accumulation and therefore output per worker. Concern about population growth led governments to adopt policies to curb it, most famously China's one-child policy.

In the 21st century, population growth has slowed in most of the world and there has been a sharp shift in thinking: economists now worry about the costs of *low* population growth. A major concern is that this development raises the number of retirees relative to the productive workforce, which is especially problematic in the presence of a pay-as-you-go social security system.[1]

Is population growth generally good or bad for an economy? Is there some optimal rate of population growth, with welfare losses from growth rates either above or below that level? If so, what factors determine optimal population growth?

The costs and benefits of population growth are analyzed in a number of recent studies, such as Kotschy and Bloom (2023) and Weil (2026). This paper contributes to this literature by studying the effects of population growth in Diamond's (1965) overlapping-generations model. This canonical growth model is a natural one to consider because it captures both the capital-eroding effect of high population growth and the adverse demographic effect of lower growth. The model is also a standard tool for studying economies with social security systems.

The Diamond model is highly stylized; it lacks the quantitative realism of many studies of population growth. The virtues of the Diamond model are its analytical tractability, transparency, and microeconomic foundations, which enable a rigorous welfare analysis. Just as this simple model has helped us understand optimal saving and the effects of fiscal policy, it can clarify the costs and benefits of population growth.[2]

[1] For 20th century worries about rapid population growth, see the 1977 speech by World Bank President McNamara, who says that "population growth is the gravest issue the world faces" except for the threat of nuclear war. For recent worries about low population growth, see the IMF's analysis of the "silver economy" in its *World Economic Outlook* (2025).

[2] A recent paper with some similarities to this one is Abel and Panageas (2025). That paper introduces

Section 2 of this paper presents a standard version of the Diamond model without a social security system, and Section 3 asks how population growth affects welfare in the model's steady state. An optimal population-growth rate does not always exist, but one does exist under fairly mild restrictions on parameters (in particular, on the capital intensity of the production function and the substitutability of consumption when young and old). Under these conditions, there is a closed-form solution for optimal population growth. For some reasonable calibrations of the model, this growth rate is negative: it is optimal for the population to shrink over time.

Section 4 adds a social security system to the model. The main result here is that the presence of social security has an ambiguous effect on optimal population growth. On one hand, rapid population growth has the benefit that a given tax on young workers finances a larger social security benefit. On the other hand, the social security system crowds out capital, which raises the benefit from increasing capital through lower population growth. For some reasonable calibrations, the net effect is to reduce optimal population growth; if it was negative without social security, it becomes more negative.

Section 5 concludes the paper. I discuss how the analysis fits into the broader debate about population growth and outline some ways in which the model might be extended. I also introduce a companion paper (Ball, 2026), which goes beyond this paper's steady-state analysis to study transitions between regimes with higher and lower population growth.

2 The Model

My specification of the Diamond model mostly follows the Romer textbook (2019, chapter 2). However, I specialize the model by assuming a Cobb-Douglas functional form for the production function, which yields analytical results that we can interpret. The Appendix presents the more general but messier case of a CES production function.

immigration of foreigners into an OLG model and derives the optimal level of immigration. Future work might consider a model in which population growth depends on both immigration and the fertility of natives.

Assumptions

The model is set in discrete time. In each period t, a cohort of L_t people is born and lives for two periods, t and $t+1$. The size of a cohort grows at a gross rate N: $L_t = NL_{t-1}$. When young at t, an individual inelastically supplies a unit of labor, earns a wage, and divides this income between current consumption and saving. Each unit of saving produces a unit of capital, which the individual rents out at $t+1$ to finance consumption when old. Capital depreciates fully after one period.[3]

A person born in period t has the utility function:

$$U_t = \frac{(C_t^y)^{1-\theta}}{1-\theta} + \beta\frac{(C_{t+1}^o)^{1-\theta}}{1-\theta}, \qquad \beta > 0, \quad \theta > 0, \tag{1}$$

where C_t^y is the person's consumption when young at t and C_{t+1}^o is their consumption when old at $t+1$. The parameter β is a discount factor, and θ is the coefficient of relative risk aversion; the intertemporal elasticity of substitution in consumption, which we will denote by η, is $\eta = 1/\theta$. This utility function is a standard assumption because it implies balanced growth.

Firms hire workers, rent capital, and produce output with a Cobb-Douglas production function with labor-augmenting technical progress. Individual firms' production functions can be aggregated to yield

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}, \qquad 0 < \alpha < 1, \tag{2}$$

where Y is aggregate output, K is capital, and A is the effectiveness of labor, which grows at rate G: $A_t = GA_{t-1}$. This assumption implies that output per effective worker, $y_t = Y_t/(A_t L_t)$, is a function of capital per effective worker, $k_t = K_t/(A_t L_t)$:

$$y_t = f(k_t) = k_t^\alpha. \tag{3}$$

Factor markets are competitive, so labor and capital are paid their marginal products.

[3]The original Diamond paper and the Romer text assume that capital does not depreciate. The assumption of full depreciation makes the model more tractable and is also more realistic when a period is half a lifetime.

The rental price of capital is $R_t = f'(k_t) = \alpha k_t^{\alpha-1}$ and the wage per effective worker is $w_t = f(k_t) - f'(k_t)k_t = (1-\alpha)k_t^{\alpha}$. Someone born at t supplies A_t units of effective labor and earns a total of $A_t w_t$. Firms' profits are zero so it does not matter who owns them.

Capital Accumulation

A person born at t chooses their saving when young to maximize lifetime utility. Let s_t be the fraction of earnings that is saved, so a person's saving is $s_t A_t w_t$. Consumption when young equals the earnings that are not saved, and consumption when old is the return on saving:

$$C_t^y = (1-s_t)A_t w_t; \tag{4}$$

$$C_{t+1}^o = R_{t+1} s_t A_t w_t. \tag{5}$$

Substituting these expressions into the utility function and maximizing with respect to s_t yields

$$s_t = \frac{\beta^{\eta}}{\beta^{\eta} + R_{t+1}^{1-\eta}}, \tag{6}$$

where again $\eta = 1/\theta$. Saving per effective worker equals $s_t w_t$ and total saving is $s_t w_t A_t L_t$.

Total capital at $t+1$ equals total saving at t, and the number of effective workers at $t+1$ is $A_{t+1}L_{t+1}$. Thus capital per effective worker at $t+1$ is $k_{t+1} = s_t w_t A_t L_t/(A_{t+1}L_{t+1}) = s_t w_t/(NG)$. Using equation (6) and the facts that $w_t = (1-\alpha)k_t^{\alpha}$ and $R_{t+1} = \alpha k_{t+1}^{\alpha-1}$, we obtain

$$k_{t+1} = \frac{(1-\alpha)k_t^{\alpha}\beta^{\eta}}{NG(\beta^{\eta} + \alpha^{1-\eta}k_{t+1}^{(\alpha-1)(1-\eta)})}, \tag{7}$$

which implicitly defines k_{t+1} as a function of k_t. Setting $k_{t+1} = k_t$ defines a steady-state level of capital per effective worker, which I denote by k without a time subscript. The condition defining k simplifies to

$$k^{1-\alpha} + \alpha^{1-\eta}\beta^{-\eta}k^{\eta(1-\alpha)} = \frac{1-\alpha}{NG}. \tag{8}$$

One can show that this equation defines a unique, stable steady state with $k > 0$.

Effects of Population Growth

Holding constant the other parameters of the model, steady state capital is a function of population growth: $k = k(N)$. A key feature of the model is that $k(N)$ is strictly decreasing:

$$dk/dN < 0. \tag{9}$$

This is a standard result in neoclassical growth models. Faster labor-force growth reduces k because it raises break-even investment—the investment needed to maintain a given level of capital per effective worker.

Because $k(N)$ is strictly decreasing, the expressions for w_t and R_t imply

$$dw/dN < 0; \tag{10}$$

$$dR/dN > 0, \tag{11}$$

where w and R are the steady-state wage and rental price of capital. Because higher population growth makes capital scarcer, it reduces the real wage and raises the rental price of capital. Since a lower wage reduces an individual's utility and a higher return on saving raises utility, the net effect on welfare is not obvious.

3 Optimal Population Growth

This section derives the relationship between the population-growth rate N and welfare in the model's steady state and asks whether there is an optimal level of N.

A Measure of Welfare

Let $c_t^y = C_t^y/A_t$ and $c_{t+1}^o = C_{t+1}^o/A_t$, where again C_t^y and C_{t+1}^o are consumption when young and old of someone born at t. In steady state, c_t^y and c_{t+1}^o are constant at some levels c^y and c^o, andC_t^y and C_{t+1}^o grow with the exogenous variable A_t. Using these facts

and equation (1), the steady-state utility of someone born at t is $(A_t^{1-\theta})U$, where

$$U = \frac{(c^y)^{1-\theta}}{1-\theta} + \beta\frac{(c^o)^{1-\theta}}{1-\theta}. \tag{12}$$

I measure steady-state welfare by U because a higher U raises utility for all generations. A population-growth rate N^* is optimal if it maximizes U given the other parameters of the model.[4]

The Main Result

The following Proposition summarizes the results of this section:

Proposition 1: In the absence of social security, an optimal population-growth rate N^* exists if and only if $\alpha < 1/2$ and $\eta < 1$. Under those conditions, N^* is defined by

$$f'(k(N^*)) = N^*G, \tag{13}$$

which implies

$$N^* = (\frac{1}{G})\beta^{\eta/(1-\eta)}(\frac{1-2\alpha}{\alpha})^{1/(1-\eta)}. \tag{14}$$

Proposition 1 is proved in the Appendix. Here I sketch the argument and discuss the different parts of the Proposition.

An Indirect Utility Function

To begin, notice that equations (4) and (5) imply $c^y = (1-s)w$ and $c^o = swR$, where s is the steady-state level of the saving rate s_t. Letting $Q = wR$, consumption levels and hence utility are determined by w, Q, and s. An individual takes w and Q as exogenous and chooses s optimally, which implies an indirect utility function of the form

$$U = V(w, Q). \tag{15}$$

[4]I follow mainstream economic theory in equating welfare with the utility of an individual. Some recent work argues that true economic welfare also depends on the number of individuals or total utility in the economy (e.g., Adhami et al., 2024; Spears and Geruso, 2025). Considering this alternative view is beyond the scope of this paper.

We can interpret w as the highest possible c^y (the level when $s = 0$) and Q as the highest possible c^o (the level when $s = 1$).

Both w and $Q = wR$ are determined by k, the steady-state level of capital per effective worker. We can therefore write

$$U = V(w(k), Q(k)). \tag{16}$$

Finally, recall that steady state capital depends on population growth: $k = k(N)$.

With this set-up, the effect of population growth on welfare is

$$dV/dN = (dV/dk)(dk/dN). \tag{17}$$

Using equation (16) to expand dV/dk,

$$dV/dN = \underset{(+)}{[V_w}\underset{(+)}{(dw/dk)} + \underset{(+)}{V_Q}\underset{(?)}{(dQ/dk)}]\underset{(-)}{(dk/dN)}, \tag{18}$$

where V_w and V_Q are the partial derivatives of $V(\cdot)$ and we show the signs of various terms. V_w and V_Q are positive (utility is increasing in both w and wR) and dk/dN is negative. dw/dk is positive: a higher capital stock raises the wage. dQ/dk is ambiguous because $Q = wR$ and a higher k raises w but reduces R.

If N^* exists, it satisfies the first-order condition that $dV/dN = 0$ at $N = N^*$. Since $dk/dN < 0$, equation (17) implies that this condition reduces to $dV/dk = 0$ at $k = k(N^*)$.

The Restrictions on α and η

We can now understand the conditions needed for N* to exist:

$\alpha < 1/2$: The production function implies $w = (1-\alpha)k^\alpha$ and $R = \alpha k^{\alpha-1}$, which together imply $Q = wR = \alpha(1-\alpha)k^{2\alpha-1}$. If $\alpha \geq 1/2$ that is, if the production function is capital-intensive, then Q is non-decreasing in k. In this case, a higher k raises utility through its effect on w and either raises utilty or has no effect through Q. As a result, dV/dk is positive for all k, and utility rises without bound as $k \to \infty$. Since $dk/dN < 0$, utility rises without bound as $N \to 0$ and no $N^* > 0$ maximizes welfare.

$\eta < 1$: This means that consumption when young and when old are not very substitutable. To understand why this condition is needed, note that, for $\alpha < 1/2$, $k \to \infty$ implies $w \to \infty$ and $Q \to 0$. In turn, this behavior of w and Q implies $c^y \to \infty$ and $c^o \to 0$ for any saving rate s between zero and one. The opposite case of $k \to 0$ implies $w \to 0$ and $Q \to \infty$, and hence $c^y \to 0$ and $c^o \to \infty$. In sum, either $k \to 0$ or $k \to \infty$ causes consumption to approach 0 in one period of life and ∞ in the other. In both cases, average consumption over the two periods approaches ∞.

If consumption is highly substitutable across periods, the effect of $k \to 0$ or $k \to \infty$ on average consumption means that utility grows without bound in either of these extreme cases. These cases arise as $N \to \infty$ or $N \to 0$, so once again no $N^* > 0$ maximizes utility. On the other hand, if consumption when young and when old are not very substitutable, then utility suffers if these two consumption levels differ greatly, which occurs for extreme values of N and k. For $\eta < 1$, the benefit from smoothing consumption across the two periods of life is great enough to imply an N^* between 0 and ∞.

For many purposes, the assumption of $\eta = 1$—logarithmic utility—is a convenient simplification of the Diamond model. For this application, however, this simplification is not innocuous: it rules out the existence of an optimal population growth rate.

How Restrictive? The restrictions that $\alpha < 1/2$ and $\eta < 1$ do not seem very strong. For the U.S., a typical calibration of α is 1/3, based on capital's share of income. In the literature on consumption, most estimates of the substitution elasticity η are less than one (or equivalently, the coefficient of relative risk aversion θ is greater than one). While estimates of η vary considerably, one meta-analysis suggests a consensus estimate of $\eta = 1/3$ (Havranek, 2015).

The Optimality Condition

The condition defining N^* is derived in the Appendix. I evaluate the components of dV/dk (the terms inside the brackets in equation (18)) for a given N and $k = k(N)$ and set $dV/dk = 0$, which leads to the simple condition $f'(k(N)) = NG$ stated in Proposition 1. Note that $f'(k) = R = \alpha k^{\alpha-1}$. Under the assumptions that $\alpha < 1/2$ and $\eta < 1$, this condition defines a unique N^* that maximizes utility.

The condition $f'(k) = NG$ is a familiar one: it defines the "golden-rule" level of capital in neoclassical growth models (Phelps, 1961). However, the condition has a somewhat different meaning from usual. The golden-rule k is usually derived assuming that N is a parameter fixed at an arbitrary level. The golden rule maximizes steady state consumption per effective worker for the given N. Here, the familiar condition defines an optimal N^* and a corresponding $k(N^*)$.

Setting $N = N^*$ does *not* maximize steady-state consumption per effective worker. Consumption per effective worker is $f(k(N))–NGk(N)$ (output minus investment), and the N that maximizes this expression is defined by the first-order condition $[f'(k(N))–NG]\cdot(dk/dN) – Gk(N) = 0$; note the $Gk(N)$ term that does not appear in the condition defining N^*. One can show that the welfare-maximizing N^* is always higher than the N that maximizes consumption per effective worker. This makes sense because, for a given level of consumption per worker, a higher N has a benefit: it raises the ratio of workers to retirees and therefore raises consumption per person.[5]

*The Solution for N**

The final step is to combine the optimality condition we have just discussed with the condition defining $k(N)$ (equation (8)). This leads to the explicit solution for N^* in Proposition 1.

The solution for N^* tells us how the model's parameters affect optimal population growth. For $\alpha < 1/2$ and $\eta < 1$,

$$\partial N^*/\partial G \; < \; 0 \tag{19}$$

$$\partial N^*/\partial \alpha \; < \; 0 \tag{20}$$

$$\partial N^*/\partial \beta \; > \; 0 \tag{21}$$

$$sign(\partial N^*/\partial \eta) \; = \; sign(\beta - \alpha - 2\alpha\beta). \tag{22}$$

[5]This intuition might suggest that the optimal N is the level that maximizes consumption per person, accounting for effects on both consumption per worker and workers per person. But that is also not the case: one can show that N^* can be either higher or lower than the N that maximizes consumption per person. Welfare depends not only on average consumption per person but also on the division of consumption between the young and old and the sizes of those two cohorts, which depend on N in somewhat complex ways.

N^* depends inversely on G. (Equivalently, there is an optimal growth rate for total output NG.) N^* is decreasing in the capital intensity of the production function and increasing in the discount factor. The effect of the substitution elasticity η is ambiguous and depends on α and β.

For common calibrations of the parameters, N^* is less than one: it is optimal for the population to shrink over time. In particular, if $\alpha = 1/3$, sufficient conditions for $N^* < 1$ are $\beta < 1$ and $G \geq 1$: there is some discounting of future utility and technology does not regress over time.

As a specific numerical example, suppose $\alpha = 1/3$ and $\eta = 1/3$. To calibrate G and β, let us interpret a period in the model as 30 years: a person works for 30 years and then lives off capital income for 30 years. Suppose the growth rate of productivity is 1 percent per year and the annual discount factor is 0.99. Compounding over 30 years yields $G = 1.35$ and $\beta = 0.74$. Substituting these parameter values into equation (14) yields $N^* = 0.64$. This level of N implies that the population falls by about 1.5 percent per year.

4 The Implications of Social Security

In discussing the effects of low population growth, economists often emphasize the strain on pay-as-you-go social security systems from a high ratio of retirees to workers. This section adds a social security system to the model and asks how it affects the optimal level of population growth.

Introducing Social Security

Assume the government levies a tax on the young which is a fraction $\tau > 0$ of their earnings, and uses the proceeds to pay social security benefits to the old. Since there are N young workers per old retiree, the benefit to each old person is a fraction $N\tau$ of a current worker's earnings. Under this policy, the expressions for consumption of someone

born in period t become

$$C_t^y = (1-\tau-s_t)A_t w_t; \tag{23}$$

$$C_{t+1}^o = R_{t+1}s_t A_t w_t + \tau N G A_t w_{t+1}, \tag{24}$$

where s_t is saving as a fraction of pre-tax earnings and I use the fact that a worker's earnings at t+1 are $A_{t+1}w_{t+1} = GA_t w_{t+1}$. Consumption when young is the part of earnings $A_t w_t$ that is neither taxed away nor saved, and consumption when old is the return on saving plus the social security benefit at $t+1$.

As before, we can derive a person's optimal choice of s_t by maximizing utility given the equations for C_t^y and C_{t+1}^o:

$$s_t = \frac{(1-\tau)\beta^\eta - \tau N G R_{t+1}^{-\eta}(w_{t+1}/w_t)}{R_{t+1}^{1-\eta} + \beta^\eta} \tag{25}$$

assuming that the tax rate τ is small enough that s_t is strictly positive. Using the fact that $k_{t+1} = s_t w_t/(NG)$ we can again determine a function $k_{t+1}(k_t)$ and then set $k_{t+1} = k_t = k$. This leads to

$$k^{1-\alpha} + \alpha^{1-\eta}\beta^{-\eta}k^{\eta(1-\alpha)} = \frac{1-\alpha}{NG}(1-\tau) - (1-\alpha)\tau\alpha^{-\eta}\beta^{-\eta}k^{\eta(1-\alpha)}, \tag{26}$$

which defines a unique, stable steady state with $k > 0$ for small enough τ.

In this version of the model, steady-state capital per worker depends on both population growth and the social security tax rate: $k = k(N,\tau)$. One can show that k is strictly decreasing in both N and τ. The effect of τ is a standard result in OLG models: a social security system crowds out capital.

The Main Result with Social Security

In this version of the model with an arbitrary tax rate τ, no simple conditions determine whether N^* exists, and there is no closed-form solution for N^*. The following is the main thing that we can prove (see Appendix):

Proposition 2: In the presence of social security, if an an optimal population growth rate exists, it is defined by:

$$[f'(k(N^*,\tau)) - N^*G](\partial k/\partial N) \; + \; \frac{\tau G}{\alpha(1-\tau)} k(N^*,\tau) \; = \; 0. \tag{27}$$

As for whether N^* exists, the restrictions $\alpha < 1/2$ and $\eta < 1$ remain sufficient for values of τ close to zero (by continuity of the relevant functions), so I will derive results when these conditions hold and τ is small. I will also do numerical calculations for larger values of τ, checking that (27) can be solved for N^* and that the second order condition for optimality is met.

The rest of this section explores the determination of N^* and asks how N^* is affected by the tax rate τ.

The Optimality Condition

Introducing social security complicates the model because both τ and N appear in the equations for C_t^y and C_{t+1}^o ((23) and (24)). This means that τ and N directly affect consumption possibilities for given levels of w and Q. The indirect utility function becomes

$$U \; = \; V(w, Q, N, \tau) \tag{28}$$

The partial derivative V_N is positive because higher population growth raises the social security benefit received by the old.

Once again $w(k)$ and $Q(k)$ are determined by the production function, and here $k = k(N,\tau)$. The effect of N on welfare is

$$dV/dN \; = \; [V_w(dw/dk) + V_Q(dQ/dk)](\partial k/\partial N) \; + \; V_N. \tag{29}$$

The Appendix shows that setting $dV/dN = 0$ leads to the condition for N^* in Proposition 2 (equation (27)). Notice that $\partial k/\partial N$ no longer drops out of the condition because it does not multiply all the terms.

In equation (27), the second term on the left is positive and $\partial k/\partial N$ is negative. These

facts imply that $f'(k(N^*,\tau)) > N^*G$: N^* is no longer determined by a version of the golden rule for capital accumulation.

The Effect of Social Security on Optimal Population Growth

How does the presence of social security affect N^*? The answer is ambiguous, because social security raises N^* through one channel and reduces it through another. The ambiguity of the net effect is established in the Appendix; here I outline the economics of the two channels. For simplicity I consider a small social security system, whose effect is determined by $dN^*/d\tau$ evaluated at $\tau = 0$.

The levels of N and τ determine all the variables in the welfare function $V(\cdot)$ either directly or through their effects on k. We can therefore write welfare as

$$U \;=\; \hat{V}(N,\tau) \;=\; V(w(k(N,\tau)), Q(k(N,\tau)), N, \tau). \tag{30}$$

With this notation, the first-order condition for N^* can be written as $\hat{V}_N = 0$. Differentiating this condition with respect to τ leads to

$$dN^*/d\tau \;=\; -\frac{\hat{V}_{N\tau}}{\hat{V}_{NN}}, \tag{31}$$

where $\hat{V}_{N\tau}$ and $\hat{V}_{NN}$ are second derivatives of $\hat{V}(\cdot)$ evaluated at $N = N^*$. By the second-order condition for N^*, $\hat{V}_{NN} < 0$, which implies

$$sign(dN^*/d\tau) \;=\; sign(\hat{V}_{N\tau}). \tag{32}$$

To analyze the cross-derivative $\hat{V}_{N\tau}$, we start with the first derivative $\hat{V}_N$, which is given by dV/dN in equation (29). Let $\partial V/\partial k$ be the total effect of k through both w and Q: $\partial V/\partial k = V_w(dw/dk) + V_Q(dQ/dk)$.Then (29) implies

$$\hat{V}_N \;=\; (\frac{\partial V}{\partial k})(\frac{\partial k}{\partial N}) \;+\; V_N. \tag{33}$$

This equation breaks $\hat{V}_N$ into two parts: the effects of N that operate through k and the

direct effect on social security benefits.

The cross-derivative $\hat{V}_{N\tau}$ is the derivative of $\hat{V}_N$ with respect to τ. As shown in the Appendix, taking this derivative and evaluating it at $\tau = 0$ (which eliminates some terms) leads to

$$\hat{V}_{N\tau}|_{\tau=0} = \underset{(-)}{(\partial^2 V/\partial k^2)}\underset{(-)}{(\partial k/\partial \tau)}\underset{(-)}{(\partial k/\partial N)} + \underset{(+)}{V_{N\tau}}. \tag{34}$$

We see that $\hat{V}_{N\tau}$ is ambiguous, and therefore $dN^*/d\tau$ is ambiguous, because $\hat{V}_{N\tau}$ is the sum of a negative term (the product of three negatives) and a positive term.

To understand this result, start with the positive term $V_{N\tau}$. This term is the effect of social security on V_N, the direct benefit from higher population growth. V_N is zero when $\tau = 0$: as shown above, with no social security N does not affect welfare for a given k. For $\tau > 0$, V_N becomes positive because a higher N raises the social security received by the old. By creating this benefit from a higher N, social security tends to raise $\hat{V}_N$ and N^*.

The negative term in equation (33) reflects the fact that social security crowds out capital ($\partial k/\partial \tau < 0$). Lower capital raises the marginal benefit of capital ($\partial^2 V/\partial k^2 < 0$). That in turn raises the benefit from reducing N, which increases capital ($\partial k/\partial N < 0$). In sum, the crowding out of capital by social security raises the benefit of increasing capital through lower population growth, and that effect tends to reduce $\hat{V}_N$ and N^*.

A Numerical Example

While there is no closed-form solution for N^*, we can determine it numerically for given parameter values including τ. We use the condition defining the steady state k, equation (26), and the condition defining N^*, equation (27), which are two equations in N^* and k. In (27), the expression for $\partial k/\partial N$ is determined by differentiating (26).[6]

As an example, assume as before that $\alpha = 1/3$, $\eta = 1/3$, $G = 1.35$, and $\beta = 0.74$. With no social security system ($\tau = 0$), these parameters imply $N^* = 0.64$. With a social security system with a 10 percent tax on wages ($\tau = 0.1$), $N^* = 0.46$ (and the second order condition is satisfied). This N^* implies that the population falls by 2.5 percent per year if a period is 30 years. In this example, the negative effect of τ on N^* in equation (34) outweighs the

[6]Differentiating (26) yields $\partial k/\partial N = \frac{1-\tau}{GN^2[k^{-\alpha}+\eta(\alpha\beta)^{-\eta}(\alpha+\tau-\alpha\tau)k^{\eta(1-\alpha)-1}]}$.

positive effect.

5 Discussion

This paper asks how population growth affects steady-state welfare in the Diamond OLG model. A welfare-maximizing growth rate exists under non-trivial but plausible restrictions on the model's parameters. I derive this optimal growth rate and find that it may be negative: it may well be optimal for the population to shrink over time. The presence of a social security system has an ambiguous effect on the optimal growth rate.

Many economists have studied the costs and benefits of population growth. Important work over the years includes Cutler et al. (1990), Lee at al. (2014), Kotschy and Bloom (2023), and Weil (2026). This research emphasizes the same issues as this paper—the effects of population growth on demographic structure, capital intensity, and social security programs. Generally the authors analyze detailed age distributions that evolve under realistic assumptions about fertility and mortality. This paper's stylized model is a step backward in quantitative realism, but it allows a rigorous welfare analysis, one that accounts for interactions between different effects of population growth and for the adjustment of saving behavior.

Future research should extend the model in this paper, retaining its microfoundations but bringing it closer to the realism of other work. One could, for example, introduce a richer demographic structure, as in Auerbach and Kotlikoff's (1987) OLG model with 55-period lives. One could include a period of childhood before a person's working years and consider changes in mortality and retirement ages as well as fertility. For practical policy applications, it would be important to introduce heterogeneity across people in labor income, saving behavior, and reliance on social security benefits.

Of course a complete analysis of optimal population growth must also consider factors ignored by neoclassical growth models, such as possible effects of population on technological progress (e.g., Kremer, 1993) and environmental externalities (e.g., McKibben, 1998).

This paper analyzes welfare in the steady state of the Diamond model. A companion

paper (Ball, 2026) considers the transition from one steady state to another if population growth falls over time, as it has recently in many countries. In the model, even if population growth is initially above the optimal level and the slowdown raises steady state welfare, not all generations gain; in particular, the cohort that is old at the start of the transition loses because slower labor-force growth reduces its capital income. The companion paper derives the effects of the population slowdown on every generation and considers possible policy responses that redistribute consumption from the winners to the losers.

Appendix

Proof of Proposition 1

Proposition 1 concerns the optimal population growth rate N^* in the model without social security. As explained in the text, welfare in that case can be written as

$$V(w(k(N)), Q(k(N))). \tag{35}$$

I establish the different parts of the Proposition by examining the first and second derivatives of $V(\cdot)$ with respect to N.

The First Order Condition for N^*: Equation (18) in the text shows that the first derivative is $dV/dN = [V_w w'(k)) + V_Q Q'(k)](dk/dN)$, where $w'(k) = dw/dk$ and $Q'(k) = dQ/dk$. I derive dV/dN by evaluating the terms on the right side of this equation. In doing so, I hold an individual's saving rate s constant, which is justified by the envelope theorem because s is chosen to maximize V. The analysis uses an individual's first order condition for optimal saving: $(C^y)^{-1/\eta} = R\beta(C^o)^{-1/\eta}$. It also uses the condition that $k = (sw)/(NG)$ in steady state.

The equations for $U(\cdot)$ and for $C_t^y = A_t c_t^y$ (equations (12) and (4)) imply

$$V_w = (\partial U/\partial c^y)(dc^y/dw) = (c^y)^{-1/\eta}(1-s). \tag{36}$$

Notice that w affects welfare only through c^y because c^o is not influenced by w for a given Q.

Similarly, equations (12) and (5) imply

$$V_Q = (\partial U/\partial c^o)/(dc^o/dQ) = \beta(c^o)^{-1/\eta}s = \frac{(c^y)^{-1/\eta}}{R}s, \tag{37}$$

where the last equality uses an individual's first order condition.

Substituting these expressions for V_w and V_Q into the expression for dV/dk yields

$$dV/dN = (c^y)^{-1/\eta}[(1-s)w'(k) + (s/R(k))Q'(k)](dk/dN). \tag{38}$$

We now use the steady-state relation $k = (sw)/(NG)$ and the fact that $Q'(k) = w(k)R'(k) + w'(k)R(k)$. Substituting these equations into equation (38) leads after algebra to

$$dV/dN = (c^y)^{-1/\eta}[w'(k) + \frac{kNGR'(k)}{R(k)}](dk/dN). \tag{39}$$

The wage and rental price of capital are $w(k) = f(k)–kf'(k)$ and $R(k) = f'(k)$. Differentiating these functions leads to the result that $w'(k) = -kR'(k)$, and substituting into (39) yields

$$dV/dN = (c^y)^{-1/\eta}w'(k)[1 - (NG)/R(k)](dk/dN). \tag{40}$$

Since both $(c^y)^{-1/\eta}w'(k)$ and dk/dN are non-zero, the first-order condition $dV/dN = 0$ reduces to

$$\frac{NG}{R(k)} = 1, \tag{41}$$

which is equivalent to the condition $f'(k) = NG$ stated in Proposition 1.

As discussed in the text, this condition cannot be satisfied for $\alpha \geq 1/2$ because dV/dk is always positive in that case. Thus there is no optimal N^* for $\alpha \geq 1/2$. I therefore assume $\alpha < 1/2$ in the rest of this analysis. For the moment I also assume $\eta \neq 1$; I return below to the knife-edge case of $\eta = 1$.

The Solution for N^*: To derive an explicit solution for N^*, substitute $f'(k) = ak^{\alpha-1}$ into the condition $f'(k) = NG$, which leads to

$$k = (\frac{\alpha}{NG})^{1/(1-\alpha)}. \tag{42}$$

Substituting this expression for k into equation (8), which defines the steady-state k, and solving for N yields the expression for N^* in the Proposition for $\alpha < 1/2$ and $\eta \neq 1$. Thus there is an N that satisfies the first order condition for N^* under these parameter restrictions.

The Second Order Condition The second order condition for N^* is that $d^2V/dN^2 < 0$ when the first order condition is satisfied. To derive this second derivative, start with the expression for dV/dN in equation (38). Using the Cobb-Douglas production function, we can derive $R(k)$, $w'(k)$ and $Q'(k)$ as functions of k. Substituting these results into (38)

leads after simplification to

$$dV/dN = (c^y)^{-1/\eta}(1-\alpha)k^{\alpha-1}(dk/dN)[\alpha - s + \alpha s]. \tag{43}$$

Differentiating this equation with respect to N and evaluating at $dV/dN = 0$ (which implies $\alpha - s + \alpha s = 0$) yields

$$\partial^2 V/\partial N^2 = (c^y)^{-1/\eta}(1-\alpha)k^{\alpha-1}(dk/dN)(\alpha - 1)(ds/dN). \tag{44}$$

Since dk/dN and $(\alpha - 1)$ are both negative,

$$sign(\partial^2 V/\partial N^2) = sign(ds/dN). \tag{45}$$

The saving rate s is $s(R(k(N)))$, with $s(R)$ given by equation (6). Therefore,

$$ds/dN = (ds/dR)(dR/dk)(dk/dN). \tag{46}$$

Since dR/dk and dk/dN are both negative,

$$sign(ds/dN) = sign(ds/dR) = sign(\eta - 1), \tag{47}$$

where the second equality follows from equation (6).

Together, these results imply that the second order condition for N^* is satisfied if and only if $\alpha < 1/2$ and $\eta < 1$. If $\alpha < 1/2$ and $\eta > 1$, the first order condition $R = NG$ determines a level of N that *minimizes* welfare.

The Case of $\eta = 1$: In the well-known case of $\eta = 1$ (log utility), the steady state condition (8) leads to an explicit solution for k, which in turn implies

$$R = \frac{\alpha(1+\beta)NG}{\beta(1-\alpha)}. \tag{48}$$

Here, R is proportional to NG. The first order condition $R = NG$ becomes

$$\frac{\alpha(1+\beta)}{\beta(1-\alpha)} = 1, \tag{49}$$

which does not depend on N. For combinations of α and β that satisfy this condition, the first order condition is satisfied for all N, and otherwise it is satisfied for no N. In either case an optimal N does not exist.

Proof of Proposition 2

With a social security system, welfare is $V(w, Q, \tau, N)$. Differentiating yields

$$dV/dN = [V_w w'(k) + V_Q Q'(k)](dk/dN) + V_N. \tag{50}$$

Paralleling our analysis without social security, using equations (23)-(24), one can derive:

$$V_w = (c^y)^{-1/\eta}(1-\tau-s) + \beta(c^o)^{-1/\eta}\tau NG; \tag{51}$$

$$V_Q = s\beta(c^o)^{-1/\eta}; \tag{52}$$

$$V_N = \beta(c^o)^{-1/\eta}\tau Gw, \tag{53}$$

where the equations for V_w and V_N include effects on social security taxes and benefits.

Once again, we can rewrite dV/dN using an individual's first order condition, the steady state condition $sw = kNG$, and the expressions for $w'(k)$ and $Q'(k)$ implied by the production function. These steps reduce the first order condition $dV/dN = 0$ to equation (27) in Proposition 2.

The Sign of $dN^*/d\tau$

Here I show that the sign of $dN^*/d\tau$ is ambiguous. It suffices to show that this derivative is ambiguous at $\tau = 0$, which implies that it is ambiguous for sufficiently small $\tau > 0$ because $dN^*/d\tau$ is continuous.

Differentiating the optimality condition (27) with respect to τ and evaluating at $\tau = 0$ yields

$$[(f''(k)(\partial k/\partial N) - G)(dN^*/d\tau) + f''(k)(\partial k/\partial \tau)](\partial k/\partial N) + \frac{Gk}{\alpha} = 0, \tag{54}$$

which leads to

$$dN^*/d\tau = \frac{-[f''(k)(\partial k/\partial \tau)(\partial k/\partial N) + \frac{Gk}{\alpha}]}{f''(k)(\partial k/\partial N) - G}. \tag{55}$$

Differentiating the steady state condition (26) and evaluating at $\tau = 0$ yields

$$\partial k/\partial N = -\frac{Gk^{2-\alpha}}{\alpha^2 + \alpha\eta(1-2\alpha)} \tag{56}$$

$$\partial k/\partial \tau = -\frac{(1-2\alpha)^X \alpha^{(\eta-2)X} \beta^{\eta X} + (1-\alpha)(1-2\alpha)^{[1+\eta(1-\alpha)]X} \alpha^{(\alpha\eta-2)X} \beta^{\eta(2-\alpha)X}}{\alpha + \eta(1-2\alpha)} \tag{57}$$

where $X = \frac{1}{(\eta-1)(1-\alpha)}$. Substituting these results into (55), one can show after algebra that the denominator in (55) is positive, so the sign of $dN^*/d\tau$ is given by the sign of the numerator. Additional algebra leads to the result that

$$sign(dN^*/d\tau) = sign((1-\alpha)^2[(1-2\alpha)\alpha^{\eta-2}\beta^{\eta}]^{1/(\eta-1)} - \alpha[\eta^2(1-2\alpha) + \alpha(2\eta-1)]). \tag{58}$$

It is easy to see that the sign is ambiguous (e.g., it is positive for $\alpha \to 1/2$ and $\eta < 1/2$ and negative for $\alpha \to 1/2$ and $\eta > 1/2$).

Derivation of $\hat{V}_{N\tau}$

It is useful here to write the utility function one more way:

$$\hat{\hat{V}}(k(N,\tau),N,\tau) = \hat{V}(N,\tau) = V(w(k(N,\tau)),Q(k(N,\tau)),N,\tau). \tag{59}$$

With this notation, $\partial V/\partial k$ as defined in the text equals $\hat{\hat{V}}_k$ and $\partial^2 V/\partial k^2$ equals $\hat{\hat{V}}_{kk}$.

We can write $\hat{V}_N$ as

$$\hat{V}_N = \hat{\hat{V}}_k(\partial k/\partial N) + \hat{\hat{V}}_N = \hat{\hat{V}}_k(\partial k/\partial N) + V_N \tag{60}$$

where the second equality uses the definitions of $\hat{\hat{V}}(\cdot)$ and $V(\cdot)$. Differentiating with re-

spect to τ yields

$$\hat{V}_{N\tau} = \hat{\hat{V}}_k(\partial^2 k/\partial N \partial \tau) + (\partial k/\partial N)[\hat{\hat{V}}_{kk}(\partial k/\partial \tau) + \hat{\hat{V}}_{k\tau}] + V_{N\tau}. \tag{61}$$

At $\tau = 0$, $\hat{\hat{V}}_k = 0$ from the first order condition for N^* with no social security, so (61) reduces to

$$\hat{V}_{N\tau} = (\partial k/\partial N)[\hat{\hat{V}}_{kk}(\partial k/\partial \tau) + \hat{\hat{V}}_{k\tau}] + V_{N\tau}. \tag{62}$$

This expression yields equation (34) in the text if $\hat{\hat{V}}_{k\tau} = 0$, so let us examine that cross-derivative evaluated at $\tau = 0$. The first derivative $\hat{\hat{V}}_k = V_w w'(k) + V_Q Q'(k)$. The derivatives V_w and V_Q are given by equations (51)-(52) and $w'(k)$ and $Q'(k)$ are again determined by the production function. We can derive

$$\hat{\hat{V}}_k(k, N, \tau) = (c^y)^{-\eta} w'(k)(1-\tau)[1 - (NG)/R(k)]. \tag{63}$$

At $\tau = 0$ and $N = N^*$, $[1 - (NG)/R(k)] = 0$ (the first order condition for N^* with no social security). In addition, the derivative $\partial[1 - (NG)/R(k)]/\partial\tau = 0$ because it is a derivative holding k and N constant. Together, these facts imply $\hat{\hat{V}}_{k\tau} = 0$, the result needed for equation (34).

Generalizing the Production Function

Here I generalize the model by assuming a CES production function. In this case, output per effective worker is

$$y = f(k) = [\alpha k^\rho + (1-\alpha)]^{1/\rho}, \qquad \rho < 1. \tag{64}$$

The elasticity of substitution between capital and labor is $\sigma = 1/(1-\rho)$. The Cobb-Douglas production function is the special case in which $\sigma = 1$ ($\rho \to 0$).

It is challenging to analyze the model with this production function, in part because, for a given N, the model may have no steady state with $k > 0$ or multiple steady states. There is at most one stable steady state, and in evaluating welfare for a given N I assume that the economy is in that steady state if it exists. I assume that an N is not optimal if

there is no stable steady state. (In that case, k converges to zero from any initial level.)

Here I assume there is no social security system. My main result is

Proposition A1: In the absence of social security, with a CES production function, the conditions $\eta \leq 1$ and $\sigma < 1$ are sufficient for an optimal population growth rate N^* to exist. Under these conditions, N^* is determined by the steady state condition

$$\frac{kNG}{w(k)} = \frac{\beta^\eta}{\beta^\eta + (R(k))^{1-\eta}}, \tag{65}$$

where $w(k) = (1-\alpha)[\alpha k^\rho + (1-\alpha)]^{(1-\rho)/\rho}$ and $R(k) = \alpha k^{\rho-1}[\alpha k^\rho + (1-\alpha)]^{(1-\rho)/\rho}$, and the condition

$$f'(k) = NG. \tag{66}$$

Notice that the parameter restrictions in this Proposition are a bit weaker when $\sigma < 1$ than they are in the Cobb-Douglas case: no restriction on α is needed, and N^* exists for $\eta = 1$ (log utility). These restrictions are sufficient but not necessary; full necessary and sufficient conditions are complex and involve the parameters α and β as well as η and σ.

I will sketch the reasoning behind this Proposition but eschew a detailed proof. As before, the steady state condition is derived from equation (6) for the saving rate and the condition $sw = NGk$, and the factor prices are implied by the CES production function. The optimality condition $f'(k) = NG$ is the same as in the Cobb-Douglas case; notice that the derivation of that condition above does not use the Cobb-Douglas assumption.[7]

The parameter restrictions in the Proposition assure that the N^* determined by equations (65) and (66) is a global optimum. With $\sigma < 1$ (low substitubility of capital and labor), as k approaches either 0 or ∞, one of $w(k)$ and $R(k)$ approaches 0 and the other approaches a finite limit, which in turn implies that either c^y or c^o approaches 0 and utility approaches $-\infty$ for any $\eta \leq 1$ (including log utility). These results for extreme values of k ensure the existence of an optimal k and hence an optimal N.

Combining Propositions 1 and A1, sufficient conditions for N^* to exist are $\eta < 1$ and

[7] A nuance: For $\sigma < 1$, one can show that the model has no steady state for N above some bound N', and for $N < N'$ there are two steady states, one that is unstable and the other (with a lower k) that is stable. However, when the steady state condition is combined with the optimality condition $f'(k) = NG$, these equations determine a unique (N, k) pair with $N < N'$ and k at its level in the stable steady state.

either $\sigma < 1$ or $\sigma = 1$ and $\alpha < 1/2$. The restriction that $\sigma \leq 1$, like the restriction on η, is non-trivial but not especially strong. For the US economy, a meta-analyis of 77 studies concludes that σ is likely between 0.45 and 0.87 (Knoblach, et al. [2020]).

The effect of σ on optimal population growth, $\partial N^*/\partial\sigma$, is ambiguous, with its sign depending in complicated ways on most of the model's parameters. As a numerical example, I set all the parameters except σ at their levels in earlier examples and compute N^* for various σ's. In this example, N^* falls monotonically as σ falls, from 0.64 at $\sigma = 1$ to 0.29 as $\sigma \to 0$. For $\sigma = 0.66$, the midpoint of the range in Knoblach et al., $N^* = 0.35$. If a period is 30 years, this value of N^* implies that the population falls by 3.4 percent per year.

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