Market Access, Entry, and Dynamic Innovation Traps: Rent Compression under Non-Convex R&D*

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Abstract

How does market access affect innovation when R&D is lumpy? We develop a dynamic model in which larger markets raise the gross prize from innovation but also induce entry that compresses frontier rents. The analysis turns on the innovation-replenishment ratio: successful innovation per unit of existing capability. A trap arises when this ratio lies below depreciation near the boundaries but exceeds depreciation at intermediate states. Endogenous entry and heterogeneous fixed R&D costs generate this shape without imposing a threshold directly. Market access then has two margins. Current frontier rents fall when the competition elasticity times the entry elasticity exceeds one. The unstable threshold rises when market access lowers the innovation flow at that branch. With strong composite rent compression from business stealing and scarce frontier inputs, larger markets can shrink rather than expand the basin of attraction of high innovation.

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"The possibility of a low-level underdevelopment trap: firms do not invest in the modern sector, because there is insufficient demand, but demand is insufficient because the sector is too small."

— *Paul Krugman*

1 Introduction

Market access is usually good news for innovators: it makes the prize larger. This paper shows why that intuition can fail when R&D is lumpy. A larger market also draws in more firms. If those firms compress frontier rents, the economy may need a higher stock of capability to keep innovation self-sustaining.

A larger market does two things at once. It increases the gross return to a successful innovation. It also draws in entrants, and those entrants compete for the rents and inputs that make frontier innovation valuable. With convex R&D, this mainly changes how much innovation firms do. With fixed R&D costs, it can change whether enough firms innovate at all, and whether the economy stays above the capability threshold.

The recursive block links entry, R&D, and future frontier rents. Potential firms observe market access and the current innovation state before deciding whether to enter. Entrants earn ordinary operating profits and then draw heterogeneous fixed R&D costs. A firm that pays the fixed cost succeeds with probability q. A successful innovator earns a frontier rent that rises with market access and implementation capability but falls with entry-induced competition. Successful innovation replenishes the aggregate innovation state, which in turn affects future frontier rents.

Market access can therefore tighten the innovation threshold. At a transverse unstable branch $z^M(A)$, the threshold rises when market access lowers the innovation flow at that branch:

$$\frac{\mathrm{d}z^M(A)}{\mathrm{d}A} > 0 \quad \Longleftrightarrow \quad \varepsilon_A^m(z^M) < 0.$$

This is not the same as a fall in current frontier rents. Current rents fall when the competition elasticity times the entry elasticity exceeds one. The threshold moves only if the induced

change in entry and continuation values lowers successful innovation at the branch. A reform can raise gross demand and still shrink the set of initial conditions from which high innovation is self-sustaining.

The same logic also says when such an unstable branch exists. The useful object is the innovation-replenishment ratio,

$$\Psi_A(z) = \frac{m_A(z)}{z},$$

which measures successful innovation per unit of existing capability. Since

$$T_A(z) - z = z\{\lambda \Psi_A(z) - \delta\},$$

a trap arises when replenishment is too weak near zero and at high states but strong enough at intermediate states. The primitives generate this shape. Heterogeneous fixed costs make the aggregate innovation schedule continuous even though firm-level R&D is non-convex. Implementation complementarities and a thick middle range of marginal R&D firms then produce a hump-shaped replenishment ratio.

The strongest reversal requires strong composite rent compression. In the benchmark specification, frontier rents take the form $AB(z)n^{-\tau}$. The condition for simultaneous current-flow reversal and threshold tightening is not simply $\tau > 1$; it is a local condition comparing the rent-compression elasticity to the mass of firms near the R&D margin. A scarce-input microfoundation gives $\tau = \sigma + \rho\nu$, where σ is product-market rent dilution and $\rho\nu$ captures input congestion. Thus the relevant elasticity can exceed one even when no single channel is extreme.

The closest work studies how competition, trade, or complementarities affect innovation incentives and long-run growth. We ask a different question: how does market access move the unstable threshold that separates low- and high-innovation basins? The channel is entry-induced rent compression at a non-convex R&D margin.

The exercise ranks reforms by their effect on the basin boundary, not by welfare.

Road map. Section 2 positions the paper in the literature. Section 3 previews the replenishment logic. Section 4 presents the environment. Section 5 derives the recursive entry–innovation block. Section 6 gives the exact-three and market-access results. Section 7 discusses policy and robustness. The appendix contains derivations, proofs, benchmark comparisons, primitive interpretation, and the numerical robustness exercises.

2 Related Literature

Competition-and-innovation models usually ask whether rivalry makes firms innovate more or less. We ask a different question: does rivalry move the boundary between a low-innovation basin and a high-innovation basin? That distinction matters because market access can raise the gross prize while lowering successful innovation at the threshold. The comparison is closest in spirit to the Schumpeter–Arrow tradeoff and the inverted-U literature, but the comparative static operates through the transition map rather than through a static innovation incentive. This connects to Schumpeter (1942), Arrow (1962), Aghion and Howitt (1992), Aghion et al. (2005), Vives (2008), and dynamic models with firm heterogeneity, rents, and reallocation such as Klette and Kortum (2004), Acemoglu and Linn (2004), Acemoglu et al. (2018), Akcigit and Kerr (2018), Aghion et al. (2023), De Loecker et al. (2020), De Ridder (2024), and Aghion et al. (2025).

Dynamic industry models with lumpy investment provide the natural firm-level benchmark. Ericson and Pakes (1995) study stochastic investments in firm state variables under endogenous entry and exit, Doraszelski and Jaumandreu (2013) estimate firm-level R&D and endogenous productivity, and Shah (2025) compares innovation and entry policies in a dynamic industry environment with fixed costs and aggregate R&D spillovers. Here the economy is more aggregated. The recursive link from current innovation decisions to future states remains, but it is used to characterize when aggregate replenishment creates an unstable threshold.

The market-access channel builds on trade and growth models in which larger markets

raise the return to innovation, including Romer (1990), Rivera-Batiz and Romer (1991), Grossman and Helpman (1991a), and Grossman and Helpman (1991b). Trade models such as Melitz (2003), Melitz and Ottaviano (2008), Atkeson and Burstein (2010), and Sampson (2016) add selection, entry, markups, and innovation responses; Buera and Oberfield (2020) and Perla et al. (2021) study international diffusion; and Bloom et al. (2016) and Coelli et al. (2022) provide evidence that trade liberalization can induce innovation and upgrading. We hold trade-induced selection fixed. Active firms are symmetric, so the mechanism isolates how a common market-size shifter induces entry, compresses frontier rents, and moves the R&D threshold.

The threshold logic also relates to work on multiple steady states and development traps. Murphy et al. (1989), Azariadis and Drazen (1990), Cooper and John (1988), and Krugman (1991) emphasize complementarities, coordination failure, and history dependence; Galor and Zeira (1993), Acemoglu and Zilibotti (1997), and Howitt and Mayer-Foulkes (2005) show how nonlinearities can generate persistent development and innovation gaps. Here the multiplicity is not in the recursive value at a given market-access level. It is in the steady states of the transition map, and the threshold is generated by innovation replenishment per unit of existing capability.

3 Mechanism Preview

The analysis turns on the innovation-replenishment ratio,

$$\Psi_A(z) = \frac{m_A(z)}{z},$$

where $m_A(z)$ is the mass of successful innovators. For $z > 0$,

$$T_A(z) - z = z\{\lambda \Psi_A(z) - \delta\}.$$

Hence the sign of the transition surplus is governed by whether innovation per unit of existing capability exceeds depreciation. An exact-three regime requires $\Psi_A(z)$ to be low near zero, high at intermediate states, and low again near the upper boundary. The primitives deliver this shape through three margins: implementation complementarities flatten innovation near zero; the fixed-cost distribution creates a thick middle range of marginal R&D firms; and saturation of the R&D cutoff limits further replenishment at high states.

This object also separates two comparative statics. Current frontier rents may fall with market access, but the basin boundary moves only if the innovation flow changes at the unstable branch. Theorem 6.1 formalizes this distinction.

4 Environment

Time is discrete. The aggregate innovation state is $z_t \in [0, 1]$. A larger z_t means that frontier implementation capabilities, specialized suppliers, and complementary know-how are more abundant. The lower boundary $z = 0$ is a no-frontier-activation state: the complementary capabilities needed to turn an R&D attempt into frontier rents are absent. The baseline model has no exogenous knowledge inflow, imitation arrival, or outside technology transfer. Hence zero capability is absorbing in the baseline system; from $z = 0$ the economy cannot spontaneously start frontier innovation. Section 7 later shows that adding a small outside knowledge inflow replaces the absorbing corner with a nearby positive low-innovation steady state.

Market access is a scalar $A > 0$. It can be read as effective demand, inverse trade costs, or the accessible size of the market. The scalar is a reduced-form market-size shifter rather than a full trade-cost system. A richer trade environment would map iceberg costs, foreign demand, or foreign entry into this object. This reduction isolates how such a market-size shock changes entry, frontier rents, and the dynamic R&D margin.

This reduced-form treatment holds trade-induced selection fixed. The environment

does not include firm-level productivity heterogeneity, export cutoffs, or the reallocation of market shares from low-productivity to high-productivity producers. Those forces can independently affect innovation by changing the composition of active firms. The analysis below therefore isolates one channel of market access—the effect of a common market-size shifter on entry, frontier rents, and the R&D margin—rather than giving a complete theory of trade liberalization. A richer trade environment with Melitz-style selection could shift the effective entry margin or the distribution of innovation costs, and may either reinforce or offset the threshold movement studied here.

Timing. At the beginning of a period, potential firms observe (A, z). They decide whether to enter and pay an entry cost $\kappa > 0$. Entrants earn ordinary operating profits and then draw a fixed R&D cost f. A firm that pays the R&D cost succeeds with probability $q \in (0, 1)$. Successful innovation raises the next-period innovation state.

4.1 Product-market rents

The ordinary operating-profit pool is proportional to market access. With n active firms, symmetric operating profit is

$$\pi^0(A, n) = \frac{\mu A}{n}, \tag{1}$$

where $\mu > 0$ is the share of accessible demand that accrues as ordinary operating profit. Equation (1) is the free-entry profit component. It is the average operating return from production before the R&D option is exercised.

A successful innovator obtains a frontier position with value $R_A(z)$, which includes current flow rents and continuation value. The current flow rent from that position is

$$\pi^F(A, z, n) = AB(z)C(n). \tag{2}$$

The function B captures implementation complementarities. Its boundary value $B(0) =$

0 formalizes the no-frontier-activation interpretation: frontier rents require some active implementation capability. The function C captures business stealing and markup or market-share dilution from entry. This specification uses the same market-access shifter as ordinary profits but allows frontier rents to have a different entry elasticity. A frontier leader's rent need not fall one-for-one with the average operating profit because leadership changes markups, residual demand, and the value of being ahead of nearby competitors.

The benchmark used for the main construction is

$$B(z) = \chi z^{\alpha}, \qquad C(n) = n^{-\tau}, \tag{3}$$

with $\chi > 0$, $\alpha > 1$, and $\tau > 0$. The exponent $\alpha > 1$ gives low-state flatness: the direct rent gain from a marginal increase in z vanishes at the origin. The elasticity τ is a composite rent-compression elasticity of frontier rents with respect to entry. Here n should be read as effective competitive pressure, not only as a count of symmetric product varieties. Entry can erode frontier rents through business stealing, duplication of R&D races, congestion in scarce frontier inputs, and poaching of frontier talent. Values $\tau \leq 1$ correspond to weak or proportional rent compression. Values $\tau > 1$ correspond to amplified frontier-rent compression, but the strong-compression condition below should be read as a local comparison between composite rent compression and the local R&D margin, not as an unrestricted power-form assumption.

The next proposition gives one source of amplified rent compression. The scarce input market clears within the period, so it does not add a new dynamic state. It only provides a primitive interpretation of the competition function $C(n)$.

Proposition 4.1 (Scarce frontier inputs and amplified rent compression)**.** *Suppose a successful frontier leader earns a flow rent*

$$\pi^{F}(A, z, n) = A\widehat{B}(z)n^{-\sigma}p_X(n)^{-\nu}, \tag{4}$$

where $\sigma > 0$ is the product-market rent-dilution elasticity, $p_X(n)$ is the competitive price of a scarce frontier input, and $\nu > 0$ is the elasticity of frontier surplus with respect to that input price. Suppose aggregate demand for the scarce input is proportional to entry,

$$X(n) = \xi n, \qquad \xi > 0,$$

and inverse input supply is

$$p_X(X) = \bar{p} X^{\rho}, \qquad \bar{p} > 0, \quad \rho > 0.$$

Then the frontier rent can be written as

$$\pi^F(A, z, n) = AB(z)n^{-\tau}, \qquad \tau = \sigma + \rho\nu, \tag{5}$$

where $B(z) = \widehat{B}(z)(\bar{p}\xi^{\rho})^{-\nu}$. Hence $\tau > 1$ can arise even when no single channel is more than unit elastic, provided product-market rent dilution and scarce-input congestion jointly satisfy $\sigma + \rho\nu > 1$.

Proof. Input-market clearing gives

$$p_X(n) = p_X(X(n)) = \bar{p}(\xi n)^{\rho} = \bar{p}\xi^{\rho} n^{\rho}.$$

Substituting into (4) yields

$$\pi^F(A, z, n) = A\widehat{B}(z)n^{-\sigma}(\bar{p}\xi^{\rho}n^{\rho})^{-\nu} = A\widehat{B}(z)(\bar{p}\xi^{\rho})^{-\nu}n^{-(\sigma+\rho\nu)}.$$

Absorb the constant $(\bar{p}\xi^{\rho})^{-\nu}$ into $B(z)$. □

Proposition 4.1 clarifies the interpretation of τ. The relevant rent-compression elasticity is composite. Product-market business stealing contributes σ, and scarce-input congestion

contributes $\rho\nu$. Thus $\tau > 1$ need not require any single channel to be extreme; moderate product-market dilution and moderate input congestion can jointly make frontier rents more than unit elastic with respect to entry. The weak-competition benchmark below shows that the threshold-tightening result does not go through when this combined elasticity is at most one.

Remark 4.2 (Positive entry spillovers)**.** The term $C(n)$ should be read as a net rent-compression term. If entry also generates agglomeration spillovers, the frontier rent can be written as

$$\pi^F(A, z, n) = AB(z)n^{-\tau}S(n), \qquad S'(n) > 0.$$

The relevant competition elasticity is then

$$\theta_{\text{net}}(n) = -\frac{\partial \log[n^{-\tau}S(n)]}{\partial \log n} = \tau - \frac{\partial \log S(n)}{\partial \log n}.$$

All local comparative-static results below apply with $\theta_C(n)$ replaced by this net elasticity. Positive spillovers therefore weaken the rent-compression channel and can eliminate threshold tightening if they are large enough. The threshold-tightening result applies to sectors in which business stealing and scarce frontier-input congestion dominate agglomeration benefits at the relevant margin.

Let $R_A(z)$ denote the value of holding the frontier position at the beginning of a period. The Bellman equation is

$$R_A(z) = AB(z)C(n_A(z)) + \beta R_A(T_A(z)), \tag{6}$$

where $\beta \in (0, 1)$ and T_A is the law of motion for the aggregate state.

4.2 Fixed R&D costs and aggregate innovation

Each entrant draws $f \in [0, \bar{f}]$. The benchmark fixed-cost distribution is the rescaled Beta$(2, 2)$ distribution

$$G(f) = 3\left(\frac{f}{\bar{f}}\right)^2 - 2\left(\frac{f}{\bar{f}}\right)^3, \qquad g(f) = \frac{6f(\bar{f} - f)}{\bar{f}^3}, \qquad f \in [0, \bar{f}]. \tag{7}$$

The zero density at both endpoints gives a small mass of marginal firms when innovation rents are very low and when almost all firms already innovate. The cubic form is used for closed-form derivations; the mechanism only requires an interior mass of marginal firms.

A firm innovates when the expected gain from attempting R&D exceeds the fixed cost:

$$qR_A(z) \geq f. \tag{8}$$

The mass of entrants that attempt innovation is therefore $n_A(z)G(qR_A(z))$. Since each attempt succeeds with probability q, the successful innovation flow is

$$m_A(z) = qn_A(z)G(qR_A(z)). \tag{9}$$

The state evolves according to

$$T_A(z) = (1 - \delta)z + \lambda m_A(z), \tag{10}$$

where $\delta \in (0, 1)$ is depreciation of frontier capability and $\lambda > 0$ converts successful innovations into next period's state. There is no additive inflow term in the baseline law of motion. This is the formal sense in which zero capability does not regenerate itself without successful frontier innovation.

The ex ante value of drawing a fixed cost below the cutoff is

$$\Omega(x) = \int_0^x (x - f)\, \mathrm{d}G(f), \qquad x \in [0, \bar{f}]. \tag{11}$$

Under (7), integration by parts gives

$$\Omega(x) = \frac{x^3}{\bar{f}^2} - \frac{x^4}{2\bar{f}^3}, \qquad \Omega'(x) = G(x), \qquad \Omega(\bar{f}) = \frac{\bar{f}}{2}. \tag{12}$$

Appendix A.1 derives (12) step by step.

Free entry equates the ordinary operating profit plus the R&D option value to the entry cost:

$$\frac{\mu A}{n_A(z)} + \Omega(qR_A(z)) = \kappa. \tag{13}$$

Solving (13) gives the entry rule

$$n_A(z) = \frac{\mu A}{\kappa - \Omega(qR_A(z))}. \tag{14}$$

Equations (6), (9), (10), and (14) jointly determine R_A, n_A, m_A, and T_A.

4.3 Primitives and equilibrium

For $L > 0$, define

$$\mathcal{X}_L^0 = \left\{ R : [0,1] \to \left[0, \frac{\bar{f}}{q}\right] : R(0) = 0,\ R \text{ is increasing and } L\text{-Lipschitz} \right\}.$$

The restriction $R(0) = 0$ is the value-function expression of the no-frontier-activation boundary. It is not imposed to manufacture a trap. At $z = 0$, $B(0) = 0$ gives zero current frontier flow, the absence of an exogenous inflow keeps the transition at the boundary, and $G(0) = 0$ implies that a zero cutoff generates no frontier R&D attempts. The recursive operator therefore

preserves $R(0) = 0$ and maps zero capability back into itself. Let

$$n_{\min}(A) = \frac{\mu A}{\kappa}, \qquad n_{\max}(A) = \frac{\mu A}{\kappa - \bar{f}/2}.$$

Assumption 4.3 (Rent and cost primitives)**.** The following conditions hold.

1. $\mu, \kappa, \bar{f}, \lambda > 0$ and $q, \beta, \delta \in (0, 1)$.

2. $B : [0, 1] \to \mathbb{R}_+$ is continuously differentiable, increasing, and bounded. Moreover $B(0) = B'(0) = 0$ and $B(z) > 0$ for $z > 0$.

3. $C : (0, \infty) \to (0, \infty)$ is continuously differentiable and decreasing. On $[n_{\min}(A), n_{\max}(A)]$, both C and $|C'|$ are bounded.

4. The fixed-cost distribution is given by (7).

Assumption 4.4 (Admissible state space)**.** For the market-access value under study,

1. $\kappa > \bar{f}/2$;

2. letting $\bar{B} = \max_{z \in [0,1]} B(z)$,
$$\frac{qA\bar{B}C(n_{\min}(A))}{1 - \beta} \leq \bar{f};$$

3.
$$\lambda q n_{\max}(A) \leq \delta.$$

The first condition keeps the entry denominator positive. The second condition keeps equilibrium R&D cutoffs inside the support of G. The third condition keeps $T_A([0, 1]) \subseteq [0, 1]$.

Assumption 4.5 (Recursive regularity)**.** For each admissible A, there exists $L_A < \infty$ such that the Bellman-entry operator

$$(\mathcal{B}_A R)(z) = AB(z)C(n(R, z; A)) + \beta R((1 - \delta)z + \lambda q n(R, z; A) G(qR(z))) \tag{15}$$

maps $\mathcal{X}^0_{L_A}$ into itself and is a contraction in the sup norm, where

$$n(R, z; A) = \frac{\mu A}{\kappa - \Omega(qR(z))}. \tag{16}$$

Assumption 4.5 is a maintained monotone-regime restriction. It is not only a contraction condition. It also ensures that the equilibrium value and transition map lie in the increasing Lipschitz class used below. Economically, the condition requires implementation complementarities to dominate the negative effect of state-induced entry on frontier rents. If this restriction fails, the current frontier-rent flow can be locally decreasing in the state, and the induced transition map need not be monotone. Such non-monotone maps may generate additional steady states, cycles, or more complex one-dimensional dynamics. The baseline analysis focuses on the monotone exact-three regime. The assumption does not impose the exact-three crossing pattern, but it does restrict attention to a monotone equilibrium class. Appendix A.3 gives primitive sufficient inequalities for this restriction.

Proposition 4.6 (Primitive regularity check)**.** *If the inequalities in Lemma A.1 hold strictly for some (A_0, L_0), then Assumption 4.5 holds on a neighborhood of A_0. In particular, recursive uniqueness is an open property of primitives. The benchmark construction in Proposition 6.6 chooses β small and κ large so that these regularity inequalities and the exact-three inequalities hold simultaneously.*

Proof. Lemma A.1 gives the primitive inequalities that make $\mathcal{B}_A$ map $\mathcal{X}^0_{L_0}$ into itself and make it a contraction. Strict inequalities persist under small changes in A and in the remaining primitives. The last statement follows from the proof of Proposition 6.6, which verifies support, state-space, contraction, and crossing conditions with slack. □

Definition 4.7 (Markov equilibrium)**.** A Markov equilibrium at market access A is a triple (R_A, n_A, T_A) with $R_A \in \mathcal{X}^0_{L_A}$ such that R_A satisfies (6), n_A satisfies (14), and T_A is generated by (9) and (10).

5 Equilibrium Block and Transition Map

Proposition 5.1 (Recursive entry-and-innovation block)**.** *Suppose Assumptions 4.3–4.5 hold. Then for every admissible A there is a unique bounded Markov equilibrium (R_A, n_A, T_A) in the maintained increasing Lipschitz class. In that equilibrium,*

$$n_A(z) = \frac{\mu A}{\kappa - \Omega(qR_A(z))}, \tag{17}$$

$$m_A(z) = qn_A(z)G(qR_A(z)), \tag{18}$$

and R_A, n_A, and m_A are continuous. Under the maintained monotone-regime restriction, they are increasing in z. Moreover,

$$R_A(0) = 0, \qquad m_A(0) = 0, \qquad T_A(0) = 0.$$

At every point where R_A is differentiable,

$$n'_A(z) = \frac{qn_A(z)G(qR_A(z))R'_A(z)}{\kappa - \Omega(qR_A(z))}, \tag{19}$$

and

$$m'_A(z) = q^2 n_A(z)R'_A(z)\left[g(qR_A(z)) + \frac{G(qR_A(z))^2}{\kappa - \Omega(qR_A(z))}\right]. \tag{20}$$

Thus, at differentiability points in the maintained monotone class,

$$T'_A(z) = 1 - \delta + \lambda m'_A(z) \geq 1 - \delta > 0.$$

Proof. The free-entry condition is strictly decreasing in n, so it has the unique solution (17). Assumption 4.5 defines the operator on the class $\mathcal{X}^0_{L_A}$ of increasing L_A-Lipschitz functions and maps this class into itself. Hence its unique fixed point in this class is increasing and satisfies $R_A(0) = 0$.

The monotonicity statement is conditional on Assumption 4.5. The primitive role of Lemma A.1 is to verify that the current-flow payoff remains increasing even though entry can compress frontier rents. To see the economic content, let $R \in \mathcal{X}^0_{L_A}$ and write

$$n(R, z; A) = \frac{\mu A}{\kappa - \Omega(qR(z))}.$$

Because $\Omega'(x) = G(x) \geq 0$, $n(R, z; A)$ is weakly increasing in z whenever R is increasing. Its derivative is bounded by

$$0 \leq n_z(R, z; A) \leq N_R(A) L_A.$$

The derivative of the current frontier-rent flow is

$$A \left[B'(z) C(n(R, z; A)) + B(z) C'(n(R, z; A)) n_z(R, z; A) \right].$$

Since $C' < 0$, this derivative is not automatically nonnegative. Lemma A.1 imposes the sufficient condition

$$B'(z) C(n_{\max}(A)) \geq B(z) \bar{C}_1 N_R(A) L_A$$

for every z, which makes the current-flow component increasing. The transition argument is increasing because

$$\frac{\mathrm{d}}{\mathrm{d}z} \{(1 - \delta) z + \lambda q n(R, z; A) G(qR(z))\} \geq 1 - \delta > 0.$$

Since R is increasing, the continuation term is increasing as well. Therefore the operator maps $\mathcal{X}^0_{L_A}$ into itself, and the fixed point R_A is increasing within the maintained monotone class.

Because $G(0) = 0$, (18) gives $m_A(0) = q n_A(0) G(0) = 0$. Equation (10) then gives $T_A(0) = 0$. Equations (18), (19), and (20) follow by differentiating (17) and (18); Appendix A.2 gives the algebra. The displayed expression for T'_A follows from $T'_A(z) = 1 - \delta + \lambda m'_A(z)$ and

$R'_A(z) \geq 0$ at differentiability points in the maintained class. □

The transition map is not imposed. It is generated by three primitive objects: the implementation function B, the competition function C, and the distribution of fixed R&D costs.

Lemma 5.2 (Local differentiability of equilibrium objects)**.** *Suppose the equilibrium objects are locally differentiable in market access A on a neighborhood of A_0, and suppose the branch under study is transverse. A sufficient set of primitive conditions is that the Bellman-entry operator is a contraction on a C^2 neighborhood of the fixed point, with B, C, and G sufficiently smooth, and that support, denominator, and state-space constraints hold with slack. In particular, suppose there is an open set $\mathcal{A}_0$ containing A_0 such that Assumptions 4.3–4.5 hold for every $A \in \mathcal{A}_0$ with a common Lipschitz bound, and suppose that on a C^2 neighborhood $\mathcal{U}^0$ of R_{A_0} inside functions satisfying $R(0) = 0$ the derivative operator has norm bounded by some $\rho < 1$, uniformly in $A \in \mathcal{A}_0$. Suppose the support and entry-denominator inequalities hold with slack:*

$$0 \leq qR(z) \leq \bar{f} - \epsilon, \qquad \kappa - \Omega(qR(z)) \geq \epsilon, \qquad T_R(z; A) \in [0, 1]$$

for some $\epsilon > 0$, all $R \in \mathcal{U}^0$, all $z \in [0, 1]$, and all $A \in \mathcal{A}_0$, where T_R is defined in Appendix A.4. Then the map $(A, R) \mapsto \mathcal{B}_A R$ is continuously Fréchet differentiable from $\mathcal{A}_0 \times \mathcal{U}^0$ into $C^1[0, 1]$. Consequently, the fixed point $A \mapsto R_A$ is continuously differentiable near A_0. The objects $n_A(z)$, $m_A(z)$, and $T_A(z)$ are continuously differentiable in A at each state z. If $H(A_0, z^*) = 0$ and $H_z(A_0, z^*) \neq 0$, then the associated steady-state branch is locally continuously differentiable in A.

Proof. The comparative statics below use only local differentiability of the equilibrium objects around a transverse branch. The sufficient smoothness conditions above are stronger than necessary, but they avoid relying on differentiability of the composition map in a merely C^1 function space. Under the stated C^2 neighborhood and slack restrictions, the entry

rule, transition rule, and continuation term are differentiable with bounded derivatives. Let $\Phi(A, R) = R - \mathcal{B}_A R$. The derivative with respect to R is $I - D_R \mathcal{B}_A$, and the uniform bound on $D_R \mathcal{B}_A$ makes this operator invertible by the Neumann series. The Banach-space implicit-function theorem gives local differentiability of the fixed point R_A. The explicit entry rule and innovation-flow formula then imply differentiability of n_A, m_A, and T_A in A. The final statement follows from the ordinary implicit-function theorem applied to $H(A, z) = T_A(z) - z$. □

Proposition 5.3 (Slope decomposition)**.** *Suppose Assumptions 4.3–4.5 hold. If R_A is differentiable at z, then*

$$T'_A(z) = 1 - \delta + \lambda q^2 n_A(z) R'_A(z) \Xi_A(z), \tag{21}$$

where

$$\Xi_A(z) = g(qR_A(z)) + \frac{G(qR_A(z))^2}{\kappa - \Omega(qR_A(z))}. \tag{22}$$

If z^* is a steady state, R_A is differentiable at z^*, and $1 - \beta T'_A(z^*) > 0$, then

$$R'_A(z^*) = \frac{A\left[B'(z^*)C(n_A(z^*)) + B(z^*)C'(n_A(z^*))n'_A(z^*)\right]}{1 - \beta T'_A(z^*)}. \tag{23}$$

Equivalently, with

$$\theta_C(n) = -\frac{\partial \log C(n)}{\partial \log n} = -\frac{nC'(n)}{C(n)}, \tag{24}$$

(23) *becomes*

$$R'_A(z^*) = \frac{AB(z^*)C(n_A(z^*))\left[\frac{B'(z^*)}{B(z^*)} - \theta_C(n_A(z^*))\frac{n'_A(z^*)}{n_A(z^*)}\right]}{1 - \beta T'_A(z^*)}. \tag{25}$$

For the benchmark (3), (25) reduces to

$$R'_A(z^*) = \frac{\chi A n_A(z^*)^{-\tau} {z^*}^{\alpha-1}\left[\alpha - \tau z^* \frac{n'_A(z^*)}{n_A(z^*)}\right]}{1 - \beta T'_A(z^*)}. \tag{26}$$

Proof. Differentiate (10) and use (20) to obtain (21). At a steady state $z^* = T_A(z^*)$, differentiating (6) gives

$$R'_A(z^*) = A\frac{\mathrm{d}}{\mathrm{d}z}\left[B(z)C(n_A(z))\right]_{z=z^*} + \beta R'_A(z^*)T'_A(z^*).$$

The derivative of the current-flow term is $B'C + BC'n'_A$. Rearranging gives (23); substituting (24) gives (25). Under (3), $B'/B = \alpha/z$ and $\theta_C = \tau$, which gives (26). □

Equation (21) shows where the three-crossing shape can come from. Low-state flatness comes from $B'(0) = 0$ and the zero density of marginal R&D firms. The middle slope is large when many firms are near the fixed-cost margin, so $g(qR_A(z))$ is large. The high-state slope falls when G is close to one and additional entry mainly compresses frontier rents through $C'(n) < 0$.

Definition 5.4 (Exact-three regime)**.** The economy is in an exact-three regime at A if the transition surplus

$$F_A(z) = T_A(z) - z$$

has exactly three zeros on $[0, 1]$, denoted $0 < z^M(A) < z^H(A)$ together with zero, and

$$F_A(z) < 0 \quad \text{for all sufficiently small } z > 0, \qquad F'_A(z^M(A)) > 0, \qquad F'_A(z^H(A)) < 0.$$

When $F'_A(0)$ exists, the first condition is implied by $F'_A(0) < 0$. The middle steady state is the unstable threshold.

6 Main Results

The formal results separate the general mechanism from the benchmark construction. The first result is local and does not use the exact-three construction: it shows how market access moves current rents and an unstable threshold. The second result gives a portable

sufficient condition for exact-three dynamics in terms of the innovation-replenishment ratio. The benchmark power-and-Beta economy then verifies that these sufficient conditions are nonempty under primitives. The final results connect the mechanism to strong composite rent compression and to hysteresis.

6.1 Market access, rents, and threshold movement

Theorem 6.1 (Market access, rents, and threshold movement)**.** *Suppose the equilibrium objects are differentiable in A on a neighborhood of a market-access value A; Lemma 5.2 gives a local sufficient condition. Let*

$$H(A, z) = T_A(z) - z.$$

Let $z^M(A)$ be a transverse unstable branch, so $H(A, z^M(A)) = 0$ and $H_z(A, z^M(A)) > 0$.

Define the entry, frontier-flow, continuation-value, and innovation-flow elasticities by

$$\begin{aligned} \varepsilon_A^n(z) &= \frac{\partial \log n_A(z)}{\partial \log A}, \quad \varepsilon_A^\pi(z) = \frac{\partial \log \pi^F(A, z, n_A(z))}{\partial \log A}, \\ \varepsilon_A^R(z) &= \frac{\partial \log R_A(z)}{\partial \log A}, \quad \varepsilon_A^m(z) = \frac{\partial \log m_A(z)}{\partial \log A}. \end{aligned} \tag{27}$$

Let $\theta_C(n)$ be defined by (24). Then

$$\varepsilon_A^\pi(z) = 1 - \theta_C(n_A(z))\varepsilon_A^n(z). \tag{28}$$

Thus current-flow frontier rents fall with market access if and only if

$$\theta_C(n_A(z))\varepsilon_A^n(z) > 1. \tag{29}$$

For the benchmark $C(n) = n^{-\tau}$, this condition is $\tau\varepsilon_A^n(z) > 1$.

The unstable threshold tightens if and only if the innovation flow falls with market access

at the threshold:

$$\frac{\mathrm{d}z^M(A)}{\mathrm{d}A} > 0 \quad \Longleftrightarrow \quad \varepsilon_A^m(z^M(A)) < 0. \tag{30}$$

Moreover, whenever $G(qR_A(z)) > 0$*,*

$$\varepsilon_A^m(z) = \varepsilon_A^n(z) + \frac{qR_A(z)g(qR_A(z))}{G(qR_A(z))}\varepsilon_A^R(z), \tag{31}$$

and the entry elasticity satisfies

$$\varepsilon_A^n(z) = 1 + \frac{qR_A(z)G(qR_A(z))}{\kappa - \Omega(qR_A(z))}\varepsilon_A^R(z). \tag{32}$$

Hence threshold tightening is equivalent to the local value-elasticity condition

$$1 + qR_A(z^M)\left[\frac{G(qR_A(z^M))}{\kappa - \Omega(qR_A(z^M))} + \frac{g(qR_A(z^M))}{G(qR_A(z^M))}\right]\varepsilon_A^R(z^M) < 0. \tag{33}$$

Proof. Log-differentiate (2) at fixed z:

$$\varepsilon_A^\pi(z) = 1 + \frac{\partial \log C(n_A(z))}{\partial \log n_A(z)}\varepsilon_A^n(z) = 1 - \theta_C(n_A(z))\varepsilon_A^n(z),$$

which gives (28). Next, $H_A(A, z) = \partial_A T_A(z) = \lambda \partial_A m_A(z)$. Since $H_z(A, z^M(A)) > 0$, the implicit-function theorem gives

$$\frac{\mathrm{d}z^M(A)}{\mathrm{d}A} = -\frac{H_A(A, z^M(A))}{H_z(A, z^M(A))}.$$

Because $H_A = (\lambda m_A/A)\varepsilon_A^m$, (30) follows. Finally, log-differentiate $m_A(z) = qn_A(z)G(qR_A(z))$ to get (31). Log-differentiate (14) to get (32). Substituting (32) into (31) gives (33). □

The theorem is the paper's general comparative-static object. Current-flow reversal is a rent condition. Threshold tightening is an innovation-flow condition. Market access can therefore raise the size of the prize and still raise the threshold when the value of innovation

falls enough at the unstable branch.

Corollary 6.2 (Weak competition cannot deliver both reversals)**.** *In the benchmark $C(n) = n^{-\tau}$ with $0 < \tau \leq 1$, suppose $G(qR_A(z)) > 0$. If the innovation-flow elasticity is negative at a state z, then the current-flow frontier rent cannot fall with market access at that state. Formally,*

$$\varepsilon_A^m(z) < 0 \quad \Longrightarrow \quad \tau \varepsilon_A^n(z) < 1.$$

Thus simultaneous current-flow reversal and threshold tightening require competition elasticity above one in the benchmark power specification; under Proposition 4.1, this means $\sigma + \rho\nu > 1$.

Proof. Let

$$a(z) = \frac{qR_A(z)G(qR_A(z))}{\kappa - \Omega(qR_A(z))} > 0, \qquad b(z) = \frac{qR_A(z)g(qR_A(z))}{G(qR_A(z))} \geq 0.$$

Equations (31) and (32) give

$$\varepsilon_A^n(z) = 1 + a(z)\varepsilon_A^R(z), \qquad \varepsilon_A^m(z) = 1 + [a(z) + b(z)]\varepsilon_A^R(z).$$

If $\varepsilon_A^m(z) < 0$, then $\varepsilon_A^R(z) < -1/[a(z) + b(z)]$. Hence

$$\varepsilon_A^n(z) < 1 - \frac{a(z)}{a(z) + b(z)} \leq 1.$$

With $0 < \tau \leq 1$, this implies $\tau\varepsilon_A^n(z) < 1$ whenever the current-flow reversal condition is evaluated at an interior cutoff. □

6.2 Innovation replenishment and exact-three dynamics

The previous result is local: it applies whenever a transverse unstable branch exists. The next theorem gives sufficient conditions for the monotone exact-three case. For $z > 0$, define

the innovation-replenishment ratio

$$\Psi_A(z) = \frac{m_A(z)}{z}. \tag{34}$$

This object measures successful innovation per unit of existing capability. Since

$$F_A(z) = T_A(z) - z = -\delta z + \lambda m_A(z) = z\{\lambda \Psi_A(z) - \delta\}, \tag{35}$$

the shape of Ψ_A determines whether innovation replenishes depreciating capability. Under Proposition 5.1, $m_A(0) = 0$ in the baseline equilibrium, so the zero-capability state is a well-defined steady state. This zero steady state should be read as the absorbing no-frontier-activation boundary of the no-inflow baseline, not as a free normalization of the frontier rent.

Theorem 6.3 (Sufficient conditions for an exact-three monotone regime)**.** *Suppose $m_A(0) = 0$, $T_A([0,1]) \subseteq [0,1]$, T_A is increasing, and Ψ_A is continuous on $(0,1]$. Suppose further that*

1. $\limsup_{z \downarrow 0} \lambda \Psi_A(z) < \delta$ *and* $\lambda \Psi_A(1) < \delta$*;*

2. *the upper contour set*

$$U_A = \{z \in (0,1) : \lambda \Psi_A(z) > \delta\}$$

   is a nonempty interval $(z^M(A), z^H(A))$;

3. *the two boundary crossings are transverse:*

$$\frac{\mathrm{d}}{\mathrm{d}z}\{\lambda \Psi_A(z) - \delta\}\Big|_{z = z^M(A)} > 0, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\{\lambda \Psi_A(z) - \delta\}\Big|_{z = z^H(A)} < 0.$$

Then the economy is in the exact-three regime of Definition 5.4. The steady states 0 and $z^H(A)$ are locally stable, and $z^M(A)$ is locally unstable.

Proof. Equation (35) implies that the sign of $F_A(z)$ for $z > 0$ is the sign of $\lambda \Psi_A(z) - \delta$. The

endpoint condition makes $F_A(z) < 0$ for all sufficiently small $z > 0$ and also gives $F_A(1) < 0$. The upper-contour condition implies $F_A(z) > 0$ exactly on $(z^M(A), z^H(A))$ and $F_A(z) < 0$ outside that interval. Hence F_A has exactly two positive zeros, $z^M(A)$ and $z^H(A)$, in addition to the zero steady state. Transversality gives $F'_A(z^M(A)) > 0$ and $F'_A(z^H(A)) < 0$. Because T_A is increasing, the sign pattern gives monotone local dynamics: points just below $z^M(A)$ move downward, points just above $z^M(A)$ move upward, points near $z^H(A)$ move toward $z^H(A)$ from both sides, and small positive states move toward zero. Thus $z^M(A)$ is locally unstable, while 0 and $z^H(A)$ are locally stable. □

The theorem is a sufficient condition, not a complete classification of the transition map. The single-interval upper-contour assumption rules out multiple humps of the replenishment ratio by construction. If Ψ_A has several separated upper-contour intervals, the same sign logic may generate additional steady states. If the transition map is non-monotone, local sign patterns no longer provide a complete basin characterization, and cycles or more complex one-dimensional dynamics may arise. The numerical robustness exercises below keep track of whether the computed equilibrium remains in the exact-three monotone regime.

A single-peaked replenishment ratio is an easy way to verify the theorem. If Ψ_A has one strict interior maximum, the maximum exceeds δ/λ, and the two endpoint values lie below δ/λ, then the upper contour set in the theorem is a single interval. This condition is more general than the benchmark Beta$(2, 2)$ construction below.

Proposition 6.4 (Primitive hump condition for exact-three dynamics)**.** *For this proposition, replace item 4 of Assumption 4.3 by an arbitrary fixed-cost distribution G on $[0, \bar{f}]$ with continuous density and $G(0) = 0$, define Ω from this distribution, and replace the upper entry bound by $n_{\max}(A) = \mu A/[\kappa - \Omega(\bar{f})]$. Consider the benchmark rent functions $B(z) = \chi z^\alpha$ and $C(n) = n^{-\tau}$ with $\alpha > 1$ and $\tau > 0$. Fix $x \in (0, \bar{f}]$ and define*

$$h_x(z) = \frac{G(xz^\alpha)}{z}, \qquad z > 0,$$

with $h_x(0) = 0$ whenever the continuous extension exists. Suppose h_x is continuous on $[0, 1]$, $h_x(1) > 0$, has a strict interior maximum, and admits a level $1/r$ such that

$$\frac{1}{\max_{z\in[0,1]} h_x(z)} < r < \min\left\{\frac{1}{h_x(1)}, 1\right\},$$

the upper contour set $\{z \in (0, 1) : h_x(z) > 1/r\}$ is a nonempty interval, and the two boundary crossings are transverse. Then there exists a nonempty open set of primitives with $\beta > 0$ and finite κ for which the equilibrium replenishment ratio satisfies the conditions of Theorem 6.3. Hence the economy is in the exact-three regime.

Proof. Pick $n_0 > 0$ and choose μ, A, κ so that $\mu A/\kappa = n_0$ while $\kappa/\bar{f}$ is large. Choose χ so that $qA\chi n_0^{-\tau} = x$ and set $\lambda = r\delta/(qn_0)$. In the limiting economy with $\beta = 0$ and $\kappa/\bar{f} \to \infty$, entry is constant at n_0 and

$$qR_A(z) \to xz^\alpha, \qquad m_A(z) \to qn_0 G(xz^\alpha), \qquad \Psi_A(z) \to qn_0 h_x(z).$$

Thus

$$\lambda\Psi_A(z) - \delta \to \delta\{rh_x(z) - 1\}.$$

The assumed contour and transversality conditions imply that the limiting expression is negative near zero and at one, positive on one interior interval, and has exactly two transverse positive boundary crossings. Uniform convergence of the value, entry, and innovation-flow objects preserves these signs and crossings for all sufficiently small $\beta > 0$ and all sufficiently large finite κ. The strict support, state-space, and contraction inequalities then persist on an open neighborhood of these finite primitives. Proposition 5.1 then gives an increasing transition map on this neighborhood. □

Corollary 6.5 (Basin characterization)**.** *Suppose the economy is in the exact-three regime and T_A is increasing. Then every trajectory with $z_0 < z^M(A)$ converges to 0, and every trajectory with $z_0 > z^M(A)$ converges to $z^H(A)$.*

Proof. In an exact-three regime, $F_A(z) < 0$ on $(0, z^M(A))$, $F_A(z) > 0$ on $(z^M(A), z^H(A))$, and $F_A(z) < 0$ on $(z^H(A), 1]$. Since T_A is increasing, monotone iteration gives convergence to the unique fixed point in the relevant interval. □

Figure 1 summarizes this exact-three geometry. The low steady state and the high steady state are locally stable, while the interior branch is unstable and separates the two basins of attraction.

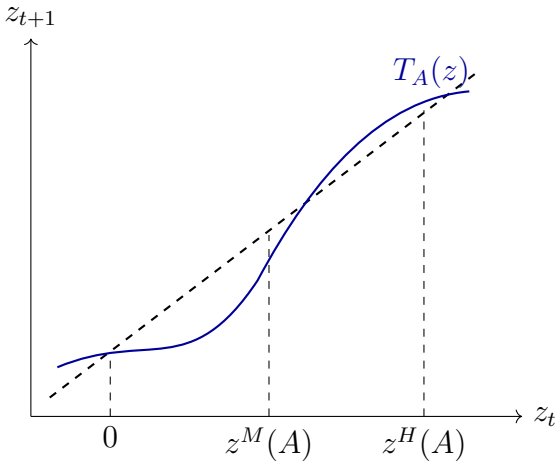


Figure 1: *Schematic exact-three transition map.*

Note. The transition map has a low stable steady state at zero, an unstable interior threshold $z^M(A)$, and a high stable steady state $z^H(A)$. The general exact-three theorem is stated in terms of the replenishment ratio $\Psi_A(z) = m_A(z)/z$. The benchmark economy delivers this geometry from implementation complementarities and a non-convex R&D margin. With a small exogenous knowledge inflow, the corner trap is replaced by a nearby positive low-innovation steady state.

6.3 Benchmark primitives and nonempty exact-three regions

Proposition 6.6 (Benchmark verification of exact-three dynamics)**.** *Consider the benchmark rent functions $B(z) = \chi z^\alpha$ and $C(n) = n^{-\tau}$ with $\alpha > 1$ and $\tau > 0$. There exists a nonempty open set of primitives*

$$(\alpha, \tau, \kappa, \lambda, \bar{f}, \beta, \delta, q, \mu, \chi, A)$$

satisfying Assumptions 4.3–4.5 such that the replenishment-ratio conditions of Theorem 6.3 hold. Hence the equilibrium transition map is in the exact-three regime.

Proof sketch. The construction verifies the primitive hump condition in Proposition 6.4 rather than imposing the transition-map shape. Start from a limiting economy with $\beta = 0$ and entry flat in the state but still responsive to market access. Choose $n_0 > 0$, choose $\bar{x} \in (0, \bar{f})$ close to $\bar{f}$, and set $qA\chi n_0^{-\tau} = \bar{x}$. Let $\kappa \to \infty$ while holding $\mu A/\kappa = n_0$. Then $n_A(z) \to n_0$ uniformly and

$$qR_A(z) \to \bar{x}z^{\alpha}, \qquad m_A(z) \to qn_0 G(\bar{x}z^{\alpha}).$$

Thus the limiting replenishment ratio is

$$\Psi^0(z) = \frac{qn_0 G(\bar{x}z^{\alpha})}{z} = qn_0 h_{\bar{x}}(z), \qquad h_{\bar{x}}(z) = \frac{G(\bar{x}z^{\alpha})}{z}. \tag{36}$$

When $\bar{x} = \bar{f}$, $h_{\bar{f}}(z) = 3z^{2\alpha-1} - 2z^{3\alpha-1}$ is single-peaked and has an interior maximum above its endpoint value. This single-peakedness persists for $\bar{x} < \bar{f}$ close to $\bar{f}$. Choose r so that

$$\frac{1}{\max_z h_{\bar{x}}(z)} < r < 1$$

and set $\lambda = r\delta/(qn_0)$. Then $\lambda\Psi^0(z) - \delta = \delta\{rh_{\bar{x}}(z) - 1\}$ is negative near zero and at one, and positive on one interior interval. Avoiding tangency values gives transverse crossings. Uniform convergence of n_A, R_A, and m_A then preserves the replenishment-ratio conditions for nearby finite economies. Finally choose $\beta_0 > 0$ small enough that $\bar{x}/(1-\beta_0) < \bar{f}$ and choose κ finite but large. Strict inequalities give an open set of finite primitives satisfying the support, state-space, regularity, and replenishment conditions. Appendix A.5 gives the full construction. □

Proposition 6.6 is a verification result. The general theorem above gives the exact-three condition; the benchmark proposition shows that the condition is nonempty under the primitives of the model. The construction uses a static-limit argument to sign the

replenishment ratio and then extends the result to a finite open neighborhood.

6.4 Composite rent compression and threshold tightening

Theorem 6.7 (Primitive threshold tightening under strong competition)**.** *Consider the benchmark $B(z) = \chi z^{\alpha}$ and let rent compression be generated by Proposition 4.1, so that $C(n) = n^{-\tau}$ with $\tau = \sigma + \rho\nu$. Suppose $\alpha > 1$ and $\sigma + \rho\nu > 1$. Fix an exact-three equilibrium generated by Proposition 6.6, with $\beta > 0$ small and κ finite but large. Let $z^M(A)$ be the unstable threshold and set*

$$x_M = qR_A(z^M(A)), \qquad D_M = \kappa - \Omega(x_M),$$

$$a_M = \frac{x_M G(x_M)}{D_M}, \qquad b_M = \frac{x_M g(x_M)}{G(x_M)}.$$

If

$$b_M(\tau - 1) > 1 + a_M, \tag{37}$$

which under Proposition 4.1 is

$$b_M(\sigma + \rho\nu - 1) > 1 + a_M, \tag{38}$$

then, after possibly restricting to a smaller open neighborhood of the baseline primitives,

$$\varepsilon_A^n(z^M) > 0, \qquad \tau\varepsilon_A^n(z^M) > 1, \qquad \varepsilon_A^m(z^M) < 0, \qquad \frac{\mathrm{d}z^M(A)}{\mathrm{d}A} > 0. \tag{39}$$

Holding entry fixed eliminates this channel: in the corresponding exogenous-entry version, market access raises current frontier rents and the innovation flow at the same branch for β sufficiently small.

Proof. The proof starts from the static limit used in Proposition 6.6. When $\beta = 0$, the value

equation is

$$R_A(z) = A\chi z^{\alpha} n_A(z)^{-\tau}.$$

At any interior cutoff $x = qR_A(z)$, define

$$a = \frac{xG(x)}{\kappa - \Omega(x)}, \qquad b = \frac{xg(x)}{G(x)}.$$

Log-differentiating the value equation and the entry rule gives

$$\varepsilon_A^R = 1 - \tau\varepsilon_A^n, \qquad \varepsilon_A^n = 1 + a\varepsilon_A^R.$$

Solving the two equations yields

$$\varepsilon_A^R = \frac{1-\tau}{1+\tau a}, \qquad \varepsilon_A^n = \frac{1+a}{1+\tau a}. \tag{40}$$

Therefore

$$\tau\varepsilon_A^n - 1 = \frac{\tau - 1}{1+\tau a} > 0,$$

so current-flow frontier rents fall with market access. Using (31),

$$\varepsilon_A^m = \frac{1 + a - b(\tau - 1)}{1+\tau a}. \tag{41}$$

Condition (37) makes (41) strictly negative at $z^M(A)$. Under Proposition 4.1, this is condition (38). Since $H_z(A, z^M) > 0$, Theorem 6.1 implies $\mathrm{d}z^M/\mathrm{d}A > 0$.

All inequalities are strict. The exact-three regime, the unstable-branch condition, and the elasticities vary continuously with primitives whenever the recursive fixed point is unique and the branch is transverse. Hence the same signs hold for all $\beta > 0$ sufficiently small and all finite but large κ in a neighborhood of the constructed primitives. Nonemptiness follows because $b_M > 0$ at any interior cutoff and a_M can be made arbitrarily small by choosing κ large. When the composite elasticity $\tau = \sigma + \rho\nu$ is changed through the product-market or scarce-input

primitives, choose χ to keep $qA\chi n_0^{-\tau} = \bar{x}$ in the limiting construction; the limiting cutoff profile and hence b_M are unchanged. Choosing (σ, ρ, ν) so that $\sigma + \rho\nu > 1 + 1/b_M$ then makes (37), equivalently (38), strict. If entry is held fixed, then in the static limit $\varepsilon_A^n = 0$, $\varepsilon_A^R = 1$, $\varepsilon_A^\pi = 1$, and $\varepsilon_A^m = b > 0$. By continuity, the exogenous-entry version has no current-flow reversal and no negative innovation-flow elasticity for small positive β. □

Remark 6.8 (Finite-parameter illustration)**.** Appendix D checks the same mechanism at finite parameter values. The appendix solves the recursive fixed point directly, rather than using the limiting construction. It shows that the replenishment ratio can be hump-shaped, the transition map can have three crossings, and the unstable threshold can rise with market access when the composite rent-compression elasticity is sufficiently large. It also reports nearby parameter checks showing that exact-three dynamics can survive weaker rent compression even when threshold tightening disappears.

Lemma 6.9 (Beta margin and feasible rent-compression thresholds)**.** *Under the Beta*$(2,2)$ *fixed-cost distribution, let* $u = x/\bar{f} \in (0,1)$*. Then*

$$b(u) = \frac{xg(x)}{G(x)} = \frac{6(1-u)}{3-2u}.$$

The function $b(u)$ *is strictly decreasing on* $(0,1)$*, with* $b(u) \uparrow 2$ *as* $u \downarrow 0$ *and* $b(u) \downarrow 0$ *as* $u \uparrow 1$*. For a given local value of* $a \geq 0$*, the condition* $b(u)(\tau - 1) > 1 + a$ *is equivalent to*

$$\tau > 1 + \frac{1+a}{b(u)}.$$

When a *is small, the implied thresholds are:*

$u = x/\bar{f}$	$b(u)$	*approximate lower bound for* τ
0.3	1.75	1.57
0.5	1.50	1.67
0.7	1.125	1.89
0.8	0.857	2.17

Thus the strong-compression condition is nonempty for finite values of the composite elasticity. Entry does not only split product-market demand; it can also bid up scarce frontier inputs, so moderate product-market dilution and moderate input congestion can jointly generate $\tau > 1$.

Proof. Using (7),

$$G(x) = 3u^2 - 2u^3 = u^2(3 - 2u), \qquad xg(x) = \bar{f}u\frac{6\bar{f}u(\bar{f} - \bar{f}u)}{\bar{f}^3} = 6u^2(1 - u).$$

Dividing gives $b(u) = 6(1 - u)/(3 - 2u)$. Differentiation yields $b'(u) = -6/(3 - 2u)^2 < 0$. The equivalence of $b(u)(\tau - 1) > 1 + a$ and the displayed lower bound follows by dividing by $b(u) > 0$. The table reports $1 + 1/b(u)$. □

Theorem 6.7 says where the reversal comes from. Multiple steady states do not require strong rent compression; the benchmark can generate them through the replenishment margin. Strong rent compression matters for the comparative static: it makes market access move the threshold in the wrong direction. With the input-margin interpretation in Proposition 4.1, the condition is not an unrestricted power-form assumption. Product-market rent dilution and scarce-input congestion must be strong relative to the local R&D margin terms a_M and b_M.

Corollary 6.10 (No endogenous entry, no competition reversal)**.** *If entry is exogenous, then* $\varepsilon_A^n(z) = 0$ *and* $\varepsilon_A^\pi(z) = 1$. *Market access raises current frontier rents at every state, and the rent-compression channel in Theorem 6.1 disappears.*

Proof. Set $\varepsilon_A^n(z) = 0$ in (28). □

6.5 Shock direction, duration, and hysteresis

Theorem 6.11 (Shock direction, duration, and hysteresis)**.** *Fix a baseline A in the maintained monotone exact-three regime, and let $z_0 \in (z^M(A), z^H(A))$. Define $I = [z^M(A), z_0]$ and*

$$\bar{H}_I(A) = \max_{z \in I} H(A, z).$$

Let $\widetilde{A}$ be a temporary market-access level, and define the downward-shift amount on I by

$$D_I(A, \widetilde{A}) = \min_{z \in I} \{H(A, z) - H(\widetilde{A}, z)\}. \tag{42}$$

If

$$D_I(A, \widetilde{A}) > \bar{H}_I(A), \tag{43}$$

then the shocked transition map satisfies

$$T_{\widetilde{A}}(z) \leq z - \eta(A, \widetilde{A}) \qquad \text{for all } z \in I, \tag{44}$$

where

$$\eta(A, \widetilde{A}) = D_I(A, \widetilde{A}) - \bar{H}_I(A) > 0. \tag{45}$$

The crossing time is bounded by

$$M(A, \widetilde{A}, z_0) \leq \left\lfloor \frac{z_0 - z^M(A)}{\eta(A, \widetilde{A})} \right\rfloor + 1. \tag{46}$$

The ceiling bound gives weak crossing. The floor-plus-one expression guarantees strict crossing below the old unstable threshold, except in cases where strict crossing is not needed because the perturbed map itself places the state strictly below the threshold earlier. If the shock lasts at least $M(A, \widetilde{A}, z_0)$ periods and market access then returns to A, the economy converges to the low-innovation steady state.

A conventional negative access shock is a sufficient case. If $\widetilde{A} = A - \Delta A$ *and*

$$H_A(a, z) \geq \underline{h} > 0 \qquad \text{for all } a \in [A - \Delta A, A] \text{ and } z \in I,$$

then $D_I(A, \widetilde{A}) \geq \Delta A\, \underline{h}$*. A threshold-tightening positive access shock is another sufficient case. If* $\widetilde{A} = A + \Delta A$ *and*

$$H_A(a, z) \leq -\underline{h} < 0 \qquad \text{for all } a \in [A, A + \Delta A] \text{ and } z \in I,$$

then again $D_I(A, \widetilde{A}) \geq \Delta A\, \underline{h}$*.*

Proof. For any $z \in I$,

$$H(\widetilde{A}, z) \leq H(A, z) - D_I(A, \widetilde{A}) \leq \bar{H}_I(A) - D_I(A, \widetilde{A}) = -\eta(A, \widetilde{A}).$$

Since $H(\widetilde{A}, z) = T_{\widetilde{A}}(z) - z$, this is (44). The state therefore falls by at least $\eta(A, \widetilde{A})$ each period until it crosses below $z^M(A)$. After $\lfloor (z_0 - z^M(A))/\eta(A, \widetilde{A}) \rfloor + 1$ periods the accumulated decline is strictly larger than $z_0 - z^M(A)$, which gives (46). Once the shock ends, Corollary 6.5 applies.

For the negative-access sufficient case,

$$H(A, z) - H(A - \Delta A, z) = \int_{A - \Delta A}^{A} H_A(a, z)\, \mathrm{d}a \geq \Delta A\, \underline{h}.$$

For the positive-access threshold-tightening case,

$$H(A, z) - H(A + \Delta A, z) = -\int_{A}^{A + \Delta A} H_A(a, z)\, \mathrm{d}a \geq \Delta A\, \underline{h}.$$

Both inequalities imply (43) whenever $\Delta A\, \underline{h} > \bar{H}_I(A)$. □

For numerical work, the conservative bound in Theorem 6.11 should be complemented by

the exact crossing time under the shocked transition map,

$$M^*(\Delta A, z_0) = \min\{t \geq 0 : T^t_{\widetilde{A}}(z_0) < z^M(A)\}.$$

For threshold-tightening positive market-access shocks, $\widetilde{A} = A + \Delta A$; for negative market-access shocks, $\widetilde{A} = A - \Delta A$. Appendix D computes $M^*(\Delta A, z_0)$ directly by iteration.

Figure 2 gives the corresponding schematic bifurcation picture. In the threshold-tightening region, higher market access raises the unstable branch and therefore shrinks the basin of attraction of the high-innovation steady state.

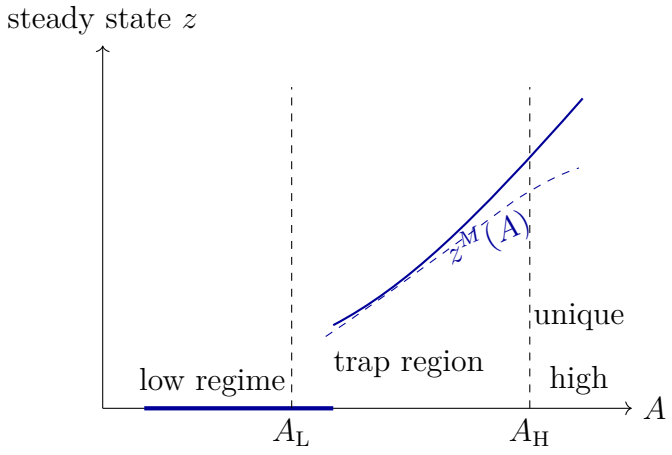


Figure 2: *Schematic market-access bifurcation diagram.*

Note. Solid branches denote locally stable steady states and the dashed branch denotes the unstable threshold. For intermediate market access the economy is in the exact-three regime. In the threshold-tightening region, a marginal increase in market access raises $z^M(A)$ and shrinks the basin of attraction of the high-innovation steady state.

7 Policy and Robustness

The results have a simple policy lesson: market access moves the transition map, not just the current payoff to innovation. A reform that raises accessible demand also changes entry. If the induced entrants compress frontier rents at the R&D margin, the reform can raise the

threshold for self-sustaining innovation. Market access is not a magic wand: it moves both demand and entry.

This is not a welfare ranking. The model does not include consumer surplus, markups, entry costs, or the value of additional varieties. The narrower question is whether a reform expands the set of initial conditions that converge to the high-innovation steady state.

Two forces can undo the threshold-tightening result. First, trade exposure may reallocate activity toward more productive innovators. Second, entry may create agglomeration spillovers that raise, rather than compress, frontier rents. The rent-compression channel is most relevant in sectors where business stealing, scarce frontier inputs, or poaching of specialized talent dominate these offsetting forces.

R&D support moves a different margin. A subsidy that lowers the fixed cost of R&D shifts firms across the innovation cutoff directly. Call a subsidy support-preserving if $qR_{A,s}(z) + s \in [0, \bar{f}]$ for all $z \in [0, 1]$. If the subsidy is support-preserving, the cutoff becomes

$$f \leq qR_{A,s}(z) + s, \tag{47}$$

and successful innovation is

$$m_{A,s}(z) = qn_{A,s}(z)G(qR_{A,s}(z) + s). \tag{48}$$

Unlike market access, this policy does not rely on entry to raise the innovation prize. It pushes on the fixed-cost margin itself. The proposition below asks when that push is large enough to remove the lower basin.

The proposition is deliberately modest. It does not solve for the optimal subsidy. It gives a threshold-removal condition: if the subsidized transition map lies above the 45-degree line throughout the old lower basin, then the trap disappears.

Proposition 7.1 (Conditional subsidy experiment)**.** *Fix A in the exact-three regime. Suppose that, for every $s \in [0, \bar{s}]$, the subsidized recursive equilibrium is well defined, support-preserving,*

continuous in (z, s), and has a transition map $T_{A,s}$ that is weakly increasing in s. Let

$$F_{A,s}(z) = T_{A,s}(z) - z.$$

If for some $\bar{s} > 0$,

$$\min_{z \in [0, z^M(A)]} F_{A,\bar{s}}(z) > 0, \tag{49}$$

then

$$s^\star(A) = \inf \left\{ s \in [0, \bar{s}] : \min_{z \in [0, z^M(A)]} F_{A,s}(z) > 0 \right\} \tag{50}$$

is well defined, and every $s > s^\star(A)$ eliminates the lower basin: $T_{A,s}(z) > z$ for all $z \in [0, z^M(A)]$. If the subsidized map has a unique stable positive steady state above $z^M(A)$, every initial condition $z_0 \in [0, z^M(A)]$ eventually enters its basin.

Proof. The set in (50) is nonempty by (49). Monotonicity in s makes it an upper interval. Therefore every $s > s^\star(A)$ belongs to the set and satisfies $F_{A,s}(z) > 0$ on $[0, z^M(A)]$. States in the former lower basin then move upward rather than toward the trap. The final convergence claim follows from monotonicity of the one-dimensional map and uniqueness of the remaining stable positive steady state. □

The strict inequality in (49) avoids the tangency case in which the old lower basin is only weakly lifted.

Proposition 7.2 (Small exogenous knowledge inflows)**.** *Suppose the baseline economy is in the exact-three regime, the three crossings are transverse, and $T_A(1) < 1$. Add a small exogenous knowledge inflow $\eta \geq 0$ to the transition law,*

$$T_A^\eta(z) = (1 - \delta)z + \lambda m_A^\eta(z) + \eta.$$

Suppose the recursive equilibrium varies continuously in η, and suppose that for all sufficiently small $\eta > 0$, the perturbed transition map remains increasing and maps $[0, 1]$ into itself. Then

the corner steady state is replaced by a locally stable low-innovation steady state $z^L(\eta) > 0$*, with*

$$z^L(\eta) \to 0 \qquad as \qquad \eta \downarrow 0.$$

The unstable threshold and high steady state also persist and remain close to their baseline values.

Proof. Because $T_A(1) < 1$, continuity of the perturbed equilibrium in η implies that $T_A^\eta(1) < 1$ for all sufficiently small $\eta > 0$. Together with the maintained monotonicity of T_A^η, this gives $T_A^\eta([0,1]) \subseteq [0,1]$.

At $\eta = 0$, the low crossing is at the boundary and is locally stable. The perturbation satisfies $T_A^\eta(0) > 0$ for $\eta > 0$. Since the baseline surplus is negative just to the right of zero and the crossing is transverse in the one-sided sense, a nearby positive low steady state exists for small $\eta > 0$. Its location converges to zero as $\eta \downarrow 0$. The unstable and high steady states persist by the ordinary implicit-function theorem because their baseline crossings are transverse. □

The absorbing zero state is also not essential. If a small outside knowledge inflow enters the transition law, zero is no longer absorbing. The low trap moves slightly inside the state space, while the unstable and high steady states remain close to their baseline values. In other words, the trap does not disappear when the corner is softened; it just stops sitting exactly at zero. With $\eta > 0$, the numerical exercise uses the natural Bellman equation on $[0,1]$, without imposing $R^\eta(0) = 0$.

8 Conclusion

Market access changes not only the prize for innovation, but also the basin in which innovation is self-sustaining. A larger market raises the gross return to success. It also attracts entry. When the entry response compresses frontier rents at the R&D margin, successful innovation

can fall exactly where it matters most: at the unstable branch.

The mechanism points to three quantities worth measuring: how much entry responds to market access, how many firms sit near the fixed R&D-cost margin, and how strongly frontier rents fall when the race gets crowded. These quantities decide whether market access expands the high-innovation basin or tightens it. Larger markets are not automatically a free lunch for innovation; sometimes entry shows up with the bill.

References

Aghion, Philippe, Nick Bloom, Richard Blundell, Rachel Griffith, and Peter Howitt. 2005. "Competition and Innovation: An Inverted-U Relationship." *Quarterly Journal of Economics* 120 (2): 701–728.

Aghion, Philippe, and Peter Howitt. 1992. "A Model of Growth through Creative Destruction." *Econometrica* 60 (2): 323–351.

Aghion, Philippe, Antonin Bergeaud, Timo Boppart, Peter J. Klenow, and Huiyu Li. 2023. "A Theory of Falling Growth and Rising Rents." *Review of Economic Studies* 90 (6): 2675–2702.

Aghion, Philippe, Antonin Bergeaud, Timo Boppart, Peter J. Klenow, and Huiyu Li. 2025. "Good Rents versus Bad Rents: R&D Misallocation and Growth." NBER Working Paper No. 34190.

Acemoglu, Daron, Ufuk Akcigit, Harun Alp, Nicholas Bloom, and William R. Kerr. 2018. "Innovation, Reallocation, and Growth." *American Economic Review* 108 (11): 3450–3491.

Acemoglu, Daron, and Joshua Linn. 2004. "Market Size in Innovation: Theory and Evidence from the Pharmaceutical Industry." *Quarterly Journal of Economics* 119 (3): 1049–1090.

Acemoglu, Daron, and Fabrizio Zilibotti. 1997. "Was Prometheus Unbound by Chance? Risk, Diversification, and Growth." *Journal of Political Economy* 105 (4): 709–751.

Akcigit, Ufuk, and William R. Kerr. 2018. "Growth through Heterogeneous Innovations." *Journal of Political Economy* 126 (4): 1374–1443.

Arrow, Kenneth J. 1962. "Economic Welfare and the Allocation of Resources for Invention." In *The Rate and Direction of Inventive Activity: Economic and Social Factors*, 609–626. Princeton, NJ: Princeton University Press.

Arkolakis, Costas, Arnaud Costinot, Dave Donaldson, and Andrés Rodríguez-Clare. 2018. "The Elusive Pro-Competitive Effects of Trade." *Review of Economic Studies* 86 (1): 46–80.

Atkeson, Andrew, and Ariel Burstein. 2010. "Innovation, Firm Dynamics, and International Trade." *Journal of Political Economy* 118 (3): 433–484.

Azariadis, Costas, and Allan Drazen. 1990. "Threshold Externalities in Economic Development." *Quarterly Journal of Economics* 105 (2): 501–526.

Bloom, Nicholas, Mirko Draca, and John Van Reenen. 2016. "Trade Induced Technical Change? The Impact of Chinese Imports on Innovation, IT and Productivity." *Review of Economic Studies* 83 (1): 87–117.

Buera, Francisco J., and Ezra Oberfield. 2020. "The Global Diffusion of Ideas." *Econometrica* 88 (1): 83–114.

Coelli, Federica, Andreas Moxnes, and Karen Helene Ulltveit-Moe. 2022. "Better, Faster, Stronger: Global Innovation and Trade Liberalization." *Review of Economics and Statistics* 104 (2): 205–216.

Cooper, Russell, and Andrew John. 1988. "Coordinating Coordination Failures in Keynesian Models." *Quarterly Journal of Economics* 103 (3): 441–463.

De Loecker, Jan, Jan Eeckhout, and Gabriel Unger. 2020. "The Rise of Market Power and the Macroeconomic Implications." *Quarterly Journal of Economics* 135 (2): 561–644.

De Ridder, Maarten. 2024. "Market Power and Innovation in the Intangible Economy." *American Economic Review* 114 (1): 199–251.

Doraszelski, Ulrich, and Jordi Jaumandreu. 2013. "R&D and Productivity: Estimating Endogenous Productivity." *Review of Economic Studies* 80 (4): 1338–1383.

Ericson, Richard, and Ariel Pakes. 1995. "Markov-Perfect Industry Dynamics: A Framework for Empirical Work." *Review of Economic Studies* 62 (1): 53–82.

Grossman, Gene M., and Elhanan Helpman. 1991a. *Innovation and Growth in the Global Economy.* Cambridge, MA: MIT Press.

Grossman, Gene M., and Elhanan Helpman. 1991b. "Quality Ladders in the Theory of Growth." *Review of Economic Studies* 58 (1): 43–61.

Howitt, Peter, and David Mayer-Foulkes. 2005. "R&D, Implementation, and Stagnation: A Schumpeterian Theory of Convergence Clubs." *Journal of Money, Credit and Banking* 37 (1): 147–177.

Galor, Oded, and Joseph Zeira. 1993. "Income Distribution and Macroeconomics." *Review of Economic Studies* 60 (1): 35–52.

Jones, Charles I. 1995. "R&D-Based Models of Economic Growth." *Journal of Political Economy* 103 (4): 759–784.

Klette, Tor Jakob, and Samuel Kortum. 2004. "Innovating Firms and Aggregate Innovation." *Journal of Political Economy* 112 (5): 986–1018.

Krugman, Paul. 1991. "History versus Expectations." *Quarterly Journal of Economics* 106 (2): 651–667.

Melitz, Marc J. 2003. "The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity." *Econometrica* 71 (6): 1695–1725.

Melitz, Marc J., and Gianmarco I. P. Ottaviano. 2008. "Market Size, Trade, and Productivity." *Review of Economic Studies* 75 (1): 295–316.

Murphy, Kevin M., Andrei Shleifer, and Robert W. Vishny. 1989. "Industrialization and the Big Push." *Journal of Political Economy* 97 (5): 1003–1026.

Perla, Jesse, Christopher Tonetti, and Michael E. Waugh. 2021. "Equilibrium Technology Diffusion, Trade, and Growth." *American Economic Review* 111 (1): 73–128.

Rivera-Batiz, Luis A., and Paul M. Romer. 1991. "Economic Integration and Endogenous Growth." *Quarterly Journal of Economics* 106 (2): 531–555.

Romer, Paul M. 1990. "Endogenous Technological Change." *Journal of Political Economy* 98 (5, Part 2): S71–S102.

Sampson, Thomas. 2016. "Dynamic Selection: An Idea Flows Theory of Entry, Trade, and Growth." *Quarterly Journal of Economics* 131 (1): 315–380.

Schumpeter, Joseph A. 1942. *Capitalism, Socialism and Democracy.* New York: Harper & Brothers.

Shah, Rohan. 2025. "Boosting Innovation or Entry: What Increases Output More?" *Economic Modelling* 153: 107314.

Vives, Xavier. 2008. "Innovation and Competitive Pressure." *Journal of Industrial Economics* 56 (3): 419–469.

Young, Alwyn. 1998. "Growth without Scale Effects." *Journal of Political Economy* 106 (1): 41–63.

A Derivations and Proof Details

A.1 Closed-form option value

The fixed-cost distribution is

$$G(f) = 3\left(\frac{f}{\bar{f}}\right)^2 - 2\left(\frac{f}{\bar{f}}\right)^3, \qquad g(f) = \frac{6f(\bar{f}-f)}{\bar{f}^3}.$$

For $x \in [0, \bar{f}]$,

$$\Omega(x) = \int_0^x (x-f)\,\mathrm{d}G(f) = \int_0^x (x-f)g(f)\,\mathrm{d}f.$$

Write

$$\Omega(x) = x\int_0^x g(f)\,\mathrm{d}f - \int_0^x fg(f)\,\mathrm{d}f = xG(x) - \int_0^x fg(f)\,\mathrm{d}f.$$

The first term is

$$xG(x) = 3\frac{x^3}{\bar{f}^2} - 2\frac{x^4}{\bar{f}^3}.$$

The second term is

$$\int_0^x fg(f)\,\mathrm{d}f = \int_0^x \left(6\frac{f^2}{\bar{f}^2} - 6\frac{f^3}{\bar{f}^3}\right)\mathrm{d}f = 2\frac{x^3}{\bar{f}^2} - \frac{3}{2}\frac{x^4}{\bar{f}^3}.$$

Therefore

$$\Omega(x) = \frac{x^3}{\bar{f}^2} - \frac{x^4}{2\bar{f}^3}.$$

Differentiating gives

$$\Omega'(x) = 3\frac{x^2}{\bar{f}^2} - 2\frac{x^3}{\bar{f}^3} = G(x),$$

and $\Omega(\bar{f}) = \bar{f}/2$.

A.2 Entry and innovation derivatives

The free-entry condition is

$$\frac{\mu A}{n_A(z)} + \Omega(qR_A(z)) = \kappa.$$

Solving for $n_A(z)$ yields

$$n_A(z) = \frac{\mu A}{\kappa - \Omega(qR_A(z))}.$$

Let $\Delta_A(z) = \kappa - \Omega(qR_A(z))$. Then

$$n'_A(z) = \mu A \frac{\Omega'(qR_A(z))qR'_A(z)}{\Delta_A(z)^2}.$$

Because $\Omega' = G$ and $n_A = \mu A/\Delta_A$,

$$n'_A(z) = \frac{qn_A(z)G(qR_A(z))R'_A(z)}{\Delta_A(z)}.$$

Since $m_A(z) = qn_A(z)G(qR_A(z))$,

$$m'_A(z) = qn'_A(z)G(qR_A(z)) + q^2 n_A(z)g(qR_A(z))R'_A(z).$$

Substituting the expression for n'_A gives

$$m'_A(z) = q^2 n_A(z)R'_A(z)\left[g(qR_A(z)) + \frac{G(qR_A(z))^2}{\Delta_A(z)}\right].$$

A.3 Primitive contraction check

Let

$$\bar{B} = \max_{z\in[0,1]} B(z), \qquad L_B = \sup_{z\in[0,1]} |B'(z)|,$$

and on $[n_{\min}(A), n_{\max}(A)]$ define

$$\bar{C} = C(n_{\min}(A)), \qquad \bar{C}_1 = \sup |C'(n)|.$$

Also define

$$g_{\max} = \max_{f \in [0,\bar{f}]} g(f) = \frac{3}{2\bar{f}}, \qquad N_R(A) = \frac{\mu A q}{(\kappa - \bar{f}/2)^2},$$

$$M_R(A) = \lambda q \left[N_R(A) + q n_{\max}(A) g_{\max} \right], \qquad F_R(A) = A\bar{B}\bar{C}_1 N_R(A).$$

Lemma A.1 (Sufficient conditions for recursive regularity)**.** *Fix A and $L > 0$. Suppose*

1. $A\bar{B}\bar{C} \leq (1-\beta)\bar{f}/q$;
2. $B'(z)C(n_{\max}(A)) \geq B(z)\bar{C}_1 N_R(A) L$ *for every* $z \in [0,1]$;
3. $AL_B\bar{C} + F_R(A)L + \beta L[(1-\delta) + M_R(A)L] \leq L$;
4. $F_R(A) + \beta + \beta L M_R(A) < 1$.

Then the operator (15) *maps $\mathcal{X}_L^0$ into itself and is a contraction.*

Proof. For any $R \in \mathcal{X}_L^0$, $qR(z) \in [0, \bar{f}]$, so $0 \leq \Omega(qR(z)) \leq \bar{f}/2$. Hence $n(R, z; A) \in [n_{\min}(A), n_{\max}(A)]$. The flow payoff is bounded above by $A\bar{B}\bar{C}$, so condition 1 keeps the fixed point inside $[0, \bar{f}/q]$. At $z = 0$, $R(0) = 0$ and $G(0) = 0$, so the transition argument equals zero; since $B(0) = 0$, $(\mathcal{B}_A R)(0) = 0$. Thus the boundary condition is preserved.

The derivative of candidate entry with respect to the scalar R is bounded by

$$\left| \frac{\partial n(R, z; A)}{\partial R} \right| = \frac{\mu A q G(qR(z))}{[\kappa - \Omega(qR(z))]^2} \leq N_R(A).$$

Thus the flow term has scalar R-derivative bounded by $F_R(A) = A\bar{B}\bar{C}_1 N_R(A)$. The transition argument has scalar R-derivative bounded by

$$M_R(A) = \lambda q [N_R(A) + q n_{\max}(A) g_{\max}].$$

It remains to check monotonicity and the Lipschitz bound. If $R \in \mathcal{X}_L^0$, then $n_z \le N_R(A)L$. The flow derivative satisfies

$$\frac{\mathrm{d}}{\mathrm{d}z}\{AB(z)C(n(R,z;A))\} = A\{B'(z)C(n) + B(z)C'(n)n_z\}.$$

Condition 2 makes this derivative nonnegative for every $R \in \mathcal{X}_L^0$, so the flow component is increasing. The transition argument is increasing because its derivative is at least $1-\delta > 0$. Hence the continuation component is also increasing. The direct z-derivative of the flow term is bounded by $AL_B\bar{C}$, and the indirect part through R is bounded by $F_R(A)L$. The continuation term has Lipschitz bound $\beta L[(1-\delta)+M_R(A)L]$. Condition 3 therefore preserves the Lipschitz class. For two candidate value functions $R_1, R_2 \in \mathcal{X}_L^0$,

$$\|\mathcal{B}_A R_1 - \mathcal{B}_A R_2\|_\infty \le [F_R(A) + \beta + \beta L M_R(A)]\|R_1 - R_2\|_\infty,$$

so condition 4 gives a contraction. □

Corollary A.2 (Open admissible regions)**.** *If the inequalities in Lemma A.1 hold strictly at (A_0, L_0), then they hold on a neighborhood of A_0 with the same L_0. Hence admissibility is an open property.*

Proof. All bounds in Lemma A.1 are continuous in A on the set where $\kappa > \bar{f}/2$. □

A.4 Local differentiability formulas

This subsection records the derivative terms used in Lemma 5.2. For a candidate value function R, write

$$\Delta_R(z) = \kappa - \Omega(qR(z)), \qquad n_R(z;A) = \frac{\mu A}{\Delta_R(z)},$$

and

$$T_R(z;A) = (1-\delta)z + \lambda q n_R(z;A)G(qR(z)).$$

Holding A fixed, the scalar derivative of entry with respect to the value function in direction h is

$$D_R n_R(z; A)[h] = \frac{\mu A q G(qR(z))}{\Delta_R(z)^2} h(z). \tag{51}$$

Holding R fixed, the market-access derivative is

$$\partial_A n_R(z; A) = \frac{\mu}{\Delta_R(z)}. \tag{52}$$

The transition derivative in direction h is

$$D_R T_R(z; A)[h] = \lambda q \left[D_R n_R(z; A)[h] G(qR(z)) + q n_R(z; A) g(qR(z)) h(z) \right]. \tag{53}$$

Holding R fixed,

$$\partial_A T_R(z; A) = \lambda q \partial_A n_R(z; A) G(qR(z)). \tag{54}$$

Under the stronger smoothness assumptions in Lemma 5.2, the formal derivative of the continuation term is

$$D_R\{R(T_R(z; A))\}[h] = h(T_R(z; A)) + R'(T_R(z; A)) D_R T_R(z; A)[h], \tag{55}$$

$$\partial_A\{R(T_R(z; A))\} = R'(T_R(z; A)) \partial_A T_R(z; A). \tag{56}$$

These are formal derivative formulas under the stronger smoothness assumptions. The paper uses them only locally around equilibria where the displayed derivatives are bounded and the fixed-point contraction is uniform.

A.5 Proof of the benchmark exact-three verification

Fix $\alpha > 1$ and consider the normalized function

$$h_{\bar{f}}(z) = \frac{G(\bar{f} z^\alpha)}{z} = 3z^{2\alpha-1} - 2z^{3\alpha-1}.$$

Its derivative is

$$h'_{\bar{f}}(z) = z^{2\alpha-2}\left[3(2\alpha-1) - 2(3\alpha-1)z^{\alpha}\right].$$

The unique critical point is

$$z_\alpha = \left(\frac{3(2\alpha-1)}{2(3\alpha-1)}\right)^{1/\alpha} \in (0,1).$$

The derivative is positive below z_α and negative above it, so z_α is the unique maximum. Moreover $h_{\bar{f}}(1) = 1$, while $h'_{\bar{f}}(1) = -1 < 0$, so the interior maximum exceeds one.

For $\bar{x} < \bar{f}$, define

$$h_{\bar{x}}(z) = \frac{G(\bar{x}z^{\alpha})}{z}.$$

On every compact subinterval of $(0,1]$, $h_{\bar{x}}$ and $h'_{\bar{x}}$ converge uniformly to $h_{\bar{f}}$ and $h'_{\bar{f}}$ as $\bar{x} \uparrow \bar{f}$. Near zero, $h_{\bar{x}}(z) = O(z^{2\alpha-1})$, so the convergence is also uniform after extending $h_{\bar{x}}(0) = 0$. Hence for $\bar{x}$ close enough to $\bar{f}$, the derivative remains positive on $[\epsilon, z_\alpha - \epsilon]$ and negative on $[z_\alpha + \epsilon, 1]$ for small $\epsilon > 0$, while the maximum remains strict and exceeds one. Thus $h_{\bar{x}}$ has a strict interior maximizer and

$$\max_{z\in[0,1]} h_{\bar{x}}(z) > 1 > h_{\bar{x}}(1) = G(\bar{x}).$$

Choose

$$\frac{1}{\max_z h_{\bar{x}}(z)} < r < 1 \tag{57}$$

and choose r away from the finite set of tangency values generated by local extrema of $h_{\bar{x}}$. The limiting surplus

$$F^0(z) = \delta z\{r h_{\bar{x}}(z) - 1\}$$

is negative for small positive z, positive at the maximizer of $h_{\bar{x}}$, and negative at $z = 1$. Because r avoids tangencies, the two positive crossings are transverse. Since $h_{\bar{x}}$ is single-peaked in this neighborhood of $\bar{f}$, the upper contour set $\{z : h_{\bar{x}}(z) > 1/r\}$ is a single interval. The sign

pattern therefore gives exactly two positive zeros.

Embed this limiting map in primitives. Pick $n_0 > 0$ and choose μ, A, κ so that $\mu A/\kappa = n_0$ while $\kappa/\bar{f}$ is arbitrarily large. Choose χ so that

$$qA\chi n_0^{-\tau} = \bar{x}.$$

Set

$$\lambda = \frac{r\delta}{qn_0}.$$

Since $r < 1$ and $n_{\max}(A) \to n_0$, the state-space condition $\lambda q n_{\max}(A) < \delta$ holds strictly for large finite κ. Choose $\beta > 0$ small enough that $\bar{x}/(1-\beta) < \bar{f}$; then the support restriction holds strictly for nearby finite primitives. Let $\beta \downarrow 0$ and $\kappa/\bar{f} \to \infty$ along this construction. Since $\Omega(qR)/\kappa \to 0$ uniformly, entry converges uniformly to n_0. Since the Bellman equation has discount factor tending to zero, the value function converges uniformly to the current payoff,

$$R_A(z) \to A\chi z^{\alpha} n_0^{-\tau}.$$

Thus

$$qR_A(z) \to \bar{x}z^{\alpha}, \qquad m_A(z) \to qn_0 G(\bar{x}z^{\alpha}),$$

and

$$F_A(z) = T_A(z) - z \to \delta z\{rh_{\bar{x}}(z) - 1\}$$

uniformly. To rule out extra crossings, take disjoint neighborhoods around the two limiting positive roots and use transversality to preserve exactly one root in each. On the compact complement of these neighborhoods and a small neighborhood to the right of zero, the limiting surplus is bounded away from zero, so uniform convergence preserves its sign there. Hence nearby finite economies have exactly two positive roots and the same stability signs.

Finally, choose a small but strictly positive β_0 and a large but finite κ for which all the preceding inequalities hold strictly. Lemma A.1 holds for β_0 small and κ large because

$N_R(A) \to 0$ and the contraction coefficient is strictly below one. Strict inequalities persist on an open neighborhood of these finite primitives. This proves a nonempty open exact-three region with positive discounting and finite entry costs.

A.6 Market-access derivative identities

Let $\Delta_A(z) = \kappa - \Omega(qR_A(z))$. From (14),

$$\log n_A(z) = \log \mu + \log A - \log \Delta_A(z).$$

Differentiate with respect to $\log A$:

$$\varepsilon_A^n(z) = 1 + \frac{AqG(qR_A(z))\partial_A R_A(z)}{\Delta_A(z)}.$$

Since $\varepsilon_A^R = A\partial_A R_A / R_A$, this is (32).

For innovation flow,

$$\log m_A(z) = \log q + \log n_A(z) + \log G(qR_A(z)).$$

Differentiating gives

$$\varepsilon_A^m(z) = \varepsilon_A^n(z) + \frac{qR_A(z)g(qR_A(z))}{G(qR_A(z))}\varepsilon_A^R(z).$$

Finally,

$$H_A(A, z) = \partial_A T_A(z) = \lambda \partial_A m_A(z) = \frac{\lambda m_A(z)}{A}\varepsilon_A^m(z).$$

At an unstable branch, $\mathrm{d}z^M/\mathrm{d}A = -H_A/H_z$, so the threshold rises exactly when $\varepsilon_A^m(z^M) < 0$.

A.7 Subsidy cutoff and monotonicity

Under subsidy s, a firm innovates when $qR_{A,s}(z) - f + s \geq 0$, or $f \leq qR_{A,s}(z) + s$. Support preservation keeps this cutoff in $[0, \bar{f}]$, so the mass of successful innovators is (48).

Proposition 7.1 defines

$$\mathcal{S} = \left\{ s \in [0, \bar{s}] : \min_{z \in [0, z^M(A)]} F_{A,s}(z) > 0 \right\}.$$

The set is nonempty by assumption. If $s \in \mathcal{S}$ and $s' > s$, monotonicity of $T_{A,s}$ in s implies $s' \in \mathcal{S}$. Hence $\mathcal{S}$ is an upper interval, and $s^\star = \inf \mathcal{S}$ has the property that all $s > s^\star$ lie in $\mathcal{S}$. No claim is made at $s = s^\star$, where a tangency can occur.

B Benchmark Comparisons

B.1 Exogenous entry

If entry is fixed, $\varepsilon_A^n(z) = 0$. Equation (28) gives $\varepsilon_A^\pi(z) = 1$. Market access then raises current frontier rents one-for-one, and the entry-induced rent-compression channel disappears.

B.2 No fixed-cost heterogeneity

Suppose $f = 0$ almost surely, so every entrant innovates. Then $G(qR_A(z)) = 1$ and $m_A(z) = qn_A(z)$. The transition derivative becomes

$$T_A'(z) = 1 - \delta + \lambda q n_A'(z).$$

If

$$\sup_{z \in [0,1]} \lambda q n_A'(z) < \delta,$$

then $T_A'(z) < 1$ for every z, so the transition map crosses the 45-degree line at most once. The density term $g(qR_A(z))$ is what creates the interior amplification in the benchmark model.

B.3 Low complementarity

Under $B(z) = \chi z^{\alpha}$, the direct state derivative of current frontier rents is proportional to $z^{\alpha-1}$. As $\alpha \downarrow 1$, this term no longer vanishes at the origin. Low complementarity weakens the damping force that helps generate the low-innovation basin.

B.4 No business-stealing dominance

For the benchmark $C(n) = n^{-\tau}$, current-flow reversal requires $\tau\varepsilon_A^n(z) > 1$. Under Proposition 4.1, $\tau = \sigma + \rho\nu$, so the same condition can be written $(\sigma + \rho\nu)\varepsilon_A^n(z) > 1$. If this inequality fails throughout the relevant state range, market access raises current frontier rents everywhere. Corollary 6.2 shows that when $\sigma + \rho\nu \leq 1$, current-flow reversal and threshold tightening cannot occur at the same interior cutoff. Strong rent compression is therefore the benchmark region in which the full market-access reversal can operate, and it can come from product-market dilution, scarce-input congestion, or both.

C Primitive Interpretation

C.1 Economic meaning of the key objects

The function $B(z)$ measures implementation capability. A steep B at intermediate states means that specialized suppliers, skills, or complementary know-how raise the productivity of frontier implementation. The function $C(n)$ measures rent compression from entry. Its elasticity $\theta_C(n)$ is the local competition elasticity of frontier rents. The distribution G measures the mass of firms close to the R&D margin. The entry cost κ controls how strongly entry reacts to the option value of innovation.

C.2 Support and state-space bounds

Since C is decreasing and $n_A(z) \geq n_{\min}(A)$,

$$AB(z)C(n_A(z)) \leq A\bar{B}C(n_{\min}(A)).$$

The Bellman equation gives

$$R_A(z) \leq \frac{A\bar{B}C(n_{\min}(A))}{1-\beta}.$$

Thus the sufficient support condition is

$$\frac{qA\bar{B}C(n_{\min}(A))}{1-\beta} \leq \bar{f}.$$

Similarly, since $G \leq 1$ and $n_A(z) \leq n_{\max}(A)$,

$$T_A(z) \leq (1-\delta)z + \lambda q n_{\max}(A) \leq (1-\delta) + \lambda q n_{\max}(A).$$

The condition $\lambda q n_{\max}(A) \leq \delta$ keeps the state in $[0,1]$.

C.3 General fixed-cost margin

The Beta$(2,2)$ distribution is a convenient sufficient case, not an essential primitive. The construction of Proposition 6.6 only uses the normalized margin profile

$$h_x(z) = \frac{G(xz^\alpha)}{z}$$

for some cutoff scale $x \in (0, \bar{f}]$. The construction extends to any fixed-cost distribution for which h_x is continuous on $[0,1]$, single-peaked, has a strict interior maximum, and that maximum is larger than both endpoint values. Equivalently, for the chosen level $1/r$, the upper contour set $\{z : h_x(z) > 1/r\}$ is a single interval. This no-extra-crossing property rules

out additional positive steady states in the limiting map. Under these conditions, any

$$\frac{1}{\max_z h_x(z)} < r < \min\left\{\frac{1}{h_x(1)}, 1\right\}$$

that avoids tangencies generates the same exact-three limiting map. The rescaled Beta$(2,2)$ benchmark satisfies this condition because the corresponding h_x is single-peaked for x close to $\bar{f}$.

C.4 What the constructive theorem adds

The benchmark verification uses two forces most directly. First, $B'(0) = 0$ and $g(0) = 0$ make the map flat near the origin. Second, the Beta$(2,2)$ density creates a large mass of marginal firms at intermediate cutoffs. Saturation of G helps flatten the high-state region. Rent compression through $C'(n) < 0$ is essential for the market-access threshold movement in Theorem 6.7, rather than for the mere existence of multiple steady states. The proof of Proposition 6.6 shows that the exact-three geometry is compatible with the contraction and bounded-state conditions.

D Numerical Benchmark

This appendix reports numerical robustness exercises for the finite-parameter benchmark. The purpose is not calibration. The purpose is to check that the exact-three geometry and the threshold-tightening comparative static are not artifacts of a single parameter vector or a single set of functional forms. We vary the rent-compression elasticity, the discount factor, entry costs, implementation complementarity, depreciation, innovation productivity, and success probability. We also replace the benchmark power rent-compression function, power implementation function, and Beta$(2,2)$ fixed-cost distribution with alternative specifications. Each exercise solves the full recursive fixed point and then classifies whether the transition map remains in the monotone exact-three regime.

D.1 Parameters and algorithm

The benchmark parameters are:

$\bar{f}$	q	β	δ	α	τ	κ	μ	A	λ	χ
1	0.8	0.02	0.1	1.2	2	50	50	1	0.12	1.2375

The choice sets $n_0 = \mu A/\kappa = 1$, uses $\bar{x} = 0.99$ and $r = 0.96$, and then sets

$$\lambda = \frac{r\delta}{qn_0} = 0.12, \qquad \chi = \frac{\bar{x}}{qAn_0^{-\tau}} = 1.2375.$$

Large κ makes entry close to the flat-entry construction, small β keeps the model close to the static benchmark, $\bar{x}$ close to $\bar{f}$ places many firms near the R&D margin at intermediate states, and $\tau = 2$ gives strong composite rent compression.

The computation uses a uniform grid of 4001 points on $[0, 1]$. Starting from $R^{(0)}(z) = 0$, it iterates

$$n^{(k)}(z) = \frac{\mu A}{\kappa - \Omega(qR^{(k)}(z))}, \qquad m^{(k)}(z) = qn^{(k)}(z)G(qR^{(k)}(z)),$$

$$T^{(k)}(z) = (1-\delta)z + \lambda m^{(k)}(z),$$
$$R^{(k+1)}(z) = A\chi z^{\alpha}\left[n^{(k)}(z)\right]^{-\tau} + \beta R^{(k)}(T^{(k)}(z)).$$

The algorithm uses linear interpolation for the continuation value and imposes $R^{(k+1)}(0) = 0$. The iteration stops when $\max_z |R^{(k+1)}(z) - R^{(k)}(z)| < 10^{-12}$. In the reported run, the final sup-norm error is 3.33×10^{-13} after eight iterations.

D.2 Exact-three geometry

Figure 3 plots the replenishment object $\lambda\Psi_A(z)$ against δ. The curve is below δ near zero and near one, but above δ at intermediate states. Thus successful innovation replenishes depreciating capability only in the middle range.

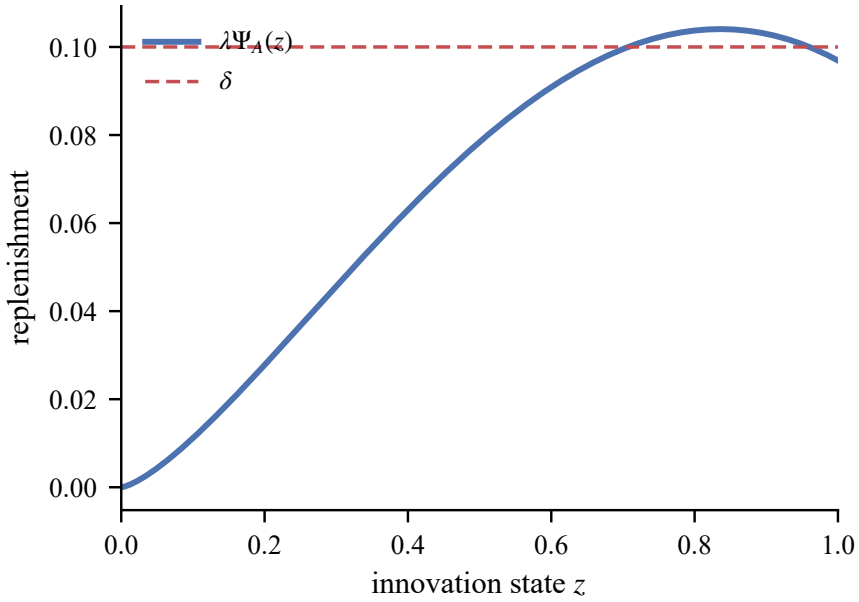


Figure 3: *Finite-parameter replenishment ratio.*

Note. The benchmark has $\lambda\Psi_A(z) < \delta$ near the lower and upper boundaries and $\lambda\Psi_A(z) > \delta$ on an interior interval.

Figure 4 plots the corresponding transition map. The three steady states are

$$0, \qquad z^M(A) = 0.7074, \qquad z^H(A) = 0.9602.$$

The middle crossing is the unstable threshold and the high crossing is the high-innovation steady state.

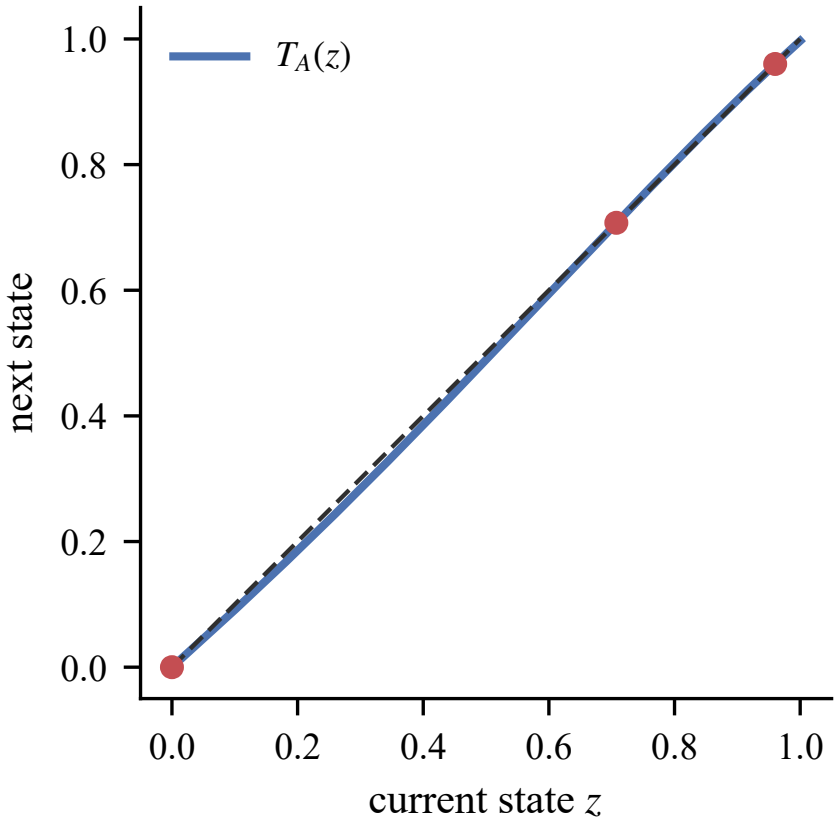


Figure 4: *Finite-parameter transition map.*

Note. The solved recursive equilibrium has exactly three crossings with the 45-degree line.

Because the transition map is close to the 45-degree line near the crossings, Figure 5 plots the transition surplus directly. The sign pattern is negative near zero, positive at intermediate states, and negative near one.

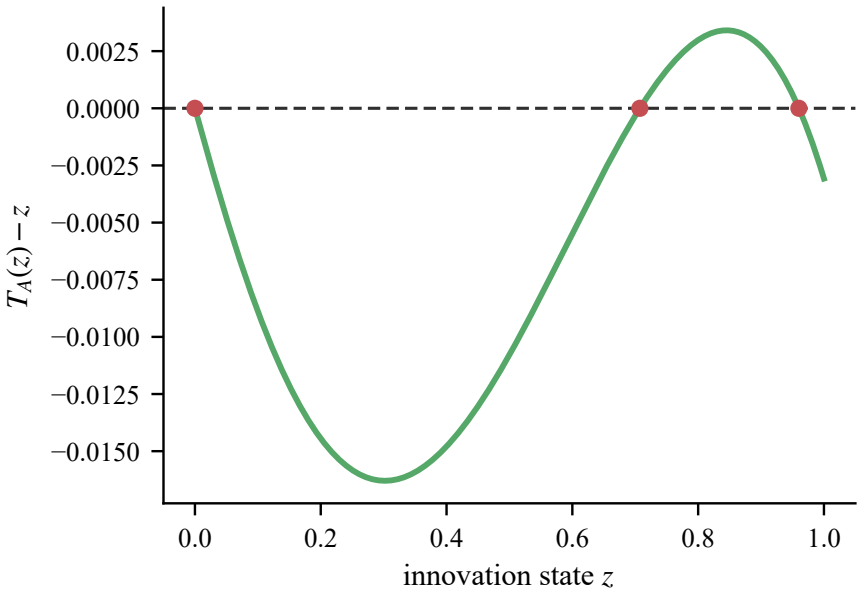


Figure 5: *Finite-parameter transition surplus.*

Note. The surplus $F_A(z) = T_A(z) - z$ crosses zero exactly at 0, $z^M(A)$, and $z^H(A)$.

D.3 Threshold tightening

For each $A \in [0.95, 1.03]$, the recursive fixed point is recomputed and the unstable threshold is located by linear interpolation of $T_A(z) - z$. Figure 6 shows that $z^M(A)$ rises with market access over this range. A linear fit gives slope 0.316, so better market access tightens the high-innovation basin in this finite benchmark.

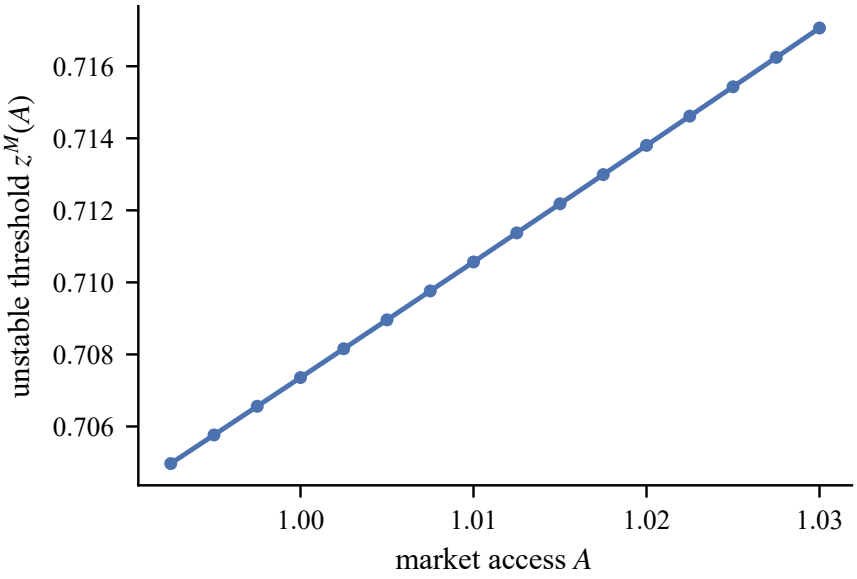


Figure 6: *Finite-parameter threshold movement.*

Note. The unstable threshold rises with market access on the reported interval.

The finite-difference elasticities are evaluated at the fixed baseline threshold $z^M(1) = 0.7074$. For each object $x_A \in \{n_A, \pi_A^F, m_A\}$, the reported number is

$$\varepsilon_A^x(z^M(1)) \approx \frac{\log x_{1+\Delta A}(z^M(1)) - \log x_{1-\Delta A}(z^M(1))}{\log(1+\Delta A) - \log(1-\Delta A)}, \qquad \Delta A = 0.001.$$

The values are:

$\varepsilon_A^n(z^M)$	$\varepsilon_A^\pi(z^M)$	$\varepsilon_A^m(z^M)$
0.990	−0.981	−0.198

Figure 7 displays the same decomposition. Market access raises entry, entry compresses frontier rents strongly enough that current-flow frontier rents fall, and the innovation flow falls at the unstable threshold. This is the numerical counterpart of the threshold-tightening condition in Theorem 6.7.

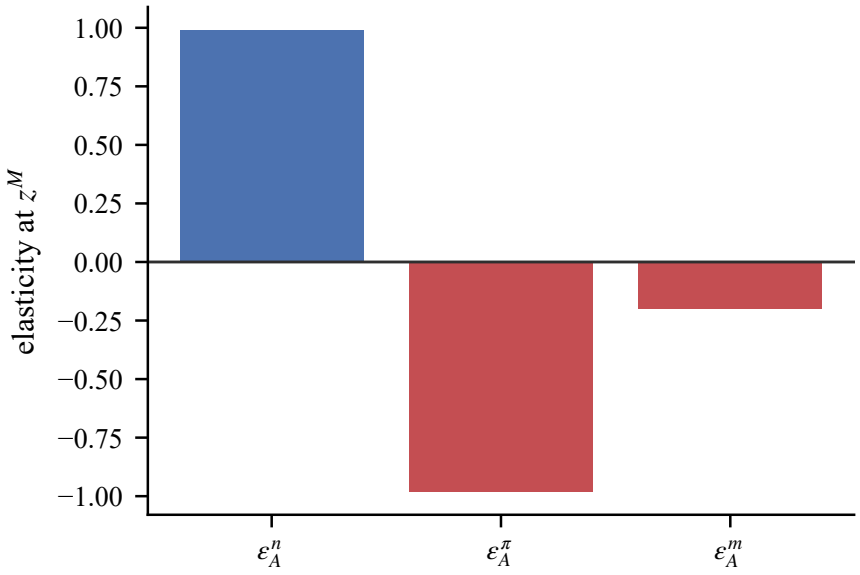


Figure 7: *Finite-parameter elasticity decomposition.*

Note. At the unstable threshold, entry rises with market access, current-flow frontier rents fall, and the innovation flow falls.

Table 1 reports nearby numerical checks. When κ is lowered, μ is lowered proportionally to keep $n_0 = \mu A/\kappa = 1$, so the perturbation isolates the finite-entry-cost margin rather than changing the baseline entry mass. The final row shows that exact-three dynamics can survive weaker rent compression even when threshold tightening does not.

Table 1: *Finite-parameter robustness checks.*

Case	β	κ	μ	τ	exact-three?	$\varepsilon_A^m(z^M)$	$\mathrm{d}z^M/\mathrm{d}A > 0$?
Baseline	0.02	50	50	2.0	yes	-0.198	yes
Higher discounting	0.05	50	50	2.0	yes	-0.282	yes
Lower entry cost, same n_0	0.02	30	30	2.0	yes	-0.182	yes
Weaker compression	0.02	50	50	1.9	yes	-0.080	yes
Insufficient compression	0.02	50	50	1.7	yes	0.157	no

Notes. Exact-three is checked at the baseline $A = 1$ and over the interval $A \in [0.95, 1.03]$ used for the threshold movement exercise. The elasticity $\varepsilon_A^m(z^M)$ is computed by the same central finite difference as Figure 7, evaluated at each row's baseline unstable threshold $z^M(1)$. The sign of $\mathrm{d}z^M/\mathrm{d}A$ is computed from a linear fit of the solved unstable thresholds over the same interval. In the lower-entry-cost row, μ is lowered with κ so that $n_0 = \mu A/\kappa$ remains equal to one.

D.4 Parameter sweeps

The first robustness exercise varies one primitive at a time and resolves the recursive equilibrium at each grid point. Figures 8–11 report the main one-dimensional sweeps. The τ sweep changes the composite rent-compression elasticity. The β sweep changes the weight on continuation values. The two entry-cost sweeps separate changes in the option-value feedback from changes in the baseline mass of entrants: the first varies κ while changing μ to keep $n_0 = \mu A/\kappa = 1$, and the second varies κ holding $\mu = 50$ fixed. Wider ranges produce many inadmissible parameter points because the support and state-space restrictions fail; the reported ranges focus on the admissible neighborhood around the benchmark.

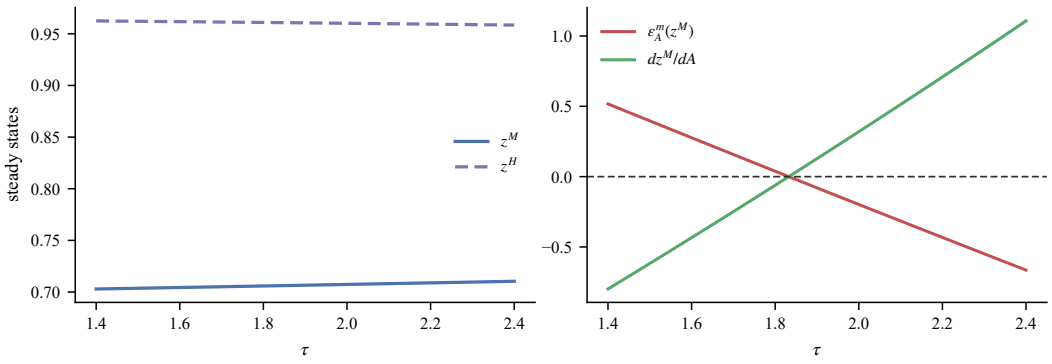


Figure 8: *Rent-compression elasticity sweep.*

Note. Each point solves the full recursive equilibrium after varying τ and holding the other benchmark parameters fixed. Threshold tightening appears when $dz^M/dA > 0$, equivalently when the innovation-flow elasticity at the unstable branch is negative up to numerical tolerance.

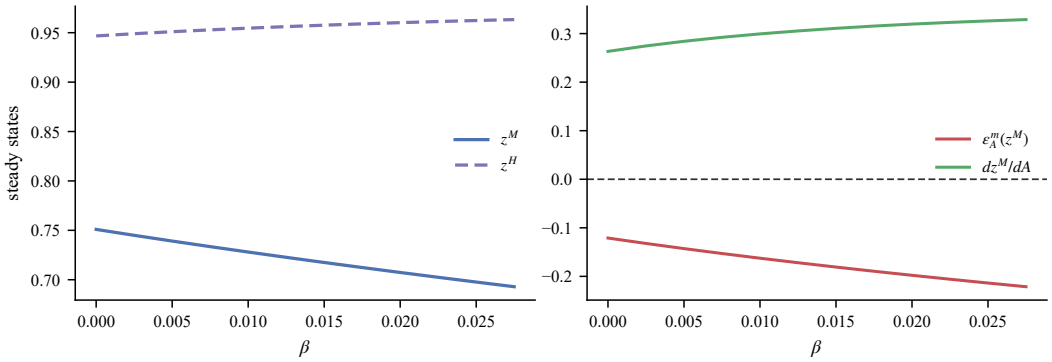


Figure 9: *Discount-factor sweep.*

Note. The sweep varies β and recomputes the equilibrium at each grid point. The figure reports the unstable threshold and the local market-access comparative static where the solved map remains admissible and exact-three.

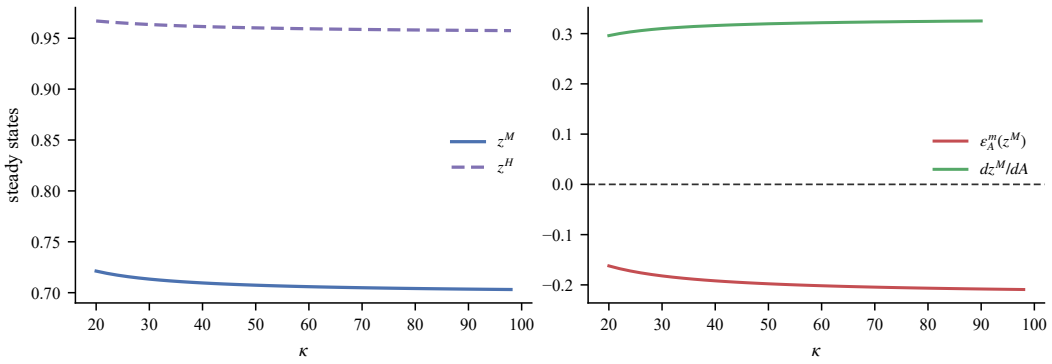


Figure 10: *Entry-cost sweep with fixed baseline entry.*

Note. The sweep varies κ while setting $\mu = \kappa$, so $n_0 = \mu A/\kappa$ remains equal to one. This isolates how the option-value feedback in the entry denominator affects the threshold.

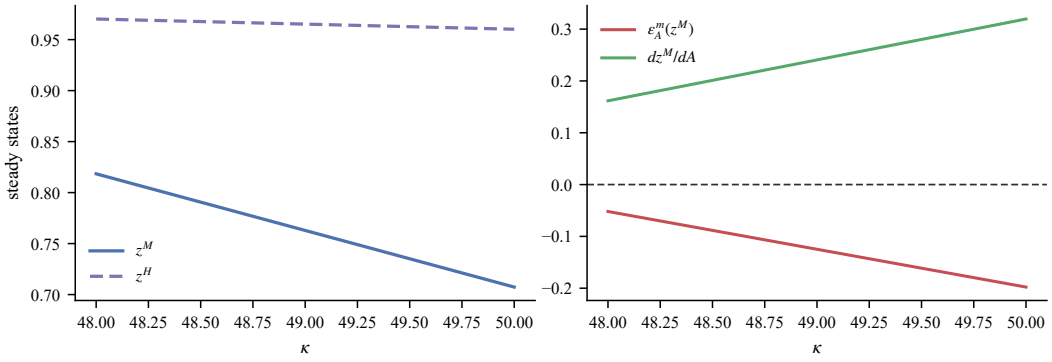


Figure 11: *Entry-cost sweep with changing baseline entry.*

Note. The sweep varies κ holding $\mu = 50$ fixed. This changes the baseline mass of entrants and separates the entry-level effect from the option-value feedback isolated in Figure 10.

Table 2: *One-dimensional parameter sweeps.*

Parameter	Range	Points	Admissible	Exact-three	Tightening	$\varepsilon_A^m < 0$	z^M range
τ	[1.4,2.4]	41	1.00	1.00	0.56	0.56	0.703–0.710
β	[0,0.08]	33	0.36	1.00	1.00	1.00	0.693–0.751
κ fixed n_0	[20,100]	41	0.98	1.00	0.90	1.00	0.703–0.721
κ fixed μ	[20,100]	41	0.39	0.12	1.00	1.00	0.707–0.818
α	[1.05,2]	39	1.00	0.64	0.56	0.56	0.585–0.911
δ	[0.05,0.2]	31	0.68	0.05	1.00	1.00	0.707–0.707
λ	[0.05,0.25]	41	0.37	0.07	1.00	1.00	0.707–0.707
q	[0.4,0.95]	34	0.74	0.08	0.50	0.50	0.707–0.829

Notes. Each row varies one parameter over the displayed range and holds all other parameters at their benchmark values, except for the fixed-n_0 entry-cost row, which varies μ with κ. A point is classified as threshold-tightening when the model is admissible, the transition map is in the exact-three monotone regime, and $dz^M/dA > 0$.

D.5 Parameter-space heat maps

Figure 12 and Figure 13 classify two-dimensional parameter regions. Each cell solves the recursive equilibrium, checks admissibility, verifies exact-three monotone dynamics, and then computes whether the unstable threshold rises with market access.

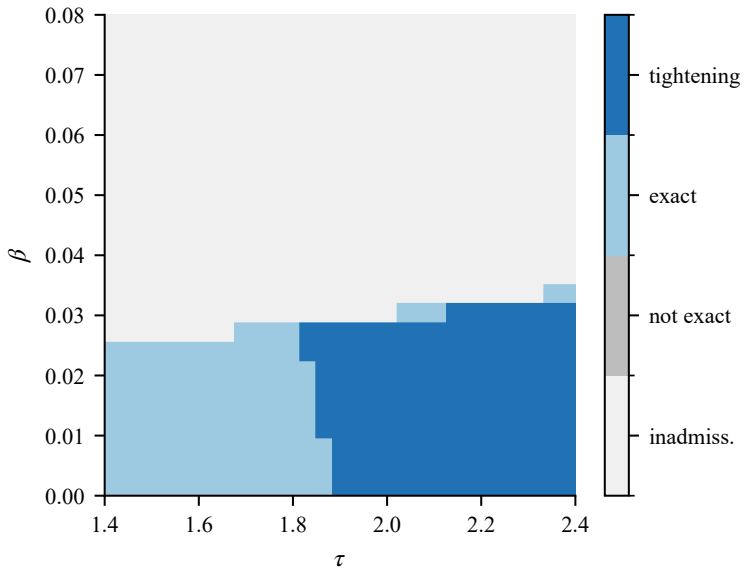


Figure 12: *Parameter-space classification: τ and β.*

Note. Each cell solves the full recursive equilibrium. Darker cells denote exact-three parameter points with threshold tightening. The figure shows that the mechanism is not confined to the single benchmark parameter vector.

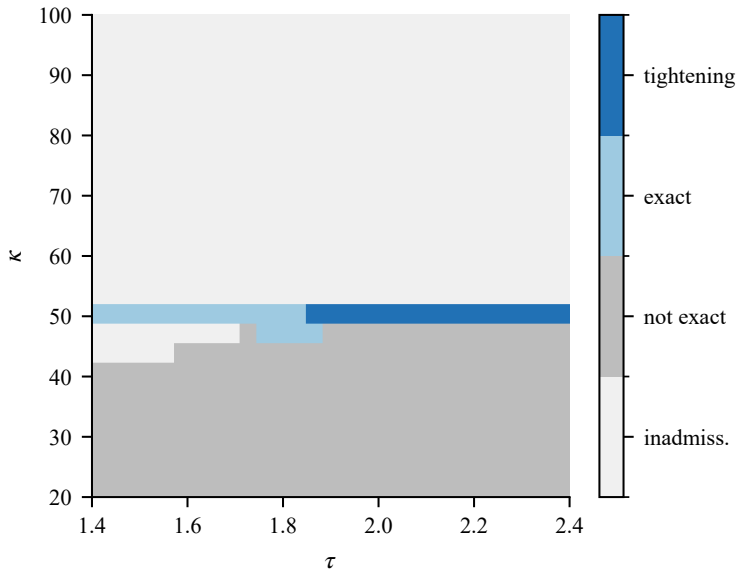


Figure 13: *Parameter-space classification: τ and κ.*

Note. The heat map varies rent compression and entry costs holding other primitives at their benchmark values. The classification uses $dz^M/dA > 0$ for threshold tightening.

D.6 Alternative functional forms

The benchmark uses $B(z) = \chi z^{\alpha}$, $C(n) = n^{-\tau}$, and a rescaled Beta$(2, 2)$ fixed-cost distribution. These choices make the mechanism transparent, but they are not essential to the numerical result. Table 3 repeats the benchmark exercise under alternative implementation, rent-compression, and fixed-cost specifications. The table reports whether the solved transition map remains in the exact-three monotone regime, whether the innovation-flow elasticity at the unstable branch is negative, and whether the unstable threshold rises with market access. The threshold-tightening result persists for the reported alternatives that remain exact-three; the Beta$(3, 2)$ row is not exact-three at the benchmark parameter vector.

Table 3: *Alternative functional-form robustness.*

Specification	Exact-three	z^M	z^H	$\varepsilon_A^m(z^M)$	dz^M/dA	Basin size
Benchmark	yes	0.707	0.960	-0.198	0.320	0.293
Shifted-power $C(n)$	yes	0.707	0.960	-0.201	0.324	0.293
Exponential $C(n)$	yes	0.707	0.960	-0.207	0.335	0.293
Hill $B(z)$	yes	0.119	0.968	-0.829	0.115	0.881
Exp.-saturation $B(z)$	yes	0.200	0.968	-0.756	0.185	0.800
Beta(2,3) G	yes	0.406	0.970	-0.434	0.244	0.594
Beta(3,2) G	no	–	–	–	–	–
Truncated lognormal G	yes	0.606	0.938	-0.344	0.340	0.394
Smoothed uniform G	yes	0.633	0.968	-0.640	0.418	0.367

Notes. Each row changes one functional-form component and holds the remaining primitives at their benchmark values, except for normalizations that match the benchmark local scale or local elasticity at $A = 1$. dz^M/dA is computed by a central finite difference in market access. The table reports whether the solved transition map remains in the exact-three monotone regime and whether threshold tightening persists.

D.7 Small inflow robustness

The absorbing zero state is not essential. If a small outside knowledge inflow enters the transition law, zero is no longer absorbing. The low trap moves slightly inside the state space, while the unstable and high steady states remain close to their baseline values. In other words, the trap does not disappear when the corner is softened; it just stops sitting exactly at zero. The threshold-tightening comparative static persists over the reported range.

Table 4: *Small exogenous knowledge inflows.*

η	z^L	z^M	z^H	$\varepsilon_A^m(z^M)$	dz^M/dA	Basin	Increasing
0e+00	0.0000	0.7074	0.9602	-0.198	0.320	0.293	yes
1e-05	0.0001	0.7071	0.9603	-0.198	0.320	0.293	yes
5e-05	0.0005	0.7062	0.9610	-0.201	0.321	0.294	yes
1e-04	0.0010	0.7051	0.9617	-0.203	0.323	0.295	yes
5e-04	0.0050	0.6962	0.9678	-0.224	0.337	0.304	yes

Notes. For $\eta > 0$, the fixed-point iteration uses the natural Bellman update at $z = 0$ and does not impose $R^\eta(0) = 0$. The table reports the three steady states when the perturbed map remains exact-three and monotone; the low steady state is positive for $\eta > 0$.

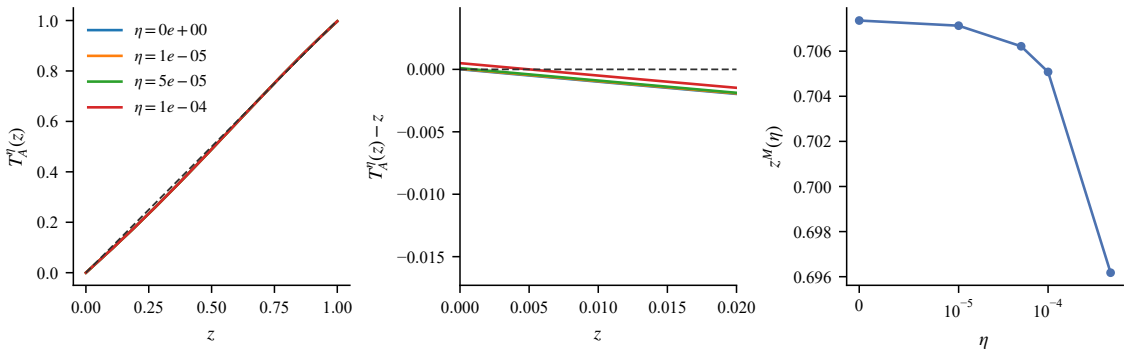


Figure 14: *Small exogenous knowledge inflows.*

Note. A small inflow replaces the absorbing zero state with a nearby positive low-innovation steady state. The unstable and high steady states remain close to the baseline values, and the threshold-tightening comparative static persists for small η.

D.8 Shock-duration simulations

The analytical hysteresis result gives a sufficient duration bound based on the smallest downward shift of the shocked transition map over the relevant interval. This bound is intentionally conservative. We therefore also compute the exact crossing time under the

shocked transition map,

$$M^*(\Delta A, z_0) = \min\{t \geq 0 : T_{\tilde{A}}^t(z_0) < z^M(A)\}.$$

The resulting duration-size curves show the tradeoff between the size of the market-access shock and the number of periods required to push the economy below the original unstable threshold. The reported curves use initial states close to the original threshold so that positive access shocks remain in the admissible local range while still producing finite crossing times.

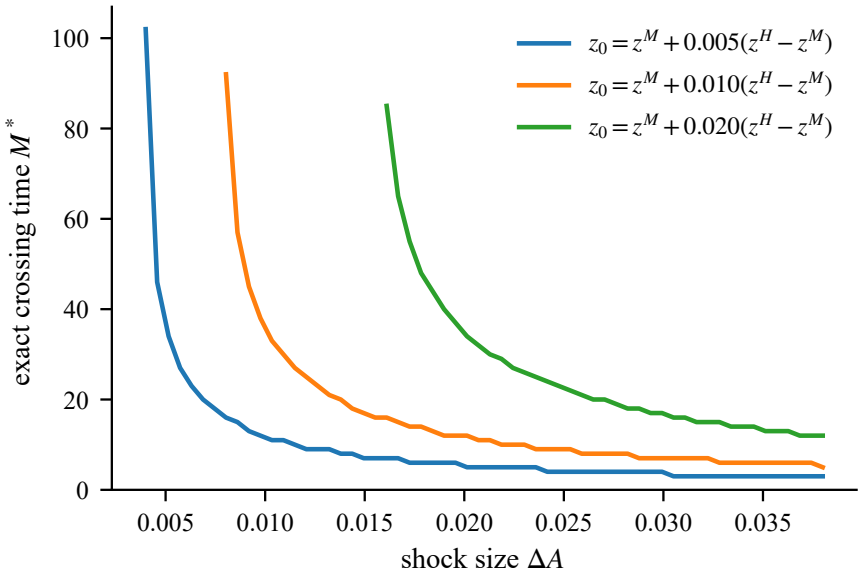


Figure 15: *Shock size and duration.*

Note. The figure reports the exact number of shocked periods required to move the state below the original unstable threshold. Larger shocks require shorter duration, while initial conditions farther inside the high-innovation basin require longer shocks.

D.9 Grid refinement and solver checks

The benchmark quantities are stable across grid refinements. The fixed-point iteration stops when the sup-norm change in the value function falls below 10^{-12}.

Table 5: *Grid-refinement check.*

Grid size	z^M	z^H	$\varepsilon_A^m(z^M)$	dz^M/dA	Residual
2001	0.7074	0.9602	-0.198	0.320	3.33e-13
4001	0.7074	0.9602	-0.198	0.320	3.33e-13

Notes. Both grids use the same fixed-point algorithm and central finite difference in market access. The benchmark run satisfies the support restriction $qR(z) \in [0, \bar{f}]$, has positive entry denominators, and maps the state space into $[0, 1]$.

Code availability. The replication code for the finite-parameter numerical exercise in Appendix D, including the fixed-point solver and scripts generating Figures 3–15 and Tables 1–5, will be made available upon publication. It is available to the editor and referees upon request during the review process.