Widening Currency Bands

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Abstract

I work out an extension of Krugman's (1991) target zone model. The extension consists of announcing a predetermined percentage increase in the width of the currency bands per period. The assumption of the pre-announced widening of the bands is motivated by the framework adopted in Argentina in 2025. I study two features: the first is the extra flexibility that this pre-announced widening of the bands provides, and I define two measures of this anticipation effect. I found that, in terms of flexibility, the widening-band case increases it by up to 40%. The second is that, somewhat surprisingly, early on, the widening bands are more likely to require intervention than the fixed ones. I found that this initial period can persist for up to a year.

Keywords: Exchange rates, Argentina, Anticipation Effect, Free-boundary problem, PDE

Introduction

In April 2025, Argentina implemented a monetary policy framework that consists of an open capital account (at least for current and future transactions) and a target exchange zone, or currency bands. A traditional fixed exchange-rate stabilization preceded this –starting with a large devaluation and a preannounced monthly depreciation rate, accompanied by a sudden, substantial, and persistent fiscal adjustment. In the initial phase of the stabilization plan, which began at the start of 2024, capital account controls and dual exchange rates were in place, inherited from a previous administration. Both types of controls have been dismantled gradually. The new framework, which began in April 2025 and was accompanied by an IMF program and support from other multilateral institutions, ended the dual exchange-rate system and most capital controls. The announced new framework will allow the exchange rate to fluctuate within pre-established bands, where the Central Bank intervenes to keep it within them. The agreement with the IMF allows the use of reserves to defend the parity at the top of the band. President Milei has declared that the Central Bank will purchase

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reserves only at the bottom of the bands. Otherwise, the money supply is announced to be roughly constant.[2] One peculiarity of the new Argentine framework is that the width of the exchange rate bands, which started at a relatively high value, 40%, is announced to widen as time goes by. In particular, the top band will increase by 1% per month, and the bottom band will decrease by 1% per month. This system is designed as a preannounced transition to a flexible exchange rate system. See Appendix Appendix A for more information on Argentina's framework, and the evolution until the time of this writing.

The objective of this note is to write an extension of Krugman's (1991) seminal stylized model of target zones for exchange rates, which incorporates the effects of the pre-announced path for the top and bottom bands. I refer to this system as one with "widening bands", and refer to the Krugman's (1991) framework as one with "fixed bands". I concentrate the analysis on two effects. I called the first one the "anticipation effect": the difference in how much more flexible the exchange rate is for a given band size due to market participants anticipating that the bands will widen in the future. I called the second, the "intertemporal intervention twist": the widening band case requires intervention with higher probability early on, but lower later on.

Krugman's (1991) proposes and works out an elegant solution for the exchange rate as a function of fundamentals for the case of fixed currency bands. He shows that the exchange rate is less sensitive to the fundamentals than under flexible exchange rates, even far from the bands. Moreover, an inextricable part of the solution is determining the range of values for the fundamentals over which the central bank does not intervene. In terms of comparative statics, the wider the bands are, the closer the solution becomes to flexible exchange rates. Unlike Krugman's (1991), when the bands widen continuously, as in the Argentinean case, the function that relates the fundamentals to the nominal exchange rate must also depend on time. In this case, the function for a fixed value of fundamentals changes through time. Moreover, the boundaries of the region for fundamentals over which the central bank does not intervene vary over time. In this paper, I compare the solution with fixed bands with the one for widening bands. I show that, even if the bands for fixed and widening bands are *currently* at the same value, the function linking the exchange rate with the fundamentals is more sensitive to fundamentals in the widening bands case. Or, put differently, with widening bands, the anticipation effect of future expansion makes the exchange rate's behavior closer to that under flexible exchange rates. I present an example calibrated to Argentina, where the behavior of the widening bands is close to halfway between fixed bands and flexible exchange rates. I also analyze conditional volatility and how foreign reserve purchases within the bands affect these results.

The model setup for the fixed- and widening-band cases has the following aspects in common. In both cases, I use the same forward-looking equation relating the exchange rate to the fundamentals, so that the exchange rate determination is similar to that of an asset price, which depends on the expected discounted values of the fundamentals. This relationship can, in turn, be derived from money demand, purchasing power parity, and uncovered interest parity conditions. Importantly, the process for the fundamentals includes the central bank's behavior. An equilibrium requires the central bank policy to implement three features. One

[2]Two caveats on this simple interpretation. The agreement with the IMF also stipulates a target for foreign reserve accumulation. Furthermore, the Argentine central bank has reportedly held positions in the exchange-rate futures market, and it has repeatedly changed reserve requirements.

is the specification of the exogenous part of the process for the fundamentals, which applies when the exchange rate is strictly inside the bands. The second feature is the requirement that the central bank intervene only when the exchange rate reaches either band. The third is that the exchange rate is never outside the bands. The difference between the fixed and widening cases is the following. In the fixed band case, the function describing the exchange rate must satisfy an ordinary differential equation, or an ODE, and the boundary of intervention must be determined. This ODE, using symmetry, is solved with two implied boundary conditions. In the widening band case, the function describing the exchange rate has two arguments: the fundamentals and time, and it satisfies a partial differential equation or a PDE The solution also involves, instead of finding a single value for the range of fundamentals where the central bank does not intervene, as in the fixed case, finding an entire path for this range as a function of time. This PDE and free boundary are solved by imposing two boundary equations that must be satisfied at each time, as well as a terminal condition. In other words, in both cases it is a free-boundary problem; the difference is that in the fixed-bands case, the boundary is a number, while in the widening-bands case, it is an entire function of time. The main technical contribution of this note is to solve the PDE for the widening band case by using a functional expansion of the free boundary case in terms of the rate at which the bands widen. I derived closed-form expressions for the first- and second-order expansions of the exchange rate as a function of fundamentals and time. For the intervention threshold, I only derived the first-order expansion, since this function is not twice differentiable. Furthermore, I proposed to use a recursive version of this expansion, which I argue is more accurate for measuring the anticipation effect. The advantage of the analytic approximations is that they provide closed-form expressions that clearly show which features depend on parameter values and which don't. I develop a finite-difference approximation for the exchange rate as a function of fundamentals and time, which, in principle, can provide an arbitrarily accurate solution for assessing the approximation's accuracy and for considering other cases, such as incorporating drifts.

I derived two closed-form expressions measuring the anticipation effect. Both measures compare the exchange rate's responsiveness to changes in fundamentals by taking the ratio of the indicator for the widening-band case to the same indicator for the fixed-band case. This ratio is defined for the same band's width. The first indicator uses the slope of the equilibrium exchange rate at the center of the band. The second indicator uses the average slope of the equilibrium exchange rate in the no-intervention zone for each case. Both measures show an anticipation effect that decreases as the bands widen, or equivalently, as time passes. This result is intuitive, since for very large bands the exchange is almost flexible in both cases. I also show that, for any band size, the anticipation effect increases the semi-elasticity of money demand. This result is quite intuitive, since the inverse of the semi-elasticity of the interest rate equals the discount rate in the asset-pricing formula. For the calibrated values, the exchange rate is about 40% more volatile than in the fixed-band case.

The second main result is the "intertemporal intervention twist". I show that, somewhat surprisingly, intervention is more likely early on in the widening band case than in the fixed band case. This is a direct consequence of the narrower range of fundamentals where intervention occurs. Later on, when the bands have expanded sufficiently, intervention is less likely in the widening-band case than in the fixed-band case. I derived a closed-form expression for the critical time below which intervention is more likely for widening bands.

This critical time is monotonically increasing with the semi-elasticity of money demand and the size of the initial bands. I found that for plausible parameters, this critical time is around a year.

Finally, I evaluate the accuracy of the first-order expansion by comparing it with the second-order and finite-difference results. I found the first-order expansion quite accurate, and the second-order almost indistinguishable from the finite-difference result. I also compute an analytical joint second-order expansion on the rate of widening of the bands and the drift of the fundamentals.

Related Literature. There is a large literature on currency bands, with Krugman (1991) widely accepted as the seminal paper. Several topics have been investigated in the theoretical literature, including the effects of non-infinitesimal interventions, limited credibility, and the possibility of realignments (i.e., changes in the bands). The literature on the effect of realignments is the closest to the topic of this paper. Examples of related papers on realignments are Svensson (1991), Bertola and Caballero (1992), and Werner (1995), among others. Svensson (1991) incorporates a constant probability of a devaluation. Bertola and Caballero (1992) proposed a Markovian process for the bands, where when the exchange rate gets closer to the current band, the central bank either defends it –by pushing it to the center–, or abandons it by creating a new band displacing it contiguously to the previous one. Werner (1995), following Bertola and Svensson (1993), considers a probability of one realignment proportional to the distance of the fundamentals to the center of the bands. These papers, as well as others, were motivated by empirical evidence on bands used primarily in Europe before the introduction of the Euro, where countries experienced recurrent realignments. In my view, Bertola and Caballero (1992) is the closest setup, in the sense that as time goes by, the variance of the fundamental increases, and hence the changes in a sequence of realignments in which the bands have widened are greater. On technical grounds, Froot and Obstfeld (1991), which clarifies the nature of the interventions, is the closest reference. As far as I know, there was no work on bands that are *announced to continually widen as time goes by*, which is Argentina's framework, as well as the topic of this paper.

Krugman's currency bands' model

In this section, I describe the basic setup in Krugman's seminal currency bands model. I go into some detail on the mathematics of the solution to measure the effect of anticipation and clarify the nature of the extension.

I let m be the log of nominal money supply, p be the log of the domestic price level, p^* be the log of the foreign price level, i the domestic interest rate, i^* the foreign interest rate, s the (log of) the nominal exchange rate, and α be the semi-elasticity of the money demand.

I start describing a simple model of the exchange rate for a small open economy. It has three basic assumptions. First, that Purchasing Power Parity (PPP) holds so that:

$$p(t) = s(t) + p^*(t)\,.$$

Second, that uncovered interest rate parity (UIP) holds:

$$i(t) = i^*(t) + \frac{1}{dt} E\left[ds(t)\right] .$$

Third, that there is equilibrium in the domestic money market with the following semi-log money demand:

$$m(t) - p(t) + v(t) = -\alpha\, i(t) .$$

where v is an exogenous velocity shock and α the semi-elasticity of the money demand.

The next proposition is a standard textbook result, which combines these equations to obtain a well-known stylized model of the exchange rate.

PROPOSITION 1. Define

$$x(t) \equiv \frac{m(t) + v(t) - p^*(t)}{\alpha} + i^*(t) \text{ and } \rho \equiv \frac{1}{\alpha}$$

Combining the UIP-PPP and equilibrium in the money market gives:

$$\rho s(t) = x(t) + \frac{1}{dt} E\left[ds(t)\right] \text{ or } s(t) = \int_t^\infty e^{-\rho(\tau - t)} \mathbb{E}_t\left[x(\tau)\right] d\tau$$

I refer to $x(t)$ as the "fundamentals" determining the exchange rate. It is immediately clear that a larger semi-elasticity of the money demand α, which corresponds to a lower ρ, makes the current exchange rate more sensitive to future values of the fundamentals. This will be crucial for the size of the anticipation effect. I assume that, whenever m is constant, the processes for $i^*(t), p^*(t)$ and $v(t)$ are such that $x(t)$ follows a drift-less random walk in continuous time, i.e.,

$$dx(t) = \sigma dW(t)$$

where W is a standard Brownian Motion. Thus, in the absence of intervention by the Central Bank, the fundamentals are assumed to follow a random walk with no drift (i.e., zero expected change per unit time) and a variance of changes per unit time equal to σ^2.

The exchange rate bands are given by $b > 0 > \underline{s} = -b$ so that if $s(t) \in (-b, b)$ then m is constant, and otherwise m is adjusted so that $s(t) \in [-b, b]$. The adjustment for m implies that as soon as $s(t)$ hits either band, then m is changed to "counterbalance the shock dW", so that the solution for s is maintained within the bands. Mathematically, one looks for a function $s(t) = F(x(t))$ which satisfies the following conditions. There is no intervention, i.e., no changes in m for a range $x \in (-X, X)$, where X is the intervention boundary to be found as part of the solution. Inside the bands, using a typical continuous time expansion on the expected changes, one obtains the following ordinary differential equation and boundary conditions:

$$\rho F(x; b) = x + F_{xx}(x)\frac{\sigma^2}{2} \text{ for } -X < x < X$$
$$F(x; b) = -b \text{ for } x \leq -X \text{ and } F(x; b) = b \text{ for } x \geq X$$

I require the function F to be once differentiable everywhere, and twice differentiable except at $x \in \{-X, X\}$. Note that the requirement that F is once differentiable gives the so-called "smooth pasting" condition $F_x(X) = 0$. Krugman (1991) goes over an intuitive explanation of this condition. I collect mostly known results about the solution in the following proposition:

PROPOSITION 2. With a fixed currency band under the maintained assumption, the exchange rate s as a function of the fundamentals is given by

$$s = F(x; b) = \frac{1}{\rho}\left[x - \frac{1}{\eta}\frac{\sinh(\eta x)}{\cosh(\eta X)}\right] \text{ for } -X \leq x \leq X$$

where $\eta = \sqrt{2\rho/\sigma^2}$. The boundary $X(b)$ solves $b = \frac{1}{\eta\rho}\left[\eta X - \tanh(\eta X(b))\right]$. The boundary $X(b)$ is increasing in b, and it can be approximated as follows

$$\frac{\partial X(b)}{\partial b} = \frac{\rho}{1 - \tanh'(\eta X)} \text{ and } X(b) \approx \begin{cases} \left(\frac{3}{2}\sigma^2 b\right)^{1/3} & \text{for small } b \\ \rho b + \frac{1}{\eta} & \text{for large } b \end{cases}$$

Note that if there will be no intervention, i.e., as fixed bands $b \to \infty$ is infinite, then $X \to \infty$ and

$$s = F(x; b) \to \frac{x}{\rho} \text{ for all } x \in (-b, b) \text{ a } b \to \infty$$

On the other hand, for fixed exchange rates, i.e., if $b = 0$ then $X = 0$, and $s = b$ at all times.

The function F is S-shaped, with positive slope in $(-X, X)$, approximately linear at $x = 0$, i.e., with zero second derivative, and with zero slope at the boundaries. At the top boundary, it is concave, while at the bottom boundary, it is convex.

The volatility of changes in the exchange rate, conditional on $x(t) = x$, is given by $\sigma F_x(x)$. The expression for F_x immediately yields two statistics of the exchange rate's flexibility.

PROPOSITION 3. The slope of F at the center of the band is:

$$F_x(0; b) = \left(\frac{1}{\rho}\right)\left(1 - \frac{1}{\cosh(\eta X(b))}\right) \tag{1}$$

which is increasing in b from 0 to $1/\rho$ as b goes from 0 to ∞. The average slope is

$$\bar{F}_x \equiv \frac{1}{X(b)}\int_0^{X(b)} F_x(x)dx = \frac{b}{X(b)} \tag{2}$$

which increases monotonically between 0 to $1/\rho$ as b goes from 0 to $+\infty$.

Proposition 3 offers two definitions of flexibility of the exchange rates, which I will keep for the general case of a widening band. The first is the volatility measured at the center of the band, $\sigma_s(0) = \sigma F_x(0)$. The second is the average slope of the function F, which uses the properties of F being increasing and antisymmetric around 0, so its slope is symmetric.

For future reference, note that the ODE evaluated at $x = X$ gives:

$$\rho b = X + F_{xx}(X; b)\frac{\sigma^2}{2} \text{ or using the solution } F_{xx}(X; b) = -\frac{\eta}{\rho}\tanh(\eta X)$$

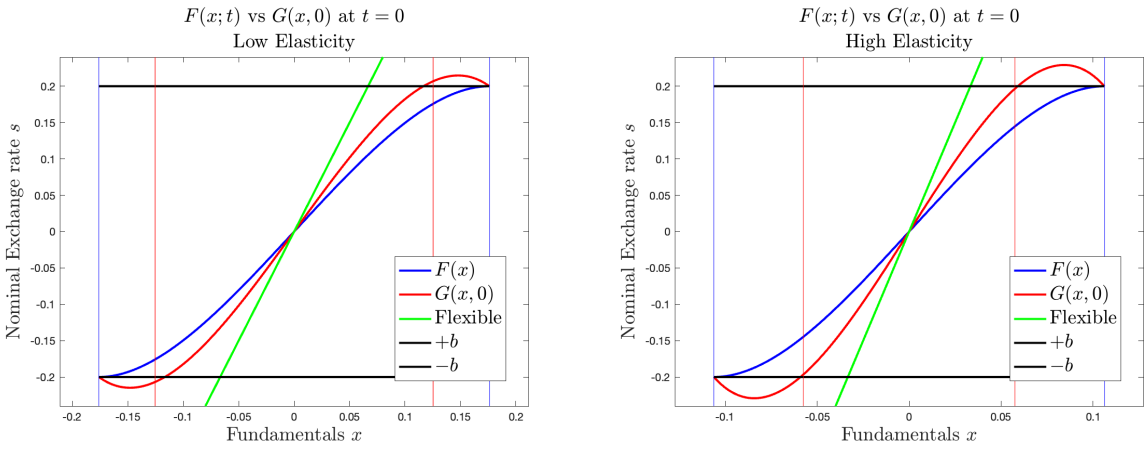


Figure 1: Fixed vs Widening Bands

Parameter values. I use two sets of parameter values for the examples below. For semi-elasticity, I report results for two values: a low elasticity of money demand, $\alpha = 3$, and a high one, $\alpha = 6$. These choices are based on an estimated elasticity of a log-log money demand for Argentina of 0.35 from Nicolini (2017) and Benati et al. (2021). In both papers, the authors show that the log-log functional form provides a reasonable fit, except when the interest rate is close to zero. The value of 0.35 for the log-log money demand corresponds to semi-elasticities of money demand of about 5, i.e., $\alpha = 5$ – see Lucas (2000) for the comparison on fit of both functional forms, and the correspondence between log-log and semi-log elasticities. Proposition 1 shows that the exchange rate depends more on the future expected values of fundamentals for larger values of the semi-elasticity of the money demand. Hence, to be conservative, I use $\rho = 1/3$ as the benchmark parameter, which corresponds to $\alpha = 3$, a relatively low elasticity of money demand. I also report results for the alternative parameter $\rho = 1/6$, which corresponds to a semi-elasticity of money demand of $\alpha = 6$. The other parameter that enters the model is σ, the volatility of the fundamentals. If the economy has flexible exchange rates, the exchange rate will follow a random walk, with volatility equal to σ/ρ. I set σ so that the annual volatility of the flexible exchange rates is 0.30. Thus, I chose $\sigma = 0.30\,\rho$ for semi-elasticities, assuming a flexible exchange rate has an annual volatility of 30%. I collect these parameters in Table 1. I report sensitivity analysis with respect to ρ and σ for the main conclusions.

Table 1: Parameter values

Parameter	Low Elasticity	High Elasticity
Semi-elasticity money demand α	3	6
Discount rate $\rho = \frac{1}{\alpha}$	1/3	1/6
Annual volatility of fundamentals $\sigma = 0.3\,\rho$	0.10	0.05

Figure 1 plots F as a function of x for the fixed bands model, given by the blue line. The left panel uses parameters for low money-demand elasticity, and the right panel uses parameters for high money-demand elasticity. There are two other functions plotted in this

figure: the straight line in green represents the relationship between the fundamental x and the exchange rate in the flexible exchange rate case. Two of the defining features of the currency bands are clear in this figure: Even at the center of the band, i.e., when $x = 0$, the slope $F_x(0)$ is smaller than the one for the green line corresponding to the flexible case, which is equal to $1/\rho$. Second, the average slope of $\bar{F}_x$, given by b/X, is smaller than the average slope of the flexible case $1/\rho$. The second property can be clearly seen since the width of the set for no intervention values of fundamentals, given by the $(-X, X)$, is wider than the one corresponding to the flexible exchange rate, i.e., $(-b\rho, b\rho)$.

A model with widening bands: set up

I start with the same "structural" model for exchange rates, but I replace the fixed (symmetric) bands $(-b, b)$ with two time-dependent functions that correspond to widening bands. For almost all analyzes, I maintain the symmetry assumptions of Krugman's model: the upper band is $\bar{s}(t)$, and the lower band is $\underline{s}(t) = -\bar{s}(t)$. For the widening case, the solution for s is a function of x and also time t, which I denote by G, i.e., $s(t) = G(x(t), t)$. As in the case of the Krugman model, I look for the boundary of the fundamental values that define the set of no-intervention for the central bank. The difference with the fixed band case is that in the widening case, the (upper) boundary is a *time varying function* which I denote by $Y(t)$, and correspondingly, for the lower bound, it is minus the upper bound, i.e., $-Y(t)$. Again, I consider the case of no intervention in the interior, and a credible intervention so that the exchange rate satisfies $\underline{s}(t) \leq s(t) = G(x(t), t) \leq \bar{s}(t)$ for all t.

Using the same logic as before, in the interior of the band, the function G solves the following partial differential equation and boundary conditions:

$$\begin{aligned}
\rho G(x,t) &= x + G_{xx}(x,t)\frac{\sigma^2}{2} + G_t(x,t) \text{ for } -Y(t) < x < Y(t) \text{ and all } t \in [0,T] \\
G(x,t) &= -\bar{s}(t) \text{ for } x \leq -Y(t) \text{ and } G(x,t) = \bar{s}(t) \text{ for } x \geq Y(t) \text{ and all } t \in [0,T] \\
G(x,T) &= F(x; b + \zeta T) \text{ for all } x
\end{aligned}$$

As explained above, G also depends on time because the currency bands are time-dependent. Note that the behavior of the bands is assumed to revert to the initial band at time T. I will comment on this particular choice for the terminal condition below.

I assume that $G(x,t)$ is once differentiable in (x,t) for all $(x,t) \in \mathbb{R} \times (0,T]$, and twice differentiable for $(x,t) : -Y(t) < x < Y(t), t \in (0,T]$. These are analogous to the assumptions in the fixed band case. Thus, I obtain the following boundary conditions:

$$G(Y(t),t) = \bar{s}(t) \text{ and } G_x(Y(t),t) = 0 \text{ for all } t \in [0,T]$$

These are, again, the so-called value-matching and smooth-pasting conditions. I also assume, as in the fixed band case, that $G(\cdot,t)$ is antisymmetric, i.e. $G(x,t) = -G(-x,t)$ for all (x,t). The terminal condition at $t = T$ corresponds to the assumption that after $t \geq T$ the boundary is constant forever at the value $\bar{s}(T) = b + \zeta T$. This is done for mathematical convenience; I will use it in the expansion introduced below. More importantly, I will be taking the limit as $T \to \infty$, so that, given that $\rho > 0$, the value of the terminal condition at T has a negligible effect on the exchange rate at t for large $T - t$.

In terms of the form of the bands for the exchange rate, I am interested in bounds of the form $\bar{s}(t) = b + \zeta t$ and likewise $-\bar{s}(t) = \underline{s}(t) = -b - \zeta t$, for some positive constant ζ. This is indeed the policy framework in Argentina, where each band's upper bound increases by 1% and the lower bound decreases by 1%. It also preserves the symmetry of the Krugman case in the analysis of the widening-band case, which simplifies the mathematics.

In graphical terms, Figure 2 shows what the solution looks like. The top panel displays the boundaries $Y(t)$ as a function of time. The bottom panel displays the exchange rate $s(t) = G(x,t)$ as a function of the fundamentals x for two time periods $t = 0$ and $t = t_1$. In blue in the two panels, I include the corresponding quantities for the Krugman model of fixed bands.

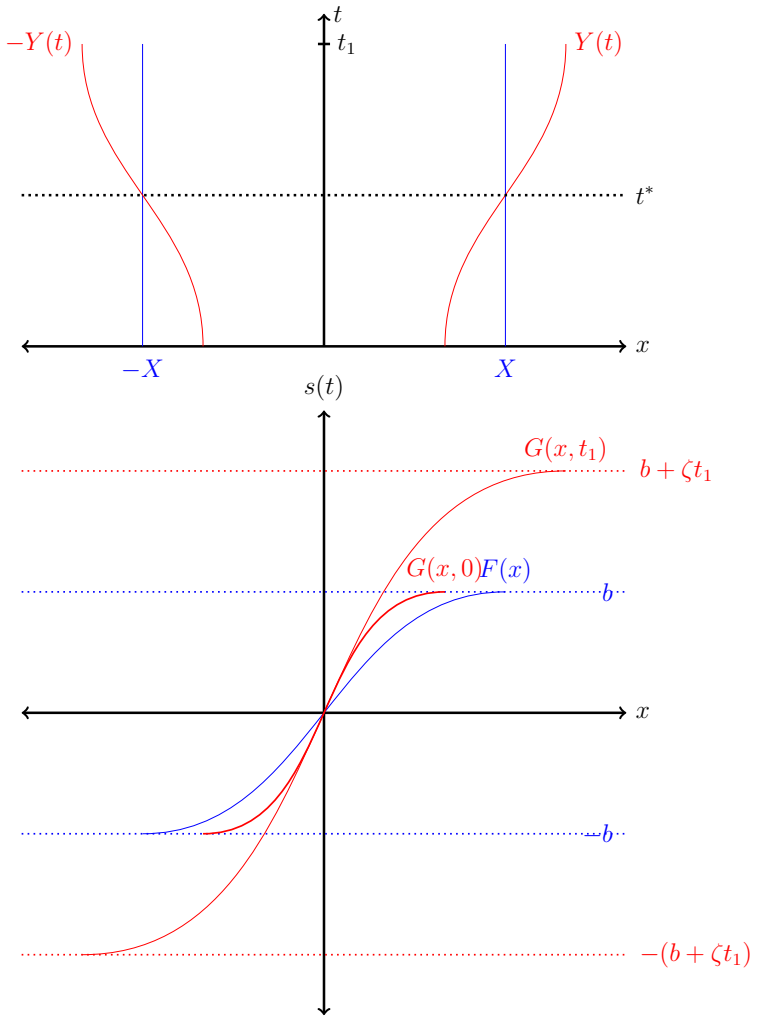


Figure 2: Y and G for widening bands

Functional Expansion for the widening band case

Solving for the function $G(x,t)$ and the boundary $Y(t)$ is a relatively difficult free boundary problem in the PDE literature. The difficulty is that PDE has to be solved simultaneously

with the free boundary $Y(t)$ for all times t, i.e., it is a moving boundary problem, as in the classical Stefan's problem. It is formally similar to the valuation problem for an American option, in that one needs to find the optimal exercise boundary. In the American option case, there are no exact or closed-form expressions for the option's boundary and price, and the literature has developed a variety of analytical approximations. Nevertheless, the boundary and terminal conditions for the widening band are different from the ones for American options, so I present and solve for a new analytical approximation below.[3]. The analytical approximation I proposed uses a first-order expansion of the PDE and its boundary conditions with respect to the parameter ζ, which measures the rate at which the bands are widening over time. In particular, I index the boundaries with a small scalar ϵ so that

$$\bar{s}(t,\epsilon) = b + \epsilon t \text{ and } \underline{s}(t) = -b - \epsilon t$$

Corresponding to each ϵ there is a function $G^\epsilon(x,t)$ solving the PDE and boundary conditions specified above. Notice that for every ϵ, the solution also specifies a path $Y^\epsilon(t)$. Recapitulating, including explicitly ϵ the PDE and boundary conditions becomes

$$\begin{aligned}
\rho G^\epsilon(x,t) &= x + G^\epsilon_{xx}(x,t)\frac{\sigma^2}{2} + G^\epsilon_t(x,t) \text{ for } -Y^\epsilon(t) < x < Y^\epsilon(t) \text{ and } t \in [0,T] \\
G^\epsilon(Y^\epsilon(t),t) &= b + \epsilon t \text{ and } G^\epsilon(-Y^\epsilon(t),t) = -b - \epsilon t \text{ for all } t \in [0,T] \\
0 &= G^\epsilon_x(Y^\epsilon(t),t) \text{ and } 0 = G^\epsilon_x(-Y^\epsilon(t),t) \text{ for all } t \in [0,T] \\
G^\epsilon(x,T) &= F(x; b + \epsilon T) \text{ for all } x
\end{aligned}$$

For future reference I note that if $\epsilon = 0$, then $G^0(x,t) = F(x;b)$ for all x and $t \in [0,T]$, and also $Y^0(t) = X$ for all $t \in [0,T]$. This is a convenient property for the expansion.

LEMMA 1. Differentiating $G^\epsilon(x,t)$ and $Y^\epsilon(t)$ with respect to ϵ, and evaluating it at $\epsilon = 0$, letting $g : [-X,X] \times [0,T] \to \mathbb{R}$ and $y : [0,T] \to \mathbb{R}$ where

$$g(x,t) \equiv \frac{\partial G^\epsilon(x,t)}{\partial \epsilon}|_{\epsilon=0} \text{ and } y(t) \equiv \frac{\partial Y^\epsilon(t)}{\partial \epsilon}|_{\epsilon=0}$$

then $g(x,t) = -g(-x,t)$ for all (x,t), and g solves:

$$\begin{aligned}
\rho g(x,t) &= \tfrac{\sigma^2}{2} g_{xx}(x,t) + g_t(x,t) \text{ for all } x \in [0,X], \text{ and all } t \in [0,T] \\
g(X,t) &= t \text{ and } g(0,t) = 0 \text{ for all } t \in [0,T] \text{ and} \\
g(x,T) &= T\frac{\partial F(x;b)}{\partial b} \text{ for all } x \in [0,X]
\end{aligned}$$

and where $y(t)$ solves: $F_{xx}(X)y(t) + g_x(X,t) = 0$ for all $t \in [0,T]$.

With these definitions, one can write the first-order expansion for G and Y as:

$$\begin{aligned}
G(x,t) \equiv G^\zeta(x,t) &= F(x) + g(x,t)\zeta + o(\zeta) \approx F(x) + g(x,t)\zeta \text{ and} \\
Y(t) \equiv Y^\zeta(t) &= X + y(t)\zeta + o(\zeta) \approx X + y(t)\zeta
\end{aligned}$$

[3]See the survey for the nature of the solution in Myneni (1992), and the survey on different analytic approximations in Li (2010).

The advantage of the approximation is that it has a closed-form solution since this PDE is a version of the heat equation in a cylinder with a lateral boundary condition of the displaced Dirichlet type. The next proposition is the key analytical result of this paper. It gives a closed-form solution for the expansion to the widening currency band for large T.

PROPOSITION 4. The solution for $g(x,t)$ as $T \to \infty$ is given by:

$$g(x,t) = \frac{x}{X}t + \frac{2}{\rho}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\pi}\frac{\eta^2X^2}{\eta^2X^2+(j\pi)^2}\left[1-\rho t-\frac{\eta^2X^2}{\eta^2X^2+(j\pi)^2}\right]\sin\left(j\pi\frac{x}{X}\right)$$

for any $t \geq 0$ and $x \in [-X, X]$.

The substantive analysis of the effect of the widening bands uses the solution in Proposition 4. I turn to that next.

Anticipation Effect

In this section, I provide an initial characterization of the anticipation effect. A later section introduces a variation on the approximation to refine it further.

I use Proposition 4 to compare the two regimes: a fixed currency band with a widening band, both of them starting with the same value of the bands at time $t = 0$, namely b. Figure 1 plots the exchange rate s as a function of the fundamentals, i.e. $s(0) = G(x(0), 0)$ at the launch of the widening bands regime, given by the red line. The left panel uses the low elasticity of money demand, and the right panel uses the high one. The blue line is, as explained above, the function $F(x(0); b)$ describing the fixed bands. Since at $t = 0$ the two bands are the same, the difference between the two functions is the *anticipation effect*, i.e., how much more flexible the exchange rate is due to the anticipation that the bands will widen in the future. It can be seen that G is roughly between F and the function with the flexible exchange rate. Furthermore, one can see that the range of fundamentals where there is no intervention is narrower for the widening bands than for the fixed band, i.e., that $Y(0) < X < 1/\rho$.

Note that the expression for $g(x,t)$ depends only on two parameters ρ, η and the value of X. Alternatively, one can say it depends on three parameters: $\rho\eta$ and b, since X is a function of ρ, η and b.

Next, I concentrate on the comparison of the slopes $G_x(x,t)$ and $F_x(x)$ as a measure of the anticipation effect. To do so, direct computation from the expression in Proposition 4 gives its first x derivative:

$$g_x(x,t) = \frac{1}{X}\left[t + \frac{2}{\rho}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{\eta^2X^2}{\eta^2X^2+(j\pi)^2}\left[1-\rho t-\frac{\eta^2X^2}{\eta^2X^2+(j\pi)^2}\right]\cos\left(j\pi\frac{x}{X}\right)\right]$$

The next proposition characterizes the value of $g_x(x,0)$ evaluated at two particular values of x.

PROPOSITION 5. Let $t = 0$, the x derivatives of g can be written in terms of exponential functions:

$$g_x(0,0) = \frac{1}{\eta\sigma^2}\left[(X\eta\coth(X\eta) - 1)\operatorname{csch}(X\eta)\right] > 0$$
$$g_x(X,0) = -\frac{1}{\eta\sigma^2}\left[\coth(X\eta) - X\eta\operatorname{csch}^2(X\eta)\right] < 0$$

This proposition has two important corollaries, which establish the existence of the anticipation effect.

1. The widening band case is more flexible than the fixed band case at the center of the band. The sensitivity of the exchange rate to fundamentals at the center of the band is given by:

   $$G_x(0,0) \approx F_x(0) + g_x(0,0)\zeta$$

   Since Proposition 5 shows that $g_x(0,0)$ is positive, then it means that, relative to the fixed bands, the exchange rate is becoming more flexible at the center of the band. In other words, *the anticipation effect of the widening of the bands makes the exchange rate more flexible*, since at time $t = 0$ the size of the band is the same as the one for the fixed band case.

2. The widening band case has a higher average slope than the fixed band case. The width of the no-intervention zone with a widening band at time $t = 0$, i.e. $Y(0)$, is smaller than the one for a fixed band for all t:

   $$y(0) = -\frac{g_x(X,0)}{F_{xx}(X)} \text{ which gives } Y(0) = X + y(0)\zeta + o(\zeta)$$

   Since $F_{xx}(X) < 0$ and Proposition 5 shows that $g_x(X,0) < 0$, then $y(0) < 0$, i.e., the intervention barrier is smaller for the widening case. Recall that the average sensitivity is given by $\bar{F}_x = \frac{b}{X}$ and $\bar{G}_x = \frac{b}{Y(0)}$ for the fixed and widening band case, respectively. In other words, *the anticipation effect of the widening of the bands makes the exchange rate more flexible on average.*

Figure 3 illustrates these two effects. The left panel uses the low elasticity of money demand, and the right one uses the high one. This figure compares volatility across different fundamentals (i.e., different values of x) in three cases: Fixed currency bands, Widening currency bands, and Flexible exchange rates. From this comparison, one can see that at the center of the band, i.e., when $x = 0$, the volatility of the widening band is closer to the flexible exchange rate case than to the fixed band case. This is true for both parameterization, but much more so for the alternative one with higher semi-elasticity of money demand. Moreover, Figure 3 also illustrates that the boundary for intervention for the widening band is narrower than the one for the fixed bands, i.e., that $Y(0) < X$.

For the remainder of this paper, it is convenient to include an extra argument in the two objects for the fixed band F and X, as well as in the two objects for the widening band G

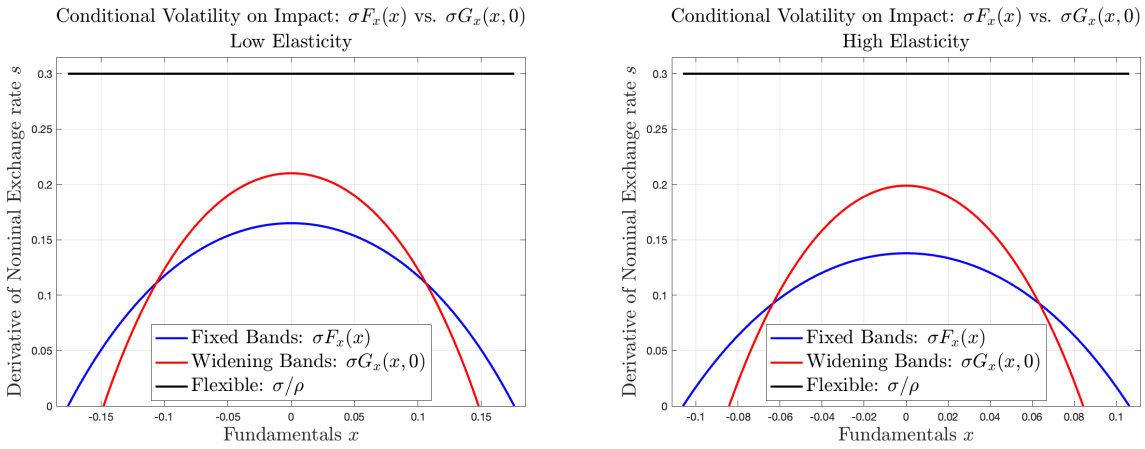


Figure 3: Anticipation Effect on Volatility

and Y. For the fixed band case, I include the size of the bands, summarized by the level of the upper bound as a last argument, so I write $F(x;b)$ and $X(b)$. For the widening band case, I include the *initial* size of the bands, summarized by the level of the upper bound as a last argument, so I write $G(x,t;b)$ and $Y(t;b)$.

Using the notation just introduced, I consider $X(b+\zeta t)$ which is the threshold for the fixed band when the band is equal to $b+\zeta t$, This threshold can be compared with $Y(0,b+\zeta t)$ which is the threshold for the moving band $Y(t,b)$, a band that started t period ago at the value b. Proposition 5 above characterized the difference between $Y(t,b)$ and $X(b+\zeta t)$.

PROPOSITION 6. Fix b. Then for all $t \geq 0$: $y(t;b) = y(0;b) + \rho \coth^2(\eta X(b))t$, thus for all $t \geq 0$ and all $\zeta \geq 0$:

$$\begin{aligned} Y(t;b) &= X(b) + y(0;b)\zeta + \rho \coth^2(\eta X(b))\ t\zeta + o(\zeta), \\ X(b+t\zeta) &= X(b) + \rho \coth^2(\eta X(b))\ t\zeta + o(\zeta), \text{ and} \\ y(0;b) &= -\frac{1}{2}\frac{\coth(\eta X(b)) - \eta X(b)\operatorname{csch}^2(\eta X(b))}{\tanh(\eta X(b))}. \end{aligned}$$

Thus, this means that, up to first-order in ζ, the paths of X and Y are parallel as function of time, since the expansion in t is the same for X and Y, i.e: $Y(t;b) - X(b+t\zeta) = y(0;b) + o(\zeta)$. In other words, the anticipation effect, measured as the average slope of $G(\cdot,t)$ vs $F(\cdot,b+\zeta t)$, is the same for all t. The difference between the two paths is given by $y(0;b)$, which was characterized in Proposition 5. The expressions for $Y(t;b)$ and $X(b+t\zeta)$ are both first-order expansions on ζ, so they do depend on the initial value b. They may lose accuracy if we evaluate them for large t. The next subsection proposes and implements a related expansion that should be more accurate for large t.

Intertemporal Intervention Twist

In this section, I compare the likelihood that intervention thresholds are met, and hence that an intervention is required, for a fixed barrier vs. a widening one. Recall that intervention

occurs when the fundamentals, which are uncontrolled and follow $dx = \sigma dW$, hit either $Y(t,b)$ and money contracted (and reserves sold), or hit $-Y(t,b)$ and money is expanded (and reserves are bought). Instead, in the fixed case, intervention occurs at either $+X(b)$ or $-X(b)$ respectively. For large t. It is clear that there is a smaller chance of hitting the barrier for the widening one case since the bands have expanded without limit. Somewhat surprisingly, for small t, there is a higher chance of hitting the barrier for the fundamentals, and thus requiring an intervention in the widening case. This is because $Y(0;b) < X(b)$, which is due to the anticipation effect. But at the same time $Y(t,b)$ is increasing in t. Thus, I define the critical time t^* for which $Y(t^*;b) = X(b)$, which separates the time horizon with the two different behaviors, which explains the name "Intertemporal Intervention Twist". The next proposition characterizes t^* for the case of small ζ.

PROPOSITION 7. There is a $t^* > 0$ such that for $t \in [0, t^*]$ it is more likely that the barrier is hit for the widening case than for the fixed case. The critical time t^* can be approximated by $\bar{t}$:

$$t^*(\zeta;b) = \bar{t}(b) + \frac{o(\zeta)}{\zeta} \text{ where } \bar{t}(b) = \frac{1}{2\rho}\left[1 - \frac{\eta X(b)}{\cosh(\eta X(b))\sinh(\eta X(b))}\right]$$

The values of $\bar{t}(b) \in [0, \frac{1}{2\rho}]$, and $\bar{t}$ is strictly increasing in b.

Figure 4 illustrates the critical time for various parameters. The left panel fixes the volatility of the corresponding flexible exchange rate at the benchmark value of 30% per year and changes the semi-elasticity of the money demand. The right panel fixes the semi-elasticity of money demand at $\alpha = 4.5$ (mid-range) and varies the volatility of the corresponding flexible exchange rate. In both cases, I plot $\bar{t}$ as a function of the initial band width, i.e., $2b$.

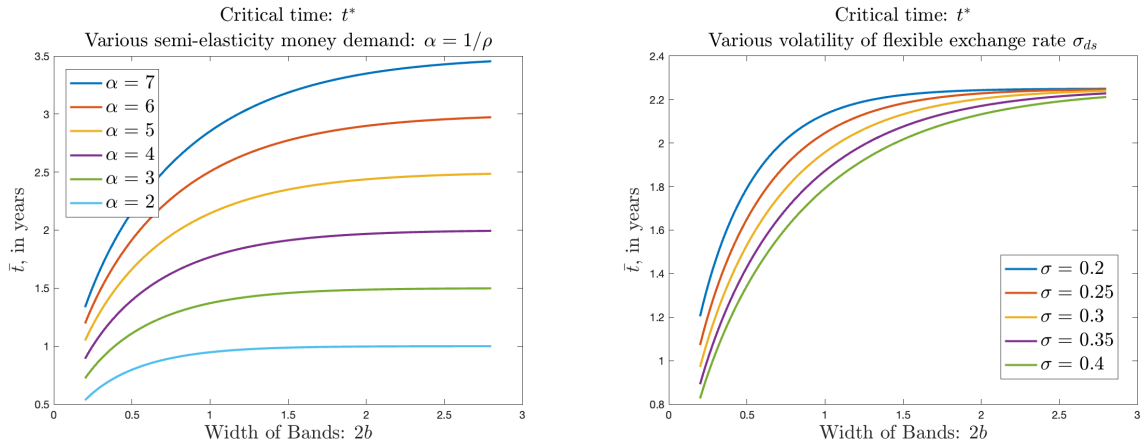


Figure 4: Critical time below which widening bands hit barriers earlier

Examining Figure 4, one sees that for currency bands with a width of about 50%, the critical time is around a year. That is, in the comparison between fixed and widening currency bands, widening bands are more likely to require an intervention in the first year.

Refining the approximation: recursive expansion.

In this section, I refine the expansion to make it more accurate for measuring the anticipation effect across different time horizons, i.e., when the bands are wider. To do so, I use the notation introduced above, where I specify the bands in the functions G and F and the boundaries Y and X. The forward-looking nature of G implies that these two functions are the same. I write the following immediate proposition, which is a direct consequence of this property:

PROPOSITION 8. Let the widening band solution at time t with a band starting at time zero at b be $G(x,t;b)$, and let the widening band solution at time 0 with a band starting at time zero at $b+\zeta t$ be $G(x,0;b+\zeta t)$. Likewise, let $g(x,t;b)$ and $g(x,0;b+\zeta t)$ be defined likewise. Let $F(x,b+\zeta t)$ be the solution of the fixed band at $b+\zeta t$, with corresponding boundary $X(b+\zeta t)$ when the size of the fixed band is $b+\zeta t$. Then:

$$G(x,t;b) = G(x,0;b+\zeta t) = F(x;b+\zeta t) + g(x,0;b+\zeta t)\ \zeta + o(\zeta)$$

where for any $b \geq 0$, the values for $X(b)$ and $F(x;b)$ solve:

$$b = \frac{1}{\eta\rho}\left[\eta X(b) - \tanh(\eta X(b))\right] \text{ and}$$
$$F(x;b) = \frac{1}{\rho}\left[x - \frac{1}{\eta}\frac{\sinh(\eta x)}{\cosh(\eta X(b))}\right] \text{ for } -X(b) \leq x \leq X(b)$$

I will use this recursion to solve for the anticipation effect for all $t \geq 0$. First, I solve the simple non-linear equation for the boundary of the fixed band $X(b+\zeta t)$ for each $t \geq 0$, or equivalently for each $\bar{s}(t)$. Then, I use the first-order approximation evaluated at $x = X(b+\ zetat)$. This should be more accurate than using the linear expansion from the initial bands for large t, where the difference with the point of approximation becomes larger. Moreover, this method does *not* use an approximation to the fixed band $X(b+\zeta t)$ for $t > 0$. I refer to this approximation as *recursive expansions*, since it combines the first-order expansion, with the recursive structure of G expressed in $G(x,t;b) = G(x,0;b+\zeta t)$. In particular, for any time $t \geq 0$, I compute:

$$Y(t;b) = X(b+\zeta t) - \frac{g_x\left(X(b+\zeta t),0;b+\zeta t\right)}{F_{xx}\left(X(b+\zeta t);X(b+\zeta t)\right)}\zeta\ + o(\zeta)$$

It is useful to consider the limit as $t \to \infty$, which for this expression is just equivalent to $b \to \infty$. Note that as $X(b) \to \infty$ then (i) $F_{xx}(X;X) \to -\frac{\eta}{\rho}$ and (ii) $g_x(X,0;X) \to -\frac{1}{\eta\sigma^2}$. Then, using the expression for η, the approximation for large $X \approx \rho b + 1/\eta$, and the expression for $Y(t)$ I get:

$$\lim_{t\to\infty}\left[Y(t;b) - X(b+\zeta t)\right] = -\frac{1}{2}\zeta \text{ and } \lim_{t\to\infty}\left|Y(t;b)/\rho - \bar{s}(t)\right| = \frac{1}{\rho}\left|\frac{1}{\eta} - \frac{\zeta}{2}\right|$$

Figure 5 plots the path of the intervention boundary for the fixed band case for each size band using the recursive expansion. I plot $X(b+\zeta t)$, labelled as myopic $X(\bar{s}(t))$ in

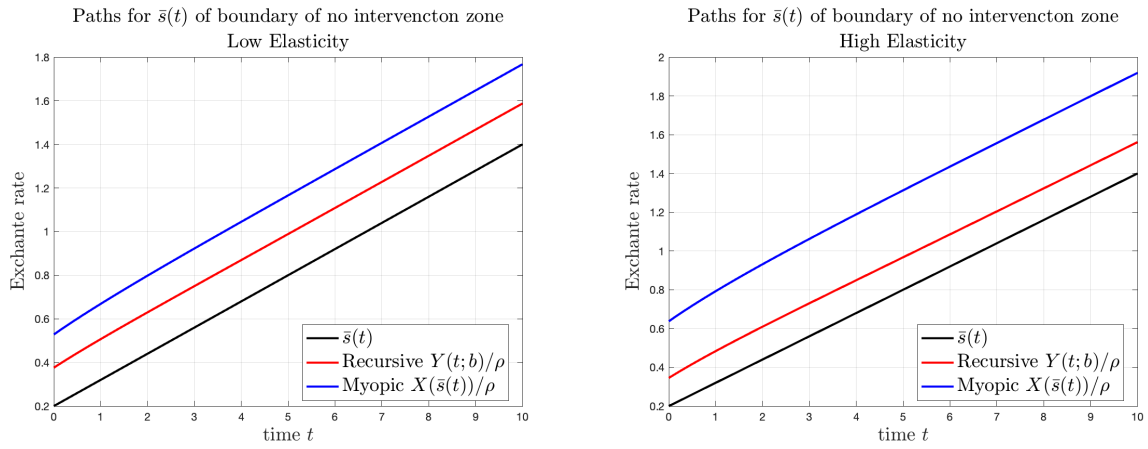


Figure 5: Path of boundaries for intervention

the figure, as well as the path of the band for the widening band case, which I label $Y(t;b)$ recursive. In both cases, I divide them by ρ to express them in units of exchange rates. Finally, the path of the upper band $\bar{s}(t)$ is plotted. As explained above, the anticipation effect makes $\bar{s}(t) \equiv b + \zeta t < Y(t;b)/\rho < X(b+\zeta t)/\rho$ for each $t \geq 0$. The left panel uses the parameter values for low semi-elasticity of money demand, and the right panel those for high semi-elasticity. The difference between the two thresholds, for those parameter values, seems almost constant. The fact that these lines are almost parallel is consistent with the results shown in Proposition 6, where the first-order expansion for all $t \geq 0$ is exactly parallel. Motivating by this almost parallel nature, I compute the limiting expression for the difference developed above, which is given by $\frac{1}{\rho}\left(\frac{1}{\eta} - \frac{\zeta}{2}\right)$, and which gives an accurate estimate even for small values of t.

Anticipation Effect: characterization. I define two measures of the anticipation effect. The first one is the ratio of the volatility of the widening bands to the volatility of the fixed band measures at the center of the bands, which I denote by $a^0 \equiv G_x(0,0)/F_x(0)$. The second measure is the ratio of the average slope for the widening bands given by $\bar{G}_x(t)$ to the average slope for the fixed band given by $\bar{F}_x$. Taking advantage of the antisymmetric nature of $G(\cdot,t)$ and $F(\cdot)$, I compute them as:

$$\bar{F}_x \equiv \frac{1}{X(b)}\int_0^{X(b)} F_x(x)dx = \frac{b}{X(b)} \text{ and } \bar{G}_x \equiv \frac{1}{Y(0;b)}\int_0^{Y(0;b)} G_x(x,t)dx = \frac{b}{Y(0;b)}$$

I define the second measure of the anticipation effect using the ratio of the average slopes as $\bar{a} \equiv \bar{G}_x/\bar{F}_x$, which is equivalent to $X/Y(0)$. Using Proposition 5, both anticipation effects are positive, i.e., these ratios are larger than one. The next proposition shows that as b increases, both measures of the anticipation effect decrease. The next proposition gives a closed-form expression for these effects as a function of X. In particular, Proposition 9 shows that as the bands widen, or equivalently as time goes by, the anticipation effect, measured as these ratios, is smaller. It also shows that for a given size of the band, the anticipation effect is increasing in the semi-elasticity of the money demand $\alpha = 1/\rho$.

PROPOSITION 9. Let's define the anticipation effect in central volatility and average slope as

$$a^0(b) \equiv \frac{G_x(0,0;b)}{F_x(0;X(b))} \text{ and } \bar{a}(b) \equiv \frac{X(b)}{Y(0;b)}$$

where b is the initial value of the band, and $X(b)$ is the fixed band corresponding to b. Then

$$a^0(b) = 1 + \frac{\eta}{2}\frac{(\eta X(b)\coth(\eta X(b)) - 1)\operatorname{csch}(\eta X(b))}{1 - \operatorname{sech}(\eta X(b))}\zeta + o(\zeta)$$
$$\bar{a}(b) = 1 + \frac{\eta}{2}\frac{\coth(\eta X(b)) - \eta X(b)\operatorname{csch}^2(\eta X(b))}{\eta X(b)\tanh(\eta X(b))}\zeta + o(\zeta)$$

where $X(b)$ is the intervention threshold corresponding to the fixed currency band b. The function a, given by either $\bar{a}$ or a^0 satisfies: (i) $\lim_{b\to 0} a(b) = \infty$, (ii) $a(b)$ is strictly decreasing in b, and (iii) $\lim_{b\to\infty} a(b) = 1$.

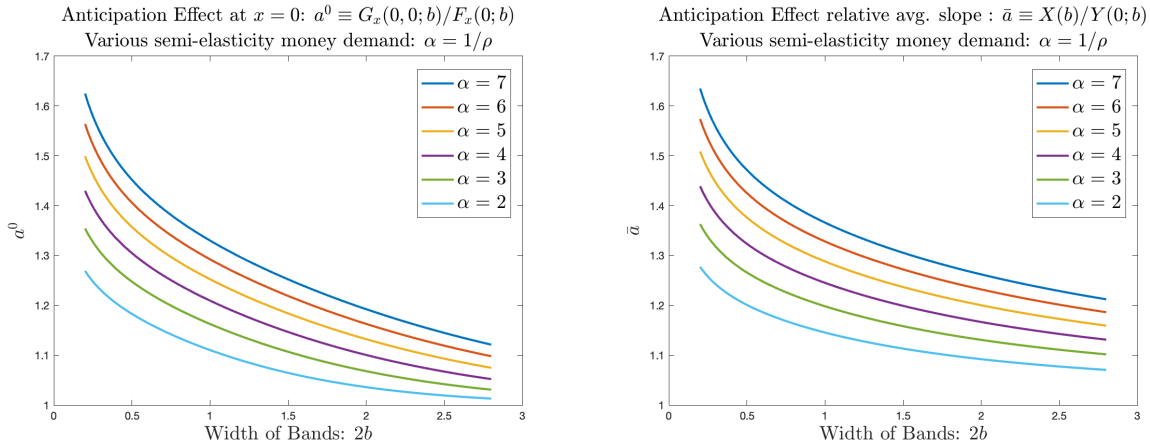


Figure 6: Anticipation Effects

Figure 6 displays the ratio $a^0(b)$ in the left panel and the ratio $\bar{a}(b)$ in the right panel, as function of b. In both cases, I plot several versions of each function for different values of the semi-elasticity of money demand, α. As Proposition 9 shows, $a^0(b)$ is larger than one, so the widening bands have more volatility in the center of the band than the fixed bands. Still, this difference decreases with band size, i.e., $a^0(b)$ decreases with b. As Proposition 9 shows, $\bar{a}(b)$ is larger than one, so the average slope of the widening band is higher than the average slope of the fixed band case. Still, this difference decreases with the size of the bands, i.e. $\bar{a}(b)$ decreases with b. For both measures, the differences are higher for a higher semi-elasticity of the money demand α. In terms of magnitudes, for a band with a width of about 40%– the starting value for Argentina–and a semi-elasticity of money demand equal to 3, both measures of flexibility for the widening band are about 35% higher than the fixed band case. For a semi-elasticity of money demand equal to 6, both measures of flexibility for the widening band are 55% higher than in the fixed-band case.

Second-order expansion

In principle, one can obtain a second-order expansion on ζ of G and Y. In this case, one differentiates the PDE and boundary conditions for $G^\epsilon(x,t)$ and $Y^\epsilon(t)$ twice with respect to ϵ. The next lemma shows the PDE and equations that these derivatives must satisfy.

LEMMA 2. Differentiating $G^\epsilon(x,t)$ and $Y^\epsilon(t)$ twice with respect to ϵ, and evaluating it at $\epsilon = 0$, letting $u : [-X, X] \times [0,T] \to \mathbb{R}$ and $w : [0,T] \to \mathbb{R}$ where

$$u(x,t) \equiv \frac{\partial^2 G^\epsilon(x,t)}{\partial \epsilon^2}|_{\epsilon=0} \text{ and } w(t) \equiv \frac{\partial^2 Y^\epsilon(t)}{\partial \epsilon^2}|_{\epsilon=0}$$

then $u(x,t) = -u(-x,t)$ for all (x,t), and u solves:

$$\begin{aligned}
\rho u(x,t) &= \tfrac{\sigma^2}{2} u_{xx}(x,t) + u_t(x,t) \text{ for all } x \in [0,X], \text{ and all } t \in [0,T] \\
u(X,t) &= F_{xx}(X)\,(y(t))^2 \text{ and } u(0,t) = 0 \text{ for all } t \in [0,T] \text{ and} \\
u(x,T) &= T^2 \frac{\partial^2 F(x;b)}{\partial b^2} \text{ for all } x \in [0,X]
\end{aligned}$$

and where $w(t)$ solves:

$$0 = u_x(X,t) + 2g_{xx}(X,t)\,y(t) + F_{xx}(X)\,w(t) + F_{xxx}(X)\,(y(t))^2 \text{ for all } t \in [0,T].$$

Using the definitions in the lemma, one can write the second-order expansions

$$\begin{aligned}
G(x,t) \equiv G^\zeta(x,t) &= F(x) + g(x,t)\zeta + \tfrac{1}{2}u(x,t)\zeta^2 + o(\zeta^2) \text{ and} \\
Y(t) \equiv Y^\zeta(t) &= X + y(t)\zeta + \tfrac{1}{2}w(t)\zeta^2 + o(\zeta^2)
\end{aligned}$$

The next proposition gives a closed-form solution for $u(x,0)$. We specialize in $t = 0$ since this is the object we use for the recursive expansion.

PROPOSITION 10. The solution for $u(x,0)$ as $T \to \infty$ is given by:

$$\begin{aligned}
&u(x,0) \\
&= \frac{F_{xx}(X)\sigma^2}{X^2} \sum_{j=1}^{\infty} (-1)^{j+1}\, j\pi\, \frac{y(0)^2\lambda_j^2 + 2y(0)\rho\coth^2(\eta X)\,\lambda_j + 2\rho^2\coth^4(\eta X)}{(\lambda_j)^3}\, \sin\left(\frac{j\pi x}{X}\right)
\end{aligned}$$

for any $t \geq 0$ and $x \in [-X, X]$, where $\lambda_j = \rho + (j\pi)^2 \frac{\sigma^2}{2X^2}$ and $y(0) = -\frac{1}{2}\frac{\coth(\eta X) - \eta X\,\mathrm{csch}^2(\eta X)}{\tanh(\eta X)}$. The function $u(\cdot,0)$ is differentiable for $x \in (-X, X)$.

A few remarks are in order. The Fourier sine series representation of $u(x,0)$ in Proposition 10 does not converge to the true boundary value at $|x| = X$, since the sine basis functions vanish there while $u(X,0) = F_{xx}(X)(y(0))^2 \neq 0$. This is an artifact of the series representation, not a discontinuity in the function itself. To compute $u_x(X,0)$—and hence $w(0)$ via Lemma 2—one can decompose $u(x,t) = r(x,t) + \tilde{u}(x,t)$, where r matches the boundary values and $\tilde{u}$ has homogeneous boundary conditions, as is done in the proof of Proposition 13. I instead use the second-order expansion of G for $|x|$ in the interior of $(-X, X)$ to compare its accuracy with that of a convergent numerical scheme based on finite differences, which I turn to next.

Adding a drift on the fundamentals

In this section, I add a drift into the fundamentals, so that in the absence of intervention $dx = \mu dt + \sigma dW$, where $\mu > 0$. This can be interpreted as a positive expected money growth, in excess of what is demanded away from the bands, or a drift on the i^* or in the money demand. In this case, the PDE for G is:

$$\begin{aligned}
\rho G(x,t) &= x + \mu G_x(x,t) + \frac{\sigma^2}{2} G_{xx}(x,t) + G_t(x,t) \text{ for } \underline{Y}(t) \leq x \leq \bar{Y}(t), \text{ and } t \in [0,T] \\
-b - \zeta t &= G(\underline{Y}(t),t)\,, b + \zeta t = G(\bar{Y}(t),t) \text{ for all } t \in [0,T], \\
0 &= G_x(\underline{Y}(t),t) = G_x(\bar{Y}(t),t) \text{ for all } t \in [0,T], \text{ and} \\
G(x,T) &= F(x) \text{ for all } x
\end{aligned}$$

where I distinguish the lower and upper bounds since there is no longer symmetry. I will follow the same steps as before, and construct a second-order approximation of $G^{\mu\zeta}(x,t)$ with respect to μ and ζ evaluated at $\mu = \zeta = 0$:

$$h(x,t) \equiv \frac{\partial G^{\mu\zeta}(x,t)}{\partial \mu}|_{\mu=\zeta=0}\,, q(x,t) \equiv \frac{\partial^2 G^{\mu\zeta}(x,t)}{\partial \mu^2}|_{\mu=\zeta=0} \text{ and } k(x,t) \equiv \frac{\partial^2 G^{\mu\zeta}(x,t)}{\partial \mu \partial \zeta}|_{\mu=\zeta=0}$$

With these definitions, I can expand to second-order

$$\begin{aligned}
G^{\zeta\mu}(x,t) &= F^0(x) + g(x,t)\zeta + h(x)\mu + \tfrac{1}{2}u(x,t)\zeta^2 + \tfrac{1}{2}q(x)\mu^2 + k(x,t)\mu\zeta + o(||(\zeta,\mu)||^2) \\
F^{\mu}(x) &= F^0(x) + h(x)\mu + \tfrac{1}{2}q(x)\mu^2 + o(\mu^2)
\end{aligned}$$

where $G^{\zeta\mu}$ is the widening band at the rate ζ and with expected fundamental growth μ, and $F^{\mu}(x)$ is the fixed band case with fundamentals growing at the rate μ. With this notation, $F^0(x)$ is the fixed band case with no drift on the fundamentals, and hence $G^{0,0}(x,t) = F^0(x)$. Note that for the anticipation effect, the only derivative with respect to μ that appears is the cross derivative, i.e.

$$G^{\zeta\mu}(x,t) - F^{\mu}(x) = g(x,t)\zeta + \tfrac{1}{2}u(x,t)\zeta^2 + k(x,t)\mu\zeta + o(||(\zeta,\mu)||^2)$$

The next two results give the first-order correction for expected money growth, which is the same for both fixed and widening bands. Following the same steps as before, the next lemma gives the conditions for the derivatives h and z. For this, I define $z(t) \equiv \frac{\partial \bar{Y}^{\mu\zeta}(t)}{\partial \mu}|_{\mu=\zeta=0}$.

LEMMA 3. The functions $h : [-X, X] \times [0,T] \to \mathbb{R}$, $z : [0,T] \to \mathbb{R}$ satisfy:

$$\begin{aligned}
\rho h(x,t) &= \frac{\sigma^2}{2} h_{xx}(x,t) + h_t(x,t) + F_x(x) \text{ for all } -X \leq x \leq X \text{ and } t \in [0,T] \\
0 &= h(-X,t) = h(X,t) \text{ for all } t \in [0,T] \text{ and} \\
0 &= h(x,T) \text{ for all } x
\end{aligned}$$

The function $h(\cdot,t)$ is symmetric. The derivatives of $\bar{Y}$ and $\underline{Y}$ are given by

$$\frac{\partial \bar{Y}^{\mu\zeta}(t)}{\partial \mu}|_{\mu=\zeta=0} = -\frac{h_x(X,t)}{F_{xx}(X)} = \frac{\partial \underline{Y}^{\mu\zeta}(t)}{\partial \mu}|_{\mu=\zeta=0} = -\frac{h_x(-X,t)}{F_{xx}(-X)} = z(t)$$

The solutions for h and z are given in the next proposition.

PROPOSITION 11. The solution of h and z when $T \to \infty$ are given by:

$$h(x,t) \equiv h(x) = \frac{128X^4\eta^2}{\rho\sigma^2} \sum_{j=1,3,5}^{\infty} \frac{1}{j\pi} \left(\frac{1}{4\eta^2X^2 + (j\pi)^2} \right)^2 \sin\left(j\pi \frac{x+X}{2X} \right)$$

for all $x \in [-X, X]$ and $t \geq 0$ and

$$z(t) \equiv z = -\frac{64X^3\eta}{\sigma^2 \tanh(\eta X)} \sum_{j=1,3,5}^{\infty} \left(\frac{1}{4\eta^2X^2 + (j\pi)^2} \right)^2 < 0 \text{ for all } t \geq 0$$

A few comments are appropriate here. First, neither $h(x)$ nor $z(t)$ depends on time. A simple way to see why is to realize that derivatives of $G(x,t)$ with respect to μ evaluated at $\zeta = 0$ are the same as the ones for $F(x)$. In other words, one can solve the simpler problem of the comparative statics in the fixed barrier problem. Note that this also implies that the second derivative $q(x,t)$ does not depend on time. This means that the functions G and F shift to the left for $\mu > 0$. But up to first-order, the correction to G and F is the same. Second, the expression for z makes it clear that z is negative, i.e., the no-intervention interval for the fundamentals shifts to the left. Third, inspection of the expressions, shows that $h_x(x,t) = h_x(x)$ is antisymmetric, with $h_x(0) = 0$ and $h_x(X) < 0$. Thus, the first order approximation for the fixed band case including the effect of the drift is $F_x(x) = \frac{1}{\rho}(1 - \cosh(\eta x)/\cosh(\eta X)) + h_x(x)\mu + o(\mu)$. The first-order approximation for the sensitivity for the widening band including the effect of the drift is $G_x(x,t) = \frac{1}{\rho}(1 - \cosh(\eta x)/\cosh(\eta X)) + g_x(x,t)\zeta + h_x(x)\mu + o(||(\mu,\zeta)||)$. Because $h_x(x,t) < 0$ for $x \in [0, X]$, then the maximum for $F_x(x)$, and $G_x(x,t)$ occurs at negative values of x, compared to the case where $\mu = 0$ which both have a maximum at $x = 0$.

Next, we analyze the second derivative with respect to μ.

LEMMA 4. The second derivative of $F(x)$ with respect to μ evaluated at $\mu = 0$ is equal to the second derivative of $G(x,t)$ with respect to μ evaluated at $\mu = \zeta = t = 0$ as $T \to \infty$, and solves:

$$\begin{aligned}
\rho q(x) &= 2h_x(x) + q_{xx}(x)\frac{\sigma^2}{2} \text{ for } -X \leq x \leq X \\
q(X) &= -zh_x(X) = \frac{h_x(X)^2}{F_{xx}(X)} \text{ and } q(-X) = -zh_x(-X) = \frac{h_x(-X)^2}{F_{xx}(-X)} \\
0 &= 2h_{xx}(-X)z + q_x(-X) + F_{xx}(-X)z_\mu + F_{xxx}(-X)\,z^2 \text{ and} \\
z_\mu &= -\left(q_x(X) + F_{xxx}(X)\,z^2\right)/F_{xx}(X)
\end{aligned}$$

Next, we solve for $q(x)$.

PROPOSITION 12. The function $q : [-X, X] \to \mathbb{R}$ that solves the ODE from in Lemma 4 is:

$$q(x) = -zh_x(X)\frac{\sinh(\eta x)}{\sinh(\eta X)} + \frac{8192X^5\eta^2}{\rho\sigma^4}\Bigg\{ \sum_{j=1}^{\infty}\sum_{k=1,3,5\ldots}^{\infty}\left(\frac{1}{4\eta^2X^2+(k\pi)^2}\right)^2\left(\frac{1}{4(j\pi)^2-(k\pi)^2}\right)\frac{(-1)^j\, j\pi}{4\eta^2X^2+4(j\pi)^2}\sin\left(j\pi\frac{x}{X}\right)\Bigg\}$$

and thus

$$q_x(X) = -zh_x(X)\eta\frac{\cosh(\eta X)}{\sinh(\eta X)} + \frac{8192X^4\eta^2}{\rho\sigma^4}\sum_{j=1}^{\infty}\sum_{k=1,3,5\ldots}^{\infty}\left(\frac{1}{4\eta^2X^2+(k\pi)^2}\right)^2\left(\frac{1}{4(j\pi)^2-(k\pi)^2}\right)\frac{(j\pi)^2}{4\eta^2X^2+4(j\pi)^2}$$

The more interesting result to characterize the anticipation effect per se is the one for the cross derivative k, since it gives the first non-zero correction for the computation of the anticipation effect. The next lemma characterizes k.

LEMMA 5. The function $k : [-X, X] \times [0, T] \to \mathbb{R}$ satisfies:

$$\begin{aligned}
&\rho k(x,t) = \frac{\sigma^2}{2}k_{xx}(x,t) + k_t(x,t) + g_x(x,t) \text{ for all } -X \le x \le X \text{ and } t \in [0,T]\\
&k(X,t) = -h_x(X,t)y(t), \quad k(-X,t) = -h_x(-X,t)y(t) \text{ for all } t \in [0,T] \text{ and}\\
&k(x,T) = 0 \text{ for all } x
\end{aligned}$$

The next proposition gives an expression for k.

PROPOSITION 13. As $T \to \infty$, the function $k : [-X, X] \times \mathbb{R}_+ \to \mathbb{R}$ that solves the PDE from in Lemma 5 is:

$$k(x,0) = k^s(x,0) + k^a(x,0)$$

where k^s is symmetric and k^a is antisymmetric, given by:

$$k^s(x,0) = \frac{16X^3}{\sigma^4}\sum_{j=1,3,5,\ldots}^{\infty}\left(\frac{1}{(j\pi)^2} + \frac{2\eta^2X^2+(j\pi)^2}{4(j\pi)^2\eta^2X^2+(j\pi)^4}\right)\frac{j\pi}{(\eta^2X^2+(j\pi)^2)^2}\sin\left(j\pi\frac{x+X}{2X}\right)$$

$$k^a(x,0) = \frac{-h_x(X)\,\sigma^2}{X^2}\sum_{j=1}^{\infty}(-1)^{j+1}\, j\pi\left[\frac{y(0)}{\nu_j} + \frac{\rho\coth^2(\eta X)}{\nu_j^2}\right]\sin\left(\frac{j\pi\, x}{X}\right)$$

for $x \in (-X, X)$. Where $\nu_j = \rho + (j\pi)^2\sigma^2/(2X^2)$, $y(0)$ is given by Proposition 6, and $h_x(X)$ is given by Proposition 11. Since k^a is antisymmetric, $k(0,0) = k^s(0,0)$.

Finite Difference Approximation

So far, I have computed a first and second-order expansion of the solution, which should be accurate for small ζ. One can check the accuracy by comparing with a finite-difference approximation to the original PDE for $G(x,t)$. I develop such a numerical approximation below, where I also allow for a positive drift in the fundamentals, i.e., $dx = \mu dt + \sigma dW$.

Set Up Let's consider times t_i index by $i = 1, 2, \ldots, I$ and x_j for $j = 1, 2, \ldots, J$. In particular $t_{i+1} = t_i + \Delta_t$ and $x_{j+1} = x_j + \Delta_x$. Thus, Δ_x and Δ_t are the (constant) size of the grid in the discretization of x and t, respectively. I replace the derivative of G in the PDE with its finite differences. I also incorporate a drift $\mu \geq 0$ into the law of motion x, which, as standard, adds a first derivative with respect to x to the PDE for G. The first difference approximation when x is *away* from the boundaries is:

$$\rho \mathcal{G}_{j,i} = x_j + \left(\frac{\mathcal{G}_{j+1,i} - \mathcal{G}_{j,i}}{\Delta_x} \right) \mu + \frac{\sigma^2}{2} \left(\frac{\mathcal{G}_{j+1,i} - 2\mathcal{G}_{j,i} + \mathcal{G}_{j-1,i}}{\Delta_x^2} \right) + \left(\frac{\mathcal{G}_{j,i} - \mathcal{G}_{j,i-1}}{\Delta_t} \right)$$

dividing by Δ_t, and rearranging one obtains:

$$\mathcal{G}_{j,i-1} = x_j \Delta_t + (1 - \Delta_t \rho) \left[p_u \mathcal{G}_{j+1,i} + p_d \mathcal{G}_{j-1,i} + (1 - p_d - p_u) \mathcal{G}_{j,i} \right]$$

where:

$$p_u = \frac{\Delta_t}{1 - \rho \Delta_t} \left(\frac{\mu}{\Delta_x} + \frac{\sigma^2}{2\Delta_x^2} \right) \text{ and } p_d = \frac{\Delta_t}{1 - \rho \Delta_t} \left(\frac{\sigma^2}{2\Delta_x^2} \right)$$

As it is standard, it is required that $\rho \Delta_t < 1$ and that Δ_x be sufficiently small, given Δ_t, so that $0 < p_u + p_d < 1$. At the highest and lowest value of the grid, which should be chosen to be larger than the intervention boundary, I impose reflection. Thus, this gives, ignoring the interventions on the exchange rate:

$$\begin{aligned}
\mathcal{G}_{1,i-1} &= x_1 \Delta_t + (1 - \Delta_t \rho) \left[p_u \mathcal{G}_{2,i} + (1 - p_u) \mathcal{G}_{1,i} \right] \\
\mathcal{G}_{j,i-1} &= x_j \Delta_t + (1 - \Delta_t \rho) \left[p_u \mathcal{G}_{j+1,i} + (1 - p_d - p_u) \mathcal{G}_{j,i} + p_d \mathcal{G}_{j-1,i} \right] \text{ for } 2 \leq j \leq J-1 \\
\mathcal{G}_{J,i-1} &= x_J \Delta_t + (1 - \Delta_t \rho) \left[(1 - p_d) \mathcal{G}_{J,i} + p_d \mathcal{G}_{J-1,i} \right]
\end{aligned}$$

I regard $\mathcal{G}_i$ and x as two J dimensional vectors, i.e.

$$\mathcal{G}_i = \begin{bmatrix} \mathcal{G}_{1,i} \\ \mathcal{G}_{2,i} \\ \vdots \\ \mathcal{G}_{J,i} \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_J \end{bmatrix}$$

and arrange the transition probabilities into a tri-diagonal $J \times J$ matrix Π:

$$\Pi = \begin{bmatrix} 1-p_u & p_u & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ p_d & 1-p_u-p_d & p_u & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & p_d & 1-p_u-p_d & p_u & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & p_d & 1-p_u-p_d & p_u & 0 & \dots & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 & p_d & 1-p_u-p_d & p_u \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & p_d & 1-p_u-p_d \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & p_d \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ \\ \vdots \\ \vdots \\ 0 \\ p_u \\ 1-p_d \end{matrix}$$

Thus Π is a matrix of zeros, except for $2 \leq j \leq J-1$, where it has $\Pi(j, j-1) = p_d$, $\Pi(j,j) = 1-p_u-p_d$ and $\Pi(j, j+1) = p_u$. For $j = 1$ then $\Pi(j,j) = 1-p_u$ and $\Pi(j,j+1) = p_u$. For $j = J$, $\Pi(j,j) = 1-p_d$ and $\Pi(j, j-1) = p_d$.

Let $\mathcal{G}_I$ be a J dimensional column vectors with the final values of $G(\cdot, T)$, and $\bar{s}$ and $\underline{s}$ be two I dimensional vector with the time path of the boundaries for the exchange rate, where $\bar{s}_i = b + \zeta \Delta_t (i-1)$ for $i = 1, 2, \dots, I$. Thus, I solve the $\{\mathcal{G}_{ij}\}$ by solving the recursion:

$$\mathcal{G}_i = \max\left\{\underline{s}_i\,,\, \min\left\{\bar{s}_i\,,\, x\Delta_t + (1-\rho\Delta_t)\,\Pi\,\mathcal{G}_{i+1}\right\}\right\} \text{ for } i = 1, 2, \dots, I-1$$

with terminal condition $G_I = G(\cdot, T)$. Note that the min and max implement the bands for the exchange rate. As $j\Delta_x \to x$ and $i\Delta_t \to t$ then $\mathcal{G}_{j,i} \to G(x,t)$.

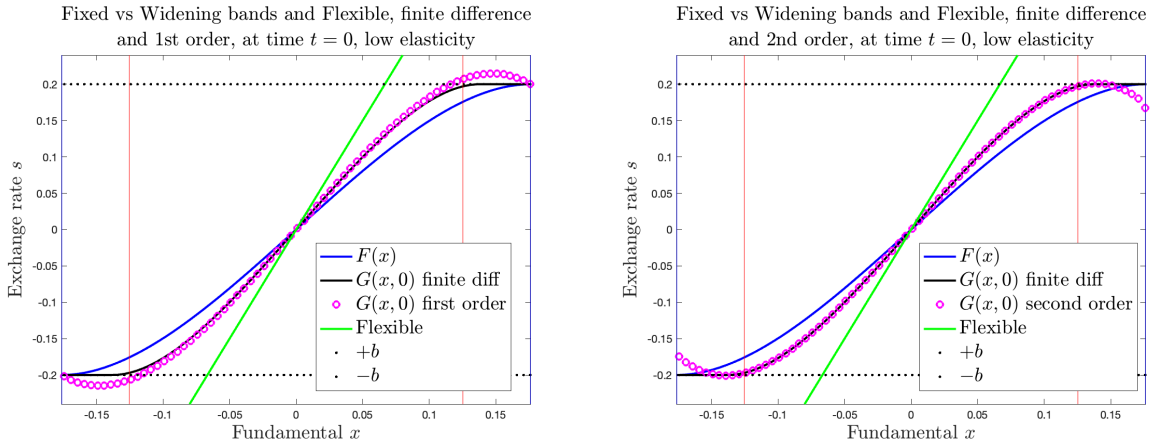


Figure 7: 1st and 2nd Order Expansion vs finite-difference, low elasticity

I use the finite-difference approximation to compute approximations to $G(x,t)$ for low values of the elasticity of money demand. In the computations, I use $\mathcal{G}_I$ to be the discretization of $F(x)$, so at time T the system returns to fixed bands of size b. The solution to the finite-difference approximation can be made arbitrarily close to the function $G(x,t)$ by choosing the grid sides Δ_x and Δ_t sufficiently small and T large. The results are plotted in Figure 7 for the low semi-elasticity of money demand. In the left panel, I compare the first-order expansion (magenta circles) with the finite-difference solution (black lines). As can be seen, the finite difference is almost identical

to the first-order approximation, with small differences when the fundamentals $|x|$ are close to X, where the barrier X is marked with a vertical line. In the right panel, I compare the second-order expansion (magenta circles) with the finite-difference solution (black lines). In this case, the two functions are indistinguishable for $x \in [-Y(0), Y(0)]$.

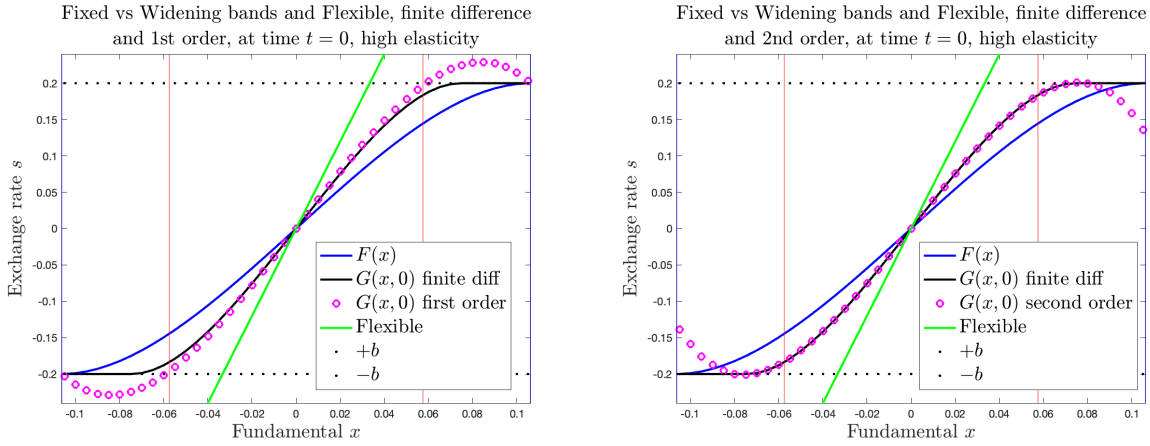


Figure 8: 1st and 2nd Order Expansion vs finite-difference, high elasticity

Adding drift: purchasing FX within the bands. In this subsection, I add a positive drift to x for the finite-difference approximation, with $\mu > 0$. The left panel of Figure 9 plots G and F with a $\mu = 0.05$. Note that the intervention thresholds are asymmetric: the left thresholds for the inaction of fundamentals $\underline{Y}(0)$ is smaller than (minus) the right one $-\bar{Y}(0)$ since for low x the positive drift $\mu > 0$ helps to self-correct, and the exchange rate returns to the interior of the band with greater ease than for large x. The left panel of Figure 9 corresponds to the low semi-elasticity of money demand, and the right panel to the high one. The difference between the two functions measures the anticipation effect. As can be seen, there is an anticipation effect, i.e., the slope of the widening-band case G (the red line) is higher than that of the fixed-band case F (the blue line).

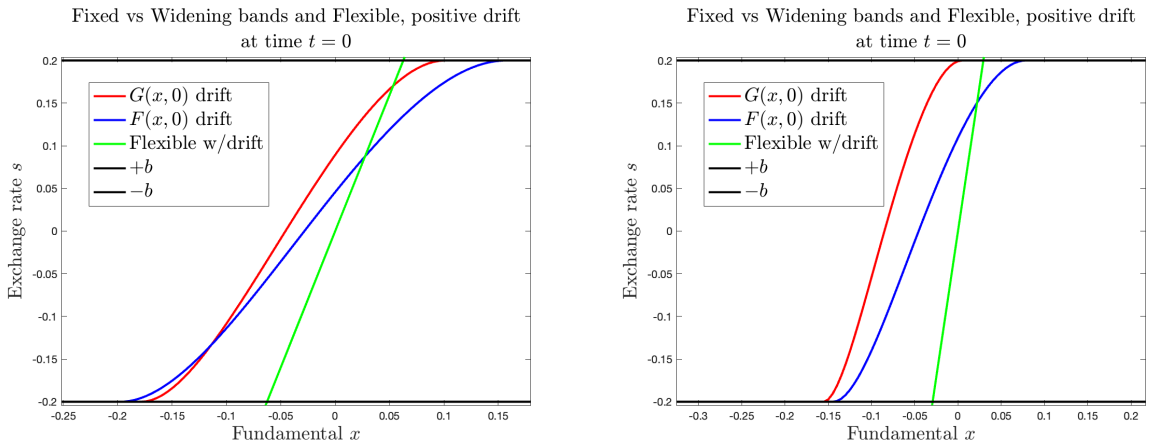


Figure 9: Fixed vs Widening band case at $t = 0$ for the case with drift

Figure 10 complements this analysis, displaying the conditional volatility at time $t = 0$ for the cases of widening bands, fixed bands, and flexible exchange rates. Again, these figures show that

while the volatility in the center of the band for the case of widening bands is higher than for the fixed, they are closer to each other, relative to the volatility of the flexible exchange rate, as compared with the case of zero drift. The left panel shows the case of low elasticity of demand, and the right panel shows the case of high elasticity.

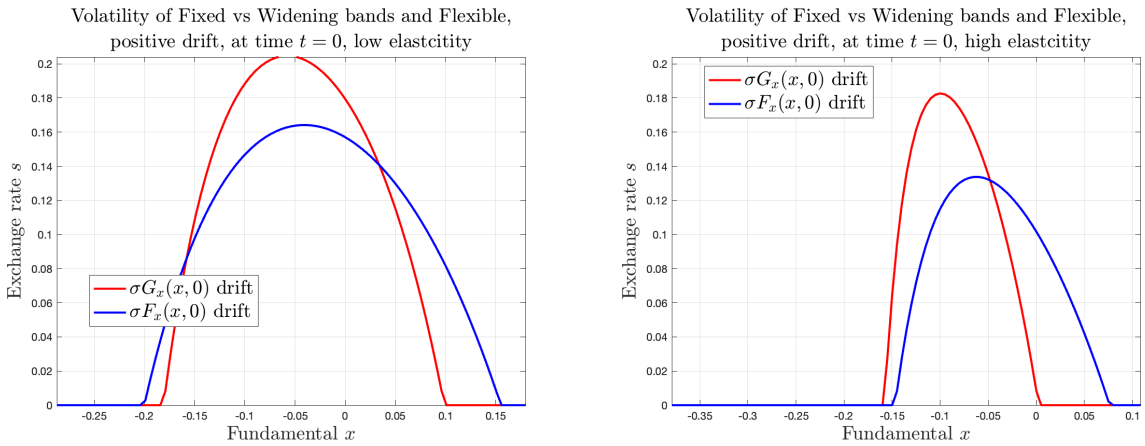


Figure 10: Conditional volatility at $t = 0$ for the case with drift

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Appendix

Widening Currency Bands

Fernando Alvarez

A Argentina's framework for currency bands

This section provides a brief description of Argentina's monetary framework. In December 2023, there was a change in the administration, where President A. Fernandez left office, and President J. Milei took power. The new administration implemented a sharp change in economic policies. Previously, there were multiple exchange rates, price controls, and capital controls.

Additionally, President Fernandez pursued a very expansive fiscal policy, especially given that voluntary capital markets for the public sector were closed to the Republic. Inflation has been increasing steadily and was above 210% during 2023. President Mile administration eliminated price controls and the multiple exchange rates used for commercial operations; i.e., all imports and exports were unified under a single "commercial" exchange rate. This elimination implied a sharp devaluation of the peso in a single day, as shown in Figure 11. Right after the large devaluation of the peso, the central bank (BCRA from now on) announced a crawling peg policy with a pre-announced depreciation of 2% per month. Capital controls were streamlined after President Milei took office, but were still maintained. The fiscal deficit of almost 5% of GDP was eliminated in January 2024, i.e., one month after President Milei took office. The 2% per month peg was maintained until the beginning of 2025, when it was announced that it would be changed to a 1% monthly preannounced depreciation rate for the peso. Inflation decreased substantially to around 2% per month.

Negotiations for a new agreement with the IMF were made public in February 2025. An agreement was signed in mid-April of 2025. This agreement rolled over the Republic's debt to the IMF from a previous loan, established targets for fiscal policy and reserve accumulation, and, importantly, gave access to IMF-lent reserves, which can be used under stated conditions. After the agreement, the Republic eliminated capital controls on new flows, keeping them in place for certain operations dated before – such as dividends from foreign companies from previous years. Additionally, the BCRA announced a new policy of currency bands. The initial levels of the upper and lower bands were 1,400 and 1,000 pesos per dollar, respectively. The upper bound was announced to increase by 1% per month, and the lower bound was announced to decrease by 1% per month. The BCRA cannot sell the IMF reserves unless the exchange rate reaches the upper band. The central bank, and indeed President Milei himself, have announced that the BCRA will not buy reserves unless the exchange rate hits the lower bound of the band. Instead, the Republic's treasury has agreed to purchase reserves.

To visualize this scheme, Figure 11 plots two exchange rates. One is a financial exchange rate, denoted by CCL ("Contado Con Liqui"), which can and has been used for financial flows even during periods of capital controls. It is computed as the ratio of the price of the same asset purchased in the Buenos Aires stock market (in pesos) to the price of the same asset sold in the US stock market (in US dollars). To compute it, I used two commonly traded assets: Apple stock and the stock of a local large Argentinian bank, Banco Galicia. The other exchange rate is the commercial exchange rate, denoted as Official XR. The two plots have the same exchange rates, expressed in pesos per dollar, and they are on a log scale. The second panel covers a longer period. During periods of capital and exchange controls, the Official XR rate at which the BCRA sold US dollars to importers and bought US dollars from exporters was substantially lower than the CCL rate. Before 2024, there were multiple values for the Official XR, and the plotted value is for imports, which was lower than that for exports.

After January 2024, the different official exchange rates have been almost completely unified. After April 2025, the Official XR is the exchange rate negotiated between buyers and sellers of dollars in the Buenos Aires markets for any purpose. Both exchange rates are expressed in pesos per dollar. As shown in Figure 11, the commercial exchange rate (XR) was pegged, increasing first by 2% and later by 1%, and after mid-April of 2025 it floated. One can clearly see the large step-up

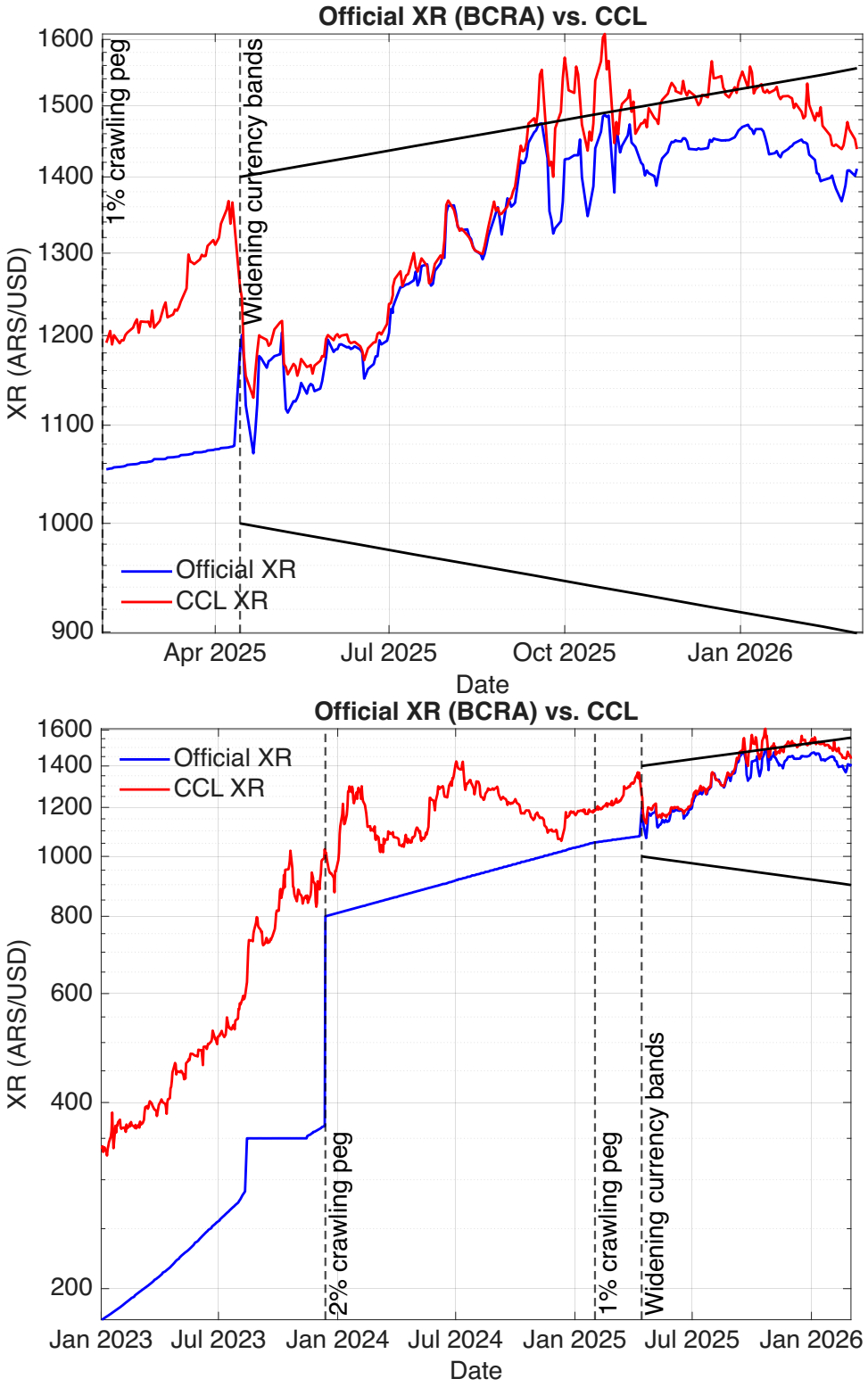


Figure 11: Exchange Rates

in devaluation at the end of December 2023, when President Milei took office. One can also see

how the exchange rate fluctuates day to day, even during periods of capital controls. The bands are also plotted starting in mid-April of 2025. Finally, after the bands were introduced, the two exchange rates converged and have stayed very close to each other, consistent with the abandonment of exchange rate controls, at least until September 2025.

Since June 2025, the opposition has passed several laws that would have implied a more expansive fiscal policy. Some of these laws were successfully vetoed by President Milei, and for some, the veto of President Milei was overwritten by Congress. Additionally, at the beginning of September 2025, local elections were held in the Republic's largest province, Buenos Aires. In this election, President Milei's party and allies lost by almost 15% to the same party that was in office before and favors very different policies. Other parties, which were previously closer to President Milei, also moved their announced policies closer to the opposition. This initiated a period of high uncertainty about future policies, especially since the national midterm elections will be held at the end of October 2025, and most polls indicate a negative result for the executive. During this period, a limited form of capital controls was reinstated, as evidenced by the divergence between the CCL and XR. Moreover, President Milei's party faced two political scandals related to corruption, one involving the first candidate for the national congress from the province of Buenos Aires. In this context, the US Secretary announced support for Argentina, as indicated in the graph. Soon, the US president met with the Argentine president, and by the end of October, an agreement for the swap was signed. Incidentally, the US Treasury had bought pesos for approximately USD 2 billion. The midterm election, contrary to most polls, turned out very favorable to the party of President Milei. The next day after the elections, there was a large drop in both CCL and the Official XR inside the bands.

B Proofs

Proof. (of Proposition 1) One can replace the money demand in the UIP to eliminate interest rates

$$\frac{p(t) - m(t) - v(t)}{\alpha} = i^*(t) + \frac{1}{dt}\mathbb{E}\left[ds(t)\right]$$

and replacing the PPP to eliminate the price level:

$$\frac{s(t) + p^*(t) - m(t) - v(t)}{\alpha} = i^*(t) + \frac{1}{dt}\mathbb{E}\left[ds(t)\right]$$

or

$$\frac{s(t)}{\alpha} = \frac{m(t) + v(t) - p^*(t)}{\alpha} + i^*(t) + \frac{1}{dt}\mathbb{E}\left[ds(t)\right]$$

One can rewrite the exchange rate as solving:

$$\rho s(t) = x(t) + \frac{1}{dt}E\left[ds(t)\right] \text{ where}$$
$$x(t) \equiv \frac{m(t) + v(t) - p^*(t)}{\alpha} + i^*(t) \text{ and } \rho \equiv \frac{1}{\alpha}$$

alternatively, solving it forward:

$$s(t) = \int_t^\infty e^{-\rho(\tau - t)}\mathbb{E}_t\left[x(\tau)\right] d\tau$$

□

Proof. (of Proposition 2) The solution of the ODE in the interior is the sum of the particular solution and a linear superposition of homogeneous solutions, i.e.:

$$F(x) = \frac{x}{\rho} + A_1 e^{\eta x} + A_2 e^{-\eta x} \text{ for } -X \leq x \leq X \text{ where } \eta \equiv \sqrt{2\rho/\sigma^2}$$

I am looking for an antisymmetric solution, so there is only one constant, and the two exponentials become the hyperbolic sine:

$$F(x) = \frac{x}{\rho} + A\sinh(\eta x) \text{ for } -X \leq x \leq X$$

The requirement that the solution be once differentiable, sometimes referred to as smooth pasting, and that it achieves the values of the band at the extremes of the non-intervention zone, sometimes referred to as value matching, implies:

$$F_x(X) = 0 \implies 0 = \frac{1}{\rho} + \eta A\cosh(\eta X), \text{ and}$$
$$F(X) = b \implies b = \frac{X}{\rho} + A\sinh(\eta X)$$

This is a system of two non-linear equations in two unknowns, namely A and X. Solving for A and substituting, I get:

$$A = -\frac{1}{\rho\eta\cosh(\eta X)} \text{ and } b = \frac{X}{\rho} - \frac{1}{\rho\eta}\frac{\sinh(\eta X)}{\cosh(\eta X)}$$

I arrive to an implicit equation for X:

$$b = \frac{1}{\eta\rho}\left[\eta X - \tanh(\eta X)\right]$$

The RHS of this expression is continuous and increasing in X, so it has a unique solution for X. For small b, I can approximate the solution as follows:

$$y - \tanh(y) = \frac{y^3}{3}, \text{ hence } b \approx \frac{\eta^2}{\rho}\frac{X^3}{3} \text{ or}$$
$$X \approx \left(3\frac{\rho}{\eta^2}b\right)^{1/3} = \left(\frac{3}{2}\sigma^2 b\right)^{1/3}$$

which should be a good approximation for small b. On the other hand, for large b, I can approximate the solution as follows:

$$y - \tanh(y) = y - 1, \text{ hence } b \approx \frac{X}{\rho} - \frac{1}{\rho\eta} \text{ or}$$
$$X \approx \rho b + \frac{1}{\eta}$$

The solution for the exchange rate as a function of the "fundamentals" is thus:

$$F(x) = \frac{1}{\rho}\left[x - \frac{1}{\eta}\frac{\sinh(\eta x)}{\cosh(\eta X)}\right] \text{ for } -X \leq x \leq X$$

Differentiating the equation that determines X I get

$$db = \frac{1}{\eta\rho}\left[\eta dX - \eta\tanh'(\eta X)dX\right] \text{ or } 0 < \frac{dX}{db} = \frac{\rho}{1 - \tanh'(\eta X)} < \rho$$

If $b \to \infty$, then:

$$\rho = \left[\frac{\eta X}{\eta b} - \frac{\tanh(\eta X)}{\eta b}\right] \to \frac{X}{b}$$

In this case:

$$F(x) = \frac{1}{\rho}\left[x - \frac{1}{\eta}\frac{\sinh(\eta x)}{\cosh(\eta X)}\right] \to \frac{x}{\rho} \text{ for } -X \leq x \leq X$$

as $X \to \infty$. □

Proof. (of Proposition 3) The first derivative $F_x(x)$ gives the sensitivity of F with respect to the fundamentals, and it is equal to:

$$F_x(x) = \frac{1}{\rho}\left[1 - \frac{\cosh(\eta x)}{\cosh(\eta X)}\right] \text{ for } -X \leq x \leq X$$

In particular, evaluating it at $x = 0$, I get:

$$F_x(0) = \frac{1}{\rho}\left[1 - \frac{1}{\cosh(\eta X)}\right], \; F_{xx}(0) = 0,$$

The volatility of the exchange rate at the center of the band is given by

$$\sigma_s^2 = \sigma^2 (F_x(0))^2 = \left(\frac{\sigma}{\rho} \left[1 - \frac{1}{\cosh(\eta X)} \right] \right)^2 \tag{3}$$

This expression depends on X, which itself is a function of b. Since X is increasing in b, and since cosh is larger than one and increasing, then $\sigma_s(0)$ is increasing in b.
The comparative static for $\bar{F}_x$ is as follows:

$$\begin{aligned} \frac{\partial \bar{F}_x}{\partial b} &= \frac{X(b) - bX'(b)}{X(b)^2} = \frac{X(b) - \frac{\rho b}{1 - \tanh'(\eta X(b))}}{X(b)^2} \\ &= \frac{\left(1 - \tanh'(\eta X(b))\right) X(b) - \rho b}{\left(1 - \tanh'(\eta X(b))\right) X(b)^2} \end{aligned}$$

But X satisfies

$$-\rho b = -X(b) + \frac{1}{\eta} \tanh(\eta X(b))$$

thus:

$$\frac{\partial \bar{F}_x}{\partial b} = \frac{\left(1 - \tanh'(\eta X(b))\right) X(b) - \rho b}{\left(1 - \tanh'(\eta X(b))\right) X(b)^2}$$

The denominator is positive since $X(b) \geq 0$ and $\tanh'(\eta X(b)) \in [0, 1]$. But X satisfies

$$-\rho b = -X(b) + \frac{1}{\eta} \tanh(\eta X(b))$$

thus the numerator of $\frac{\partial \bar{F}_x}{\partial b}$ can be written as:

$$\begin{aligned} &\left(1 - \tanh'(\eta X(b))\right) X(b) - X(b) + \frac{1}{\eta} \tanh(\eta X(b)) \\ &\qquad - \tanh'(\eta X(b)) X(b) + \frac{1}{\eta} \tanh(\eta X(b)) \end{aligned}$$

But using the expressions for $\tanh(y)$ and its derivative $\tanh'(y) = \operatorname{sech}(y)^2$ one verifies that the numerator is positive because

$$\begin{aligned} \frac{1}{\eta} \tanh(\eta X(b)) - \tanh'(\eta X(b)) X(b) &> 0 \\ \frac{1}{\eta} \tanh(\eta X(b)) &> \tanh'(\eta X(b)) X(b) \\ \frac{1}{\eta} \frac{\sinh(\eta X(b))}{\cosh(\eta X(b))} &> \operatorname{sech}^2(\eta X(b)) X(b) \\ \frac{\sinh(\eta X(b))}{\cosh(\eta X(b))} &> \frac{\eta X(b)}{\cosh^2(\eta X(b))} \\ \cosh(\eta X(b)) \sinh(\eta X(b)) &> \eta X(b) \\ \frac{1}{2} \sinh(2\eta X(b)) &> \eta X(b) \\ \sinh(2\eta X(b)) &> 2\eta X(b) \end{aligned}$$

Where I used that $\cosh(y)\sinh(y) = \frac{1}{2}\sinh(2y)$ and lastly $\sinh(y) > y$. $\square$

Proof. (of Lemma 1) The first step is to differentiate the PDE at every (x,t) with respect to ϵ,

$$\rho G^{\epsilon}_{\epsilon}(x,t) = G^{\epsilon}_{xx\epsilon}(x,t)\frac{\sigma^2}{2} + G^{\epsilon}_{\epsilon t}(x,t) \text{ for } -Y^{\epsilon}(t) < x < Y^{\epsilon}(t) \text{ and all } t \in [0,T]$$

and evaluating it at $\epsilon = 0$, which gives

$$\rho g(x,t) = g_{xx}(x,t)\frac{\sigma^2}{2} + g_t(x,t) \text{ for } -X < x < X \text{ and all } t \in [0,T]$$

Consider the lateral upper boundary conditions (value matching and smooth pasting)

$$G^{\epsilon}(Y^{\epsilon}(t),t) = b + \epsilon t \text{ and } G^{\epsilon}(-Y^{\epsilon}(t),t) = -b - \epsilon t$$
$$0 = G^{\epsilon}_x(Y^{\epsilon}(t),t) = -G^{\epsilon}_x(-Y^{\epsilon}(t),t)$$

Differentiating them with respect to ϵ, and concentrating in the upper boundary:

$$t = G^{\epsilon}_{\epsilon}(Y^{\epsilon}(t),t) + G^{\epsilon}_x(Y^{\epsilon}(t),t)\,Y^{\epsilon}_{\epsilon}(t) \text{ and } 0 = G^{\epsilon}_{x\epsilon}(Y^{\epsilon}(t),t) + G^{\epsilon}_{xx}(Y^{\epsilon}(t),t)\,Y^{\epsilon}_{\epsilon}(t)$$

evaluating at $\epsilon = 0$, using that $G^0_{\epsilon}(Y^0(t),t) = g(X,t)$, $G^0_x(Y^0(t),t) = 0$, $G^0_{x\epsilon}(Y^0(t),t) = g_x(X,t)$, $G^0_{xx}(Y^0(t),t) = F_{xx}(X)$ and $Y^0_{\epsilon}(t) = y(t)$, then I get

$$g(X,t) = t \text{ and } F_{xx}(X)y(t) + g_x(X,t) = 0 \text{ and all } t \in [0,T]$$

Finally, the derivative of the terminal condition with respect to ϵ gives:

$$g(x,T) = T\frac{\partial F(x,b)}{\partial b} \text{ for all } -X < x < X$$

Following the same steps at the lower boundary, one obtains $g(-X,t) = -t$. Given the antisymmetric nature of the lateral boundaries and of the terminal condition, it is easy to show, using the maximum principle, that the solution for $g(x,t)$ is antisymmetric. Since g is antisymmetric, it is equivalent to solve for g on $[0,X]\times[0,T]$, and add the lateral boundary condition $g(0,t) = 0$. Then the solution is extended to $[-X,X]\times[0,T]$ using the antisymmetry of g.
$\square$

Proof. (of Proposition 4) The proof proceeds in steps. Step 0 uses the superposition principle to solve for the terminal condition. Step 1 derives the solution of the heat equation (or a PDE) in terms of a Fourier series. Step 2 uses computes the solution. Step 3, I take the limit at $T \to \infty$ and rearrange.

Step 0. I use the superposition principle and solve the PDE as the sum of two functions. One function is the solution of PDE with lateral boundary condition equal to $g(X,t) = g(-X,t) = 0$ for all $t \in [0,T]$, and terminal condition $g(x,T) = T\partial F(x,b)/\partial b$ for all x. The other is the solution of the PDE with lateral boundary conditions $g(X,t) = t$, and $g(-X,t) - t$ for all $t \in [0,T]$ and terminal condition $g(x,T) = 0$ for all x which I will turn in the next steps. Using Feynman-Kak, the solution of the PDE for the first function satisfies:

$$g(x,t) = \mathbb{E}\left[e^{-\rho(T-t)}1_{\{\tau \geq T\}}\,T\,\frac{\partial F(x,b)}{\partial b}\,|\,x(t) = x\right]$$

where τ is the first time that x hits either X or $-X$, and where the expected value uses the distribution of the standard Brownian motion with variance per unit of time σ. Since $\frac{\partial F(x,b)}{\partial b}$ is bounded in $x \in [-X, X]$, and $\rho \geq 0$, then as $T \to \infty$, then $g(x.t) \to 0$ for any t.

Step 1. To solve this PDE with terminal condition $g(x, T) = 0$, I restrict the domain $x \in [0, X]$ and let:

$$\begin{aligned}
g(x,t) &= r(x,t) + \sum_{j=1}^{\infty} a_j(t)\varphi_j(x) \\
r(x,t) &= \frac{x}{X}t \\
\varphi_j(x) &= \sin\left(j\pi\frac{x}{X}\right) \\
q_j(t) &= \frac{\langle \varphi_j, r_t(\cdot,t) - \rho r(\cdot,t)\rangle}{\langle \varphi_j, \varphi_j\rangle} \\
\lambda_j &= \rho + \frac{(j\pi)^2}{X^2}\frac{\sigma^2}{2} \\
a_j(t) &= a_j(T)e^{-\lambda_j(T-t)} + \int_t^T q_j(\tau)e^{\lambda_j(t-\tau)}d\tau \\
a_j(T) &= \frac{\langle \varphi_j, -r(\cdot,T)\rangle}{\langle \varphi_j, \varphi_j\rangle}
\end{aligned}$$

The restrictions to the domain $[0, X]$ ensure that the set of functions $\{\varphi_j\}_{j=1}^{\infty}$ forms an orthonormal base for functions with such a domain. On this domain:

1. For each t, then $\varphi_j(0) = \varphi_j(X) = 0$ so $g(X,t) = r(X,t) = t$, and $g(0,t) = r(0,t) = 0$, so it satisfies the lateral boundary conditions.

2. At $t = T$, then $g(x,T) = r(x,T) + \sum_{j=1}^{\infty} a_j(T)\varphi_j(x) = 0$ since $a_j(T)$ are the coefficients of the projection of $-r(\cdot,T)$ using the orthonormal base $\{\varphi_j\}_{j=1}^{\infty}$.

3. The last condition to verify is that the PDE holds for $x \in (0, X)$ and $t \in [0, T]$. For each j, let $f_j(x,t) \equiv a_j(t)\varphi_j(x)$, and compute

$$\begin{aligned}
h^j(x,t) &\equiv \rho f^j(x,t) - \frac{\sigma^2}{2}f^j_{xx}(x,t) - f^j_t(x,t) \\
&= a_j(t)\varphi_j(x)\left[\rho + \frac{\sigma^2}{2}\frac{(j\pi)^2}{X^2} - \frac{1}{a_j(t)}\frac{d}{dt}a_j(t)\right] \\
&= a_j(t)\varphi_j(x)\left[\lambda_j - \frac{1}{a_j(t)}\frac{d}{dt}a_j(t)\right]
\end{aligned}$$

   Using the definition of $a_j(t)$, direct computation gives

$$\begin{aligned}
\frac{d}{dt}a_j(t) &= \lambda_j\left[a_j(T)e^{-\lambda_j(T-t)} + \int_t^T q_j(\tau)e^{\lambda_j(t-\tau)}d\tau\right] - q_j(t) \\
&= \lambda_j a_j(t) - q_j(t)
\end{aligned}$$

   Thus

$$h^j(x,t) = a_j(t)\varphi_j(x)\left[\lambda_j - \lambda_j + \frac{q_j(t)}{a_j(t)}\right] = q_j(t)\varphi_j(x)$$

Hence, replacing back each h^j into the PDE :

$$\rho g(x,t) - g_{xx}(x,t)\frac{\sigma^2}{2} - g_t(x,t) = \rho r(x,t) - r_t(x,t) + \sum_{j=1}^{\infty} q_j(t)\varphi_j(x)$$

But, $q_j(t)$ are the coefficients of the expansion of $-(\rho r(\cdot,t) - r_t(\cdot,t))$, so this expression is zero for all (x,t) for $x \in (0,X)$ and $0 \leq t < T$, and the PDE holds in an L^2 sense.

Step 2 To complete the solution I compute $q_j(t)$, and $a_j(t)$. I first get

$$\begin{aligned} q_j(t) &= \frac{\langle \varphi_j, r_t(\cdot,t) - \rho r(\cdot,t)\rangle}{\langle \varphi_j, \varphi_j\rangle} \\ &= \frac{\langle \varphi_j, r_t(\cdot,t)\rangle}{\langle \varphi_j, \varphi_j\rangle} - \rho\frac{\langle \varphi_j, r(\cdot,t)\rangle}{\langle \varphi_j, \varphi_j\rangle} = \frac{1}{X}\left[\frac{\langle \varphi_j, x\rangle}{\langle \varphi_j, \varphi_j\rangle} - \rho t\frac{\langle \varphi_j, x\rangle}{\langle \varphi_j, \varphi_j\rangle}\right] \\ &= \frac{(1-\rho t)}{X}\frac{\langle \varphi_j, x\rangle}{\langle \varphi_j, \varphi_j\rangle} \end{aligned}$$

Note that

$$\begin{aligned} \frac{X^2(-1)^{j+1}}{j\pi} = \int_0^X \varphi_j(x)x dx &= \int_0^X \sin(j\pi x/X)\, x\, dx \\ \frac{X}{2} = \int_0^X (\varphi_j(x))^2\, dx &= \int_0^X (\sin(j\pi x/X))^2\, dx \end{aligned}$$

Thus:

$$\begin{aligned} q_j(t) &= \frac{2(1-\rho t)}{X^2}\langle \varphi_j, x\rangle = \frac{2(1-\rho t)}{X^2}\frac{X^2(-1)^{j+1}}{j\pi} \\ &= 2(1-\rho t)\frac{(-1)^{j+1}}{j\pi} \end{aligned}$$

Furthermore

$$\begin{aligned} a_j(T) &= -\frac{\langle \varphi_j, r(\cdot,T)\rangle}{\langle \varphi_j, \varphi_j\rangle} = -\frac{2T}{X^2}\langle \varphi_j, x\rangle = -\frac{2T}{X^2}\frac{X^2(-1)^{j+1}}{j\pi} \\ &= -2T\frac{(-1)^{j+1}}{j\pi} \end{aligned}$$

Hence, I can now compute the expression for $a_j(t)$ as

$$\begin{aligned} a_j(t) &= a_j(T)e^{-\lambda_j(T-t)} + \int_t^T q_j(\tau)e^{\lambda_j(t-\tau)}d\tau \\ &= -2T\frac{(-1)^{j+1}}{j\pi}e^{-\lambda_j(T-t)} + \int_t^T 2(1-\rho\tau)\frac{(-1)^{j+1}}{j\pi}e^{\lambda_j(t-\tau)}d\tau \\ &= \frac{(-1)^{j+1}}{j\pi}\left[-2Te^{-\lambda_j(T-t)} + \int_t^T 2(1-\rho\tau)e^{\lambda_j(t-\tau)}d\tau\right] \end{aligned}$$

Solving the integral:

$$a_j(t) = \frac{(-1)^{j+1}}{j\pi}\left(-2Te^{-\lambda_j(T-t)} + \frac{2}{\lambda_j}\left[1-\rho t - \frac{\rho}{\lambda_j} + e^{\lambda_j(t-T)}\left(-1+\rho T + \frac{\rho}{\lambda_j}\right)\right]\right)$$

Step 3. Note that the terms of $a_j(t)$ that depends on T are:

$$a_j(t;T) = e^{-\lambda_j(T-t)}\frac{(-1)^{j+1}}{j\pi}\left[-2T + \frac{2}{\lambda_j}\left(-1+\rho T + \frac{\rho}{\lambda_j}\right)\right]$$

Now I take the limit as $T \to \infty$ on the expressions for $a_j(t)$. Fixing any $0 \leq t < T$, and letting $T \to \infty$, using that $\lambda_j > 0$

$$a_j(t) = \frac{(-1)^{j+1}}{j\pi}\frac{2}{\lambda_j}\left[1-\rho t - \frac{\rho}{\lambda_j}\right] \text{ as } T \to \infty$$

Thus, as $T \to \infty$, the expression for $g(x,t)$ is

$$g(x,t) = \frac{x}{X}t + \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\pi}\frac{2}{\lambda_j}\left[1-\rho t - \frac{\rho}{\lambda_j}\right]\sin\left(j\pi\frac{x}{X}\right)$$

Replacing λ_j, I get the final expression:

$$g(x,t) = \frac{x}{X}t + 2\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\pi}\frac{1}{\rho + \frac{(j\pi)^2\sigma^2}{2X^2}}\left[1-\rho t - \frac{\rho}{\rho + \frac{(j\pi)^2\sigma^2}{2X^2}}\right]\sin\left(j\pi\frac{x}{X}\right)$$

or

$$\begin{aligned}
g(x,t) &= \frac{x}{X}t + 2\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\pi}\frac{\frac{2}{\sigma^2}X^2}{\frac{2\rho}{\sigma^2}X^2 + (j\pi)^2}\left[1-\rho t - \frac{\frac{2\rho}{\sigma^2}X^2}{\frac{2\rho}{\sigma^2}X^2 + (j\pi)^2}\right]\sin\left(j\pi\frac{x}{X}\right) \\
&= \frac{x}{X}t + \frac{2}{\rho}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\pi}\frac{\frac{2\rho}{\sigma^2}X^2}{\frac{2\rho}{\sigma^2}X^2 + (j\pi)^2}\left[1-\rho t - \frac{\frac{2\rho}{\sigma^2}X^2}{\frac{2\rho}{\sigma^2}X^2 + (j\pi)^2}\right]\sin\left(j\pi\frac{x}{X}\right) \\
&= \frac{x}{X}t + \frac{2}{\rho}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\pi}\frac{\eta^2X^2}{\eta^2X^2 + (j\pi)^2}\left[1-\rho t - \frac{\eta^2X^2}{\eta^2X^2 + (j\pi)^2}\right]\sin\left(j\pi\frac{x}{X}\right)
\end{aligned}$$

for $x \in [-X, X]$, and for $t \geq 0$. □

Proof. (of Proposition 5) Evaluating g_x at time $t = 0$, and using that $\cos(0) = 1$, then

$$\begin{aligned}
g_x(0,0) &= \frac{2}{\rho X}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{X^2\eta^2}{X^2\eta^2 + (j\pi)^2}\left[1 - \frac{X^2\eta^2}{X^2\eta^2 + (j\pi)^2}\right]\cos(0) \\
&= \frac{2}{\rho}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{X\eta^2}{X^2\eta^2 + (j\pi)^2}\left[\frac{(j\pi)^2}{X^2\eta^2 + (j\pi)^2}\right] \\
&= \frac{4X}{\sigma^2}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{1}{X^2\eta^2 + (j\pi)^2}\left[\frac{(j\pi)^2}{X^2\eta^2 + (j\pi)^2}\right] \\
&= \frac{4X}{\sigma^2}\left(\frac{(a\coth(a) - 1)\operatorname{csch}(a)}{4a}\right) \text{ where } a^2 \equiv X^2\eta^2 \text{ or } a = X\eta \\
&= \frac{4X}{\sigma^2}\left(\frac{(X\eta\coth(X\eta) - 1)\operatorname{csch}(X\eta)}{4X\eta}\right) \\
&= \frac{1}{\sigma^2\eta}\left[(X\eta\coth(X\eta) - 1)\operatorname{csch}(X\eta)\right]
\end{aligned}$$

Likewise, for $x = X$, using that $\cos(j\pi) = -1^j$ then

$$\begin{aligned}
g_x(X,0) &= \frac{2}{\rho X}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{X^2\eta^2}{X^2\eta^2+(j\pi)^2}\left[1-\frac{X^2\eta^2}{X^2\eta^2+(j\pi)^2}\right]\cos(j\pi) \\
&= -\frac{2}{\rho}\sum_{j=1}^{\infty}\frac{X\eta^2}{X^2\eta^2+(j\pi)^2}\left[\frac{(j\pi)^2}{X^2\eta^2+(j\pi)^2}\right] \\
&= -\frac{4X}{\sigma^2}\sum_{j=1}^{\infty}\frac{1}{X^2\eta^2+(j\pi)^2}\left[\frac{(j\pi)^2}{X^2\eta^2+(j\pi)^2}\right] \\
&= -\frac{4X}{\sigma^2}\left(\frac{\coth(a)-a\,\mathrm{csch}^2(a)}{4a}\right) \text{ where } a^2 \equiv X^2\eta^2 \text{ or } a = X\eta \\
&= -\frac{4X}{\sigma^2}\left(\frac{\coth(X\eta)-X\eta\,\mathrm{csch}^2(X\eta)}{4X\eta}\right) \\
&= -\frac{1}{\eta\sigma^2}\left[\coth(X\eta)-X\eta\,\mathrm{csch}^2(X\eta)\right]
\end{aligned}$$

□

Proof. (of Proposition 6)

$$\begin{aligned}
\frac{d}{dt}y(t) &= -\frac{g_{xt}(X,t)}{F_{xx}(X)} \text{ with} \\
g_{xt}(X,t) &= \frac{1}{X}+\frac{2}{\rho X}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{X^2\eta^2}{X^2\eta^2+(j\pi)^2}(-\rho)\cos(j\pi) \\
&= \frac{1}{X}\left(1-2\sum_{j=1}^{\infty}(-1)^{j+1}\frac{X^2\eta^2}{X^2\eta^2+(j\pi)^2}(-1)^j\right) \\
&= \frac{1}{X}\left(1+2\sum_{j=1}^{\infty}\frac{X^2\eta^2}{X^2\eta^2+(j\pi)^2}\right) \\
&= \frac{1}{X}\left(1+2X^2\eta^2\sum_{j=1}^{\infty}\frac{1}{X^2\eta^2+(j\pi)^2}\right) \\
&= \frac{1}{X}\left(1+2X^2\eta^2\left(\frac{\coth(\eta X)}{2\eta X}-\frac{1}{2\eta^2X^2}\right)\right) \\
&= \frac{1}{X}\eta X\coth(\eta X)
\end{aligned}$$

Thus

$$\begin{aligned}\frac{d}{dt}y(t) &= \frac{-g_{xt}(X,t)}{F_{xx}(X)} \\ &= -\frac{1}{F_{xx}(X)X}\eta X \coth(\eta X) \\ &= \frac{\rho}{\eta X \tanh(\eta X)}\eta X \coth(\eta X) \\ &= \frac{\rho}{\tanh(\eta X)}\coth(\eta X) \\ &= \rho \coth^2(\eta X)\end{aligned}$$

Since $g_x(X,t)$ is linear in t I have:

$$Y(t) - X = y(t)\zeta = y(0)\zeta + \rho \coth^2(\eta X)\, t\zeta$$

Using the results from Proposition 5

$$\begin{aligned}Y(t) - X &= y(0)\zeta + \rho \coth^2(\eta X)\, t\zeta \\ &= -\frac{1}{2}\frac{\coth(X\eta) - X\eta \operatorname{csch}^2(X\eta)}{\tanh(\eta X)}\zeta + \rho \coth^2(\eta X)\, t\zeta + o(\zeta)\end{aligned}$$

Then, the derivative of $X(b+\zeta t)$ with respect to t is

$$\frac{dX(b+\zeta t)}{dt} = \frac{\rho}{1 - \tanh'(\eta X(b))}\zeta$$

Using $\tanh'(y) = \operatorname{sech}^2(y)$ I have $1/(1-\tanh'(\eta X)) = \coth^2(\eta X)$. □

Proof. (of Proposition 7) Proposition 6 gives

$$Y(t;b) = X(b) + \left(y(0;b) + \rho \coth^2(\eta X(b))\ t\right)\zeta + o(\zeta)$$

Using the definition of t^* is $Y(t^*;b) = X(b)$,

$$\begin{aligned}0 &= \left(y(0;b) + \rho \coth^2(\eta X(b))\ t^*(\zeta)\right)\zeta + o(\zeta) \text{ or} \\ t^*(\zeta) &= -\frac{y(0;b)}{\rho \coth^2(\eta X(b))} + o(\zeta)/\zeta\end{aligned}$$

using that from Proposition 6

$$-y(0;b) = \frac{1}{2}\coth^2(\eta X(b)) - \frac{1}{2}\frac{\eta X(b)\operatorname{csch}^2(\eta X(b))}{\tanh(\eta X(b))}$$

then

$$\begin{aligned}t^*(\zeta) &= \frac{1}{2\rho}\left[1 - \frac{\eta X(b)\operatorname{csch}^2(\eta X(b))}{\tanh(\eta X(b))\coth^2(\eta X(b))}\right] + o(\zeta)/\zeta \\ &= \frac{1}{2\rho}\left[1 - \frac{\eta X(b)}{\cosh(\eta X(b))\sinh(\eta X(b))}\right] + o(\zeta)/\zeta\end{aligned}$$

The properties of $\bar{t}$ follow directly from the analysis of the function $h(x) = 1 - x/(\sinh(x)\cosh(x))$ and the determination of $\eta X(b)$ in $\rho\eta b = X(b)\eta - \tanh(\eta X(b))$. □

Proof. (of Proposition 8) As the PDE has constant coefficients, I can shift the time t to 0 while also adjusting the starting point of the boundaries. Hence, these are $b + \zeta t$ and $-(b + \zeta t)$, i.e., the widening–band problem starting at time 0 with band $b + \zeta t$. By uniqueness,

$$G(x, t; b) = G(x, 0; b + \zeta t)$$

For any starting band B at time 0, the ζ–expansion yields

$$G(x, 0; B) = F(x; B) + g(x, 0; B)\,\zeta + o(\zeta).$$

With $B = b + \zeta t$ from the first part

$$G(x, t; b) = G(x, 0; b + \zeta t) = F(x; b + \zeta t) + g(x, 0; b + \zeta t)\,\zeta + o(\zeta).$$

Proposition 2 shows that the solution to the fixed band problem is

$$F(x) = \frac{1}{\rho}\left[x - \frac{1}{\eta}\frac{\sinh(\eta x)}{\cosh(\eta X(b))}\right] \quad \text{for } -X(b) \le x \le X(b)$$

where $X(b)$ is implicitly defined by

$$b = \frac{1}{\eta\rho}\left[\eta X(b) - \tanh(\eta X(b))\right]$$

□

Proof. (of Proposition 9) As a first step define $a^0(X)$ as the anticipation effect, as function of X. Using the expression for $F_x(0) = (1 - \operatorname{sech}(\eta X))/\rho$, and the expression for approximation above for $G_x(0,0) = F_x(0) + g_x(0,0)\zeta$ for $g_x(0,0)$ from Proposition 5, defining

$$\begin{aligned}
a^0(X) &= \frac{(1 - \operatorname{sech}(\eta X))/\rho + g_x(0,0)\zeta}{(1 - \operatorname{sech}(\eta X))/\rho} + o(\zeta) \\
&= 1 + \frac{g_x(0,0)\zeta}{(1 - \operatorname{sech}(\eta X))/\rho} + o(\zeta) \\
&= 1 + \frac{\rho}{\eta\sigma^2}\frac{(\eta X \coth(\eta X) - 1)\operatorname{csch}(\eta X)}{(1 - \operatorname{sech}(\eta X))}\zeta + o(\zeta) \\
&= 1 + \frac{\eta}{2}\frac{(\eta X \coth(\eta X) - 1)\operatorname{csch}(\eta X)}{(1 - \operatorname{sech}(\eta X))}\zeta + o(\zeta)
\end{aligned}$$

where the second to last line uses the expression for $g_x(0,0)$, and where the last line uses the definition of η. The three properties of $a^0(X)$ follow directly from the explicit expression for a^0 in terms of the hyperbolic functions. In particular, it follows from the analysis of

$$h^V(y) = \frac{(y\coth(y) - 1)\operatorname{csch}(y)}{(1 - \operatorname{sech}(y))}$$

This function is positive, decreasing, diverges to $+\infty$ as $y \to 0$, and converges to 0 as $y \to \infty$. This shows that properties of $a^0(X)$. Now I use that $a^0(b) = a^0(X(b))$. Since $X(b)$ is a strictly monotone function of b, with $X(0) = 0$, and $X(\infty) = \bar{\infty}$, then one obtains the desired results for $a^0(b)$.

Likewise, I define $\bar{a}(X)$ using the expression for $F_{xx}(X) = -\eta \tanh(\eta X)/\rho$, and the expression for approximation above for $Y(0) = X - g_x(X,0)/F_{xx}(X)\zeta + o(\zeta)$ for $g_x(X,0)$ from Proposition 5 defining

$$\begin{aligned}\bar{a}(X) &= \frac{X}{X - g_x(X,0)/F_{xx}(X)\zeta + o(\zeta)} \\ &= \frac{1}{1 - g_x(X,0)/(XF_{xx}(X))\zeta + o(\zeta)} = 1 - \left(-\frac{g_x(X,0)}{XF_{xx}(X)}\right)\zeta + o(\zeta) \\ &= 1 + \frac{g_x(X,0)}{XF_{xx}(X)}\zeta + o(\zeta)\end{aligned}$$

where we have use that $1/(1-x) = 1 + x + o(x)$. Thus,

$$\begin{aligned}\bar{a}(X) &= 1 - \frac{1}{\eta\sigma^2}\left[\coth(X\eta) - X\eta\,\mathrm{csch}^2(X\eta)\right]\frac{1}{-\eta X\tanh(\eta X)/\rho}\zeta + o(\zeta) \\ &= 1 + \frac{\rho}{\eta\sigma^2}\frac{\coth(X\eta) - X\eta\,\mathrm{csch}^2(X\eta)}{\eta X\tanh(\eta X)}\zeta + o(\zeta) \\ &= 1 + \frac{\eta}{2}\frac{\coth(X\eta) - X\eta\,\mathrm{csch}^2(X\eta)}{\eta X\tanh(\eta X)}\zeta + o(\zeta)\end{aligned}$$

where the second to last line uses the expression for $g_x(X,0)$, and where the last line uses the definition of η. The three properties of $\bar{a}(X)$ follow directly from the explicit expression for $\bar{a}$ in terms of the hyperbolic functions. In particular, it follows from the analysis of

$$h^X(y) = \frac{\coth(y) - y\,\mathrm{csch}^2(y)}{y\tanh(y)}$$

This function is positive, decreasing, diverges to $+\infty$ as $y \to 0$, and converges to 0 as $y \to \infty$. This shows that properties of $\bar{a}(X)$. Now I use that $\bar{a}(b) = \bar{a}(X(b))$. Since $X(b)$ is a strictly monotone function of b, with $X(0) = 0$, and $X(\infty) = \bar{\infty}$, then one obtains the desired results for $\bar{a}(b)$.

$$\begin{aligned}\bar{a}(X) \equiv \frac{X}{Y(\zeta)} &= \frac{X}{Y(0)} + \frac{\partial}{\partial\zeta}\left(\frac{X}{Y(\zeta)}\right)|_{\zeta=0}\,\zeta + o(\zeta) = 1 - \frac{X}{(Y(0))^2}\frac{\partial Y(\zeta)}{\partial\zeta}|_{\zeta=0}\,\zeta + o(\zeta) \\ &= 1 - \frac{1}{X}\frac{\partial Y(\zeta)}{\partial\zeta}|_{\zeta=0}\,\zeta + o(\zeta) = 1 + \frac{1}{X}\frac{g_x(X,0)}{F_{xx}(X)}\,\zeta + o(\zeta)\end{aligned}$$

Now I use the expression for $g_x(X,0)$ in Proposition 5 and the one for $F_{xx}(X) = -\eta\tanh(\eta X)/\rho$ to obtain:

$$\begin{aligned}\bar{a}(X) &= 1 + \frac{1}{X}\frac{g_x(X,0)}{F_{xx}(X)}\,\zeta + o(\zeta) \\ &= 1 + \frac{1}{\eta\sigma^2}\left[\coth(X\eta) - X\eta\,\mathrm{csch}^2(X\eta)\right]\frac{1}{-\eta X\tanh(\eta X)/\rho}\zeta + o(\zeta) \\ &= 1 + \frac{\rho}{\eta\sigma^2}\frac{\coth(X\eta) - X\eta\,\mathrm{csch}^2(X\eta)}{\eta X\tanh(\eta X)}\zeta + o(\zeta) \\ &= 1 + \frac{\eta}{2}\frac{\coth(X\eta) - X\eta\,\mathrm{csch}^2(X\eta)}{\eta X\tanh(\eta X)}\zeta + o(\zeta)\end{aligned}$$

□

Proof. (of Lemma 2) Differentiating the PDE again with respect to ϵ, and evaluating it at $\epsilon = 0$, using the notation introduced I get

$$\rho u(x,t) = u_{xx}(x,t)\frac{\sigma^2}{2} + u_t(x,t) \text{ for all } -X < x < X \text{ and all } t \in [0,T]$$

Differentiating with respect to ϵ value matching and smooth pasting:

$$\begin{aligned} t &= G^\epsilon_\epsilon(Y^\epsilon(t),t) + G^\epsilon_x(Y^\epsilon(t),t)\, Y^\epsilon_\epsilon(t) \\ -t &= G^\epsilon_\epsilon(-Y^\epsilon(t),t) - G^\epsilon_x(-Y^\epsilon(t),t)\, Y^\epsilon_\epsilon(t) \text{ and} \\ 0 &= G^\epsilon_{x\epsilon}(Y^\epsilon(t),t) + G^\epsilon_{xx}(Y^\epsilon(t),t)\, Y^\epsilon_\epsilon(t) \end{aligned}$$

differentiate once again to obtain:

$$\begin{aligned} 0 &= G^\epsilon_{\epsilon\epsilon}(Y^\epsilon(t),t) + G^\epsilon_{x\epsilon}(Y^\epsilon(t),t)\, Y^\epsilon_\epsilon(t) \\ &\quad + G^\epsilon_x(Y^\epsilon(t),t)\, Y^\epsilon_{\epsilon\epsilon}(t) + G^\epsilon_{\epsilon x}(Y^\epsilon(t),t) Y^\epsilon_\epsilon(t) + G^\epsilon_{xx}(Y^\epsilon(t),t)\ (Y^\epsilon_\epsilon(t))^2 \\ 0 &= G^\epsilon_{\epsilon\epsilon}(-Y^\epsilon(t),t) - G^\epsilon_{x\epsilon}(-Y^\epsilon(t),t)\, Y^\epsilon_\epsilon(t) \\ &\quad - G^\epsilon_x(-Y^\epsilon(t),t)\, Y^\epsilon_{\epsilon\epsilon}(t) - G^\epsilon_{\epsilon x}(-Y^\epsilon(t),t) Y^\epsilon_\epsilon(t) + G^\epsilon_{xx}(-Y^\epsilon(t),t)\ (Y^\epsilon_\epsilon(t))^2 \\ 0 &= G^\epsilon_{x\epsilon\epsilon}(Y^\epsilon(t),t) + G^\epsilon_{xx\epsilon}(Y^\epsilon(t),t)\, Y^\epsilon_\epsilon(t) + G^\epsilon_{xx}(Y^\epsilon(t),t)\, Y^\epsilon_{\epsilon\epsilon}(t) \\ &\quad + G^\epsilon_{xx\epsilon}(Y^\epsilon(t),t) Y^\epsilon_\epsilon(t) + G^\epsilon_{xxx}(Y^\epsilon(t),t)\ (Y^\epsilon_\epsilon(t))^2 \end{aligned}$$

Evaluating them at $\epsilon = 0$, I obtain:

$$\begin{aligned} 0 &= u(X,t) + 2g_x(X,t)\, y(t) + F_{xx}(X)\ (y(t))^2 \\ 0 &= u(-X,t) - 2g_x(-X,t)\, y(t) + F_{xx}(-X)\ (y(t))^2 \\ 0 &= u_x(X,t) + 2g_{xx}(X,t)\, y(t) + F_{xx}(X)\, w(t) + F_{xxx}(X)\ (y(t))^2 \end{aligned}$$

where I have used that $F_x(X) = F_x(-X) = 0$. Likewise, differentiating the terminal conditions with respect to ϵ and evaluating it at $\epsilon = 0$, I obtain $u(x,T) = 0$ for all x.
Next, I rewrite the lateral boundary conditions. I use that $g_x(X,t) = -F_{xx}(X)y(t)$ and that $g(\cdot,t)$ and $F(\cdot)$ are antisymmetric so $g_x(-X,t) = F_{xx}(-X)y(t)$ hence we can write:

$$\begin{aligned} 0 &= u(X,t) - 2\,F_{xx}(X)\,(y(t))^2 + F_{xx}(X)\ (y(t))^2 \implies u(X,t) = F_{xx}(X)\,(y(t))^2 \\ 0 &= u(-X,t) - 2F_{xx}(-X)\ (y(t))^2 + F_{xx}(-X)\ (y(t))^2 \implies u(-X,t) = F_{xx}(-X)\ (y(t))^2 \end{aligned}$$

Thus u has to solve:

$$\begin{aligned} &\rho u(x,t) = u_{xx}(x,t)\tfrac{\sigma^2}{2} + u_t(x,t) \text{ for all } -X < x < X \text{ and all } t \in [0,T] \\ &u(X,t) = F_{xx}(X)\ (y(t))^2 \text{ and } u(-X,t) = -F_{xx}(X)\ (y(t))^2 \text{ for all } t \in [0,T] \\ &u(x,T) = T^2\frac{\partial^2 F(x;b)}{\partial b^2} \text{ for all } -X < x < X \end{aligned}$$

The step to show that u is antisymmetric, and the boundaries in half of the domain are identical to the ones in the lemma for g. □

Proof. (of Proposition 10) I follow similar steps as in the derivation of the solution for g. The solution for u on the domain $x \in [0, X]$ is

$$\begin{aligned}
u(x,t) &= r(x,t) + \sum_{j=1}^{\infty} a_j(t)\varphi_j(x) \\
r(x,t) &= \frac{x}{X} F_{xx}(X)\ (y(t))^2 \\
\varphi_j(x) &= \sin\left(j\pi \frac{x}{X}\right) \\
q_j(t) &= \frac{\langle \varphi_j, r_t(\cdot,t) - \rho r(\cdot,t)\rangle}{\langle \varphi_j, \varphi_j\rangle} \\
\lambda_j &= \rho + \frac{(j\pi)^2}{X^2}\frac{\sigma^2}{2} \\
a_j(t) &= a_j(T)e^{-\lambda_j(T-t)} + \int_t^T q_j(\tau)e^{\lambda_j(t-\tau)}d\tau \\
a_j(T) &= \frac{\langle \varphi_j, -r(\cdot,T)\rangle}{\langle \varphi_j, \varphi_j\rangle}
\end{aligned}$$

The expression for $q_j(t)$ can be written as:

$$q_j(t) = \frac{\langle \varphi_j, r_t(\cdot,t) - \rho r(\cdot,t)\rangle}{\langle \varphi_j, \varphi_j\rangle} = \frac{\langle \varphi_j, x\rangle}{\langle \varphi_j, \varphi_j\rangle}\frac{F_{xx}(X)}{X}\left(\frac{d}{dt}y(t)^2 - \rho y(t)^2\right)$$

Using

$$\begin{aligned}
\frac{X^2(-1)^{j+1}}{j\pi} &= \int_0^X \varphi_j(x)x dx = \int_0^X \sin(j\pi x/X)\, x\, dx \\
\frac{X}{2} &= \int_0^X (\varphi_j(x))^2\, dx = \int_0^X (\sin(j\pi x/X))^2\, dx
\end{aligned}$$

Then

$$q_j(t) = \frac{\frac{X^2(-1)^{j+1}}{j\pi}}{\frac{X}{2}}\frac{F_{xx}(X)}{X}\left(\frac{d}{dt}y(t)^2 - \rho y(t)^2\right) = \frac{2\,(-1)^{j+1}F_{xx}(X)}{j\pi}\left(\frac{d}{dt}y(t)^2 - \rho y(t)^2\right)$$

Since $y(t)$ is quadratic t, as $T \to \infty$ we have:$\lim_{T\to\infty} a_j(T)e^{-\lambda_j(T-t)} = 0$. Thus,

$$\lim_{T\to\infty} a_j(t) = \int_t^{\infty} q_j(\tau)e^{\lambda_j(t-\tau)}d\tau$$

and in particular, setting $t = 0$:

$$\lim_{T\to\infty} a_j(0) = \int_0^{\infty} q_j(\tau)e^{\lambda_j-\tau}d\tau = \frac{2\,(-1)^{j+1}F_{xx}(X)}{j\pi}\int_0^{\infty} e^{-\lambda_j\tau}\left(\frac{d}{d\tau}y(\tau)^2 - \rho y(\tau)^2\right)d\tau$$

Integrating by parts:

$$\int_0^{\infty} e^{-\lambda_j\tau}\left(\frac{d}{d\tau}y(\tau)^2 - \rho y(\tau)^2\right)d\tau = -y(0)^2 + \int_0^{\infty} e^{-\lambda_j\tau}\left(\lambda_j y(\tau)^2 - \rho y(\tau)^2\right)d\tau$$

using the definition of $\lambda_j = \rho + \frac{\sigma^2(j\pi)^2}{2X^2}$:

$$\lim_{T\to\infty} a_j(0) = \frac{2\,(-1)^{j+1}F_{xx}(X)}{j\pi}\left[-y(0)^2 + \frac{\sigma^2(j\pi)^2}{2X^2}\int_0^\infty e^{-\lambda_j\tau}y(\tau)^2 d\tau\right]$$

Using the expression for $y(t)$ from Proposition 6

$$y(t) = y(0) + \rho\coth^2(\eta X)\ t$$

Then

$$\begin{aligned}\int_0^\infty e^{-\lambda_j t}y(t)^2 dt &= e^{-\lambda_j t}\left(y(0) + \rho\coth^2(\eta X)\ t\right)^2 dt \\ &= \frac{y(0)^2\lambda_j^2 + 2y(0)\rho\coth^2(\eta X)\,\lambda_j + 2\rho^2\coth^4(\eta X)}{(\lambda_j)^3}\end{aligned}$$

Hence:

$$\begin{aligned}u(x,0) &= \frac{x}{X}F_{xx}(X)\ (y(0))^2 \\ &+ \sum_{j=1}^\infty \frac{2\,(-1)^{j+1}F_{xx}(X)}{j\pi}\left[-y(0)^2 + \frac{\sigma^2(j\pi)^2}{2X^2}\int_0^\infty e^{-\lambda_j\tau}y(\tau)^2 d\tau\right]\sin\left(\frac{j\pi x}{X}\right) \\ &= \frac{x}{X}F_{xx}(X)\ (y(0))^2 + \sum_{j=1}^\infty \frac{2\,(-1)^{j+1}F_{xx}(X)}{j\pi} \\ &\times\left[-y(0)^2 + \frac{\sigma^2(j\pi)^2}{2X^2}\frac{y(0)^2\lambda_j^2 + 2y(0)\rho\coth^2(\eta X)\,\lambda_j + 2\rho^2\coth^4(\eta X)}{(\lambda_j)^3}\right]\sin\left(\frac{j\pi x}{X}\right)\end{aligned}$$

Note that

$$\sum_{j=1}^\infty \frac{2\,(-1)^{j+1}}{j\pi}\sin\left(\frac{j\pi x}{X}\right) = \frac{x}{X}$$

so

$$u(x,0) = \frac{F_{xx}(X)\sigma^2}{X^2}\sum_{j=1}^\infty (-1)^{j+1}\,j\pi\,\frac{y(0)^2\lambda_j^2 + 2y(0)\rho\coth^2(\eta X)\,\lambda_j + 2\rho^2\coth^4(\eta X)}{(\lambda_j)^3}\ \sin\left(\frac{j\pi x}{X}\right)$$

Finally, I show that $u(\cdot,0)$ is differentiable for $x\in(-X,X)$. For this, it is sufficient to show that $\sum_{j=1}^\infty j|b_j| < 0$ where $\{b_j\}_{j=1}^\infty$ are the Fourier coefficients. The Fourier coefficients of the solution are proportional to

$$c_j = (-1)^{j+1}\,j\pi\left(\frac{y(0)^2}{\lambda_j} + \frac{2y(0)\rho\coth^2(\eta X)}{(\lambda_j)^2} + \frac{2\rho^2\coth^4(\eta X)}{(\lambda_j)^3}\right)$$

Thus, it suffices to show that the desired property is satisfied by the term $b_j = (-1)^{j+1}\,j\pi/(\eta^2X^2 + (j\pi)^2)$. The other two terms clearly satisfy the desired property. I note the well known property that $d_j = (-1)^{j+1}/(j\pi)$ are the Fourier coefficients of $x/2$, which is a differentiable function. So it is

enough to show that the coefficients $e_j = d_j - b_j$ satisfy the desired property. Direct computations give $e_j = (-1)^{j+1}\frac{\eta^2 X^2}{(j\pi)(\eta^2 X^2+(j\pi)^2)}$. But:

$$\sum_{j=1}^{\infty} j|e_j| = \sum_{j=1}^{\infty} \frac{\eta^2 X^2}{\pi(\eta^2 X^2 + (j\pi)^2)} = \frac{\eta X \coth(\eta X) - 1}{2\pi}$$

Thus, $u(\cdot, 0)$ is differentiable in $x \in (-X, X)$. □

Proof. (of Lemma 3) Differentiating the PDE for G with respect to μ we get:

$$\rho G^{\mu}_{\mu}(x,t) = G^{\mu}_{x}(x,t) + \mu G^{\mu}_{x\mu}(x,t) + \frac{\sigma^2}{2} G^{\mu}_{xx\mu}(x,t) + G^{\mu}_{t\mu}(x,t)$$

which evaluating it at $\mu = 0$, noting that $G^0_x(x,t) = F_x(x)$ gives the PDE for h. Differentiating $s_0 + \zeta t = G^{\mu}(\bar{Y}^{\mu}(t), t)$ with respect to μ

$$0 = G^{\mu}_{\mu}(\bar{Y}^{\mu}(t), t) + G^{\mu}_{x}(\bar{Y}^{\mu}(t), t)\, \bar{Y}^{\mu}_{\mu}(t)$$

Evaluating it at $\mu = 0$ and using that $G^0_x(\bar{Y}^0(t), t) = F_x(X) = 0$, then it corresponds to $0 = h(X, t)$. The same argument holds at the lower bound. Next I differentiate $0 = G^{\mu}_{x}(\bar{Y}^{\mu}(t), t)$ with respect to μ obtaining:

$$0 = G^{\mu}_{x\mu}(\bar{Y}^{\mu}(t), t) + G^{\mu}_{xx}(\bar{Y}^{\mu}(t), t)\, \bar{Y}^{\mu}_{\mu}(t)$$

Evaluating it at $\mu = 0$, I get:

$$0 = h_x(X, t) + F_{xx}(X) z(t)$$

□

Proof. (of Proposition 11) I will show that the solution is of the form:

$$\begin{aligned}
h(x,t) &= \sum_{j=1}^{\infty} a_j(t)\varphi_j(x) \\
\varphi_j(x) &= \sin\left(j\pi \frac{x+X}{2X}\right) \text{ for } j = 1, 2, \dots \\
q_j(t) &= \frac{\langle \varphi_j, F_x(x)\rangle}{\langle \varphi_j, \varphi_j\rangle} \text{ for } j = 1, 2, \dots \\
\lambda_j &= \rho + \frac{1}{4}\frac{(j\pi)^2}{X^2}\frac{\sigma^2}{2} \text{ for } j = 1, 2, \dots \\
a_j(t) &= \int_t^T q_j(\tau) e^{\lambda_j (t-\tau)} d\tau
\end{aligned}$$

The next steps verify that this is indeed the solution.

1. I check that this solves the PDE. Define $D(x,t)$ as

$$D(x,t) \equiv \rho h(x,t) - h_{xx}(x,t)\frac{\sigma^2}{2} - h_t(x,t) - F_x(x)$$

The aim is to show that $D(x,t)=0$. Direct computation gives

$$\begin{aligned} D(x,t) &= \rho\sum_{j=1}^{\infty} a_j(t)\varphi_j(x) - \sum_{j=1}^{\infty} a_j(t)\frac{d^2}{dx^2}\varphi_j(x) - \sum_{j=1}^{\infty}\frac{d}{dt}a_j(t)\varphi_j(x) - F_x(x) \\ &= \left[\lambda_j a_j(t) - q_j(t)\right]\varphi_j(x) - F_x(x) \end{aligned}$$

where I use that $\frac{d^2}{dx^2}\varphi_j(x) = -\varphi_j(x)(j\pi)^2/(4X^2)$, and $\frac{d}{dt}a_j(t) = \lambda_j a_j(t) - q_j(t)$. Using the expression for λ_j:

$$D(x,t) = \sum_{j=1}^{\infty} q_j(t)\varphi_j(x) - F_x(x) = \sum_{j=1}^{\infty}\frac{\langle\varphi_j, F_x\rangle}{\langle\varphi_j,\varphi_j\rangle}\varphi_j(x) - F_x(x)$$

and using the $\{\varphi_j\}_{j=1}^{\infty}$ is an orthogonal base for function $F_x : [-X, X]$, a and thus $D(x,t)=0$. Hence, the PDE is solved for

$$\begin{aligned} a_j(t) &= e^{\lambda_j t}\int_t^T e^{-\lambda_j\tau}q_j(\tau)d\tau = \frac{e^{\lambda_j t}\left(e^{-\lambda_j t} - e^{-\lambda_j T}\right)}{\lambda_j}\frac{\langle\varphi_j, F_x\rangle}{\langle\varphi_j,\varphi_j\rangle} \\ &= \frac{\left(1 - e^{-\lambda_j(T-t)}\right)}{\lambda_j}\frac{\langle\varphi_j, F_x\rangle}{\langle\varphi_j,\varphi_j\rangle} \end{aligned}$$

Taking the limit:

$$\lim_{T\to\infty} a_j(t) = \frac{1}{\lambda_j}\frac{\langle\varphi_j, F_x\rangle}{\langle\varphi_j,\varphi_j\rangle}$$

2. The lateral boundary conditions $h(X,t) = h(-X,t) = 0$ are satisfied because since $\varphi_j(X) = \varphi_j(-X) = 0$.

Finally, I compute the expression for $q_j(t)$ and $a_j(t)$. For this consider:

$$\begin{aligned} \langle\varphi_j, F_x\rangle &= \begin{cases} \int_{-X}^{X}\sin\left(j\pi\frac{x+X}{2X}\right)F_x(x)dx & \text{if } j \text{ is odd} \\ 0 & \text{if } j \text{ is even} \end{cases} \\ \langle\varphi_j,\varphi_j\rangle &= \int_{-X}^{X}\sin\left(j\pi\frac{x+X}{2X}\right)^2 dx = X \end{aligned}$$

For odd j odd we have:

$$\begin{aligned} q_j(t) &= \frac{\langle\varphi_j, F_x\rangle}{\langle\varphi_j,\varphi_j\rangle} = \frac{1}{\rho X}\left[\int_{-X}^{X}\sin\left(j\pi\frac{x+X}{2X}\right)dx - \int_{-X}^{X}\frac{\sin\left(j\pi\frac{x+X}{2X}\right)\cosh(\eta x)}{\cosh(\eta X)}dx\right] \\ &= \frac{1}{\rho X}\left[\frac{4X}{j\pi} - \frac{4Xj\pi}{4X^2\eta^2 + (j\pi)^2}\right] = \frac{4}{\rho}\left[\frac{j\pi}{(j\pi)^2} - \frac{j\pi}{4X^2\eta^2 + (j\pi)^2}\right] \\ &= \frac{4j\pi}{\rho}\left[\frac{1}{(j\pi)^2} - \frac{1}{4X^2\eta^2 + (j\pi)^2}\right] \end{aligned}$$

and thus

$$\lim_{T\to\infty} a_j(t) = \left(\frac{1}{\rho + \frac{1}{4}\frac{(j\pi)^2\sigma^2}{2X^2}}\right)\frac{4j\pi}{\rho}\left[\frac{1}{(j\pi)^2} - \frac{1}{4X^2\eta^2 + (j\pi)^2}\right]$$

This gives the following

$$\begin{aligned}
h(x,t) &= \sum_{j=1,3,5}^{\infty}\left(\frac{1}{\rho+\frac{1}{4}\frac{(j\pi)^2\sigma^2}{2X^2}}\right)\frac{4j\pi}{\rho}\left[\frac{1}{(j\pi)^2}-\frac{1}{4X^2\eta^2+(j\pi)^2}\right]\sin\left(j\pi\frac{x+X}{2X}\right)\\
&= \sum_{j=1,3,5}^{\infty}\left(\frac{\frac{2X^2}{\sigma^2}}{\eta^2X^2+\frac{1}{4}(j\pi)^2}\right)\frac{4j\pi}{\rho}\left[\frac{1}{(j\pi)^2}-\frac{1}{4X^2\eta^2+(j\pi)^2}\right]\sin\left(j\pi\frac{x+X}{2X}\right)\\
&= \frac{8X^2}{\rho\sigma^2}\sum_{j=1,3,5}^{\infty}\left(\frac{1}{\eta^2X^2+\frac{1}{4}(j\pi)^2}\right)(j\pi)\left[\frac{1}{(j\pi)^2}-\frac{1}{4X^2\eta^2+(j\pi)^2}\right]\sin\left(j\pi\frac{x+X}{2X}\right)
\end{aligned}$$

I can rewrite it as:

$$\begin{aligned}
h(x,t) &= \frac{8X^2}{\rho\sigma^2}\sum_{j=1,3,5}^{\infty}\left(\frac{j\pi}{\eta^2X^2+\frac{1}{4}(j\pi)^2}\right)\left[\frac{1}{(j\pi)^2}-\frac{1}{4X^2\eta^2+(j\pi)^2}\right]\sin\left(j\pi\frac{x+X}{2X}\right)\\
&= \frac{32X^4\eta^2}{\rho\sigma^2}\sum_{j=1,3,5}^{\infty}\frac{1}{j\pi}\left(\frac{1}{\eta^2X^2+\frac{1}{4}(j\pi)^2}\right)\left(\frac{1}{4X^2\eta^2+(j\pi)^2}\right)\sin\left(j\pi\frac{x+X}{2X}\right)\\
&= \frac{128X^4\eta^2}{\rho\sigma^2}\sum_{j=1,3,5}^{\infty}\frac{1}{j\pi}\left(\frac{1}{4\eta^2X^2+(j\pi)^2}\right)\left(\frac{1}{4X^2\eta^2+(j\pi)^2}\right)\sin\left(j\pi\frac{x+X}{2X}\right)\\
&= \frac{128X^4\eta^2}{\rho\sigma^2}\sum_{j=1,3,5}^{\infty}\frac{1}{j\pi}\left(\frac{1}{4\eta^2X^2+(j\pi)^2}\right)^2\sin\left(j\pi\frac{x+X}{2X}\right)
\end{aligned}$$

Now, I compute $z(t)$. To compute $z(t)$ note that:

$$\begin{aligned}
h_x(X,t) &= \frac{128X^4\eta^2}{\rho\sigma^2}\frac{1}{2X}\sum_{j=1,3,5}^{\infty}\left(\frac{1}{4\eta^2X^2+(j\pi)^2}\right)^2\cos\left(j\pi\frac{X+X}{2X}\right)\\
&= \frac{64X^3\eta^2}{\rho\sigma^2}\sum_{j=1,3,5}^{\infty}\left(\frac{1}{4\eta^2X^2+(j\pi)^2}\right)^2\cos\left(j\pi\right)\\
&= -\frac{64X^3\eta^2}{\rho\sigma^2}\sum_{j=1,3,5}^{\infty}\left(\frac{1}{4\eta^2X^2+(j\pi)^2}\right)^2
\end{aligned}$$

Thus, using that $F_{xx}(X) = -(\eta/\rho)\tanh(\eta X)$, then

$$z(t) = -\frac{h_x(X,t)}{F_{xx}(X)} = -\frac{64X^3\eta}{\sigma^2\tanh(\eta X)}\sum_{j=1,3,5}^{\infty}\left(\frac{1}{4\eta^2X^2+(j\pi)^2}\right)^2$$

An alternative proof of this result uses that $h(x,)$ is also the derivative of the case with fixed barriers, i.e.: $h(x,t) = \frac{\partial F(x;\mu)}{\partial\mu}|_{\mu=0}$, where $F(x,\mu)$ and the boundaries $\bar{X}(\mu), \underline{X}(\mu)$ solve the ODE and boundary conditions:

$$\begin{aligned}
\rho F(x,\mu) &= x + \mu F_x(x;\mu) + F_{xx}(x;\mu)\frac{\sigma^2}{2} \text{ for } \underline{X}(\mu) \le x \le \bar{X}(\mu).\\
-b &= F(\underline{X}(\mu),\mu), b = F(\bar{X}(\mu),\mu) \text{ and}\\
0 &= F_x(\underline{X}(\mu),\mu) = F_x(\bar{X}(\mu),\mu)
\end{aligned}$$

Differentiating this system with respect to μ

$$\begin{aligned}\rho F_\mu(x,\mu) &= F_x(x;\mu) + \mu F_{x\mu}(x,\mu) + F_{xx\mu}(x;\mu)\frac{\sigma^2}{2} \text{ for } \underline{X}(\mu) \le x \le \bar{X}(\mu).\\ 0 &= F_x(\underline{X}(\mu),\mu)\underline{X}_\mu(\mu) + F_\mu(\underline{X}(\mu),\mu) = F_x(\bar{X}(\mu),\mu)\bar{X}_\mu(\mu) + F_\mu(\bar{X}(\mu),\mu) \text{ and}\\ 0 &= F_{xx}(\underline{X}(\mu),\mu)\underline{X}_\mu(\mu) + F_{x\mu}(\underline{X}(\mu),\mu) = F_{xx}(\bar{X}(\mu),\mu)\bar{X}_\mu(\mu) + F_{x\mu}(\bar{X}(\mu),\mu)\end{aligned}$$

Evaluating it at $\mu = 0$ gives:

$$\begin{aligned}\rho h(x) &= F_x(x) + h_{xx}(x;\mu)\frac{\sigma^2}{2} \text{ for } -X \le x \le X \text{ where}\\ 0 &= h(-X) = h(X) \text{ and}\\ 0 &= F_{xx}(-X)z + h_x(-X) = F_{xx}(X)z + h_x(X)\end{aligned}$$

We first guess and verify the solution of this ODE for h, where

$$h(x) = \sum_{j=1,3,5,\ldots} \frac{\langle \varphi_j, F_x(\cdot)\rangle}{\langle \varphi_j, \varphi_j\rangle}\frac{1}{\lambda_j}\varphi_j(x)$$

Define $d(x)$ as follows:

$$d(x) = \rho h(x) - F_x(x) - \frac{\sigma^2}{2}h_{xx}(x)$$

and use that $F_x(x) = \sum_{j=1,3,5,\ldots} \frac{\langle \varphi_j, F_x(\cdot)\rangle}{\langle \varphi_j, \varphi_j\rangle}\varphi_j(x)$, then

$$\begin{aligned}d(x) &= \rho \sum_{j=1,3,5,\ldots} \frac{\langle \varphi_j, F_x(\cdot)\rangle}{\langle \varphi_j, \varphi_j\rangle}\frac{1}{\lambda_j}\varphi_j(x) - \sum_{j=1,3,5,\ldots} \frac{\langle \varphi_j, F_x(\cdot)\rangle}{\langle \varphi_j, \varphi_j\rangle}\varphi_j(x) - \frac{\sigma^2}{2}\sum_{j=1,3,5,\ldots} \frac{\langle \varphi_j, F_x(\cdot)\rangle}{\langle \varphi_j, \varphi_j\rangle}\frac{1}{\lambda_j}\varphi_j''(x)\\ &= \rho \sum_{j=1,3,5,\ldots} \frac{\langle \varphi_j, F_x(\cdot)\rangle}{\langle \varphi_j, \varphi_j\rangle}\frac{1}{\lambda_j}\varphi_j(x) - \sum_{j=1,3,5,\ldots} \frac{\langle \varphi_j, F_x(\cdot)\rangle}{\langle \varphi_j, \varphi_j\rangle}\varphi_j(x)\\ &+ \sum_{j=1,3,5,\ldots} \frac{\langle \varphi_j, F_x(\cdot)\rangle}{\langle \varphi_j, \varphi_j\rangle}\frac{\sigma^2}{2}\frac{(j\pi)^2}{(2X)^2\lambda_j}\varphi_j(x)\\ &= \sum_{j=1,3,5,\ldots} \frac{\langle \varphi_j, F_x(\cdot)\rangle}{\langle \varphi_j, \varphi_j\rangle}\frac{1}{\lambda_j}\varphi_j(x)\left[\rho - \lambda_j + \frac{\sigma^2}{2}\frac{(j\pi)^2}{(2X)^2}\right]\end{aligned}$$

But $\lambda_j = \rho + \frac{\sigma^2}{2}\frac{(j\pi)^2}{(2X)^2}$ for all j, so $d(x) = 0$. Additionally, since $\varphi_j(-X) = \varphi_j(X) = 0$, then it also satisfies the boundary condition. □

Proof. (of Lemma 4) To obtain the second derivative of $F(x,\mu)$ with respect to μ, we differentiate the expression below with respect to μ

$$\begin{aligned}\rho F_\mu(x,\mu) &= F_x(x;\mu) + \mu F_{x\mu}(x,\mu) + F_{xx\mu}(x;\mu)\frac{\sigma^2}{2} \text{ for } \underline{X}(\mu) \le x \le \bar{X}(\mu).\\ 0 &= F_x(\underline{X}(\mu),\mu)\underline{X}_\mu(\mu) + F_\mu(\underline{X}(\mu),\mu) = F_x(\bar{X}(\mu),\mu)\bar{X}_\mu(\mu) + F_\mu(\bar{X}(\mu),\mu) \text{ and}\\ 0 &= F_{xx}(\underline{X}(\mu),\mu)\underline{X}_\mu(\mu) + F_{x\mu}(\underline{X}(\mu),\mu) = F_{xx}(\bar{X}(\mu),\mu)\bar{X}_\mu(\mu) + F_{x\mu}(\bar{X}(\mu),\mu)\end{aligned}$$

obtaining:

$$\begin{aligned}
\rho F_{\mu\mu}(x,\mu) &= 2F_{x\mu}(x,\mu) + \mu F_{x\mu\mu}(x,\mu) + F_{xx\mu\mu}(x,\mu)\frac{\sigma^2}{2} \text{ for } \underline{X}(\mu) \leq x \leq \bar{X}(\mu) \\
0 &= F_{x\mu}(\underline{X}(\mu),\mu)\underline{X}_\mu(\mu) + F_x(\underline{X}(\mu),\mu)\underline{X}_{\mu\mu}(\mu) + F_{\mu\mu}(\underline{X}(\mu),\mu) \\
&\quad + F_{xx}(\underline{X}(\mu),\mu)(\underline{X}_\mu(\mu))^2 + F_{x\mu}(\underline{X}(\mu),\mu)\underline{X}_\mu(\mu) \\
0 &= F_{xx\mu}(\underline{X}(\mu),\mu)\underline{X}_\mu(\mu) + F_{x\mu\mu}(\underline{X}(\mu),\mu) + F_{xx}(\underline{X}(\mu),\mu)\underline{X}_{\mu\mu}(\mu) \\
&\quad + F_{xxx}(\underline{X}(\mu),\mu)(\underline{X}_\mu(\mu))^2 + F_{xx\mu}(\underline{X}(\mu),\mu)\underline{X}_\mu(\mu)
\end{aligned}$$

Letting $q(x) = F_{\mu\mu}(x,\mu)|_{\mu=0}$ we can write:

$$\begin{aligned}
\rho q(x) &= 2h_x(x) + q_{xx}(x)\frac{\sigma^2}{2} \text{ for } -X \leq x \leq X \\
0 &= 2h_x(-X)z + q(-X) + F_{xx}(-X)\, z^2 \\
0 &= 2h_{xx}(-X)z + q_x(-X) + F_{xx}(-X)z_\mu + F_{xxx}(-X)\, z^2
\end{aligned}$$

Likewise, at the upper boundary:

$$0 = 2h_{xx}(X)z + q_x(X) + F_{xx}(X)z_\mu + F_{xxx}(X), z^2,$$

hence

$$z_\mu = -\left(q_x(X) + F_{xxx}(X), z^2\right)/F_{xx}(X).$$

Thus, I am looking for the solution to:

$$\begin{aligned}
\rho q(x) &= 2h_x(x) + q_{xx}(x)\frac{\sigma^2}{2} \text{ for } -X \leq x \leq X \\
q(X) &= -z\left(2h_x(X) + F_{xx}(X)z\right) \text{ and } q(-X) = -z\left(2h_x(-X) + F_{xx}(-X)z\right)
\end{aligned}$$

Note that $z = -h_x(X)/F_{xx}(X)$, so that

$$\begin{aligned}
\rho q(x) &= 2h_x(x) + q_{xx}(x)\frac{\sigma^2}{2} \text{ for } -X \leq x \leq X \\
q(X) &= -zh_x(X) = \frac{h_x(X)^2}{F_{xx}(X)} \text{ and } q(-X) = -zh_x(-X) = \frac{h_x(-X)^2}{F_{xx}(-X)}
\end{aligned}$$

□

Proof. (of Proposition 12) I first solve for a particular solution to the ODE for q. I claim that the particular solution is:

$$q^p(x) = \sum_{j=1}^{\infty} \frac{\langle 2h_x(\cdot), \varphi_j\rangle}{\langle \varphi_j, \varphi_j\rangle}\frac{1}{\lambda_j}\varphi_j(x)$$

where

$$\varphi_j(x) = \sin\left((j\pi)\frac{x}{X}\right) \text{ and } \lambda_j = \rho + \frac{(j\pi)^2\sigma^2}{2X^2}$$

and

$$2h_x(x) = \frac{128X^3\eta^2}{\rho\sigma^2} \sum_{k=1,3,5}^{\infty} \left(\frac{1}{4\eta^2X^2 + (k\pi)^2}\right)^2 \cos\left(k\pi\frac{x+X}{2X}\right)$$

To compute the projection coefficients, note that:

$$X = \langle\varphi_j, \varphi_j\rangle = \int_{-X}^{X} \sin\left((j\pi)\frac{x}{X}\right)^2 dx$$

and for integer j and odd integer k:

$$\begin{aligned}\int_{-X}^{X} \sin\left((j\pi)\frac{x}{X}\right)\cos\left(k\pi\frac{x+X}{2X}\right)dx &= X\frac{j8\cos(\pi j)\sin(\pi k/2)^2}{\pi(4j^2-k^2)} = X\frac{j8(-1)^j}{\pi(4j^2-k^2)} \\ &= X\,8\,j\pi\,(-1)^j\frac{1}{(4(j\pi)^2-(k\pi)^2)}\end{aligned}$$

Thus

$$\frac{\langle 2h_x(\cdot), \varphi_j\rangle}{\langle\varphi_j,\varphi_j\rangle} = \frac{128X^3\eta^2 8}{\rho\sigma^2}(-1)^j j\pi \sum_{k=1,3,5...}^{\infty} \frac{1}{4(j\pi)^2-(k\pi)^2}\left(\frac{1}{4\eta^2X^2+(k\pi)^2}\right)^2$$

The particular solution is thus:

$$\begin{aligned}q^p(x) &= \frac{128X^3\eta^2 8}{\rho\sigma^2}\sum_{j=1}^{\infty}\frac{(-1)^j\,j\pi}{\rho+\frac{(j\pi)^2\sigma^2}{2X^2}}\sum_{k=1,3,5...}^{\infty}\frac{1}{4(j\pi)^2-(k\pi)^2}\left(\frac{1}{4\eta^2X^2+(k\pi)^2}\right)^2\sin\left(j\pi\frac{x}{X}\right) \\ &= \frac{128X^3\eta^2 8}{\rho\sigma^2}\sum_{j=1}^{\infty}\sum_{k=1,3,5...}^{\infty}\left(\frac{1}{4\eta^2X^2+(k\pi)^2}\right)^2\frac{1}{4(j\pi)^2-(k\pi)^2}\frac{(-1)^j\,j\pi}{\rho+\frac{(j\pi)^2\sigma^2}{2X^2}}\sin\left(j\pi\frac{x}{X}\right) \\ &= \frac{128X^3\eta^2 8}{\rho\sigma^2}\sum_{j=1}^{\infty}\sum_{k=1,3,5...}^{\infty}\left(\frac{1}{4\eta^2X^2+(k\pi)^2}\right)^2\frac{8X^2}{\sigma^2}\frac{1}{4(j\pi)^2-(k\pi)^2}\frac{(-1)^j\,j\pi}{4\frac{2\rho}{\sigma^2}X^2+4(j\pi)^2}\sin\left(j\pi\frac{x}{X}\right) \\ &= \frac{1024X^3\eta^2}{\rho\sigma^2}\sum_{j=1}^{\infty}\sum_{k=1,3,5...}^{\infty}\left(\frac{1}{4\eta^2X^2+(k\pi)^2}\right)^2\frac{8X^2}{\sigma^2}\frac{1}{4(j\pi)^2-(k\pi)^2}\frac{(-1)^j\,j\pi}{4\eta^2X^2+4(j\pi)^2}\sin\left(j\pi\frac{x}{X}\right) \\ &= \frac{8192X^5\eta^2}{\rho\sigma^4}\sum_{j=1}^{\infty}\sum_{k=1,3,5...}^{\infty}\left(\frac{1}{4\eta^2X^2+(k\pi)^2}\right)^2\left(\frac{1}{4(j\pi)^2-(k\pi)^2}\right)\frac{(-1)^j\,j\pi}{4\eta^2X^2+4(j\pi)^2}\sin\left(j\pi\frac{x}{X}\right)\end{aligned}$$

Thus $q^p(X) = q^p(-X) = 0$.
Since h_x and thus $q^p(x)$ are antisymmetric, we look for an antisymmetric solution. Note that for any constant A, the function $A\sinh(\eta x)$ is antisymmetric and solves the homogeneous equation $\rho q^h(x) = q^h_{xx}(x)\sigma^2/2$. Thus, the solution is

$$q(x) = A\sinh(\eta x) + q^p(x)$$

where A is chosen so that

$$q(X) = A\sinh(\eta X) + q^p(X) = -zh_x(X) \text{ or } q(X) = A\sinh(\eta X) + q^p(X) = -zh_x(X)$$

and since $q^p(X) = 0$,

$$A = -z\frac{h_x(X)}{\sinh(\eta X)}.$$

Thus, the solution is:

$$q(x) = -zh_x(X)\frac{\sinh(\eta x)}{\sinh(\eta X)} + q^p(x)$$

We also have

$$\begin{aligned} h_x(X) &= \frac{64X^3\eta^2}{\rho\sigma^2}\sum_{k=1,3,5}^{\infty}\left(\frac{1}{4\eta^2X^2+(k\pi)^2}\right)^2\cos(k\pi) \\ &= -\frac{64X^3\eta^2}{\rho\sigma^2}\sum_{k=1,3,5}^{\infty}\left(\frac{1}{4\eta^2X^2+(k\pi)^2}\right)^2 \end{aligned}$$

Hence, the solution is:

$$\begin{aligned} q(x) = \sinh(\eta x)\frac{z}{\sinh(\eta X)}&\left(\frac{64X^3\eta^2}{\rho\sigma^2}\sum_{k=1,3,5}^{\infty}\left(\frac{1}{4\eta^2X^2+(k\pi)^2}\right)^2\right) \\ &+ \frac{8192X^5\eta^2}{\rho\sigma^4}\sum_{j=1}^{\infty}\sum_{k=1,3,5\ldots}^{\infty}\left(\frac{1}{4\eta^2X^2+(k\pi)^2}\right)^2\left(\frac{1}{4(j\pi)^2-(k\pi)^2}\right)\frac{(-1)^j\,j\pi}{4\eta^2X^2+4(j\pi)^2}\sin\left(j\pi\frac{x}{X}\right) \end{aligned}$$

By antisymmetry of both terms, the lower-bound condition at $-X$ is also satisfied. □

Proof. (of Lemma 5) I start with the PDE for $G_\mu^{\mu\zeta}$ derived in the proof for Lemma 3 with respect to μ:

$$\rho G_\mu^{\mu\zeta}(x,t) = G_x^{\mu\zeta}(x,t) + \mu G_{x\mu}^{\mu\zeta}(x,t) + \frac{\sigma^2}{2}G_{xx\mu}^{\mu\zeta}(x,t) + G_{t\mu}^{\mu\zeta}(x,t)$$

Then, I differentiate it with respect to ζ, obtaining:

$$\rho G_{\mu\zeta}^{\mu\zeta}(x,t) = G_{x\zeta}^{\mu\zeta}(x,t) + \mu G_{x\mu\zeta}^{\mu\zeta}(x,t) + \frac{\sigma^2}{2}G_{xx\mu\zeta}^{\mu\zeta}(x,t) + G_{t\mu\zeta}^{\mu\zeta}(x,t)$$

which evaluating it at $\mu = \zeta = 0$, noting that $G_{x\zeta}^{00}(x,t) = g_x(x,t)$ gives the PDE for k. Now I turn to the boundary conditions. Let's start with the boundary condition coming from value matching:

$$0 = G_\mu^{\mu\zeta}(\bar{Y}^{\mu\zeta}(t),t) + G_x^{\mu\zeta}(\bar{Y}^{\mu\zeta}(t),t)\,\bar{Y}_\mu^{\mu\zeta}(t)$$

Differentiating it with respect to ζ

$$\begin{aligned} 0 = {} & G_{\mu\zeta}^{\mu\zeta}(\bar{Y}^{\mu\zeta}(t),t) + G_{x\zeta}^{\mu\zeta}(\bar{Y}^{\mu\zeta}(t),t)\,\bar{Y}_\mu^{\mu\zeta}(t) + G_x^{\mu\zeta}(\bar{Y}^{\mu\zeta}(t),t)\,\bar{Y}_{\mu\zeta}^{\mu\zeta}(t) \\ & + G_{\mu x}^{\mu\zeta}(\bar{Y}^{\mu\zeta}(t),t)\bar{Y}_\zeta^{\mu\zeta}(t) + G_{xx}^{\mu\zeta}(\bar{Y}^{\mu\zeta}(t),t)\,\bar{Y}_\mu^{\mu\zeta}(t)\bar{Y}_\zeta^{\mu\zeta}(t) \end{aligned}$$

Evaluating it at $\mu = \zeta = 0$

$$0 = k(X,t) + g_x(X,t)\,z(t) + F_x(X)\,\bar{Y}_{\mu\zeta}^{00}(t) + h_x(X,t)y(t) + F_{xx}(X)\,y(t)z(t)$$

Using $F_x(X) = 0$, I get:

$$0 = k(X,t) + g_x(X,t)\, z(t) + h_x(X,t) y(t) + F_{xx}(X)\, y(t) z(t)$$

As a check, let's invert the order of differentiation. I will start with the derivatives of the G with respect to ζ:

$$\rho G^{\mu\zeta}_{\zeta}(x,t) = \mu G^{\mu\zeta}_{x\zeta}(x,t) + \frac{\sigma^2}{2} G^{\mu\zeta}_{xx\zeta}(x,t) + G^{\mu\zeta}_{t\zeta}(x,t)$$

Differentiating it with respect to μ:

$$\rho G^{\mu\zeta}_{\zeta\mu}(x,t) = G^{\mu\zeta}_{x\zeta}(x,t) + \mu G^{\mu\zeta}_{x\zeta\mu}(x,t) + \frac{\sigma^2}{2} G^{\mu\zeta}_{xx\zeta\mu}(x,t) + G^{\mu\zeta}_{t\zeta\mu}(x,t)$$

Evaluating at $\mu = \zeta = 0$ I get:

$$\rho k(x,t) = G^{00}_{x\zeta}(x,t) + \frac{\sigma^2}{2} k_{xx}(x,t) + k_t(x,t)$$

or

$$\rho k(x,t) = g_x(x,t) + \frac{\sigma^2}{2} k_{xx}(x,t) + k_t(x,t)$$

Likewise, starting with the lateral boundary condition coming from differentiating value matching:

$$t = G^{\mu\zeta}_{\zeta}(\bar{Y}^{\mu\zeta}(t),t) + G^{\mu\zeta}_{x}(\bar{Y}^{\mu\zeta}(t),t)\, \bar{Y}^{\mu\zeta}_{\zeta}(t)$$

Differentiating this expression with respect to μ

$$\begin{aligned} 0 = {} & G^{\mu\zeta}_{\zeta\mu}(\bar{Y}^{\mu\zeta}(t),t) + G^{\mu\zeta}_{x\mu}(\bar{Y}^{\mu\zeta}(t),t)\, \bar{Y}^{\mu\zeta}_{\zeta}(t) + G^{\mu\zeta}_{x}(\bar{Y}^{\mu\zeta}(t),t)\, \bar{Y}^{\mu\zeta}_{\zeta\mu}(t) \\ & + G^{\mu\zeta}_{\zeta x}(\bar{Y}^{\mu\zeta}(t),t) \bar{Y}^{\mu\zeta}_{\mu}(t) + G^{\mu\zeta}_{xx}(\bar{Y}^{\mu\zeta}(t),t)\, \bar{Y}^{\mu\zeta}_{\zeta}(t) \bar{Y}^{\mu\zeta}_{\mu}(t) \end{aligned}$$

Evaluating it at $\mu = \zeta = 0$, I obtain:

$$\begin{aligned} 0 = {} & k(X,t) + h_x(X,t)\, y(t) + F_x(X)\, \bar{Y}^{\mu\zeta}_{\zeta\mu}(t) \\ & + g_x(X,t) z(t) + F_{xx}(X)\, y(t) z(t) \end{aligned}$$

and using that $F_x(X) = 0$, then

$$0 = k(X,t) + h_x(X,t)\, y(t) + g_x(X,t) z(t) + F_{xx}(X)\, y(t) z(t)$$

Finally, I use that $y(t) = -\frac{g_x(X,t)}{F_{xx}(X)}$ then

$$0 = z(t)\left[g_x(X,t) + F_{xx}(X)\, y(t)\right] = z(t)\left[-F_{xx}(X)\, y(t) + F_{xx}(X)\, y(t)\right]$$

Hence:

$$\begin{aligned} k(X,t) &= -h_x(X,t) y(t), \\ k(-X,t) &= -h_x(-X,t) y(t). \end{aligned}$$

□

Proof. (of Proposition 13) I decompose $k = k^s + k^a$ where:

- k^s is symmetric and solves the PDE with source g_x and zero lateral boundary conditions:

$$\rho k^s(x,t) = \tfrac{\sigma^2}{2} k^s_{xx}(x,t) + k^s_t(x,t) + g_x(x,t), \quad k^s(\pm X,t) = 0, \quad k^s(x,T) = 0.$$

- k^a is antisymmetric and solves the homogeneous PDE with the non-zero lateral boundary conditions:

$$\begin{aligned} &\rho k^a(x,t) = \tfrac{\sigma^2}{2} k^a_{xx}(x,t) + k^a_t(x,t), \\ &k^a(X,t) = -h_x(X)\, y(t),\ k^a(-X,t) = h_x(X)\, y(t), \quad k^a(x,T) = 0. \end{aligned}$$

This decomposition is valid since the PDE is linear. The source $g_x(\cdot,t)$ is symmetric (since g is antisymmetric), and the lateral boundary conditions are antisymmetric (since h_x is antisymmetric, so $h_x(-X) = -h_x(X)$), which is consistent with the symmetry properties of each component.
Part 1: Solution for $k^s(x,0)$. The function k^s has zero lateral boundary conditions and a symmetric source g_x. I expand it using the basis $\{\varphi_j\}_{j=1}^{\infty}$ where $\varphi_j(x) = \sin\!\big(j\pi \frac{x+X}{2X}\big)$, which vanishes at $x = \pm X$. Only odd j contribute, since $g_x(\cdot,t)$ is symmetric. The solution is:

$$\begin{aligned} k^s(x,t) &= \sum_{j=1}^{\infty} a_j(t)\, \varphi_j(x), \\ q_j(t) &= \frac{\langle \varphi_j,\, g_x(\cdot,t)\rangle}{\langle \varphi_j,\, \varphi_j\rangle}, \qquad \lambda_j = \rho + \frac{(j\pi)^2}{(2X)^2}\,\frac{\sigma^2}{2}, \\ a_j(t) &= \int_t^T q_j(\tau)\, e^{\lambda_j(t-\tau)}\, d\tau. \end{aligned}$$

I verify that this solves the PDE. Define $D(x,t)$ as

$$D(x,t) \equiv \rho\, k^s(x,t) - (k^s)_{xx}(x,t)\,\frac{\sigma^2}{2} - (k^s)_t(x,t) - g_x(x,t).$$

Direct computation, using $\frac{d^2}{dx^2}\varphi_j(x) = -\varphi_j(x)\,(j\pi)^2/(2X)^2$ and $\frac{d}{dt}a_j(t) = \lambda_j a_j(t) - q_j(t)$, gives

$$D(x,t) = \sum_{j=1}^{\infty} q_j(t)\,\varphi_j(x) - g_x(x,t) = 0,$$

since $\{\varphi_j\}_{j=1}^{\infty}$ is an orthogonal basis for functions on $[-X,X]$ vanishing at the endpoints. The lateral boundary conditions $k^s(\pm X,t) = 0$ hold because $\varphi_j(\pm X) = 0$ for all j.
Since $g_x(x,t)$ is linear in t, write $g_x(x,t) = g_x(x,0) + g_{xt}(x,0)\,t$, with

$$\begin{aligned} g_x(x,0) &= \frac{1}{X}\,\frac{2}{\rho}\sum_{r=1}^{\infty}(-1)^{r+1}\frac{\eta^2X^2}{\eta^2X^2+(r\pi)^2}\,\frac{(r\pi)^2}{\eta^2X^2+(r\pi)^2}\,\cos\!\Big(r\pi\frac{x}{X}\Big), \\ g_{xt}(x,0) &= \frac{1}{X}\left[1 - 2\sum_{r=1}^{\infty}(-1)^{r+1}\frac{\eta^2X^2}{\eta^2X^2+(r\pi)^2}\,\cos\!\Big(r\pi\frac{x}{X}\Big)\right]. \end{aligned}$$

Taking $T \to \infty$, the terminal-condition contribution vanishes, and

$$a_j(t) = \frac{1}{\lambda_j}\left[\frac{\langle \varphi_j,\, g_x(\cdot,0)\rangle}{\langle \varphi_j,\, \varphi_j\rangle} + \frac{\langle \varphi_j,\, g_{xt}(\cdot,0)\rangle}{\langle \varphi_j,\, \varphi_j\rangle}\left(t + \frac{1}{\lambda_j}\right)\right].$$

Using $\langle \varphi_j,\, \varphi_j \rangle = X$ and, for odd j,

$$\int_{-X}^{X} \sin\Big(j\pi \frac{x+X}{2X}\Big)\, \cos\Big(r\pi \frac{x}{X}\Big)\, dx = \frac{4Xj\,(-1)^r}{\pi(j^2-4r^2)},$$

the inner products are:

$$\langle \varphi_j,\, g_x(\cdot,0) \rangle = -(j\pi)\, \frac{8\eta^2 X^2}{\rho} \sum_{r=1}^{\infty} \frac{(r\pi)^2}{\big(\eta^2 X^2 + (r\pi)^2\big)^2}\, \frac{1}{(j\pi)^2 - 4(r\pi)^2},$$
$$\langle \varphi_j,\, g_{xt}(\cdot,0) \rangle = \frac{4}{j\pi} + (j\pi)\, 8\eta^2 X^2 \sum_{r=1}^{\infty} \frac{1}{\eta^2 X^2 + (r\pi)^2}\, \frac{1}{(j\pi)^2 - 4(r\pi)^2}.$$

Using the closed-form series representations

$$\sum_{r=1}^{\infty} \frac{(r\pi)^2}{\big(\eta^2 X^2 + (r\pi)^2\big)^2}\, \frac{1}{(j\pi)^2 - 4(r\pi)^2} = -\frac{\eta X \coth(\eta X)}{8\eta^2 X^2 + 2(j\pi)^2} + \frac{\eta^2 X^2}{\big(4\eta^2 X^2 + (j\pi)^2\big)^2},$$
$$\sum_{r=1}^{\infty} \frac{1}{\eta^2 X^2 + (r\pi)^2}\, \frac{1}{(j\pi)^2 - 4(r\pi)^2} = \frac{-2\eta^2 X^2 + (j\pi)^2 \coth(\eta X)\eta X - (j\pi)^2}{8(j\pi)^2\eta^4 X^4 + 2(j\pi)^4 \eta^2 X^2},$$

and combining the two inner-product terms (the $\coth(\eta X)\,\eta X$ contributions cancel upon adding), I obtain

$$\langle \varphi_j,\, g_x(\cdot,0) \rangle + \langle \varphi_j,\, g_{xt}(\cdot,0) \rangle\, \frac{1}{\lambda_j} = \frac{8X^2}{\sigma^2} \left(\frac{1}{(j\pi)^2} + \frac{2\eta^2 X^2 + (j\pi)^2}{4(j\pi)^2\eta^2 X^2 + (j\pi)^4} \right) \frac{j\pi}{\eta^2 X^2 + (j\pi)^2}.$$

Hence,

$$k^s(x,0) = \frac{16X^3}{\sigma^4} \sum_{j=1,3,5,\ldots} \left(\frac{1}{(j\pi)^2} + \frac{2\eta^2 X^2 + (j\pi)^2}{4(j\pi)^2\eta^2 X^2 + (j\pi)^4} \right) \frac{j\pi}{\big(\eta^2 X^2 + (j\pi)^2\big)^2}\, \sin\Big(j\pi \frac{x+X}{2X}\Big). \tag{4}$$

Part 2: Solution for $k^a(x,0)$. The function k^a is antisymmetric, so it suffices to solve on $[0, X]$ with $k^a(0,t) = 0$ and $k^a(X,t) = -h_x(X)\, y(t)$. I introduce the auxiliary function

$$r(x,t) = -h_x(X)\, y(t)\, \frac{x}{X},$$

which satisfies $r(0,t) = 0$, $r(X,t) = -h_x(X)\, y(t)$, and $r_{xx} = 0$. Writing $\tilde{k}(x,t) = k^a(x,t) - r(x,t)$, the function $\tilde{k}$ has zero lateral boundary conditions and solves

$$\rho\, \tilde{k} = \frac{\sigma^2}{2}\, \tilde{k}_{xx} + \tilde{k}_t + \big[r_t(x,t) - \rho\, r(x,t)\big].$$

Computing the source term using $y(t) = y(0) + \rho \coth^2(\eta X)\, t$ from Proposition 6:

$$\begin{aligned} r_t - \rho\, r &= -h_x(X)\, \frac{x}{X} \big[y'(t) - \rho\, y(t)\big] \\ &= -h_x(X)\, \frac{x}{X} \big[C_0 + C_1\, t\big], \end{aligned}$$

where

$$C_0 \equiv \rho\big[\coth^2(\eta X) - y(0)\big], \qquad C_1 \equiv -\rho^2 \coth^2(\eta X).$$

I expand $\tilde{k}$ using the basis $\phi_j(x) = \sin(j\pi\, x/X)$ on $[0, X]$, with eigenvalues

$$\nu_j = \rho + \frac{(j\pi)^2\sigma^2}{2X^2}.$$

The Fourier coefficients of x/X on $[0, X]$ in this basis are

$$\frac{\langle \phi_j,\, x/X\rangle}{\langle \phi_j,\, \phi_j\rangle} = \frac{2(-1)^{j+1}}{j\pi},$$

so the projection of the source is

$$q_j^a(t) = -h_x(X)\,\frac{2(-1)^{j+1}}{j\pi}\,(C_0 + C_1\, t).$$

Taking $T \to \infty$ (the terminal-condition contribution vanishes since $y(T)\, e^{-\nu_j T} \to 0$):

$$\tilde{a}_j(0) = \int_0^\infty q_j^a(\tau)\, e^{-\nu_j \tau}\, d\tau = -h_x(X)\,\frac{2(-1)^{j+1}}{j\pi}\left[\frac{C_0}{\nu_j} + \frac{C_1}{\nu_j^2}\right].$$

Reassembling $k^a(x,0) = r(x,0) + \tilde{k}(x,0)$ and using the following Fourier expansion

$$x/X = \sum_{j=1}^{\infty} \frac{2(-1)^{j+1}}{j\pi}\sin(j\pi\, x/X)$$

to absorb $r(x,0) = -h_x(X)\, y(0)\, x/X$ into the series:

$$k^a(x,0) = -h_x(X)\sum_{j=1}^{\infty}\frac{2(-1)^{j+1}}{j\pi}\left[y(0) + \frac{C_0}{\nu_j} + \frac{C_1}{\nu_j^2}\right]\sin\!\Big(\frac{j\pi\, x}{X}\Big).$$

The bracket simplifies by substituting the expressions for C_0 and C_1:

$$\begin{aligned} y(0) + \frac{C_0}{\nu_j} + \frac{C_1}{\nu_j^2} &= y(0) + \frac{\rho[\coth^2(\eta X) - y(0)]}{\nu_j} - \frac{\rho^2\coth^2(\eta X)}{\nu_j^2} \\ &= y(0)\left(1 - \frac{\rho}{\nu_j}\right) + \frac{\rho\coth^2(\eta X)}{\nu_j}\left(1 - \frac{\rho}{\nu_j}\right) \\ &= \frac{(j\pi)^2\sigma^2}{2X^2\,\nu_j}\left[y(0) + \frac{\rho\coth^2(\eta X)}{\nu_j}\right], \end{aligned}$$

where the last line uses $1 - \rho/\nu_j = (j\pi)^2\sigma^2/(2X^2\nu_j)$. Substituting back:

$$k^a(x,0) = \frac{-h_x(X)\,\sigma^2}{X^2}\sum_{j=1}^{\infty}(-1)^{j+1}\, j\pi\left[\frac{y(0)}{\nu_j} + \frac{\rho\coth^2(\eta X)}{\nu_j^2}\right]\sin\!\Big(\frac{j\pi\, x}{X}\Big), \tag{5}$$

where $\nu_j = \rho + (j\pi)^2\sigma^2/(2X^2)$, $y(0)$ is given by Proposition 6, and $h_x(X)$ is given by Proposition 11. This expression is extended to $[-X, X]$ by antisymmetry: $k^a(x,0) = -k^a(-x,0)$.

Part 3: Combining. The full solution is

$$
\begin{aligned}
k(x,0) &= k^s(x,0) + k^a(x,0) \\
&= \frac{16X^3}{\sigma^4} \sum_{j=1,3,5,\ldots} \left(\frac{1}{(j\pi)^2} + \frac{2\eta^2 X^2 + (j\pi)^2}{4(j\pi)^2\eta^2 X^2 + (j\pi)^4} \right) \frac{j\pi}{\left(\eta^2 X^2 + (j\pi)^2\right)^2} \sin\left(j\pi \frac{x+X}{2X} \right) - \\
&\quad \frac{h_x(X)\,\sigma^2}{X^2} \sum_{j=1}^{\infty} (-1)^{j+1}\, j\pi \left[\frac{y(0)}{\nu_j} + \frac{\rho \coth^2(\eta X)}{\nu_j^2} \right] \sin\left(\frac{j\pi\, x}{X} \right)
\end{aligned}
$$

with $\nu_j = \rho + (j\pi)^2\sigma^2/(2X^2)$, $y(0)$ given by Proposition 6, and $h_x(X)$ given by Proposition 11. Two comments on the properties of $k^a(\cdot,0)$ are in order.

- Regularity of $k^a(\cdot,0)$: The Fourier coefficients of k^a are proportional to $j\pi/\nu_j^2 \sim j^{-3}$ for large j, so the series for k^a and its x-derivative both converge absolutely. In particular, $k^a(\cdot,0)$ is differentiable on $(-X,X)$.
- Remark on the anticipation effect at the center: Since k^a is antisymmetric, $k^a(0,0) = 0$, so $k(0,0) = k^s(0,0)$. Thus, the correction from the non-zero boundary conditions does not affect the anticipation effect measured at the center of the band ($x = 0$). The correction does affect k at $x \neq 0$ and, importantly, the derivative $k_x(X,0)$, which enters the second-order approximation of the (now asymmetric) boundaries $\bar{Y}$ and $\underline{Y}$.

□