Dynamic Investment and Product Market Rivalry: The Network Q Model

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Abstract

We present a new dynamic model of corporate investment in imperfectly-competitive product markets, extending the neoclassical (Q) theory of capital to a multi-firm, multi-product, fully-structural model. Our model embeds a state-of-the-art hedonic demand system, endogenizes firms' markups and generalizes Tobin's Q to a matrix (or network) of product market spillovers, which captures how each firm's investment affects that of its rivals. We provide equilibrium existence and uniqueness results along with global analytical solutions for the firms' investment policies. We then take our model to the data for the universe of U.S. public companies and obtain five novel insights: 1) product market competition is a key driver of aggregate investment and capital allocation; 2) capital stocks became more persistent (or "sticky") over the past 25 years; 3) monopoly rents account for a large, rising share of firms' value; 4) shocks to firms' cost of capital generate highly heterogeneous investment and markups responses across firms, and thus impact industry concentration; 5) mergers consummated since 1995 have led to a modest decline in aggregate capital formation, yet firm-level markup increases have been highly heterogeneous.

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1 Introduction

How does product market competition affect individual firms' strategic investment decisions and enterprise values? This question is central to policy makers, researchers and practitioners.

Billionaire investor Warren Buffett, for example, has often emphasized product market dynamics (particularly substitution) as a determinant of firms' investment intensity and rates of return. In his annual letters to the shareholders of Berkshire Hathaway [1], he has repeatedly used airlines to exemplify an industry populated by many competitors that produce near-perfect substitutes (airline seats): in his view, such intense competition made the industry a "*bottomless pit*" for capital, plagued by investment arms races and low rates of return, "*since the days of the Wright Brothers*".

When it comes to understanding corporate investment, the neoclassical (Q) model, developed sequentially by Jorgenson (1963), Tobin (1969), Abel (1980) and Hayashi (1982), stands out as the work-horse both in terms of influence and wide range of applications. However, the Q model was developed for a single firm in isolation. In practice, as noted by Buffett, firms do not operate in isolation, but within a competitive environment, and their investment decisions are interrelated: the marginal rate of return on a firm's capital investment depends on the consumer's demand for the firm products, and this is in turn affected by the investments of its competitors.

These strategic interactions remain hard to analyze at scale, both theoretically and empirically. On the one end, in order to preserve tractability, theoretical papers typically focus on dynamic investment models of duopoly or symmetric oligopoly, which lack broad empirical relevance.[2] On the other end, the applied empirical literature faces the difficulty of identifying product market investment spillovers in reduced form, or structurally estimating complex models of strategic behavior.[3]

In this paper, we provide a highly tractable and scalable Q model of investment in which firms that produce differentiated products compete in oligopolistic product markets. Leveraging the Generalized Hedonic-Linear demand system (GHL) of Pellegrino (2025), we model the firms' product market interactions as a network: in this (weighted) network firms are nodes, and the pairwise degree of substitution between the goods and services they supply are the edges. We call this the Network-Q model.

Our model is, simultaneously, a neoclassical model *and* a dynamic game of oligopoly. Our equilibrium concept is Markov, appropriately refined by the standard restriction that best response functions are linear and that the resulting dynamic system of capital stocks is dynamically stable. These restrictions imply that in equilibrium firms' capital stocks follow a Vector Auto Regressive (VAR) process. We characterize all Stable Markov Linear equilibria (SLME) of the model, which include Open-Loop and Closed-Loop equilibria, and we discuss their properties.

For our quantitative analysis, we focus on the Open-Loop Stable Markov Equilibrium in Linear strategies (OSLME), which imposes minimal plausible assumptions about managerial strategic behavior, and for which we obtain an explicit solution in closed-form, and we establish both existence and uniqueness. We also provide closed-form expressions for firms' investment rates, enterprise values and marginal Q's in terms of the underlying primitives (discount rates, adjustment cost param-

[1] See in particular the 1991, 2004, and 2007 letters at www.berkshirehathaway.com/letters/letters.html

[2] See the seminal papers Maskin and Tirole (1987), Maskin and Tirole (1988a) and Maskin and Tirole (1988b), as well as the neoclassical symmetric duopoly model with adjustment costs by Reynolds (1991).

[3] For studies on product market peer effects and investment spillovers, see for instance Bloom, Schankerman, and Van Reenen (2013) and Bustamante and Fresard (2021).

eters, etc...). This is unusual for both neoclassical investment models as well as dynamic oligopoly games, and underscores the tractability of our framework.

In line with previous neoclassical studies considering imperfect competition (e.g Hayashi, 1982 and Crouzet and Eberly, 2023), we show that the wedge between each firm's average Q and marginal Q is associated with monopoly rents. However, our model advances this literature in one important respect: markups in our model are endogenous, and co-determined with investment in equilibrium. A firm's investment is thus affected by its market power; vice-versa a firm's investment affects its market power and that of its peers. Because capital expenses vary with firms' discount rates, we further show that firm-level and aggregate fluctuations in discount rates affect the dynamics of markups and industry concentration.

Our analytical characterization yields a fresh insight into how firms make investment decisions: namely, product market competition plays a key role in determining the rate of capital accumulation at the firm level. This is not only true in the steady state, but also along the transition dynamics. In determining the speed of capital adjustments, firms trade off capital adjustment costs against the profit loss that would be incurred by *not* adjusting capacity in response to rivals' investment. Firms that face mild competitive pressures (as captured by the concavity of their profit function), relative to adjustment costs (captured by the convexity of the capital installation cost function) will converge more slowly to their steady state level of capital. Vice-versa, firms that face intense product market competition will converge faster. As a result, in the aggregate economy, the sparsity of the product market network and the initial allocation of capital among firms jointly determine how fast the economy grows its capital stock over time.

The intensity of competition faced by each individual firm in turn depends on how similar or dissimilar their product is to those produced by its rivals. Through the lenses of the GHL framework, this in turn depends on the attributes of all of the products that the consumer can purchase.

A second interesting theoretical consequence of modeling the investment of a multiplicity of firms that compete with each other is that Tobin's Q becomes a matrix (or, equivalently, a network): specifically, it is the Jacobian of the firms' values with respect to the vector of capital stocks. The diagonal entries of this matrix correspond to marginal Q, as defined by Hayashi (1982). Meanwhile, the off-diagonal terms capture the spillover of firms' investment on their rival firms' values. Relatedly, we derive a closed form expression for an alternative metric of investment spillovers, namely the effect of a unit increase in a firm's capital on a rival's investment (Bloom, Schankerman, and Van Reenen, 2013). For both of these measures of spillovers, we show how they relate to the degree of competitive pressure that firms are subject to in equilibrium.

After completing the theoretical exposition of our model, we take it to the data for the universe of U.S. public companies from 1995 to 2021. Following Pellegrino (2025), we use the text-based product similarities dataset created by Hoberg and Phillips (2016), which covers the universe of publicly-traded firms, to parameterize our massive hedonic demand system (for an average year, it contains several millions of elasticities).

Our empirical implementation departs radically from the previous corporate finance literature, where empirical applications of the Q model are dominated by reduced-form predictive regressions and structural estimation of key model parameters. By contrast, our model is mapped to the data firm-by-firm (each firm in the model corresponds to an actual real firm in the data) and is implemented fully structurally: that is, we recover all model primitives at the firm level. In this respect our empirical approach is much closer to empirical dynamic oligopoly models from the industrial organization

literature. However, unlike IO oligopoly model, which typically must be solved numerically and only apply to a few firms or a few tens of firms, our model is implemented fully analytically and can be seamlessly scaled up to the entire Compustat universe.

Because our model is significantly over-identified, we begin our quantitative analysis by assessing the model's goodness of fit. In particular, we show that the model performs well in matching the cross-section of variables that can't be perfectly fitted - namely: 1) firm revenues; 2) investment rates; 3) firm valuations. We then proceed to utilize our model to deliver five key novel insights.

Our framework opens up a wide range of new quantitative applications for the Q model, such as policy simulations and counterfactual analyses. In the fourth section of the paper, we deploy our model to produce a number of novel insights about corporate investment and product market competition.

First, we show that the transition dynamics of capital accumulation have significantly slowed down during the past decades: in other words, capital became stickier, primarily due to technological factors (production became more capital dependent, increasing the relative importance of adjustment costs). We demonstrate this by performing a spectral decomposition of the transition matrix that governs the capital accumulation process. Its eigenvalues (which by construction are comprised in the unit interval) have systematically shifted towards 1 between 1995 and 2021, indicating a higher persistence of corporate capital stocks.

As our second quantitative exercise, we project all firms' steady-state valuations, and (through the lens of our model) break them down into two fundamental additive components. The first captures the replacement value of capital. The second is the present value of all future monopoly rents. We find that aggregate firm value has risen between 1995 and 2021, and monopoly rents have contributed significantly to the rise of firm valuations over time. The model decomposition also suggests that monopoly rents have risen significantly as a proportion of aggregate firm value: monopoly rents make up 47.27% of aggregate firm value in 1995, and increase to 71.50% of aggregate firm value in 2021. Our results confirm the importance of market power in determining firms' valuations, as noted by Crouzet and Eberly (2023) and Eeckhout (2025), among others.

We then consider multiple counterfactuals, which highlight the ability of our model to address relevant questions for both researchers and policymakers. By means of a counterfactual on firms' competitive conduct, we confirm product market competition is a key force driving aggregate investment. Our baseline estimation focuses on the canonical Cournot case, in which firms internalize the downward sloping residual demand, but do not internalize their own product market spillovers on peer firms. We compare this case to a cartel scenario in which all product market spillovers are internalized. Moving from Cournot competition to full cartelization reduces aggregate steady-state capital by 32.27%. The result confirms that investment spillovers significantly contribute to aggregate investment.

Our next quantitative exercise is motivated by the fact that, in our model, firms pass their costs of capital onto their customers through their markups. In this exercise, we show that shocks to the discount rate affect markups and concentration, and alter the inter-temporal allocation of capital. We show this by simulating a 1pp unanticipated, persistent drop in the discount rate. We find that the investment response of firms is high heterogeneous, and so is the response of markups. Firms in general respond by moving to an alternative OSLME path characterized by lower industry concentration and markups. While our model does not feature monetary policy in the strict sense, this finding suggests that monetary policy loosening (implemented as a persistent decrease in the nominal interest rate) may alter product market competition among firms.

Because our model also allows us to simulate mergers, in our last quantitative exercise we assess the impact of takeover activity on aggregate investment between 1995 and 2021. Compared to the counterfactual case of no takeover activity after 1995, we infer that the documented takeover activity has led to a modest reduction of 1.36% of all corporate investment on aggregate. Simultaneously, however, we observe significant variation in the impact of mergers in the cross section: while the median merging firm barely alters its price, the merging firm at the 99th percentile range of the distribution increases markups by approximately 1.11% within five years' time of the merger.

The paper concludes with a sensitivity analysis of our quantitative results (where we study alternative parametrizations of the model) and with a discussion of extensions and alternative applications of our network Q model. We show that the model can be extended to incorporate more complex forms of strategic interaction beyond simple product market competition, including input-output relations and common ownership. While we focus on the Cournot outcome in which firms compete in capacity, the model can be re-interpreted to allow for competition in prices and sticky adjustment menu costs. While our baseline model focuses on a single adjustment cost function for all firms, we show in an extension how to incorporate heterogeneous adjustment costs and discount rates. Lastly, although our core empirical application focuses on firm level data, our formulation allows for multi-product firms and can be readily applied to study investment data at the product or establishment level.

Previous Literature. Our study relates to a broad range of contributions in finance, macroeconomics and industrial organization. The most closely-related work is two recent contributions by Okumura (2025) and Hopenhayn and Okumura (2025), who develop a model of endogenous growth that is based on the same demand system (GHL) and mode of competition used in our own model. Our contributions are complementary: they both make important progress in developing dynamic extensions of the static network oligopoly model of Pellegrino (2025), but focus on different applications and research questions. While Okumura (2025) and Hopenhayn and Okumura (2025) focus on growth dynamics in a continuous-time model where firms' R&D expenditures enhance productivity, we develop instead a discrete-time neoclassical investment application, where firms invest to scale up. More broadly, our model contributes to the rapidly growing literature on network oligopoly and GHL demand (e.g. Eeckhout and Veldkamp, 2023, Bizzarri and Vega-Redondo, 2024, Voelkening, 2024, Ederer and Pellegrino, 2025, Mukherjee et al., 2025, Miyashita, 2026).

As mentioned above, our model falls squarely in the tradition of neoclassical models, including the seminal contributions of Jorgenson (1963), Tobin (1969), Abel (1980), Hayashi (1982) and Abel and Eberly (1994). This literature has been recently revived by the recent debate around the rise of market power (Syverson, 2019; Syverson, 2024) and the role of intangible capital (Eisfeldt, Kim, and Papanikolaou, 2022; Eisfeldt and Papanikolaou, 2013), with key contributions by Peters and Taylor (2017), and Crouzet and Eberly (2021, 2023). Eberly and Wang (2009) study capital reallocation in a neoclassical setting, whereas two recent contributions by Chodorow-Reich, Smith, Zidar, and Zwick (2024) and Chodorow-Reich (2025) examine the role of tax policy on investment. Our quantitative analyses also complement existing methodological studies on the estimation of neoclassical investment models, including Erickson and Whited (2000), Erickson and Whited (2002) and Cooper and Haltiwanger (2006), among others.

Our model has its roots in a long-standing theoretical literature on dynamic duopoly and oligopoly games with linear demand, dating back to Maskin and Tirole (1987). Reynolds (1987, 1991) pioneered early models of dynamic duopoly and symmetric oligopoly with capacity adjustment costs.

We also contribute to a literature at the boundary between finance, macroeconomics and industrial

organization, including studies on the secular change in industry concentration and markups (Autor, Dorn, Katz, Patterson, and Van Reenen, 2020; De Loecker, Eeckhout, and Unger, 2020; Covarrubias, Gutiérrez, and Philippon, 2020; Kwon, Ma, and Zimmermann, 2024; Hoberg and Phillips, 2025), and the relationship between market power and stock market valuations (Bustamante and Donangelo, 2017; Cho, Grotteria, Kremens, and Kung, 2023; Eeckhout, 2025; Corhay, Kung, and Schmid, 2025; Boppart, Klenow, Laski, and Li, 2025).

Because the model we describe is a type of network game, our paper connects to a booming literature on networks, which spans both microeconomic theory and macroeconomics. On the micro side, we note the work of Ushchev and Zenou (2018), Galeotti, Golub, and Goyal (2020) and Galeotti, Golub, Goyal, Talamas, and Tamuz (2024). The literature on networks in macroeconomics includes (among many others): Acemoglu et al. (2012), Carvalho and Gabaix (2013), Bloom, Schankerman, and Van Reenen (2013), Carvalho, Nirei, Saito, and Tahbaz-Salehi (2021) Baqaee and Farhi (2020, 2024), Baqaee (2018) and Flynn, Patterson, and Sturm (2022). Liu and Tsyvinski (2024) develop a dynamic model that incorporates adjustment costs of changing inputs. Vom Lehn and Winberry (2022) describe the "investment network" and analyze its role in business cycle fluctuations. In our quantitative analysis, we exploit the regularity that productivity distributions exhibit heavy right tails which are well-approximated by Pareto distributions, consistent with Gabaix (2009).

Last, we add to a recent literature that enriches macroeconomic models with oligopolistic competition, which includes (among others) Atkeson and Burstein (2008), Grassi (2017), Covarrubias (2022), Wang and Werning (2022) and Burstein, Carvalho, and Grassi (2025).

2 The Network Q Model

In this section, we provide an exposition of our theory of corporate investment in oligopolistic product markets. We begin by describing the hedonic demand system that underpins product market competition between the firms in our model.

2.1 Hedonic Consumer Demand

Products and Characteristics. Our model is populated by generically-defined *business units*, which can be interpreted as establishments, plants, product lines or single-product firms, depending on the application. We model the supply and investment decision at the level of business units, which may be either independent from each other or instead owned by the same firm, as we elaborate below.

Time is discrete and indexed by t. We index each business unit with the subscript $n \in \{1,2,...,N\}$. Consumer demand follows the GHL demand framework of Pellegrino (2025). We model each business unit's product n as a bundle of characteristics, sorted into two distinct types. Characteristics that are common across all products are indexed by $j = 1,2,...,J$, whereas product-specific characteristics are directly indexed by their product label n. For each product n, we define a_{jn} as the quantity of common characteristic j provided by that product. Every product is represented by a J-dimensional, strictly-positive column vector $\mathbf{a}_i$ with unit length, expressed formally as:

$$\mathbf{a}_n = \begin{bmatrix} a_{1n} & a_{2n} & \dots & a_{Jn} \end{bmatrix}' \tag{1}$$

$$\text{with} \qquad \sum_{j=1}^{J} a_{jn}^2 = 1 \quad \forall\, n \in \{1,2,...,N\} \tag{2}$$

This vector $\mathbf{a}_n$ locates product n within the common characteristic space. Considering all products at the same time, these location vectors can be organized into a single $J \times N$ matrix $\mathbf{A}$:

$$\mathbf{A} \equiv \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{J1} & a_{J2} & \cdots & a_{JN} \end{bmatrix} \tag{3}$$

Let y_{nt} represent the volume of product n that is produced and consumed at time t, and denote by $\mathbf{y}_t$ the consumption bundle demanded by the representative consumer at time t. In accordance with hedonic demand theory, consumers aggregate characteristics from various products through linear combination, with utility derived from these characteristics rather than the products themselves. If we denote the total consumption of common characteristic j at time t as x_{jt}, then:

$$x_{jt} = \sum_{n=1}^{N} a_{jn}\, y_{nt} \qquad \text{or (in linear algebra notation)} \qquad \mathbf{x}_t = \mathbf{A}\mathbf{y}_t \tag{4}$$

For product-specific characteristics, we assume that each unit of product n delivers exactly one unit of its corresponding unique characteristic. Hence, y_n is both the output of product line n as well as the total quantity of the product-specific characteristic delivered by product line n.

Consumer Utility. Our model employs a representative consumer framework, with utility that is quadratic in both the common characteristics (x_{jt}) and the idiosyncratic characteristics (equal to output y_{nt}) associated with each of $n = 1,...,N$ products in any period t. In matrix notation, this corresponds to the function:

$$U(\mathbf{x}_t, \mathbf{y}_t, L_t) \overset{\text{def}}{=} \alpha\left(\mathbf{x}_t'\mathbf{b}^x - \frac{1}{2}\mathbf{x}_t'\mathbf{x}_t\right) + (1-\alpha)\left(\mathbf{y}_t'\mathbf{b}^y - \frac{1}{2}\mathbf{y}_t'\mathbf{y}_t\right) - L_t \tag{5}$$

where $\alpha \in (0,1)$ is the utility weight assigned to common characteristics and the vectors $\mathbf{b}^x$ and $\mathbf{b}^y$ represent characteristic-specific preference shifters and L_t is the total number of work hours supplied by the representative consumer at time t. The representative consumer faces a vector of product prices $\mathbf{p}_t = [p_{1t}, p_{2t}, ..., p_{Nt}]'$ and takes it as given. The consumer is also endowed with the shares of all the firms in the economy and thus he/she faces the following budget constraint:

$$\sum_{n=1}^{N} p_{nt} y_{nt} \leq L_t + \Pi_t \tag{6}$$

where Π_t is aggregate profits and the price of labor is normalized to 1 (labor is the numéraire good). The consumer's demand for each product n maximizes (5) subject to (6). As shown by Pellegrino (2025), in matrix form, the inverse demand function resulting from the consumer's optimization problem is given by:

$$\mathbf{p}_t = \mathbf{b} - (\mathbf{I} + \boldsymbol{\Sigma})\mathbf{y}_t \tag{7}$$

where $\mathbf{b}$ is defined as a time-invariant $N \times 1$ vector of product-specific demand shifters

$$\mathbf{b} \overset{\text{def}}{=} \alpha\mathbf{A}'\mathbf{b}^x + (1-\alpha)\mathbf{b}^y \tag{8}$$

$\mathbf{I}$ is an $N \times N$ identity matrix and the matrix $\mathbf{\Sigma}$ is symmetric and by definition equal to:

$$\mathbf{\Sigma} \overset{\text{def}}{=} \alpha\left(\mathbf{A}'\mathbf{A} - \mathbf{I}\right) \tag{9}$$

$\alpha \in (0,1)$ is the same preference parameter from equation (5). The (n,m) entry of $\mathbf{\Sigma}$ measures the degree of substitutability between products n and m, denoted σ_{nm}. When products n and m are more similar in their characteristic bundles, σ_{nm} is larger, implying that an increase in the supply of product n leads to a larger decline in the price of product m.

Cosine Similarity. The key input to construct $\mathbf{\Sigma}$ is the matrix $\mathbf{A}'\mathbf{A}$, whose (n,m) entry equals the inner product $\mathbf{a}_n'\mathbf{a}_m$. This inner product has a natural geometric interpretation: it equals the cosine of the angle between the characteristic vectors $\mathbf{a}_n$ and $\mathbf{a}_m$ in the J-dimensional space of common characteristics. For this reason, $\mathbf{a}_n'\mathbf{a}_m$ is called the *cosine similarity* between products n and m. The cosine similarity ranges from zero (when products share no common characteristics and are thus orthogonal in characteristic space) to one (when products have identical characteristic bundles). Products with higher cosine similarity provide more overlapping mixes of characteristics to consumers, making them closer substitutes in demand.

The parameter α serves two distinct roles in our framework. First, as shown in equation (5), it determines the relative weight consumers place on common versus product-specific characteristics when evaluating utility. Second, through equation (9), it controls the overall degree of substitution across all products. A higher value of α implies that cross-price effects are stronger for any given level of similarity in the space of product characteristics.

This network approach to modeling product market competition offers a critical advantage over traditional methods: it allows the data to define product markets endogenously, rather than imposing market boundaries through administrative classifications.

2.2 Production and Investment

Production Technology. Each product in the economy is produced by a single business unit n, which in turn it is administered by a manager whom we also index by n. The optimization problem of the manager builds on the neoclassical literature: each business unit uses capital and a composite variable input which we label "labor" for convenience to produce a perishable amount y_{nt} of product n. Production exhibits constant returns to scale. Consistent with recent quasi-experimental evidence on firm-level capital-labor substitution (Moreau, 2019; Benzarti and Harju, 2021) we assume a Leontief production function:[4] in order to produce a unit of product n, its producer requires the supply of $1/z_n$ units of capital and c_n units of labor, so that:

$$y_{nt} = z_n k_{nt}\,; \qquad \ell_{nt} = c_n y_{nt} \tag{10}$$

where ℓ_{nt} captures the number of hours allocated to business unit n by the representative consumer, z_n represents the productivity associated of capital of business unit n, and c_n represents the unit labor cost of operating business unit n. We interpret k_{nt} as a composite of heterogeneous types of capital held by business unit n (i.e. it includes both tangible and intangible types of capital).

[4] Aggregate substitution between capital and labor arises instead from between-firm reallocation, which our model captures through the equilibrium transition dynamics.

Law of Motion of Capital and Installation Costs. In each period t, manager n starts with capital stock k_{nt} and decides how much to invest and thus what output will be at time $t+1$. We define i_{nt} as the level of investment in capital of business unit n in period t. The law of motion for capital is then given by:

$$k_{nt+1} = i_{nt} + (1-\delta_n)k_{nt} \tag{11}$$

where δ_n is the rate at which the capital stock k_{nt} depreciates. Investment entails costs $\psi_n(k_{nt+1};k_{nt})$ which drain the business unit's available cash flows each period. Consistent with Abel and Eberly (1994, among others), we assume that the function $\psi_n(k_{nt+1};k_{nt})$ is convex. Furthermore, we assume that it takes the functional form:

$$\psi_n(k_{nt+1};k_{nt}) \stackrel{\text{def}}{=} p_n^k i_{nt} + \frac{\theta}{2}(k_{nt+1}-k_{nt})^2 \tag{12}$$

The first term on the right hand side of (12) is the cost of purchasing new capital; p_n^k is the price of capital for firm n, assumed to be exogenous. The second term is a quadratic adjustment cost with curvature parameter $\theta > 0$. We interpret the function $\psi_n(k_{nt},k_{nt+1})$ as the business unit's total costs associated with capital expenditures in period t.

Notice that our specification of equation (12) assumes a common adjustment cost parameter θ for all business units. While this may seem restrictive at first sight, in practice our setting allows for plenty of heterogeneity in the effective burden of adjustment costs, which vary substantially across business units through differences in the productivity of capital z_n. All else equal, business units with low productivity z_n find it relatively costlier to adjust their stock of capital, making them sluggish in responding to changing competitive conditions. Conversely, business units with high productivity z_n attain the same level of output with less capital, allowing them to adjust their capital stock more nimbly over time. Furthermore, in Section 5 we show how the model can be extended to allow for firm-specific adjustment cost parameters θ_n.

Operating Profits. Next we define π_n , the operating profit of business unit n, as follows:

$$\pi_n \stackrel{\text{def}}{=} (p_{nt}-c_n)y_{nt} \tag{13}$$

Substituting the price p_{nt} in terms of the output y_{nt} (imposing consumer maximization) and y_{nt} in terms of the capital stock k_{nt}, we can re-write π_n as a (linear quadratic) function of the vector of capital stocks:

$$\pi_n(\mathbf{k}_t) = (b_n-c_n)z_n k_{nt} - z_n^2 k_{nt}^2 - \sum_{m\neq n}\sigma_{nm} z_n z_m k_{nt} k_{mt} \tag{14}$$

where $\mathbf{k}_t$ is the vector of capital stocks (one row for each business unit) and σ_{nm} is the (n,m) entry of the adjacency matrix $\mathbf{\Sigma}$ in (7). Alternatively, in matrix form, operating profits are expressed as:

$$\boldsymbol{\pi}(\mathbf{k}_t) = \mathbf{K}_t\mathbf{Z}(\mathbf{b}-\mathbf{c}) - \mathbf{K}_t\mathbf{Z}(\mathbf{I}+\mathbf{\Sigma})\mathbf{Z}\mathbf{k}_t \tag{15}$$

where $\mathbf{Z}$ and $\mathbf{K}_t$ are simply the vectors $\mathbf{z}$ and $\mathbf{k}_t$ rearranged as a diagonal matrices:

$$\mathbf{Z} \stackrel{\text{def}}{=} \begin{bmatrix} z_1, & 0, & \dots & 0 \\ 0 & z_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z_N \end{bmatrix}; \qquad \mathbf{K}_t \stackrel{\text{def}}{=} \begin{bmatrix} k_{1t}, & 0, & \dots & 0 \\ 0 & k_{2t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k_{Nt} \end{bmatrix} \tag{16}$$

Investors discount cashflows in period $t+1$ to period t with the discount rate r. We define a business unit's cash flows as the operating profits in equation (13) net of investment costs in equation (12). We next proceed to define the manager's objective. In doing so, we allow the manager of business unit n to internalize the spillovers of her supply choices on some other business unit $m \neq n$. This flexibility allows to model complex ownership structures (such as multi-product firms) and study competitive arrangements across business units or firms (such as cartels). Two business units that operate under the same umbrella firm or which are part of the same cartel are assumed to fully internalize each other's product market spillovers (Baker and Bresnahan, 1985), so that there is no cannibalization among co-owned business units.

The Managers' Objective. Let $\mathscr{M}_n$ be the set of business units that are owned by the same firm as n (including business unit n itself). Then, the optimization problem of the manager of business unit n at time t is to maximize the sum from time t onward of the discounted cashflows of all business units associated with the firm owning business unit n, which we index hereafter by $m \in \mathscr{M}_n$. Specifically, the manager maximizes the joint shareholder value of all business units $m \in \mathscr{M}_n$ belonging to the same firm (or cartel), by choosing optimally the capital stock k_{nT} at every future time T. In so doing, manager n takes the capital choices of all business units different from n as given:

$$v_n(\mathbf{k}_t) \stackrel{\text{def}}{=} \max_{(k_{nt+1}, k_{nt+2}, \dots)} \sum_{T=t}^{\infty} \sum_{m \in \mathscr{M}_n} \frac{\pi_m(\mathbf{k}_T) - \psi_m(k_{mT+1}; k_{mT})}{(1+r)^{T-t}} \tag{17}$$

where $v_n(\mathbf{k}_t)$ represents the total firm value relevant for the manager of business unit n, as a function of the vector of capital stock $\mathbf{k}_t$. Each manager then solves equation (17) subject to equations (11), (12), (13), and the transversality condition:

$$\lim_{T \to \infty} (1+r)^{-(T-t)} v_n(\mathbf{k}_T) = 0 \tag{18}$$

By construction, managers of business units within the same firm maximize identical objective functions, and yet the control variable of each manager within the same firm is the investment decision associated with their specific business unit. The formulation in (17) implies firm value at time t is expressed *cum-dividend*.

The dynamic oligopoly game that we have set up is a type of *dynamic network game.* A network is defined as a set of nodes and relationships between the nodes, defined by an *adjacency matrix*. In this setting, business units are nodes, and the (scaled) product similarity matrix $\mathbf{\Sigma}$ can be thought of as the adjacency matrix. We also define a network of ownership/collusion between business units. Let γ_{mn} be an indicator/dummy variable that equals 1 if $m \in \mathscr{M}_n$ and $n \in \mathscr{M}_m$. Then the symmetric matrix $\mathbf{\Gamma}$, whose (m,n) entry is γ_{mn}, is an adjacency matrix that describes shared ownership or collusion between two alternative business units.

2.3 Marginal Q: Definition

Ahead of solving the dynamic network game, we formally introduce the concept of marginal Q, consistent with the seminal contribution of Hayashi (1982). The marginal contribution of a business unit's own capital to the value of its associated firm at time t is defined as:

$$q_{nn}(\mathbf{k}_t) \overset{\text{def}}{=} \frac{\partial v_n(\mathbf{k}_t)}{\partial k_{nt}} \tag{19}$$

The double subscript nn is intentional: because the economic environment is populated by multiple business units we can equally well define the marginal effect on the value of the firm associated with business unit n of an additional unit of capital accumulated by another business unit m:

$$q_{nm}(\mathbf{k}_t) \overset{\text{def}}{=} \frac{\partial v_n(\mathbf{k}_t)}{\partial k_{mt}} \tag{20}$$

Our model thus generalizes Hayashi's marginal- Q from the single firm case to a network of business units that interact with each other. In our network game, Q is no longer a scalar, but rather a matrix. Specifically, we define $\mathbf{Q}(\mathbf{k}_t)$ as the Jacobian of the vector of firm values $\mathbf{v}(\mathbf{k}_t)$:

$$\mathbf{Q}(\mathbf{k}_t) \overset{\text{def}}{=} \frac{\partial \mathbf{v}(\mathbf{k}_t)}{\partial \mathbf{k}_t} \equiv \begin{bmatrix} q_{11}(\mathbf{k}_t) & q_{12}(\mathbf{k}_t), & \dots & q_{1N}(\mathbf{k}_t) \\ q_{21}(\mathbf{k}_t) & q_{22}(\mathbf{k}_t) & \dots & q_{2N}(\mathbf{k}_t) \\ \vdots & \vdots & \ddots & \vdots \\ q_{N1}(\mathbf{k}_t) & q_{N2}(\mathbf{k}_t) & \dots & q_{NN}(\mathbf{k}_t) \end{bmatrix} \tag{21}$$

Due to the presence of product market externalities, the capital accumulation decision of any business unit within firm n can affect the market value of any firm m. If we think of $\mathbf{Q}(\mathbf{k}_t)$ as an adjacency matrix, it describes a network of product market spillovers on enterprise values. With some abuse of notation, we can use the vector $\mathbf{q}(\mathbf{k}_t)$ to denote the diagonal entries of matrix $\mathbf{Q}(\mathbf{k}_t)$

$$\mathbf{q}(\mathbf{k}_t) = \text{diag}[\mathbf{Q}(\mathbf{k}_t)] \equiv \begin{bmatrix} q_{11}(\mathbf{k}_t) & q_{22}(\mathbf{k}_t) & \cdots & q_{NN}(\mathbf{k}_t) \end{bmatrix}' \tag{22}$$

which correspond to Hayashi's familiar definition of a firm's own marginal Q. In what follows, we use marginal Q to write the managers' optimality conditions to derive the equilibrium. Armed with the corresponding equilibrium value functions, we shall later derive $\mathbf{Q}(\mathbf{k}_t)$ in closed form.

2.4 Equilibrium Concepts and Dynamic Stability

The equilibrium concept that we use is Markov-Nash. This refinement of the Nash Equilibrium requires that the optimal investment policy of business unit n takes the form:

$$k_{nt+1} = f_n(\mathbf{k}_t) \tag{23}$$

It follows that k_{nt+1} is only a function of the vector of contemporaneous capital stocks: the previous history of play is irrelevant for the future evolution of the system.

In what follows, we derive the optimal investment decision of all managers n at time t by maximizing the N value functions in (17) in stacked form. Markov equilibria are associated to a vector of optimized value functions and the associated investment policies for all business units $n = 1, 2, ..., N$,

defined as:

$$\mathbf{v}(\mathbf{k}) \equiv [v_1(\mathbf{k}), v_2(\mathbf{k}), ..., v_N(\mathbf{k})]' ; \qquad \mathbf{f}(\mathbf{k}) \equiv [f_1(\mathbf{k}), f_2(\mathbf{k}), ..., f_N(\mathbf{k})]' \tag{24}$$

Throughout our analysis, the manager n optimizes on k_{nt+1} while taking as given the decision of all other business units, which we denote collectively with the subscript $-n$.

Bellman Equation. We can re-write the maximization problem of manager n in recursive form as the following dynamic program:

$$v_n(\mathbf{k}) = \max_{f_n(\mathbf{k})} \sum_{m \in \mathcal{M}_n} \left[\pi_m(\mathbf{k}) - \psi_m(f_m(\mathbf{k}); k_m) \right] + \frac{v_n(\mathbf{f}(\mathbf{k}))}{1+r} \tag{25}$$

We can thus obtain the equilibrium vector $\mathbf{k}_{t+1}$ by solving simultaneously a system of n stacked Bellman equations, where each equation represents a business unit facing the optimization problem in equation (25).

Definition 1. *For any Markov Equilibrium, we define a steady-state as a vector of capital stocks* $\mathbb{k}$ *such that* $\mathbf{f}(\mathbb{k}) = \mathbb{k}$.

We follow Jun and Vives (2004) and adopt a refinement that focuses on linear best responses and imposes the dynamic stability of the process that describes the evolution of the state vector $\mathbf{k}$ along the equilibrium path.

Definition 2. *A Stable Linear Markov Equilibrium (SLME) is a Markov Equilibrium with linear investment policies where* $\mathbf{f}(\mathbf{k})$ *takes the form*[5]

$$\mathbf{f}(\mathbf{k}) - \mathbb{k} = \mathbf{\Phi}(\mathbf{k} - \mathbb{k}) \tag{26}$$

where $\mathbf{\Phi}_{nn}^{-1} > 0$ *for all business units* $n = 1, 2, ..., N$, *all eigenvalues of matrix* $\mathbf{\Phi}$ *lie in the interval* $(0,1)$, *and the vector* $\mathbb{k}$ *is equal to the unique steady state of the model.*

While the restriction to linear strategies and dynamic stability may appear restrictive at first sight, it is not arbitrary. In Proposition 3 we show that our game's SLME (appropriately refined) arises naturally as the point-wise limit of a sequence of unique (non-necessarily linear) equilibria of the corresponding finite-horizon games. In these truncated games, backward induction pins down a unique equilibrium without any auxiliary refinement; the linearity of investment policies emerges endogenously in the limit as the horizon recedes to infinity.

Dynamic Stability. As we elaborate further below, the matrix $\mathbf{\Phi}$ governs the dynamics of the capital accumulation process for all business units, as well as the spillover effects of a manager's own investment decision on the investment of its peers over time. By construction, along the equilibrium path, the capital stocks of business units follow a vector auto-regression (VAR) process. The equilibrium is dynamically stable if such VAR process is stable in a time-series sense (technically, the eigenvalues of matrix $\mathbf{\Phi}$ lie in the interval (0,1)). This is an important property of the equilibrium: it ensures that, for every initial vector of capital stocks, operating profits do not diverge, thus ensuring

[5] Notice that this definition of "linear" is without loss of generality as long as the matrix $(\mathbf{I} - \mathbf{\Phi})$ is invertible, which we verify later on. Specifically, expressing linear strategies as in (26) is the same as: 1) writing the more general expression $f(\mathbf{k}_t) = \boldsymbol{\iota} + \mathbf{\Phi}\mathbf{k}_t$; 2) noticing that the unique steady state is given by the vector $\mathbb{k} = (\mathbf{I} - \mathbf{\Phi})^{-1}\boldsymbol{\iota}$; 3) substituting vector $\boldsymbol{\iota}$ in terms of $\mathbb{k}$ inside $f(\mathbf{k}_t) = \boldsymbol{\iota} + \mathbf{\Phi}\mathbf{k}_t$ to obtain (26).

the transversality condition (18) hold. It follows that the vector of capital stocks gradually converges to its steady state level along the equilibrium path:

$$\lim_{t\to\infty} \mathbf{k}_t \;=\; \mathbb{k} \tag{27}$$

For this reason, with some abuse of notation, we shall use $k_{n\infty}$ to denote the steady-state capital vector of firm n (the n^{th} entry of $\mathbb{k}$), so that:

$$\mathbb{k} \overset{\text{def}}{=} \left[\begin{array}{cccc} k_{1\infty} & k_{2\infty} & \dots & k_{N\infty} \end{array} \right] \tag{28}$$

SLME Investment Policy. By combining equation (26) with the law of motion of capital (equation 11), we obtain the investment policies of all business units as a system of N linear equations:

$$\boldsymbol{i}_t(\mathbf{k}_t) \;=\; \boldsymbol{\Delta}\mathbf{k}_t + (\mathbf{I}-\boldsymbol{\Phi})(\mathbb{k}-\mathbf{k}_t) \tag{29}$$

where $\boldsymbol{\Delta}$ is the vector of depreciation rates $\boldsymbol{\delta}$ rearranged as a diagonal matrix:

$$\boldsymbol{\Delta} \overset{\text{def}}{=} \left[\begin{array}{cccc} \delta_1, & 0, & \dots & 0 \\ 0 & \delta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \delta_N \end{array} \right] \tag{30}$$

It is known in the previous literature (see Reynolds, 1987, 1991) that, even after restricting the analysis to stable linear Markov strategies, dynamic investment games still generally admit multiple equilibria. In particular, the literature has divided those equilibria into two types: Open-Loop and Closed Loop. These two types of equilibria differ by their underlying assumptions about what firms internalize or take as given as they make strategic investment decisions. We next provide a formal definition for both.

Definition 3. *An Open-Loop Stable Linear Markov Equilibrium (OSLME) is a vector of value functions* $\mathbf{v}(\mathbf{k})$ *and a vector of investment policies* $\mathbf{f}(\mathbf{k})$ from Definition 2 *that solve the maximization problem in equation (17) for all business units* $n = 1,2,...,N$*, subject to*

$$\mathbf{k}_{t+1} = \boldsymbol{i}_t + (\mathbf{I}-\boldsymbol{\Delta})\mathbf{k}_t \tag{31}$$

taking the sequence of competitors' capital stocks $\left(\mathbf{k}_{-n,t+1}, \mathbf{k}_{-n,t+2}, ...\right)$ *as given.*

Definition 4. *A Closed-Loop Stable Linear Markov Equilibrium (CSLME) consists of a vector of value functions* $\mathbf{v}(\mathbf{k})$ *and a vector of investment policies* $\mathbf{f}(\mathbf{k})$ from Definition 2 *that solve the Bellman equation (25) maximization problem for all business units* $n = 1,2,...,N$*, subject to*

$$k_{mt+1} \;=\; \begin{cases} i_{mt} + (1-\delta_m)k_{mt} & \text{for} \quad m = n \\ k_{m\infty} + \boldsymbol{\varphi}'_m(\mathbf{k}_t - \mathbb{k}) & \text{for} \quad m \neq n \end{cases} \tag{32}$$

taking the competitors' policy functions $\mathbf{f}_{-n}(\mathbf{k})$ *as given. Moreover we require that, for all business units* $n = 1,2,...,N$*,* $\boldsymbol{\Phi}_{-n,-n}$ *has a spectral radius lower than* $1+r$ *(*$\boldsymbol{\Phi}_{-n,-n}$ *is the matrix obtained by removing row and column* n *from* $\boldsymbol{\Phi}$*).*

The key difference between these two equilibrium concepts lies in the fact that Closed-Loop equilibrium investment policies take as given competitors' investment policies ($\mathbf{f}_{-n}$), instead of the future evolution of rivals' capital stocks ($\mathbf{k}_{-nt+1}, \mathbf{k}_{-nt+2}, ...$). This implies that managers plan investment with an extreme degree of strategic sophistication, and that their strategy profiles are subgame-perfect over the infinite-horizon of the game. Paraphrasing, managers fully account for how any unilateral deviation from the equilibrium path may affect any of their competitors' capital stocks at any point in time in the future. By contrast, in an Open-Loop equilibrium, managers do not anticipate how all of their N potential competitors will respond to an unilateral deviation, and instead take as given the entire future evolution of their competitors' capital stocks. Open-Loop equilibrium strategies are only optimal along the equilibrium path.

In the next sections, we gradually work our way to a solution for the steady state vector of capital stocks $\mathbb{k}$ and the transition matrix $\mathbf{\Phi}$, that is valid for both types of equilibria.

2.5 Optimality Conditions and the Competitive Pressure Matrix Ω

First Order Condition. We begin by solving for the optimal investment decision of each manager. Using the definition of marginal Q in equation (19), we can re-write the n^{th} first order condition of (25) with respect to k_{nt+1} as:

$$\frac{q_{nn}(\mathbf{k}_{t+1})}{1+r} = p_n^k + \theta(k_{nt+1} - k_{nt}) \tag{33}$$

where the left hand side captures the discounted marginal benefit of investment next period, and the right hand side captures the marginal adjustment costs of investment today. Importantly, notice that the function $q_{nn}(\mathbf{k}_{t+1})$ depends not only on the investment of business unit n itself, but also on the investment choices of all the other business units in the product market network.

In order to provide a general characterization of all SLMEs, including Open-Loop as well as Closed-Loop equilibria, we use a dummy variable – $\mathbb{1}_{\text{CL}}$ – which equals zero in the Open-Loop equilibrium conditions, and one in the Closed-Loop equilibrium conditions. $q_{nn}(\mathbf{k}_{t+1})$ – the left hand side of the first order condition in (33) – naturally depends on $\mathbb{1}_{\text{CL}}$. It is equal to:

$$q_{nn}(\mathbf{k}_{t+1}) = \frac{\partial \pi_n(\mathbf{k}_{t+1})}{\partial k_{nt+1}} + (1-\delta_n)p_n^k + \theta(k_{nt+2} - k_{nt+1}) + \frac{\mathbb{1}_{\text{CL}}}{1+r} \cdot \frac{\partial v_n(\mathbf{k}_{t+2})}{\partial \mathbf{k}_{-n,t+2}} \cdot \frac{\partial \mathbf{k}_{-n,t+2}}{\partial k_{n,t+1}} \tag{34}$$

The first additive term of this expression is the the profit generated by an additional unit of n-capital at time $t+1$. The second is equal to the resale value at $t+1$ of the undepreciated portion of the same unit of capital. The third term is the marginal change in adjustment costs at $t+1$.

More importantly, the last term encapsulates the economic distinction behind Open-Loop and Closed-Loop equilibria in the model. The term reflects the indirect (feedback) effect of the investment decision of business unit n in period t on its continuation value in period $t+2$, mediated by the reactions of all its rivals. In the Open-Loop equilibrium ($\mathbb{1}_{\text{CL}} = 0$) managers do not internalize such feedback effect: they take the full evolution of their peers' capital stocks as given. Consequently, in an Open-Loop equilibrium, the term collapses to zero. By contrast, in Closed-Loop equilibria ($\mathbb{1}_{\text{CL}} = 1$), managers are (infinitely) sequentially rational: they actively take into account how their deviating from the equilibrium path would the impact of their investment choice on that of their rivals from period $t+1$ until infinity, similarly to as if they were playing a Stackelberg game.

The Competitive Pressure Matrix. Stacking all firms in vector notation, we can then express the first term of equation (33) as follows:

$$\left[\frac{\partial \pi_1(\mathbf{k}_{t+1})}{\partial k_{1t+1}}, \frac{\partial \pi_2(\mathbf{k}_{t+1})}{\partial k_{2t+1}}, ..., \frac{\partial \pi_N(\mathbf{k}_{t+1})}{\partial k_{Nt+1}}\right]' = \mathbf{Z}(\mathbf{b}-\mathbf{c}) - \mathbf{\Omega}\mathbf{k}_{t+1} \tag{35}$$

The matrix $\mathbf{\Omega}$, which we call hereafter the *Competitive Pressure* matrix, is defined as:

$$\mathbf{\Omega} \stackrel{\text{def}}{=} \mathbf{Z}(\mathbf{I} + \mathbf{\Sigma} + \mathbf{D} + \mathbf{\Gamma} \circ \mathbf{\Sigma})\mathbf{Z} \tag{36}$$

where the symbol "$\circ$" denotes the Hadamard (entry-by-entry) product, and $\mathbf{D}$ is a diagonal matrix of conduct parameters (Weyl and Fabinger, 2013) with binary diagonal entries which are equal to one if the corresponding business unit acts as an oligopolist (and, thus, internalizes she is facing a downward sloping demand curve):

$$\mathbf{D} \stackrel{\text{def}}{=} \begin{bmatrix} \mathscr{D}_1 & 0 & \dots & 0 \\ 0 & \mathscr{D}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathscr{D}_N \end{bmatrix}, \qquad \mathscr{D}_n \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if firm } n \text{ is a price-taker} \\ 1 & \text{if firm } n \text{ is an oligopolist} \end{cases} \tag{37}$$

The entries γ_{nm} of the adjacency matrix $\mathbf{\Gamma}$ are dummy variables capturing cross-internalization, and hence are equal to one if business units n and m are part of the same firm or cartel – formally:

$$\mathbf{\Gamma} \stackrel{\text{def}}{=} \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1N} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{N1} & \gamma_{N2} & \cdots & \gamma_{NN} \end{bmatrix} \qquad \gamma_{nm} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } n \notin \mathscr{M}_m, m \notin \mathscr{M}_n \\ 1 & \text{if } n \in \mathscr{M}_m, m \in \mathscr{M}_n \end{cases} \tag{38}$$

The formulation of matrix $\mathbf{\Omega}$ is flexible enough to allow for a multiplicity of competitive conduct cases and policy-relevant counterfactuals. For instance, if we let $\mathbf{D} = \mathbf{\Gamma} = \mathbf{I}$ we have the canonical Cournot conduct (single-product firms that internalize downward sloping demand curves, but not product market spillovers); if we consider instead $\mathbf{D} = \mathbf{I}$, $\mathbf{\Gamma} = \mathbf{1}\mathbf{1}'$ we are simulating an economy where all spillovers are internalized – a large cartel that encompasses all business units; finally, if $\text{diag}(\mathbf{D}) = \mathbf{0}$ and $\mathbf{\Gamma} = \mathbf{I}$ we have a competitive economy where the first welfare theorem holds.

The Competitive Pressure matrix $\mathbf{\Omega}$ is key to our modeling framework: its (n,m) entry is by definition the second-order derivative

$$\frac{\partial^2 \left(\sum_{n' \in \mathscr{M}_n} \pi_{n't+1}\right)}{\partial k_{nt+1} \partial k_{mt+1}} = -\omega_{nm} \tag{39}$$

$\mathbf{\Omega}$ is therefore a measure of the *concavity* of the firm's (internalized) profit functions: its n^{th} diagonal entries equals $2z_n^2$ for the canonical Cournot conduct, and z_n^2 when firms are price takers. The off-diagonal entries measure the effect of a unit change in a competitor's capital stock on the focal firm's marginal rate of return on capital. By the implicit function theorem, $\frac{1}{4}\omega_{nm}^2$ is also the $t+1$ dollar loss that is incurred (internalized) by business unit n if it fails to adjust its capacity in response to a 1 unit change in the capital of its peer m.

In what follows, we shall assume that $\mathbf{\Omega}$ is positive definite: All of its eigenvalues are strictly positive.

This assumption is always satisfied in our empirical implementation.

2.6 Stable Linear Markov Equilibria: Characterization, Existence, Uniqueness

In this subsection, we first characterize all stable linear Markov equilibria of the model using a common analytical framework. We then proceed to establish the existence and uniqueness of the linear stable OSLME, for which we obtain a convenient analytical solution for firms' investment strategies.

Proposition 1 (Characterization of Stable Linear Markov Equilibria)**.** *Consider the linear investment policy in equation (26). All Open-Loop and Closed-Loop Linear Stable Markov Equilibria are characterized by the following expression for the steady state capital stock vector*

$$\mathbb{k} \;=\; [\mathbf{\Omega} + \mathbb{1}_{\text{CL}} \cdot \mathbf{S}_1(\mathbf{\Phi})]^{-1} \left[\mathbf{Z}(\mathbf{b} - \mathbf{c}) - (r\mathbf{I} + \mathbf{\Delta})\mathbf{p}^k \right] \tag{40}$$

and the following matrix equation, which pins down $\mathbf{\Phi}$ *as an implicit function of* $(\mathbf{\Omega}, \theta, r)$

$$\left[\mathbf{I} - \frac{\mathbb{1}_{\text{CL}}}{\theta} \cdot \mathbf{S}_2(\mathbf{\Phi}) \right] \mathbf{\Phi}^2 - \left[\frac{1}{\theta}\mathbf{\Omega} + (2+r)\mathbf{I} \right] \mathbf{\Phi} + (1+r)\mathbf{I} = \mathbf{0} \tag{41}$$

where the matrices $\mathbf{S}_1(\mathbf{\Phi})$ *and* $\mathbf{S}_2(\mathbf{\Phi})$ *are non-linear functions of* $\mathbf{\Phi}$ *that capture closed-loop feedback effects, for which we provide formulas in Appendix A.*

Proof. See Appendix A. □

Proposition 1 shows that CSLME and OSLME share the same functional form for the transition matrix and steady state capital, with the exception of feedback effects embedded in the matrices $\mathbf{S}_1(\mathbf{\Phi})$ and $\mathbf{S}_2(\mathbf{\Phi})$. By construction, feedback effects only appear in the equations that characterize the Closed-Loop equilibria. The dummy variable $\mathbb{1}_{\text{CL}}$ "turns on/off" such feedback effects.

Considering that product substitution typically results in negative feedback effects, results imply that, at a high level, the steady state capital stocks are typically higher on average for CSLME than for OSLME investment strategies. The argument is consistent with the findings in Reynolds (1991), who compares Open-Loop and Closed-Loop investment equilibria in a dynamic symmetric oligopoly model of investment with adjustment costs.

Throughout the rest of the paper, we focus our analysis to Open-Loop equilibria. We do so for three key reasons. The first and most important is that Open-Loop equilibria are based on behavioral assumptions that are plausible in our settings, while Closed-Loop equilibria impose an unrealistic degree of strategic sophistication from the agents that populate the model. The array that governs feedback interactions in Closed-Loop equilibria has a dimensionality of N^3, which amounts to $N^3 > 100$ billion in a typical year of our working sample. As a consequence, the Closed-Loop implementation of our model's SLME is not robust to the recent critique by Moll (2025).

As we show next a second key advantage of focusing Open-Loop equilibria is that we are able to establish both existence and uniqueness, whereas the model admits an unknown number of CSLME. The contrast between uniqueness in Open-Loop and multiplicity in Closed-Loop equilibria aligns with the findings by Reynolds (1991) and, in option exercise games, Dai, Jiang, and Wang (2024).

Last, a third and final factor that determines our choice to work with OSLME is that, unlike CSLME, it produces fully analytical solutions for $\mathbb{k}$ and $\mathbf{\Phi}$, thus allowing us to dissect the underlying economic forces at work in the model.

We proceed to establish the existence and uniqueness of the linear stable OSLME of our model, and then characterize it analytically.

Proposition 2 (Existence and Uniqueness of OSLME)**.** *Consider the linear investment policy in equation (26). There exists a unique Open Loop Stable Linear Markov Equilibrium, characterized by the following explicit expressions for the steady state capital stock vector*

$$\mathbb{k} \;=\; \mathbf{\Omega}^{-1}\left[\mathbf{Z}(\mathbf{b}-\mathbf{c})-(r\mathbf{I}+\mathbf{\Delta})\mathbf{p}^{k}\right] \tag{42}$$

and the transition matrix $\mathbf{\Phi}$*:*

$$\mathbf{\Phi} \;=\; \frac{1}{2}\left[\frac{1}{\theta}\mathbf{\Omega}+(2+r)\mathbf{I}\right]-\sqrt[+]{\frac{1}{4}\left[\frac{1}{\theta}\mathbf{\Omega}+(2+r)\mathbf{I}\right]^{2}-(1+r)\mathbf{I}} \tag{43}$$

where $\sqrt[+]{\mathbf{M}}$ *denotes, for a symmetric square matrix* $\mathbf{M}$ *with all positive eigenvalues, the "principal" square root matrix – that is, the unique square root matrix with all positive eigenvalues.*[6]

Proof. The steady state vector of capital stocks is trivially found by imposing $\mathbb{1}_{\text{CL}} = 0$ and $\mathbf{k}_{t+2} = \mathbf{k}_{t+1} = \mathbf{k}_t = \mathbb{k}$ inside equation (34). See Appendix A for the rest of the proof. □

The equilibrium transition matrix $\mathbf{\Phi}$ in equation (43) is the unique, explicit solution of a Non-symmetric Algebraic Riccati Equation (NARE). Neoclassical models typically solve for a relationship between investment policies as a function of firms' marginal Q. They rarely provide explicit solutions for firms' investment policies, value functions and marginal Q. The OSLME of our model is uniquely tractable, in that we can solve for the steady-state vector $\mathbb{k}$ and the transition matrix $\mathbf{\Phi}$.

The diagonal entries of matrix $\mathbf{\Phi}$ in equation (43) capture the how fast firm n's capital stock gets pulled toward its steady state, whereas the off-diagonal terms capture instead the equilibrium investment spillovers across business units.

As in the static game of Pellegrino (2025), the steady state equilibrium capital stock vector $\mathbb{k}$ in equation (42) admits an intuitive interpretation from network theory. The competitive pressure matrix $\mathbf{\Omega}$ can also be used (like $\mathbf{\Sigma}$ and $\mathbf{\Gamma}$) as an adjacency matrix to define a network. Then $\mathbb{k}$ can be interpreted as a metric of network centrality – specifically, the metric developed by Katz (1953) and Bonacich (1987). A business unit that is "peripheral" in the network – that is, a business unit that is shielded from competitive pressures (e.g. because it has few productive competitors) – will attain, all else equal, a higher level of capital in the steady-state.

As is well known from the literature on dynamic games, the infinite-horizon game may admit not only linear but also non-linear Markov equilibria, as well as equilibria with non-Markovian (e.g. trigger) strategies. A natural question is therefore whether our focus on the linear strategies is arbitrary. The following proposition shows that it is not: it turns out that our OSLME arises naturally as the limit of a sequence of unique (non-necessarily linear) Open-Loop Markov Equilibria (OME) of

[6] See Higham (2008, Chapter 1, Theorem 1.29).

the corresponding finite-horizon game. This justification for linear strategies follows Green and Liu (2021), who used it first the context of a dynamic borrowing and lending game.[7]

Proposition 3 (Linear Stable OSLME as the limit of a sequence of a finite-horizon OMEs.)**.** *Consider the truncated game with an exogenous finite horizon $T \in \mathbb{N}$, in which business units produce for the last time at date T, sell their undepreciated capital at price $\mathbf{p}^k$, and make no further investment. Then, for all T, the truncated game admits a unique Open-Loop Markov Equilibrium (OME) characterized by the following linear strategy profiles:*

$$\mathbf{k}_{t+1} = \boldsymbol{\iota}_t^T + \boldsymbol{\Phi}_t^T \mathbf{k}_t \tag{44}$$

these OME policies converge to the linear stable OSLME policy of Proposition 2. That is, for every t:

$$\lim_{T\to\infty} \boldsymbol{\Phi}_t^T = \boldsymbol{\Phi} \qquad \text{and} \qquad \lim_{T\to\infty} \boldsymbol{\iota}_t^T = (\mathbf{I} - \boldsymbol{\Phi})\mathbb{k} \tag{45}$$

Proof. See Appendix A. In the proof, we provide analytical recursive expressions for $\boldsymbol{\iota}_t^T$ and $\boldsymbol{\Phi}_t^T$. □

In the following subsections, we use equations (42) and (43) associated to the OSLME of the dynamic game to explicitly characterize explicitly the value function of the firm, investment rates and the associated marginal Q's.

2.7 Transition Dynamics and Spectral Analysis of $\boldsymbol{\Phi}$

In this subsection we present and discuss a novel and unique theoretical insight that emerges from our model: it turns out that the intensity of product market competition affects not only the equilibrium steady state level of capital (42), but also the speed of convergence to the steady state.

As is well known from macroeconometrics literature[8], the speed of the transition along the equilibrium path is governed by the eigenvalues of the matrix $\boldsymbol{\Phi}$. Because business units differ by their capital productivity (z_n) and by the substitutability of the goods they produce (as measured by $\boldsymbol{\Sigma}$), the competitive pressure matrix $\boldsymbol{\Omega}$ represents a highly asymmetric network: some firms are significantly more exposed to competitive pressure than others. It follows that the transition matrix $\boldsymbol{\Phi}$ inherits rich spectral properties from $\boldsymbol{\Omega}$ – most importantly, vast dispersion in its eigenvalues.

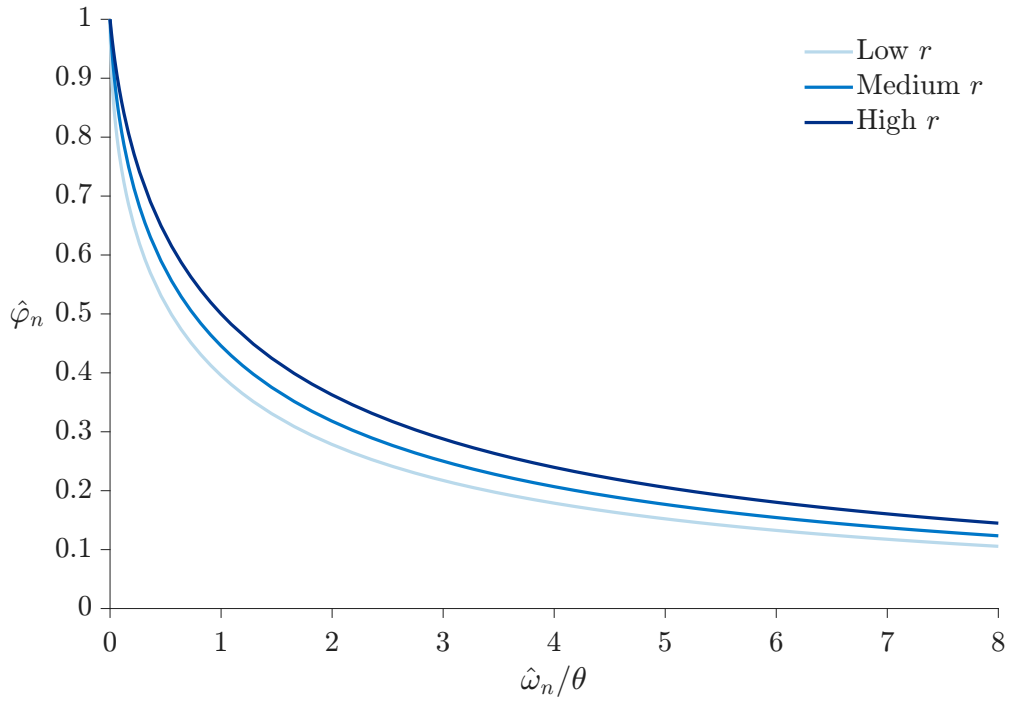
Balanced Growth Paths. To analyze the spectral properties of matrix $\boldsymbol{\Phi}$, we introduce some more notation. Let $\mathbf{e}_n$ be the n^{th} eigenvector of $\boldsymbol{\Phi}$, and let $\hat{\varphi}_n$ be the corresponding n^{th} eigenvalue. By initializing the capital stock vector at $\mathbf{k}_0$, which is removed from $\mathbb{k}$ by an amount that is proportional to any of the N eigenvectors $\mathbf{e}_n$, we can construct N "balanced growth paths". Along these balanced growth paths, all firms' capital stock converge towards the steady state at a rate that is exactly equal to eigenvalue $\hat{\varphi}_n$:

$$(\mathbf{k}_0 - \mathbb{k}) \propto \mathbf{e}_n \qquad \Longrightarrow \qquad (\mathbf{k}_t - \mathbb{k}) = \hat{\varphi}_n^t \, (\mathbf{k}_0 - \mathbb{k}) \tag{46}$$

Due to the rich structure of $\boldsymbol{\Omega}$, the transition matrix $\boldsymbol{\Phi}$ will generally have N different eigenvalues: this implies that, in our model, the OSLME transition speed is *state-dependent* – i.e. it is determined by the allocation of capital (along which direction $\mathbf{k}_t$ deviates from the steady state). Fixing an aggregate level of capital stock K, there are capital allocations (states) that are highly "transitory",

[7] We thank Ernest Liu and Joel Flynn for conjecturing this Proposition at a meeting of the NYC Macro Theory Group.
[8] See Ljungqvist and Sargent (2018) for a review.

FIGURE 1: EIGENVALUES OF THE TRANSITION MATRIX



in the sense that firms will swiftly transition away from them to migrate to the steady state $\mathbb{k}$. Other capital allocations are instead sticky (i.e. persistent), in the sense that firms will transition slowly to the steady state.

Firm-level Trade-off and Persistence. The rich spectral properties of $\mathbf{\Phi}$ also imply that, outside of the balanced growth path, individual firms will adjust their capital stock at markedly different speeds. It also means that the speed itself at which each firm adjusts its capacity will change as the transition unfolds. The speed at which firm n adjusts its capital stock depends on whether (at any point in time) firm n is associated with high or low eigenvalues. In turn, we predict that such speed also depends on the configuration of the product market. Specifically, firms that are shielded from competitive pressure (in a certain mathematical sense) load on higher eigenvalues of $\mathbf{\Phi}$, and thus adjust their capacity more slowly. Vice-versa, firms that are *central* in the network of competitive pressures will adjust their capital stock much faster.

We start building intuition for this claim by noting that the analytical solution for $\mathbf{\Phi}$ (equation 43) depends directly on the competitive pressure matrix $\mathbf{\Omega}$. However, equation (43) includes matrix powers and roots that obfuscate somewhat the relation between the two matrices. To make the link between these two matrices more transparent, we can leverage the properties of $\mathbf{\Omega}$ and a key result from matrix theory.

Corollary (to Proposition 2)**.** *Let $\hat{\boldsymbol{\omega}}$ the vector of eigenvalues of $\mathbf{\Omega}$ and $\mathbf{E}$ be the corresponding (orthogonal) matrix of eigenvectors, so that*

$$\mathbf{\Omega} \;=\; \mathbf{E}\,\mathrm{diag}(\hat{\boldsymbol{\omega}})\,\mathbf{E}^{*} \tag{47}$$

and $\mathbf{E}^* \equiv \mathbf{E}' \equiv \mathbf{E}^{-1}$ *is the transpose or inverse*[9] *of* $\mathbf{E}$ *. Also, let* $\hat{\boldsymbol{\varphi}}$ *be the vector eigenvalues of the OSLME transition matrix* $\boldsymbol{\Phi}$*. Then,* $\boldsymbol{\Phi}$ *and* $\boldsymbol{\Omega}$ *are simultaneously diagonalizable.*[10]

$$\boldsymbol{\Phi} \;=\; \mathbf{E}\,\mathrm{diag}\left(\hat{\boldsymbol{\varphi}}\right)\mathbf{E}^* \tag{48}$$

and their respective n^{th} *eigenvalues are related by the following the scalar function:*

$$\hat{\varphi}_n \;=\; \frac{1}{2}\left(\frac{\hat{\omega}_n}{\theta}+2+r\right)-\sqrt{\frac{1}{4}\left(\frac{\hat{\omega}_n}{\theta}+2+r\right)^2-(1+r)} \tag{49}$$

which is simply the scalar equivalent of equation (43)*.*

Proof. Embedded in the proof of Proposition 2, in Appendix A. □

This corollary allows us to unpack the economic link between the competitive pressure matrix $\boldsymbol{\Omega}$ and the OSLME transition matrix $\boldsymbol{\Phi}$. Figure 1 displays $\hat{\varphi}_n$ as a function of $\hat{\omega}_n/\theta$ and r. First, consistent with previous studies such as Chodorow-Reich (2025), we find that all eigenvalues $\hat{\varphi}_n$ are increasing functions of the curvature of adjustment costs θ and of the discount rate r. That is, higher adjustment costs and higher discount rates slow down the speed of convergence to the steady state. The intuition for why higher discount rates also induce more persistence in capital is that, in deciding future capacity, business units evaluate adjustment costs that are paid in the current period against profits that are realized in future periods. A higher discount rate increases the relative importance adjustment costs, leading to higher persistence of capital.

As a novel insight, our model shows that $\hat{\varphi}_n$ is a decreasing function of the ratio of $\hat{\omega}_n/\theta$. From a quantitative standpoint, the eigenvalue $\hat{\omega}_n$ stands out as the "antagonist" of the adjustment cost parameter θ in determining the speed of transition. To build economic intuition behind this result, let us recall the economic interpretation of the competitive pressure matrix $\boldsymbol{\Omega}$: it measures the concavity of the firms' profit functions – or, equivalently the economic loss from failing to adjust capacity in response to a change in a rival's capital stock.

This observation takes us to the heart of the dynamic trade-off faced by business units: when deciding how much to invest, managers trade off the direct cost of adjusting the current capital stock (captured by the convexity of the installation function ψ_n) against the future loss in profits that they incur by *not* adjusting the capital stock (captured by the concavity of the profit function). The convexity of the adjustment cost function, modulated by θ, is the economic force that "pulls" capital stocks towards the current level $\mathbf{k}_t$. The concavity of the profit function vector, modulated by the eigenvalues of $\boldsymbol{\Omega}$, is the economic force that "pushes" capital stocks towards the steady state $\mathbb{k}$. The OSLME presented in Proposition 2 is the outcome of these two economic forces balancing each other out as a result of the firms' forward-looking decision-making.

Special Case: Symmetry. The matrix $\boldsymbol{\Omega}$ and its eigenvalues are themselves non-trivial functions of N (the number of firms), $\mathbf{z}$ (the capital productivity) and the substitution matrix $\boldsymbol{\Sigma}$. While it is interesting to ask how the transition matrix $\boldsymbol{\Phi}$ depends on these individual objects, it is impossible to derive analytical results for the general case. Therefore, to further investigate the relationship between $\boldsymbol{\Phi}$ and these more fundamental objects, we narrow down our focus to the special (symmetric)

[9] In this case the two coincide due to $\mathbf{E}$ being orthogonal.

[10] This is equivalent to saying that $\boldsymbol{\Phi}$ and $\boldsymbol{\Omega}$ share a basis or that they commute.

case where all firms are identical and engage in standard Cournot competition. For this case, we can obtain analytical results. As we show in Appendix A, even in this special case the competitive pressure matrix $\mathbf{\Omega}$ has multiple eigenvalues: yet, only one of such eigenvalues is associated with the unit eigenvector, and is therefore of interest.

Proposition 4. *In a Cournot oligopoly with* $\mathbf{D} = \mathbf{\Gamma} = \mathbf{I}$ *and* N *identical single product firms, the unique eigenvalue of matrix* $\mathbf{\Omega}$ *associated with the unit eigenvector* $\mathbf{1}$*, which we denote* $\hat{\omega}_{\mathbf{1}}$*, is given by*

$$\hat{\omega}_{\mathbf{1}} = z^2[2+(N-1)\sigma] \tag{50}$$

Denoting $\hat{\varphi}_{\mathbf{1}}$ *the corresponding eigenvalue of the OSLME transition matrix* $\mathbf{\Phi}$*, it follows that*

$$\frac{\partial\hat{\varphi}_{\mathbf{1}}}{\partial z} < 0 \qquad \frac{\partial\hat{\varphi}_{\mathbf{1}}}{\partial\sigma} < 0 \qquad \frac{\partial\hat{\varphi}_{\mathbf{1}}}{\partial N} < 0 \tag{51}$$

Proof. See Appendix A. □

The proposition shows that lower dependence on capital (higher z), stronger substitution among products (σ), or a higher number of firms (N), all result in lower persistence in capital, and a faster convergence to the steady state.

In sum, our model delivers the novel insight that the rate of capital accumulation is not only affected by adjustment costs or the discount rate, as well-known from the previous literature: it is also regulated by the degree of product market competition.

2.8 Equilibrium Enterprise Values

We now utilize the OSLME investment policy to solve for the vector of firm-level value functions $\mathbf{v}(\cdot)$, a crucial component of the equilibrium. As a stepping stone, we first derive the vector of marginal Q. Leveraging the expression for $\mathbb{k}$ that we previously obtained, we can re-write $\mathbf{q}(\mathbf{k}_t)$ as a function of the price of capital and of the deviation of the vector of capital stocks $\mathbf{k}_t$ from the steady state.

Proposition 5. *The OSLME vector of marginal Qs is equal to*:

$$\mathbf{q}(\mathbf{k}_t) = (1+r)\mathbf{p}^k + \mathbf{\Lambda}(\mathbb{k}-\mathbf{k}_t) \tag{52}$$

where the matrix $\mathbf{\Lambda}$ *is defined as*

$$\mathbf{\Lambda} \overset{\text{def}}{=} \theta(\mathbf{I}-\mathbf{\Phi})+\mathbf{\Omega} \tag{53}$$

this implies that the corresponding steady state value is

$$\mathbf{q}(\mathbb{k}) = (1+r)\mathbf{p}^k \tag{54}$$

Proof. See Appendix A. □

Equation (52) has two important implications. First, it tells us that along the equilibrium path the matrix $\mathbf{\Lambda}$ defines a time-invariant linear mapping between marginal Q and the deviation of the state vector $\mathbf{k}_t$ from its steady-state. In other words, if $\mathbf{\Lambda}$ is known, $\mathbf{q}(\mathbf{k}_t)$ reveals each firm's capital stock's deviation from the steady state value. Second, because $q_{nn}(k_{nt};\mathbf{k}_{-nt})$ is the derivative of $v_n(k_{nt};\mathbf{k}_{-nt})$

with respect to a firm's own capital stock, (52) is also a system of differential equations, which we can solve to recover enterprise value $\mathbf{v}(\cdot)$.

To obtain an exact solution for the Bellman Equation, however, we also need a boundary condition. We obtain it by evaluating equation (25) at the steady state. First, notice that in the steady state the cum-dividend value of the firm is given by:

$$v_n(\mathbb{k}) = \frac{1+r}{r} \sum_{m \in \mathcal{M}_n} \left[\pi_m(\mathbb{k}) - \delta_m p_m^k k_{m\infty} \right] \tag{55}$$

The expression for firm value under steady state is straightforward. The value of the firm equals the present value of its operating profits across all business units, less the cost of maintaining its capital stock. By expanding this expression and substituting the previously-derived vector of steady-state capital stocks ($\mathbb{k}$) we obtain an even more compact expression for the steady-state firm value.

Proposition 6. *In the OSLME steady-state, the value of the firm $v_n(\mathbb{k})$ is equal to the sum of the resale value of the steady-state capital stock and ρ_n – the present discounted value of future monopoly rents*:

$$v_n(\mathbb{k}) = \sum_{m \in \mathcal{M}_n} \left[(1+r) p_m^k k_{m\infty} + \rho_m \right] \tag{56}$$

where ρ_n is the present value of oligopoly rents in steady state, defined as follows:

$$\rho_n \overset{\text{def}}{=} \sum_{T=t}^{\infty} \frac{1}{(1+r)^{(T-t)}} \left[p_n(\mathbb{k}) - \frac{\partial p_n y_n}{\partial y_n}(\mathbb{k}) \right] y_n(\mathbb{k}) = \frac{1+r}{r} \cdot z_n^2 k_{n\infty}^2 \tag{57}$$

Proof. See Appendix A. □

Armed with the boundary condition in equation (56), we can now proceed to recover the value function vector $\mathbf{v}(\cdot)$ by solving the system of differential equations in (52).

Proposition 7. *The vector of maximized Bellman equations in (25), corresponding to the OSLME enterprise values, is solved by*:

$$\mathbf{v}(\mathbf{k}_t) = \mathbf{v}(\mathbb{k}) + (\mathbf{k}_t - \mathbb{k}) \circ \left[(1+r)\mathbf{p}^k - \left(\mathbf{\Lambda} - \frac{1}{2}\mathbf{I} \circ \mathbf{\Lambda} \right) (\mathbf{k}_t - \mathbb{k}) \right] \tag{58}$$

in scalar notation:

$$v_n(\mathbf{k}_t) = v_n(\mathbb{k}) + (k_{nt} - k_{n\infty}) \left[(1+r) p_n^k - \sum_{m=1}^{N} \frac{\lambda_{nm}}{2^{\mathbb{I}\{m=n\}}} (k_{mt} - k_{m\infty}) \right] \tag{59}$$

Proof. Differentiating each row of (59) with respect to k_i we verify that it satisfies (52). To verify that it satisfies the boundary condition (56) we plug $\mathbf{k}_t = \mathbb{k}$. □

2.9 Product Market Spillovers and the Equilibrium Q Network

There is a substantial literature on product market spillovers and peer effects in corporate investment. Within this literature, we find a variety of different measures of product market spillovers. One such measure, used in the seminal paper of Bloom, Schankerman, and Van Reenen (2013, henceforth BSV), is the marginal effect of an additional unit of firm m capital on the investment of firm n.

In our framework, this is simply the Jacobian the vector of firm's investment policies from equation (29):

$$\frac{\partial i_{nt}}{\partial k_{mt}} = \begin{cases} -(1-\delta_n)+\varphi_{nn} & \text{for } m = n \\ \varphi_{nm} & \text{for } m \neq n \end{cases} \tag{60}$$

In plain language, the off-diagonal terms of the transition matrix $\mathbf{\Phi}$ capture product market investment spillovers in our model, consistent with BSV.

An alternative measure of product market spillovers is the marginal effect of an additional unit of firm m capital on the enterprise value of firm n. This is simply the $\mathbf{Q}$ matrix, which we previously derived in subsection 2.3. With knowledge of the value function (equation 59) we can now recover an analytical formula for it.

Proposition 8. *The OSLME matrix $\mathbf{Q}(\mathbf{k}_t)$ is given by:*

$$\mathbf{Q}(\mathbf{k}_t) = (1+r)\mathbf{P}^k + \text{diag}[\mathbf{\Lambda}(\mathbb{k}-\mathbf{k}_t)] + (\mathbb{k}-\mathbf{k}_t)\mathbf{1}' \circ \left(\mathbf{1}\mathbf{1}' - \mathbf{I}\right) \circ \mathbf{\Lambda} \tag{61}$$

and thus the corresponding steady-state value is:

$$\mathbf{Q}(\mathbb{k}) = (1+r)\mathbf{P}^k \tag{62}$$

where $\mathbf{P}^k$ is simply $\mathbf{p}^k$ rearranged as a diagonal matrix:

$$\mathbf{P}^k \overset{\text{def}}{=} \begin{bmatrix} p_1^k & 0, & \dots & 0 \\ 0 & p_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_N^k \end{bmatrix} \tag{63}$$

Proof. Follows trivially from the definition of $\mathbf{Q}(\mathbf{k}_t)$ in equation (21) and knowledge of the solution to the Bellman equation (59). □

Notice that $\mathbf{Q}(\mathbf{k}_t)$ becomes a diagonal matrix in the steady state. This means that a small increase in the capital stock of a business unit will only have a first order effect on other firms' values *outside* the steady state, not *at* the steady state. Around the steady-state, all product market spillovers on firm valuations become second-order effects. These second-order spillover effects are described by the matrix $\mathbf{\Lambda}$, the Jacobian of the firms' own marginal Q vector:

$$\frac{\partial \mathbf{q}(\mathbf{k}_t)}{\partial \mathbf{k}_t} \equiv -\mathbf{\Lambda} \tag{64}$$

In sum, the three matrices $\mathbf{\Phi}$, $\mathbf{Q}$ and $\mathbf{\Lambda}$ capture product market spillovers in our network Q model. Each of these three matrices plays a role along a different margin.

2.10 Neoclassical Investment Equations

Next, we re-express our optimal investment rule in equation (29) as a vector of stacked neoclassical investment equations (Hayashi, 1982).

Proposition 9. *The vector of business units' investment policies in the OSLME obeys the following neoclassical investment equation:*

$$\boldsymbol{i}_t(\mathbf{k}_t) \;=\; \boldsymbol{\Delta}\mathbf{k}_t + \frac{1}{\theta}\boldsymbol{\Phi}\left[\frac{1}{1+r}\mathbf{q}(\mathbf{k}_t) - \mathbf{p}^k\right] \tag{65}$$

Proof. See Appendix A. □

The functional form of equation (65) evokes Hayashi (1982) and subsequent studies. This literature establishes marginal Q as a "sufficient statistic" to predict investment. One implication of Proposition 9 is that, in our setting, knowledge of a business unit's own contemporaneous marginal Q is insufficient to predict its investment rate. While there is still an elegant linear mapping between the vector of investment $\boldsymbol{i}_t(\mathbf{k}_t)$ and the vector of marginal $\mathbf{q}(\mathbf{k}_t)$, the presence of product market spillovers implies that we must know the full vector $\mathbf{q}(\mathbf{k}_t)$ in order to predict any entry of $\boldsymbol{i}_t(\mathbf{k}_t)$.

2.11 Equilibrium Markups

An important contribution of our model is to investigate the relationship between markups and investment, two objects that in our model are endogenous and co-determined as part of the equilibrium.

Having solved for the OSLME investment policy, we can now derive an explicit expression for the endogenous markup charged by each business at time t. Here, we naturally focus on the single-product Cournot case ($\mathbf{D} = \boldsymbol{\Gamma} = \mathbf{I}$). We define $\mu_n(\mathbf{k}_t)$ as the markup charged by business unit n at state $\mathbf{k}_t$ as the ratio of the output price to the unit labor cost:

$$\mu_n(\mathbf{k}_t) \;\stackrel{\text{def}}{=}\; \frac{p_n(\mathbf{k}_t)}{c_n} \tag{66}$$

Using the solution for the steady state vector $\mathbb{k}$ derived in Proposition 2, we derive an explicit expression for the equilibrium vector of markups or any state $\mathbf{k}_t$.

Proposition 10. *The equilibrium vector of markups associated with the OSLME in Proposition 2, for any state $\mathbf{k}_t$, is equal to*

$$\boldsymbol{\mu}(\mathbf{k}_t) \;=\; \boldsymbol{\mu}(\mathbb{k}) + \mathbf{C}^{-1}(\mathbf{I}+\boldsymbol{\Sigma})\mathbf{Z}(\mathbb{k}-\mathbf{k}_t) \tag{67}$$

where the vector of equilibrium markups at the steady state is equal to:

$$\boldsymbol{\mu}(\mathbb{k}) \;=\; \mathbf{C}^{-1}\left\{\mathbf{b} + (\mathbf{I}+\boldsymbol{\Sigma})\,\mathbf{Z}\boldsymbol{\Omega}^{-1}\left[(r\mathbf{I}+\boldsymbol{\Delta})\mathbf{p}^k - \mathbf{Z}(\mathbf{b}-\mathbf{c})\right]\right\} \tag{68}$$

and matrix $\mathbf{C}$ is simply the vector of unit labor costs rearranged as a diagonal matrix:

$$\mathbf{C} \;\stackrel{\text{def}}{=}\; \begin{bmatrix} c_1 & 0, & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_N \end{bmatrix} \tag{69}$$

Proof. See Appendix A. □

The proposition above provides two important insights. First, equation (67) shows that some of the cross-sectional variation in markups is transitory: a business unit that is operating below steady-state capacity will optimally reduce its markup over time as its stock of capital increases.

Second, equation (68) shows that the markup charged over variable costs will embed, over the long run, a compensation for capital costs. The matrix $\mathbf{C}^{-1}(\mathbf{I}+\boldsymbol{\Sigma})\mathbf{Z}\boldsymbol{\Omega}^{-1}$ in equation (68) captures the extent to which business units pass the user cost of capital through to their customers via the markup.

2.12 Single Product Cournot Case: Wedge between Marginal and Average Q

As a final step in our theoretical analysis, before moving onto the quantification of the model, we characterize the relation between marginal Q in equation (65) (which is empirically unobservable) and the associated market to book asset ratio (or average Q). The result wedge between both quantities has been thoroughly studied in the neoclassical literature, for practical reasons (see, e.g., Hayashi, 1982, Erickson and Whited, 2000, Hennessy, 2004). In our setting, this wedge reveals the extent to which firms internalize their investment on output prices.

For analytical convenience and in alignment with our quantitative application, we focus on the single-product Cournot case ($\mathbf{D}=\boldsymbol{\Gamma}=\mathbf{I}$), where each business unit represents a single-product firm. In this case, we define the average Q of firm n as follows:

$$\bar{q}_n(\mathbf{k}_t) \overset{\text{def}}{=} \frac{v_n(\mathbf{k}_t)}{k_{nt}} \tag{70}$$

The wedge between average and marginal Q is then given by:

$$\bar{q}_n(\mathbf{k}_t) - q_n(\mathbf{k}_t) \;=\; \frac{\rho_n}{k_{nt}} + \frac{\lambda_{nn}}{2k_{nt}}(k_{nt}-k_{n\infty})^2 + \frac{k_{n\infty}}{k_{nt}}\sum_{m\neq n}\lambda_{nm}(k_{mt}-k_{m\infty}) \tag{71}$$

Using a first-order Taylor approximation around the steady state of vector $\mathbf{k}_t$, the expression then simplifies to:

$$\bar{q}_n(\mathbf{k}_t) - q_n(\mathbf{k}_t) \;\approx\; \frac{\rho_n}{k_{nt}} + \sum_{m=1}^{N}\lambda_{nm}(k_{mt}-k_{m\infty}) \tag{72}$$

At any point in time, as long as firms have market power, the first term in equation (71) is positive and implies that average Q overstates marginal Q, where ρ_n captures the present value of oligopoly rents for firm n in the (long-run) steady state. The second term captures changes in oligopoly rents due to short-term adjustments in the firm's own stock of capital. The third term captures changes in oligopoly rents due to short-term adjustments in the stock of capital of competing firms.

More simply, at the steady state, the wedge between average and marginal Q is equal to the present value of future monopoly rents per unit of capital:

$$\bar{q}_n(\mathbb{k}) - q_n(\mathbb{k}) \;=\; \frac{\rho_n}{k_{n\infty}} \tag{73}$$

The result is consistent with the argument that average Q overstates marginal Q when firms have monopolistic power in Hayashi (1982). We extend the implications of this early study on monopoly firms to the general case of oligopoly, while solving explicitly for firms' endogenous markups in equilibrium.

2.13 Investment Sensitivity and Marginal Propensity to Invest

A growing literature in macroeconomics and finance studies how businesses adjust investment in response to changes in discount rates or shocks to fundamentals. We solve analytically for two statistics that summarize these responses, and use them to characterize how product markets shape investment policies in the cross section.

The first statistic is the Investment Sensitivity (IS), which measures the response of investment to changes in discount rates and borrowing costs. It plays a central role in the transmission of monetary policy (Ottonello and Winberry, 2020; Best, Born, and Menkhoff, 2026).

Proposition 11. *Let the Investment Sensitivity (IS) of business unit n be defined as the change in the investment of business unit n following a 1% drop in the discount rate. In the OSLME, for any state* $\mathbf{k}_t$*, the vector of IS then equals:*

$$-1\%\frac{\mathrm{d}\,\boldsymbol{i}(\mathbf{k}_t)}{\mathrm{d}\,r} \;=\; 1\%\left[(\mathbf{I}-\boldsymbol{\Phi})\,\boldsymbol{\Omega}^{-1}\mathbf{p}^{k}+\frac{\mathrm{d}\,\boldsymbol{\Phi}}{\mathrm{d}\,r}\,(\mathbb{k}-\mathbf{k}_t)\right] \tag{74}$$

where the derivative of matrix $\boldsymbol{\Phi}$ *with respect to r is provided in the Appendix. At the steady state it is equal to*:

$$-1\%\frac{\mathrm{d}\,\boldsymbol{i}(\mathbb{k})}{\mathrm{d}\,r} \;=\; 1\%\,\boldsymbol{\Delta}\,\boldsymbol{\Omega}^{-1}\mathbf{p}^{k} \tag{75}$$

Proof. See Appendix A. □

Equation (75) shows that competitive pressure in product markets, captured by matrix $\boldsymbol{\Omega}$, dampens the steady state IS: all else equal, the steady state investment of business unit n is less responsive to a 1% drop in the discount rate when the unit is subject to greater competitive pressure. Out of steady state (equation 74), two additional channels operate: a lower discount rate raises the steady state marginal product of capital (first term), and it accelerates the transition of the capital stock towards the steady state, both directly and through the investment spillovers of rivals (second term).

The second relevant statistic is firms' Marginal Propensity to Invest (MPI), which measures how a change in operating cash flows translates into additional capital expenditures. The MPI has received growing attention in the recent macroeconomics literature (e.g. Martin-Baillon, 2021; Winberry, Auclert, Rognlie, and Straub, 2025). Earlier studies in economics characterize the impact of investment frictions on the MPI (e.g. Abel and Eberly, 1994; Cooper and Haltiwanger, 2006), whereas finance studies further incorporate financing frictions (e.g. Hennessy and Whited, 2005; Bolton et al., 2011; Bustamante, 2016).

In our setting, the usual definition of the MPI — the change in investment following a one-dollar increase in operating cash flows — requires refinement. A one-dollar increase in operating cash flows is not a uniquely defined experiment: different perturbations of the model primitives generate the same change in cash flows with different implications for investment; absent financing frictions, transitory cash flows have no effect on investment; and shocks to one business unit spill over to the cash flows of its competitors through product markets. We therefore consider a specific perturbation of the primitives $(\mathbf{b},\mathbf{c})$ which permanently increases, through $k_{n\infty}$, the steady state value of business unit n — the present value of its current and future operating cash flows — by one dollar, while leaving the steady state values of all other units unchanged. The resulting statistic — which we label

the marginal propensity to invest out of growth opportunities, or MPI^{G} — summarizes how readily a business unit converts improvements in its growth opportunities into capital accumulation.

Proposition 12. *Let the Marginal Propensity to Invest out of Growth Opportunities* (MPI^{G}) *of business unit n be the change in its investment expenditure following a specific perturbation in the primitives* $(\mathbf{b},\mathbf{c})$ *which increases the steady state value of business unit n by one dollar, yet leaving its competitors' steady state values unaffected. In the OSLME, for any state $\mathbf{k}_t$, the* MPI^{G} *of business unit n then equals:*

$$\text{MPI}^{\text{G}}_n \overset{\text{def}}{=} \frac{\partial\left(p^k_n\, i_{nt}\right)}{\partial k_{n\infty}} \Bigg/ \frac{\partial v_n(\mathbb{k})}{\partial k_{n\infty}} = \frac{1-\varphi_{nn}}{(1+r)+2\left[\bar{q}_n(\mathbb{k})-q_n(\mathbb{k})\right]/p^k_n} \tag{76}$$

it follows that $\text{MPI}^{\text{G}}_n \in \left(0, \frac{1}{1+r}\right)$ *for any business unit n.*

Proof. See Appendix A. □

Equation (76) admits a natural economic interpretation. The numerator, $(1-\varphi_{nn})$, is the speed at which business unit n closes the gap between its current and desired capital stocks: units with highly persistent capital (φ_{nn} close to 1) display a low MPI^{G}.

The denominator is equal to the steady state marginal Q per dollar of installed capital, $(1+r)$, plus twice the wedge between average and marginal Q — the present value of monopoly rents per dollar of capital (Section 2.12). The value of high-rent business units rises steeply with capital: a one dollar value gain requires little expansion and elicits little investment. Competitive units must instead expand substantially to generate the same dollar, and invest more on impact; for price takers the wedge vanishes and the MPI^{G} attains its maximum, $\left(1-\varphi_{nn}\right)/(1+r)$.

The MPI^{G} is thus increasing in the speed of capital adjustment and decreasing in monopoly rents — the two central forces of our model: the implication is that firms that have more market power and that face larger adjustment frictions will be slower to react to growth opportunities.

3 Data and Calibration

In this section, we discuss how we take our model to the data: how we assemble the data, calibrate the model, recover the firm-level primitives and test the model fit. The dataset that we use spans the near-universe of US publicly-traded corporations from 1995 to 2021.

3.1 Data

Compustat/CRSP. We rely on two key data sources. The first is balance sheet data from Compustat, which we download from WRDS. From that database we use sales/revenues, which are mapped to revenues firm in our model ($p_i y_i$). We also use Costs of Goods Sold, which are mapped to labor costs ($c_i y_i$). Dollar totals are deflated using GDP deflators. Finally, we also use Capex, PPE, R&D Expenditures and Selling, General and Administrative Costs (SGA), which we use to construct a panel data of total capital stocks, as explained in the next subsection.

Under the assumption that COGS equal unit labor costs of production times output, we later compute markups empirically as revenues divided by COGS, in line with previous work by De Loecker,

Eeckhout, and Unger (2020). We also compute, in order to validate the model, the market value of assets as total assets, minus the book value of equity, plus the market value of equity. The latter is obtained by sourcing stock price data from CRSP, and multiplying by the number of shares outstanding. We then compute Average Q as the ratio of market to book value of assets.

Hoberg and Phillips (2016) Product Similarity. To measure the matrix of product cosine similarities $\mathbf{A}'\mathbf{A}$, we follow Pellegrino (2025) and rely on the text-based product cosine similarity data of Hoberg and Phillips (2016, henceforth HP). The HP dataset applies text analysis methods to the business description section (Item 1) of annual 10-K regulatory filings. For each firm-year, HP extract a vector indicating the frequency with which different words appear in the product description. Their word selection focuses on substantive terms—specifically, nouns and proper nouns that are not ubiquitous across filings—yielding a vocabulary of approximately 61,000 unique words. The similarity score for any firm pair equals the cosine of the angle between their respective word vectors, bounded between zero and one. Unlike traditional classifications such as NAICS or SIC that group firms based on common production technologies, the HP approach captures product-level competition by identifying firms that describe their offerings using similar language. Additionally, the HP data offer several practical advantages: annual updates that reflect evolving product portfolios, graduated similarity measures rather than binary industry membership, and direct derivation from legally binding disclosures that firms have strong incentives to report accurately. The dataset spans nearly all Compustat firms and is available from 1995 onward.

Validation of the HP Cosine Similarities. Pellegrino (2025) provides multiple validation tests supporting the use of HP similarity scores within the GHL demand framework. A first test examines whether firms naturally group into coherent product markets. When the HP network is visualized using standard algorithms, the resulting clusters align closely with independently defined product-market categories from S&P's Global Industry Classification system, suggesting the text data successfully capture competitive relationships at the market level. A second set of tests, originally conducted by HP and confirmed by Pellegrino (2025), compare the HP-derived industry classifications (TNIC) against conventional approaches. When benchmarked against firms' own disclosures of their main competitors in Capital IQ filings, the text-based classifications prove more accurate than traditional SIC or NAICS groupings. Most importantly for our empirical strategy, Pellegrino (2025) shows that demand elasticities derived from the GHL system—when parameterized using HP data—align well with structural estimates from prior industrial organization studies. Drawing on established work in automobiles , breakfast cereals , and personal computers , he finds that model-implied elasticities track the patterns in these microeconometric benchmarks. Collectively, these validation exercises justify our interpretation of HP cosine scores as empirical analogs to the theoretical product similarity matrix $\mathbf{A}'\mathbf{A}$.

In every year, our sample is composed of firms that appear in both Compustat and HP's database and, either have no missing data from the original Compustat dataset (including capital for the previous and the next year) or variables have been reconciled from accounting equations and linear interpolation. Moreover, there are no outliers (defined as firms that experience a change in the capital stock by a factor of over 2). This results in a sample size of 5,719 in 1995 and 3,202 in 2021.

Construction of the Total Capital Panel Dataset. Our measured state variable, the stock of capital k_{nt}, combines both tangible and intangible capital. To implement our model empirically, we build a new database of total capital stocks for US publicly-traded firms, building on the foundational methodologies of Peters and Taylor (2017, henceforth PT) and Eisfeldt, Kim, and Papanikolaou (2022,

TABLE 1: SUMMARY STATISTICS (1995-2021)

Description	**Notation**	**Mean**	**Std. Dev.**	**p.10**	**p.90**
Revenues (2021 US$mln)	$p_{nt}y_{nt}$	3,841	12,099	14	8,175
Costs of Goods Sold (2021 US$mln)	$c_{nt}y_{nt}$	2,421	7,892	5	5,086
Market Value of Assets (2021 US$mln)	v_{nt}	12,896	47,982	32	22,699
Total Capital Stock (2021 US$mln)	k_{nt}	4,157	14,060	22	8,121
Price of Capital	p_n^k	1.177	0.100	1.035	1.276
Depreciation Rate	δ_n	0.154	0.048	0.076	0.198
Investment Rate	i_{nt}/k_{nt}	0.277	0.353	0.085	0.475
Average Q (log)	$\log \bar{q}_{nt}$	0.740	1.413	-0.816	2.672
Capital Requirement (log)	$\log z_n$	-4.144	1.139	-5.344	-2.696

henceforth EKP). Specifically, we measure total capital as the sum of three components: knowledge capital (accumulated R&D spending), organization capital (accumulated SG&A spending), and physical capital (property, plant, and equipment). For each component, we apply the perpetual inventory method using industry-specific depreciation rates from the Bureau of Economic Analysis, combined with component-specific price deflators from multiple sources including the BEA, FRED, and EUKLEMS & INTANProd databases. For knowledge capital, we use industry-specific BEA depreciation rates, with a default depreciation rate of 15% for industries without coverage, while for organization capital, we use a depreciation rate of 20% for all firms. For physical capital, we use industry-specific BEA depreciation rates that vary across equipment and structures.

A key innovation in our approach is the construction of firm-specific chain price indexes for total capital. Rather than deflating all capital components with a single economy-wide deflator—as is common in the literature—we construct separate price indexes for each capital component and then aggregate them using time-varying, firm-specific weights based on each component's nominal share in the firm's total capital stock. For knowledge capital, we use the BEA's R&D price deflator. For organization capital, we use price indexes from EUKLEMS & INTANProd, spliced with the BEA's intellectual property products (IPP) deflator to extend coverage back to 1950 and forward to 2021. For physical capital, we construct a chain-weighted index from separate BEA price deflators for equipment and structures, using the relative investment shares as weights. This approach allows us to account for the fact that firms with different capital compositions face different effective price changes over time, which is particularly important given the divergent price trends across tangible and intangible assets over our sample period. These deflators are themselves finally deflated using the GDP deflator, so that all dollar amounts are expressed in consistent units.

Summary Statistics and Additional Details. Table 1 reports the summary statistics of our working sample before merging with the HP data. The distributions of firm's stock of capital, investment and depreciation rates are consistent with those reported in the alternative working samples studied by PT and EKP. The log values of firms' productivities of capital ($\log z_n$) are inferred from our own calibration exercise, on which we elaborate next.

We provide additional details on data sources and construction in Appendix B.

3.2 Calibration

We now describe our calibration strategy for the model parameters. We begin by considering externally calibrated parameters that are either standard in the literature or directly observable from the data, then turn to the calibration of productivity ($\mathbf{z}$), adjustment costs (θ), and the steady-state capital distribution ($\mathbb{k}$). The latter require careful treatment due to the high degree of dimensionality of our parameter space is ($\mathbf{z}$ has thousands of entries) and the even larger number of equality and inequality restrictions imposed by the model, which imply our model is over-identified.

Externally calibrated parameters. We set the utility weight on common characteristics to $\alpha = 0.12$, following Pellegrino (2025), who calibrates this parameter by matching model-implied demand elasticities to micro econometric estimates from the industrial organization literature. Firm-specific depreciation rates δ_n are computed as the time-varying weighted average of component-specific BEA depreciation rates, as described in Section 3.2. The discount rate r is year-specific and set equal to the estimate by Gormsen and Huber (2025).

Set-identification and point-calibration of the adjustment cost parameter θ. The curvature parameter θ governs the convexity of adjustment costs. We identify an upper bound for θ by imposing two economically motivated restrictions.[11] First, the implied distribution of $z_{nt}^2 k_{nt}$ must have a positive mean. This yields the following upper bound:

$$\bar{\theta}_1 = \frac{\sum_{n=1}^{N}\sum_{t\in\mathcal{T}_n} \pi_{nt}/k_{nt} - (r_t+\delta_{nt})p_{nt}^k}{\sum_{n=1}^{N}\sum_{t\in\mathcal{T}_n} (1+r_t)(k_{nt}-k_{nt-1}) - (k_{nt+1}-k_{nt})} \tag{77}$$

where $\mathcal{T}_n$ is the set of years where data for firm n is available. Second, we require that adjustment costs cannot exceed operating profits net of the user cost of capital, as this would imply firms have no economic reason to exist. This yields a second upper bound:

$$\bar{\theta}_2 = \frac{\sum_{n=1}^{N}\sum_{t\in\mathcal{T}_n} \pi_{nt} - (r_t+\delta_{nt})p_{nt}^k k_{nt}}{\sum_{n=1}^{N}\sum_{t\in\mathcal{T}_n} \frac{1}{2}(k_{nt+1}-k_{nt})^2} \tag{78}$$

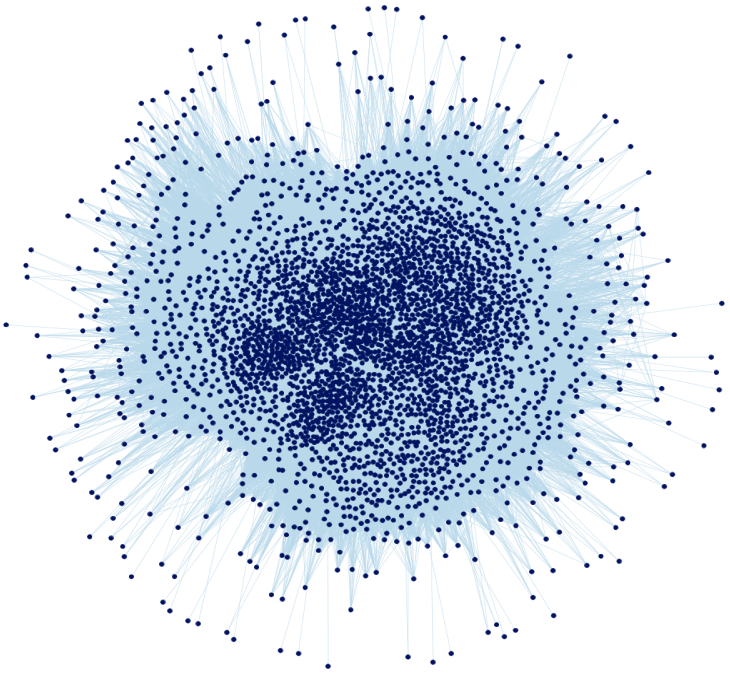
In practice, the second constraint is the binding one in our data, therefore it must be that $\theta \in \left(0, \bar{\theta}_2\right)$. In our setting, $\bar{\theta}_2 = 6.000 \times 10^{-4}$.

Within this set-identified interval, we specify a point estimate of θ by matching the aggregate magnitude of adjustment costs to the empirical benchmark of Cooper and Haltiwanger (2006), who structurally estimate adjustment costs using plant-level manufacturing data. The authors' preferred specification estimates adjustment costs to be approximately 3.1% of operating profits. We calibrate θ such that aggregate adjustment costs in our model equal 3.1% of aggregate operating profits. The calibration yields $\theta = 1.081 \times 10^{-4}$, which falls comfortably within the set-identified bounds. The approach ensures that our adjustment cost parameter both satisfies economically motivated restrictions and aligns with credible estimates from the neoclassical investment literature.

Estimation of capital productivity z_n. The key identification challenge that we face is recovering the high (N)-dimensional capital productivity parameter vector $\mathbf{z}$ (where N is the number of firm-level observations in each cross section of the working sample, 5,719 in 1995 and 3,202 in 2021). For

[11] While the discount rate is constant in the model, its empirical counterpart varies over time; the calibration formulas below therefore use the year-specific series r_t.

FIGURE 2: EMPIRICAL INVESTMENT SPILLOVER NETWORK



each firm in the working sample, we notice that the productivity parameter z_n appears squared in the first-order condition (35). Rearranging this condition, we can express:

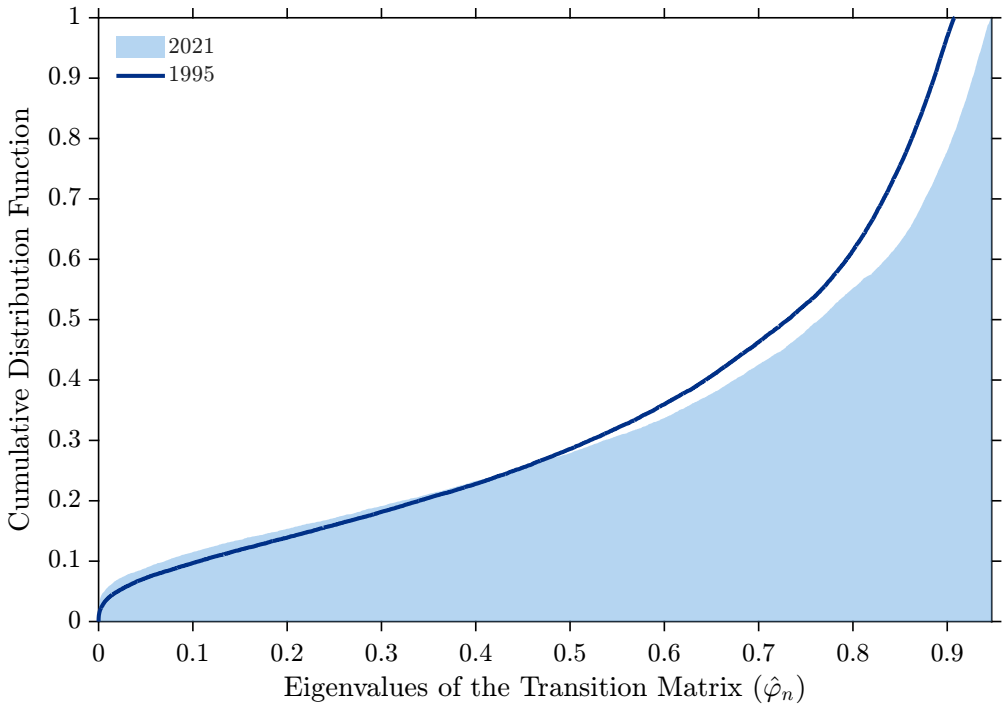
$$z_n^2 = \frac{\pi_{nt}}{k_{nt}^2} - (r_t + \delta_{nt}) \frac{p_{nt}^k}{k_{nt}} - \theta \left[(1 + r_t) \frac{k_{nt} - k_{nt-1}}{k_{nt}} - \frac{k_{nt+1} - k_{nt}}{k_{nt}} \right] \tag{79}$$

In principle, equation (35) would allow us, conditional on a calibrated value of θ, to infer z_n^2 from observable data. However, because our model is over-identified (it has more equality and inequality restrictions than parameters), it cannot perfectly fit equation (79). As a result, its right-hand side can take negative values in the data. These negative values are inadmissible since z_n^2 is positive by construction.

To extract a robust cross-section of firm-level productivities, we employ a statistical approach, and exploit the well-established regularity that productivity distributions exhibit heavy right tails that are well-approximated by Pareto distributions (Gabaix, 2009; Luttmer, 2007). This empirical fact holds robustly for the noisy estimate that we can recover from equation (79). We therefore assume that the underlying cross-section of true capital productivity z_n follows a Pareto distribution. Hence, z_n^2 also follows a Pareto distribution: we obtain the corresponding empirical estimate of z_n by fitting a Pareto distribution to the empirical noisy estimate of z_n^2 obtained from equation (79). After fitting the Pareto distribution, we rescale the entire productivity vector by a scalar factor chosen to match the aggregate sum of model-predicted firm values to the aggregate sum of observed market values. We provide additional details of these steps in Appendix (B).

Enforcement of positive steady-state capital. While $\mathbb{k}$ is an equilibrium object rather than a primitive, we must ensure it satisfies $\mathbb{k} > 0$: this condition is both intuitive (we expect firms that

FIGURE 3: EMPIRICAL TRANSITION MATRIX: EIGENVALUES



we observe in the data to remain active in the long-term) as well as a requirement for tractability (taking the model to the data requires that the data generating process be an interior equilibrium). Yet, the fact that our model is over-identified, and that (despite our best efforts) capital stocks will always be measured imperfectly, leads to the direct estimate of $\mathbb{k}$ from the transition equation (26) to be negative for a small subset of firms (<10%). To enforce positivity while preserving the cross-sectional distribution, we fit a Gompertz distribution[12] to the predicted ratio $k_{n\infty}/k_{nt+1}$ and use it to estimate positive steady-state values. We provide additional details in Appendix B.

Recovery of demand and cost parameters. Given $\mathbf{z}$, θ, and $\mathbb{k}$, we recover the remaining parameters directly from the data and equilibrium conditions. Unit labor costs are set to $c_n = \text{COGS}_n/(y_{nt})$. The demand intercept vector $\mathbf{b}$ is recovered by inverting the steady-state equation (42).

3.3 Estimated Transition Matrix

Figure 2 displays the matrix of investment spillovers (equation 60) or, equivalently, the off-diagonal entries of the transition matrix $\mathbf{\Phi}$, as a weighted network. Each node represents a firm in compustat as of 2021; each edge represents the magnitude investment spillovers (see equation 60 between a firm pair. Specifically, we plot edges corresponding to the top 320,000 entries of $-\mathbf{\Phi}$ (multiplied by negative one so that competitive spillovers—which dominate empirically—appear as positive weights), with edge thickness proportional to spillover strength. The network exhibits a characteristic core-periphery structure. Central firms, positioned densely in the interior, experience strong bilateral

[12] We use a Gompertz distribution because it matches the empirical properties of the distribution of the ratio $k_{n\infty}/k_{nt+1}$, which is negatively skewed and in logs has a fat left tail, while maintaining non-negative support. As a robustness check, we also fit a Weibull distribution using the same moment-matching procedure; results are virtually identical.

interactions with numerous other firms, reflecting either intense product market competition with many rivals or high capital productivity z_n. The latter is equivalent to low reliance on capital: it reduces the incidence capital adjustment frictions and amplifies responsiveness to competitive conditions. Peripheral firms, scattered along the exterior with sparse connections, exhibit weak investment spillovers either because they occupy isolated niches in the product market network, or because low capital productivity (high capital dependence) generates relatively large adjustment costs that dampen their investment response to competitive pressures.

The network displays a moderate degree of clustering: this suggests that investment spillovers organize firms into loosely defined groups of firms that affect each other's investment choices. This cluster structure proves less pronounced than the one observed in the underlying Hoberg-Phillips product similarity network, as documented in Pellegrino (2025). This attenuation reflects the fact that $\mathbf{\Phi}$ incorporates not only product market similarity—captured by the substitution matrix $\mathbf{\Sigma}$—but also firm-level heterogeneity in capital productivities z_n. Firms with low z_n exhibit weak investment responses even to closely positioned competitors, whereas firms with high z_n respond vigorously to product market conditions. In addition, $\mathbf{\Phi}$ also reflect indirect linkages between firms. This additional heterogeneity obscures the clean industry boundaries visible in pure product similarity networks.

Figure 3 presents the cumulative distribution of eigenvalues $\hat{\varphi}_n$ of the estimated transition matrix $\mathbf{\Phi}$ for 1995 and 2021. These eigenvalues govern the persistence of the capital stocks: values near zero indicate rapid convergence to the steady state, whereas values approaching unity imply instead slow adjustment. We observe substantial mass at both extremes of the cumulative distribution: many eigenvalues cluster near zero (indicating rapid convergence along those modes) while others concentrate near the maximum (indicating much slower adjustment).

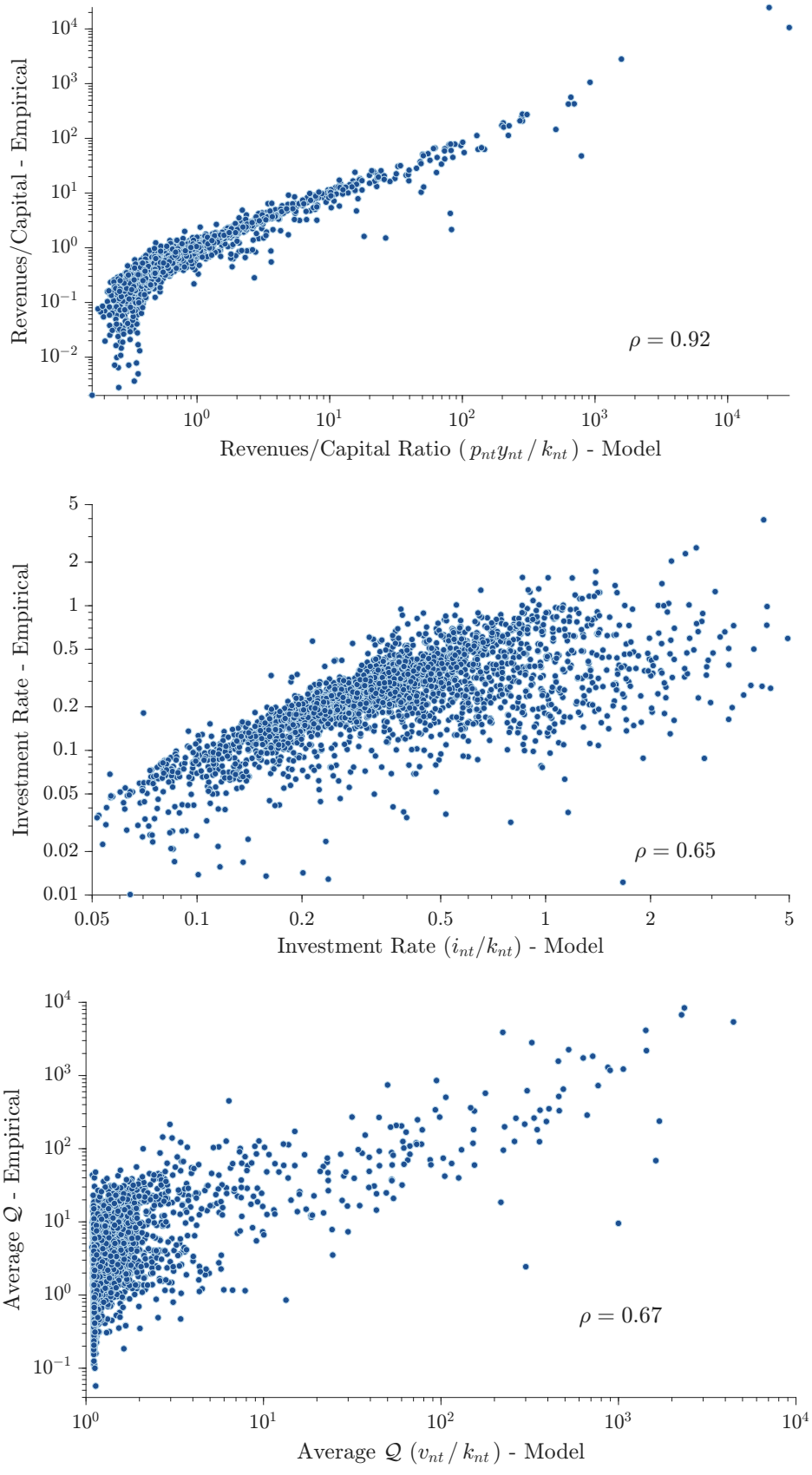
The large variation in eigenvalues implies that predicting the speed of convergence for capital on aggregate is inherently difficult. The actual adjustment speed depends on the current capital distribution ($\mathbf{k}_t - \mathbb{k}$): if the economy happens to be displaced primarily along fast-adjusting eigenvector directions (associated with firms with high productivity or high product market centrality), convergence will be rapid; if the displacement aligns with slow-adjusting modes (associated with firms with low productivity or high market power), aggregate capital will be stickier. This combination of heterogeneity and state-dependence distinguishes our network model from simpler frameworks where convergence rates are uniform across all initial conditions, and underlines the importance of product markets in determining the economy's as well as the individual firms' response to shocks.

Another fact that stands out from Figure 3 is that the 2021 distribution (approximately) first-order stochastically dominates the 1995 distribution. The rightward shift indicates that capital stocks have become more persistent over time. The increased persistence reflects rising capital dependence across firms: firms in 2021 use more capital than their 1995 counterparts (due to a reduction in z_n). This decreases the eigenvalues $\hat{\omega}_n$ of matrix $\mathbf{\Omega}$, and this leads in turn to higher persistence in firms' capital stocks $\hat{\varphi}_n$.

3.4 Model Fit

Next, we assess the model's performance by evaluating its fit against the firm-level data used in calibration. In order to do so, we start by noting that our structural model is over-identified: it has more structural equation restrictions (equalities and inequalities) than degrees of freedom. Given

FIGURE 4: MODEL FIT



observed revenues, investment, and capital for each firm-year, perfectly matching both observed revenues and investment rates would require inferring, for some firms, negative productivity ($z_n^2 < 0$) or steady-state capital ($k_{n\infty} < 0$). After we impose the economically necessary restrictions $z_n > 0$ and $\mathbb{k} > 0$, it becomes clear that (by construction) the model cannot simultaneously fit all observed data perfectly. This is a feature, not a bug.

Specifically, the variables that the model won't match perfectly are revenues and investment rates. By comparing the model-generated versions of these variables against the corresponding empirical ones and gauging the resulting fit, we can get a sense of how binding these over-identifying restrictions are in practice.

Figure 4 displays three key validation exercises, plotting model-predicted values against their empirical counterparts for all firm of our sample in 2021. The first two panels examine the model's ability to match targeted moments. The top panel shows the revenues-to-capital ratio ($p_{nt}y_{nt}/k_{nt}$), achieving a correlation of ρ =0.93 between model and data. This is reassuring: revenues are one of the most accurately measured variables in accounting data, and it is important for our model to match this variable as closely to perfectly as possible.

The middle panel displays the investment rate (i_{nt}/k_{nt}), with a correlation of ρ =0.57. The goodness of fit captured in the middle panel corresponds to an R-square of 0.4, which compares favorably to that of estimated investment equations in the existing neoclassical literature. For instance PT, whose paper incorporates intangible capital and corrects for measurement error, reports R-squares ranging from 0.233 to 0.374.

The bottom panel provides an important validation using the cross-sectional distribution of average Q, defined as the market value of the firm's assets divided by its capital stock (v_{nt}/k_{nt}). Average Q is an untargeted quantity in our calibration—the model does not use this moment in recovering any parameter. Nevertheless, the model achieves a correlation of ρ =0.55 with the empirical average Q. This provides evidence that the model's structural predictions extend beyond the specific moments used in calibration, and therefore that the economic forces that underlie our model are important to explain firm heterogeneity in valuations.

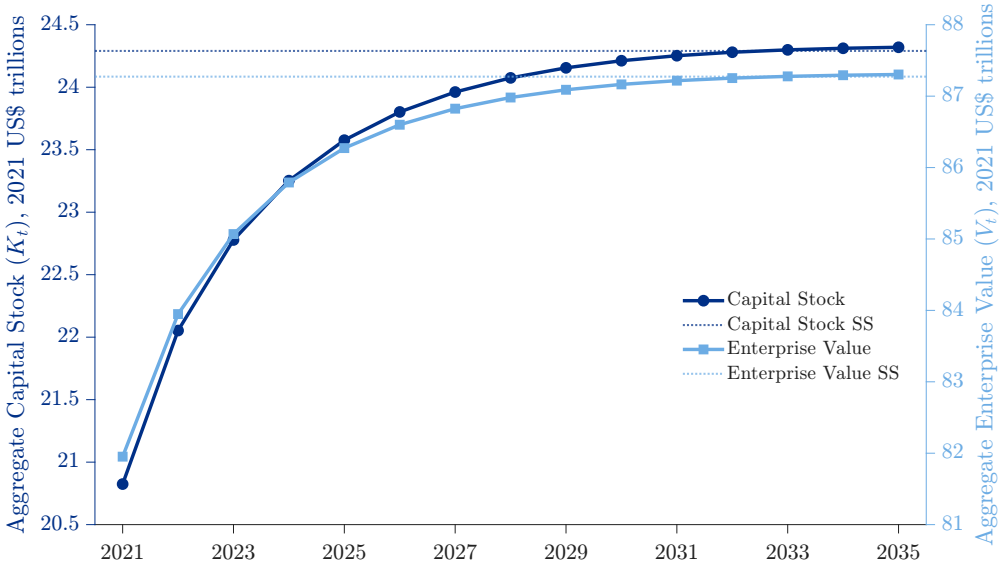
4 Model Quantification

In this section, we bring our calibrated model to bear on a series of quantitative exercises. Our baseline implementation assumes simple Cournot competition, where each firm in our Compustat sample corresponds to an independent business unit that internalizes its own downward-sloping residual demand but not its product market spillovers on rivals. We begin by analyzing the properties of this factual equilibrium, and then turn to counterfactual exercises that illustrate the model's capacity for policy-relevant analysis.

4.1 Analysis of the Factual Equilibrium

We start by taking the model to the data in 2021 and examining the properties of the calibrated factual Cournot equilibrium. We focus on three sets of results: the aggregate and firm-level transition dynamics of capital, the relationship between average and marginal Q, and the decomposition of steady-state enterprise values.

FIGURE 5: TRANSITION DYNAMICS OF THE AGGREGATE CAPITAL STOCK



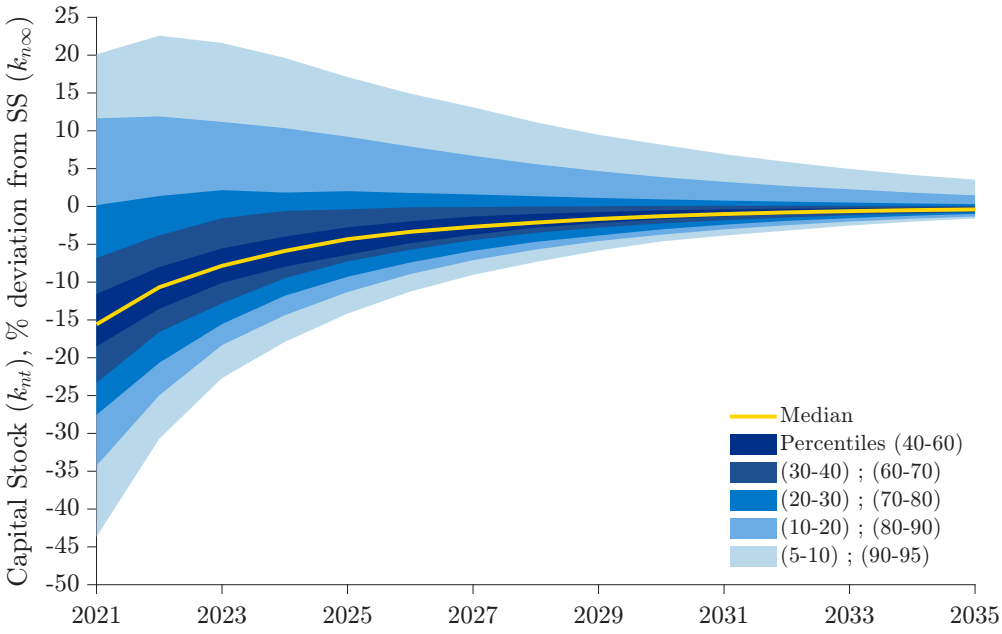
Transition Dynamics. We begin our quantitative analysis by studying the properties of the Open Loop Markov Equilibrium path predicted by the model, with an emphasis on the transition dynamics. Figure 5 displays the evolution of aggregate capital stock in the US economy over a 15-year forecast horizon, starting from the observed 2021 cross-section. The model predicts a smooth increase from approximately $18.12 trillion in 2021, approaching its steady-state level of $21.84 trillion by 2035, representing aggregate capital growth of roughly $3.71 trillion over the period. This trajectory reflects the collective adjustment of thousands of individual firms as they converge toward their steady-state capital levels, each responding to their own capital intensity, adjustment costs, and competitive environment.

The concave shape of the aggregate capital path—with faster growth in earlier years that gradually decelerates—is characteristic of VAR models. Starting from 2021, when many firms are displaced from their long-run equilibrium positions, the economy experiences relatively rapid capital accumulation as firms close the gap to steady state. As firms approach their steady state level of capital stock, the investment rate naturally slows, even though adjustment continues.

Over the transition, the aggregate capital stock increases on average by approximately 1.37% of its 2021level per year—a pace that reflects both the fundamental forces driving capital accumulation (capital productivity, demand conditions, and competitive interactions) and the frictions that slow adjustment (convex adjustment costs parameterized by θ).

Figure 6 reveals the firm-level heterogeneity masked by the aggregate trajectory shown in Figure 5. The fan chart displays the cross-sectional distribution of capital deviations from steady state, measured as percentage gaps. Observations are weighted by 2021 capital stocks. In 2021, firms exhibit substantial dispersion, with the 5th percentile approximately -42.54% below steady state and the 95th percentile roughly 22.59% above. The distribution is notably asymmetric: more firms operate below their steady state than above. The median firm itself starts approximately -21.96% below its long-run equilibrium.

FIGURE 6: FIRM-LEVEL TRANSITION DYNAMICS



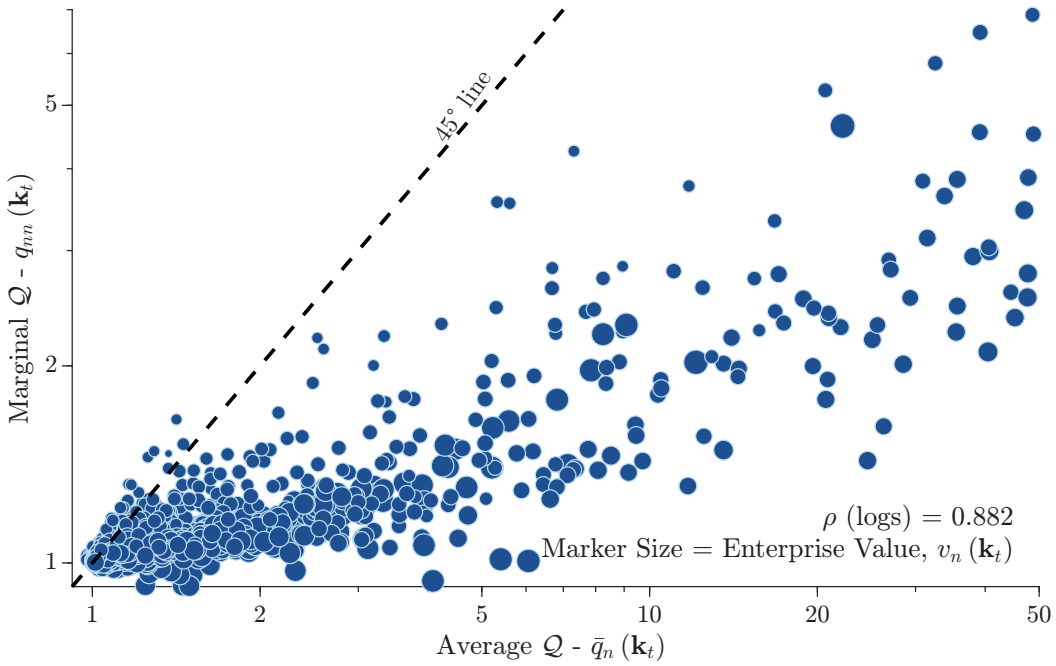
The percentile bands systematically narrow and drift toward zero: over the forecast horizon, the model predicts gradual convergence of each firm towards its steady state level of capital. Firms substantially below steady state invest heavily to close the distance, whereas those above their steady state allow capital to depreciate faster than they replace it (or divest). By 2030, the range between the 40th and 60th percentile bands has tightened considerably, spanning only about ±2.64% around steady state.

The heterogeneous adjustment speeds visible in Figure 6 directly reflect the eigenvalue distribution analyzed in Figure 3. Firms whose current deviations align with fast-adjusting eigenvector directions (corresponding to small eigenvalues of $\mathbf{\Phi}$) converge rapidly, while those aligned with slow-adjusting modes take much longer to reach steady state. Overall, the graph reveals an economy characterized by rich transitional dynamics, where firms that start with a stock of capital above their steady state tend to adjust their capital more slowly.

Average and Marginal Q. Figure 7 plots the model-implied firm-level estimates of average Q as of 2021 against the corresponding estimates for marginal Q. In spite the relatively high correlation of 0.82, the fit between both quantities masks substantial heterogeneity in the cross section. The wedge between average and marginal Q is such that most observations lie strictly below the 45-degree line, confirming that monopoly rents significantly contribute to enterprise values for most firms. Further validating the workings of the model, firms associated with relatively larger dots in the plot (and, thus, with higher market capitalization) are typically further away from the 45-degree line: a higher wedge between average and marginal Q is naturally associated with higher enterprise value, driven by with higher markups and, therefore, higher market power.

Firm Value Decomposition. A recent literature has explored the link between the stock markets and market power, suggesting a link between the rising market power of US publicly traded firms and their rising valuations (see Eeckhout, 2025; Cho, Grotteria, Kremens, and Kung, 2023).

FIGURE 7: AVERAGE Q AND MARGINAL Q



Because our model features endogenous markups and enterprise values, it can help inform this debate. In particular, Equation (55) provides a useful decomposition of firm n's steady state market value in terms of two components: the replacement value of capital ($p_n^k k_{n\infty}$) and the present value of monopoly rents (ρ_n). We implement this decomposition in Figure 8, which depicts the combined steady state market value of all firms in our sample for 1995 and 2021 as two pie charts.

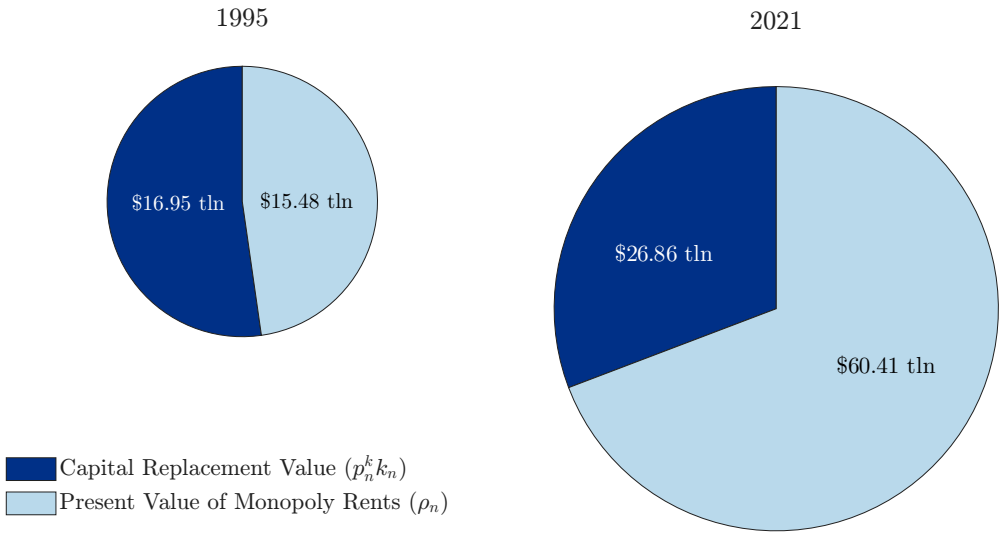
The first thing that we observe from the figure is that the absolute scale of firm value in steady state grew dramatically, from $27.66 trillion in 1995 to $84.74 trillion in 2021. This increase reflects *both* capital accumulation as well as the increase in monopoly rents associated with rising market concentration.

Next, we consider the shares of the two component. In 1995, the replacement value of capital accounted for 52.73% of total enterprise value in steady state, whereas monopoly rents comprised 47.27%. By 2021, this composition changed drastically: the replacement value of capital decreased to just 28.50% of firm value in steady state, whereas monopoly rents have risen to 71.50%.

In sum, two interesting facts emerge from this analysis: 1) monopoly rents account for a significant fraction of firms' market values; 2) the relative share of this rent component has increased significantly over time. This in turn suggests that rising monopoly rents can help explain the rapidly rising market valuations of US firms.

Our findings from Figure 8 are therefore consistent with the recent findings of Cho et al. (2023), who find that markups account for a large share of rising US stock market valuations and that expected markups correlate with intangible investments. The latter are included in our definition of total capital. While our model does not distinguish between rents accruing to intangible capital from those accruing to tangible capital, intangible capital accounts for an increasing share of the aggregate

FIGURE 8: DECOMPOSITION OF STEADY STATE AGGREGATE FIRM VALUE



capital stock over our sample (Eisfeldt and Papanikolaou, 2014; Crouzet and Eberly, 2021). Therefore both our findings are consistent with rising markups providing, at least in part, compensation for accumulated past intangible investments, as evinced by (Crouzet and Eberly, 2023).

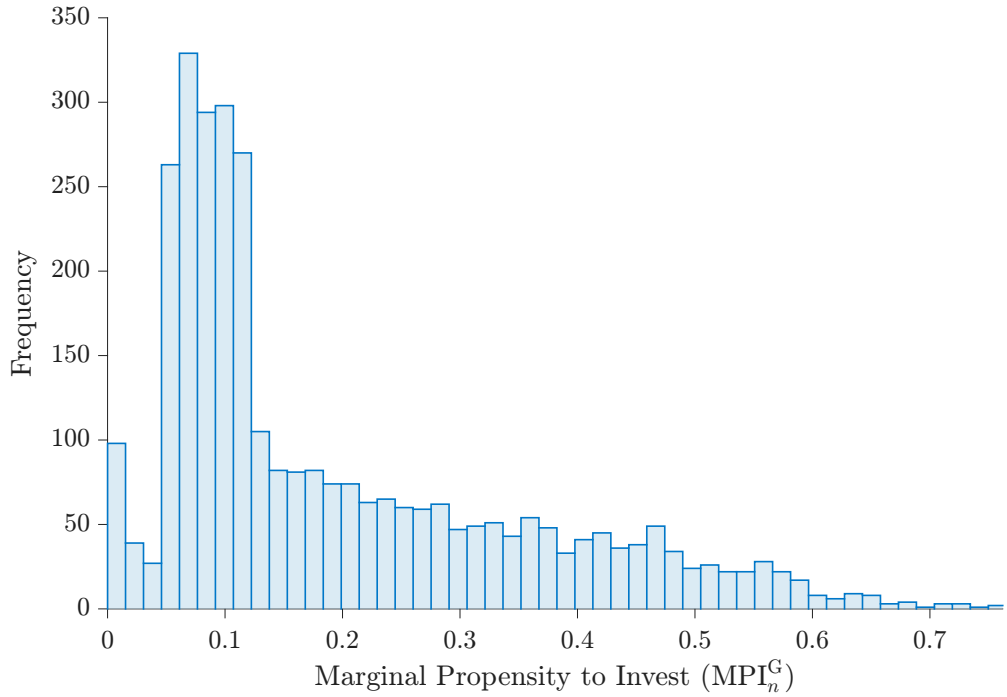
Marginal Propensity to Invest. Figure 9 reports the cross-sectional distribution of the marginal propensity to invest out of growth opportunities, computed from equation (76). For the typical business unit, investment responsiveness is modest: most of the mass lies near 0.1, meaning that a permanent shock that raises steady state firm value by one dollar increases current investment expenditure by roughly ten cents. The distribution is strongly right-skewed: a long tail of business units — those that face intense competitive pressure and, as a consequence, a low incidence of adjustment costs — converts value gains into capital expenditure at several times the typical rate, approaching the price-taker benchmark of $\left(1-\varphi_{nn}\right)/(1+r)$. At the opposite extreme, a small cluster of business units exhibits an MPI^{G} close to zero: for these firms the wedge between average and marginal Q is so large that additional value accrues almost entirely as rents on existing capacity, generating little incentive to expand. This dispersion illustrates a central implication of our framework: product market power weakens the pass-through from value creation to capital formation, so that the business units that gain the most value are not necessarily those that invest the most.

4.2 Counterfactual Analysis

In the next subsection, we exploit the counterfactual-building capabilities of our model to provide additional insights on the role of product market competition, discount rate shocks, and M&A activities. Before we do so, we provide a technical clarification on the practical implementation of the counterfactual exercises.

How to Implement Counterfactuals with Non-negativity Constraints. One minor challenge that must be dealt with when performing counterfactual exercises (mergers, changes in r or unit labor costs, etc.), is that the interior steady state implied by Proposition 2 does not preclude capital stocks from becoming negative for some business units. To perform counterfactuals, we develop a

FIGURE 9: MARGINAL PROPENSITY TO INVEST OUT OF GROWTH OPPORTUNITIES



method to enforce a non-negativity constraint on $\mathbb{k}$. To build intuition for our method, we notice that the steady state presented in equation (42) corresponds to the Nash equilibrium of the corresponding *static* game obtained by shutting down adjustment costs (i.e., setting $\theta = 0$) - specifically, for the single-product Cournot case, $\mathbb{k}$ solves:

$$k_{n\infty} \equiv \underset{k_n}{\text{argmax}}\, z_n k_n (b_n - c_n) - (r+\delta_n) p_n^k k_n - z_n^2 k_n^2 - \sum_{m=1}^{N} z_n z_m \sigma_{nm} k_n k_m \tag{80}$$

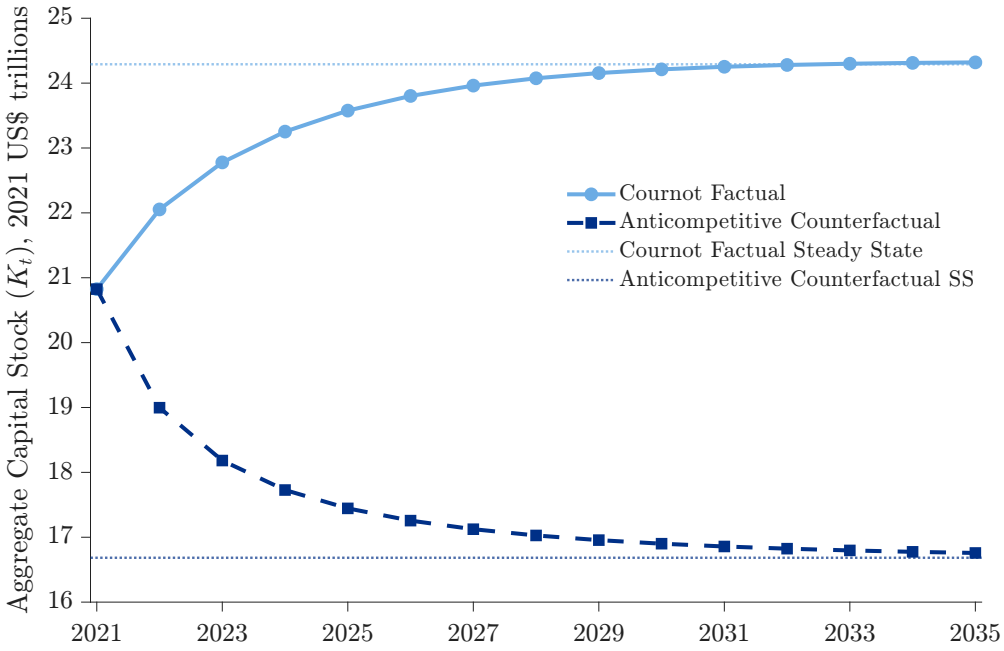
As discussed in Pellegrino (2025), this static game is a type of *Potential* game, which can be represented by a scalar function $\mathscr{P}(\mathbf{k})$ or *Potential*, defined as follows:

$$\mathscr{P}(\mathbf{k}) \overset{\text{def}}{=} \mathbf{k}' \left[\mathbf{Z}(\mathbf{b}-\mathbf{c}) - (r\mathbf{I}+\boldsymbol{\Delta})\mathbf{p}^k \right] - \frac{1}{2}\mathbf{k}'\boldsymbol{\Omega}\mathbf{k} \tag{81}$$

Regardless of whether a non-negativity constraint applies to $\mathbf{k}$, the solution of the static game (80) is also the maximizer of the potential function (81), as the Kuhn-Tucker Conditions are the same.

To compute counterfactuals, we can therefore find the steady state capital stock by maximizing the potential function $\mathscr{P}(\mathbf{k})$, with all primitives ($\boldsymbol{\Omega}$, $\mathbf{b}$, $\mathbf{c}$, etc...) evaluated according to the counterfactual, and subject to the non-negativity constraint $\mathbf{k} \geq \mathbf{0}$. While this constrained steady state does not generally admit a closed-form expression, it is a convex quadratic program with linear constraints; in our implementation we solve it using `quadprog` in MATLAB: it completes the optimization in a few seconds on an ordinary laptop even when the number of firms is in the thousands. This makes it straightforward to compute the counterfactual $\mathbb{k}$, while ensuring its non-negativity. A separate discussion is required for the computation of the counterfactual transition dynamics: here, we present

FIGURE 10: ANTICOMPETITIVE COUNTERFACTUAL: AGGREGATE CAPITAL STOCK



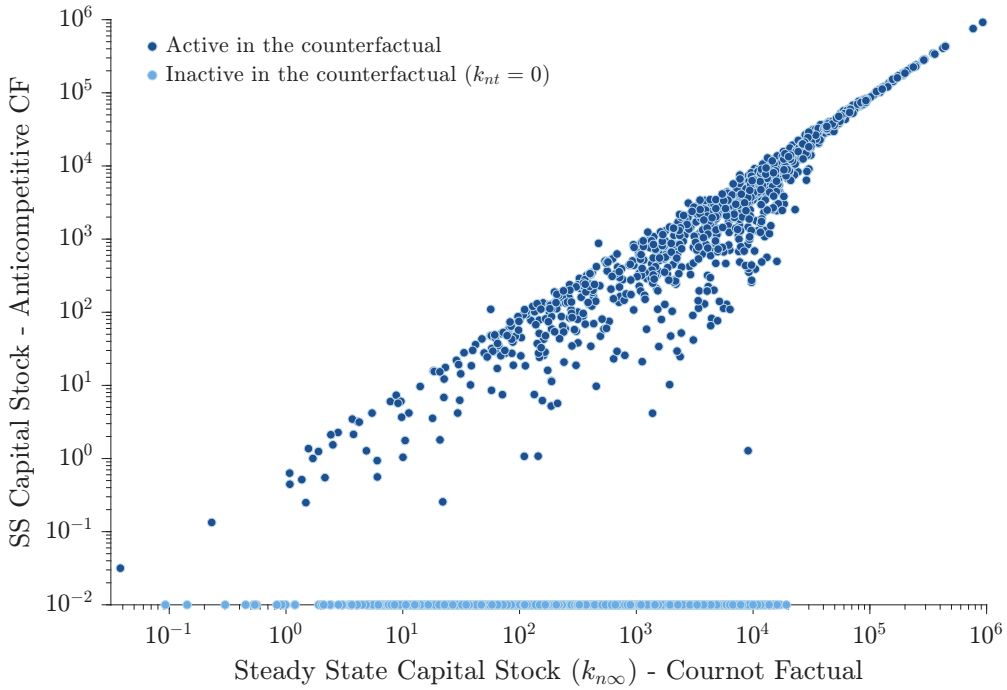
two possible ways to proceed.

Active Set implementation. The first method relies on restricting the oligopoly game to subset of firms that are guaranteed to be active in the steady state counterfactual. Specifically, let $\mathscr{A} \equiv \{n : k_{n,\infty} > 0\}$ be the set of firms that operate with positive capital stock in the steady state counterfactual that solves (80). We restrict the counterfactual economy to $\mathscr{A}$ (retain the $\mathscr{A} \times \mathscr{A}$ sub-matrices of the primitives and set $k_{nt} = 0$ for all $n \notin \mathscr{A}$ and all t), and compute the transition matrix on the reduced system using the analytical interior formula (43). The advantage of this approach is internal consistency: conditional on the active set $\mathscr{A}$, the interior solution in equation (42) holds exactly and non-negativity is enforced by construction. The drawback is that extensive-margin adjustment is "front-loaded": firms with $k_{n,\infty} = 0$ are treated as inactive from $t = 0$, ruling out gradual exit dynamics.

Full Set (interior-$\mathbf{\Phi}$) implementation. An alternative solution is to keep all firms in the state vector, use the constrained $\mathbb{k}$ from the static quadratic programming (allowing some components to be zero), and compute $\mathbf{\Phi}$ from the interior formula (43) on the full system. The main advantage of this solution is that it preserves the full network of interactions and yields smooth dynamics; the obvious drawback is that it is best interpreted as a heuristic approximation: the transition dynamics so computed do not correspond exactly to the genuine OSLME of the counterfactual because the law of motion consistent with equations (26) and (43) needs not satisfy the Kuhn–Tucker conditions of the constrained dynamic game.

In order to make the counterfactuals as comparable as possible to the factual equilibrium, we implement the second method (the "full set" implementation) in computing our counterfactuals. This choice does not affect any of our steady-state results, which follow from the constrained problem (80) alone; it matters only for the shape of the counterfactual transition paths.

FIGURE 11: ANTICOMPETITIVE COUNTERFACTUAL: CROSS-SECTION

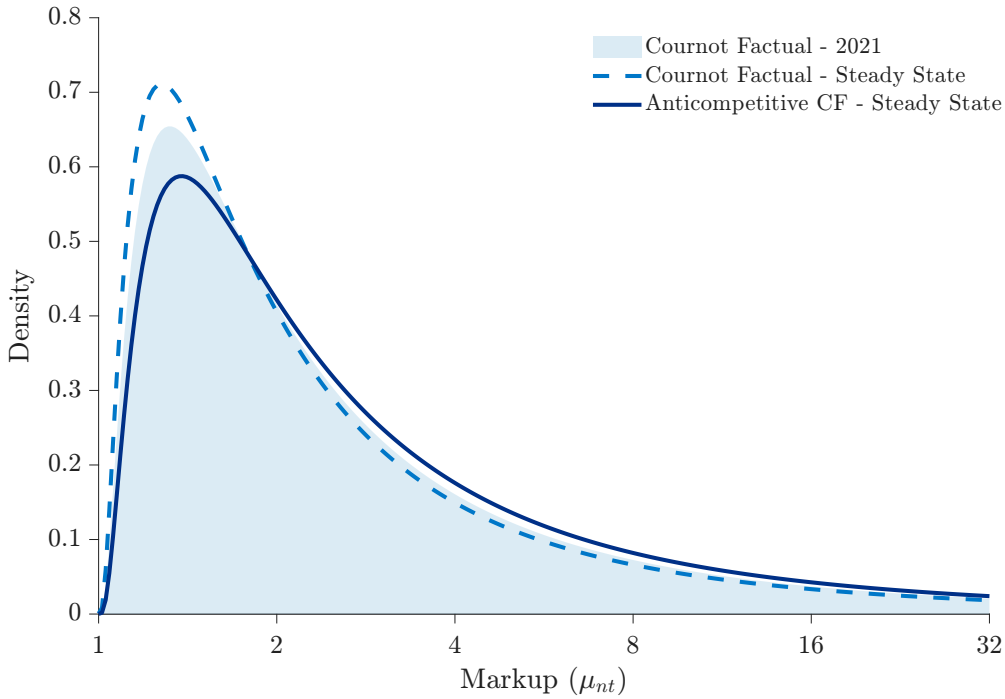


Counterfactual 1: Competition and Investment. A central question in finance and macroeconomics is the extent to which product market competition affects capital accumulation. While the literature has long recognized that product markets affects static allocative efficiency, quantifying the dynamic effects of competition on aggregate investment has proven challenging. In canonical macroeconomic models featuring monopolistic competition or perfect competition, the concept of "shutting down competition" lacks a clear operational definition for counterfactual analysis. Our framework's explicit parameterization of competitive conduct—through the matrices $\mathbf{D}$ and $\mathbf{\Gamma}$, which govern how much firms internalize product market spillovers—renders this question tractable. We can precisely compare alternative competitive regimes while holding constant all other economic primitives.

We quantify the impact of product market competition on corporate investment by comparing the factual Cournot equilibrium to an anti-competitive counterfactual in which all firms fully internalize product market spillovers. Under the Cournot factual equilibrium, firms internalize their downward-sloping residual demand curves but do not account for strategic spillovers on rival firms. In our anti-competitive counterfactual all business units operate as a single cartel: we set $\mathbf{\Gamma} = \mathbf{1}\mathbf{1}'$ such that the competitive pressure matrix becomes $\mathbf{\Omega} = 2\mathbf{Z}(\mathbf{I} + \mathbf{\Sigma})\mathbf{Z}$. Under this regime, firms coordinate their investment decisions to maximize joint profits, fully internalizing the negative externalities their capital accumulation imposes on competitors through increased output and lower equilibrium prices. The answer to our central question is economically significant: moving from Cournot competition to full cartelization reduces aggregate steady-state capital by 32.27%.

Figure 10 displays the aggregate implications. Commencing from the observed 2021 capital stock of \$18.12 trillion, the factual Cournot economy converges monotonically to its steady state of \$21.84 trillion over approximately 10 years. The anti-competitive counterfactual, in contrast, exhibits a pronounced contraction, with aggregate capital declining by 32.27% (relative to the factual steady state)

FIGURE 12: DISTRIBUTION OF MARKUPS



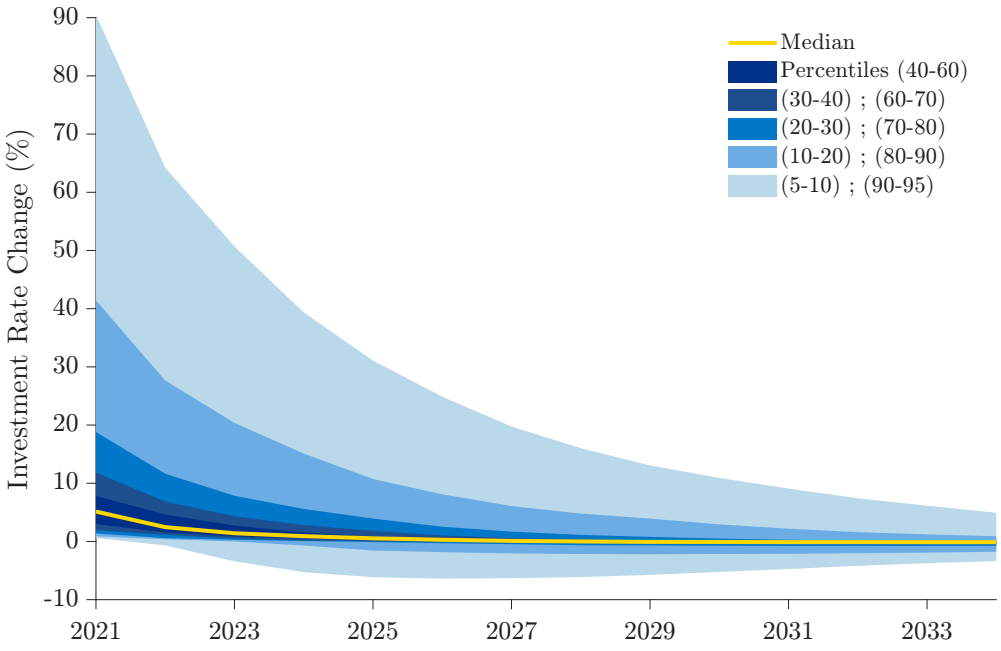
to \$14.79 trillion in steady state. This substantial reduction reflects the coordinated restriction of capacity to sustain elevated markups: the canonical cartel incentive to limit supply extends naturally to dynamic investment decisions. The transition path reveals that this adjustment occurs relatively rapidly, with the bulk of the convergence completed within 5 years.

The cross-sectional heterogeneity in firm-level responses, illustrated in Figure 11, reveals important additional details. The figure plots each firm's steady-state capital in the counterfactual against its factual steady state, with observations below the 45-degree line indicating investment reductions. Nearly all active firms reduce their capital stocks substantially under cartel coordination, though the magnitude of adjustment varies considerably across the size distribution. Notably, a non-trivial fraction of firms—depicted in light blue—exit entirely in the counterfactual steady state, with their capital stocks falling to zero.[13] These exits occur predominantly among smaller firms with high product similarity to larger competitors, for whom the cartel finds it optimal to consolidate capacity into fewer, larger producers. This pattern indicates that anti-competitive coordination not only suppresses aggregate investment but also fundamentally reshapes market structure toward greater concentration.

Figure 12 illustrates the markup consequences of this investment reallocation. The distribution of markups in 2021 (light blue shaded region) exhibits moderate central tendency, with a median markup of approximately 2.10. The factual steady state (dashed line) displays modest compression towards a median markup of 1.97 as the typical firm increases capital and supply and the economy

[13] For visibility on the logarithmic scale, exiting firms are displayed at a nominal floor in Figure 11. These firms are retained in the counterfactual markup distribution of Figure 12 and are assigned their model-implied markups, which remain well-defined at zero output because the equilibrium price is well-defined.

FIGURE 13: RESPONSE OF INVESTMENTS RATES TO DISCOUNT RATE SHOCK



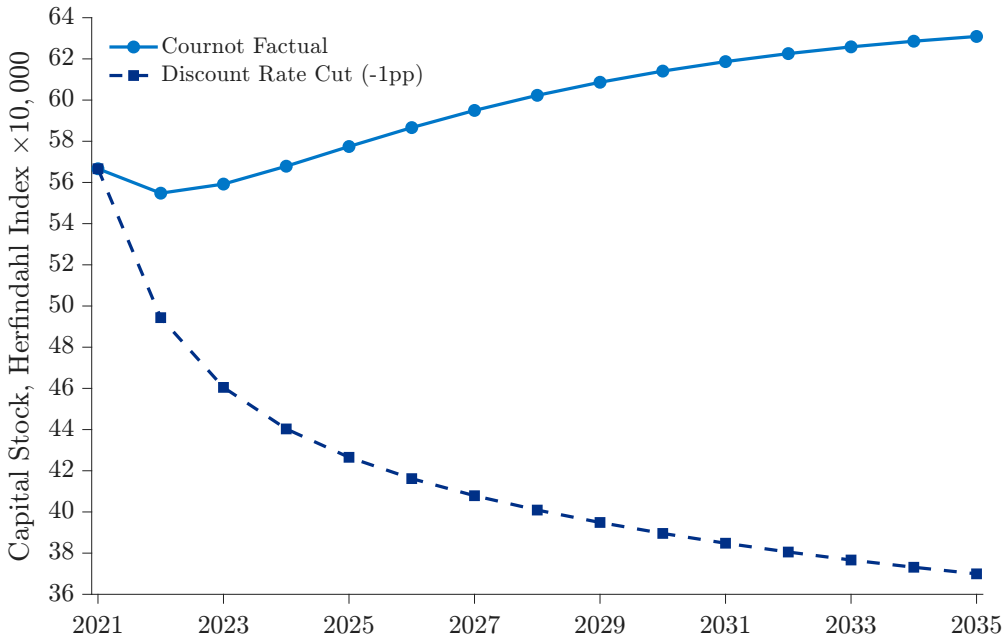
converges to its long-run equilibrium. The anti-competitive counterfactual (solid line), however, generates a pronounced shift of the entire distribution. Markups increase for the majority of firms, with the median rising to approximately 2.29—a 16.35% increase—and the right tail extending comfortably into double digits. This reflects of coordinated output restriction. The marked contrast between the factual and counterfactual markup distributions underscores how competitive conduct shapes not only the level but also the cross-sectional dispersion of market power.

These results indicate that imperfect competition in product markets has quantitatively substantial implications for aggregate and firm-level capital accumulation. This finding complements existing work on the welfare costs of oligopoly by highlighting a distinct dynamic channel: beyond the static allocative distortions from markup pricing (Pellegrino, 2025), imperfect competition distorts the inter-temporal allocation of resources by attenuating incentives for capital formation.

Counterfactual 2: Discount Rates and Market Structure. Our second counterfactual examines the market structure implications of changing the discount rate. Specifically, we study the case of a one percentage point permanent, unanticipated decrease in the aggregate discount rate ($r \to r - 0.01$), holding all other primitives constant, including the competitive conduct regime. This experiment helps understand how a change in aggregate financial conditions interacts with an oligopolistic market structure to affect the equilibrium path, abstracting from changes in competitive behavior.

We start by studying the impact of the shock on the investment rates. Figure 13 shows the effect of the discount-rate cut on firms' investment rates. The plot displays the % change in firm-level investment rates in the counterfactual scenario against the corresponding investment rate in the Cournot case, again grouping firms into percentiles by the magnitude of their response. It reveals significant semi-elasticities and striking cross-firm heterogeneity in the first few years following ths shock: the median firm adjusts its investment rate by about 2.86%; the majority of firms' investment response on impact is within 1.07% and 8.63%. However, there is a thick right tail: as we move

FIGURE 14: EVOLUTION OF CONCENTRATION

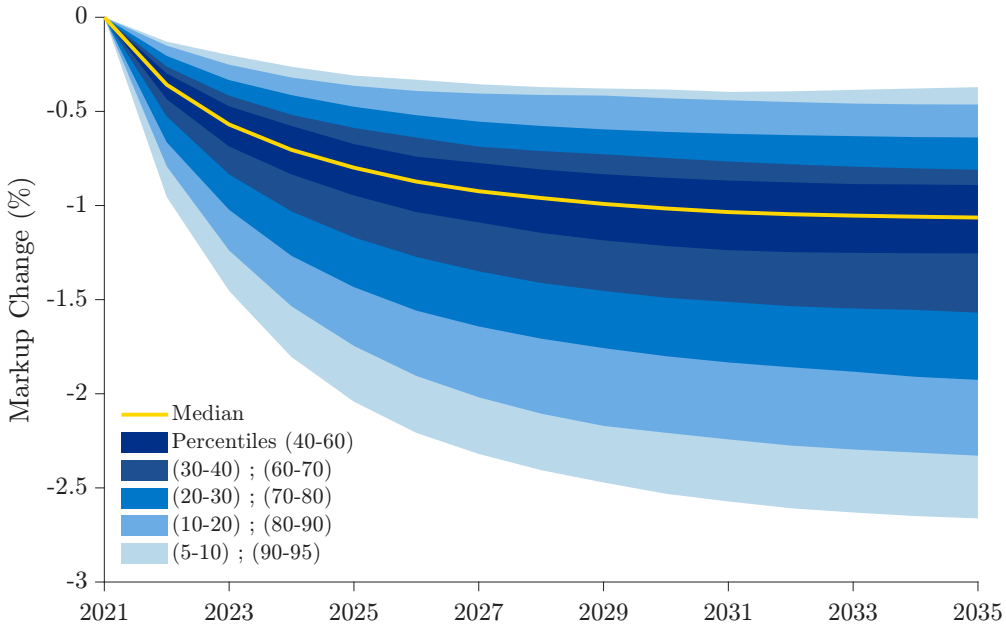


up in the higher percentiles of the distribution, we encounter firms that respond much more, with the 90th percentile adjusting the investment rate by 16.80% and the 95th nearly doubling it. Yet, these effects are temporary: every firm's investment rate eventually reverts towards depreciation rate as the model converges to its new steady state. Interestingly, due to the complex product market interactions that are present in our model, some firms end up reducing their investment along the equilibrium path. Through these our model to contribute to an emerging literature that investigates firms' heterogeneous responses to discount rates fluctuations (Ottonello and Winberry, 2020; Winberry et al., 2025; Best et al., 2026).

These asymmetric investment responses have implications for concentration, which we measure as the Herfindahl index of capital stock. Figure 14 displays the evolution of concentration, measured under both scenarios. The factual Cournot economy exhibits an initial decline in the Herfindahl index from 29.57 in 2021 to approximately 28.40 by 2022, followed by a gradual increase toward 31.14 in steady state. Given the initial configuration of capital stocks, product market spillovers generate interesting transition dynamics, including a non-monotonic evolution of concentration. The discount rate shock counterfactual, by contrast, generates a substantially different trajectory. Concentration falls immediately and monotonically, reaching 26.20 in steady state –significantly lower than the factual outcome. This pronounced decrease indicates that lower aggregate discount rates fundamentally alter the competitive landscape by differentially affecting firms across the size distribution.

The mechanism underlying this concentration effect operates through heterogeneous responses to the aggregate discount rate shock. The discount rate shock increases the present value of future profits, strengthening investment incentives for all firms. However, this effect is not uniform: smaller firms, which charge thinner markups and operate closer to their exit thresholds, raise investment more sharply than larger incumbents. The resulting reallocation of market share toward smaller firms manifests as decreased concentration. It is again important to underscore that this channel

FIGURE 15: RESPONSE OF MARKUPS TO DISCOUNT RATE SHOCK



operates holding constant the competitive conduct regime—the decrease in concentration strictly relates to the differential impact of financial conditions across the firm size distribution, rather than changes in strategic behavior.

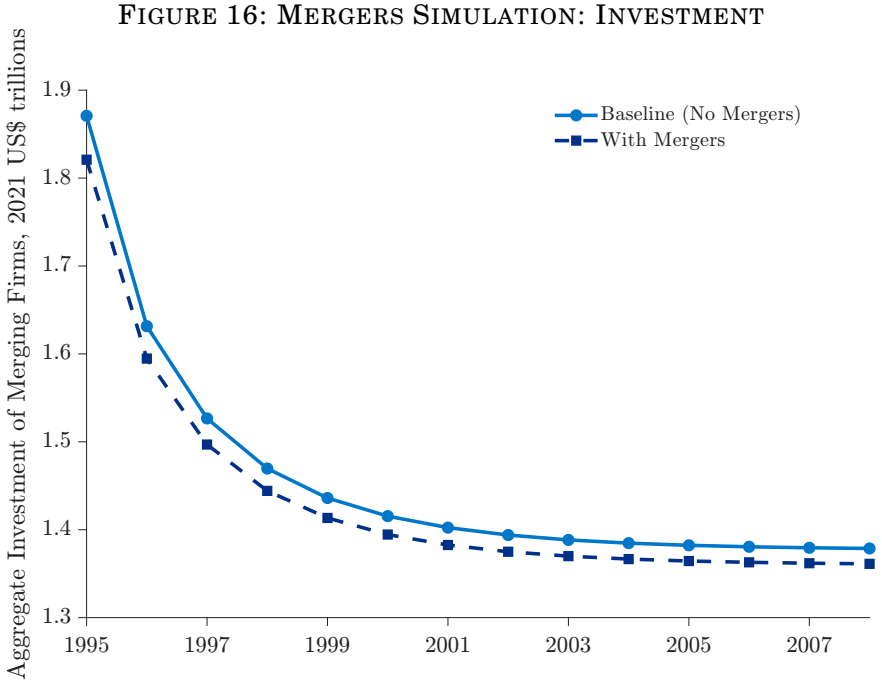
Figure 15 illustrates the markup consequences of this reallocation effect. The fan chart plots the % change in firm level markups in the counterfactual with respect to the corresponding firm-level markup in the Cournot factual. The median firm experiences a modest markup decrease, falling gradually to -0.87% by 2027. However, this aggregate pattern masks substantial heterogeneity. The lower percentiles of the distribution exhibit considerably greater markup decreases, with firms in the 5th-10th percentile experiencing decreases exceeding -1.73%. By contrast, firms at the 90th-95th percentile barely adjust markups.

This dispersion reflects the oligopolistic nature of competition as well as the properties of GHL demand, which generate vast asymmetry across firms in the cost of capital passthrough. The widening fan over time indicates that these markup effects amplify along the transition path, as the cumulative impact of differential investment decisions compounds.

These findings demonstrate that discount rate changes have asymmetric effects across the firm distribution in oligopolistic markets, with implications extending beyond aggregate investment to market structure and pricing behavior. This in turn suggests that policies that affect the discount rate may inadvertently affect product market competition, with persistently lowered discount rates contributing to decreased market concentration and smaller markups. This mechanism is absent from standard models with perfect competition or monopolistic competition and CES demand.

Counterfactual 3: Mergers. In a recent monograph, Philippon (2019) identifies rising market power as a potential explanatory factor for sluggish aggregate investment in the United States, and suggests that intense M&A activity could be one of the driver of rising market power. Gutierrez and Philippon (2023) further examine diverging trends in antitrust policy between EU and the US,

FIGURE 16: MERGERS SIMULATION: INVESTMENT



and suggest that timid merger enforcement in the United States is a contributing factor to rising concentration and market power.

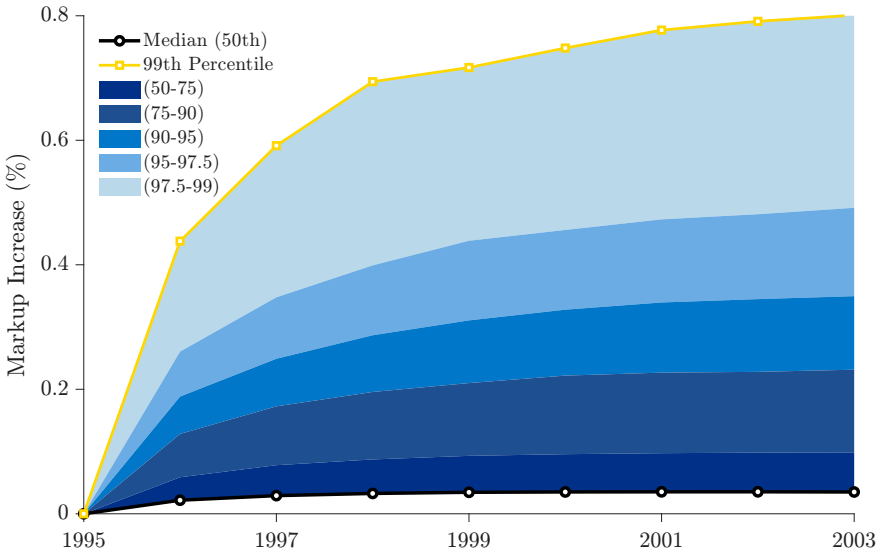
An interesting question that arises from the existing literature is therefore what has been the real impact of M&A activity on aggregate investment and prices over the last 25 years. This is the question that our third counterfactual exercise seeks to answer. We take the model to the data in 1995 and simulate the impact of all mergers between firms in our sample observed in the SDC Platinum database through 2021.[14] This counterfactual isolates the effect of common ownership on investment decisions by comparing an economy where merged entities internalize product market spillovers (setting $\mathbf{\Gamma}$ according to observed merger activity) against a baseline where all firms remain independent. The analysis encompasses 4,455 mergers involving 3,289 firms—representing 57.51% of our 1995 sample—providing substantial variation in merger exposure across the firm distribution.

Two important remarks merit emphasis on how we perform the exercise. First, we assume mergers generate no marginal cost reductions or product quality improvements : these could potentially render some mergers welfare-enhancing. Second, we treat all observed mergers as horizontal transactions between competing firms. In practice, some transactions may represent vertical integration (Frésard, Hoberg, and Phillips, 2020) or conglomerate mergers that could benefit consumers through the reduction of double marginalization. Hence, our results should be interpreted as an upper bound on the anticompetitive harm caused by merger activity.

Figure 16 displays the aggregate investment dynamics of merging firms under both scenarios. In the baseline economy without mergers, aggregate investment among these firms exhibits a pronounced

[14] We retain completed, controlling-stake (50% or more) transactions between firms in our 1995sample, excluding minority-stake purchases, buybacks, self-tenders, management buyouts, and subsidiary deals, and we collapse repeat transactions to unique acquirer–target pairs. All qualifying mergers are imposed simultaneously and permanently in 1995: the corresponding entries of $\mathbf{\Gamma}$ are set to one, acquired targets continue to operate as separate business units, and ownership links are not extended transitively across chains of deals.

FIGURE 17: MERGERS SIMULATION: MARKUPS



decline from $1.69 trillion in 1995 to approximately $1.19 trillion by 2003, reflecting the natural adjustment dynamics as firms converge toward their steady-state capital stocks. The merger counterfactual, however, generates a different trajectory: every year along the transition path the aggregate investment of merging firms is lower, reaching $1.17 trillion by 2003.

The associated investment suppression observed in Figure 16 thus reflects the canonical mechanism identified in our model: when firms internalize product market spillovers through common ownership, they account for the negative externality their capital accumulation imposes on affiliated entities through increased output and lower equilibrium prices. Interestingly, however, while the observed magnitude of the decline in aggregate investment is roughly $16.22 billion and therefore highly significant in absolute values, it represents a modest decline in relative terms, as it captures a 1.36% reduction with respect to the no-merger baseline. This finding suggests that the increased merger activity in recent decades in the US is not the likely driver of the sharp reduction in aggregate investment.

Analyzing merger activity more granularly at the firm level, we next observe striking heterogeneity on the impact of mergers on markups. Figure 17 illustrates this dispersion through a percentile fan chart focused on the upper half of the distribution, plotting the ratio of markups in the case with mergers to markups in the alternative counterfactual scenario without mergers. The median merging firm experiences a modest initial markup increase, rising gradually to 0.07% by 2003. However, this central tendency masks dramatic variation in the upper tail. Firms between the 90th and 95th percentiles exhibit markup increases above 0.36% in 2003, while those in the 95th-99th percentile range experience increases exceeding 0.56% in 2003. Most remarkably, mergers in the 99th percentile range generate a markup increase of approximately 1.11% in 2003, indicating significant anti-competitive effects in the relevant product market.

Put together, our findings suggest that even if mergers on aggregate have had a modest distortion on aggregate investment, a few mergers may have had very significant effects on markups.

5 Extensions and Sensitivity Analysis

In this section, we consider various extensions and generalizations of the Network Q model presented in Section 2, alongside a series of sensitivity analyses, where we re-run our quantitative exercises as we vary key model parameters. We begin by showing that our model can easily be generalized to accommodate arbitrarily rich investment spillovers between firms.

5.1 Arbitrarily Rich Spillovers: the General-Form Network Q Model

In the model presented in Section 2, strategic interactions between firms are governed by the matrix of product substitution $\mathbf{\Sigma}$. However, it is reasonable to ask whether the model is amenable to extensions that can accommodate additional layers of strategic interaction among firms, such as input-output linkages (Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi, 2012), labor market power (Azar and Marinescu, 2024), etc...

In this subsection, we show how the OSLME solution to the dynamic game presented in Section 2 actually solves an entire class of dynamic network oligopoly games, which can incorporate arbitrary richness in strategic interactions. The model solution remains virtually identical, as long as operating profits remain quadratic in the vector of capital stocks.

The General-Form Network Q Model. Consider now a more general version of the network Q model presented in Section 2, where firms' operating profits are given by the quadratic function:

$$\boldsymbol{\pi}(\mathbf{k}_t) \;=\; \mathbf{K}_t\mathbf{Z}\boldsymbol{\beta} - \mathbf{K}_t\mathbf{Z}\mathbf{G}\mathbf{Z}\mathbf{k}_t \tag{82}$$

where the vector $\boldsymbol{\beta}$ and the matrix $\mathbf{G}$ are defined generically[15]. The matrix $\mathbf{G}$ captures arbitrary investment spillovers. The production block of the model is left unchanged. The key requirement for tractability is that we preserve the linear quadratic functional form of the operating profits function, so that the first-order conditions remain linear and the equilibrium policies are explicitly defined.

Then, identical derivations (which we omit for brevity) lead to the following equations for the Open Loop Markov Equilibrium path. The steady state is given by the following variant of equation (42).

$$\mathbb{k} \;=\; \mathbf{\Omega}^{-1}\left[\mathbf{Z}\boldsymbol{\beta} - (r\mathbf{I}+\mathbf{\Delta})\mathbf{p}^k\right] \tag{83}$$

while the transition matrix is unchanged from equation (43). The key difference is that the competitive pressure matrix $\mathbf{\Omega}$ takes the more general form

$$\mathbf{\Omega} \;=\; \mathbf{Z}\left[\mathbf{G}\circ\left(\mathbf{1}\mathbf{1}' - \mathbf{I} + \mathbf{D} + \mathbf{\Gamma}\right)\right]\mathbf{Z} \tag{84}$$

it still captures how an increase in capital by any other peer firms affect each firm's incentive to invest in the next period.

It is easy to see that the model presented in Section 2 provides a product market microfoundation such that $\boldsymbol{\beta} = (\mathbf{b}-\mathbf{c})$ and $\mathbf{G} = (\mathbf{I}+\mathbf{\Sigma})$. In this general formulation, the Network Q model can alternatively be extended and applied to several other types of networks. We next provide a couple of concrete examples.

[15] We still require the resulting competitive pressure matrix $\mathbf{\Omega}$ to be symmetric and positive definite.

Input-Output Linkages. We next show how this general form game can be used to accommodate an alternative microfoundation where firms are also linked through supply chain relationships (as opposed to just final consumer demand) and where producers exercise market power in input as well as output market.

To implement this alternative microfoundation we follow Pellegrino (2025) and modify the firms' Leontief production function to incorporate not only labor but other inputs bought from peer firms. Specifically, let o_{nm} be the quantity of input m required per unit of gross output of firm n and let $\mathbf{y}_t^c$ denote the vector of firm output that is destined to final consumption. The consumer utility in equation (5) is then re-written in terms of the vector of final output $\mathbf{y}_t^c$ instead of $\mathbf{y}_t$ so that final consumer demand is given by

$$\mathbf{p}_t = \mathbf{b} - (\mathbf{I} + \boldsymbol{\Sigma})\mathbf{y}_t^c \tag{85}$$

Collecting all o_{nm} inside the $N \times N$ matrix $\mathbf{O}$, we can write the following linear relationship between final output and total output:

$$\mathbf{y}_t^c = (\mathbf{I} - \mathbf{O})' \mathbf{y}_t \tag{86}$$

The operating profits of firm n are given by:

$$\pi_n(\mathbf{k}_t) = \left[p_n(\mathbf{k}_t) - c_n - \sum_{m=1}^{N} o_{nm}\, p_m(\mathbf{k}_t) \right] y_n(\mathbf{k}_t) \tag{87}$$

By plugging the inverse demand function (85) inside the operating profit function (87) we obtain the following expressions for $\boldsymbol{\beta}$ and $\mathbf{G}$:

$$\boldsymbol{\beta} = (\mathbf{I} - \mathbf{O})\mathbf{b} - \mathbf{c} \qquad \mathbf{G} = (\mathbf{I} - \mathbf{O})(\mathbf{I} + \boldsymbol{\Sigma})(\mathbf{I} - \mathbf{O})' \tag{88}$$

the network $\mathbf{G}$ now combines both product market as well as input-output linkages. This extended version of the model predicts that the input-output network affects the long-term level distribution of firms' capital in steady state through vector $\boldsymbol{\beta}$, as well as the transition dynamics of aggregate investment through the matrix $\boldsymbol{\Omega}$, which in turn depends on $\mathbf{G}$.

More Complex Ownership Structures. Another relevant extension involves relaxing the binary nature of the ownership matrix $\boldsymbol{\Gamma}$. In the baseline model, the entry γ_{mn} of this matrix take values of either 0 or 1, indicating whether business units m and n are owned by the same firm. By allowing γ_{mn} to take continuous values between 0 and 1, the model can capture more nuanced ownership and control arrangements through matrix $\boldsymbol{\Omega}$. For instance, partial cross-holdings between firms can be modeled by setting γ_{mn} to the fraction of firm n owned by the entity controlling firm m. This extension naturally accommodates the analysis of cross-shareholdings, joint ventures, and strategic alliances where firms partially internalize spillovers without full common ownership.[16]

Exploiting the case the elements γ_{mn} range between 0 and 1, the model also lends itself to studying common ownership structures, where institutional investors hold significant stakes in multiple competing firms. Following Ederer and Pellegrino (2025), the degree of common ownership can be captured through appropriate calibration of the $\boldsymbol{\Gamma}$ matrix, allowing the analysis of how institutional ownership concentration affects investment incentives and market outcomes. By inspection of ma-

[16] If ownership stakes are not reciprocal ($\gamma_{mn} \neq \gamma_{nm}$), $\boldsymbol{\Omega}$ becomes asymmetric: the interior solution applies as long as $\boldsymbol{\Omega}$ is diagonalizable with real, positive eigenvalues, but the potential game formulation that we use to compute constrained steady states requires symmetry.

trix $\mathbf{\Omega}$, we infer that common ownership, cross-shareholdings, join ventures and strategic alliances will all affect the OSLME path.

These extensions preserve the model's tractability while significantly expanding its scope for analyzing complex organizational forms and market structures. The key insight remains that the pattern of strategic interactions—whether through product markets, ownership structures, or input-output relationships—determines both the steady-state allocation of capital and the dynamic adjustment process of firms' stocks of capital.

5.2 Heterogeneous Adjustment Costs and Discount Rates

Our baseline model assumes homogeneous adjustment costs and discount rates across all business units, with θ and r being constant parameters. This subsection demonstrates how the model can be generalized to accommodate firm-specific heterogeneity in these parameters, where each business unit n faces its own adjustment cost parameter θ_n and discount rate r_n. While heterogeneous θ_n are redundant in our setting (as we explained in Section 2, z_n already generates massive firm heterogeneity in adjustment costs), this extension might be useful for alternative empirical applications.

Under this generalization, the investment cost function in equation (12) becomes:

$$\psi_n(k_{nt+1};k_{nt}) \overset{\text{def}}{=} p_n^k i_{nt} + \frac{\theta_n}{2}(k_{nt+1}-k_{nt})^2 \tag{89}$$

The main cost of this added heterogeneity is that we lose the elegant closed-form solution for the transition matrix $\mathbf{\Phi}$ presented in equation (53). As mentioned earlier, that expression is obtained as the solution of a Non-Symmetric Algebraic Riccati Equation (NARE), which we derive in Appendix A. It the general case with heterogeneous discount rates and adjustment costs, the NARE is:

$$\mathbf{\Phi}^2 - \left[\mathbf{\Theta}^{-1}\mathbf{\Omega} + (2\mathbf{I}+\mathbf{R})\right]\mathbf{\Phi} + (\mathbf{I}+\mathbf{R}) = \mathbf{0} \tag{90}$$

where $\mathbf{\Theta}$ and $\mathbf{R}$ are diagonal matrices containing the firm-specific adjustment costs and discount rates. This NARE generally lacks a closed-form solution and must be solved numerically through iterative procedures. An added disadvantage of this extension is that $\mathbf{\Phi}$ and $\mathbf{\Omega}$ are no longer simultaneously diagonalizable.

Despite this added computational complexity, the model still retains a remarkable degree of tractability, as well as its fundamental economic insights: product market spillovers continue to generate rich network effects in investment decisions, and the steady-state capital allocation still reflects firms' positions in the product similarity network.

5.3 Bertrand Competition and Sticky Prices

While our baseline model is based on quantity competition (Cournot) and capital adjustment costs, the modeling framework can be adapted to work with competition in prices and nominal rigidities. It is well known that these two classes of model bear many similarities. Here we briefly review a variant of our model based on Bertrand competition with price adjustment costs à la Rotemberg (1982). We start by re-writing the vector of profits in terms of prices

$$\boldsymbol{\pi}_t(\mathbf{p}_t) \;=\; (\mathbf{p}_t - \mathbf{c}) \circ \left[(\mathbf{I}+\mathbf{\Sigma})^{-1}(\mathbf{b}-\mathbf{p}_t)\right] \tag{91}$$

Next, we re-define the adjustment cost function ψ in terms of prices, as opposed to capital stock:

$$\psi(p_{nt+1}, p_{nt}) = \frac{\theta}{2}(p_{nt+1} - p_{nt})^2 \tag{92}$$

As in the Cournot case, business units maximize the present discounted value of of profits, net of adjustment costs. Their optimization leads the following system of stacked first-order conditions:

$$\frac{1}{1+r}\left[(\mathbf{I}+\mathbf{\Sigma})^{-1}(\mathbf{b}-\mathbf{p}_{t+1}) - \mathbf{\Upsilon}\mathbf{p}_{t+1} + \mathbf{\Upsilon}\mathbf{c} + \theta(\mathbf{p}_{t+2}-\mathbf{p}_{t+1})\right] = \theta(\mathbf{p}_{t+1}-\mathbf{p}_t) \tag{93}$$

where $\mathbf{\Upsilon}$ is a diagonal matrix that shares its diagonal entries with the inverse of $(\mathbf{I}+\mathbf{\Sigma})$.

As for the Cournot case, we posit a linear Stable Open Loop Markov equilibrium. Here the policy functions are next-period prices that are linear in current prices (as opposed to next period capital stocks that are linear in current capital stocks):

$$\mathbf{p}_{t+1} - \mathbb{p} = \mathbf{\Phi}(\mathbf{p}_t - \mathbb{p}) \tag{94}$$

By setting $\mathbf{p}_{t+2} = \mathbf{p}_{t+1} = \mathbb{p}$ we find the OSLME steady-state price vector.

$$\mathbb{p} - \mathbf{c} = [\mathbf{I} + (\mathbf{I}+\mathbf{\Sigma})\mathbf{\Upsilon}]^{-1}(\mathbf{b}-\mathbf{c}) \tag{95}$$

As in Section 2, the equilibrium transition matrix is obtained as the solution of a Non-symmetric Algebraic Riccati Equation (NARE):

$$\mathbf{\Phi} = \frac{1}{2}\mathcal{V} - \sqrt[+]{\frac{1}{4}\mathcal{V}^2 - (1+r)\mathbf{I}} \qquad \text{where} \qquad \mathcal{V} \overset{\text{def}}{=} \frac{1}{\theta}\left(\mathbf{1}\mathbf{1}' + \mathbf{I}\right) \circ (\mathbf{I}+\mathbf{\Sigma})^{-1} + (2+r)\mathbf{I} \tag{96}$$

This Bertrand variant demonstrates the versatility of our Network Q framework: by substituting price adjustment costs for capital adjustment costs and strategic price-setting for quantity competition, we obtain an analogous dynamic equilibrium with the same mathematical structure, suggesting that our core insights about network effects and strategic spillovers extend beyond the specific modeling choices of our baseline Cournot specification. Consistent with Wang and Werning (2022), this price-based formulation opens the door to additional applications in monetary economics, where the interaction between nominal rigidities and strategic firm behavior could provide new insights into the transmission of monetary policy shocks.

5.4 Sensitivity Analysis

It is natural to ask how the quantitative results in Section 4 vary under alternative calibration strategies. Specifically, the adjustment cost parameter θ and the discount rate r play a central role in our model: the former governs the speed at which firms reallocate capital in response to shocks, while the latter determines how forward-looking firms are and, consequently, how strongly future competitive conditions shape current investment decisions. Hence, our core quantitative findings and counterfactual exercises should be sensitive to the calibration of these parameters.

We address this concern systematically in Appendix C, where we reproduce all the figures from Sections 3 and 4 under three alternative specifications. In Appendix C.1, we relax the assumption that θ is constant over time and instead allow it to be calibrated independently for each year, accommo-

dating potential time variation in the frictions governing capital reallocation. In Appendix C.2, we replace our baseline discount rate with the 10-year US Treasury rate plus 5 percentage points. In Appendix C.3, we use analyst-based estimates of the implied cost of capital from Lee, So, and Wang (2021) as an alternative measure of r. In Appendix C.4, we multiply the baseline estimate of θ by 1.5. Across all four specifications, the qualitative patterns documented in our baseline calibration remain robust.

As an additional exercise and for the sake of completeness, we also winsorize the input variables of our calibration at 1% on each tail of the distribution and present the associated results in Appendix C.5. This is common practice in the neoclassical literature, which explicitly removes outliers with the goal of explaining the behavior of the average firm in the working sample. Note however that while qualitative patterns are also preserved in this case, we refrain from this practice in our baseline calibration because winsorization mechanically and arbitrarily caps the values reported by the largest and most important firms in our working sample. Our model is estimated at the firm level, and unlike previous studies we require unrestricted cross sectional variation in revenues, total capital and firm value to derive accurate inferences on key objects such as the degree of concentration or the cross-sectional distribution of markups.

6 Conclusions

In this paper we have introduced a new dynamic model of corporate investment in oligopolistic product markets, extending the neoclassical Q theory of capital to a multi-firm, multi-product, fully-structural model. By embedding an hedonic demand system into a neoclassical investment setting, we endogenize markups and characterize in matrix form the network of investment spillovers. After obtaining an explicit solution for firms' investment policies, we then apply the model to US firm-level data: our product market network features thousands of firms and dozens of millions of firm pairs.

The quantitative analyses in the paper illustrate the multiple practical applications of the model, as a tool for both researchers and policy makers alike. Specifically, we make inferences on the aggregate effects on investment of specific corporate events—such as cartels or mergers—and run exercises that are relevant for policy-makers—such as studying the impact of a decrease in aggregate discount rates on markups and concentration, when firms compete in an oligopoly.

Combined together, the theoretical predictions and the quantitative findings of our paper confirm that product market competition is a key force driving investment and capital allocation, both at the micro and macro levels. One key insight of our product market network approach to Q-theory is to show how firms' strategic behavior (at the micro level) ultimately affects important macro objects such as aggregate capital formation or the distribution of markups.

Our network Q model can be extended and applied to many types of networks. These alternative networks include, but are not limited to, input-output linkages, common ownership, and strategic interactions in production technologies and input markets. The many advantages and potential extensions of our model suggest a wide array of future applications in multiple fields of economics, including macroeconomics, corporate finance, and international economics.

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ONLINE APPENDIX:

DYNAMIC INVESTMENT AND PRODUCT MARKET RIVALRY: THE NETWORK Q MODEL

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A Additional Proofs

Proof of Proposition 1 (continued). We begin by explicitly solving the feedback effects internalized by managers in the CSLME. We then move on to formally deriving the steady state capital and transition matrix for all types of SLME, using a common analytical approach.

As a first step, we solve for the feedback effect of the CSLME investment strategy of a specific business unit n, using a forward iteration of the linear investment policy in equation (26):

$$\frac{\partial v_n(\mathbf{k}_{t+2})}{\partial \mathbf{k}_{-n,t+2}} \cdot \frac{\partial \mathbf{k}_{-n,t+2}}{\partial k_{n,t+1}} = -\sum_{T=t+2}^{\infty} (1+r)^{-(T-t-2)} h_{nT} k_{nT} \tag{97}$$

where the scalar h_{nT} aggregates over peer reactions at time T, to infer how the product market network jointly reacts to the investment of manager n in such period. Hence, we define vectors and matrices that implicitly remove business unit n from the computation of aggregate peer reactions, namely:

$$h_{nT} \overset{\text{def}}{=} \boldsymbol{\omega}_{n,-n} \boldsymbol{\Phi}_{-n,-n}^{T-t-2} \boldsymbol{\varphi}_{-n,n} \tag{98}$$

where $\boldsymbol{\omega}_{n,-n}$ is the row vector that is obtained by deleting column n and all rows except n from the competitive pressure matrix $\boldsymbol{\Omega}$; $\boldsymbol{\Phi}_{-n,-n}$ is the sub-matrix of the transition matrix $\boldsymbol{\Phi}$ that excludes the n^{th} row and the n^{th} column; and $\boldsymbol{\varphi}_{-n,n}$ is the column vector that is obtained by deleting row n and all columns except n from the transition matrix $\boldsymbol{\Phi}$.

We next define the vector $\mathbf{h}_t$ and matrix $\mathbf{H}_T$ as follows:

$$\mathbf{h}_t \overset{\text{def}}{=} \begin{bmatrix} h_{1t} & h_{2t} & \dots & h_{Nt} \end{bmatrix} \qquad \mathbf{H}_t \overset{\text{def}}{=} \text{diag}(\mathbf{h}_t) \tag{99}$$

The diagonal of matrix $\mathbf{H}_T$ thus collects the feedback effects associated with the investment decision of each business unit n at time T. Having defined matrix $\mathbf{H}_T$, we move next to express in a general form the system of stacked marginal Q as:

$$\mathbf{q}(\mathbf{k}_{t+1}) = \mathbf{Z}(\mathbf{b}-\mathbf{c}) - \boldsymbol{\Omega}\mathbf{k}_{t+1} + (\mathbf{I}-\boldsymbol{\Delta})\mathbf{p}^k + \theta(\mathbf{k}_{t+2}-\mathbf{k}_{t+1}) - \mathbb{1}_{\text{CL}} \sum_{T=t+2}^{\infty} \frac{\mathbf{H}_T \mathbf{k}_T}{(1+r)^{(T-t-1)}} \tag{100}$$

where feedback effects are only relevant to managers in the case of the CSLME, for which the dummy variable $\mathbb{1}_{\text{CL}}$ is equal to one. The dummy variable $\mathbb{1}_{\text{CL}}$ equals instead zero (and therefore shuts down feedback effects) in the OSLME.

Evaluating the first order condition at the steady state, we then obtain

$$\frac{1}{1+r}\left[\mathbf{Z}(\mathbf{b}-\mathbf{c})-\mathbf{\Omega}\mathbb{k}+(\mathbf{I}-\mathbf{\Delta})\mathbf{p}^{k}-\mathbb{1}_{\text{CL}}\mathbf{S}_{1}(\mathbf{\Phi})\mathbb{k}\right] \;=\; \mathbf{p}^{k} \tag{101}$$

where $\mathbf{S}_1(\mathbf{\Phi})$ is a diagonal matrix that is the infinite discounted sum of $\mathbf{H}_T$, and depends on the transition matrix $\mathbf{\Phi}$:

$$\mathbf{S}_{1}(\mathbf{\Phi}) \;=\; \sum_{T=t+2}^{\infty}\frac{\mathbf{H}_{T}}{(1+r)^{(T-t-1)}} \tag{102}$$

We can then solve for the steady state capital vector $\mathbb{k}$ as shown in Proposition 1:

$$\mathbb{k} \;=\; [\mathbf{\Omega}+\mathbb{1}_{\text{CL}}\mathbf{S}_{1}(\mathbf{\Phi})]^{-1}\left[\mathbf{Z}(\mathbf{b}-\mathbf{c})-(r\,\mathbf{I}+\mathbf{\Delta})\mathbf{p}^{k}\right] \tag{103}$$

We proceed to determine the equilibrium transition matrix $\mathbf{\Phi}$. As a preliminary step, we define the infinite sum matrix:

$$\mathbf{S}_{2}(\mathbf{\Phi}) \;=\; \sum_{T=t+2}^{\infty}\frac{\mathbf{H}_{T}\mathbf{\Phi}^{T-t-2}}{(1+r)^{T-t-1}} \tag{104}$$

Evaluating the first order condition in terms of differences with respect to the steady-state, we next obtain:

$$-\mathbf{\Omega}(\mathbf{k}_{t+1}-\mathbb{k})+\theta(\mathbf{k}_{t+2}-\mathbf{k}_{t+1})-\mathbb{1}_{\text{CL}}\cdot\mathbf{S}_{2}(\mathbf{\Phi})\cdot\mathbf{\Phi}(\mathbf{k}_{t+1}-\mathbb{k})=\theta(1+r)(\mathbf{k}_{t+1}-\mathbf{k}_{t}) \tag{105}$$

Using our guess on the linear investment rule, we have that

$$(\mathbf{k}_{t+2}-\mathbf{k}_{t+1}) \;=\; (\mathbf{\Phi}-\mathbf{I})(\mathbf{k}_{t+1}-\mathbb{k}) \tag{106}$$

Using the previous result, we re-write the first-order condition of investment as:

$$-\mathbf{\Omega}(\mathbf{k}_{t+1}-\mathbb{k})+\theta(\mathbf{\Phi}-\mathbf{I})(\mathbf{k}_{t+1}-\mathbb{k})-\mathbb{1}_{\text{CL}}\cdot\mathbf{S}_{2}(\mathbf{\Phi})\cdot\mathbf{\Phi}(\mathbf{k}_{t+1}-\mathbb{k})=\theta(1+r)\left(\mathbf{I}-\mathbf{\Phi}^{-1}\right)(\mathbf{k}_{t+1}-\mathbb{k}) \tag{107}$$

Re-ordering terms, we conclude that:

$$-\frac{1}{\theta}\mathbf{\Omega}+(\mathbf{\Phi}-\mathbf{I})-\frac{\mathbb{1}_{\text{CL}}}{\theta}\cdot\mathbf{S}_{2}(\mathbf{\Phi})\cdot\mathbf{\Phi}=(1+r)\left(\mathbf{I}-\mathbf{\Phi}^{-1}\right) \tag{108}$$

Multiplying by $\mathbf{\Phi}$ and rearranging, we obtain the following matrix equation, as shown in Proposition 1, which implicitly pins down the equilibrium transition matrix $\mathbf{\Phi}$::

$$\left[\mathbf{I}-\frac{\mathbb{1}_{\text{CL}}}{\theta}\cdot\mathbf{S}_{2}(\mathbf{\Phi})\right]\mathbf{\Phi}^{2}-\left[\frac{1}{\theta}\mathbf{\Omega}+(2+r)\mathbf{I}\right]\mathbf{\Phi}+(1+r)\mathbf{I}=\mathbf{0} \tag{109}$$

Having solved for both the steady state capital vector $\mathbb{k}$ and the transition matrix $\mathbf{\Phi}$ as shown in Proposition 1, we conclude the proof by verifying the existence of the associated SLME.

The verification consists in two steps. As a first step in the analysis, we prove the value function of

each business unit n is indeed maximized when evaluated at the equilibrium investment strategies. Given the state variable $\mathbf{k}_t$ the optimization problem of each business unit n involves a scalar maximization over k_{nt+1} Hence, to verify optimality, we require strict concavity of the value function of business unit n with respect to k_{nt+1}.

Given the stacked equations in (100), we derive the marginal value of capital of each business unit n with respect to k_{nt+1}. to obtain:

$$\frac{\partial q_{nn}(\mathbf{k}_{t+1})}{\partial k_{nt+1}} = -\mathbf{\Omega}_{nn} + \theta(\varphi_{nn} - 1) - \mathbb{1}_{\text{CL}}[\mathbf{S}_2(\mathbf{\Phi})\mathbf{\Phi}]_{nn} \tag{110}$$

Simultaneously, using the expression for matrix $\mathbf{\Phi}$ as defined in equation (108), and re-ordering terms, we obtain:

$$-\theta\mathbf{I} + \frac{1}{1+r}[-\mathbf{\Omega} + \theta(\mathbf{\Phi} - \mathbf{I}) - \mathbb{1}_{\text{CL}}\mathbf{S}_2(\mathbf{\Phi})\mathbf{\Phi}] = -\theta\mathbf{\Phi}^{-1} \tag{111}$$

Taking the nn^{th} element from the matrix above, we infer:

$$-\theta + \frac{1}{(1+r)}\left[-\mathbf{\Omega}_{nn} + \theta(\varphi_{nn} - 1) - \mathbb{1}_{\text{CL}}[\mathbf{S}_2(\mathbf{\Phi})\mathbf{\Phi}]_{nn}\right] = -\theta\mathbf{\Phi}_{nn}^{-1} \tag{112}$$

It follows that the value function of each business unit is concave in the direction of k_{nt+1} for either the CSLME or OSLME of the model as long as:

$$-\theta + \frac{1}{(1+r)}\frac{\partial q_{nn}(\mathbf{k}_{t+1})}{\partial k_{nt+1}} = -\theta\mathbf{\Phi}_{nn}^{-1} < 0 \tag{113}$$

where the condition $\mathbf{\Phi}_{nn}^{-1} > 0$ holds by assumption. Heuristically, the condition implies that the serial correlation or persistence of a business unit's own capital is positive in equilibrium.

As a second step to ensure existence, we require that the transition matrix $\mathbf{\Phi}$ is stable. By assumption, any SLME is defined such that the transition matrix $\mathbf{\Phi}$ has a spectral radius lower than one. The condition then implies that the eigenvalues of the transition matrix $\mathbf{\Phi}$ are below one. We confirm the latter implication with an explicit solution for the eigenvalues of matrix $\mathbf{\Phi}$ for the OSLME of the model, in the proof to Proposition 2 below.

In the case of the OSLME, the requirement that $\mathbf{\Phi}$ has a spectral radius lower than one ensures that the transversality condition of the model holds, and therefore the value functions of all business units are bounded. In the alternative case of the CSLME, we further require that $\mathbf{\Phi}_{-n,-n}$ has spectral radius lower than $1+r$, for any business unit $n = 1, 2, ..., N$. The condition ensures that the present value of feedback effects is finite. □

Proof of Proposition 2 (continued). We derive most of the proof using matrices $\mathbf{\Theta}$ and $\mathbf{R}$ in general form, to accommodate for the cases of heterogeneous adjustment costs and firm-specific discount rates discussed in an extension to the baseline model in Section 5. To obtain the explicit solution in Proposition 2, we further impose below $\mathbf{\Theta} = \theta\mathbf{I}$ and $\mathbf{R} = r\mathbf{I}$.

As a first step, we divide both side by replace $\mathbf{k}_{t+2}$ using the investment rule (26) and re-express the system of stacked first order conditions (35) and imposing $\mathscr{L} = 0$:

$$\begin{aligned}\mathbf{Z}(\mathbf{b}-\mathbf{c}) - \mathbf{\Omega}\mathbf{k}_{t+1} + (\mathbf{I}-\mathbf{\Delta})\mathbf{p}^k + \mathbf{\Theta}[\mathbb{k} + \mathbf{\Phi}(\mathbf{k}_{t+1}-\mathbb{k}) - \mathbf{k}_{t+1}] = \\ = (\mathbf{I}+\mathbf{R})\left[\mathbf{p}^k + \mathbf{\Theta}(\mathbf{k}_{t+1}-\mathbf{k}_t)\right]\end{aligned} \tag{114}$$

and distribute:

$$\begin{aligned}\mathbf{Z}(\mathbf{b}-\mathbf{c}) - \mathbf{\Omega}\mathbf{k}_{t+1} + (\mathbf{I}-\mathbf{\Delta})\mathbf{p}^k + \mathbf{\Theta}(\mathbf{I}-\mathbf{\Phi})\mathbb{k} - \mathbf{\Theta}(\mathbf{I}-\mathbf{\Phi})\mathbf{k}_{t+1} \quad = \\ = \quad (\mathbf{I}+\mathbf{R})\mathbf{p}^k + (\mathbf{I}+\mathbf{R})\mathbf{\Theta}\mathbf{k}_{t+1} - (\mathbf{I}+\mathbf{R})\mathbf{\Theta}\mathbf{k}_t\end{aligned} \tag{115}$$

We now rearrange the equation so that all the multiples of $\mathbf{k}_{t+1}$ are on the left:

$$\begin{aligned}(\mathbf{I}+\mathbf{R})\mathbf{\Theta}\mathbf{k}_{t+1} + \mathbf{\Omega}\mathbf{k}_{t+1} + \mathbf{\Theta}(\mathbf{I}-\mathbf{\Phi})\mathbf{k}_{t+1} \quad = \\ = \quad \mathbf{Z}(\mathbf{b}-\mathbf{c}) - (\mathbf{R}+\mathbf{\Delta})\mathbf{p}^k + \mathbf{\Theta}(\mathbf{I}-\mathbf{\Phi})\mathbb{k} + (\mathbf{I}+\mathbf{R})\mathbf{\Theta}\mathbf{k}_t\end{aligned} \tag{116}$$

next, we rewrite the right hand side of the equation in terms of $\mathbf{\Omega}\mathbb{k}$ on the left hand side:

$$\begin{aligned}(\mathbf{I}+\mathbf{R})\mathbf{\Theta}\mathbf{k}_{t+1} + \mathbf{\Omega}\mathbf{k}_{t+1} + \mathbf{\Theta}(\mathbf{I}-\mathbf{\Phi})\mathbf{k}_{t+1} \quad = \\ = \quad \mathbf{\Omega}\mathbb{k} + \mathbf{\Theta}(\mathbf{I}-\mathbf{\Phi})\mathbb{k} + (\mathbf{I}+\mathbf{R})\mathbf{\Theta}\mathbf{k}_t\end{aligned} \tag{117}$$

Reordering terms appropriately we obtain:

$$[\mathbf{\Omega} + \mathbf{\Theta}(\mathbf{I}-\mathbf{\Phi}) + (\mathbf{I}+\mathbf{R})\mathbf{\Theta}](\mathbf{k}_{t+1}-\mathbb{k}) = (\mathbf{I}+\mathbf{R})\mathbf{\Theta}(\mathbf{k}_t-\mathbb{k}) \tag{118}$$

if we pre-multiply by $\mathbf{\Theta}^{-1}$ and invert the matrix in square brackets this equation can then be re-written as:

$$(\mathbf{k}_{t+1}-\mathbb{k}) = \left[\mathbf{\Theta}^{-1}\mathbf{\Omega} + (2\mathbf{I}+\mathbf{R})\mathbf{I} - \mathbf{\Phi}\right]^{-1}(\mathbf{I}+\mathbf{R})(\mathbf{k}_t-\mathbb{k}) \tag{119}$$

Then, comparing with (26), we see that $\mathbf{\Phi}$ must satisfy:

$$\left[\mathbf{\Theta}^{-1}\mathbf{\Omega} + (2\mathbf{I}+\mathbf{R})\mathbf{I} - \mathbf{\Phi}\right]^{-1}(\mathbf{I}+\mathbf{R}) = \mathbf{\Phi} \tag{120}$$

rearranging we obtain:

$$\mathbf{\Phi}^2 - \left[\mathbf{\Theta}^{-1}\mathbf{\Omega} + (2\mathbf{I}+\mathbf{R})\right]\mathbf{\Phi} + (\mathbf{I}+\mathbf{R}) = \mathbf{0} \tag{121}$$

Equation (121) is a Non-symmetric Algebraic Riccati Equation (NARE), a variation of the Algebraic Riccati Equation (ARE) that is commonly encountered in linear-quadratic regulator problems from the optimal control literature. In general, neither the canonical AREs from linear quadratic optimal control, nor the NARE that we derived in (121) admit a closed-form solution. It is also known that they can admit multiple solutions, or none.

Our specific version is a quadratic matrix equation of the type $\mathbf{X}^2 + \mathbf{M}_1\mathbf{X} + \mathbf{M}_2 = \mathbf{0}$. This equation was first studied by Sylvester (1909)[17] He proved that when $\mathbf{M}_1\mathbf{M}_2 = \mathbf{M}_2\mathbf{M}_1$ and $\mathbf{M}_1^2 - 4\mathbf{M}_2$ has a square root, then all solutions take the form:

$$\mathbf{X} = -\frac{1}{2}\mathbf{M}_1 \pm \sqrt{\frac{1}{4}\mathbf{M}_1^2 - \mathbf{M}_2} \tag{122}$$

[17] See also the review of Higham (2014).

This condition is satisfied in our case if $\mathbf{\Theta} = \theta\mathbf{I}$ and $\mathbf{R} = r\mathbf{I}$. Applying this solution to (121) yields:

$$\mathbf{\Phi} = \frac{1}{2}\left[\frac{1}{\theta}\mathbf{\Omega} + (2+r)\mathbf{I}\right] \pm \sqrt{\frac{1}{4}\left[\frac{1}{\theta}\mathbf{\Omega} + (2+r)\mathbf{I}\right]^2 - (1+r)\mathbf{I}} \tag{123}$$

As discussed by Higham (2008), the square root of a matrix with positive eigenvalues may admit up to $2^{\mathscr{E}}$ solutions, where $\mathscr{E}$ is the number of unique eigenvalues of the radicand. This is because the square roots of a matrix can be found by taking the square root of its eigenvalues (which in our setting are guaranteed to be positive). Those square roots in turn each admit both a positive and a negative solution. Considering all sign combinations we obtain $2^{\mathscr{E}}$ potential solutions.

We show next that focusing on the unique principal solution (with all positive eigenvalues) preceded by a leading minus sign is both necessary and sufficient for $\mathbf{\Phi}$ to be stable. To do so, we exploit the fact that all candidate solutions $\mathbf{\Phi}$ are simultaneously diagonalizable with $\mathbf{\Omega}$. That is, let $\hat{\omega}_n$ be the n^{th} eigenvalue of $\mathbf{\Omega}$ and let $\mathbf{E}$ be the matrix of eigenvectors of $\mathbf{\Omega}$, so that its spectral decomposition is

$$\mathbf{\Omega} = \mathbf{E}\,\mathrm{diag}(\hat{\boldsymbol{\omega}})\,\mathbf{E}^{-1} \tag{124}$$

also, let $\hat{\varphi}_n$ be the n^{th} eigenvalue of $\mathbf{\Phi}$ .The matrices $\mathbf{\Omega}$ and $\mathbf{\Phi}$ are simultaneously diagonalizable, that is:

$$\mathbf{\Phi} = \mathbf{E}\,\mathrm{diag}\left(\hat{\boldsymbol{\varphi}}\right)\mathbf{E}^{-1} \tag{125}$$

and their respective n^{th} eigenvalues are related by the following function, the scalar equivalent of equation (43)

$$\hat{\varphi}_n = \frac{1}{2}\left(\frac{\hat{\omega}_n}{\theta} + 2 + r\right) \pm \sqrt{\frac{1}{4}\left(\frac{\hat{\omega}_n}{\theta} + 2 + r\right)^2 - (1+r)} \tag{126}$$

This is a known result in matrix theory that applies when $\mathbf{\Omega}$ is symmetric (it is by construction) and when $\mathbf{\Phi}$ is a function of $\mathbf{\Omega}$ that results from the concatenation of 1) summation to multiples of identity matrices; 2) scalar multiplications and 3) symmetric matrix power functions. This latter condition can be verified by examining Equation (123).

Because for all $\hat{\omega}_n > 0$, $\theta > 0$ and $r > 0$ we have

$$\frac{1}{2}\left(\frac{\hat{\omega}_n}{\theta} + 2 + r\right) + \sqrt{\frac{1}{4}\left(\frac{\hat{\omega}_n}{\theta} + 2 + r\right)^2 - (1+r)} \geq 1 \tag{127}$$

$$\frac{1}{2}\left(\frac{\hat{\omega}_n}{\theta} + 2 + r\right) - \sqrt{\frac{1}{4}\left(\frac{\hat{\omega}_n}{\theta} + 2 + r\right)^2 - (1+r)} \leq 1 \tag{128}$$

it follows that the unique principal square root matrix, preceded by the minus sign, yields the only stable solution $\mathbf{\Phi}$.

The stable solution in Equation (123) then ensures the transversality condition in Equation (18) holds in equilibrium, because firm value is quadratic in capital and capital converges geometrically to the steady state. □

Proof of Proposition (3). Fix an exogenous finite horizon T. We solve the truncated game by backward induction. In particular, we start by proving that backward induction yields unique linear investment policies

$$\mathbf{k}_{t+1} = \boldsymbol{\iota}_t^T + \boldsymbol{\Phi}_t^T \mathbf{k}_t \tag{129}$$

In the truncated game, at terminal date T the firms' first order conditions become:

$$\frac{1}{1+r}\left[\mathbf{Z}(\mathbf{b}-\mathbf{c}) - \boldsymbol{\Omega}\mathbf{k}_T + (\mathbf{I}-\boldsymbol{\Delta})\mathbf{p}^k\right] = \mathbf{p}^k + \theta(\mathbf{k}_T - \mathbf{k}_{T-1}) \tag{130}$$

because this is a system of linear equations for $\mathbf{k}_T$ in $\mathbf{k}_{T-1}$, linearity is easily established at terminal date T. Substituting (129) into the system of stacked first-order conditions (35) we obtain:

$$\frac{1}{1+r}\left[\mathbf{Z}(\mathbf{b}-\mathbf{c}) - \boldsymbol{\Omega}\mathbf{k}_{t+1} + (\mathbf{I}-\boldsymbol{\Delta})\mathbf{p}^k + \theta\left(\boldsymbol{\iota}_{t+1}^T + \boldsymbol{\Phi}_{t+1}^T\mathbf{k}_{t+1} - \mathbf{k}_{t+1}\right)\right] = \mathbf{p}^k + \theta(\mathbf{k}_{t+1} - \mathbf{k}_t) \tag{131}$$

In so doing, we obtain another system of linear equations for $\mathbf{k}_t$ in $\mathbf{k}_{t-1}$, and have thus established that for all dates $t < T$, investment policies are linear.

The next steps consists of characterizing recursively the investment policies intercepts $\boldsymbol{\iota}_t^T$ and $\boldsymbol{\Phi}_t^T$. By construction (capital is sold at terminal date T) we have that $\boldsymbol{\iota}_T^T = \mathbf{0}$ and $\boldsymbol{\Phi}_T^T = \mathbf{0}\mathbf{0}'$. To obtain $\boldsymbol{\iota}_t^T$ and $\boldsymbol{\Phi}_t^T$ at $t < T$, we collect all terms in $\mathbf{k}_{t+1}$ on the left and all remaining terms on the right:

$$\left\{\boldsymbol{\Omega} + \theta\left[(2+r)\mathbf{I} - \boldsymbol{\Phi}_{t+1}^T\right]\right\}\mathbf{k}_{t+1} = \left[\mathbf{Z}(\mathbf{b}-\mathbf{c}) - (r\mathbf{I}+\boldsymbol{\Delta})\mathbf{p}^k + \theta\boldsymbol{\iota}_{t+1}^T\right] + (1+r)\theta\mathbf{k}_t \tag{132}$$

Since $\boldsymbol{\Omega}$ is symmetric positive definite and $\boldsymbol{\Phi}_{t+1}^T$ has non-negative eigenvalues bounded above by 1 (verified by induction below), the coefficient matrix in square brackets is positive definite, hence invertible. The backward recursion for $\boldsymbol{\iota}_t^T$ and $\boldsymbol{\Phi}_t^T$ therefore yields:

$$\boldsymbol{\Phi}_t^T = (1+r)\left[\frac{1}{\theta}\boldsymbol{\Omega} + (2+r)\mathbf{I} - \boldsymbol{\Phi}_{t+1}^T\right]^{-1}\mathbf{I} \tag{133}$$

$$\boldsymbol{\iota}_t^T = \left\{\boldsymbol{\Omega} + \theta\left[(2+r)\mathbf{I} - \boldsymbol{\Phi}_{t+1}^T\right]\right\}^{-1}\left[\mathbf{Z}(\mathbf{b}-\mathbf{c}) - (r\mathbf{I}+\boldsymbol{\Delta})\mathbf{p}^k + \theta\boldsymbol{\iota}_{t+1}^T\right] \tag{134}$$

The third step consists of proving the point-wise convergence of $\boldsymbol{\Phi}_t^T$. Because $\boldsymbol{\Omega}$ is symmetric positive definite, every iterate $\boldsymbol{\Phi}_t^T$ defined by (133) is simultaneously diagonalizable with $\boldsymbol{\Omega}$. Letting $\hat{\omega}_n > 0$ denote an eigenvalue of $\boldsymbol{\Omega}$ and $\hat{\varphi}_{n,t}^T$ the corresponding eigenvalue of $\boldsymbol{\Phi}_t^T$, the matrix recursion (133) decouples into N independent scalar recursions

$$\hat{\varphi}_{n,t}^T = g_n\left(\hat{\varphi}_{n,t+1}^T\right) \tag{135}$$

where

$$g_n(x) \overset{\text{def}}{=} (1+r)\left(\frac{\hat{\omega}_n}{\theta} + 2 + r - x\right)^{-1} \tag{136}$$

with terminal condition $\hat{\varphi}_{n,T}^T = 0$. Therefore we have

$$\hat{\varphi}_{n,t}^T = g_n^{T-t}(0) \tag{137}$$

and the fixed point $g_n(x) = x$ solves the scalar analogue of the NARE underlying Proposition 2. The unique fixed point in the unit circle $\hat{\varphi}_n \in (0,1)$ is given by equation (49):

$$\hat{\varphi}_n \;=\; \frac{1}{2}\left(\frac{\hat{\omega}_n}{\theta}+2+r\right)-\sqrt{\frac{1}{4}\left(\frac{\hat{\omega}_n}{\theta}+2+r\right)^2-(1+r)} \tag{138}$$

Because we obtain each term of the sequence $\hat{\varphi}_{n,t}^T$ by applying the mapping g_n iteratively, starting at 0, it follows that if we can prove that g_n maps $[0,\hat{\varphi}_n]$ into itself and is a contraction on this interval, the contraction mapping theorem will imply that

$$\lim_{T\to\infty}\hat{\varphi}_{n,t}^T \;=\; \hat{\varphi}_n \qquad \forall\, t<T \tag{139}$$

We claim that indeed g_n maps $[0,\hat{\varphi}_n]$ into itself and is a contraction on this interval. First, $g_n(0) = (1+r)\left(\frac{\hat{\omega}_n}{\theta}+2+r\right)^{-1} > 0$ and $g_n(\hat{\varphi}_n) = \hat{\varphi}_n$ (by the fixed-point definition). Since g_n is strictly increasing on $(-\infty,1)$ and $\hat{\varphi}_n < 1$, the interval $[0,\hat{\varphi}_n]$ is mapped into itself. Next, notice that

$$g_n'(x) \;=\; (1+r)\left(\frac{\hat{\omega}_n}{\theta}+2+r-x\right)^{-2} \tag{140}$$

is increasing, so the maximal slope on $[0,\hat{\varphi}_n]$ is attained at $x = \hat{\varphi}_n$, where

$$g_n'\big(\hat{\varphi}_n\big) \;=\; (1+r)\left(\frac{\hat{\omega}_n}{\theta}+2+r-\hat{\varphi}_n\right)^{-2} \;=\; \frac{\hat{\varphi}_n^2}{1+r} \;<\; 1 \tag{141}$$

where the second equality uses the fixed-point relation

$$\hat{\varphi}_n\left(\frac{\hat{\omega}_n}{\theta}+2+r-\hat{\varphi}_n\right) = 1+r \tag{142}$$

and the inequality follows from $\hat{\varphi}_n \in (0,1)$. Hence g_n is a contraction on $[0,\hat{\varphi}_n]$ with Lipschitz constant $\hat{\varphi}_n^2/(1+r) < 1$. Reassembling the eigendecomposition we have that

$$\lim_{T\to\infty}\boldsymbol{\Phi}_t^T \;=\; \lim_{T\to\infty}\mathbf{E}\,\mathrm{diag}\Big(\hat{\boldsymbol{\varphi}}^T\Big)\mathbf{E}' \;=\; \mathbf{E}\,\mathrm{diag}\big(\hat{\boldsymbol{\varphi}}\big)\mathbf{E}' \;=\; \boldsymbol{\Phi} \tag{143}$$

The last step in the proof consists of proving the convergence of $\boldsymbol{\iota}_t^T$. Substituting the steady-state identity $\mathbf{Z}(\mathbf{b}-\mathbf{c})-(r\,\mathbf{I}+\boldsymbol{\Delta})\mathbf{p}^k = \boldsymbol{\Omega}\mathbb{k}$ from equation (42) into the intercept recursion (134):

$$\boldsymbol{\iota}_t^T \;=\; \Big[\boldsymbol{\Omega}+\theta\big((2+r)\mathbf{I}-\boldsymbol{\Phi}_{t+1}^T\big)\Big]^{-1}\Big[\boldsymbol{\Omega}\,\mathbb{k}+\theta\,\boldsymbol{\iota}_{t+1}^T\Big] \tag{144}$$

Consider first the stationary counterpart of the recursion (144), obtained by replacing $\boldsymbol{\Phi}_{t+1}^T$ with its limit $\boldsymbol{\Phi}$ and both $\boldsymbol{\iota}_t^T$ and $\boldsymbol{\iota}_{t+1}^T$ with a stationary value $\boldsymbol{\iota}$:

$$\boldsymbol{\iota} \;=\; \{\boldsymbol{\Omega}+\theta\,[(2+r)\mathbf{I}-\boldsymbol{\Phi}]\}^{-1}(\boldsymbol{\Omega}\,\mathbb{k}+\theta\,\boldsymbol{\iota}) \tag{145}$$

which can be conveniently rearranged as:

$$\boldsymbol{\iota} \;=\; \{\boldsymbol{\Omega}+\theta\,[(1+r)\mathbf{I}-\boldsymbol{\Phi}]\}^{-1}\boldsymbol{\Omega}\,\mathbb{k} \tag{146}$$

Now, recall that $\mathbf{\Phi}$ satisfies the NARE (121) which (for $\mathbf{\Theta} = \theta\mathbf{I}$ and $\mathbf{R} = r\mathbf{I}$) can be rearranged as

$$\{\mathbf{\Omega} + \theta[(1+r)\mathbf{I} - \mathbf{\Phi}]\}^{-1}\mathbf{\Omega} = \mathbf{I} - \mathbf{\Phi} \tag{147}$$

implying that $\boldsymbol{\iota} = (\mathbf{I} - \mathbf{\Phi})\mathbb{k}$ solves the equation. Since the matrix in curly brackets in (146) is positive definite (we verified it has positive eigenvalues), this solution is unique. It remains to establish that $\boldsymbol{\iota}_t^T$ converges to this fixed point. As every matrix in equation (144) is simultaneously diagonalizable with $\mathbf{\Omega}$, the recursion decouples in the eigenbasis of $\mathbf{\Omega}$: denoting by $\hat{\imath}_{n,t}^T$, $\hat{k}_n$ and $\hat{\imath}_n$ the n-th entries of $\mathbf{E}'\boldsymbol{\iota}_t^T$, $\mathbf{E}'\mathbb{k}$ and $\mathbf{E}'(\mathbf{I} - \mathbf{\Phi})\mathbb{k}$, we have:

$$\hat{\imath}_{n,t}^T = \frac{\hat{\omega}_n \hat{k}_n + \theta\,\hat{\imath}_{n,t+1}^T}{\hat{\omega}_n + \theta(2+r) - \theta\,\hat{\varphi}_{n,t+1}^T} \qquad \text{with} \qquad \hat{\imath}_{n,T}^T = 0 \tag{148}$$

Because $\hat{\varphi}_{n,t+1}^T \in [0, \hat{\varphi}_n] \subset [0,1)$, the coefficient multiplying $\hat{\imath}_{n,t+1}^T$ is at most $\theta/[\hat{\omega}_n + \theta(1+r)] < (1+r)^{-1} < 1$, uniformly in n, t and T. Writing $e_{n,t}^T \overset{\text{def}}{=} \hat{\imath}_{n,t}^T - \hat{\imath}_n$ and subtracting the stationary recursion from the display above:

$$\left|e_{n,t}^T\right| \leq \frac{1}{1+r}\left|e_{n,t+1}^T\right| + \delta_{n,t}^T \tag{149}$$

where $\delta_{n,t}^T$ collects the terms proportional to $\hat{\varphi}_{n,t+1}^T - \hat{\varphi}_n$ and therefore vanishes geometrically in $T - t$ by the contraction property established for the slopes. Iterating the inequality backward from the terminal condition and letting $T \to \infty$ at any fixed t yields $e_{n,t}^T \to 0$. Therefore

$$\lim_{T\to\infty} \boldsymbol{\iota}_t^T = (\mathbf{I} - \mathbf{\Phi})\mathbb{k} \tag{150}$$

□

Proof of Proposition (4). Consider first the single-product, symmetric Cournot oligopoly case. Matrix $\mathbf{\Omega}$ is then equal to a $N \times N$ real symmetric matrix whose diagonal entries are all equal to $2z^2$ and whose off-diagonal entries are all equal to $z^2\sigma$, namely:

$$\mathbf{\Omega} = z^2\sigma\mathbf{1}\mathbf{1}' + z^2(2-\sigma)\mathbf{I}. \tag{151}$$

Consider the two terms on the right hand side of the equation. In the first term, matrix $\mathbf{1}\mathbf{1}'$ has rank 1, so its eigenvalues are either equal to N (for the eigenvector $\mathbf{1}$) or 0 (for the $N-1$ eigenvectors orthogonal to vector $\mathbf{1}$). The second term shifts every eigenvalue by $z^2(2-\sigma)$. The symmetric case in which all firms are endowed with the same level of capital is associated with the eigenvector $\mathbf{1}$. Hence, the unique eigenvalue of matrix $\mathbf{\Omega}$ for the symmetric, single product Cournot oligopoly case is equal to $z^2[2 + (N-1)\sigma]$, as stated in equation (50). □

Proof of Proposition 5. Rearrange equation (26) as follows

$$\mathbf{k}_{t+1} - \mathbf{k}_t = (\mathbf{\Phi} - \mathbf{I})(\mathbf{k}_t - \mathbb{k}) \tag{152}$$

and plug it in the definition of $\mathbf{q}(\mathbf{k}_t)$

$$\mathbf{q}(\mathbf{k}_t) \;=\; \mathbf{Z}(\mathbf{b}-\mathbf{c}) - \mathbf{\Omega}\mathbf{k}_t + (\mathbf{I}-\mathbf{\Delta})\mathbf{p}^k + \theta(\mathbf{k}_{t+1}-\mathbf{k}_t) \tag{153}$$

to obtain an expression in terms of the contemporaneous state vector:

$$\mathbf{q}(\mathbf{k}_t) \;=\; \mathbf{Z}(\mathbf{b}-\mathbf{c}) - \mathbf{\Omega}\mathbf{k}_t - \mathbf{\Delta}\mathbf{p}^k + \mathbf{p}^k + \theta(\mathbf{\Phi}-\mathbf{I})(\mathbf{k}_t-\mathbb{k}) \tag{154}$$

Add and subtract $r\mathbf{p}^k$ to obtain:

$$\mathbf{q}(\mathbf{k}_t) \;=\; \left[\mathbf{Z}(\mathbf{b}-\mathbf{c}) - (r\mathbf{I}+\mathbf{\Delta})\mathbf{p}^k\right] - \mathbf{\Omega}\mathbf{k}_t + (1+r)\mathbf{p}^k + \theta(\mathbf{\Phi}-\mathbf{I})(\mathbf{k}_t-\mathbb{k}) \tag{155}$$

rearrange equation (42) as follows:

$$\mathbf{Z}(\mathbf{b}-\mathbf{c}) - (r\mathbf{I}+\mathbf{\Delta})\mathbf{p}^k \;=\; \mathbf{\Omega}\,\mathbb{k} \tag{156}$$

and substitute inside equation (155) to obtain:

$$\mathbf{q}(\mathbf{k}_t) \;=\; \mathbf{\Omega}(\mathbb{k}-\mathbf{k}_t) + (1+r)\mathbf{p}^k + \theta(\mathbf{I}-\mathbf{\Phi})(\mathbb{k}-\mathbf{k}_t) \tag{157}$$

finally, collect terms to obtain the following version of equation (52):

$$\mathbf{q}(\mathbf{k}_t) \;=\; (1+r)\mathbf{p}^k + [\theta(\mathbf{I}-\mathbf{\Phi}) + \mathbf{\Omega}](\mathbb{k}-\mathbf{k}_t) \tag{158}$$

□

Proof of Proposition 6 expand the expression for $\boldsymbol{\pi}(\mathbb{k})$ and collect $\mathbb{K}$:

$$\frac{r}{1+r}\mathbf{v}(\mathbb{k}) \;=\; \mathbb{K}\left[\mathbf{Z}(\mathbf{b}-\mathbf{c}) - \mathbf{Z}(\mathbf{I}+\mathbf{\Sigma})\mathbf{Z}\mathbb{k} - \mathbf{\Delta}\mathbf{p}^k\right] \tag{159}$$

Then rearrange equation (42)

$$\mathbf{Z}(2\mathbf{I}+\mathbf{\Sigma})\mathbf{Z}\mathbb{k} \;=\; \mathbf{Z}(\mathbf{b}-\mathbf{c}) - (r\mathbf{I}+\mathbf{\Delta})\mathbf{p}^k \tag{160}$$

to obtain the term in square brackets in equation

$$\mathbf{Z}(\mathbf{b}-\mathbf{c}) - \mathbf{Z}(\mathbf{I}+\mathbf{\Sigma})\mathbf{Z}\mathbb{k} - \mathbf{\Delta}\mathbf{p}^k \;=\; r\mathbf{p}^k + \mathbf{Z}^2\mathbb{k} \tag{161}$$

And replace it inside equation (159) to obtain:

$$\frac{r}{1+r}\mathbf{v}(\mathbb{k}) \quad = \quad r\,\mathbf{P}^k\,\mathbb{k} + \mathbb{K}\mathbf{Z}^2\mathbb{k} \tag{162}$$

Then divide both sides by $r/(1+r)$. □

Proof of Proposition 9. Rewrite the first order condition (35) as

$$\mathbf{q}(\mathbf{k}_t) \;=\; (1+r)\Big[\mathbf{p}^k + \theta(\mathbf{k}_t - \mathbf{k}_{t-1})\Big] \tag{163}$$

next, use the fact that

$$\mathbf{k}_t - \mathbf{k}_{t-1} \;=\; \mathbf{\Phi}^{-1}(\mathbf{k}_{t+1} - \mathbf{k}_t) \tag{164}$$

substitute inside 163 and divide by $(1+r)$ to obtain 65. □

Proof of Proposition 10. Consider the derivation of steady state markups in Equation (68). Multiply both sides of the steady state level of capital by $\mathbf{Z}$

$$\mathbf{y}(\mathbb{k}) \;=\; \mathbf{Z}\mathbf{\Omega}^{-1}\Big[\mathbf{Z}(\mathbf{b}-\mathbf{c}) - (r\mathbf{I}+\mathbf{\Delta})\mathbf{p}^k\Big] \tag{165}$$

substituting the expression in the inverse demand function $\mathbf{p}(\mathbb{k})$:

$$\mathbf{b} - (\mathbf{I}+\mathbf{\Sigma})\mathbf{y}(\mathbb{k}) \;=\; \mathbf{b} - (\mathbf{I}+\mathbf{\Sigma})\mathbf{Z}\mathbf{\Omega}^{-1}\Big[\mathbf{Z}(\mathbf{b}-\mathbf{c}) - (r\mathbf{I}+\mathbf{\Delta})\mathbf{p}^k\Big] \tag{166}$$

finally we multiply both sides by $\mathbf{C}^{-1}$ and rearrange to obtain Equation (68):

$$\boldsymbol{\mu}(\mathbb{k}) \;=\; \mathbf{C}^{-1}\mathbf{b} + \mathbf{C}^{-1}(\mathbf{I}+\mathbf{\Sigma})\mathbf{Z}\mathbf{\Omega}^{-1}\Big[(r\mathbf{I}+\mathbf{\Delta})\mathbf{p}^k - \mathbf{Z}(\mathbf{b}-\mathbf{c})\Big] \tag{167}$$

The solution outside steady state in (67) follows from combining the definition of markup with the law of motion of investment in equilibrium, and rearranging terms. □

Proof of Proposition 11. Consider the law of motion of capital in any SLME:

$$\boldsymbol{i}(\mathbf{k}_t) = \mathbf{\Delta}\mathbf{k}_t + (\mathbf{\Phi}-\mathbf{I})(\mathbf{k}_t - \mathbb{k}) \tag{168}$$

Considering the dependence of matrix $\mathbf{\Phi}$ and vector $\mathbb{k}$ on the discount rate, and holding the current state $\mathbf{k}_t$ fixed, we have that:

$$\frac{\mathrm{d}\,\boldsymbol{i}(\mathbf{k}_t)}{\mathrm{d}\,r} \;=\; (\mathbf{I}-\mathbf{\Phi})\,\frac{\mathrm{d}\,\mathbb{k}}{\mathrm{d}\,r} - \frac{\mathrm{d}\,\mathbf{\Phi}}{\mathrm{d}\,r}\,(\mathbb{k}-\mathbf{k}_t) \tag{169}$$

In the OSLME, the derivative of vector $\mathbb{k}$ with respect to the discount rate is negative — steady state capital falls as the discount rate rises — and equals:

$$\frac{\mathrm{d}\,\mathbb{k}}{\mathrm{d}\,r} \;=\; -\mathbf{\Omega}^{-1}\mathbf{p}^k \tag{170}$$

whereas the derivative of matrix $\mathbf{\Phi}$ with respect to the discount rate equals:

$$\frac{\mathrm{d}\,\mathbf{\Phi}}{\mathrm{d}\,r} \;=\; \frac{1}{2}\mathbf{I} - \left[\frac{1}{4}\left(\frac{1}{\theta}\mathbf{\Omega} + (2+\mathrm{r})\mathbf{I}\right) - \frac{1}{2}\mathbf{I}\right]\left[\frac{1}{4}\left(\frac{1}{\theta}\mathbf{\Omega} + (2+\mathrm{r})\mathbf{I}\right)^{\mathbf{2}} - (1+r)\mathbf{I}\right]^{-\frac{1}{2}} \tag{171}$$

Substituting the two derivatives above into the expression for $\mathrm{d}\boldsymbol{i}(\mathbf{k}_t)/\mathrm{d}r$ and multiplying by -1% yields equation 74. Equation 75 follows from noting that steady state investment equals $\boldsymbol{i}(\mathbb{k}) = \boldsymbol{\Delta}\mathbb{k}$, so that $-1\% \frac{\mathrm{d}\boldsymbol{i}(\mathbb{k})}{\mathrm{d}r} = -1\% \boldsymbol{\Delta} \frac{\mathrm{d}\mathbb{k}}{\mathrm{d}r} = 1\% \boldsymbol{\Delta}\boldsymbol{\Omega}^{-1}\mathbf{p}^k$. □

Proof of Proposition 12. Consider again equation 168. At the firm level, the derivative of the investment expenditure of business unit n with respect to its own steady state capital stock $k_{n\infty}$ equals:

$$\frac{\mathrm{d}\left[p_n^k i_n(\mathbf{k}_t)\right]}{\mathrm{d}k_{n\infty}} = p_n^k\left(1-\varphi_{nn}\right) \tag{172}$$

which corresponds to the numerator of the $\mathrm{MPI}^{\mathrm{G}}$ of business unit n at time t. Note that matrix $\boldsymbol{\Phi}$ and the price of capital p_n^k do not depend on the primitives $(\mathbf{b},\mathbf{c})$, and are therefore unaffected by the perturbation, which moves $\mathbb{k}$ only. To solve the expression in the denominator, consider equation 56, by which the steady state value of business unit n equals $v_n(\mathbb{k}) = (1+r)p_n^k k_{n\infty} + \rho_n$, with $\rho_n = \frac{1+r}{r} z_n^2 k_{n\infty}^2$ by the definition of ρ_n. Note that the steady state value of each business unit depends only on its own steady state capital stock; hence, the perturbation leaves competitors' steady state values unaffected if and only if it leaves their steady state capital stocks unaffected. Differentiating with respect to $k_{n\infty}$:

$$\frac{\mathrm{d}v_n(\mathbb{k})}{\mathrm{d}k_{n\infty}} = (1+r)p_n^k + \frac{2\rho_n}{k_{n\infty}} = (1+r)p_n^k + 2\left[\bar{q}_n(\mathbb{k}) - q_n(\mathbb{k})\right] \tag{173}$$

where the last equality follows from equation 71 evaluated at the steady state, by which $\bar{q}_n(\mathbb{k}) - q_n(\mathbb{k}) = \rho_n/k_{n\infty}$. This gives the result. □

B Additional Details on Data and Calibration

Calibration of productivity vector z. Our empirical estimate of z_n^2 is obtained in four steps. First, we obtain a noisy estimate of z_n^2 as the residual from the first order condition (equation 79).

Second, we smooth transitory noise in firm-level productivity estimates by applying a centered 7-period moving average to the raw estimate of z_n^2 computed from equation (79). Third, for each year t in our sample, we compute each firm's percentile rank and estimate the Pareto scale and shape parameters by regressing the noisy estimate of $\log z_{nt}$ on the corresponding log of 1 minus the corresponding quantile, using only firms in the top 5% of the distribution: the year-specific intercept identifies the scale parameter, while the slope identifies the shape parameter.

Finally, we use the estimated coefficients to predict $\log z_n$ for all firms in year t, including those below the top 5%. This effectively extrapolates the Pareto tail across the entire distribution. The logic behind this approach is that while measurement error may severely distort z_n estimates for low- and medium-productivity firms, the highest-z_n firms are more reliably identified.

By allowing the regression coefficients to vary by year, we accommodate potential time trends in productivity dispersion over our 1995–2021 sample period. Within the estimation sample, the Pareto assumption provides an excellent fit: the R^2 of the regression is typically 98 to 99%, with a typical sample size of about 200 observations.

Rescaling productivity to match aggregate enterprise values. The Pareto-fitted productivity estimates yield a distribution of z_n that captures the cross-sectional shape but may not match the aggregate level of firm values in the data. We therefore apply a scalar rescaling factor $\mathcal{Z}$ to the entire productivity vector, choosing λ_z to satisfy the aggregate moment condition:

$$\sum_{n=1}^{n} v_n(\mathbf{k}_t; \mathcal{Z}\mathbf{z}) = \sum_{n}(1+r)\hat{v}_{nt} \tag{174}$$

where $v_n(\mathbf{k}_t; \lambda_z \mathbf{z})$ denotes the model-predicted enterprise value for firm n as a function of the rescaled productivity vector, and $\hat{v}_{nt}$ is the observed market value of assets. The factor $(1+r)$ accounts for the cum-dividend adjustment in the enterprise value definition. We solve for $\mathcal{Z}$ numerically using a root-finding algorithm (`fzero` in MATLAB). The final calibrated productivity vector is used in all subsequent model calculations and counterfactual simulations.

Calibration of steady-state capital $\mathbb{k}$. The steady-state capital distribution $\mathbb{k}$ is an equilibrium object rather than a primitive parameter, but we must ensure it satisfies the positivity constraint $\mathbb{k} > 0$. From the transition dynamics in Proposition 2, we can express the steady state as:

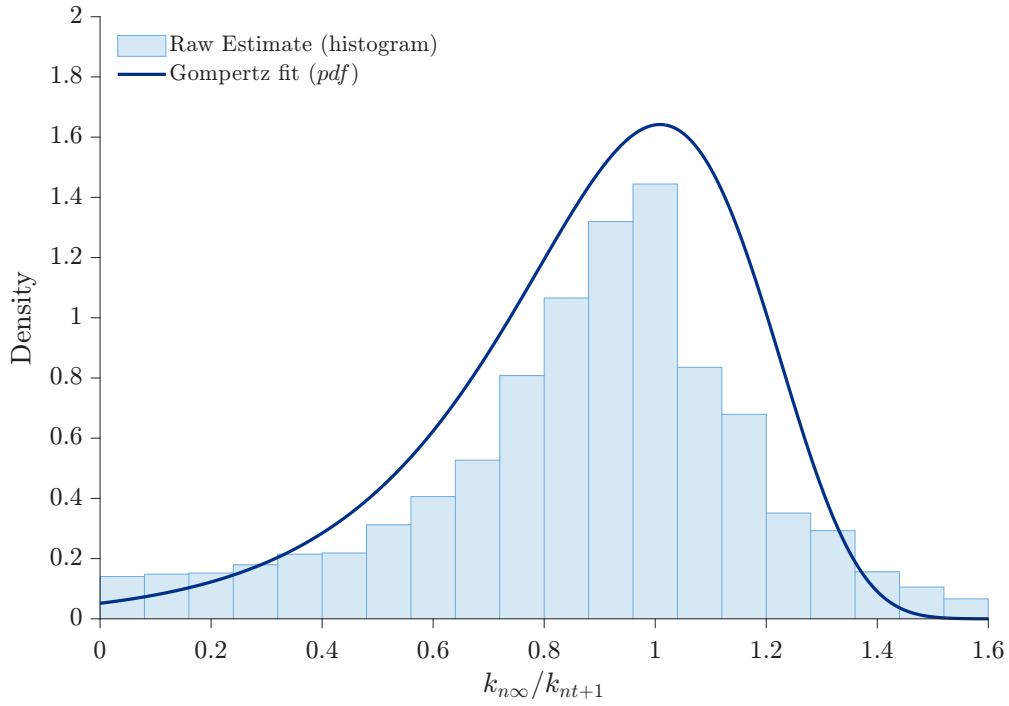
$$\mathbb{k} = (\mathbf{I} - \mathbf{\Phi})^{-1}(\mathbf{k}_{t+1} - \mathbf{\Phi}\mathbf{k}_t) \tag{175}$$

As for $\mathbf{z}$, this "raw" calculation can yield negative values due to measurement error in observed capital stocks. To ensure positivity while preserving the cross-sectional distribution, we fit a Gompertz distribution to the ratio $k_{n\infty}/k_{nt+1}$ (matching its median and interquartile range) and then use the fitted distribution to assign strictly positive steady-state values to each firm. Results are robust to replacing the Gompertz with a Weibull distribution fitted using the same procedure.

Figure 18 shows the histogram of the raw ratios $k_{n\infty}/k_{nt+1}$, together with the fitted Gompertz density.

Recovery of demand and cost parameters. Given the calibrated values of $\mathbf{z}$, θ, and $\mathbb{k}$, we recover

FIGURE 18: DISTRIBUTION OF STEADY-STATE CAPITAL STOCK



the remaining structural parameters. Output y_{nt} is simply obtained as $z_n k_{nt}$. The unit labor cost c_n is equal to the ratio of costs of goods sold (as reported in Compustat) to output y_{nt} in each firm-year. This assumes that all reported operating costs are variable, consistent with our Leontief production technology. The demand intercept vector $\mathbf{b}$ is then recovered by inverting the steady-state condition from equation (42):

$$\mathbf{b} = \mathbf{c} + \mathbf{Z}^{-1}\left[\mathbf{\Omega}\mathbb{k} + (r\mathbf{I} + \mathbf{\Delta})\mathbf{p}^{k}\right] \tag{176}$$

where $\mathbf{\Omega} = \mathbf{Z}(2\mathbf{I} + \mathbf{\Sigma})\mathbf{Z}$ is the equilibrium network effects matrix. The price vector $\mathbf{p}$ is then obtained from the inverse demand function (7).

Notice that at this point, if we compute revenues as the newly obtained vector of prices ($\mathbf{p}$) times output ($\mathbf{y}$) we will find it to be different from the vector of revenues observed in the Compustat data. This is to be expected: because our model is significantly over-identified (it features more structural equation restrictions—equalities and inequalities—than degrees of freedom) it cannot perfectly fit the observed revenues (model revenues absorb the residual variation). The same goes for the vector of next period capital $\mathbf{k}_{t+1}$, which also cannot be perfectly fit by the model. The last step therefore consists of updating the vector of revenues $p_{nt}y_{nt}$ with the model-consistent estimate.

This calibration strategy balances several competing objectives: (1) using observable data where possible, (2) imposing economically interpretative restrictions to handle measurement error, (3) ensuring all parameters satisfy their theoretical constraints (particularly positivity), and (4) maintaining internal consistency with the model's equilibrium conditions. The distributional approach for $\mathbf{z}$ and $\mathbb{k}$ is particularly important, as it allows us to extract robust cross-sectional patterns from noisy firm-level data while ensuring the model's predictions remain economically sensible.