On Apparent Relative Risk Aversion*

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Abstract

For HARA utility satisfying the precautionary saving motive, apparent relative risk aversion—the value function's relative risk aversion, which governs the optimal risky portfolio share—lacks a general characterization under concave consumption. Given a constant $1/(k-1)$, it is characterized by the elasticity of supernumerary consumption with respect to wealth, bounded above by unity. Linear consumption implies that apparent relative risk aversion is increasing and concave, such that the optimal risky portfolio share is decreasing and convex in the Merton model, a pattern difficult to reconcile with limited participation and underdiversification. This paper derives necessary and sufficient conditions for the monotonicity and concavity of apparent relative risk aversion under concave consumption. These conditions provide a benchmark to assess whether deviations rationalize heterogeneous investor behavior.

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1 Introduction

In financial economics, an important objective is understanding how households allocate their wealth between risky and risk-free assets. Following Merton (1969), the general classical formula for the risky portfolio share is proportional to the reciprocal of the relative risk aversion, $\omega_t \propto -\frac{V'}{a_t V''}$. The value function intertemporally aggregates instantaneous utility, and its concavity fully captures any non-linearity arising in the consumption function. The literature explores investor heterogeneity in characteristics, beliefs, and preferences, but the heterogeneity in risk aversion of the value function remains notably under-investigated.

Since Arrow (1965) and Pratt (1964), the functions $-f''/f'$ and $-xf''/f'$ have been used as the measures of absolute and relative risk aversion to study the properties of the utility function. Kimball (1990a) introduces the term *apparent* to distinguish the properties of the value function from those of the utility function. Specifically, *apparent absolute risk aversion* is defined as the absolute risk aversion of the value function, which is the direct determinant of the optimal level of investment in risky assets, $\theta_t := -V''/V'$.

By analogy, we define *apparent relative risk aversion* as the relative risk aversion of the value function, which is the direct determinant of the optimal risky portfolio share, $\eta_t := -a_t V''/V'$. It is the elasticity of supernumerary consumption with respect to wealth multiplied by the constant $\frac{1}{k-1}$. Supernumerary consumption refers to the optimal consumption level after excluding subsistence consumption.[1] Hence, given a constant $\frac{1}{k-1}$, this elasticity directly determines the investor's apparent relative risk aversion with respect to portfolio risk, dictating the optimal risky share.

In household finance, two important stylized facts, limited participation(Bertaut & Starr-McCluer, 2002; Campbell, 2006; Haliassos & Bertaut, 1995; Tracy et al., 1999) and underdiversification(Blume & Friend, 1975; Calvet et al., 2007; Goetzmann & Kumar, 2008; Grinblatt et al., 2011), are classified as "mistakes" by Campbell (2006). These behaviors suggest that households at the lower end of the wealth distribution exhibit high apparent relative risk aversion, while across the wealth distribution, apparent relative risk aversion is

[1] In order to ensure positive relative risk aversion, subsistence consumption should be non-negative because the negative level of consumption may otherwise emerge. The subsistence consumption is implicit in Merton (1971)'s analytical solution but is not identified there as an economic quantity; it is recovered via Carroll and Kimball (1996)'s characterization. See Appendix A for details.

decreasing. An economic intuition for household portfolio choice, developed in works such as Campbell and Viceira (2002),[2] is that apparent relative risk aversion is increasing in wealth, which implies that the optimal risky portfolio share is decreasing. In the absence of income risk, for utility functions in the Hyperbolic Absolute Risk Aversion (HARA) class, the Merton model has a closed-form analytical solution, thereby constituting the special case in which the optimal consumption rule is linear. This obtains if income exceeds subsistence consumption.

Campbell (2018) and Campbell and Ramadorai (2026) propose that these behaviors are difficult to reconcile with rational optimizing models. Rationalizing these behaviors would require apparent relative risk aversion to be decreasing from a very high level; however, for HARA utility functions with $k > 1$, this paper shows that apparent relative risk aversion is necessarily increasing and concave under a linear consumption rule.

In the presence of uninsurable income risk, the household problem generally yields a concave consumption function, except under very special conditions. This paper investigates the general case in which the optimal consumption rule is concave, a condition that necessitates numerical solutions for portfolio choice problems. We identify the necessary and sufficient conditions under which the monotonicity and concavity of apparent relative risk aversion hold, generalizing to the special case. These conditions provide a benchmark for numerical simulation and parameter estimation, allowing us to assess whether deviations from these conditions can rationalize the aforementioned phenomena.

The HARA functions are characterized by a linear relationship between absolute prudence and absolute risk aversion, where *(absolute) prudence* is defined as $-f'''/f''$.[3] For HARA utility functions satisfying the precautionary saving motive ($k > 0$),[4] the optimal consumption rule is concave (or linear) if and only if apparent prudence is greater than (or equal to) k times apparent absolute risk aversion. This equivalence allows us to express the

[2]This work summarizes research from the late 1990s and early 2000s that operationalizes the insights of Merton (1973) about portfolio choice with time-varying investment opportunities.

[3]For more research on prudence, please refer to Kimball (1990b), Kimball (1990a), Kimball (1993).

[4]In partial equilibrium, precautionary saving in response to risk is associated with a positive third derivative of a von Neumann-Morgenstern utility function (Leland, 1968), or convexity of the marginal utility function (Sandmo, 1970). Precautionary saving problems have been further extended by a substantial body of literature, including contributions from Zeldes (1989), Caballero (1990), Kimball (1990b), Kimball (1993), and Carroll and Kimball (1996).

relationship between them more intuitively through the concavity of consumption.

By the fundamental properties of value functions and the precautionary saving motive, and ruling out the singularity at $k = 1$, the admissible parameter range is restricted to $k > 1$. This range corresponds to constant relative risk aversion (CRRA) utility functions with shifted-origin variants. Subsequent results depend on this consumption concavity and the parameter range.

Given that CRRA utility functions are the decreasing absolute risk aversion (DARA) subset of the HARA functions, apparent absolute risk aversion is strictly decreasing in wealth.[5] Moreover, a convex marginal propensity to consume makes apparent absolute risk aversion strictly convex in wealth. These two properties dictate that the optimal level of risky assets is increasing and concave in wealth.

The concavity of consumption implies that the elasticity of supernumerary consumption with respect to wealth is less than unity; equivalently, apparent relative risk aversion has a constant upper bound $\frac{1}{k-1}$. This upper bound corresponds to a positive lower bound of the optimal risky portfolio share at the upper bound of wealth, which is a subset of the elasticity condition where the elasticity of supernumerary consumption equals unity. This constraint underscores the difficulty of explaining limited participation within a rational framework. If the subsistence consumption is non-negative, then the elasticity of consumption with respect to wealth is less than unity; equivalently, apparent relative risk aversion is less than relative risk aversion.

Apparent relative risk aversion is increasing in wealth if and only if the sum of elasticity of supernumerary consumption and that of the marginal propensity to consume with respect to wealth is less than unity; equivalently, it has the upper bound $\frac{1}{\phi-1}$, where ϕ is the ratio of apparent prudence to apparent absolute risk aversion. Under a linear consumption rule, ϕ collapses to k. This ratio captures the elasticity of the marginal propensity to consume.

In a neighborhood of the borrowing constraint, apparent relative risk aversion is strictly increasing if the optimal consumption rule diverges strictly slower than $1/a$ and $\sup c_t' < \infty$. As wealth approaches the upper bound and the borrowing constraint, apparent relative risk

[5] A static analysis reveals that, holding absolute risk aversion constant, an agent's apparent absolute risk aversion decreases with their marginal propensity to consume out of wealth(Kimball, 1990a).

aversion approaches $\frac{1}{k-1}$ and zero, respectively, with the latter subject to the condition that the marginal propensity to consume diverges strictly slower than $1/a$.

Finally, this paper establishes a necessary and sufficient condition for the concavity of apparent relative risk aversion. Although the property of decreasing prudence is not preserved by apparent prudence (Kimball, 1990a), we establish that a decline in the strength of the consumption concavity implies that apparent prudence is strictly decreasing in wealth. Hence, apparent temperance is strictly greater than apparent prudence, where *(absolute) temperance* is defined as $-f''''/f'''$.[6] Specifically, we derive both an unconditional sufficient condition and a conditional necessary and sufficient condition under the property of decreasing apparent prudence.

2 Proof

Assume that the utility function takes the HARA form:

$$u_t = \frac{1-k}{(2-k)A}\left[Ac_t + B\right]^{\frac{2-k}{1-k}} + C, \tag{1}$$

where c_t is consumption and C is a constant.

The level of **subsistence consumption** is defined as $\underline{c} := -B/A$. The household faces the consumption constraint $c_t > \underline{c}$.[7] We can obtain the expression for relative risk aversion:

$$\gamma(c_t) := -c_t\frac{u''}{u'} = \frac{1}{k-1}\frac{c_t}{c_t - \underline{c}}. \tag{2}$$

We define **apparent relative risk aversion** as the relative risk aversion of the value function:

$$\eta(a_t) := -a_t\frac{V''}{V'}. \tag{3}$$

The household faces the borrowing constraint $a_t \geq 0$.

[6] For research on temperance, please refer to Kimball (1990a), Kimball (1993), Gollier and Pratt (1996).

[7] Carroll and Kimball (1996) requires $Ax + B \geq 0$ in order to be able to raise this expression to a fractional power, but when we focus on risk aversion, the denominator cannot be zero.

2.1 The Merton Model as a Special Case

A notable feature of the Merton model is that it admits a closed-form solution, where the optimal consumption rule is linear. In this subsection, we adopt the framework specified by Merton (1971) for HARA utility functions. Assume that there are two assets: the risk-free asset with return r_f and the risky asset with return μ_s and volatility σ_s, whose price is log-normally distributed. The return and volatility may be subjective or time-varying, we adopt they are objective and exogenous. Assume that income y is independent of wealth. The household's wealth evolves as

$$da_t = \{[\omega_t(\mu_s - r_f) + r_f]a_t + y - c_t\}\, dt + \omega_t a_t \sigma_s dW_t,$$

where ω_t is the optimal share of risky portfolio.

The classical household's problem is

$$\rho V - \frac{\partial V}{\partial t} = u_t + \{[\omega_t(\mu_s - r_f) + r_f]a_t + y - c_t\}V' + \frac{1}{2}(\omega_t a_t \sigma_s)^2 V''. \tag{4}$$

Proposition 1. *For utility functions in the HARA class satisfying $k > 1$, if an investor's income exceeds the subsistence consumption, $y - \underline{c} \geq 0$, then apparent relative risk aversion is increasing and concave in wealth, i.e. $\eta' \geq 0$ and $\eta'' \leq 0$.*

Proof. The optimal rule is[8]

$$c_t^* = \frac{\delta}{1 - e^{-\delta(T-t)}}\left[a_t + \frac{y - \underline{c}}{r_f}\left(1 - e^{-r_f(T-t)}\right)\right] + \underline{c}$$

and

$$\omega_t^* = (k-1)\frac{\mu_s - r_f}{\sigma_s^2} \cdot \left[1 + \frac{(y - \underline{c})\left(1 - e^{-r_f(T-t)}\right)/r_f}{a_t}\right]. \tag{5}$$

The closed-form solution for apparent relative risk aversion is

$$\eta_t = \frac{1}{k-1}\frac{a_t}{a_t + \frac{y-\underline{c}}{r_f}\left(1 - e^{-r_f(T-t)}\right)}. \tag{6}$$

[8]The detailed algebraic verifications and coefficient matching are relegated to Appendix C. Merton (1971) provides an analytical solution of optimal consumption-portfolio choice for HARA utility functions.

We need to require $a_t + \frac{y-\underline{c}}{r_f}\left(1 - e^{-r_f(T-t)}\right) > 0$ in order to be able to raise this expression to a fractional power. In other words, an investor should have a positive endowment at any time, including the present value of financial wealth and non-financial net surplus.

If income exceeds the subsistence consumption, thereby leaving the household with positive discretionary resources for intertemporal allocation, we have

$$Y(t) = \frac{y-\underline{c}}{r_f}\left(1 - e^{-r_f(T-t)}\right) \geq 0.$$

Then

$$\eta' = \frac{1}{k-1}\frac{Y(t)}{\left(a_t + Y(t)\right)^2} \geq 0, \qquad \eta'' = -\frac{1}{k-1}\frac{2Y(t)}{\left(a_t + Y(t)\right)^3} \leq 0.$$

□

At the borrowing constraint, the condition always holds. The asset dynamics evolve as $da = (y - c_t)dt$. Since the borrowing constraint requires $da \geq 0$, we have $y \geq c_t$. With the subsistence consumption constraint $c_t > \underline{c}$, we have $y > \underline{c}$.

According to the analytical solution (6) and (5), we can obtain that, given a wealth level, the greater non-financial net income available for intertemporal allocation or the longer the investment horizon, the lower apparent relative risk aversion and the higher the risky portfolio share.

Corollary 1. *If an investor's income exceeds the subsistence consumption, then the optimal risky portfolio share is decreasing and convex with respect to wealth, i.e.* $\omega_t^{*\prime} \leq 0$ *and* $\omega_t^{*\prime\prime} \geq 0$.

Proof. For $a > 0$, we have $\frac{1}{\eta} = (k-1)\frac{a_t + Y(t)}{a_t} = (k-1)\left(1 + \frac{Y(t)}{a_t}\right)$.

Then

$$\omega_t^{*\prime} = \frac{\partial}{\partial a}\left(\frac{\mu_s - r_f}{\sigma_s^2}\cdot\frac{1}{\eta}\right) = (k-1)\frac{\mu_s - r_f}{\sigma_s^2}\cdot\left(-\frac{Y(t)}{a_t^2}\right) \leq 0.$$

$$\omega_t^{*\prime\prime} = \frac{\partial^2}{\partial a^2}\left(\frac{\mu_s - r_f}{\sigma_s^2}\frac{1}{\eta}\right) = (k-1)\frac{\mu_s - r_f}{\sigma_s^2}\cdot\frac{2Y(t)}{a_t^3} \geq 0.$$

□

For the boundary values of apparent relative risk aversion in this model, please refer to the Appendix B.

2.2 The General Case

We now consider the more general case, where the optimal consumption rule is concave. Assume that the value function $V \in C^4$, $V'' < 0$, the optimal consumption function $c^* \in C^3$, and the borrowing constraint is non-binding on the domain of interest ($\sup a(I) > 0$). The **apparent absolute risk aversion** is defined as $\theta(a_t) := -V''/V'$. We have $\eta_t \geq 0$ and $\theta_t > 0$.

Let the elasticity of consumption with respect to wealth be $\varepsilon(a_t) := \frac{a_t}{c_t} c_t{}'$. By the envelope condition, apparent relative risk aversion can be expressed as[9]

$$\eta_t = -a_t V''/V' = -a_t \frac{u''}{u'} c_t{}' = \left(\frac{a_t}{c_t} c_t{}'\right)\left(-c_t \frac{u''}{u'}\right) = \varepsilon_t \gamma_t. \tag{7}$$

The **elasticity of supernumerary consumption with respect to wealth** is defined as $\tilde{\varepsilon}(a_t) = \frac{a_t}{c_t - \underline{c}} c_t{}'$. Apparent relative risk aversion can be expressed as

$$\eta_t = \left(\frac{a_t}{c_t} c_t{}'\right)\left(-c_t \frac{u''}{u'}\right) = \left(\frac{a_t}{c_t} c_t{}'\right)\left(\frac{1}{k-1} \frac{c_t}{c_t - \underline{c}}\right) = \frac{1}{k-1}\left(\frac{a_t}{c_t - \underline{c}} c_t{}'\right) = \frac{1}{k-1} \tilde{\varepsilon}_t. \tag{8}$$

The HARA functions could be described as the class of functions that satisfy the condition $f'''f'/f''^2 = k$, which means that its prudence equals k times its absolute risk aversion.

Lemma 1 establishes a necessary and sufficient condition for the optimal consumption rule under intertemporal aggregation via the envelope theorem, without relying on any specific model assumptions. Subsequently, this allows us to leverage the consumption concavity to bound the ratio of apparent prudence to apparent absolute risk aversion from below, which determines the monotonicity of apparent relative risk aversion.

This is distinct from Carroll and Kimball (1996)'s Lemma 3, which states that if $V_t'''V_t'/V_t''^2 \geq k$ and $u'''u'/u''^2 = k$, then the optimal consumption rule is concave. They focus on the preservation of prudence properties for the value function through the vertical and horizontal addition of marginal utilities within a discrete two-period framework.

Lemma 1. *For HARA utility functions satisfying $k > 0$, the optimal consumption rule is concave (or linear) if and only if apparent prudence is greater than (or equal to) k times*

[9] Originally derived by Merton (1973).

apparent absolute risk aversion.

Proof. Since $V'' < 0$ and $V' > 0$, we have $-V''/V' > 0$, so

$$-\frac{V'''}{V''} \geq k\left(-\frac{V''}{V'}\right) \iff \frac{V'''V'}{V''^2} \geq k.$$

By envelope theorem, we have $V' = u'$, $V'' = u''c_t'$, and $V''' = u'''c_t'^2 + u''c_t^{*\prime\prime}$, so

$$\frac{V'''V'}{V''^2} = \frac{(u'''c_t'^2 + u''c_t^{*\prime\prime})u'}{(u''c_t')^2} = \frac{u'''u'}{u''^2} + \frac{u'c_t^{*\prime\prime}}{u''c_t'^2} = k + \frac{u'c_t^{*\prime\prime}}{u''c_t'^2}.$$

Since $-u''/u' > 0$, we have

$$\frac{V'''V'}{V''^2} \geq k \iff c_t^{*\prime\prime} \leq 0.$$

□

Note that HARA value functions do not necessarily mean that the optimal consumption rule is linear. Let the parameters of the utility function and the value function be k_1 and k_2, respectively. The value function takes the following form: $V' = (C_1 a + g)^{\frac{1}{1-k_2}}$, where g is a function independent of wealth a. Since $V' = u' = (Ac_t + B)^{\frac{1}{1-k_1}}$, we have $c_t^* = \frac{1}{A}(C_1 a + g)^{\alpha} + \underline{c}$, where $\alpha = \frac{1-k_1}{1-k_2}$. Then, $c_t^{*\prime\prime} = \frac{C_1^2\alpha(\alpha-1)}{A}(C_1 a + g)^{\alpha-2}$. If $\alpha = 1$, then $c_t^{*\prime\prime} = 0$. If $\alpha \neq 1$, then the optimal consumption rule is a non-linear power function.

Lemma 2. *For HARA utility functions satisfies $k > 0$, if $k \neq 1$, then $k > 1$ and $A > 0$.*

Proof. Since $a_t \geq 0$, $c_t - \underline{c} > 0$, and $c_t' > 0$, we have $\tilde{\varepsilon}_t \geq 0$, with $\tilde{\varepsilon}_t > 0$ wherever $a_t > 0$. By the equation (8) and $\eta_t = -a_t V''/V' = a_t\theta_t \geq 0$, we have $k > 1$ wherever $a_t > 0$. Following Carroll and Kimball (1996), to satisfy a strictly concave utility function for which $u'' < 0$, we must have $[A/(1-k)] < 0$. Then, $A > 0$. □

Proposition 2. *For HARA utility functions satisfying $k > 1$, if the optimal consumption rule is concave, then apparent absolute risk aversion is strictly decreasing in wealth, i.e. $\theta' < 0$.*

Proof. By the chain rule, we have

$$\theta' = \frac{d}{da}\left[\left(-\frac{u''}{u'}\right){c_t}'\right] = \left(-\frac{u''}{u'}\right)c_t^{*\prime\prime} + \frac{d}{dc}\left(-\frac{u''}{u'}\right){c_t}'^2.$$

For HARA utility functions satisfying $k > 1$, we have $-u''/u' = \frac{1}{k-1}\frac{1}{c_t - \underline{c}} > 0$, so $\frac{d}{dc}\left(-\frac{u''}{u'}\right) = -\frac{1}{(k-1)(c_t-\underline{c})^2} < 0$. Since $c_t^{*\prime\prime} \leq 0$ and $c' > 0$, we have $\theta' < 0$. □

Lemma 3. *For HARA utility functions satisfying $k > 1$, if the optimal consumption rule is concave, then the elasticity of supernumerary consumption with respect to wealth is less than unity, i.e. $\tilde{\varepsilon}_t \leq 1$; equivalently, apparent relative risk aversion has a constant upper bound, i.e. $\eta_t \leq \frac{1}{k-1}$.*

Proof. Suppose that $I \in \mathbb{R}_+$ is a perfect interval and $f : I \to \mathbb{R}_+$ is concave and differentiable. For all $a, b \in I$ such that $a < b$,

$$\frac{f(x) - f(a)}{x - a} \geq \frac{f(b) - f(x)}{b - x}, \qquad a < x < b.$$

Let $b \to x$, we have

$$\lim_{b \to x} \frac{f(b) - f(x)}{b - x} = f'(x).$$

Let $a = 0$, we have

$$\frac{f(x)}{x} \geq f'(x) + \frac{f(0)}{x}.$$

Since $f(0) > 0$, we have

$$\frac{f(x)}{x} > f'(x).$$

Let $c_t - \underline{c} : a \mapsto f(x)$. Since $c_t^{*\prime\prime} \leq 0$ and $c_t > \underline{c}$, we have

$${c_t}' < \frac{c_t - \underline{c}}{a_t}.$$

Then

$$\tilde{\varepsilon}_t = \frac{a_t}{c_t - \underline{c}}{c_t}' < 1, \qquad \forall a \in \mathbb{R}_+.$$

Let $f : I \to \mathbb{R}$ be monotone, $\alpha := \inf I$ and $\beta := \sup I$. Then $\lim_{x \to \alpha+} f(x) = \sup f(I)$ and $\lim_{x \to \beta-} f(x) = \inf f(I)$ if f is decreasing and $\lim_{x \to \alpha+} f(x) = \inf f(I)$ and

$\lim_{x\to\beta-} f(x) = \sup f(I)$ if f is increasing.

Let ${c_t}' : a \mapsto f(x)$. Since $\sup I = \infty$ and ${c_t^*}'' \leq 0$, we have $\lim_{a\to\infty} {c_t}'(a) = \inf {c_t}'(I)$.

Suppose that $f, g : (a, b) \to \mathbb{R}$ are differentiable and $g(x) \neq 0$ for all $x \in (a, b)$. Suppose also that either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} g(x) = \pm\infty$. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$,, if the limit on the right exists in $\bar{\mathbb{R}}$.

Let $a \mapsto g(x)$ and $c : a \mapsto f(x)$. Since $\lim_{a\to\infty} a = \infty$ and the limit on the right exists, we have $\lim_{a\to\infty} \frac{c(a)}{a} = \lim_{a\to\infty} \frac{{c_t}'(a)}{1} = \inf {c_t}'(I)$. Since ${c_t}'(I) > 0$ and $\lim_{a\to\infty} c_t = \infty$, we have

$$\lim_{a_t\to\infty} \tilde{\varepsilon}_t - 1 = \frac{\lim_{a_t\to\infty} {c_t}'(a)}{\lim_{a_t\to\infty} \frac{c_t - \underline{c}}{a_t}} - 1 = \frac{\inf {c_t}'(I)}{\inf {c_t}'(I)} - 1 = 0.$$

Then, we have

$$\tilde{\varepsilon}_t \leq 1 \iff \eta_t \leq \frac{1}{k-1}, \qquad \forall a \in \bar{\mathbb{R}}_+.$$

□

The elasticity of consumption with respect to wealth can be expressed as the elasticity of supernumerary consumption: $\varepsilon_t = \tilde{\varepsilon}_t(1 - \frac{\underline{c}}{c_t})$.

Remark 1. For HARA utility functions satisfying $k > 1$, if the optimal consumption rule is concave and the subsistence consumption is non-negative, then the elasticity of consumption with respect to wealth is less than unity, $\varepsilon_t \leq \tilde{\varepsilon}_t \leq 1$; apparent relative risk aversion is less than relative risk aversion, i.e. $\eta_t \leq \gamma_t$.

We define **apparent (absolute) prudence** as the absolute prudence of the value function, $\zeta(a_t) := -V'''/V''$. Let the ratio of apparent prudence to apparent absolute risk aversion be $\phi := \zeta_t/\theta_t$. Let the elasticity of the marginal propensity to consume with respect to wealth be $\hat{\varepsilon}(a_t) := a_t|{c_t^*}''|/{c_t}'$.

Theorem 1. *For HARA utility functions satisfying $k > 1$, if the optimal consumption rule is concave, then apparent relative risk aversion is increasing in wealth if and only if apparent relative risk aversion has an upper bound $\frac{1}{\phi-1}$; equivalently, the sum of elasticity of supernumerary consumption and that of marginal propensity to consume with respect to wealth is less than unity, i.e. $\tilde{\varepsilon}_t + \hat{\varepsilon}_t \leq 1$.*

Proof. By $\eta_t = a_t\theta_t$, we have

$$\theta' = -\frac{V'''}{V'} + \frac{(V'')^2}{(V')^2} = \zeta_t \frac{V''}{V'} + \theta_t^2 = -\zeta_t\theta_t + \theta_t^2.$$

Furthermore,

$$\eta' = \theta_t + a_t\theta' = \theta_t\left[1 - a_t\left(\zeta_t - \theta_t\right)\right]$$

Since $\theta_t > 0$ for all $a \geq 0$, the sign of η' equals the sign of $[1 - a_t\left(\zeta_t - \theta_t\right)]$.

Then

$$\eta' \geq 0 \iff 1 - a_t\left(\zeta_t - \theta_t\right) \geq 0 \iff a_t\left(\zeta_t - \theta_t\right) \leq 1.$$

By Lemma 1, we have

$$\phi = \frac{V'''V'}{V''^2} \geq k > 1.$$

Then

$$\eta' \geq 0 \iff \eta_t\left(\phi - 1\right) \leq 1 \iff \eta_t \leq \frac{1}{\phi - 1}.$$

By the definitions of θ_t and ζ_t, we have

$$\theta_t = -\frac{V''}{V'} = -\frac{u''c_t'}{u'} = \frac{1}{k-1}\frac{1}{c_t - \underline{c}}c_t'$$

$$\zeta_t = -\frac{V'''}{V''} = -\frac{u'''c_t'^2}{u''c_t'} - \frac{u''c_t^{*\prime\prime}}{u''c_t'} = \frac{k}{k-1}\frac{1}{c_t - \underline{c}}c_t' - \frac{c_t^{*\prime\prime}}{c_t'}$$

Furthermore, we have

$$\eta_t(\phi - 1) = a_t(\zeta_t - \theta_t) = a_t\left(\frac{1}{c_t - \underline{c}}c_t' - \frac{c_t^{*\prime\prime}}{c_t'}\right).$$

Under the concavity $c_t^{*\prime\prime} \leq 0$, we have $|c_t^{*\prime\prime}| = -c_t^{*\prime\prime} \geq 0$, so $\hat{\varepsilon}_t = a_t|c_t^{*\prime\prime}|/c' = -a_t c_t^{*\prime\prime}/c' \geq 0$.

Then

$$\eta' \geq 0 \iff \tilde{\varepsilon}_t + \hat{\varepsilon}_t \leq 1.$$

□

Corollary 2. *For HARA utility functions satisfying $k > 1$, if the optimal consumption rule is linear, then apparent relative risk aversion is increasing in wealth.*

Proof. Since $c_t^{*\prime\prime} = 0$, we have $\hat{\varepsilon}_t = 0$. Hence, the necessary and sufficient condition is that $\tilde{\varepsilon}_t + \hat{\varepsilon}_t = \tilde{\varepsilon}_t \leq 1$. According to Lemma 3, this condition holds. □

Remark 2. In a neighborhood of the borrowing constraint, if the optimal consumption rule is concave and diverges strictly slower than $1/a$ and $\sup c_t{}' < \infty$, then apparent relative risk aversion is strictly increasing.

Proof. Following Lemma 3 and Theorem 1, since $\inf I = 0$, $c_t{}' > 0$, and $c_t^{*\prime\prime} \leq 0$, we have $\lim_{a\to 0+} c_t{}' = \sup c_t{}'(I)$ and $\lim_{a\to 0+} c = \inf c(I)$.

$$\lim_{a\to 0+} \eta_t(\phi - 1) = \lim_{a\to 0+} a_t \left(\frac{\sup c_t{}'(I)}{\inf c(I) - \underline{c}} - \frac{c_t^{*\prime\prime}}{\sup c_t{}'(I)} \right).$$

The second term satisfies $\lim_{a\to 0+} a \cdot (-c_t^{*\prime\prime})/c_t{}' = 0$ if and only if $c_t^{*\prime\prime} \sim o(1/a)$. If $c_t^{*\prime\prime}$ diverges strictly slower than $1/a$ at the borrowing constraint, we have

$$\lim_{a\to 0+} \eta_t(\phi - 1) = 0.$$

□

Remark 3. For HARA utility functions satisfying $k > 1$, if the optimal consumption rule is concave and the marginal propensity to consume diverges strictly slower than $1/a$, then apparent relative risk aversion approaches zero as wealth is close to the borrowing constraint.

Proof. Following Lemma 3, we have $\lim_{a\to 0+} c(a) = \inf c(I)$.

Then

$$\lim_{a\to 0+} \eta_t = \lim_{a\to 0+} \frac{a_t}{(k-1)(c_t - \underline{c})} c_t{}' = \lim_{a\to 0+} \frac{a_t}{(k-1)(\inf c(I) - \underline{c})} c_t{}' = 0.$$

□

Remark 4. For HARA utility functions satisfying $k > 1$, if the optimal consumption rule is concave, then apparent relative risk aversion approaches $\frac{1}{k-1}$ as wealth approaches the upper bound.

Proof. By Lemma 3, we have $\lim_{a\to\infty} \tilde{\varepsilon}_t = 1$. Hence, $\lim_{a\to\infty} \eta_t = \lim_{a\to\infty} \frac{1}{k-1}\tilde{\varepsilon}_t = \frac{1}{k-1}$. □

We define the **apparent temperance** as the absolute temperance of the value function, $\xi_t := -V''''/V'''$. Let the ratio of apparent temperance to apparent prudence be $\psi := \xi_t/\theta_t$.

Lemma 4. *For HARA utility functions satisfying $k > 1$, if the optimal consumption rule is concave and the strength of the consumption concavity is decreasing in wealth, i.e. $\left(-\frac{c_t^{*\prime\prime}}{c_t{}'}\right)' \leq 0$, then apparent prudence is strictly decreasing in wealth.*

Proof. Since $\phi \geq k > 1$, we have $\zeta_t > \theta_t$ and $V''' > 0$ and $V'' < 0$, so

$$\left(-\frac{V'''}{V''}\right)' < 0 \iff -\frac{V''''V'' - V'''^2}{V''^2} < 0 \iff V''''V'' > V'''^2. \tag{9}$$

Furthermore,

$$V''''V'' - V'''^2 = (u''''u'' - u'''^2){c_t}'^4 + (u'''u''){c_t}'^2 c_t^{*\prime\prime} + u''^2({c_t}' c_t^{*\prime\prime\prime} - c_t^{*\prime\prime 2}) > 0. \tag{10}$$

For HARA utility functions satisfying $k > 1$, we have that

$$-\frac{u''''}{u'''} = \frac{2k-1}{k-1}\frac{1}{c_t - \underline{c}} > \frac{k}{k-1}\frac{1}{c_t - \underline{c}} = -\frac{u'''}{u''} > 0,$$

which implies positive but decreasing absolute prudence, and

$$-\frac{u'''}{u''} = \frac{k}{k-1}\frac{1}{c_t - \underline{c}} > \frac{1}{k-1} = -\frac{u''}{u'} > 0,$$

which implies decreasing absolute risk aversion.

Hence, we have $u'''' < 0$, $u''' > 0$, and $u''''u'' - u'''^2 > 0$.

Since $c_t^{*\prime\prime} \leq 0$ and $c' > 0$, we have

$$(u''''u'' - u'''^2){c_t}'^4 > 0$$

and

$$(u'''u''){c_t}'^2 c_t^{*\prime\prime} \geq 0.$$

Then, a sufficient condition for $\xi_t > \zeta_t$ to hold is

$$c_t{}' c_t^{*\prime\prime\prime} - c_t^{*\prime\prime 2} \geq 0.$$

Since $c_t^{*\prime\prime} \leq 0$, we have $-c_t^{*\prime\prime}/c_t{}' \geq 0$.

Then

$$\left(-\frac{c_t^{*\prime\prime}}{c_t{}'}\right)' \leq 0 \iff -\frac{c_t^{*\prime\prime\prime} c_t{}' - c_t^{*\prime\prime 2}}{c_t{}'^2} \leq 0 \iff c_t{}' c_t^{*\prime\prime\prime} - c_t^{*\prime\prime 2} \geq 0.$$

□

As demonstrated by Kimball (1990a), the property of decreasing absolute prudence is not preserved by the value function under intertemporal aggregation; the violation of decreasing prudence arises from the curvature of the inverse marginal utility functions.

For HARA utility functions, the analytical tractability of the value function allows us to verify this property explicitly. Lemma 4 provides a structural condition for the curvature of marginal utility functions, ensuring the alignment between temperance and prudence: positive apparent prudence is strictly decreasing in wealth if and only if apparent temperance is strictly greater than apparent prudence, i.e. $\left(-\frac{V'''}{V''}\right)' < 0 \iff -\frac{V''''}{V'''} > -\frac{V'''}{V''}$.

Proposition 3. *For HARA utility functions satisfying $k > 1$, if the optimal consumption rule is concave and the marginal propensity to consume is convex, then apparent absolute risk aversion is strictly convex in wealth, i.e. $\theta'' > 0$.*

Proof. By the chain rule, we have

$$\begin{aligned}
\theta'' =& \frac{d}{da}\left[\left(-\frac{u''}{u'}\right) c_t^{*\prime\prime} + \frac{d}{dc}\left(-\frac{u''}{u'}\right) c_t{}'^2\right] \\
=& \frac{d}{dc}\left(-\frac{u''}{u'}\right) c_t^{*\prime\prime} c' + \left(-\frac{u''}{u'}\right) c_t^{*\prime\prime\prime} + \frac{d^2}{dc^2}\left(-\frac{u''}{u'}\right) c_t{}'^3 + 2\frac{d}{dc}\left(-\frac{u''}{u'}\right) c_t^{*\prime\prime} c' \\
=& 3\frac{d}{dc}\left(-\frac{u''}{u'}\right) c_t^{*\prime\prime} c_t{}' + \left(-\frac{u''}{u'}\right) c_t^{*\prime\prime\prime} + \frac{d^2}{dc^2}\left(-\frac{u''}{u'}\right) c_t{}'^3 > 0.
\end{aligned}$$

For HARA utility functions satisfying $k > 1$, we have $-u''/u' = \frac{1}{k-1}\frac{1}{c_t - \underline{c}} > 0$, $\frac{d}{dc}\left(-\frac{u''}{u'}\right) = -\frac{1}{(k-1)(c_t-\underline{c})^2} < 0$, and $\frac{d^2}{dc^2}\left(-\frac{u''}{u'}\right) = \frac{2}{(k-1)(c_t-\underline{c})^3} > 0$. Since $c_t^{*\prime\prime} \leq 0$, $c' > 0$, and $c_t^{*\prime\prime\prime} \geq 0$, we

have $\theta'' > 0$. Additionally, the strength of the consumption concavity is decreasing in wealth implies that the marginal propensity to consume is convex. □

Theorem 2. *For HARA utility functions satisfying $k > 1$, if the optimal consumption rule is concave, apparent relative risk aversion is concave if and only if*

$$2(\phi - 1) + \eta_t(\phi(3 - \psi) - 2) \ \geq \ 0 \iff 2(\phi - 1) - \eta_t \theta_t^{-3} \theta'' \ \geq \ 0.$$

Specifically, if positive apparent prudence is strictly decreasing, then

1. (*Sufficient, unconditional.*) *If $k \in (1, 2)$ and $\psi \leq 3 - \frac{2}{k}$, then apparent absolute risk aversion is concave and apparent relative risk aversion is strictly concave for any η_t.*

2. (*Necessary and sufficient, conditional.*) *If apparent absolute risk aversion is strictly convex, then apparent relative risk aversion is concave if and only if*

$$\eta_t \leq \frac{2(\phi - 1)}{|\phi(3 - \psi) - 2|} \iff \eta_t \leq \frac{2(\phi - 1)}{\theta''} \theta_t^3.$$

Proof. By the definition of $\eta_t = -a_t \frac{V''}{V'}$, we have

$$\eta' = -\frac{V''}{V'} - a_t \frac{V''' V' - V''^2}{V'^2}. \tag{11}$$

Furthermore,

$$\eta'' = -2\frac{V''' V' - V''^2}{V'^2} - a_t \left[\frac{V'''' }{V'} - 3\frac{V''' V''}{V'^2} + 2\frac{V''^3}{V'^3} \right].$$

Let $R := \frac{V''' V' - V''^2}{V'^2}$ and $Q := \frac{V''''}{V'} - 3\frac{V''' V''}{V'^2} + 2\frac{V''^3}{V'^3}$. We have $\eta'' = -2R - a_t Q$. Hence,

$$\eta'' \leq 0 \iff 2R + a_t Q \geq 0.$$

We have $V'' = -\theta_t V'$, $V''' = -\zeta_t V'' = \theta_t \zeta_t V'$, and $V'''' = -\xi_t V''' = -\theta_t \zeta_t \xi_t V'$.

$$R = \frac{\theta_t \zeta_t V'^2 - \theta_t^2 V'^2}{V'^2} = \theta_t(\zeta_t - \theta_t).$$

$$Q = \frac{-\theta_t\zeta_t\xi_t V'}{V'} - 3\frac{\theta_t\zeta_t V'\cdot(-\theta_t V')}{V'^2} + 2\frac{-\theta_t^3 V'^3}{V'^3} = -\theta_t\zeta_t\xi_t + 3\theta_t^2\zeta_t - 2\theta_t^3.$$

$$2R + a_t Q = 2\theta_t(\zeta_t - \theta_t) + a_t\theta_t[-\zeta_t\xi_t + 3\theta_t\zeta_t - 2\theta_t^2]$$

Then, we obtain

$$\begin{aligned} 2R + a_t Q =& 2\theta_t^2\left(\frac{\zeta_t}{\theta_t} - 1\right) + \eta_t\theta_t^2\left(-\frac{\zeta_t}{\theta_t}\frac{\xi_t}{\theta_t} + 3\frac{\zeta_t}{\theta_t} - 2\right) \\ =& \theta_t^2\Big[2\big(\phi - 1\big) + \eta_t(-\phi\psi + 3\phi - 2)\Big]. \end{aligned} \tag{12}$$

Let $\Gamma := \phi(3-\psi) - 2$. Since $\theta_t^2 > 0$, the condition $2R + a_t Q \geq 0$ is equivalent to

$$2(\phi - 1) + \eta_t\Gamma \;\geq\; 0. \tag{$\star$}$$

We consider two cases according to the sign of Γ. Since $\xi_t > \zeta_t > \theta_t$, we have $\psi > \phi$.

Case 1: $\Gamma \geq 0$**.** Since $\phi \geq k > 1$, we have $2(\phi - 1) \geq 2(k-1) > 0$. Since $\eta_t \geq 0$ and $\Gamma \geq 0$, the second term satisfies $\eta_t\Gamma \geq 0$. Hence ($\star$) holds for any η_t. This constitutes a sufficient condition for $\eta'' \leq 0$.

For $\Gamma \geq 0$ to be consistent with $\psi > \phi$, note that $\psi > \phi$ implies $2 - 3\phi + \phi^2 < 2 - 3\phi + \psi\phi$, while $\Gamma \geq 0$ requires $2 - 3\phi + \psi\phi \leq 0$. Combining yields $2 - 3\phi + \phi^2 < 0$, i.e. $\phi \in (1, 2)$, which in turn requires $k < 2$. A sufficient condition is that, when $k \in (1, 2)$ and $\psi \leq 3 - 2/k$, then $\Gamma = \phi(3-\psi) - 2 \geq k(3-\psi) - 2 \geq 0$. The region $\psi \in \left(\phi,\, 3 - \frac{2}{k}\right)$ is non-empty if and only if $k \in (1, 2)$. For CRRA utility with $\underline{c} = 0$, $k \in (1, 2)$ if and only if the coefficient of relative risk aversion $\gamma = 1/(k-1) > 1$.

Case 2: $\Gamma < 0$**.** The second term $\eta_t\Gamma \leq 0$, so ($\star$) is equivalent to

$$\eta_t \;\leq\; \frac{2(\phi - 1)}{|\Gamma|}. \tag{13}$$

This is *necessary and sufficient* for $\eta'' \leq 0$ in this case. Since $\phi \geq k > 1$, the bound $\frac{2(\phi-1)}{|\Gamma|} \geq \frac{2(k-1)}{|\Gamma|} > 0$, so there is the feasible region for η_t.

1. $\psi \leq 3$*:* Here $|\Gamma| = 2-\phi(3-\psi) \in (0,2]$, so $\frac{2(\phi-1)}{|\Gamma|} \geq \phi-1 > 0$. The condition $\eta_t \leq \phi-1$ is therefore *sufficient but not necessary* for (13).

2. $\psi > 3$*:* Here $|\Gamma| = 2+\phi(\psi-3) > 2$, so $\frac{2(\phi-1)}{|\Gamma|} < \phi-1$. The exact condition (13) is necessary and sufficient; $\eta_t \leq \phi-1$ is *necessary but not sufficient*.

Since

$$\theta'' = -\left[\frac{V''''}{V'} - 3\frac{V'''V''}{V'^2} + 2\frac{V''^3}{V'^3}\right],$$

we have

$$\theta'' = -Q = -\frac{1}{a_t}\theta_t^2\eta_t\Gamma = -\theta_t^3\Gamma.$$

If $\Gamma \geq 0$, then $\theta'' \leq 0$. If $\Gamma < 0$, then $\theta'' > 0$. Substituting these equations yields the conclusion. □

Corollary 3. *For HARA utility functions satisfying $k > 1$, if the optimal consumption rule is linear, then apparent relative risk aversion is concave in wealth.*

Proof. Following Theorem 1, we have

$$\theta_t = -\frac{V''}{V'} = -\frac{u''c_t'}{u'} = \frac{1}{k-1}\frac{1}{c_t-\underline{c}}c_t'$$

$$\zeta_t = -\frac{V'''}{V''} = -\frac{u'''c_t'^2}{u''c_t'} - \frac{u''c_t^{*\prime\prime}}{u''c_t'} = \frac{k}{k-1}\frac{1}{c_t-\underline{c}}c_t' - \frac{c_t^{*\prime\prime}}{c_t'}$$

Let absolute risk aversion be $\kappa_t := -\frac{u''}{u'} = \frac{1}{k-1}\frac{1}{c_t-\underline{c}}$. By the definitions of ξ_t, we have

$$\begin{aligned}
\xi_t = -\frac{V''''}{V'''} &= -\frac{u''''c_t'^3}{u'''c_t'^2 + u''c_t^{*\prime\prime}} - \frac{3u'''c_t'c_t^{*\prime\prime}}{u'''c_t'^2 + u''c_t^{*\prime\prime}} - \frac{u''c_t^{*\prime\prime\prime}}{u'''c_t'^2 + u''c_t^{*\prime\prime}} \\
&= -\frac{u''''c_t'^3/u'''}{c_t'^2 + u''c_t^{*\prime\prime}/u'''} - \frac{3u'''c_t'c_t^{*\prime\prime}/u''}{u'''c_t'^2/u'' + c_t^{*\prime\prime}} - \frac{c_t^{*\prime\prime\prime}}{u'''c_t'^2/u'' + c_t^{*\prime\prime}} \\
&= \frac{(2k-1)\kappa_t c_t'^3}{c_t'^2 - c_t^{*\prime\prime}/(k\kappa_t)} + \frac{3k\kappa_t c_t'c_t^{*\prime\prime}}{c_t^{*\prime\prime} - k\kappa_t c_t'^2} + \frac{c_t^{*\prime\prime\prime}}{k\kappa_t c_t'^2 - c_t^{*\prime\prime}}.
\end{aligned}$$

Finally, we have

$$\xi_t = \frac{k(2k-1)\kappa_t^2 c_t'^3 - 3k\kappa_t c_t'c_t^{*\prime\prime} + c_t^{*\prime\prime\prime}}{k\kappa_t c_t'^2 - c_t^{*\prime\prime}}$$

Based on the above analysis, we can conclude that apparent prudence equals k times apparent absolute risk aversion if and only if $c_t^{*\prime\prime} = 0$. Hence, we have $\theta_t = \kappa_t {c_t}'$, $\zeta_t = k\kappa_t {c_t}'$, and $\xi_t = (2k-1)\kappa_t {c_t}'$.

Substituting these equations and $\theta_t = \kappa_t {c_t}'$ into the necessary and sufficient condition in Theorem 2

$$2(\zeta_t - \theta_t) + a_t[-\zeta_t\xi_t + 3\theta_t\zeta_t - 2\theta_t^2] \geq 0,$$

we have

$$\begin{aligned}
\text{LHS} =& 2(k-1)\theta_t + a_t[-(2k-1)k\theta_t^2 + 3k\theta_t^2 - 2\theta_t^2] \\
=& 2(k-1)\theta_t + a_t\theta_t^2[-(2k-1)k + 3k - 2] \\
=& 2(k-1)\theta_t - 2a_t(k-1)^2\theta_t^2 \\
=& 2(k-1)\theta_t[1 - \eta_t(k-1)]
\end{aligned}$$

Then, we have

$$2(\zeta_t - \theta_t) + a_t[-\zeta_t\xi_t + 3\theta_t\zeta_t - 2\theta_t^2] \geq 0 \iff \eta_t \leq \frac{1}{k-1}.$$

Hence, Lemma 3 is sufficient to guarantee that the condition is met. □

3 Conclusion

The paper focuses on the $k \neq 1$ and therefore excludes the CARA utility ($k = 1$). Tractable closed-form characterizations of Merton models with income risk have relied on this specific utility specification. In that setting, the properties of apparent relative risk aversion are readily derived from the analytical solutions in Merton (1971).

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Appendix

A

Following Carroll and Kimball (1996), we have the equivalent forms for the HARA utility function:

$$f = \frac{1-k}{(2-k)A}\left[Ax+B\right]^{\frac{2-k}{1-k}} + C,$$

$$f' = \left[Ax+B\right]^{\frac{1}{1-k}},$$

$$f'' = \frac{A}{1-k}\left[Ax+B\right]^{\frac{k}{1-k}},$$

$$f''' = \frac{A^2 k}{(1-k)^2}\left[Ax+B\right]^{\frac{2k-1}{1-k}},$$

and

$$f'''' = \frac{A^3 k(2k-1)}{(1-k)^3}\left[Ax+B\right]^{\frac{3k-2}{1-k}}.$$

We have the coefficient of relative risk aversion:

$$\gamma = -x\frac{f''}{f'} = \frac{1}{k-1}\frac{Ax}{Ax+B}.$$

B

Remark 5. If an investor's wealth is close to the borrowing constraint, then apparent relative risk aversion approaches zero.

Proof.

$$\lim_{a_t \to 0+} \eta_t = \lim_{a_t \to 0+} \frac{1}{k-1}\frac{a_t}{a_t + \frac{y-\underline{c}}{r_f}\left(1-e^{-r_f(T-t)}\right)} = 0.$$

□

Remark 6. If an investor's wealth approaches the upper bound, then apparent relative risk aversion approaches $\frac{1}{k-1}$.

Proof.

$$\lim_{a_t \to \infty} \eta_t = \lim_{a_t \to \infty} \frac{1}{k-1}\frac{a_t}{a_t + \frac{y-\underline{c}}{r_f}\left(1-e^{-r_f(T-t)}\right)} = \frac{1}{k-1}.$$

□

C

Proof. We adopt the standard formulation where the value function is defined as the present value of expected discounted utility:

$$V(a,t) = \max_{\{c_t,\omega_t\}} \mathbb{E}_t \left[\int_t^T e^{-\rho(s-t)} u(c_s)\, ds \right], \tag{14}$$

with the terminal condition $V(a,T) = 0$.

By the definition of utility function in equation (1) and the envelope condition $u' = V'$, we have

$$u' = [Ac_t + B]^{\frac{1}{1-k}}, \qquad V' = [Ac_t + B]^{\frac{1}{1-k}}.$$

Solving the above equations, we have

$$c_t^* = \frac{1}{A}\left[(V')^{1-k} - B \right].$$

Substituting the FOC condition back into the equation (4), we have

$$\begin{aligned} \rho V - \frac{\partial V}{\partial t} =& \frac{1-k}{2-k}\frac{1}{A}(V')^{2-k} + C + \left\{ \left[\frac{\mu_s - r_f}{\sigma_s^2}\frac{1}{\eta_t}(\mu_s - r_f) + r_f \right] a_t + y - \underline{c} \right\} V' \\ & - \frac{1}{A}(V')^{2-k} + \frac{1}{2}\left(\frac{\mu_s - r_f}{\sigma_s}\frac{1}{\eta_t} a_t \right)^2 V''. \end{aligned} \tag{15}$$

Following Merton (1992)'s Theorem 5.4 that $u \in \text{HARA} \iff V \in \text{HARA}$, the value function takes the HARA form:

$$V(a,t) = X(t)\Big(a + Y(t)\Big)^{\frac{2-k}{1-k}}, \tag{16}$$

where $X(t)$ and $Y(t)$ are undetermined functions of time.

The terminal condition is $V(a,T) = 0$, which implies

$$X(T)\Big(a + Y(T)\Big)^{\frac{2-k}{1-k}} = 0.$$

Since this equation hold for all a, we need $X(T) = 0$.

With the equation (16), we have

$$V' = X(t)\frac{2-k}{1-k}\Big(a + Y(t)\Big)^{\frac{1}{1-k}}, \qquad V'' = X(t)\frac{2-k}{(1-k)^2}\Big(a + Y(t)\Big)^{\frac{k}{1-k}},$$

$$\frac{\partial V}{\partial t} = \dot{X}(t)\Big(a + Y(t)\Big)^{\frac{2-k}{1-k}} + X(t)\frac{2-k}{1-k}\Big(a + Y(t)\Big)^{\frac{1}{1-k}}\dot{Y}(t).$$

$$\eta = -a_t\frac{V''}{V'} = -a_t\frac{X(t)\frac{2-k}{(1-k)^2}\Big(a_t + Y(t)\Big)^{\frac{k}{1-k}}}{X(t)\frac{2-k}{1-k}\Big(a_t + Y(t)\Big)^{\frac{1}{1-k}}} = \frac{1}{k-1}\frac{a_t}{a_t + Y(t)}.$$

Substituting these expressions into the equation (15), we have

$$\begin{aligned}
&\rho X(t)\Big(a_t + Y(t)\Big)^{\frac{2-k}{1-k}} - \dot{X}(t)\Big(a_t + Y(t)\Big)^{\frac{2-k}{1-k}} - X(t)\frac{2-k}{1-k}\Big(a_t + Y(t)\Big)^{\frac{1}{1-k}}\dot{Y}(t) \\
=&\frac{1}{k-2}\frac{1}{A}\left[X(t)\frac{2-k}{1-k}\Big(a + Y(t)\Big)^{\frac{1}{1-k}}\right]^{2-k} + C + \mathcal{S}^2(k-1)X(t)\frac{2-k}{1-k}\Big(a + Y(t)\Big)^{\frac{2-k}{1-k}} \\
&+ \left(r_f a_t + y - \underline{c}\right)X(t)\frac{2-k}{1-k}\Big(a + Y(t)\Big)^{\frac{1}{1-k}} + \frac{1}{2}\mathcal{S}^2 X(t)(2-k)\Big(a + Y(t)\Big)^{\frac{2-k}{1-k}}.
\end{aligned} \tag{17}$$

Here we need to transform the term $(r_f a_t + y - \underline{c})$:

$$\begin{aligned}
&\left(r_f a_t + y - \underline{c}\right)X(t)\frac{2-k}{1-k}\Big(a + Y(t)\Big)^{\frac{1}{1-k}} \\
=&\left(r_f(a_t + Y(t)) - r_f Y(t) + y - \underline{c}\right)X(t)\frac{2-k}{1-k}\Big(a + Y(t)\Big)^{\frac{1}{1-k}} \\
=&r_f\Big(a_t + Y(t)\Big)X(t)\frac{2-k}{1-k}\Big(a + Y(t)\Big)^{\frac{1}{1-k}} + \left(-r_f Y(t) + y - \underline{c}\right)X(t)\frac{2-k}{1-k}\Big(a + Y(t)\Big)^{\frac{1}{1-k}} \\
=&r_f X(t)\frac{2-k}{1-k}\Big(a + Y(t)\Big)^{\frac{2-k}{1-k}} + \left(-r_f Y(t) + y - \underline{c}\right)X(t)\frac{2-k}{1-k}\Big(a + Y(t)\Big)^{\frac{1}{1-k}}
\end{aligned}$$

We extract the terms with coefficient $\frac{2-k}{1-k}\Big(a_t + Y(t)\Big)^{\frac{1}{1-k}}X(t)$:

$$\dot{Y}(t) - r_f Y(t) + y - \underline{c} = 0.$$

$$\dot{Y}(t) = r_f Y(t) - (y - \underline{c}).$$

$$\frac{d}{dt}\left(Y(t)e^{-r_f t}\right) = -\left(y - \underline{c}\right)e^{-r_f t}.$$

$$\int_t^t \frac{d}{ds}\left(Y(s)e^{-r_f s}\right)ds = Y(T)e^{-r_f T} - Y(t)e^{-r_f t}.$$

$$-\int_t^T (y-\underline{c})\, e^{-r_f s} ds = -(y-\underline{c})\, \frac{e^{-r_f T} - e^{-r_f t}}{r_f}.$$

$$Y(t) = Y(T) e^{-r_f (T-t)} + (y-\underline{c})\, \frac{1 - e^{-r_f (T-t)}}{r_f}.$$

Similarly, we extract the terms with coefficient $\left(a_t + Y(t)\right)^{\frac{2-k}{1-k}}$:

$$-\rho X(t) + \dot{X}(t) - \frac{1}{A} X(t)^{2-k} \frac{(2-k)^{1-k}}{(1-k)^{2-k}} - \frac{1}{2}\mathcal{S}^2 X(t)(2-k) + r_f X(t) \frac{2-k}{1-k} = 0.$$

$$\dot{X}(t) = \left(\rho + \mathcal{S}^2 \frac{2-k}{2} - r_f \frac{2-k}{1-k}\right) X(t) + \frac{1}{A} \frac{(2-k)^{1-k}}{(1-k)^{2-k}} X(t)^{2-k}.$$

Let $n = 2-k$. Let $v = X^{1-n}$. We have $X = v^{\frac{1}{1-n}} = v^{\frac{1}{k-1}}$, $\dot{X} = \frac{1}{k-1} v^{\frac{2-k}{k-1}} \dot{v}$, and $X^n = v^{\frac{2-k}{k-1}}$. Furthermore,

$$\frac{1}{k-1} v^{\frac{2-k}{k-1}} \dot{v} = \left(\rho + \mathcal{S}^2 \frac{2-k}{2} - r_f \frac{2-k}{1-k}\right) v^{\frac{1}{k-1}} + \frac{1}{A} \frac{(2-k)^{1-k}}{(1-k)^{2-k}} v^{\frac{2-k}{k-1}}.$$

$$\dot{v} = (k-1)\left(\rho + \mathcal{S}^2 \frac{2-k}{2} - r_f \frac{2-k}{1-k}\right) v - \frac{1}{A}\left(\frac{2-k}{1-k}\right)^{1-k}.$$

Let $\kappa := \frac{1}{A}\left(\frac{2-k}{1-k}\right)^{1-k}$. Let

$$\delta := (k-1)\left(\rho + \mathcal{S}^2 \frac{2-k}{2} - r_f \frac{2-k}{1-k}\right).$$

We have

$$\dot{v}(t) - \delta v(t) = -\kappa.$$

$$\frac{d}{dt}[v(t) e^{-\delta t}] = -\kappa e^{-\delta t}.$$

$$\int_t^T \frac{d}{ds}[v(s) e^{-\delta s}] ds = -\kappa \int_t^T e^{-\delta s} ds.$$

$$\int_t^T \frac{d}{ds}[v(s) e^{-\delta s}] ds = v(T) e^{-\delta T} - v(t) e^{-\delta t}.$$

$$-\kappa \int_t^T e^{-\delta s} ds = -\kappa \left[\frac{e^{-\delta s}}{-\delta} \right]_t^T = \frac{\kappa}{\delta}(e^{-\delta t} - e^{-\delta T})$$

$$v(t)e^{-\delta t} = v(T)e^{-\delta T} + \frac{\kappa}{\delta}(e^{-\delta t} - e^{-\delta T})$$

$$v(t) = v(T)e^{-\delta(t-T)} + \frac{\kappa}{\delta}(1 - e^{-\delta(t-T)})$$

Since $X(T) = 0$, we have $v(T) = 0$.

Furthermore,

$$\begin{aligned} X(t) &= \left(\frac{\kappa}{\delta}(1 - e^{-\delta(t-T)})\right)^{\frac{1}{k-1}} \\ &= \left(\frac{1}{A}\left(\frac{2-k}{1-k}\right)^{1-k}\frac{1}{\delta}(1 - e^{-\delta(t-T)})\right)^{\frac{1}{k-1}} \\ &= \frac{1-k}{2-k}\left(\frac{1}{A}\frac{1}{\delta}(1 - e^{-\delta(t-T)})\right)^{\frac{1}{k-1}}. \end{aligned} \tag{18}$$

$$V(a,t) = \cdot\frac{1-k}{2-k}\left(\frac{1}{A}\frac{1}{\delta}(1 - e^{-\delta(t-T)})\right)^{\frac{1}{k-1}} \cdot \left(a + \frac{y-\underline{c}}{r_f}\left(1 - e^{-r_f(T-t)}\right)\right)^{\frac{2-k}{1-k}} \tag{19}$$

$$V' = \left[\frac{1}{A\delta}(1 - e^{-\delta(t-T)})\right]^{\frac{1}{k-1}} \left[a_t + \frac{y-\underline{c}}{r_f}\left(1 - e^{-r_f(T-t)}\right)\right]^{\frac{1}{1-k}}$$

$$c_t^* = \frac{\delta}{1 - e^{-\delta(T-t)}}\left[a_t + \frac{y-\underline{c}}{r_f}\left(1 - e^{-r_f(T-t)}\right)\right] + \underline{c}.$$

The consumption is an affine function.

Let the Sharpe ratio be $\mathcal{S} := \frac{\mu_s - r_f}{\sigma_s}$. The discount rate ρ should satisfy the condition

$$\rho > \frac{k-2}{k-1}r_f - \frac{k-2}{2}\mathcal{S}^2.$$

If the rate satisfies $r_f \leq \frac{k-1}{2}\mathcal{S}^2$ or the investor's utility function is logarithmic ($k = 2$), it always holds.

Proof. By the envelope condition, we have $u''c_t{}' = V''$. Since $V'' < 0$ and $u'' < 0$, we have that the marginal propensity to consume out of wealth is strictly positive, i.e.

$$c_t{}' = \frac{\delta}{1 - e^{-\delta(T-t)}} > 0.$$

Furthermore, we have

$$\begin{aligned}\delta =&(k-1)\left(\rho+\mathcal{S}^2\frac{2-k}{2}-r_f\frac{2-k}{1-k}\right)\\ =&(k-1)\rho-(k-2)r_f+\frac{(k-1)(k-2)}{2}\mathcal{S}^2>0.\end{aligned}$$

For logarithmic utility functions ($k=2$), the inequation always holds.

For HARA class with $k\neq 2$, we require

$$\rho>\frac{k-2}{k-1}r_f-\frac{k-2}{2}\mathcal{S}^2.$$

If

$$r_f\leq\frac{k-1}{2}\mathcal{S}^2,$$

the requirement always holds.

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