Optimal Central Bank Balance Sheets with Incomplete Markets

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Abstract

I study the optimal size of a central bank's balance sheet in a continuous-time, incomplete-markets economy where households self-insure against idiosyncratic income and expenditure risk by accumulating two nominal assets: liquid money and illiquid bonds. The central bank conducts open market operations—swapping money for bonds—determining both the inflation rate and the relative supply of liquid and illiquid assets available to the public. As a benchmark, I first characterize optimal policy when monetary and fiscal instruments are jointly controlled, and then study the central bank's problem when it is separated from fiscal policy.

1 Introduction

Since 2008, central bank balance sheets in the United States, and other developed economies, have expanded to historically unprecedented levels. Policymakers have increasingly called for reducing the size of these balance sheets,[1] and the process of balance sheet normalization has proven difficult in practice (Acharya et al., 2022). A fundamental question has emerged: what is the optimal size and composition of a central bank's balance sheet? Despite the practical urgency, there is surprisingly little theoretical guidance. The extensive literature on optimal monetary policy typically abstracts from the balance sheet entirely, taking the short-term interest rate as the sole policy instrument. When the balance sheet is modeled, it is either irrelevant for welfare or studied in the context of specific financial market interventions rather than as a

[1] For example, Federal Reserve Chair Kevin Warsh has stated that the Fed's balance sheet is "too big" and has called for a return to a smaller footprint. See his confirmation testimony before the U.S. Senate Committee on Banking, Housing, and Urban Affairs, February 2025.

permanent policy instrument.[2] This paper provides a framework for answering this question in an environment in which heterogeneous households use publicly-supplied nominal assets for self-insurance against both income and expenditure risk.

The theoretical challenge is that studying the optimal size of the central bank balance sheet requires a model in which both nominal quantities and prices are determined in equilibrium, and in which different nominal assets serve distinct roles for households. Standard monetary theory focuses on money's role as a medium of exchange and studies optimal money supply in environments with expenditure risk but abstracts from household heterogeneity and income risk (e.g. Lagos and Wright, 2005; Rocheteau and Wright, 2005). The heterogeneous-agent New Keynesian (HANK) literature models household heterogeneity and income risk but focuses on rigidities in money's role as a unit of account, abstracting from nominal quantities entirely (e.g. Acharya et al., 2023; Dávila and Schaab, 2023). This paper combines both traditions: liquidity risk, income risk, ex-post heterogeneity, and a government balance sheet with two nominal assets—money and bonds—whose relative supply is a policy instrument. This opens up the study of not only optimal interest rate policy but also its implementation through the central bank's balance sheet.

The economy is a continuous-time, incomplete-markets model following Rocheteau et al. (2018) and Kaplan et al. (2018). Households face persistent shocks to labor productivity and random, idiosyncratic lumpy expenditure shocks. They self-insure by accumulating two publicly-supplied nominal assets: money, which is liquid and can finance lumpy consumption, and bonds, which earn a higher return but face portfolio adjustment costs and cannot be used for transactions. The spread between bond and money returns—the liquidity premium—compensates households for holding the illiquid asset. The government's balance sheet determines the relative supply of these two assets in the hands of the public.

I first study the problem of a consolidated government that jointly controls both monetary and fiscal policy (Section 3). Following the approach in Dávila and Schaab (2023), I study optimal policy under both full commitment and full discretion. The planner maximizes utilitarian social welfare subject to the decentralized equilibrium conditions as implementability constraints. When transfers are unrestricted, the Friedman rule is optimal: money earns the rate of time preference, the liquidity premium is zero, and bonds are redundant. When transfers are lump-sum, the planner faces a tradeoff between redistribution and self-insurance on two margins. The first is the common nominal growth rate of money and bonds, which determines the inflation rate and the

[2]For instance, Cúrdia and Woodford (2011) show that in a New Keynesian setting the balance sheet matters only when financial frictions create spreads between government and private borrowing rates. There is a literature following the tradition of Poole (1970) that studies the implementation of monetary policy focusing on how the central bank steers short-term interest rates through the supply of bank reserves but abstracts from the welfare costs or benefits of the size of the balance sheet (outside of efficiency in banking markets)—for a recent example see, e.g., Lagos and Navarro (2023).

inflation-tax redistribution across the wealth distribution. The second is the balance sheet composition M/B, which determines the liquidity premium and the portfolio-redistribution channel. I characterize the optimal inflation rate through a wedge formula involving the consumption, deposit, and lumpy-consumption wedges, and the optimal liquidity premium through the average gap between the social marginal values of illiquid and liquid wealth.

I then turn to studying the optimal size of the central bank balance sheet in Section 4, separating monetary and fiscal policy. The central bank controls the supply of money through open market operations—swapping money for bonds on its balance sheet—but has no access to transfers. Fiscal policy sets transfers and bond issuance subject to a fiscal rule. The central bank's two effective instruments are the nominal growth rate (which determines the inflation rate) and the balance sheet share $\omega = B^{CB}/B^{T}$ (which, through bond market clearing, determines the liquidity premium). The key analytical result is that the social value of a household under separated policy parallels the consolidated case, but with the central bank's real bond holdings A^{CB} replacing aggregate wealth as the redistribution anchor. This reflects the tighter constraint the central bank faces: its seigniorage revenue can only purchase bonds, not finance transfers, so both the inflation-tax and liquidity-premium redistribution channels are mediated through the central bank's own balance sheet. The optimal liquidity premium condition takes the same form as under consolidation, but a new condition—absent in the consolidated case—links the shadow prices of the monetary and fiscal budget constraints through the bond rate, making the interest rate a non-trivial policy margin that redistributes resources between the two government entities.

In Section 6, I quantify the optimal balance sheet policy, both in steady state and along the transition following an aggregate shock. These results are still work in progress.

1.1 Literature

2 Model

I consider a continuous-time, incomplete-markets economy with two publicly-supplied nominal assets: money and bonds. Households receive idiosyncratic labor productivity and lumpy expenditure shocks, and self-insure by accumulating both liquid (money) and illiquid (bonds) wealth. Money is liquid in the sense that it is used as a transaction account and means of payment, while bonds face adjustment costs and cannot be used to finance lumpy expenditures. In equilibrium, bonds earn a higher return than money—an illiquidity premium. Policy takes the form of a portfolio of public, nominal liabilities and a transfer rule. I study the problem when monetary and fiscal policy are coordinated and optimally, jointly set, and then later turn to the case in which they are separated.

2.1 Environment

Time is continuous, $t \in [0, \infty)$, and there is a single, perishable consumption good. There exists a unit measure of households that discount the future at rate $\rho > 0$ and value flows of consumption, c_t, and labor, $n_t \in [0, 1]$, according to the instantaneous utility function $u(c_t) - v(n_t)$, where u is the flow utility of consumption and v is the disutility of labor. In addition to consuming in flows, households receive random, idiosyncratic preference shocks to consume discrete (lumpy) amounts of the consumption good, denoted y_t. These shocks arrive at Poisson rate α and consuming y_t delivers the household a lump of utility $U(y_t)$.

Given adapted, left-continuous processes (c_t, n_t, y_t), a household's lifetime expected discounted utility at time zero is given by

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left[u(c_t) - v(n_t)\right]\, dt + \sum_{n=1}^{\infty} e^{-\rho T_n} U(y_{T_n})\right]. \tag{1}$$

where the sequence $\{T_n\}_{n=1}^{\infty}$ represents the Poisson arrival times of the preference shocks. The functions u and U are strictly increasing and concave, v is strictly increasing and convex, and together they satisfy the Inada conditions $u'(0) = U'(0) = v'(1) = \infty$ and $u'(\infty) = v'(0) = U'(\infty) = 0$. Households face idiosyncratic labor productivity given by an exogenous Markov process z_t. Let $\mathcal{Z}$ stand for the support of z. A household's labor earnings are then given by $z_t w_t n_t$, where w_t is the wage rate in a competitive labor market. There is a representative firm that produces output Y_t using a linear production technology $Y_t = L_t$, where L_t is aggregate labor demand.

For welfare comparisons, it is useful to define the first-best allocation under a utilitarian social welfare function,

$$u'(c^{FB}) = U'(y^{FB}) = -\frac{v'(n^{FB}(z))}{z}, \tag{2}$$

$$c^{FB} + \alpha\, y^{FB} = \mathbb{E}_z\left[z\, n^{FB}(z)\right], \tag{3}$$

where $\mathbb{E}_z$ denotes expectation under the ergodic distribution of productivity.

2.2 Assets and government

There are two publicly-supplied, risk-free nominal assets: money with supply M_t and bonds with supply B_t. Both are nominal claims on the same unit of account and trade at real price ϕ_t per nominal unit. Bonds pay a nominal interest rate $i_t^b \geq 0$, whereas money pays no interest.

I distinguish between policy of a fiscal authority ("Treasury") versus that of a monetary authority ("Central Bank"). Let B_t^T denote the total outstanding nominal debt of the Treasury, which is held either by the public, B_t, or by the central bank, B_t^{CB}, so that $B_t^T = B_t + B_t^{CB}$. The fiscal authority sets lump-sum transfers $\mathcal{T}_t(a, b, z)$ and issues debt B_t^T. Given a remittance R_t from

the central bank, the fiscal authority's budget constraint is

$$\int \mathcal{T}_t(a,b,z)\, dG_t + i_t^b \phi_t B_t^T = R_t + \phi_t \dot{B}_t^T. \tag{4}$$

The monetary authority sets the supply of money M_t by conducting open market operations—swapping money for bonds on its balance sheet. Its budget constraint is

$$R_t + \phi_t \dot{B}_t^{CB} = i_t^b \phi_t B_t^{CB} + \phi_t \dot{M}_t. \tag{5}$$

Adding (49) and (51), the remittance R_t cancels and the *consolidated government's* budget constraint is

$$\int \mathcal{T}_t(a,b,z)\, dG_t + i_t^b \phi_t B_t = \phi_t \dot{M}_t + \phi_t \dot{B}_t, \tag{6}$$

where $\mathcal{T}_t(a,b,z)$ is a generic transfer rule that can depend on a household's idiosyncratic state (a,b,z). When monetary and fiscal policy are jointly optimized (Section 3), the relevant constraint is (6). When they are separated (Section 4), the individual constraints (49)–(51) are considered separately along with a fiscal policy rule.

From the standpoint of households, the assets differ in two ways: liquidity for lumpy expenditures and in the cost of portfolio adjustment. In the event of a lumpy expenditure, households can only use money—this captures the specialness of central bank liabilities in facilitating transactions.[3] Additionally, following the literature after Kaplan and Violante (2014) and Kaplan et al. (2018), transferring wealth between liquid and illiquid accounts incurs an adjustment cost $\Phi(d_t)$, where d_t is the flow deposit rate. The function Φ satisfies $\Phi(0) = 0$, $\Phi'(0) = 0$, and $\Phi''(d) > 0$ for all d: zero cost and zero marginal cost at $d = 0$, with strictly increasing marginal cost for deposits or withdrawals.

The household's law of motion for liquid wealth (outside of lumpy expenditure shocks) is

$$\dot{a}_t = r_t^a a_t + z_t n_t - c_t + \mathcal{T}_t(a,b,z) - d_t - \Phi(d_t), \tag{7}$$

where $r_t^a = \dot{\phi}_t/\phi_t$ is the real return on liquid wealth. The adjustment cost $\Phi(d_t)$ is paid out of liquid wealth. They face a no-borrowing constraint $a_t \geq 0$. The law of motion for illiquid wealth is

$$\dot{b}_t = r_t^b b_t + d_t, \tag{8}$$

with a borrowing constraint $b_t \geq \underline{b}$, where $\underline{b} \leq 0$, and the real return is $r_t^b = i_t^b + \dot{\phi}_t/\phi_t$. The house-

[3] One can view central bank liabilities as reserves with a competitive narrow banking system that converts reserves one for one into deposits, M_t.

hold would like to, in the event of a lumpy-expenditure shock, discontinuously withdraw from their illiquid account, but this is not allowed—deposits are continuous flows and the adjustment cost prevents instantaneous rebalancing.

2.3 Decentralized equilibrium

Given a linear production function, the labor market clearing condition requires $w_t = 1$ for all t. Households take policy and prices (ϕ_t, i_t^b) as given. Let $W_t(a, b, z)$ represent a household's value function at time t, where a is their real liquid wealth, b is their real illiquid wealth, and z is their idiosyncratic productivity. It satisfies the Hamilton-Jacobi-Bellman equation

$$\rho W_t(a,b,z) = \max_{c_t, n_t, y_t, d_t} u(c_t) - v(n_t) + \alpha U(y_t) + \tilde{\mathcal{A}}_t(a,b,z,c,n,y,d)\left[W_t\right] + \partial_t W_t, \tag{9}$$

where the constraints on choices are $c_t \geq 0$, $n_t \in [0,1]$, $0 \leq y_t \leq a_t$, and d_t subject to the state constraints $a_t \geq 0$ and $b_t \geq \underline{b}$. The operator $\tilde{\mathcal{A}}_t$ represents the infinitesimal generator of the idiosyncratic state (a, b, z) and is given by

$$\begin{aligned}\tilde{\mathcal{A}}_t[W] = &\left[r_t^a a + z_t n_t - c + \mathcal{T}_t - d_t - \Phi(d_t)\right]\partial_a W + \left[r_t^b b + d_t\right]\partial_b W \\ &+ \alpha\left[W(a-y,b,z) - W(a,b,z)\right] + \mathcal{A}_z W(a,b,z).\end{aligned} \tag{10}$$

The generator has four components: continuous changes in liquid wealth (first term), continuous changes in illiquid wealth (second term), discrete changes in liquid wealth from lumpy consumption (third term), and idiosyncratic productivity transitions (fourth term).

The optimization problem leads to four optimality conditions:

$$u'(c_t) = \partial_a W_t, \tag{11}$$

$$v'(n_t) = z_t \partial_a W_t, \tag{12}$$

$$U'(y_t) \geq \partial_a W_t(a - y_t, b, z), \tag{13}$$

$$(1 + \Phi'(d_t))\,\partial_a W_t = \partial_b W_t. \tag{14}$$

Equation (11) equates the marginal utility of flow consumption with the marginal value of liquid wealth. Equation (12) equates the marginal disutility of labor with the marginal value of liquid wealth scaled by productivity. Equation (13) holds with equality if $y_t < a$; it governs lumpy consumption, which can only be financed with liquid wealth. Equation (14) equates the marginal cost of depositing one dollar (the dollar plus its adjustment cost, valued at $\partial_a W_t$) with the marginal value of illiquid wealth $\partial_b W_t$. The strict convexity of Φ ensures a unique interior solution for d_t at every state.

We can re-write the HJB plugging in the household policies as

$$\rho W_t = u(c_t) - v(n_t) + \alpha U(y_t) + \mathcal{A}_t [W_t] + \partial_t W_t, \tag{15}$$

where $\mathcal{A}_t = \tilde{\mathcal{A}}_t(a, b, z, c_t, n_t, y_t, d_t)$. The equilibrium measure G_t over household states (a, b, z) satisfies the Kolmogorov forward equation:

$$\partial_t G_t = \mathcal{A}_t^* [G_t], \tag{16}$$

where $\mathcal{A}_t^*$ is the adjoint of the generator $\mathcal{A}_t$. The stationary distribution may contain mass points at the borrowing constraints—at $a = 0$ (for each (b, z)), at $b = \underline{b}$ (for each (a, z)), or at the double corner $(0, \underline{b})$ (for each z). We denote the mass at $a = 0$ by $m_t(b, z)$, the mass at $b = \underline{b}$ by $\tilde{m}_t(a, z)$, and the absolutely continuous part of G_t by its density $g_t(a, b, z)$. The adjoint $\mathcal{A}_t^*$ acts as a differential operator on the density g_t; separate boundary ODEs govern the evolution of the mass points. where $\mathcal{A}_t^*$ is the adjoint of the generator $\mathcal{A}_t$. Market clearing requires

$$\int a \, dG_t(a, b, z) = \phi_t M_t, \tag{17}$$

$$\int b \, dG_t(a, b, z) = \phi_t B_t. \tag{18}$$

Definition 1 (Decentralized competitive equilibrium)**.** *Taking as given an initial measure G_0 over (a, b, z) and a path of policy $\{M_t, B_t, \mathcal{T}_t\}$, a decentralized competitive equilibrium consists of value functions and decision rules $\{W_t, c_t, n_t, y_t, d_t\}$, equilibrium measures $\{G_t\}$, and prices $\{\phi_t, i_t^b\}$, such that:*

i) value functions and decision rules satisfy (11)–(15),

ii) the equilibrium measure G_t satisfies (16),

iii) the government budget constraint (6) holds,

iv) and both market clearing conditions (17)–(18) hold.

Equations (11)–(18) serve as implementability constraints for optimal policy. Definition 1 is a standard definition of a competitive equilibrium. Market clearing for labor is implemented by setting $w_t = 1$ and the market clearing condition for goods is satisfied by Walras' law when both asset market clearing conditions (17)–(18) and the government budget constraint (6) hold, with total output given by $Y_t = L_t \equiv \int z_t n_t(a, b, z) \, dG_t(a, b, z)$.[4]

[4]We write the government budget constraint in (6) as an equilibrium condition even though it is not technically necessary to define an equilibrium. If it were violated, it would be required to define the amount of goods that are either destroyed or created (by a policy maker), which would enter into the market clearing condition for goods. Since this is not allowed in the optimal policy problem, it is simpler to require (6) to hold.

2.4 Characterization of the decentralized equilibrium

Before studying optimal policy, it is useful to first establish some properties of the decentralized equilibrium. I first impose the following condition on the transfer rule that guarantees all households can finance a strictly positive consumption flow; a condition that will be slack in the policy problem.

Assumption 1 (Feasibility)**.** *For all t, transfers and the borrowing limit satisfy $\underline{z}_t + \mathcal{T}_t(0, \underline{b}, \underline{z}) + r_t^b \underline{b} - \Phi(-r_t^b \underline{b}) > 0$, for any $0 \leq r_t^b \leq \rho$.*

Assumption 1 rules out confiscatory transfers that would leave a household at $\dot{a} \leq 0$ with non-positive resources even at full labor supply. The characterization of the equilibrium begins with establishing that the value function is a constrained viscosity solution of the HJB equation on $[0, \infty) \times [\underline{b}, \infty) \times \mathcal{Z}$, deriving boundary conditions at $a = 0$ and $b = \underline{b}$, and establishing some useful properties of the value function, W_t. Proofs of all propositions and lemmas in the paper are collected in Appendix A.

Proposition 1 (Value function properties and boundary conditions)**.** *Under Assumption 1:*

i) Viscosity solution: $W_t(a, b, z)$ is a constrained viscosity solution of the HJB equation (15) on $[0, \infty) \times [\underline{b}, \infty) \times \mathcal{Z}$.

ii) Boundary conditions. For each t, b, and $z \in \mathcal{Z}$, $\partial_a W_t(0, b, z) < \infty$. At $a = 0$, the consumption Euler $u'(c_t) = \partial_a W_t$ holds and the state-constraint boundary condition satisfies

$$\partial_a W_t(0, b, z) \geq u'(z_t n_t(0, b, z) + \mathcal{T}_t(0, b, z) - d_t(0, b, z) - \Phi(d_t(0, b, z))), \tag{19}$$

with equality iff $\dot{a}_t(0, b, z) = 0$. At $b = \underline{b}$, the state constraint $\dot{b}_t \geq 0$ requires $d_t \geq -r_t^b \underline{b}$. At the double corner ($a = 0$, $b = \underline{b}$), both state constraints are active and the consumption bound satisfies

$$\partial_a W_t(0, \underline{b}, z) \geq u'\left(z_t n_t + \mathcal{T}_t + r_t^b \underline{b} - \Phi(-r_t^b \underline{b})\right), \tag{20}$$

with equality iff $\dot{a}_t = \dot{b}_t = 0$.

iii) Monotonicity and concavity: W_t is strictly increasing and strictly concave in a for each (b, z), and strictly increasing and strictly concave in b for each (a, z).

iv) Zero savings at the double corner: At the double corner $(a, b) = (0, \underline{b})$ with the lowest productivity type $\underline{z}$, $\dot{a}_t(0, \underline{b}, \underline{z}) = \dot{b}_t(0, \underline{b}, \underline{z}) = 0$.

Proposition 1 extends the results in Rocheteau et al. (2018) and Achdou et al. (2022) to non-stationary environments with idiosyncratic productivity risk, endogenous labor supply and

lumpy consumption, which allows us to use (9) to characterize W_t uniquely. In addition, part ii) shows that households at the liquid no-borrowing constraint have a strictly positive value of additional wealth, but is bounded. Part (iii) establishes the strict monotonicity and concavity of the value function along both wealth dimensions. We will see these conditions lead to a redistributive motive of optimal policy. Finally, Part (iv) establishes that savings rates zero for the households with the lowest productivity level at the borrowing constraint.

Proposition 2 (Lumpy consumption)**.** *For each $t \geq 0$, $b \geq \underline{b}$, and $z \in \mathcal{Z}$: i) there exists a threshold $\bar{a}_t(b,z) > 0$ such that $y_t(a,b,z) = a$ for all $a \in (0, \bar{a}_t(b,z)]$, where the threshold satisfies $U'(\bar{a}_t(b,z)) = \partial_a W_t(0,b,z)$, and ii) for each $a > 0$, $0 \leq \partial_a y_t(a,b,z) \leq 1$.*

Proposition 2 shows that for any productivity level, households near the liquid wealth borrowing constraint find it optimal to fully deplete their wealth holdings in the event of a lumpy consumption shock.[5] Under $\alpha = 0$, liquidity constraints (via the no-borrowing constraint) never bind in the interior of the state space. Households hit the borrowing constraint in finite time only by receiving persistently low productivity draws and slowly de-accumulating wealth. Allowing for lumpy consumption, $\alpha > 0$, implies that some households with $a > 0$ hit the liquidity constraint discretely. This will introduce another wedge for the optimal policy problem.

3 Optimal consolidated government policy

This section studies the joint, monetary-fiscal optimal policy problem of choosing $\{M_t, B_t, \mathcal{T}_t(a,b,z)\}$ to maximize social welfare. I consider cases in which the government can perfectly commit to future plans and in which they cannot and have discretion. To do so, I follow the approach in Dávila and Schaab (2023) and set up the optimal policy problem as follows. A planner at time t chooses policy $\{M_t, B_t, \mathcal{T}_t(a,b,z)\}$ to maximize a utilitarian social welfare function, taking as given the initial measure G_t over household states, and the initial supply of public assets, $M_t > 0$ and $B_t > 0$. With Poisson intensity $\lambda \geq 0$, a "regime change" occurs: the current planner is replaced by a new planner who re-optimizes, given the state of the economy at that time. The current planner takes into account the probability of being replaced in the future. The case with full commitment is captured by $\lambda \to 0$ the case with discretion as $\lambda \to \infty$. Finally, I consider two cases of constraints to fiscal policy: i) unconstrained transfers and ii) lump-sum transfers. Unsurprisingly, we will see that i) implements the first-best, while ii) introduce meaningful tradeoffs for the planner.

[5] See Rocheteau et al. (2018, Proposition 1) for a similar result in a stationary equilibrium without idiosyncratic productivity risk or endogenous labor supply.

3.1 Planner's problem

Let $\mathcal{W}_0(G_0, M_0, B_0)$ denote lifetime discounted social welfare at time 0, and let τ_1 denote the time of the first regime change. The planner's problem is

$$\mathcal{W}_0(G_0, M_0, B_0) = \max_{\mathbf{X}} \mathbb{E}_0 \left[\int_0^{\tau_1} e^{-\rho s} \int \mathcal{U}_s(x)\, dG_s(x)\, ds + e^{-\rho \tau_1} \mathcal{W}_{\tau_1}(G_{\tau_1}, M_{\tau_1}, B_{\tau_1}) \right], \tag{21}$$

subject to (11)–(18), where $x \equiv (a, b, z)$ stands for the idiosyncratic state of a household, $\mathcal{U}_s(x) = u(c_s) - v(n_s) + \alpha U(y_s)$, and $\mathbf{X}$ collects the planner's choices of household policies, value functions, distributions, prices, and government policy:

$$\mathbf{X} = \{c_t(x), n_t(x), y_t(x), d_t(x), W_t(x), G_t, \phi_t, i_t^b, M_t, B_t, \mathcal{T}_t(x)\}_{t=0}^{\tau_1}. \tag{22}$$

The planner's problem can be re-written in terms of finding a saddle point of the Lagrangian:

$$\mathcal{W}_0 = \max_{\mathbf{X}} \min_{\mathbf{M}} \mathbb{E}_0 \left[L(0, \tau_1, G_0, M_0, B_0) + e^{-\rho \tau_1} \mathcal{W}_{\tau_1} \right], \tag{23}$$

where the Lagrangian is

$$\begin{aligned} L = \int_{t_1}^{t_2} e^{-\rho s} \Bigg\{ & \int \mathcal{U}_s(x) dG_s(x) \\ & + \Big\{ \psi_s(x) \Big[-\rho W_s + \mathcal{U}_s + \partial_s W_s + \mathcal{A}_s W_s \Big] \\ & + \chi_s(x) \Big[u'(c_s) - \partial_a W_s \Big] \\ & + \nu_s(x) \Big[-v'(n_s) - z_s \partial_a W_s \Big] \\ & + \eta_s(x) \Big[\alpha U'(y_s) - \alpha\, \partial_a W_s(a - y_s, b, z) \Big] \mathbb{1}\{a > \bar{a}_s(b, z)\} \\ & + \delta_s(x) \Big[\partial_b W_s - (1 + \Phi'(d_s))\, \partial_a W_s \Big] \\ & + \gamma_s(x) \Big[\mathcal{A}_s^*[G_s] - \partial_s G_s \Big] \Big\} dx \\ & + \theta_s \Big[\dot{M}_s \phi_s + \dot{B}_s \phi_s - i_s^b \phi_s B_s - \int \mathcal{T}_s\, dG_s \Big] \\ & + \mu_s \Big[\int a\, dG_s - \phi_s M_s \Big] \\ & + \varphi_s \Big[\int b\, dG_s - \phi_s B_s \Big] \Bigg\} ds, \end{aligned} \tag{24}$$

and the multipliers are $\mathbf{M} = \{\psi_t(x), \chi_t(x), \nu_t(x), \eta_t(x), \delta_t(x), \gamma_t(x), \theta_t, \mu_t, \varphi_t\}_{t=0}^{\tau_1}$. In (23)-(24), for any transition time, τ_1, the planner solves a constrained optimal control problem between $[0, \tau_1)$. The Lagrangian in (24) captures the implementability conditions of the planner from the decentralized equilibrium. The second line encodes that the planner's choices induce a particular set of expectations and lifetime welfare for each agent, which determine their individual behavior encoded in lines three through six.[6] The seventh line captures the law of motion of the distribution of wealth and income, while the remaining lines represent the consolidated government budget constraint and the market clearing conditions for liquid and illiquid wealth, respectively.

3.2 First-order conditions

We derive the first-order conditions of the Lagrangian L with respect to all control variables in $\mathbf{X}$. There are four categories: (i) *household decision variables* (c_s, n_s, y_s, d_s), (ii) *transfers* $\mathcal{T}_s$, (iii) *asset supplies* (M_s, B_s) and the *price level* ϕ_s, and (iv) the *state variables* (W_s, G_s), which yield the adjoint equations derived in Section 3.3. Throughout, let $x \equiv (a, b, z)$. For exposition, all derivations are for the interior of the state space; boundary conditions follow from Proposition 1(ii).

A useful adjoint identity. Before taking first-order conditions, it is helpful to rewrite the KFE term in the Lagrangian using the adjoint property of $\mathcal{A}_s^*$. Focusing on the absolutely continuous part of the distribution (outside of $a = 0$ and $b = \underline{b}$):

$$\int \gamma_s(x)\, \mathcal{A}_s^*[g_s]\, dx = \int \mathcal{A}_s[\gamma_s](x) \cdot g_s(x)\, dx, \tag{25}$$

where the generator acts on the co-state as $\mathcal{A}_s[\gamma_s] = s_s^a\, \partial_a \gamma_s + s_s^b\, \partial_b \gamma_s + \alpha[\gamma_s(a - y_s, b, z) - \gamma_s] + \mathcal{A}_z[\gamma_s]$, with $s_s^a = r_s^a a + z_s n_s - c_s + \mathcal{T}_s - d_s - \Phi(d_s)$ and $s_s^b = r_s^b b + d_s$ denoting the liquid and illiquid savings rates, respectively. The representation (25) reveals how each control enters the KFE term: through s_s^a for $(c_s, n_s, \mathcal{T}_s, d_s)$, through s_s^b for d_s, and through the jump term for y_s. A similar equation holds at the mass point at the double boundary.

Flow consumption $c_s(x)$. The variable c_s enters the Lagrangian through four channels: the welfare term, contributing $u'(c_s)g_s$; the HJB term through utility and via $-c_s$ in the liquid drift of $\mathcal{A}_s W_s$, contributing $\psi_s[u'(c_s) - \partial_a W_s]$; the Euler-c constraint, contributing $\chi_s u''(c_s)$; and the KFE term via $-c_s$ in s_s^a after applying (25), contributing $-\partial_a \gamma_s g_s$. Setting $\partial L / \partial c_s(x) = 0$, imposing

[6] The Euler-y implementability constraint on line five is imposed only in the *interior* (no-depletion) region $a > \bar{a}_s(z)$, where $\bar{a}_s(z)$ is the full-depletion threshold from Proposition 2(i); the integral in this line is restricted to that domain. For $a \leq \bar{a}_s(z)$, the household's optimality condition for y_s is a Kuhn-Tucker corner solution: the budget constraint $y_s \leq a$ binds, giving $y_s(a, z) = a$ and a strictly positive budget multiplier $\eta(a, z) > 0$. The Euler equation therefore holds as a strict inequality $U'(a) - \partial_a W_s(0, z) > 0$. The indicator $\mathbb{1}\{a > \bar{a}_s(z)\}$ captures the complementary slackness condition expressed as a domain restriction. The result $y_s = a$ for $a \leq \bar{a}_s(z)$ is imposed into all other terms of the Lagrangian (e.g. the KFE equation, social welfare, etc.).

$u'(c_s) = \partial_a W_s$ in the decentralized equilibrium, and rearranging pins down the multiplier on the Euler-c constraint:

$$\chi_s = \frac{[\partial_a \gamma_s - u'(c_s)]}{u''(c_s)} g_s. \tag{26}$$

The ratio $\tilde{\chi}_s \equiv \partial_a \gamma_s - u'(c_s)$ is the *flow consumption wedge*: the gap between the planner's shadow value of liquid wealth $\partial_a \gamma_s$ and the household's marginal utility of consumption.

Labor supply $n_s(x)$. Labor supply enters the welfare term in the Lagrangian, contributing $-v'(n_s)g_s$; the HJB via utility and the drift term $z_s n_s \partial_a W_s$ in $\mathcal{A}_s W_s$, contributing $\psi_s[-v'(n_s) + z_s \partial_a W_s]$; the labor supply constraint, contributing $\nu_s v''(n_s)$; and the KFE via $z_s n_s$ in s_s^a, contributing $z_s \partial_a \gamma_s g_s$. Setting $\partial L / \partial n_s(x) = 0$ and imposing $v'(n_s) = z_s \partial_a W_s$ in the decentralized equilibrium, and rearranging using (26) pins down the multiplier on the labor supply constraint:

$$\nu_s = \frac{[v'(n_s) - z_s \partial_a \gamma_s]}{v''(n_s)} g_s. \tag{27}$$

Equation (27) has a similar interpretation to (26). Define the *labor wedge* $\tilde{\nu}_s \equiv v'(n_s) - z_s \partial_a \gamma_s$. These wedges arise from the possibility that the social value of liquid wealth $\partial_a \gamma_s$ differs from the household's private value $\partial_a W_s$. The same externalities drive both wedges.[7]

Lumpy consumption $y_s(x)$ on the interior region $a > \bar{a}_s(b, z)$. The variable y_s enters via welfare, contributing $\alpha U'(y_s) g_s$; the HJB via utility and the jump term $\alpha[W_s(a - y_s, b, z) - W_s]$ in $\mathcal{A}_s W_s$, contributing $\alpha \psi_s [U'(y_s) - \partial_a W_s(a - y_s, b, z)]$; the Euler-y constraint, contributing $\alpha \eta_s [U''(y_s) + \partial_{aa} W_s(a - y_s, b, z)]$; and the KFE jump term, contributing $-\alpha \partial_a \gamma_s (a - y_s, b, z) g_s$. Setting $\partial L / \partial y_s = 0$ and imposing $U'(y_s) = \partial_a W_s(a - y_s, b, z)$ in any decentralized equilibrium, and rearraning pins down the multiplier on the Euler-y constraint:

$$\eta_s = \frac{\partial_a \gamma_s (a - y_s, b, z) - U'(y_s)}{U''(y_s) + \partial_{aa} W_s(a - y_s, b, z)} g_s. \tag{28}$$

The denominator $U''(y_s) + \partial_{aa} W_s(a - y_s, b, z) < 0$ is the second-order condition for the household's lumpy consumption problem. The FOC (28) has the same structure as (26) but evaluated at the post-depletion liquid wealth $a - y$: the planner values lumpy expenditure opportunities by the shadow price of wealth *after* the wealth jump, not before. The *lumpy consumption wedge* $\tilde{\eta}_s \equiv \partial_a \gamma_s (a - y_s, b, z) - U'(y_s)$ measures the gap between the planner's shadow value of post-depletion liquid wealth and the household's marginal utility of lumpy consumption. For $a \leq \bar{a}_s(b, z)$, the planner is constrained to $y_s = a$.

[7] See Bethune and Rocheteau (2023) for a model similar to the one in this paper but that includes a frictional labor market which incorporates additional externalities.

Illiquid deposits $d_s(x)$. The deposit flow d_s enters the HJB through the drift terms, contributing $\psi_s[-(1+\Phi'(d_s))\partial_a W_s + \partial_b W_s]$; the deposit implementability constraint, contributing $-\delta_s \Phi''(d_s)\,\partial_a W_s$ (since d_s enters the constraint only through $\Phi'(d_s)$); and the KFE via the savings rates, contributing $[-(1+\Phi'(d_s))\partial_a \gamma_s + \partial_b \gamma_s]g_s$. Setting $\partial L/\partial d_s(x) = 0$ and imposing the household deposit FOC $(1+\Phi'(d_s))\partial_a W_s = \partial_b W_s$ (which zeros out the ψ_s term):

$$\delta_s = \frac{(1+\Phi'(d_s))\partial_a \gamma_s - \partial_b \gamma_s}{\Phi''(d_s)\,\partial_a W_s}\, g_s. \tag{29}$$

Define the *deposit wedge* $\tilde{\delta}_s \equiv (1+\Phi'(d_s))\partial_a \gamma_s - \partial_b \gamma_s$: the gap between the planner's shadow value of liquid wealth (scaled by the marginal cost of depositing) and the shadow value of illiquid wealth. The denominator $\Phi''(d_s)\partial_a W_s > 0$ is the negative of the household's second-order condition for the deposit problem ($-\Phi''(d)\partial_a W < 0$).

Transfers $\mathcal{T}_s(x)$. Transfers enter the Lagrangian through the HJB drift, contributing $\psi_s \partial_a W_s$; in the KFE equation, contributing $+\partial_a \gamma_s g_s$ after applying (25); and negatively in the government budget constraint, contributing $-\theta_s g_s$. Under unrestricted transfers, setting $\partial L/\partial \mathcal{T}_s(x) = 0$, using $\partial_a W_s = u'(c_s)$, and dividing by $g_s > 0$ gives:

$$\theta_s = \partial_a \gamma_s + \frac{\psi_s}{g_s}\, u'(c_s), \tag{30}$$

for all x. The scalar θ_s is the shadow cost of a marginal unit of government revenue. If transfers can condition on (a, b, z) then (30) requires the adjusted shadow value of liquid wealth to be equalized across all agents. Under lump-sum transfers ($\mathcal{T}_s = T_s$), (30) aggregates to

$$\theta_s = \int \left[\partial_a \gamma_s\, g_s + \psi_s\, \partial_a W_s\right] dx. \tag{31}$$

Under lump-sum transfers, the optimal θ_s must equal the population-wide average of two valuation channels. The first, $\partial_a \gamma_s g_s$, is the measure-weighted social lifetime value of additional wealth at each point in the distribution. The second, $\psi_s \partial_a W_s$, is a distributional correction: $\psi_s(a, z)$ is the Lagrange multiplier on the HJB implementability constraint and measures the social cost of being bound by household optimality at each (a, z). Intuitively, $\psi_s(a, z) > 0$ wherever the planner would like to shift more resources toward households at that wealth-income pair (relative to what they choose privately), and $\psi_s(a, z) < 0$ wherever the reverse holds.

Money supply M_s. The money supply enters the market clearing condition, contributing $-\mu_s \phi_s$ to the coefficient on M_s, and through $\dot{M}_s$ in the government budget constraint. Because the government budget constraint involves $\dot{M}_s$, it is useful to integrate by parts in s before differentiating. Using $\dot{\phi}_s = r_s^a \phi_s$, the effective present-value coefficient on M_s from the government

budget constraint term becomes $(\rho - r_s^a)\theta_s\phi_s - \dot{\theta}_s\phi_s$.[8] For an interior optimum with $r_s^a < \rho$, setting $\partial L/\partial M_s = 0$ and dividing by $\phi_s > 0$:

$$\dot{\theta}_s = (\rho - r_s^a)\,\theta_s - \mu_s. \tag{32}$$

Equation (32) is the co-state equation for the shadow price of government revenue, valid for $s \in [0, \tau_1)$; the terminal value θ_{τ_1} is pinned down by a separate transversality condition derived in Appendix A. The term $(\rho - r_s^a)\theta_s$ captures the rate of "interest" on the shadow cost: θ_s compounds at the impatience rate ρ and is eroded by the real return r_s^a on money holdings. The term $-\mu_s$ is the social cost of a unit increase in the real supply of money, through the market clearing shadow value.

Bond supply B_s. The bond supply enters the market clearing condition for illiquid wealth, contributing $-\varphi_s\phi_s$, and through $\dot{B}_s$ in the government budget constraint (with an additional $-i_s^b\phi_s B_s$ term from interest payments). Integrating by parts in s and setting $\partial L/\partial B_s = 0$:

$$\dot{\theta}_s = (\rho - r_s^b)\,\theta_s - \varphi_s, \tag{33}$$

where $r_s^b = r_s^a + i_s^b$. Note the parallel structure: M_s and B_s each yield a co-state equation for θ_s with the respective real return (r_s^a for money, r_s^b for bonds). Combining (32) and (33):

$$\mu_s - \varphi_s = i_s^b\,\theta_s. \tag{34}$$

Equation (34) is a key equation linking monetary and fiscal policy through the shadow values of the two market clearing conditions and the liquidity premium i_s^b. When $i_s^b > 0$, the shadow value of liquid market clearing (μ_s) exceeds the shadow value of illiquid market clearing (φ_s) by $i_s^b\theta_s$—the fiscal shadow cost of the interest spread.

Price level ϕ_s. The price ϕ_s enters the Lagrangian in multiple places: directly in the government budget constraint, contributing $\theta_s(\dot{M}_s + \dot{B}_s - i_s^b B_s)$ per unit $\delta\phi_s$; directly in both market clearing constraints, contributing $-\mu_s M_s - \varphi_s B_s$ per unit $\delta\phi_s$; and implicitly through the real return $r_s^a = \dot{\phi}_s/\phi_s$ in the savings drift s_s^a, which appears in both the HJB and KFE Lagrange terms. The r_s^a-dependent terms require an Euler–Lagrange variation, and ϕ_s additionally enters through the transfer rule when transfers depend on prices. The resulting condition is algebraically equivalent

[8] Specifically, integrate by parts in the Lagrangian (24): $\int_0^{\tau_1} e^{-\rho s}\theta_s\phi_s\dot{M}_s\,ds = [e^{-\rho s}\theta_s\phi_s M_s]_0^{\tau_1} + \int_0^{\tau_1} e^{-\rho s}M_s\phi_s([\rho - r_s^a]\theta_s - \dot{\theta}_s)ds$. The lower boundary term vanishes because M_0 is a fixed initial condition ($\delta M_0 = 0$). The upper boundary term at $s = \tau_1$ does *not* vanish: M_{τ_1} is a free terminal state chosen by the current planner and inherited by the next. Varying the full objective with respect to M_{τ_1} yields the terminal condition (37). The interior integral supplies the coefficient on M_s used to derive (32).

to (32) by Walras' law: the four aggregate constraints (government BC, two market clearing conditions, and the implied goods market clearing) are linearly dependent, so the ϕ_s FOC provides no independent information beyond the money supply co-state equation.

Nominal bond rate i_s^b. The nominal bond rate i_s^b enters the Lagrangian in two places: the government budget constraint, contributing $-\theta_s \phi_s B_s$ per unit δi_s^b; and the illiquid savings drift $s_s^b = r_s^b b + d_s$ (through $r_s^b = r_s^a + i_s^b$), which appears in the HJB and KFE terms. However, the i_s^b variation is redundant by the same logic as the ϕ_s variation: the bond market clearing condition (18) and the government budget constraint (6) are linearly dependent with the B_s FOC through Walras' law. The FOC $\partial L / \partial i_s^b = 0$ therefore yields no independent condition beyond (33) and (34), which already characterize the optimal liquidity premium.

Spatial boundary conditions for the first-order conditions. All conditions above are derived for the interior of the state space, $(a, b) \in (0, \infty) \times (\underline{b}, \infty)$. At the boundaries $a = 0$ and $b = \underline{b}$, the household FOCs (11)–(14) continue to hold (the consumption Euler, labor supply, and deposit FOC are always interior by the Inada conditions and strict convexity of Φ), and the lumpy consumption condition (13) reduces to $y_s = a$ in the full-depletion region. The implementability multipliers $(\chi_s, \nu_s, \eta_s, \delta_s)$ are determined by the same first-order logic as in the interior, with the density g_s replaced by the appropriate mass at $a = 0$ or $b = \underline{b}$. The transfer FOC (30) (unrestricted) or (31) (lump-sum) holds at the boundaries with the same replacement. The boundary conditions from Proposition 1(ii) constrain the savings drifts $\dot{a}_s$ and $\dot{b}_s$ at the state constraints, but do not alter the structure of the planner's FOCs.

3.3 Adjoint equations

Varying the Lagrangian L with respect to the two state variables—the value function $W_s(a, z)$ and the distribution $g_s(a, z)$—yields a pair of *adjoint equations*: a backward HJB equation for the social lifetime value γ_s and a forward Kolmogorov equation for utility promises ψ_s. We first consider variations on the interior of the state space, then separately consider corner solutions.

Backward HJB for the social lifetime value. Setting $\partial L / \partial g_s(x) = 0$ and integrating by parts in time using the KFE Lagrange term $\gamma_s[\mathcal{A}_s^*[g_s] - \partial_t g_s]$ gives:

$$\rho \gamma_s = \mathcal{U}_s - \theta_s \mathcal{T}_s + \mu_s a + \varphi_s b + \mathcal{A}_s[\gamma_s] + \partial_t \gamma_s. \tag{35}$$

Equation (35) is a backward HJB for $\gamma_s(x)$, the social value of having one additional household at (a, b, z) in time period s. It is the planner's analog to the private value $W_s(x)$. The social value shares some components of the private value: it has the same generator $\mathcal{A}_s$, discount rate ρ, and

flow utility $\mathcal{U}_s(x)$. However, there are three additional terms in (35) that contribute to flow welfare: the fiscal cost of transferring resources to the household, $-\theta_s \mathcal{T}_s(x)$; the intertemporal cost of the household's demand for liquid wealth through market clearing, $\mu_s a$; and the analogous cost for illiquid wealth, $\varphi_s b$.

Forward KFE for utility promises. Households' private values, $W_s(x)$, contribute to the Lagrangian through five terms: in the private HJB equation and in the four private first-order conditions for c, n, y, and d. Setting $\partial L/\partial W_s(x) = 0$ and integrating by parts in time (via the $\partial_s W_s$ term in the HJB) and in the wealth dimensions (via the $\partial_a W_s$ and $\partial_b W_s$ terms in the first-order conditions) gives:

$$\partial_t \psi_s = \mathcal{A}_s^*[\psi_s] + \partial_a \chi_s + z_s \, \partial_a \nu_s + \partial_a \left[\frac{\alpha \, \eta_s(a^+(x), b, z)}{1 - \partial_a y_s(a^+(x), b, z)} \right] + \nabla \cdot [\delta_s \, \nabla_{ab}], \tag{36}$$

where $a^+(x)$ is the pre-depletion wealth satisfying $a^+(x) - y_s(a^+(x), b, z) = a$. Equation (36) is a forward Kolmogorov equation for ψ_s, the shadow cost to the planner of the utility promises implicitly delivered to household (a, b, z) by past policy choices.[9] The first term $\mathcal{A}_s^*[\psi_s]$ accounts for the drift, identical in structure to the KFE for g_s. The remaining terms are generated by the implementability constraints. The gradient ∂_a appears in the consumption, labor, and lumpy consumption source terms because these constraints involve $\partial_a W_s$—the marginal value of liquid wealth—and integration by parts moves this derivative onto the multiplier. The term $\partial_a \chi_s$ captures how the social gain from relaxing the flow consumption Euler equation varies with liquid wealth, $z_s \, \partial_a \nu_s$ captures the analogous gradient for labor supply, the third term represents the lumpy consumption distortion transformed into post-depletion wealth (where the denominator is the Jacobian of the map $a^+ \mapsto a^+ - y_s(a^+, b, z)$), and the final term $\nabla \cdot [\delta_s \, \nabla_{ab}]$ captures the deposit implementability constraint—the integration-by-parts residuals in both the a and b directions from varying $\partial_a W_s$ and $\partial_b W_s$ in the deposit FOC.

Spatial boundary conditions for the adjoint equations. Both adjoint equations require boundary conditions at $a = 0$ and $b = \underline{b}$. For the backward HJB (35), the boundary condition at $a = 0$ is obtained from variation with respect to the mass at $a = 0$, $\partial L/\partial m_s(b, z) = 0$. The resulting condition is a backward HJB for γ_s at the liquid borrowing constraint, with two simplifications relative to the interior: agents at $a = 0$ cannot make lumpy consumption ($y_s(0, b, z) = 0$), so the

[9] This interpretation connects to the recursive contracts literature (Marcet and Marimon, 2019): the planner carries utility promises as state variables, the envelope condition on those promises is the promise-keeping constraint, and the co-state on that constraint is the shadow price of the promise. Here $W_s(x)$ plays the role of the promise—the continuation value implicitly delivered to household (a, b, z) by past policy choices—and the HJB equation is its law of motion. The multiplier $\psi_s(x)$ is therefore the shadow cost of that promise. The closest analogue in a heterogeneous-agent setting is the distributional penalty of Dávila and Schaab (2023), who derive an equivalent forward equation from the same first-order condition $\partial L/\partial W_s = 0$.

Poisson jump term in the generator drops out; and the effective discount rate is $\rho + \alpha$ rather than ρ, because agents at the mass point cycle through Poisson meetings at rate α without changing their liquid state. The social value at the constraint is governed by the flow welfare $\mathcal{U}_s(0, b, z)$, the fiscal cost of transfers, the market clearing costs $\mu_s \cdot 0 + \varphi_s \cdot b$, and the productivity- and illiquid-wealth-transition dynamics—all discounted at the Poisson-adjusted rate. An analogous condition holds at $b = \underline{b}$, where the illiquid state constraint binds and the deposit flow is constrained by $d_s \geq -r_s^b \underline{b}$.

For the forward KFE (36), spatial boundary conditions at $a = 0$ and $b = \underline{b}$ are obtained from $\partial L / \partial W_s(0, b, z) = 0$ and the corresponding variation at $b = \underline{b}$. At $a = 0$, the condition collects the residuals that the integration-by-parts steps in the a-direction leave at $a = 0^+$: a contribution from the HJB liquid drift (proportional to the shadow cost of the utility promise at the constraint times the exit rate $\dot{a}_s(0, b, z)$), plus shadow-cost terms from the flow consumption, labor supply, lumpy consumption, and deposit implementability constraints. An additional contribution arises from the endogenous full-depletion threshold $\bar{a}_s(b, z) = (U')^{-1}[\partial_a W_s(0, b, z)]$: any change in $\partial_a W_s(0, b, z)$ shifts $\bar{a}_s(b, z)$ and alters the Poisson mass flow of agents depleting into the constraint—the social value of this response enters the boundary condition. At $b = \underline{b}$, the integration-by-parts residuals in the b-direction generate an analogous boundary condition involving the deposit implementability multiplier δ_s.

3.4 Terminal and initial conditions, and the role of commitment

The backward equations (32), (33), and (35) require terminal conditions, while the forward equation (36) requires an initial condition. The terminal condition for the optimal asset supplies is given by optimality conditions for setting M_{τ_1} and B_{τ_1}, choices of the current planner that constrain the future planner. Each involves a boundary term from the integration by parts of $\dot{M}_s$ and $\dot{B}_s$ in the government budget constraint, together with the continuation value $\mathcal{W}_{\tau_1}(G_{\tau_1}, M_{\tau_1}, B_{\tau_1})$. The first-order conditions are

$$\theta_{\tau_1} = -\frac{1}{\phi_{\tau_1}} \frac{\partial \mathcal{W}_{\tau_1}}{\partial M_{\tau_1}} = -\frac{1}{\phi_{\tau_1}} \frac{\partial \mathcal{W}_{\tau_1}}{\partial B_{\tau_1}}. \tag{37}$$

The shadow cost of government revenue at the handoff date equals, in magnitude, the next planner's marginal value of the inherited nominal asset stocks deflated by the price level. The two terminal conditions yield the same θ_{τ_1}, implying $\partial \mathcal{W}_{\tau_1} / \partial M_{\tau_1} = \partial \mathcal{W}_{\tau_1} / \partial B_{\tau_1}$: the successor values money and bonds equally at the margin at the handoff date. This is because the consolidated planner can freely adjust the composition of nominal liabilities after the handoff—only the total real value of public liabilities matters, not its split between money and bonds. The interest term

i_s^b enters the *interior* co-state equation (33) but not the terminal condition.[10]

Now consider the variation in $G_{\tau_1}(x)$. Integrating the KFE Lagrange terms (both interior and boundary) by parts in time generates the boundary residual at τ_1:

$$e^{-\rho\tau_1}\left[\int\left[\gamma_{\tau_1}(x)-\frac{\delta\mathcal{W}_{\tau_1}}{\delta g_{\tau_1}(x)}\right]g_{\tau_1}\,dx+\int\left[\gamma_{\tau_1}(0,b,z)-\frac{\delta\mathcal{W}_{\tau_1}}{\delta m_{\tau_1}(b,z)}\right]m_{\tau_1}(b,z)\,dbdz+\ldots\right]=0,$$

where the ellipsis denotes analogous terms from the mass at $b=\underline{b}$ and the double corner. Since this holds for arbitrary $(g_{\tau_1}, m_{\tau_1}, \tilde{m}_{\tau_1})$, we obtain a unified terminal condition on γ_{τ_1}: for all (a,b,z),

$$\gamma_{\tau_1}(x)=\frac{\partial\,\mathcal{W}_{\tau_1}}{\partial\,G_{\tau_1}(x)}, \tag{38}$$

where the functional derivative is with respect to the full measure G_{τ_1}—evaluated as $\delta\mathcal{W}_{\tau_1}/\delta g_{\tau_1}(x)$ in the interior, $\delta\mathcal{W}_{\tau_1}/\delta m_{\tau_1}(b,z)$ at $a=0$, and $\delta\mathcal{W}_{\tau_1}/\delta\tilde{m}_{\tau_1}(a,z)$ at $b=\underline{b}$. The social lifetime value at regime change equals the marginal social welfare of the successor from an additional agent at (a,b,z), whether that agent is in the interior or at a borrowing constraint.

Now consider the variation in $W_0(x)$. Integrating the HJB multiplier term $-e^{-\rho s}\psi_s g_s\partial_s W_s$ by parts in time over $[0,\tau_1]$ generates a boundary residual at $s=0$, by the same logic that produced the residual at τ_1 above. Setting the variation with respect to $W_0(x)$ to zero gives:

$$\int\left[\psi_0(x)-\frac{\delta\mathcal{W}_0}{\delta W_0(x)}\right]W_0\,g_0\,dx+\int\left[\psi_0(0,b,z)-\frac{\delta\mathcal{W}_0}{\delta W_0(0,b,z)}\right]W_0(0,b,z)\,m_0(b,z)\,dbdz+\ldots=0,$$

which must vanish. Since this holds for arbitrary W_0 and G_0, we obtain the general initial condition for all (a,b,z),

$$\psi_0(x)=\frac{\partial\mathcal{W}_0}{\partial W_0(x)}. \tag{39}$$

Equation (39) is the exact dual of (38): the shadow cost of utility promises at the start of the current planner's tenure equals the marginal continuation value that the successor equilibrium assigns to those same promises.

The planner's initial condition for promises depends on the extent of their commitment power—the Poisson handoff rate λ *only* enters the set of optimality conditions through (39), via its effect on $\mathcal{W}_0$. Under full discretion, $\lambda\to\infty$, the planner is replaced every instant and cannot internalize effects on future value functions; the HJB penalty collapses at every date, so $\psi_s(x)=0$ for all s and x. Under full commitment, $\lambda\to 0$, there is no predecessor planner, so W_0 is a free initial choice rather than an inherited promise; the boundary term at $s=0$ carries no Lagrange multiplier, giving $\psi_0(x)=0$ for all x—this is the well-known "time=0" problem identified

[10] The composition of the balance sheet at the handoff date *does* matter when monetary and fiscal policy are separated (Section 4), because the monetary authority cannot adjust the fiscal authority's bond issuance.

in Woodford (1999). Under limited commitment, $\lambda \in (0, \infty)$, this occurs at each handoff date; the new planner can re-optimize and is only constrained by the current state.

3.5 The optimality system

The first-order conditions of Section 3.2, the adjoint equations of Section 3.3, and the terminal and initial conditions of Section 3.4 combine into a coupled forward–backward system that fully characterizes optimal policy for any $\lambda \geq 0$. Proposition 3 collects these conditions into a single result. The cases $\lambda \to 0$ and $\lambda \to \infty$ are considered separately in the following subsections.

Proposition 3 (Optimality system)**.** *Fix $\lambda \geq 0$. The necessary conditions for optimal policy $(M_s, B_s, \mathcal{T}_s)_{s\in[0,\tau_1]}$ are:*

i) Implementability multipliers. At each (x, s), the household implementability multipliers $(\chi_s, \nu_s, \eta_s, \delta_s)$ satisfy (26)–(29). The government revenue shadow price θ_s satisfies the transfer first-order condition (30); under lump-sum transfers, (30) is replaced by (31).

ii) Asset supply co-states. The government revenue shadow price θ_s satisfies the co-state equation (32). The bond supply co-state (33) is algebraically related through $\mu_s - \varphi_s = i_s^b \theta_s$ from (34), defining the market-clearing shadow values as $\mu_s = (\rho - r_s^a)\theta_s - \dot{\theta}_s$ and $\varphi_s = \mu_s - i_s^b \theta_s$. The FOCs for ϕ_s and i_s^b are redundant by Walras' law.

iii) Social lifetime value. The co-state $\gamma_s(x)$ solves (35) for all $(a, b) \in [0, \infty) \times [\underline{b}, \infty)$, with terminal condition (38).

iv) Utility promises. The co-state $\psi_s(x)$ solves (36), with initial condition ψ_0 determined by λ via (39).

v) Equilibrium state equations. The value function W_s, equilibrium measure G_s, prices (ϕ_s, i_s^b), household decisions (c_s, n_s, y_s, d_s), and policy $(M_s, B_s, \mathcal{T}_s)$ jointly satisfy the competitive equilibrium conditions of Definition 1.

It is helpful to view the optimality system as a forward–backward system of partial differential equations: the state variables $(g_s, M_s, B_s, \phi_s, \psi_s)$ evolve forward from τ_0, while $(\gamma_s, \theta_s, W_s)$ evolve backward from τ_1. The system is coupled through the implementability multipliers $(\chi_s, \nu_s, \eta_s, \delta_s, \theta_s, \mu_s, \varphi_s)$, which appear in both the backward HJB (35) and the forward KFE (36). The parameter λ does not appear in the interior equations—it enters only through the initial condition on ψ_s, so different policy regimes are distinguished solely by a boundary condition.

3.6 Optimal consolidated policy with unrestricted transfers

First consider optimal policy with unrestricted transfers $\mathcal{T}_t(x)$, in which the planner can condition the transfer on each agent's idiosyncratic state (a, b, z). The following Lemma shows that the

multiplier on utility promises vanishes for either $\lambda \to 0$ or ∞, and the planner can achieve the first-best allocation.

Lemma 1 (Commitment premium vanishes under unrestricted transfers)**.** *Under full discretion ($\lambda \to \infty$) or full commitment ($\lambda \to 0$), if transfers are unrestricted then $\psi_t(x) = 0$ for all (x, t), and the transfer first-order condition* (30) *reduces to*

$$\theta_t = \partial_a \gamma_t(x) \quad \text{for all } (x, t). \tag{40}$$

When transfers are unrestricted, the planner equates the marginal social gain of transfers across agents. The proof shows that this implies the constraints on utility promises are completely relaxed for all time and for all agents, $\psi_t(x) = 0$. This holds not only under full discretion $\lambda \to \infty$, where each instantaneous planner has no past promise obligations by assumption, but also under full commitment $\lambda \to 0$. Intuitively, if the planner can equate marginal social values across agents through transfers, terms that differentiate promises across agents in (36) collapse to zero. In the two-asset model, the deposit wedge $\tilde{\delta}_s$ also vanishes because the planner equates social values across both the liquid and illiquid wealth dimensions.

By equating the marginal social value of wealth, the fiscal side of policy achieves efficient redistribution across agents, both in flows for c_t and n_t, and in lumpy consumption y_t, and in deposits d_t, by equating marginal private values of wealth across both accounts. The monetary side of policy can then directly target household liquidity constraints, $y_t \leq a_t$, which entails running the Friedman rule.

Proposition 4 (Friedman rule under unrestricted transfers)**.** *Under unrestricted transfers and either full discretion ($\lambda \to \infty$) or full commitment ($\lambda \to 0$), the optimal policy sets $B_t = 0$ and $M_t > 0$ with the real return on money at the Friedman rule, $r_t^a = \rho$. The liquidity premium is zero ($i_t^b = 0$) and the allocation is time and state invariant, coinciding with the first-best in* (2)–(3)*.*

Three aspects of this result deserve emphasis. First, bonds are redundant at the Friedman rule. With $r_t^a = r_t^b = \rho$ ($i_t^b = 0$), bonds offer no return advantage over money but lack money's liquidity for lumpy expenditures. Any $B_t > 0$ would induce households to withdraw from bonds into money, incurring the adjustment cost $\Phi(d) > 0$—a deadweight loss. Setting $B_t = 0$ eliminates the illiquid account entirely: all households hold $b_t = 0$, $d_t = 0$, and $\Phi(0) = 0$. The two-asset model collapses to a pure-money economy at the first best.

Second, perfect insurance over *flow* consumption does not eliminate the demand for money. Even when $u'(c_t) = \theta_t$ is equalized across agents, households still need to accumulate liquid wealth to finance *lumpy* purchases: the Poisson expenditure shock arrives discretely, and the borrowing constraint $a \geq 0$ means an agent must hold sufficient real balances before the opportunity arrives. The planner cannot undo this friction since transfers are not allowed to be

lumpy. The wealth distribution G_t therefore remains non-degenerate at the first best—driven by the transactions liquidity motive rather than the precautionary savings motive.

Second, it is precisely this transactions motive that the nominal interest rate and liquidity premium distort. A positive $(\rho - r_t^a)$ imposes an implicit tax on money holdings, eroding the real return on liquid wealth. A positive i_t^b steers household savings toward the illiquid asset, reducing their liquid buffer for lumpy expenditures. With unrestricted transfers having already removed every distributional rationale for inflation, these distortions are unambiguously welfare-reducing. The Friedman rule ($r_s^a = \rho$) and zero liquidity premium ($i_s^b = 0$) restore efficiency on the margins that the fiscal instrument cannot reach.[11]

3.7 Optimal consolidated policy with discretion and lump-sum transfers

Under full discretion ($\lambda \to \infty$), the multiplier on utility promises collapses to $\psi_t(x) \equiv 0$ at every instant—each planner is only constrained by the state they inherit at time t, (G_t, M_t, B_t), but not by past promises. What remains is a considerably simpler system. The implementability multipliers on households' first-order conditions drop out of the optimality system (they are non-zero but do not determine the allocation). We focus on the stationary equilibrium throughout this subsection. In a stationary equilibrium, real aggregates are constant ($\dot{A}_{ss}^a = \dot{A}_{ss}^b = 0$) while nominal quantities grow at the inflation rate: $\dot{M}_{ss}/M_{ss} = \dot{B}_{ss}/B_{ss} = -r_{ss}^a$. Both nominal liabilities must grow at the same rate for real wealth demands to remain stationary. The planner's two instruments reduce to (i) the common nominal growth rate $-r_{ss}^a$, which determines the real return on money, and (ii) the balance sheet composition M_{ss}/B_{ss}, which determines the liquidity premium i_{ss}^b through portfolio market clearing.

Under lump-sum transfers, the marginal cost of government revenue θ_t equals the average social marginal value of liquid wealth:

$$\theta_t = \int \partial_a \gamma_t(x)\, dG_t(x). \tag{41}$$

The social value of wealth solves a reduced backward HJB. In steady state ($\partial_t \gamma = \dot{\theta} = 0$), the co-state equations give $\mu_{ss} = (\rho - r_{ss}^a)\theta_{ss}$ and $\varphi_{ss} = (\rho - r_{ss}^a - i_{ss}^b)\theta_{ss}$ from (32)–(34). To eliminate T_{ss}, note that real aggregates are stationary ($\dot{A}_{ss}^a = \dot{A}_{ss}^b = 0$) while nominal quantities grow: $\dot{A}^a = r^a A^a + \phi \dot{M} = 0$ implies $\phi_{ss}\dot{M}_{ss} = -r_{ss}^a A_{ss}^a$, and likewise $\phi_{ss}\dot{B}_{ss} = -r_{ss}^a A_{ss}^b$. The consolidated government budget constraint then gives $T_{ss} = -r_{ss}^a A_{ss} - i_{ss}^b A_{ss}^b$, where $A_{ss} \equiv A_{ss}^a + A_{ss}^b$. Substi-

[11] Unrestricted transfers also restore the *feasibility* of the Friedman rule. With lump-sum transfers, aggregate wealth diverges as $(\rho - r_t^a) \to 0^+$ due to the precautionary savings motive. With unrestricted transfers, the planner controls the savings drift $s_t^a(x)$ pointwise and can pin aggregate wealth at a finite level even at $r_t^a = \rho$.

tuting into (35):

$$\rho\gamma_{ss} = \mathcal{U}_{ss} + \rho\theta_{ss}A_{ss} + \underbrace{(\rho - r^a_{ss})\theta_{ss}[(a+b) - A_{ss}]}_{\text{inflation-tax redistribution}} - \underbrace{i^b_{ss}\,\theta_{ss}(b - A^b_{ss})}_{\text{liquidity-premium redistribution}} + \mathcal{A}_{ss}[\gamma_{ss}], \tag{42}$$

where $A^a_{ss} = \phi_{ss}M_{ss}$ and $A^b_{ss} = \phi_{ss}B_{ss}$. The social value of wealth differs from households' private value by three policy-dependent terms. The term $\rho\theta_{ss}A_{ss}$ captures the average social benefit of aggregate real wealth. The inflation-tax redistribution term $(\rho - r^a_{ss})\theta_{ss}[(a+b) - A_{ss}]$ measures the net social cost of inflation to agent (a, b, z): since the inflation tax falls proportionally on total real balances $(a+b)$ and lump-sum transfers distribute the revenue uniformly, agents with above-average total wealth bear a net cost while those below the mean receive a net benefit. The liquidity-premium redistribution term $-i^b_{ss}\theta_{ss}(b - A^b_{ss})$ captures the additional cost borne by agents with above-average illiquid wealth from the bond-money spread. Under lump-sum transfers, the planner's choice reduces to two aggregate scalars: the steady-state real return on money r^a_{ss} and the bond rate i^b_{ss}.

Assumption 2 (Bounded relative risk aversion)**.** *There exists $\bar{\gamma} < \infty$ such that $-cu''(c)/u'(c) \leq \bar{\gamma}$ for all $c > 0$. In addition, absolute risk aversion $-u''(c)/u'(c)$ remains finite as $c \downarrow 0$.*

Proposition 5 (Optimal consolidated policy under full discretion)**.** *Under Assumptions 1 and 2 and full discretion with lump-sum transfers, the optimal steady-state allocation satisfies $r^{a*}_{ss} < \rho$, with the optimal real return on money (equivalently, the common nominal growth rate $\dot{M}/M = \dot{B}/B = -r^{a*}_{ss}$) determined by*

$$(\rho - r^{a*}_{ss})\int u'(c_{ss})\,dG_{ss} = \int[\partial_a c_{ss} - z\,\partial_a n_{ss}]\tilde{\chi}_{ss}\,dG_{ss} + \int \partial_a d_{ss}\;\tilde{\delta}_{ss}\,dG_{ss} + \alpha\int \partial_a y_{ss}\;\tilde{\eta}_{ss}\,dG_{ss}, \tag{43}$$

where $\tilde{\chi}_{ss}$, $\tilde{\delta}_{ss}$, and $\tilde{\eta}_{ss}$ are the consumption, deposit, and lumpy consumption wedges defined in (26)–(28). The optimal balance sheet composition M_{ss}/B_{ss} (equivalently, the optimal liquidity premium i^{b*}_{ss}) satisfies

$$i^{b*}_{ss}\,\theta_{ss} = \int[\partial_b\gamma_{ss} - \partial_a\gamma_{ss}]\,dG_{ss}. \tag{44}$$

Equation (43) characterizes the optimal inflation rate. The left-hand side is the marginal fiscal cost of reducing the real return on money, weighted by average marginal utility. The right-hand side decomposes the marginal redistribution benefit into three channels. The first term captures how the inflation tax alters consumption and labor supply across the wealth distribution, weighted by the consumption wedge $\tilde{\chi}_{ss}$—the same channel present in the one-asset model. The second term is new: it captures how inflation-induced changes in liquid wealth affect portfolio rebalancing, weighted by the deposit wedge $\tilde{\delta}_{ss}$. When the social value of liquid wealth exceeds the adjustment-cost-scaled social value of illiquid wealth ($\tilde{\delta}_{ss} > 0$), agents who increase deposits

in response to liquid wealth ($\partial_a d > 0$) contribute positively to the redistribution benefit. The third term captures the lumpy consumption channel.

Equation (44) characterizes the optimal balance sheet composition M_{ss}/B_{ss} through the equilibrium liquidity premium. The liquidity premium i^b_{ss} clears the portfolio market at the government-supplied ratio M_{ss}/B_{ss}: increasing M_{ss}/B_{ss} reduces bond scarcity and lowers i^b_{ss}. The condition equates the fiscal cost of the liquidity premium ($i^{b*}_{ss}\theta_{ss}$) with the average gap between the social marginal values of illiquid and liquid wealth. When $\partial_b\gamma_{ss} < \partial_a\gamma_{ss}$ on average (liquid wealth is socially more valuable), the optimal $i^{b*}_{ss} < 0$—the planner tilts the balance sheet toward money to steer households toward liquidity.

3.8 Optimal consolidated policy with commitment and lump-sum transfers

The central difference between commitment and discretion is the distributional penalty ψ_t. Under discretion, $\psi_t = 0$: the planner takes the current distribution G_t as a state variable and optimizes without regard for how current policy affects the distribution that future planners inherit. Under commitment, $\psi_t \neq 0$ in general, and the planner internalizes that current and future policy shape future household savings decisions through the forward KFE (36). The distribution G_t is therefore no longer just a state—it is also an instrument, and ψ_t is the shadow price on the planner's ability to engineer it through committed policy.

As in the discretion case, I focus on the stationary equilibrium, setting all time derivatives to zero: $\partial_t\gamma = \partial_t\psi = \partial_t g = \dot{\theta} = 0$. The stationary system jointly determines $(\gamma_{ss}, \psi_{ss}, g_{ss}, \theta_{ss})$ as a fixed point of the optimality conditions in Proposition 3, with ψ_{ss} now generically nonzero. The key departure from the discretion case is the aggregate transfer first-order condition. From (31), evaluated at the stationary equilibrium using $\partial_a W_{ss} = u'(c_{ss})$:

$$\theta_{ss} = \int \left[\partial_a\gamma_{ss}(x) + \frac{\psi_{ss}(x)}{g_{ss}(x)}\, u'(c_{ss}(x))\right] g_{ss}(x)\, dx. \tag{45}$$

Define the **commitment premium**

$$\Psi_{ss} \;\equiv\; \int \psi_{ss}(x)\, u'(c_{ss}(x))\, dG_{ss}(x), \tag{46}$$

so that $\theta_{ss} = \overline{\partial_a\gamma_{ss}} + \Psi_{ss}$. The first term is the distributional average of the social marginal value of liquid wealth, which exhausts the fiscal shadow price under discretion. The commitment premium captures the additional social value of the promise not to deviate: it is positive when the committed policy systematically benefits agents with high marginal utility (typically, agents with low wealth), and negative when it benefits the wealthy. When $\Psi_{ss} > 0$, the commitment planner assigns a higher overall shadow price to government revenue than the discretion planner, reflecting the value of promised redistribution toward poor households.

The backward HJB for γ_{ss} is structurally identical to the discretion case (42)—ψ_{ss} does not enter γ_{ss} directly, only through the equilibrium objects $(c_{ss}, n_{ss}, y_{ss}, d_{ss}, g_{ss}, \theta_{ss})$. Since ψ_{ss} affects θ_{ss} (through (45)) and θ_{ss} in turn affects household decisions through lump-sum transfers, the commitment and discretion steady states differ not in the functional form of the backward HJB but in the equilibrium values of $(c_{ss}, n_{ss}, y_{ss}, d_{ss}, g_{ss}, \theta_{ss})$ that solve the coupled system. The stationary forward KFE for ψ_{ss} is

$$\mathcal{A}_{ss}^{*}[\psi_{ss}] + \partial_a \chi_{ss} + z_{ss}\, \partial_a \nu_{ss} + \partial_a \left[\frac{\alpha\, \eta_{ss}(a^+, b, z)}{1 - \partial_a y_{ss}(a^+, b, z)} \right] + \nabla \cdot [\delta_{ss}\, \nabla_{ab}] = 0. \tag{47}$$

The source terms encode the distortions introduced by the commitment policy relative to the decentralized optimum: $\partial_a \chi_{ss}$ captures the consumption distortion, $z_{ss}\partial_a \nu_{ss}$ the labor distortion, the third term the lumpy consumption distortion, and the final term the deposit distortion (new relative to the one-asset model). Together, these source terms drive the distribution of ψ_{ss}: it accumulates mass where the commitment policy most aggressively departs from the household's unconstrained optimum.

Proposition 6 (Commitment with lump-sum transfers)**.** *Under full commitment and lump-sum transfers in the stationary equilibrium:*

i) The backward HJB for γ_{ss} is structurally identical to the discretion case (42).

ii) The fiscal shadow price satisfies $\theta_{ss} = \overline{\partial_a \gamma_{ss}} + \Psi_{ss}$, where the commitment premium $\Psi_{ss} = \int \psi_{ss}\, u'(c_{ss})\, dG_{ss}$.

iii) The optimal bond rate satisfies

$$i_{ss}^{b,\text{comm}}\, \theta_{ss} = \int [\partial_b \gamma_{ss} - \partial_a \gamma_{ss}]\, dG_{ss} + \Psi_{ss}^{b}, \tag{48}$$

where $\Psi_{ss}^{b} \equiv \int \psi_{ss}[\partial_b W_{ss} - u'(c_{ss})]\, dG_{ss}$ is the portfolio commitment premium*: the shadow value of the committed policy's ability to distort the portfolio margin between liquid and illiquid wealth.*

iv) ψ_{ss} satisfies the stationary forward KFE (47).

Remark 1 (The commitment premium and welfare gain)**.** *The commitment premium Ψ_{ss} directly indexes the welfare gain of commitment over discretion. Under discretion, the orthogonality condition constrains the planner: the excess social marginal value of wealth must average to zero, limiting the degree to which policy can systematically favor poor households. Under commitment, this constraint is relaxed by Ψ_{ss}, allowing a systematic distributional tilt. The welfare gain has two components: the inflation channel (indexed by Ψ_{ss}) and the portfolio channel (indexed by Ψ_{ss}^{b}, which is new relative to the one-asset model). When $\Psi_{ss}^{b} > 0$, the committed planner sets a lower liquidity premium than the discretion planner: commitment to future redistribution reduces the need to use the liquidity premium as a current*

redistributive tool. Economies with greater dispersion in both liquid and illiquid wealth benefit more from central bank credibility.

4 Optimal central bank balance sheets

We now separate monetary and fiscal policy, each with its own budget constraint and policy instruments. We focus on the case with lump-sum transfers. Specifically, the consolidated government budget constraint (6) decomposes as follows.

Fiscal policy. Fiscal policy in reality sets taxes and spending and determines the liability structure of government debt. In our setting, this amounts to setting lump-sum transfers, T_t, and the growth rate of government bonds outstanding, $\dot{B}_t^T$, where "T" indicates it is debt outstanding for the "Treasury". Given a remittance from the central bank, R_t, the fiscal authority'sbudget constraint is

$$T_t + i_t^b \phi_t B_t^T = R_t + \phi_t \dot{B}_t^T. \tag{49}$$

Notice, $B_t^T = B_t + B_t^{CB}$, or that the fiscal authority's debt outstanding must either be held in the hands of the public, B_t, or by the central bank, B_t^{CB}.

To study the problem of the central bank in isolation, I assume that the fiscal authority follows a passive fiscal rule in the sense of Leeper (1991). Specifically, the fiscal authority sets lump-sum transfers according to a debt-stabilizing rule:

$$T_t = \bar{T} + \kappa \left(\bar{b} - \phi_t B_t^T \right), \tag{50}$$

where $\bar{T}$ is the steady-state level of transfers, $\bar{b}$ is a target for the real value of outstanding government debt, and $\kappa > 0$ governs the speed of fiscal adjustment. When real debt $\phi_t B_t^T$ exceeds its target, the fiscal authority decreases transfers, which—together with higher debt service—is financed by additional bond issuance. The parameter κ must be large enough relative to the real interest rate on debt to ensure that the fiscal rule is *passive*: the real value of debt $\phi_t B_t^T$ is stationary under (50) for any path of monetary policy. Given the fiscal rule (50), the fiscal budget constraint (49), determines the path of Treasury debt B_t^T:

Monetary policy. Monetary policy sets the supply of money, $\dot{M}_t$, but unlike before does not have access to taxes or transfers. Instead, to increase the supply of money, they must trade in financial markets, swapping money for bonds. The size of the central bank's balance is the policy

tool to target a particular rate outcome. The central bank's budget constraint is

$$R_t + \phi_t \dot{B}_t^{CB} = i_t^b \phi_t B_t^{CB} + \phi_t \dot{M}_t. \tag{51}$$

In (51), the central bank receives interest payments on Treasury debt and seigniorage revenue through their liabilities. They pay remittances to the Treasury and save or dis-save debt.

Discussion This fiscal rule has two useful properties. First, it isolates the monetary policy channel: the central bank's only instrument is the portfolio composition of money and bonds held by the public, and the nominal bond rate i_t^b adjusts endogenously to clear the bond market. Second, the consolidated government problem of Section 3 provides a natural welfare upper bound: there, the planner jointly optimized public liabilities and transfers. Comparing welfare across the two problems quantifies the cost of separating monetary and fiscal policy—equivalently, how much of the consolidated optimum can be achieved with open market operations alone.

4.1 Monetary authority's problem

We constrain the central bank to set remittances to $R_t = 0$ for all t. Not only is this a realistic (albeit, suboptimal) feature of monetary policy, but it also simplifies the choices of the central bank to only the size of their balance sheet, via $\{\dot{M}_t, \dot{B}_t^{CB}\}_t$. The central bank's problem is given by the analog of (22)–(24), extended to incorporate the behavior of fiscal policy. The saddle-point problem is

$$\mathcal{W}_0(G_0, M_0, B_0^T, B_0^{CB}) = \max_{\mathbf{X}} \min_{\mathbf{M}} \mathbb{E}_0 \left[L(0, \tau_1, G_0, M_0, B_0^T, B_0^{CB}) + e^{-\rho\tau_1} \mathcal{W}_{\tau_1}(G_{\tau_1}, M_{\tau_1}, B_{\tau_1}^T, B_{\tau_1}^{CB}) \right], \tag{52}$$

where I denote $x \equiv (a, b, z)$ as the idiosyncratic state of a household. The controls are $\mathbf{X} = \{c_t(x), n_t(x), y_t(x), d_t(x), W_t(x), G_t(x), \phi_t, i_t^b, \dot{B}_t^T, \dot{B}_t^{CB}, \dot{M}_t\}_{t=0}^{\tau_1}$, and the multipliers are
$\mathbf{M} = \{\psi_t(x), \chi_t(x), \nu_t(x), \eta_t(x), \delta_t(x), \gamma_t(x), \zeta_t, \theta_t, \mu_t, \varphi_t\}_{t=0}^{\tau_1}$. Note that the fiscal rule (50) has been substituted into the household budget constraints, so the generators $\mathcal{A}_s$ and $\mathcal{A}_s^*$ depend on the

aggregate states ϕ_s and B_s^T through transfers $T_s = \bar{T} + \kappa(\bar{b} - \phi_s B_s^T)$. The Lagrangian is given by:

$$
\begin{aligned}
L(t_1, t_2, G_t, M_t, B_t^T, B_t^{CB}) = \int_{t_1}^{t_2} e^{-\rho s} \Bigg\{ & \int \mathcal{U}_s(x) dG_s(x) \\
& + \Big\{ \psi_s(x) \Big[-\rho W_s(x) + \mathcal{U}_s(x) + \partial_s W_s(x) + \mathcal{A}_s(x) W_s(x) \Big] \\
& + \chi_s(x) \Big[\partial_c \mathcal{U}_s(x) - \partial_a W_s(x) \Big] \\
& + \nu_s(x) \Big[-\partial_n \mathcal{U}_s(x) - z_s \partial_a W_s(x) \Big] \\
& + \eta_s(x) \Big[\partial_y \mathcal{U}_s(x) - \alpha\, \partial_a W_s(a - y_s(x), b, z) \Big] \mathbb{1}\{a > \bar{a}_s(z)\} \\
& + \delta_s(x) \Big[\partial_b W_s(x) - (1 + \Phi'(d_s(x)))\, \partial_a W_s(x) \Big] \\
& + \gamma_s(x) \Big[\mathcal{A}_s^*(x)[G_s(x)] - \partial_s G_s(x) \Big] \Big\} dx \\
& + \zeta_s \Big[\phi_s \dot{B}_s^T - \bar{T} - \kappa \left(\bar{b} - \phi_s B_s^T \right) - i_s^b \phi_s B_s^T \Big] \\
& + \theta_s \Big[\dot{M}_s \phi_s + i_s^b \phi_s B_s^{CB} - \phi_s \dot{B}_s^{CB} \Big] \\
& + \mu_s \Big[\int a\, dG_s(x) - \phi_s M_s \Big] \\
& + \varphi_s \Big[\int b\, dG_s(x) - \phi_s (B_s^T - B_s^{CB}) \Big] \Bigg\} ds
\end{aligned}
\tag{53}
$$

Compared to the consolidated Lagrangian (24), the only structural addition is the fiscal budget constraint (multiplier ζ_s), which replaces the consolidated government budget constraint. The monetary budget constraint (multiplier θ_s) and the two market-clearing conditions (multipliers μ_s, φ_s) carry over unchanged. All household implementability constraints are identical.

4.2 First-order conditions

We derive the first-order conditions of the Lagrangian (53) following the same approach as Section 3.2. There are four categories: (i) *household decision variables* (c_s, n_s, y_s, d_s), (ii) *aggregate policy and price variables* ($\dot{M}_s, \dot{B}_s^{CB}, \dot{B}_s^T, \phi_s, i_s^b$), (iii) the state variables (W_s, G_s) yielding the adjoint equations, and (iv) the fiscal state B_s^T. All derivations are for the interior of the state space $(a, b) \in (0, \infty) \times (\underline{b}, \infty)$; boundary conditions at $a = 0$ and $b = \underline{b}$ follow from Proposition 1. Throughout, let $x \equiv (a, b, z)$ and recall that the generators $\mathcal{A}_s$, $\mathcal{A}_s^*$ depend on ϕ_s and B_s^T through the fiscal rule $T_s = \bar{T} + \kappa(\bar{b} - \phi_s B_s^T)$.

Household decision variables c_s, n_s, y_s, d_s. The first-order conditions for flow consumption, labor supply, lumpy consumption, and deposits are identical to those in Section 3.2, since each of these controls enters only through household-level terms in the Lagrangian (the HJB, the implementability constraints, and the KFE drift), none of which depend on whether the government is consolidated or separated. The implementability wedges are:

$$\tilde{\chi}_s \equiv \partial_a \gamma_s - u'(c_s), \tag{54}$$

$$\tilde{v}_s \equiv v'(n_s) - z_s \partial_a \gamma_s, \tag{55}$$

$$\tilde{\eta}_s \equiv \partial_a \gamma_s(a - y_s, b, z) - U'(y_s), \tag{56}$$

$$\tilde{\delta}_s \equiv (1 + \Phi'(d_s))\partial_a \gamma_s - \partial_b \gamma_s, \tag{57}$$

denoting the flow consumption wedge, the labor wedge, the lumpy consumption wedge, and the deposit wedge, respectively. The first three measure the gap between the central bank's shadow value of liquid wealth and the household's private marginal valuation. The deposit wedge measures the gap between the shadow value of liquid wealth (scaled by the marginal cost of depositing) and the shadow value of illiquid wealth; when $\tilde{\delta}_s > 0$, the planner values liquid wealth more than the household does relative to illiquid, reflecting the social value of the liquidity service.

Money supply $\dot{M}_s$. The money supply co-state equation is identical in form to the consolidated case (32):

$$\dot{\theta}_s = (\rho - r_s^a)\,\theta_s - \mu_s. \tag{58}$$

However, the interpretation of θ_s differs: it is now the shadow price of the *central bank's* budget constraint, not the consolidated government's. It measures the social value of an additional dollar of seigniorage revenue available to the central bank—revenue that can only be used to purchase bonds, not to finance transfers. This distinction is central to the separation of monetary and fiscal policy: the central bank's resource constraint is tighter than the consolidated government's, and θ_s reflects this.

Central bank bond holdings $\dot{B}_s^{CB}$. The bond supply co-state equation is identical in form to the consolidated case (33), with θ_s again representing the central bank's shadow price:

$$\dot{\theta}_s = (\rho - r_s^a - i_s^b)\,\theta_s - \varphi_s. \tag{59}$$

Combining with (58) yields $\mu_s - \varphi_s = i_s^b\,\theta_s$, the same relationship as (34).

Treasury debt $\dot{B}_s^T$. The variable $\dot{B}_s^T$ enters the fiscal budget constraint, contributing $\zeta_s \phi_s$, and the illiquid market clearing condition (through $B_s = B_s^T - B_s^{CB}$), contributing $-\varphi_s \phi_s$. Integrating by parts in s and setting $\partial L / \partial B_s^T = 0$:

$$\dot{\zeta}_s = (\rho - r_s^a)\,\zeta_s + (\kappa + i_s^b)\zeta_s + \varphi_s + \Xi_s, \tag{60}$$

where $\Xi_s \equiv \int \kappa(\psi_s\, \partial_a W_s + \partial_a \gamma_s\, g_s)\, dx$ captures the effect of fiscal policy on household welfare through the dependence of T_s on B_s^T: a marginal increase in B_s^T reduces transfers by $\kappa \phi_s$, which tightens household budgets and affects both private and social values of wealth. When $\kappa = 0$ (passive fiscal policy with fixed transfers), $\Xi_s = 0$ and the fiscal rule imposes no feedback from debt to transfers. In this case (60) reduces to $\dot{\zeta}_s = (\rho - r_s^a + i_s^b)\zeta_s + \varphi_s$: the fiscal shadow price ζ_s evolves in the same way as θ_s—the central bank shadow price.

Price level ϕ_s. The price level enters the Lagrangian through the monetary and fiscal budget constraints, both market clearing conditions, the savings drifts (via $r_s^a = \dot{\phi}_s / \phi_s$), and the fiscal rule (via $T_s = \bar{T} + \kappa(\bar{b} - \phi_s B_s^T)$). Despite these multiple channels, the ϕ_s FOC is redundant by Walras' law: the four aggregate constraints in the Lagrangian are linearly dependent, and the chain rule decomposes $\partial L / \partial \phi_s$ into a linear combination of the quantity FOCs for M_s, B_s^{CB}, and B_s^T, which vanish at the optimum. The derivation is in Appendix C.

Nominal bond rate i_s^b. The nominal bond rate enters the fiscal budget constraint, contributing $-\zeta_s \phi_s B_s^T$, and the central bank's budget constraint, contributing $\theta_s \phi_s B_s^{CB}$. Setting $\partial L / \partial i_s^b = 0$:

$$\theta_s B_s^{CB} = \zeta_s B_s^T. \tag{61}$$

This is a static optimality condition that links the shadow values of the two government budget constraints through the bond holdings on each balance sheet. It says that the marginal cost of a higher i_s^b to the central bank (more interest paid on CB-held bonds B_s^{CB}, valued at θ_s) must equal the marginal benefit to the fiscal authority (more interest received on outstanding debt B_s^T, valued at ζ_s). In the consolidated case, the i_s^b FOC is redundant by Walras' law: interest payments are internal to a single government budget, so the direct fiscal effect of i_s^b nets to zero and the bond rate affects welfare only through the household side (portfolio allocation and the liquidity premium). With separated policy, i_s^b additionally redistributes resources between two distinct budgets—the central bank earns $i_s^b \phi_s B_s^{CB}$ while the Treasury pays $i_s^b \phi_s B_s^T$—and these hit different constraints with potentially different shadow values. Condition (61) balances this inter-governmental transfer, a margin that is absent under consolidation.

4.3 Adjoint equations

Varying the Lagrangian with respect to the state variables $G_s(x)$ and $W_s(x)$ yields the adjoint equations, which extend those of Section 3.3 to the two-asset setting.

Backward HJB for the social lifetime value. Setting $\partial L/\partial g_s(x) = 0$ and integrating by parts in time:

$$\rho\gamma_s = \mathcal{U}_s - \zeta_s T_s + \mu_s a + \varphi_s b + \mathcal{A}_s[\gamma_s] + \partial_t \gamma_s. \tag{62}$$

This extends (35) in two ways. First, the fiscal cost term is now $-\zeta_s T_s$ rather than $-\theta_s \mathcal{T}_s$: the shadow price of fiscal revenue ζ_s replaces the consolidated shadow price θ_s, reflecting the separation of monetary and fiscal policy. Second, there is an additional market clearing term $\varphi_s b$ from the illiquid wealth market clearing condition, alongside the liquid term $\mu_s a$. The social value of an additional household at (a, b, z) now accounts for their demand for both liquid and illiquid wealth.

Forward KFE for utility promises. The forward KFE for ψ_s is identical to the consolidated case (36), since it derives entirely from household-level terms (W_s enters only through the HJB and the implementability constraints, none of which depend on the government's institutional structure).

Terminal, initial, and boundary conditions. The terminal conditions on $(\theta_s, \zeta_s, \gamma_s)$ and the initial condition on ψ_s extend those of Section 3.4 to the separated setting: $\theta_{\tau_1} = -(1/\phi_{\tau_1})\partial \mathcal{W}_{\tau_1}/\partial M_{\tau_1}$, $\zeta_{\tau_1} = -(1/\phi_{\tau_1})\partial \mathcal{W}_{\tau_1}/\partial B^T_{\tau_1}$, and $\gamma_{\tau_1}(x) = \delta \mathcal{W}_{\tau_1}/\delta G_{\tau_1}(x)$. The initial condition on ψ_s takes the same form as (39), with λ governing the degree of commitment. The spatial boundary conditions at $a = 0$ and $b = \underline{b}$ follow from the same arguments as Proposition 1. The detailed derivations are deferred to an appendix.

4.4 The optimality system

The first-order conditions of Section 4.2 and the adjoint equations of Section 4.3 combine into the optimality system for monetary policy in isolation. Three simplifications reduce the system relative to the full set of FOCs.

First, the ϕ_s FOC is redundant by Walras' law. Second, the bond-rate condition (61) links ζ_s to θ_s:

$$\zeta_s = \frac{\theta_s B^{CB}_s}{B^T_s}. \tag{63}$$

Third, combining the M_s and B_s^{CB} co-state equations (58)–(59) eliminates φ_s:

$$\varphi_s = \mu_s - i_s^b \, \theta_s. \tag{64}$$

These three reductions leave θ_s and μ_s as the two independent aggregate shadow prices.

Proposition 7 (Optimality system for monetary policy in isolation)**.** *Fix $\lambda \geq 0$. The necessary conditions for optimal monetary policy $(\dot{M}_s, \dot{B}_s^{CB})_{s\in[0,\tau_1]}$ given the fiscal rule (50) are:*

i) Household implementability multipliers. At each (x, s), the wedges $(\tilde{\chi}_s, \tilde{v}_s, \tilde{\eta}_s, \tilde{\delta}_s)$ satisfy (54)–(57).

ii) Money supply co-state. The shadow price θ_s satisfies

$$\dot{\theta}_s = (\rho - r_s^a)\, \theta_s - \mu_s, \tag{65}$$

with terminal condition $\theta_{\tau_1} = -(1/\phi_{\tau_1})\partial \mathcal{W}_{\tau_1}/\partial M_{\tau_1}$. The fiscal shadow price ζ_s and illiquid market clearing shadow value φ_s are determined by (63) and (64).

iii) Social lifetime value. The co-state $\gamma_s(x)$ solves the backward HJB

$$\rho\gamma_s = \mathcal{U}_s - \theta_s \frac{B_s^{CB}}{B_s^T} T_s + \mu_s\,(a+b) - i_s^b \theta_s\, b + \mathcal{A}_s[\gamma_s] + \partial_t \gamma_s, \tag{66}$$

for all $(a, b) \in [0, \infty) \times [\underline{b}, \infty)$, with terminal condition $\gamma_{\tau_1}(x) = \delta \mathcal{W}_{\tau_1}/\delta G_{\tau_1}(x)$, where $T_s = \bar{T} + \kappa(\bar{b} - \phi_s B_s^T)$.

*iv) Utility promises. The co-state $\psi_s(x)$ solves the forward KFE (**??**), with initial condition ψ_0 determined by λ via (39) (extended to the two-asset state).*

v) Equilibrium state equations. The value function W_s, distribution G_s, prices (ϕ_s, i_s^b), household decisions (c_s, n_s, y_s, d_s), and fiscal variables (T_s, B_s^T) jointly satisfy the two-asset competitive equilibrium (Definition 1) and the fiscal rule (50).

Compared to the one-asset backward HJB (35), the social value (66) now has three additional features. First, the fiscal cost term $-(\theta_s B_s^{CB}/B_s^T)T_s$ scales with the central bank's share of government debt: the larger the central bank's balance sheet relative to total debt, the greater the fiscal shadow cost per household. Second, the term $(\mu_s - i_s^b\theta_s)b = \varphi_s b$ captures the social value of the household's illiquid wealth, weighted by the illiquid market clearing shadow value $\varphi_s = \mu_s - i_s^b\theta_s$. Third, the liquidity premium i_s^b now enters the social value directly: a larger spread between bond and money returns increases the social cost of households holding illiquid rather than liquid wealth.

The system retains the same forward–backward structure as the one-asset model: (g_s, B_s^T, ψ_s) evolve forward, $(\gamma_s, \theta_s, W_s)$ evolve backward, and the system is coupled through the implementability multipliers and the fiscal rule. The parameter λ enters only through the initial condition on ψ_s.

4.5 Optimal central bank balance sheets with full discretion

Under full discretion ($\psi_s \equiv 0$), the optimality system of Proposition 7 simplifies. The household implementability wedges $(\tilde{\chi}_{ss}, \tilde{v}_{ss}, \tilde{\eta}_{ss}, \tilde{\delta}_{ss})$ are nonzero but do not determine the allocation directly—they are determined residually once the equilibrium is known. In steady state ($\partial_t \gamma_s = \dot{\theta}_s = 0$), the co-state equation (65) gives $\mu_{ss} = (\rho - r_{ss}^a)\theta_{ss}$. To eliminate T_{ss}, note that real Treasury debt is stationary ($\dot{A}_{ss}^{BT} = 0$, where $A_{ss}^{BT} = \phi_{ss} B_{ss}^T$), so $\phi_{ss} \dot{B}_{ss}^T = -r_{ss}^a A_{ss}^{BT}$. The fiscal budget constraint then gives $T_{ss} = -(r_{ss}^a + i_{ss}^b) A_{ss}^{BT}$, and the fiscal cost term becomes $-\omega_{ss}\theta_{ss} T_{ss} = (r_{ss}^a + i_{ss}^b)\theta_{ss} A_{ss}^{CB}$, where $A_{ss}^{CB} = \phi_{ss} B_{ss}^{CB}$ is the real value of the central bank's bond holdings. Substituting into (66) and regrouping:

$$\rho \gamma_{ss} = \mathcal{U}_{ss} + \rho \theta_{ss} A_{ss}^{CB} + \underbrace{(\rho - r_{ss}^a)\theta_{ss}[(a+b) - A_{ss}^{CB}]}_{\text{inflation-tax redistribution}} - \underbrace{i_{ss}^b\, \theta_{ss}(b - A_{ss}^{CB})}_{\text{liquidity-premium redistribution}} + \mathcal{A}_{ss}[\gamma_{ss}]. \tag{67}$$

This parallels the consolidated case (42), but with a revealing structural difference. In the consolidated case, θ_{ss} is the shadow price of government revenue (including transfers), and the two redistribution channels have *different* anchors: the inflation tax centers on total real wealth A_{ss}, while the liquidity premium centers on illiquid wealth A_{ss}^b, reflecting the government's full view of its liability structure. Here, θ_{ss} is the shadow price of central bank seigniorage—revenue that can only purchase bonds, not finance transfers—and *both* channels share the single anchor A_{ss}^{CB}. This collapse from two anchors to one is a direct consequence of the CB's budget constraint, which ties money creation to bond purchases: the CB cannot independently affect the inflation channel and the liquidity channel, so both are mediated through the balance sheet. The term $\rho\theta_{ss} A_{ss}^{CB}$ captures the steady-state social value of the CB's seigniorage and interest income. The Treasury's share of fiscal costs is absorbed by the fiscal rule and does not enter the CB's optimization.

Proposition 8 (Optimal liquidity premium and balance sheet under full discretion)**.** *Under Assumptions 1 and 2 and full discretion with lump-sum transfers in steady state, the optimal liquidity premium satisfies*

$$i_{ss}^{b*}\, \theta_{ss} = \mu_{ss}^* - \varphi_{ss}^* = \int [\partial_b \gamma_{ss} - \partial_a \gamma_{ss}]\, dG_{ss}, \tag{68}$$

where θ_{ss} is the shadow price of the central bank's budget constraint. The optimal balance sheet share

$\omega^*_{ss} = B^{CB*}_{ss}/B^T_{ss}$ *implements* i^{b*}_{ss} *through bond market clearing:*

$$\int b\, dG_{ss}(\cdot\,;\, i^{b*}_{ss}) = \phi_{ss}(1-\omega^*_{ss})B^T_{ss}. \tag{69}$$

Equation (68) equates the fiscal cost of the liquidity premium ($i^{b*}_{ss}\theta_{ss}$) with the average gap between the social marginal values of illiquid and liquid wealth. The optimal i^{b*}_{ss} balances the cost of distorting portfolio allocation away from money against the benefit of the return differential on bonds. Equation (69) then pins down the central bank's balance sheet: the CB chooses ω^*_{ss} so that the supply of bonds in public hands $(1-\omega^*_{ss})B^T_{ss}$ clears the bond market at the optimal spread. This two-step structure—*optimal spread, then implementation via balance sheet*—parallels the consolidated case where the planner chooses the optimal inflation rate and the money supply adjusts to implement it. The optimal policy redistributes along two margins: the inflation tax on total wealth ($r^a_{ss} < \rho$) and the liquidity premium, which steers portfolio composition between liquid and illiquid assets.

4.6 Optimal central bank balance sheet with full commitment

[TBD]

5 Quantitative Results and Transitional Dynamics

[TBD]

6 Conclusion

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A Proofs of Propositions and Lemmas

A.1 Proof of Proposition 1

We establish the viscosity solution property, boundary conditions, concavity, and smooth pasting on the state space $[0,\infty)\times[\underline{b},\infty)\times\mathcal{Z}$. Define the HJB operator $\mathcal{H}_t : [0,\infty)\times[\underline{b},\infty)\times\mathcal{Z}\times\mathbb{R}^2\to\mathbb{R}$ by

$$\begin{aligned}\mathcal{H}_t(a,b,z,p_a,p_b) \;\equiv\; & \rho W_t - u(c_t) + v(n_t) - \alpha U(y_t) \\ & - \left[r_t^a a + z_t n_t - c_t + \mathcal{T}_t - d_t - \Phi(d_t)\right] p_a - \left[r_t^b b + d_t\right] p_b \\ & - \alpha\left[W_t(a-y_t,b,z) - W_t(a,b,z)\right] - \mathcal{A}_z W_t - \partial_t W_t, \end{aligned} \tag{70}$$

so that the HJB equation (15) reads $\mathcal{H}_t(a,b,z,\partial_a W_t,\partial_b W_t)=0$.

A continuous function v is a *constrained viscosity subsolution* of (15) if, for each $(a_0,b_0,z_0)\in(0,\infty)\times(\underline{b},\infty)\times\mathcal{Z}$ and each smooth ϕ such that $v-\phi$ attains a local maximum at (a_0,b_0,z_0) with $v(a_0,b_0,z_0)=\phi(a_0,b_0,z_0)$:

$$\mathcal{H}_t(a_0,b_0,z_0,\partial_a\phi(a_0,b_0),\partial_b\phi(a_0,b_0))\le 0. \tag{71}$$

A continuous function v is a *constrained viscosity supersolution* of (15) if, for each $(a_0,b_0,z_0)\in[0,\infty)\times[\underline{b},\infty)\times\mathcal{Z}$ (including the boundaries $a_0=0$ and $b_0=\underline{b}$) and each smooth ϕ such that $v-\phi$ attains a local minimum at (a_0,b_0,z_0) with $v(a_0,b_0,z_0)=\phi(a_0,b_0,z_0)$:

$$\mathcal{H}_t(a_0,b_0,z_0,\partial_a\phi(a_0,b_0),\partial_b\phi(a_0,b_0))\ge 0. \tag{72}$$

Note the asymmetry: the subsolution condition (71) is required only on the interior, while the supersolution condition (72) is required on the full domain including the boundaries. This is the standard definition for state-constrained viscosity solutions; see Achdou et al. (2022, Appendix D).

Part (i): W_t is a constrained viscosity solution. Here we prove that W_t is both a constrained viscosity sub- and super-solution.

Step 1: W_t is a constrained viscosity subsolution. Fix $(a_0,b_0,z_0)\in(0,\infty)\times(\underline{b},\infty)\times\mathcal{Z}$ and let ϕ be a smooth test function such that $W_t-\phi$ attains a local maximum at (a_0,b_0,z_0) with $W_t(a_0,b_0,z_0)=\phi(a_0,b_0,z_0)$, so $W_t(a,b,z)\le\phi(a,b,z)$ in the neighborhood of (a_0,b_0,z_0). By the

Dynamic Programming Principle (DPP) applied to W_t, for any $h > 0$:[12]

$$W_t(a_0, b_0, z_0) = \mathbb{E}_{a_0,b_0,z_0}\left[\int_t^{t+h} e^{-\rho(s-t)}\mathcal{U}_s^*\, ds + e^{-\rho h} W_{t+h}(a_{t+h}^*, b_{t+h}^*, z_{t+h})\right], \tag{73}$$

where $\mathcal{U}_s^* = u(c_s^*) - v(n_s^*) + \alpha U(y_s^*)$ and starred variables denote optimal controls. Applying Dynkin's formula to $e^{-\rho s}\phi(a_s, b_s, z_s)$ over $[t, t+h]$, noting:

(i) $e^{-\rho h} W_{t+h}(a_{t+h}^*, b_{t+h}^*, z_{t+h}) \leq e^{-\rho h}\phi(a_{t+h}^*, b_{t+h}^*, z_{t+h})$ from the local maximum property;

(ii) for each Poisson jump at time $\tau \in [t, t+h]$: $W_t(a_\tau^* - y_\tau^*, b_\tau^*, z_\tau) \leq \phi(a_\tau^* - y_\tau^*, b_\tau^*, z_\tau)$, since ϕ touches W_t from above (lumpy consumption only depletes a, leaving b unchanged),

substituting into (73), dividing by h, and sending $h \to 0$ yields $\mathcal{H}_t(a_0, b_0, z_0, \partial_a\phi, \partial_b\phi) \leq 0$, establishing (71).

Step 2: W_t is a constrained viscosity supersolution. Fix $(a_0, b_0, z_0) \in [0, \infty) \times [\underline{b}, \infty) \times \mathcal{Z}$ and let ϕ be smooth with $W_t - \phi$ attaining a local minimum at (a_0, b_0, z_0), so $W_t(a, b, z) \geq \phi(a, b, z)$ near (a_0, b_0, z_0). For any admissible control (c, n, y, d) consistent with $a_s \geq 0$ and $b_s \geq \underline{b}$ for all $s \geq t$, the DPP gives:

$$W_t(a_0, b_0, z_0) \geq \mathbb{E}_{a_0,b_0,z_0}\left[\int_t^{t+h} e^{-\rho(s-t)}\mathcal{U}_s\, ds + e^{-\rho h} W_{t+h}(a_{t+h}, b_{t+h}, z_{t+h})\right]. \tag{74}$$

Using $W_t \geq \phi$ throughout (including after Poisson jumps: $W_t(a - y, b, z) \geq \phi(a - y, b, z)$) and applying Dynkin's formula to ϕ, then dividing by h and sending $h \to 0$, yields $\mathcal{H}_t(a_0, b_0, z_0, \partial_a\phi, \partial_b\phi) \geq 0$. This holds for all (a_0, b_0, z_0) in the interior. For $a_0 = 0$ or $b_0 = \underline{b}$: the admissible control set is restricted to policies that maintain $a_s \geq 0$ and $b_s \geq \underline{b}$. The DPP inequality (74) continues to hold with this restricted set, and the Dynkin argument goes through unchanged since the test function ϕ need not be extended outside the domain. Hence (72) holds at the boundaries as well.

Part (ii): Boundary conditions.

Finiteness of $\partial_a W_t(0, b, z)$. At $(0, b, z)$, Assumption 1 guarantees positive resources. The Inada condition $u'(0) = \infty$ ensures $c_t(0, b, z) > 0$ (consuming zero is never optimal when resources are available). The Euler condition $\partial_a W_t(0, b, z) = u'(c_t(0, b, z)) < u'(0) = \infty$ then gives finiteness.

Liquid boundary ($a = 0$). At $a = 0$, the constraint $y_t \leq a = 0$ forces $y_t(0, b, z) = 0$. The no-borrowing constraint $\dot{a} \geq 0$ requires $c_t \leq z_t n_t + \mathcal{T}_t - d_t - \Phi(d_t)$, where d_t is determined by the

[12] Invoking the DPP and Dynkin's formula requires that optimal drifts $|\dot{a}_s^*|$ and $|\dot{b}_s^*|$ are bounded along optimal paths. The liquid drift is bounded because c^*, n^*, and d^* are finite (the last ensured by the strictly convex adjustment cost Φ). The illiquid drift $r^b b + d^*$ is bounded because d^* is finite. These extend the regularity results of Rocheteau et al. (2018, Lemmas 1–2) to the two-asset setting.

deposit FOC (14). Since $u'(c_t) = \partial_a W_t$ and $u'' < 0$:

$$\partial_a W_t(0, b, z) = u'(c_t) \geq u'(z_t n_t + \mathcal{T}_t - d_t - \Phi(d_t)),$$

establishing (19). Equality holds iff $\dot{a}_t = 0$.

Illiquid boundary ($b = \underline{b}$). The state constraint $\dot{b}_t = r_t^b \underline{b} + d_t \geq 0$ directly requires $d_t \geq -r_t^b \underline{b}$.

Double corner ($a = 0$, $b = \underline{b}$). Both $\dot{a}_t \geq 0$ and $\dot{b}_t \geq 0$ must hold. The illiquid constraint gives $d_t \geq -r_t^b \underline{b}$. Substituting into the liquid constraint: $c_t \leq z_t n_t + \mathcal{T}_t - d_t - \Phi(d_t)$. When both bind ($\dot{a} = \dot{b} = 0$): $d_t = -r_t^b \underline{b}$ and $c_t = z_t n_t + \mathcal{T}_t + r_t^b \underline{b} - \Phi(-r_t^b \underline{b})$. Assumption 1 guarantees this is positive. The Euler equation gives (20).

Part (iii): Monotonicity and concavity.

Strict concavity in a. Fix t, (b, z), $\lambda \in (0, 1)$, and $a^0 \neq a^1$ with $\hat{a} = \lambda a^0 + (1 - \lambda) a^1$. Let $\pi^i = (c_s^i, n_s^i, y_s^i, d_s^i)_{s \geq t}$ denote the optimal plan from (a^i, b, z), and define the mixture plan $\hat{\pi} \equiv \lambda \pi^0 + (1 - \lambda)\pi^1$. Since the liquid budget (7) is affine in (a, c, n, d) for given (b, z), the mixture liquid wealth $\hat{a}_s \equiv \lambda a_s^0 + (1 - \lambda) a_s^1$ satisfies the budget constraint under $\hat{\pi}$.[13] The illiquid budget is affine in (b, d), so $\hat{b}_s$ is also feasible. By strict concavity of u and U and strict convexity of v, the mixture plan yields strictly higher utility than the convex combination of π^0 and π^1 on a set of positive measure (since $a^0 \neq a^1$ implies the paths differ near t). Therefore $W_t(\hat{a}, b, z) > \lambda W_t(a^0, b, z) + (1 - \lambda) W_t(a^1, b, z)$.

Strict concavity in b. The argument is identical with the roles of a and b exchanged: fix (a, z), vary $b^0 \neq b^1$. The illiquid budget is affine in b, the liquid budget is affine in (a, c, n, d), and the convexity of Φ ensures feasibility of the mixture plan. Strict concavity of preferences gives the strict inequality.

Strict monotonicity. In a: strictly more liquid wealth relaxes the budget constraint (the household can always consume the extra dollar or save it), so W_t is strictly increasing in a. In b: strictly more illiquid wealth allows the household to withdraw more (or service debt more easily), relaxing the effective budget constraint via d_t, so W_t is strictly increasing in b.

Part (iv): Zero savings at the double corner for $\underline{z}$.

At $(a, b) = (0, \underline{b})$ with $z = \underline{z}$, the argument extends the phase-diagram analysis of the one-asset model. At the double corner, $y_t = 0$ (no lumpy consumption), $d_t = -r_t^b \underline{b}$ (mandatory deposit to service illiquid debt), and the HJB reduces to:

$$\begin{aligned}(\rho + \alpha + \Lambda_{\underline{z}}) W_t(0, \underline{b}, \underline{z}) = u(c_t) - v(n_t) + \alpha U(0) + \dot{a}_t \, \partial_a W_t(0, \underline{b}, \underline{z}) \\ + \partial_t W_t + \sum_{z' > \underline{z}} \lambda_{\underline{z}z'} W_t(0, \underline{b}, z'),\end{aligned}$$

[13] The adjustment cost Φ is convex, so $\lambda \Phi(d^0) + (1 - \lambda)\Phi(d^1) \geq \Phi(\hat{d})$, meaning the mixture consumes *more* in adjustment costs than the mixed deposit. This makes the mixture feasible (it wastes resources, weakly).

where $\dot{a}_t = z_t n_t - c_t + \mathcal{T}_t + r_t^b \underline{b} - \Phi(-r_t^b \underline{b})$ is the liquid savings rate at the double corner. Differentiating with respect to a at $a = 0$ and applying the envelope theorem (the FOCs for c and n eliminate the $\partial_a c$ and $\partial_a n$ terms):

$$\dot{a}_t \, \partial_{aa} W_t(0, \underline{b}, \underline{z}) = (\rho + \alpha + \Lambda_{\underline{z}} - r_t^a - \partial_a \mathcal{T}_t) \, \partial_a W_t(0, \underline{b}, \underline{z}) - \partial_{ta} W_t - \sum_{z' > \underline{z}} \lambda_{\underline{z}z'} \partial_a W_t(0, \underline{b}, z').$$

The same phase-diagram argument as in Achdou et al. (2022, Section 4.1) applies: if $\dot{a}_t > 0$ at $(0, \underline{b}, \underline{z})$, the coupled ODE system for $(\partial_a W_t, \dot{a}_t)$ near $a = 0$ lies on an unstable manifold that diverges, inconsistent with a globally well-defined value function. The unique consistent solution requires $\dot{a}_t(0, \underline{b}, \underline{z}) = 0$. Since $d_t = -r_t^b \underline{b}$ is fixed at the double corner (from $\dot{b} = 0$), this also implies $\dot{b}_t = 0$. □

A.2 Proof of Proposition 2

At state (a, b, z), the optimal lumpy consumption solves $\max_{0 \le y \le a} \{U(y) + W_t(a - y, b, z)\}$. Since U is strictly concave and $W_t(\cdot, b, z)$ is strictly concave in a (Proposition 1(iii)), the objective is strictly concave in y and the maximizer is unique.

Part (i): Full depletion.

The corner $y = a$ is optimal iff $U'(a) \ge \partial_a W_t(0, b, z)$. By the Inada condition $U'(0) = \infty$ and the finiteness $\partial_a W_t(0, b, z) < \infty$ (Proposition 1(ii)), the intermediate value theorem guarantees a unique $\bar{a}_t(b, z) > 0$ solving $U'(\bar{a}_t(b, z)) = \partial_a W_t(0, b, z)$. Uniqueness follows from $U'' < 0$. The threshold is strictly positive because $U'(0) = \infty > \partial_a W_t(0, b, z)$.

Fix any $a \in (0, \bar{a}_t(b, z)]$ and suppose for contradiction that $y_t(a, b, z) < a$. Then the Euler equation (13) holds with equality: $U'(y_t) = \partial_a W_t(a - y_t, b, z)$. Since $a - y_t > 0$ and $\partial_a W_t(\cdot, b, z)$ is strictly decreasing (Proposition 1(iii)),

$$U'(y_t) = \partial_a W_t(a - y_t, b, z) < \partial_a W_t(0, b, z) = U'(\bar{a}_t(b, z)).$$

Since $U'' < 0$, this gives $y_t > \bar{a}_t(b, z) \ge a$, contradicting $y_t < a$. Therefore $y_t(a, b, z) = a$ for all $a \in (0, \bar{a}_t(b, z)]$.

Part (ii): Monotonicity.

For $a \in (0, \bar{a}_t(b, z)]$: $y_t(a, b, z) = a$, so $\partial_a y_t = 1$. For $a > \bar{a}_t(b, z)$: the interior FOC $U'(y_t) = \partial_a W_t(a - y_t, b, z)$ holds. Differentiating with respect to a:

$$U''(y_t) \partial_a y_t = \partial_{aa} W_t(a - y_t, b, z)(1 - \partial_a y_t).$$

Since $U'' < 0$ and $\partial_{aa} W_t < 0$:

$$\partial_a y_t = \frac{\partial_{aa} W_t(a - y_t, b, z)}{U''(y_t) + \partial_{aa} W_t(a - y_t, b, z)}.$$

Both numerator and denominator are negative, so $\partial_a y_t > 0$. Moreover $|U''| > 0$ implies $|U'' + \partial_{aa} W_t| > |\partial_{aa} W_t|$, so $\partial_a y_t < 1$. □

A.3 Proof of Proposition 3

Assumption 3 (Primitive regularity)**.** *(i) The initial distribution g_0 is absolutely continuous on $[0, \infty) \times [\underline{b}, \infty)$ with finite first moments, for each $z \in \mathcal{Z}$. (ii) Preference functions u, v, U are C^2, Φ is C^2, and transfers $\mathcal{T}_t(x)$ are Lipschitz in (a, b) with at most linear growth. (iii) The terminal welfare functional $\mathcal{W}_{\tau_1}[G]$ is C^1 in the distribution.*

Under Assumption 3, the equilibrium density g_s, social lifetime value γ_s, and distributional penalty ψ_s are all C^1 in (a, b), and boundary terms in all integration-by-parts steps below vanish. The derivation of these properties from primitives is in the technical supplement (available upon request). The optimality system follows from computing the Gâteaux derivatives of the Lagrangian L with respect to each variable in $\mathbf{X}$. Assumption 3 ensures all integration-by-parts steps are valid. The detailed derivations are in Section 3.2–3.4; here we collect the key algebraic steps.

Part (i): Implementability multipliers.

Consumption Euler (χ_s). The variable $c_s(x)$ enters L through: (a) the welfare integral contributing $u'(c_s)g_s$; (b) the HJB Lagrange term contributing $\psi_s[u'(c_s) - \partial_a W_s]$ (via $-c_s$ in $\mathcal{U}_s$ and the liquid drift); (c) the Euler-c constraint multiplied by χ_s, contributing $\chi_s u''(c_s)$; and (d) the KFE term via the adjoint identity (25), contributing $-\partial_a \gamma_s g_s$. Setting $\delta L/\delta c_s = 0$ and using $u'(c_s) = \partial_a W_s$ gives (26). Analogous variations give (27) and (28).

Deposit FOC (δ_s). The variable d_s enters through: (a) the HJB via the liquid drift $(-(1 + \Phi'(d_s))\partial_a W_s)$ and illiquid drift $(+\partial_b W_s)$, contributing $\psi_s[-(1 + \Phi')\partial_a W_s + \partial_b W_s]$; (b) the deposit constraint, contributing $-\delta_s \Phi''(d_s)\partial_a W_s$; and (c) the KFE, contributing $[-(1 + \Phi')\partial_a \gamma_s + \partial_b \gamma_s]g_s$. Setting $\delta L/\delta d_s = 0$, using the household FOC $(1 + \Phi')\partial_a W_s = \partial_b W_s$ to cancel the ψ_s term, gives (29).

Transfer FOC (θ_s). The transfer $\mathcal{T}_s$ enters via the HJB drift $(\psi_s u'(c_s))$, KFE drift $(\partial_a \gamma_s g_s)$, and the government BC $(-\theta_s g_s)$. Setting $\delta L/\delta \mathcal{T}_s = 0$ gives (30), which under lump-sum transfers aggregates to (31).

Part (ii): Co-state equations for θ_s.

Integrating the government BC terms $\theta_s \phi_s \dot{M}_s$ and $\theta_s \phi_s \dot{B}_s$ by parts in s and using $\dot{\phi}_s = r_s^a \phi_s$ generates the co-state equations (32) and (33). The market-clearing terms contribute $-\mu_s \phi_s$ (liq-

uid) and $-\varphi_s\phi_s$ (illiquid) per unit M_s and B_s, respectively. The ϕ_s and i_s^b FOCs are redundant by Walras' law. The terminal conditions follow from the boundary terms of the integration by parts.

Part (iii): Social lifetime value γ_s.

Varying with respect to g_s (interior) and the boundary masses $m_s(b,z)$ and $\tilde{m}_s(a,z)$ separately: each variation contributes the welfare term $\mathcal{U}_s$, the government BC term $-\theta_s\mathcal{T}_s$, the market-clearing terms $\mu_s a + \varphi_s b$, and the KFE/boundary-ODE term $\partial_t\gamma_s - \rho\gamma_s + \mathcal{A}_s[\gamma_s]$ (after the adjoint identity and integration by parts in time). Setting these to zero gives (35). The terminal condition (38) follows from the time-boundary residuals.

Part (iv): Distributional penalty ψ_s.

Varying with respect to W_s: the HJB term, after integration by parts in time, contributes $-\partial_t\psi_s + \mathcal{A}_s^*[\psi_s]$. The implementability constraint terms $-\chi_s\partial_a W_s$, $-\nu_s z_s\partial_a W_s$, $-\eta_s\partial_a W_s(a - y_s, b, z)$, and $-\delta_s[(1+\Phi'(d_s))\partial_a W_s - \partial_b W_s]$ each contribute $\partial_a\chi_s$, $z_s\partial_a\nu_s$, $\partial_a[\eta_s(a^+(x),b,z)/(1-\partial_a y_s(a^+(x),b,z))]$, and $\nabla\cdot[\delta_s\nabla_{ab}]$ after integration by parts in (a,b) (Assumption 3), where the Jacobian in the lumpy consumption term is from Proposition 2(ii). Setting $\delta L/\delta W_s(x) = 0$ for (a,b) in the interior gives the interior equation (36).

At the boundary $a = 0$, the variation $\delta L/\delta W_s(0,b,z) = 0$ additionally receives a contribution from the boundary ODE for the mass $m_s(b,z)$: the depletion threshold $\bar{a}_s(b,z) = (U')^{-1}(\partial_a W_s(0,b,z))$ enters the flux term $\alpha\int_0^{\bar{a}_s} g_s\,da$ through the smooth-pasting condition $U'(\bar{a}_s) = \partial_a W_s(0,b,z)$, so differentiating with respect to $\partial_a W_s(0,b,z)$ generates the extra term $-\alpha\gamma_s(0,b,z)g_s(\bar{a}_s,b,z)/U''(\bar{a}_s)$. Collecting all boundary contributions—from the χ_s, ν_s, η_s, and δ_s IBP residuals and the boundary ODE—gives the spatial boundary condition for ψ_s at $a = 0$:

$$s_s^a(0,b,z)\,\psi_s(0,b,z) = \chi_s(0,b,z) + z_s\,\nu_s(0,b,z) + \delta_s(0,b,z)\Phi''(d_s) - \frac{\alpha\,\eta_s(\bar{a}_s(b,z),b,z)}{1-\partial_a y_s(\bar{a}_s(b,z),b,z)} - \frac{\alpha\,\gamma_s(0,b,z)\,g_s(\bar{a}_s(b,z),b,z)}{U''(\bar{a}_s(b,z))}. \tag{75}$$

An analogous boundary condition holds at $b = \underline{b}$, where the integration-by-parts residuals in the b-direction generate terms involving δ_s and the illiquid state constraint. The initial condition on ψ_s follows from the Poisson handoff structure and λ; see Dávila and Schaab (2023).

Part (v) restates the competitive equilibrium conditions imposed as constraints throughout. □

A.4 Proof of Lemma 1

The proof has two steps. Step 1 establishes that $\psi_s = 0$ is *self-sustaining*: whenever it holds, all source terms in the forward KFE (36) vanish, so $\psi_s = 0$ remains consistent going forward. Step 2 establishes the initial condition $\psi_0 = 0$ under each regime, and then invokes forward uniqueness of the linear PDE to conclude $\psi_s \equiv 0$ for all s.

Step 1 (Self-sustaining property). Suppose $\psi_s(x) = 0$ at date s. The transfer FOC (30) immediately gives $\partial_a \gamma_s(x) = \theta_s$ for all x: the social shadow value of liquid wealth is uniform across the state space. We now show this forces $\chi_s = \nu_s = \eta_s = \delta_s = 0$.

Flow consumption (χ_s): From (26), the flow consumption wedge is $\tilde{\chi}_s \equiv \chi_s u''(c_s)/g_s = \partial_a \gamma_s - u'(c_s)$. Here $\partial_a \gamma_s = \theta_s$ is the planner's fiscal shadow price—the social value of a marginal unit of liquid wealth at x—while $u'(c_s)$ is the household's marginal utility of consumption at that point. With an unrestricted transfer, the planner can independently adjust the consumption of each agent x by varying $\mathcal{T}_s(x)$. Optimality of the transfer therefore requires the planner to equate these: $u'(c_s) = \partial_a \gamma_s = \theta_s$, so $\chi_s = 0$. Any agent whose $u'(c_s) \neq \theta_s$ would present an unexploited social gain—the planner could improve welfare by shifting their consumption—that an unrestricted transfer can realize at cost θ_s.

Lumpy consumption (η_s): From (28), the lumpy consumption wedge is $\tilde{\eta}_s \equiv \partial_a \gamma_s(a - y_s, b, z) - U'(y_s)$. Here $\partial_a \gamma_s(a - y_s, b, z) = \theta_s$ is the social value of the post-purchase liquid wealth, and $U'(y_s)$ is the marginal utility of the purchase. The same logic as for c_s applies: the planner uses the transfer to implement $U'(y_s) = \theta_s$ for all x, so $\eta_s = 0$.

Labor supply (ν_s): From (27), $\nu_s = -z_s \chi_s \, u''(c_s)/v''(n_s) = 0$ since $\chi_s = 0$.

Deposits (δ_s): From (29), the deposit wedge is $\tilde{\delta}_s = (1 + \Phi'(d_s))\partial_a \gamma_s - \partial_b \gamma_s$. With $\partial_a \gamma_s = \theta_s$ uniform across x, the planner's shadow value of liquid wealth is equalized. The backward HJB (35) with $\mathcal{T}_s$ unrestricted allows the planner to also equalize $\partial_b \gamma_s$ across x: varying $\mathcal{T}_s$ shifts the liquid drift s_s^a pointwise, which through the generator $\mathcal{A}_s[\gamma_s]$ adjusts $\partial_b \gamma_s$ until $\partial_b \gamma_s = (1 + \Phi'(d_s))\theta_s$ for all x. At this point, the deposit wedge vanishes: $\tilde{\delta}_s = (1 + \Phi')\theta_s - (1 + \Phi')\theta_s = 0$, so $\delta_s = 0$. Intuitively, when the planner can freely redistribute resources across all agents, the social values of both liquid and illiquid wealth are equalized at the margin, eliminating any portfolio distortion.

With $\chi_s = \nu_s = \eta_s = \delta_s = 0$ at every x, all source terms in (36) vanish. The KFE reduces to $\partial_t \psi_s = \mathcal{A}_s^*[\psi_s]$, which is satisfied by $\psi_s = 0$.

Step 2 (Initial conditions and uniqueness). Step 1 shows that $\psi_s = 0$ is self-sustaining. It remains to establish $\psi_0 = 0$ in each case.

Full discretion ($\lambda \to \infty$). $\psi_s \equiv 0$ by definition.

Full commitment ($\lambda \to 0$), standard Ramsey. The initial condition (39) gives $\psi_0 = 0$. With zero source terms (Step 1), the forward KFE propagates $\psi_s = 0$ uniquely.

Full commitment ($\lambda \to 0$), stationary equilibrium. The stationary ψ_{ss} satisfies $0 = \mathcal{A}_{ss}^*[\psi_{ss}] +$ (source terms). Step 1 shows the source terms vanish when $\psi_{ss} = 0$, so $\mathcal{A}_{ss}^*[\psi_{ss}] = 0$, whose unique solution is $\psi_{ss} = 0$.

In all cases $\psi_s \equiv 0$. Substituting into (30) yields (40). □

A.5 Proof of Proposition 4

By Lemma 1, $\psi_s \equiv 0$ under both full discretion and full commitment with unrestricted transfers. The optimality conditions therefore coincide across regimes.

With $\psi_s = 0$, the transfer FOC (40) gives $\partial_a \gamma_s = \theta_s$ for all x. The planner uses $\mathcal{T}_s(x)$ to enforce $u'(c_s(x)) = \theta_s$ for all x, so consumption is equalized across all wealth levels and $\partial_a c_s = \partial_b c_s = 0$. Differentiating the household HJB with respect to a and applying the envelope theorem gives $\dot{\theta}_s = (\rho - r_s^a)\theta_s = (\rho - r_s^a)\theta_s$. Combined with the money supply FOC (32), this requires $\mu_s = 0$.

By the same logic applied to the b-derivative of the HJB: $\dot{\theta}_s = (\rho - r_s^b)\theta_s$, which combined with the bond supply FOC (33) requires $\varphi_s = 0$. From (34): $\mu_s - \varphi_s = i_s^b \theta_s$, and with $\mu_s = \varphi_s = 0$, this gives $i_s^b = 0$.

Since $u'(c_s) = \theta_s$ holds at all x, the equilibrium allocations are determined pointwise by θ_s alone: $c_s = c^*(\theta_s)$, $y_s = y^*(\theta_s)$, $n_s(z) = n^*(z, \theta_s)$, and $d_s = 0$ (since $\partial_a W_s = \partial_b W_s$ at the first best, and $\Phi'(0) = 0$). Goods-market clearing requires $c^*(\theta_s) + \alpha\, y^*(\theta_s) + \Phi(0) = \mathbb{E}_z[z\, n^*(z, \theta_s)]$. Since $\Phi(0) = 0$, this reduces to $c^*(\theta_s) + \alpha\, y^*(\theta_s) = \mathbb{E}_z[z\, n^*(z, \theta_s)]$, which uniquely determines $\theta_s = \theta^{FB}$ (time invariant). Since $\dot{\theta}_s = 0$, $(\rho - r_s^a)\theta^{FB} = 0$ forces $r_s^a = \rho$. The conditions (2)–(3) are satisfied.

Finally, $B_t = 0$ is optimal: with $r_t^a = r_t^b = \rho$ ($i_t^b = 0$), bonds offer no return advantage over money but lack money's liquidity for lumpy expenditures. Any $B_t > 0$ would require households to hold $b > 0$, inducing rebalancing flows $d \neq 0$ and wasting resources through $\Phi(d) > 0$. Setting $B_t = 0$ gives $b_t = 0$ and $d_t = 0$ for all households, eliminating adjustment costs entirely. □

A.6 Proof of Proposition 5

Under full discretion ($\psi_s \equiv 0$) in steady state ($\partial_t \gamma = \partial_t W = \partial_t G = \dot{\theta} = 0$), the optimality system of Proposition 3 reduces to: $\theta = \int \partial_a \gamma \, dG$, $\mu = (\rho - r^a)\theta$, $\varphi = (\rho - r^a - i^b)\theta$, and the stationary backward HJB, private HJB, and KFE.

Part: $r_{ss}^{a*} < \rho$**.** The proof follows Achdou et al. (2022) Proposition 4, adapted to our setting with lumpy consumption and two assets. Differentiating the private HJB with respect to a (envelope condition) and using $\partial_a W = u'(c)$:

$$(\rho - r^a)\, u'(c) = u''(c)\, (\partial_a c)\, s^a + \underbrace{\mathcal{A}_z[u'(c)]}_{\text{income risk}} + \underbrace{\alpha[u'(c(a-y,b,z)) - u'(c)]}_{\text{lumpy precaution}} + \underbrace{(1+\Phi'(d))^{-1}\partial_b[u'(c)]\, s^b}_{\text{illiquid drift}}, \quad (76)$$

where $s^a = r^a a + zn - c + \mathcal{T} - d - \Phi(d)$ and $s^b = r^b b + d$. Set $r^a = \rho$ (Friedman rule for money) and $z = \bar{z}$ (highest productivity). The left-hand side is zero. The income-risk and lumpy-precaution terms are strictly positive (by the same arguments as in the one-asset case). Hence $u''(c)(\partial_a c)s^a < 0$ (accounting for the illiquid drift term, which is bounded). Since $u'' < 0$ and $\partial_a c > 0$ (Assumption 2), $s^a(a, b, \bar{z}; r^a = \rho) > 0$ for all (a, b) with $a > 0$.

As $r^a \uparrow \rho$, the upper bound of the support of the liquid wealth distribution diverges ($a_{\max}(r^a) \to \infty$), and aggregate liquid wealth demand $A^a(r^a) = \int a\, dG \to \infty$. A stationary equilibrium requires $A^a = \phi M$ finite, so no finite ϕ clears the money market at $r^a = \rho$. The Friedman rule for money is therefore infeasible under lump-sum transfers, establishing $r^{a*}_{ss} < \rho$.

Part: Optimal r^{a*}_{ss} characterization. For any smooth $f(a, b, z)$, the drift operator satisfies the commutation identity

$$\partial_a \mathcal{A}[f] = \mathcal{A}[\partial_a f] + (\partial_a s^a)\, \partial_a f + (\partial_a s^b)\, \partial_b f - \alpha\, \partial_a y\, \partial_a f(a - y, b, z). \tag{77}$$

Differentiating the social backward HJB (35) with respect to a and applying (77):

$$\rho\, \partial_a \gamma = \partial_a \mathcal{U} + \mu + \mathcal{A}[\partial_a \gamma] + (\partial_a s^a)\, \partial_a \gamma + (\partial_a s^b)\, \partial_b \gamma - \alpha\, \partial_a y\, \partial_a \gamma(a - y).$$

Differentiating the private HJB with respect to a (using the envelope theorem, which cancels the effects of optimally chosen controls) gives the same structure without μ:

$$\rho\, \partial_a W = \partial_a \mathcal{U} + \mathcal{A}[\partial_a W] + (\partial_a s^a)\, \partial_a W + (\partial_a s^b)\, \partial_b W - \alpha\, \partial_a y\, \partial_a W(a - y).$$

Subtracting and using $\tilde{\eta} = \tilde{\chi}(a - y, b, z)$ (since $U'(y) = \partial_a W(a - y)$):

$$\rho\, \tilde{\chi} = \mu + \mathcal{A}[\tilde{\chi}] + (\partial_a s^a)\, \tilde{\chi} + (\partial_a s^b)\, (\partial_b \gamma - \partial_b W) - \alpha\, \partial_a y\, \tilde{\eta}. \tag{78}$$

Integrating (78) against dG_{ss} and using $\int \mathcal{A}[\tilde{\chi}]\, dG = 0$ (stationary KFE):

$$\mu = \int (\rho - \partial_a s^a)\, \tilde{\chi}\, dG - \int \partial_a d\, (\partial_b \gamma - \partial_b W)\, dG + \alpha \int \partial_a y\, \tilde{\eta}\, dG,$$

where $\partial_a s^b = \partial_a d$. Now $\partial_a s^a = r^a + z\, \partial_a n - \partial_a c - (1{+}\Phi')\partial_a d$, so the terms involving $\partial_a d$ combine as

$$(1{+}\Phi')\, \partial_a d\, \tilde{\chi} - \partial_a d\, (\partial_b \gamma - \partial_b W) = \partial_a d[(1{+}\Phi')(\partial_a \gamma - u'(c)) - \partial_b \gamma + \partial_b W] = \partial_a d\, \tilde{\delta},$$

where the last equality uses the household deposit FOC $(1{+}\Phi')u'(c) = \partial_b W$ to collapse the cross-asset terms into the deposit wedge $\tilde{\delta} = (1{+}\Phi')\partial_a \gamma - \partial_b \gamma$. Using $\mu = (\rho - r^a)\theta$ and $\theta = \int \tilde{\chi}\, dG + \int u'(c)\, dG$ gives (43).

Part: Optimal i^{b*}_{ss}. In steady state, the bond supply FOC (33) with $\dot{\theta} = 0$ gives $\varphi_{ss} = (\rho - r^a_{ss} - i^b_{ss})\theta_{ss}$. From the money supply FOC: $\mu_{ss} = (\rho - r^a_{ss})\theta_{ss}$. Substituting into the backward HJB (35) and integrating $\partial_b \gamma_{ss} - \partial_a \gamma_{ss}$ against dG_{ss} using the transfer FOC $\theta_{ss} = \int \partial_a \gamma_{ss}\, dG_{ss}$:

$$i^{b*}_{ss} \theta_{ss} = \int [\partial_b \gamma_{ss} - \partial_a \gamma_{ss}]\, dG_{ss},$$

establishing (44). □

A.7 Proof of Proposition 6

We work in the stationary equilibrium of the commitment problem with lump-sum transfers. The key departure from Proposition 5 is that ψ_{ss} is generically nonzero.

Part (ii): Fiscal shadow price. From (31) using $\partial_a W_{ss} = u'(c_{ss})$:

$$\theta_{ss} = \int \left[\partial_a \gamma_{ss} + \frac{\psi_{ss}}{g_{ss}} u'(c_{ss})\right] g_{ss}\, dx = \overline{\partial_a \gamma_{ss}} + \underbrace{\int \psi_{ss}\, u'(c_{ss})\, dx}_{\Psi_{ss}}.$$

Part (i): Backward HJB. The backward HJB (35) does not contain ψ_{ss} directly. Stationarity and the government budget constraint give $\mu_{ss} = (\rho - r^a_{ss})\theta_{ss}$ and $\varphi_{ss} = (\rho - r^a_{ss} - i^b_{ss})\theta_{ss}$ by the same argument as Proposition 5. These depend only on stationarity and the government budget constraint, not on ψ_{ss}. The backward HJB (42) is therefore structurally identical to the discretion case.

Part (iii): Optimal bond rate. Under commitment, the transfer FOC includes ψ_{ss}. The bond supply FOC, combined with the transfer FOC, gives:

$$i^{b,\text{comm}}_{ss}\theta_{ss} = \int [\partial_b \gamma_{ss} - \partial_a \gamma_{ss}]\, dG_{ss} + \int \psi_{ss}[\partial_b W_{ss} - u'(c_{ss})]\, dG_{ss}.$$

The first term is the discretion condition (44). The second term is the portfolio commitment premium $\Psi^b_{ss} = \int \psi_{ss}[\partial_b W_{ss} - u'(c_{ss})]\, dG_{ss}$, which arises because the committed planner values the distributional effect of the bond rate through the utility promise channel ψ_{ss}. The term $\partial_b W_{ss} - u'(c_{ss})$ measures the household's private portfolio wedge: the gap between the marginal value of illiquid wealth and the marginal utility of consumption. Under commitment, the planner internalizes how this wedge interacts with the distribution of promise values.

Part (iv): Forward KFE. Setting $\partial_t \psi_s = 0$ in (36) immediately gives the stationary forward KFE (47). □

A.8 Proof of Proposition 7

The proof derives the optimality system for monetary policy in isolation from the Lagrangian (53), applying three reductions to eliminate redundant variables.

Part (i): Household implementability. The household decision variables (c_s, n_s, y_s, d_s) enter the Lagrangian only through household-level terms: the HJB constraint, the implementability constraints, and the KFE drift. None of these depend on how the government is organized. The FOCs

are therefore identical to the consolidated case (Section 3.2), yielding the same wedge definitions (54)–(57).

Part (ii): Money supply co-state and reductions. The co-state equation (58) follows from integrating the $\dot{M}_s$ term in the monetary BC by parts in s, exactly as in the consolidated case. The bond supply co-state (59) follows from integrating the $\dot{B}_s^{CB}$ term similarly: the monetary BC contributes $[\dot{\theta}_s - (\rho - r_s^a)\theta_s + i_s^b\theta_s]\phi_s$ and the illiquid MC contributes $+\varphi_s\phi_s$, giving $\dot{\theta}_s = (\rho - r_s^a - i_s^b)\theta_s - \varphi_s$.

Reduction 1 (ϕ_s redundancy): The ϕ_s FOC is redundant by Walras' law (Appendix C).

Reduction 2 (ζ_s elimination): The i_s^b FOC (61) gives $\theta_s B_s^{CB} = \zeta_s B_s^T$, so $\zeta_s = \theta_s B_s^{CB}/B_s^T$.

Reduction 3 (φ_s elimination): Equating (58) and (59): $(\rho - r_s^a)\theta_s - \mu_s = (\rho - r_s^a - i_s^b)\theta_s - \varphi_s$, giving $\varphi_s = \mu_s - i_s^b\theta_s$.

These leave θ_s and μ_s as the two independent aggregate shadow prices.

Part (iii): Social lifetime value. The backward HJB for γ_s is derived by setting $\partial L/\partial g_s(x) = 0$, as in Section 3.3. The variation picks up: flow utility $\mathcal{U}_s$; the fiscal BC contribution $-\zeta_s T_s$ (since T_s is received by each household); the liquid MC contribution $\mu_s a$; the illiquid MC contribution $\varphi_s b$; and the drift operator $\mathcal{A}_s[\gamma_s]$ from the KFE constraint. This gives (62). Substituting $\zeta_s = \theta_s B_s^{CB}/B_s^T$ and $\varphi_s = \mu_s - i_s^b\theta_s$:

$$\mu_s a + \varphi_s b = \mu_s(a+b) - i_s^b\theta_s b,$$

yielding (66).

Part (iv): Utility promises. The forward KFE for ψ_s derives from $\partial L/\partial W_s(x) = 0$. Since W_s enters only through household-level terms, the equation is identical to the consolidated case (36).

Part (v): Equilibrium state equations. The competitive equilibrium conditions and fiscal rule are constraints in the Lagrangian, not optimality conditions. They carry over unchanged. □

A.9 Proof of Proposition 8

Under full discretion ($\psi_s \equiv 0$) in steady state ($\partial_t\gamma = \dot{\theta} = 0$), the optimality system of Proposition 7 applies with $\mu_{ss} = (\rho - r_{ss}^a)\theta_{ss}$ and $\varphi_{ss} = \mu_{ss} - i_{ss}^b\theta_{ss}$.

Optimal i_{ss}^{b*}. From $\varphi_{ss} = \mu_{ss} - i_{ss}^b\theta_{ss}$ (eq. (64)), $i_{ss}^b\theta_{ss} = \mu_{ss} - \varphi_{ss}$. The backward HJB (62) in steady state has $\mu_{ss}a + \varphi_{ss}b$ as the market-clearing contributions. Differentiating with respect to b and a separately, integrating each against dG_{ss}, and using the stationary KFE ($\int \mathcal{A}[f]\,dG = 0$) and the transfer FOC $\theta_{ss} = \int \partial_a\gamma_{ss}\,dG_{ss}$ gives $\mu_{ss} - \varphi_{ss} = \int(\partial_b\gamma_{ss} - \partial_a\gamma_{ss})\,dG_{ss}$, establishing (68). The constant term $-\zeta_{ss}T_{ss}$ drops out upon differentiation in (a, b).

Optimal ω_{ss}^*. The bond market clearing condition in steady state is $\int b\,dG_{ss} = \phi_{ss}B_{ss}$ where $B_{ss} = B_{ss}^T - B_{ss}^{CB} = (1 - \omega_{ss})B_{ss}^T$. Household bond demand depends on i_{ss}^b through the deposit

FOC: a higher i^b_{ss} makes bonds more attractive, increasing $\int b\, dG_{ss}$. Given i^{b*}_{ss}, (69) pins down ω^*_{ss} as the supply that clears the bond market at the optimal spread. □

B Derivation of Kolmogorov Forward Equation and Adjoint Operator

We derive the Kolmogorov forward equation (16) and the adjoint operator $\mathcal{A}_t^*$ by taking limits of a discrete-time economy as the length of a period goes to zero, following closely to Achdou et al. (2022). The derivation is presented for the one-asset (liquid wealth only) case for clarity; the extension to the two-asset model follows analogously with additional drift terms for the illiquid account. I focus on the case of two income states, but the argument extends straightforwardly to any finite, discrete state space for z. Assume time is discrete and the length of a period is Δ. We assume there are only two productivity states, where the transition rate from state $z_t = z_j$ to state $z_{t+\Delta} = z_{-j}$ is $\Delta\lambda_j$. The probability of a lumpy expenditure shock is $\Delta\alpha$. Given beginning of period t wealth $\tilde{a}_t$, households first make their flow savings/consumption decisions, $\Delta s_t(\tilde{a}_t, z_t)$, then learn about their lumpy expenditure shock and choose an amount of lumpy consumption $y_t(\hat{a}_t, z_t)$, where $\hat{a}_t$ is wealth after flow savings/consumption, and then finally households learn about their productivity shock.

A household's beginning of period $t + \Delta$ wealth, $\tilde{a}_{t+\Delta}$, depends on if they receive a lumpy expenditure shock at time t. The law of motion of their wealth is

$$\tilde{a}_{t+\Delta} = \tilde{a}_t + \Delta s_t(\tilde{a}_t, z_t), \text{ or} \tag{79}$$

$$\tilde{a}_{t+\Delta} = \tilde{a}_t + \Delta s_t(\tilde{a}_t, z_t) - y_t(\hat{a}_t, z_t), \tag{80}$$

where $\hat{a}_t \equiv \tilde{a}_t + \Delta s_t(\tilde{a}_t, z_t)$. Notice, there is a important distinction between flow savings, Δs, that depends on the length of a period versus lumpy (dis)savings, y, that does not. We can alternatively write the law of motion (79)-(80) backwards in time as

$$\tilde{a}_t = \tilde{a}_{t+\Delta} - \Delta s_{t+\Delta}(\tilde{a}_{t+\Delta}, z_t), \text{ or} \tag{81}$$

$$\tilde{a}_t = \tilde{a}_{t+\Delta} - \Delta s_{t+\Delta}(\tilde{a}_{t+\Delta}, z_t) + y_t(\hat{a}_{t+\Delta}, z_t). \tag{82}$$

Let $G_t(a, z_j) = Pr(\tilde{a}_t \leq a, \tilde{z}_t = z_j)$ denote the CDF of the wealth distribution of type-j households at time t. We can write

$$\begin{aligned} Pr(\tilde{a}_{t+\Delta} < a, \tilde{z}_{t+\Delta} = z_j) =& (1 - \Delta\lambda_j)(1 - \Delta\alpha) Pr(\tilde{a}_t \leq a - \Delta s_{t+\Delta}(a, z_j), \tilde{z}_t = z_j) \\ &+ + (1 - \Delta\lambda_j)\Delta\alpha Pr(\tilde{a}_t \leq a - \Delta s_{t+\Delta}(a, z_j) + y_{t+\Delta}(a, z_j), \tilde{z}_t = z_j) \\ &+ \Delta\lambda_{-j}(1 - \Delta\alpha) Pr(\tilde{a}_t \leq a - \Delta s_{t+\Delta}(a, z_{-j}), \tilde{z}_t = z_{-j}) \\ &+ \Delta\lambda_{-j}\Delta\alpha Pr(\tilde{a}_t \leq a - \Delta s_{t+\Delta}(a, z_{-j}) + y_{t+\Delta}(a, z_{-j}), \tilde{z}_t = z_{-j}). \end{aligned} \tag{83}$$

In (83), the first term captures the event when the household a type $(\tilde{a}_t, \tilde{z}_t)$ household does not receive either a productivity or lumpy expenditure shock. The second and third terms capture

the events when the household only receives a lumpy expenditure or productivity shock, respectively. The last term captures the event when both shocks occur. Notice, in (83) we implicitly replaced $\hat{a}_{t+\Delta}$ with $\tilde{a}_{t+\Delta}$ in $y_{t+\Delta}$ which is a good approximation when we take limits as $\Delta \to 0$. Using the definition of the CDF, we can write

$$\begin{aligned} G_{t+\Delta}(a, z_j) =& (1-\Delta\lambda_j)(1-\Delta\alpha) G_t(a - \Delta s_{t+\Delta}(a, z_j), z_j) \\ &+ + (1-\Delta\lambda_j)\Delta\alpha G_t(a - \Delta s_{t+\Delta}(a, z_j) + y_{t+\Delta}(a, z_j)) \\ &+ \Delta\lambda_{-j}(1-\Delta\alpha) G_t(a - \Delta s_{t+\Delta}(a, z_{-j}), z_{-j}) \\ &+ \Delta\lambda_{-j}\Delta\alpha G_t(a - \Delta s_{t+\Delta}(a, z_{-j}) + y_{t+\Delta}(a, z_{-j}), z_{-j}). \end{aligned} \tag{84}$$

Subtracting $G_t(a, z_j)$ from both sides and taking limits as $\Delta \to 0$, we get

$$\begin{aligned} \partial_t G_t(a, z_j) = & -s_t(a, z_j)\partial_a G_t(a, z_j) - \alpha \left[G_t(a, z_j) - G_t(a + y_t(a, z_j), z_j) \right] \\ & -\lambda_j G_t(a, z_j) + \lambda_{-j} G_t(a, z_{-j}). \end{aligned} \tag{85}$$

Differentiating (85) with respect to a, using $g_t(a, z) = \partial_a G_t(a, z)$, yields the KFE (16). The drift term gives $-\partial_a[s_t g_t]$ directly. For the jump term, the chain rule applied to $G_t(a + y_t(a, z), z)$ produces $g_t(a + y_t(a, z), z) \cdot (1 + \partial_a y_t(a, z))$, which is the source of the Jacobian correction $(1 + \partial_a y_t)$ in the adjoint generator. This correction is generically non-zero wherever $y_t(a, z)$ varies with a.

C Price Level FOC Redundancy

This appendix shows that $\partial L / \partial \phi_s = 0$ is automatically satisfied whenever the FOCs for M_s, B_s^{CB}, B_s^T, and i_s^b hold, so it yields no independent condition. The argument proceeds by showing the four aggregate constraints in the Lagrangian are linearly dependent (Walras' law).

Step 1: Walras' law. The four aggregate constraints in (53) are:

$$\text{Monetary BC:} \quad \dot{M}_s \phi_s + i_s^b \phi_s B_s^{CB} - \phi_s \dot{B}_s^{CB} = 0, \tag{86}$$

$$\text{Fiscal BC:} \quad \phi_s \dot{B}_s^T - T_s - i_s^b \phi_s B_s^T = 0, \tag{87}$$

$$\text{Liquid MC:} \quad \int a \, dG_s - \phi_s M_s = 0, \tag{88}$$

$$\text{Illiquid MC:} \quad \int b \, dG_s - \phi_s B_s = 0. \tag{89}$$

Aggregating the household liquid budget over G_s and using (88):

$$\phi_s \dot{M}_s = Y_s - C_s + T_s - D_s - \bar{\Phi}_s, \tag{90}$$

where $Y_s = \int z_s n_s \, dG_s$, $C_s = \int c_s \, dG_s$, $D_s = \int d_s \, dG_s$, and $\bar{\Phi}_s = \int \Phi(d_s) \, dG_s$. Similarly, aggregating the illiquid budget and using (89):

$$\phi_s \dot{B}_s = i_s^b \phi_s B_s + D_s. \tag{91}$$

Adding (86) and (87) gives the consolidated government BC: $\phi_s \dot{M}_s + \phi_s \dot{B}_s = T_s + i_s^b \phi_s B_s$. Substituting (90)–(91):

$$(Y_s - C_s + T_s - D_s - \bar{\Phi}_s) + (i_s^b \phi_s B_s + D_s) = T_s + i_s^b \phi_s B_s,$$

which simplifies to $Y_s = C_s + \bar{\Phi}_s$: goods market clearing. Therefore the four constraints are linearly dependent.

Step 2: Chain rule decomposition. The price level ϕ_s enters each aggregate constraint through the nominal-to-real conversion $\phi_s \times$ (nominal quantity). It also enters through $r_s^a = \dot{\phi}_s / \phi_s$ in the savings drifts and through $T_s = \bar{T} + \kappa(\bar{b} - \phi_s B_s^T)$ in the fiscal rule. Define the normalized quantity FOCs:

$$F_s^M \equiv \frac{1}{\phi_s} \frac{\partial L}{\partial M_s}, \qquad F_s^{B^{CB}} \equiv \frac{1}{\phi_s} \frac{\partial L}{\partial B_s^{CB}}, \qquad F_s^{B^T} \equiv \frac{1}{\phi_s} \frac{\partial L}{\partial B_s^T}.$$

The chain rule gives:

$$\left. \frac{\partial L}{\partial \phi_s} \right|_{\text{aggregate}} = M_s \cdot F_s^M + B_s^{CB} \cdot F_s^{B^{CB}} + B_s^T \cdot F_s^{B^T}. \tag{92}$$

This extends to the full Lagrangian including drift terms: the Euler–Lagrange variation of $r_s^a = \dot{\phi}_s / \phi_s$ in the HJB/KFE drift generates $(\rho R_s^a - \dot{R}_s^a)/\phi_s$, where $R_s^a = \int a(\psi_s \partial_a W_s + \partial_a \gamma_s g_s) \, dx$. The same quantity R_s^a appears in the M_s FOC, weighted by M_s. The fiscal rule's effect $-\kappa B_s^T \int (\psi_s \partial_a W_s + \partial_a \gamma_s g_s) \, dx$ similarly matches terms in the B_s^T FOC.

Conclusion. Since $F_s^M = F_s^{B^{CB}} = F_s^{B^T} = 0$ at the optimum, (92) gives $\partial L / \partial \phi_s = 0$ automatically. The ϕ_s FOC provides no independent information beyond the money supply co-state equation (58).