An Equilibrium Model with Buy and Hold Investors

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Abstract

This is the first paper to analyze the effects of buy and hold investors on equilibrium security price dynamics while previous studies mainly dealt with less general restricted participation case. The empirical literature suggests that many investors follow buy and hold strategies by rarely changing asset and flow allocations due to information costs or other frictions. Similar strategies are documented for institutional investors. A buy and hold investor effectively faces an incomplete market and differs in her pricing of risk from a dynamic asset allocator. The equilibrium is solved through the construction of a representative agent with state-dependent utility. The fraction of the stock held by the buy and hold investor emerges as an additional state variable. Characterizations of equilibrium quantities are given analytically as functions of the state variables. Stock return volatility is solved endogenously in this paper. A simple calibration of our model shows that the economy with buy and hold investors can simultaneously produce a low interest rate and a high Sharpe ratio for the stock. In addition, the buy and hold economy can deliver stock return volatility more than twice that in the limited participation economy, because the stock price is potentially more sensitive to dividend shocks in the buy and hold economy. Moreover, the buy and hold economy achieves this while keeping interest rate volatility at reasonably low levels at the same time. The intuition is related to the smaller impact of dividend shocks on the welfare weight in the buy and hold economy as both types of agents hold stocks.

Key Words: General Equilibrium, portfolio constraint, incomplete market, equity premium/risk-free rate puzzle.

* "Buy and hold" as defined in this paper allows for additional purchases of the stock, unlike the one-shot buy and hold strategy.

Introduction

Researchers have recently been exploring the extent to which limited stock market participation might help explain the "equity premium puzzle" of Mehra and Prescott (1985) and the "risk-free rate puzzle" of Weil (1989). For example, using the 1984 Panel Study of Income Dynamics (PSID) data, Mankiw and Zeldes (1991) document that only about a quarter of the domestic households directly invest in stocks. They also note that the aggregate consumption of stockholders is more volatile and more correlated with excess stock market returns than that of non-stockholders. Using stockholders' consumption data in a traditional asset pricing model such as that of Breeden (1979) can help explain the size of the equity premium with a reasonable level of the relative risk aversion.

Indeed, according to the Survey of Consumer Finances (2001), the percentages of U.S. households which hold stocks through brokerage and mutual fund accounts are only 21.3% and 17.7%, respectively. However, 52.2% of the U.S. households invest in retirement accounts. Empirical studies suggest that the contributions and portfolios of many of these retirement accounts are infrequently revised. The contributors to these accounts effectively follow a buy and hold strategy, possibly due to information costs or other frictions. In this paper, we show that an equilibrium model with buy and hold investors can shed new light on some asset pricing puzzles.

In 2000, 2.5 trillion dollars were invested in private sector defined contribution plans[1], representing the largest pool of money invested in capital markets.[2] Of this amount, about three quarters were invested in equities.[3] (The US equity market capitalization is on the

[1] U.S. Department of Labor data.

[2] See "Hewitt Announces Launch of New 401(k) Index? at http://was.hewitt.com/hewitt /services/401k/observ/news.htm.

[3] "Perspective", Investment Company Institute, Vol. 7, No. 5, November 2001.

order of ten trillion.) Poterba, Venti and Wise (2001) document that the annual contribution flow to 401(k) programs now exceeds a hundred billion. In addition, the study using TIAA-CREF data from 1986 to 1996 by Ameriks and Zeldes (2004) reports, "47 percent of individuals made no change to the contribution flow during the ten year period and another 21 percent only made one change. Roughly 73 percent made no change to asset allocations over the entire 10-year period and another 14 percent made one change. A full 44 percent of the population made no changes whatsoever to either the contribution flow or asset allocation, and another 17.2 percent made one change to either stocks or flows... The finding that participants rarely change either asset or flow allocations is consistent with earlier evidence reported in Samuelson and Zeckhauser (1998)." Choi et. al. (2002) also find that investors tend to choose "the path of least resistance" by doing nothing to their flow and asset allocations. Such behaviors are not limited to defined contribution investors. Many investors who rarely make changes to their allocations follow buy and hold[4], an investment strategy that has been recommended by a lot of financial advisors for many years historically and is in no way restricted to recent retirement investing. For example, Burton Malkiel, in his famous book, *A Random Walk Down Wall Street*, argues that "...returns are gained by the steady strategy of *buying* and *holding*..."[5]

Buy and hold strategy is by no means employed only by individual investors. It is also popular among many institutional investors, such as pension funds, endowments and foundations, who "have static or slow-changing goals; invest large pools of assets that are difficult and expensive to move; are governed by boards with complicated decision making; and es-

[4]Our model also covers the case where some of the investors dynamically trade stocks through a separate brokerage account (although it may be optimal to trade stocks in the retirement account because of tax consequences, which is beyond the scope of this paper) – they simply become dynamic asset allocators. Moreover, according to the Survey of Consumer Finances (2001), a large fraction of households invest only through retirement accounts.

[5]The focus of this paper is to study the buy and hold strategy, so we abstract away from differential taxes, borrowing constraint, and non-diversification, etc.

chew market timing."[6] The popularity of buy and hold strategy by institutional investors was documented as early as Cottle and Whitman (1950). According to Statman (1995), "there is no evidence of a decline in that popularity." However, little is known about the equilibrium impact of buy and hold investors on asset price dynamics.

To address this question, we introduce buy and hold agents into a continuous-time version of the Lucas (1978) economy with finite horizon. This approach offers tractability as well as easy comparisons with similar benchmark models in the dynamic asset pricing literature. There are two types of agents in our economy: dynamic asset allocators (unrestricted agents) and buy and hold investors (restricted agents). Both types of agents have CRRA and can dynamically trade a locally riskless asset. This corresponds well to reality where all investors can save or borrow to smooth consumption over time. In addition, the asset allocator dynamically trades a stock, which is a claim to an exogenously specified dividend process. She first chooses her stock investment. Then she decides how much to consume and invests the rest (or borrows) in riskless bond. Due to information costs and other frictions[7], the buy and hold investor, on the other hand, can only contribute continuously to an account that is invested in the stock[8]. Her investment in the stock, in this specific sense, is constrained. Her decision in each period is therefore choosing how much to consume by adjusting borrowing and lending, just as many investors do so in real life. The stock and riskless bond in the buy and hold economy are however, priced jointly by *both* agents' consumption streams as shown in the next two sections. Since the focus of this study is on the buy and hold

[6]See "Institutional Investors Buy and Hold Stocks", *Amarillo Business Journal*, April 1, 2002.

[7]Other potential motivations for buy and hold strategy include investor in-attention, capital gains lock-in, etc. Like Basak and Cuoco (1998) and Shapiro (2002), we do not explicitly model such frictions, but rather take them as given and focus on their asset pricing implications. To endogenize such frictions is beyond the scope of this study and therefore left for future research.

[8]It will become clear in later sections, that for realistic levels of contribution rate, the equilibrium impact is primarily driven by agent's "holding" rather than additional "buying". In fact, all results will go through if we set the faction of stock held by the buy and hold investor constant and contribution rate to zero. This special case also provides a nice "long run" steady state interpretation of the model.

strategy, we do not impose borrowing constraint or other additional forms of constraints or frictions. The bond is in zero net supply[9] and the stock is in positive net supply. The optimal consumption allocations, the interest rate process, and the stock price process are all determined endogenously.

The equilibrium is solved through the construction of a representative investor assigning stochastic welfare weights[10] to the two types of agents. The buy and hold feature leads to path dependence for the economy. By introducing the fraction of the stock held by the buy and hold investor as another state variable, in addition to aggregate dividend and the stochastic welfare weight, we are able to preserve the Markovian structure of the economy. Characterizations of the equilibrium quantities are then given analytically as functions of the state variables, time, the dynamic asset allocator's wealth, and the stock price. The latter two are shown to satisfy a system of coupled partial differential equations.

The main results of this paper are presented by comparing the buy and hold economy with 1) an otherwise identical complete market economy, where both agents can trade the bond and the stock dynamically, and 2) a limited stock market participation economy where the non-stockholder can only trade the bond. A simple calibration of our model indicates that unlike the complete market economy, the buy and hold economy can simultaneously produce a low interest rate and a high Sharpe ratio for the stock. Moreover, the buy and hold economy can deliver a stock return volatility more than twice that in the limited participation economy while keeping interest rate volatility at reasonably low levels. Intuition for these results is also provided.

[9]In other words, aggregate borrowing must equal aggregate lending in the economy. This condition is necessary for interest rate to be determined endogenously.

[10]Cuoco and He (1994a,b) first introduced this approach to characterize equilibriums with incomplete market. A partial list of recent papers that applied this method includes Basak and Cuoco (1998) on limited stock market participation, Basak (2000) on heterogenous beliefs, Shapiro (2002) on the investor recognition hypothesis.

To the best of our knowledge, this is the first model to analyze the effects of buy and hold investors on equilibrium asset prices while the previous literature mainly dealt with less general restricted participation case. Saito (1996) presents a limited stock market participation model in a continuous-time production economy. In his model, non-stock holders invest all their wealth in the riskfree asset. Such higher demand (compared with the complete market case) lowers the return on the riskfree asset. The equity premium is increased as a result of the reduction in the risk-free rate. However, in a production economy, the moments of return for the stock are determined exogenously by the technology, rather than being endogenously determined as in an exchange economy.

Basak and Cuoco (1998) demonstrate that in a continuous-time exchange economy, limited stock market participation can produce realistic values for the risk-free rate and market Sharpe ratio with reasonable relative risk aversion coefficients. Therefore, if the stock return volatility is *exogenously* taken to be the observed historical value, the limited participation model yields the correct size for the equity premium (when multiplying the Sharpe ratio by the correct level of volatility). In contrast, we solve for the equilibrium stock return volatility *endogenously*. We show that the stock return volatility implied by the limited participation model is much lower than that observed in the data. Therefore, the limited participation model does *not* resolve the "equity premium puzzle". The limited participation model also implies interest rate volatility much higher than that observed in the data when the model matches the market Sharpe ratio.[11] On the other hand, the buy and hold model is capable of producing a much higher stock return volatility than the limited participation model because the stock price can potentially be more sensitive to dividend shocks in the buy and

[11]Equilibrium interest rate, Sharpe ratio, and stock return volatility in our model are expressed as deterministic functions of observable state variables. Therefore, differently from Mehra and Prescott (1985) and Weil (1989), we report equilibrium quantities conditional on the empirical values of the state variables, rather than their expected values under the stationary distribution of the state variables. See also Basak and Cuoco (1998).

hold case. Moreover, the buy and hold economy can achieve a higher stock return volatility while keeping interest rate volatility at reasonably low levels at the same time. The intuition is related to the smaller impact of dividend shocks on the welfare weight in the buy and hold economy than that in the limited participation economy as both types of agents hold stocks in the former. [12]

The rest of the paper is structured as follows. Section I introduces the economic setup. In Section II we provide the main results on the characterization of the equilibrium in a buy and hold economy when both agents have constant relative risk aversion preferences. Section III collects the results on the corresponding limited participation economy and complete market economy for comparison purposes. In Section IV, we discuss the impact of buy and hold on the equilibrium by comparing the buy and hold economy with the complete market economy and the limited participation economy, under logarithmic preferences. Section V calibrates the model under the more general constant relative risk aversion preferences. Section VI concludes the paper.

I. The Economy

We present a continuous-time economy on the finite time horizon $[0, T]$. Uncertainty in the economy is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathcal{P})$, on which is defined a one-dimensional Brownian motion w. The common information is given by the augmented filtration $\mathbf{F} = \{\mathcal{F}_t\}$ generated by w under the probability measure $\mathcal{P}$. The sigma-field

[12] Constantinides and Duffie (1996) have shown that the "equity premium puzzle" can be resolved with incomplete market where agents face uninsurable income shocks. However, Heaton and Lucas (1996) have provided some evidence suggesting that the actual size and persistence of income shifts might be insufficient to generate realistic values for the equity premium. Alternative explanations have been based on non-time-additive preferences and market frictions [see, e.g., Aiyagari and Gertler (1991), and Heaton and Lucas (1996)]. Additional papers on limited stock market participation include Allen and Gale (1994), Cao, Wang, and Zhang (2005), Linnainmaa (2005), Gormley, Liu and Zhou (2010).

$\mathcal{F}_t$ represents information available up to time t and $\mathcal{P}$ represents agents' common beliefs. All stochastic processes are progressively measurable with respect to $\mathbf{F}$; all the equalities involving random variables are understood to hold $\mathcal{P}$-almost surely.

A. Consumption Space

There is a single perishable consumption good (the numeraire). The agents' consumption space $\mathcal{C}$ is given by the set of nonnegative consumption-rate process c with $\int_0^T |c(t)|dt < \infty$.

B. Securities Market

The investment opportunities are represented by a locally riskless bond[13] earning an instantaneous interest rate r, and one share of risky stock[14]. The bond is in zero net supply. Its initial price is scaled to one so that the bond price satisfies

$$B(t) = \exp\left(\int_0^t r(s)ds\right).$$

The stock is a claim to an exogenously specified, strictly positive dividend process δ, which is described by the stochastic differential equation

$$d\delta(t) = \mu_\delta(t)dt + \sigma_\delta(t)dw(t), \quad \delta(0) = \delta_0. \tag{1}$$

where μ_δ and σ_δ are stochastic processes. The stock pays dividends at the rate $\delta(t)$ over $[0, T]$. In equilibrium, it will be shown that the stock price S follows an Itô process:

$$dS(t) = (\mu(t)S(t) - \delta(t))dt + \sigma(t)S(t)dw(t). \tag{2}$$

Note that in models with such Lucas-type economies, e.g. Basak and Cuoco (1998), labor income is assumed to be riskless and its present value is included in agents' holding of

[13]The "bond" here can be understood as a money market account.
[14]The stock can be understood to be the market portfolio.

riskless assets. The case of risky labor income is beyond the scope of this paper and left for future research. In addition, to focus on the effects of buy and hold constraint, we do not impose other frictions, such as borrowing constraint.

C. Trading Strategies

An admissible trading strategy is given by a vector process (α, θ), where $\alpha(t)$ and $\theta(t)$ represent the amounts invested in the bond and the stock at time t, respectively, with

$$\int_0^T |\alpha(t)r(t) + \theta(t)\mu(t)|dt + \int_0^T |\theta(t)\sigma(t)|^2 dt < \infty$$

and

$$\alpha(t) + \theta(t) \geq 0 \ \ \forall t \in [0, T].$$

The set of admissible trading strategies is denoted by Θ.

A trading strategy $(\alpha, \theta) \in \Theta$ is said to finance a consumption plan $c \in \mathcal{C}$ if the wealth process $W = \alpha + \theta$ satisfies the budget constraint

$$dW(t) = (\alpha(t)r(t) + \theta(t)\mu(t) - c(t))dt + \theta(t)\sigma(t)dw(t).$$

D. Agent Types and Constraints on Trading Strategies

There are two types of agents in the economy. Agent (type) 1 can invest in the stock as well as the bond without constraints. Her trading strategy at time t is denoted by the pair $(\alpha_1(t), \theta_1(t))$.

Agent (type) 2's trading strategy in the stock follows buy and hold, i.e., contributing an amount $x(t)dt$ per time interval dt, from her wealth (originally invested in the riskless bond)

into an account that is invested in the stock[15]. Buy and hold introduces path dependence in the economy. In other words, the state of the economy depends on the historical paths of contributions and stock prices in addition to other state variables. To preserve the Markovian structure and to exploit the connection to partial differential equations, we introduce η, the fraction of the stock held by agent 2, as an additional state variable. Its value is given by

$$\eta(t) = \eta(0) + \int_0^t \frac{x(s)}{S(s)} ds. \tag{3}$$

Equation (3) indicates that η has no diffusion term. Let $\theta_2(t)$ denote agent 2's time t amount in the stock. By definition,

$$\theta_2(t) = \eta(t) S(t). \tag{4}$$

Therefore,

$$\theta_2(t) = \left[\frac{\theta_2(0)}{S(0)} + \int_0^t \frac{x(s)}{S(s)} ds \right] S(t). \tag{5}$$

Then Equation (5) intuitively describes the value of an account with initial investment of $\theta_2(0)$ and subsequent investments in the stock S at the rate $x(s)$, $\forall s, 0 \leq s \leq t$.

In addition, agent 2 faces no constraint in investing in the riskless bond. We use $\alpha_2(t)$ to denote her bond position.

E. Agents' Preferences and Endowments

Preference for agent i $(i = 1, 2)$ is given by a time-additive expected utility function

$$U_i(c) = \mathrm{E}\left[\int_0^T e^{-\rho t} u_i(c(t)) dt \right].$$

[15] Again, please refer to footnote 8 on page 3.

In particular, we assume that agents have constant relative risk aversion, i.e. $u_1(c_1) = c_1^{1-\gamma_1}/(1-\gamma_1)$ and $u_2(c_2) = c_2^{1-\gamma_2}/(1-\gamma_2)$. If γ_i equals 1, the corresponding utility function is taken to be the logarithmic preference of $u_i(c_i) = \log(c_i)$.

At time 0, agent 1 is endowed with $1-\eta(0)$ share of the stock and a short position in b shares of the bond. Agent 2 is endowed with $\eta(0)$ share of the stock and b shares of the bond.

F. Equilibrium

An equilibrium for the economy is a price process (B, S) or equivalently, an interest rate and stock price process (r, S), and a set $c_i^*, (\alpha_i^*, \theta_i^*)$ of consumption and admissible trading strategies for the two agents such that

(i) (α_i^*, θ_i^*) finances c_i^* for $i = 1, 2$;

(ii) c_1^* maximizes U_1 over the set of consumptions plans $c \in \mathcal{C}$ which are financed by an admissible trading strategy $(\alpha, \theta) \in \Theta$, with $\alpha(0) + \theta(0) = (1-\eta(0))S(0) - b$;

(iii) c_2^* maximizes U_2 over the set of consumptions plans $c \in \mathcal{C}$ which are financed by an admissible trading strategy $(\alpha, \theta) \in \Theta$, with $\alpha(0) + \theta(0) = \eta(0)S(0) + b$, and $\theta_2(t)$ given by Equation (5);

(iv) all markets clear, that is, $c_1^* + c_2^* = \delta$, $\alpha_1^* + \alpha_2^* = 0$, and $\theta_1^* + \theta_2^* = S$.

II. Equilibrium with Buy and Hold Investors

A. Portfolio Constraint and Agents' State Price Densities

Agent 1 faces a complete market. Therefore, her state price density is given by

$$\pi_1(t) = B(t)^{-1}\xi_1(t),$$

where $\xi_1(t)$ is defined by

$$\xi_1(t) = \exp\left(-\int_0^t \kappa(s)dw(s) - \frac{1}{2}\int_0^t \kappa(s)^2 ds\right).$$

$\kappa(t)$ is the market price of risk given by

$$\kappa(t) = \frac{\mu(t) - r(t)}{\sigma(t)}. \tag{6}$$

By Itô's lemma, ξ_1 follows

$$d\xi_1(t) = -\kappa(t)\xi_1(t)dw(t). \tag{7}$$

Agent 2 can borrow or lend by trading the riskless bond dynamically. But her stock investment is constrained to follow a buy and hold strategy. It is well known that the constrained consumption and portfolio choice problem can be transformed into that in a fictitious economy with no constraint but with a modified market price of risk and interest rate[16]. In particular, the constraint can be written as $(\alpha_t, \theta_t) \in A_t$, where $A_t = \{(\alpha(t) \in \Re, \theta(t) = \left[\frac{\theta_2(0)}{S(0)} + \int_0^t \frac{x(s)}{S(s)}ds\right] S(t) = \theta_2(t)\}$. For $(\nu_0, \nu_-) \in \Re^2$, let

$$\Delta_t(\nu_0, \nu_-) = \sup_{(\alpha,\theta) \in A_t} -(\alpha\nu_0 + \theta\nu_-)$$

denote the support function of $-A_t$ and let

$$\tilde{A}_t = \{(\nu_0, \nu_-) \in \Re^2 : \Delta_t(\nu_0, \nu_-) < \infty\}$$

[16]See, for example, He and Pearson (1991), Cvitanic and Karatzas (1992), and Cuoco (1997).

denote its effective domain. It is easily verified that in the case of buy and hold constraint,

$$\tilde{A}_t = \tilde{A} = \{(0, \nu_-), \nu_- \in \Re\}.$$

In addition, let $\nu(t) = -\sigma(t)^{-1}\nu_-$, $\nu(t) \in \Re$,

$$\Delta_t(\nu_0, \nu_-) = -\theta_2(t)\nu_- = \theta_2(t)\sigma(t)\nu(t). \tag{8}$$

Moreover, agent 2's state price density is given by

$$\pi_2(t) = \beta_\nu(t)\xi_2(t),$$

where

$$\beta_\nu(t) = \exp\left(-\int_0^t (r(s) + \nu_0(s))ds\right) = \exp\left(-\int_0^t r(s)ds\right) = B(t)^{-1},$$

$$\xi_2(t) = \exp\left(-\int_0^t \kappa_\nu(s)dw(s) - \frac{1}{2}\int_0^t \kappa_\nu(s)^2 ds\right),$$

and

$$\kappa_\nu(t) = \sigma(t)^{-1}(\mu(t) - r(t) + \nu_-(t)) = \kappa(t) + \sigma(t)^{-1}\nu_-(t) = \kappa(t) - \nu(t). \tag{9}$$

By Itô's lemma, ξ_2 follows

$$d\xi_2(t) = -\kappa_\nu(t)\xi_2(t)dw(t). \tag{10}$$

Since the constraint is on agent 2's stock position, and not on her bond position, in the fictitious economy, only the market price of risk is affected, but not the interest rate.

B. Stochastic Welfare Weight, Interest Rate, Market Price of Risk, and Equity Premium

Consider the utility function of the representative agent

$$U(c; \lambda) = \mathrm{E}\left[\int_0^T e^{-\rho t}u(c(t), \lambda(t))dt\right],$$

where

$$u(c(t), \lambda(t)) = \max_{c_1 + c_2 = c} u_1(c_1) + \lambda(t) u_2(c_2). \tag{11}$$

Equation (11) implies the representative agent's marginal utility

$$u_c(\delta(t), \lambda(t)) = u_1'(c_1^*(t)) = \lambda(t) u_2'(c_2^*(t)). \tag{12}$$

The representative agent's marginal utility now has two sources of randomness: one comes from the aggregate dividend, the other comes from the stochastic welfare weight. By Equation (12) and the first order conditions of both agents, the stochastic weight is given by

$$\lambda(t) = \frac{u_1'(t)}{u_2'(t)} = \frac{\psi_1 \xi_1(t)}{\psi_2 \xi_2(t)}, \tag{13}$$

where ψ_1 and ψ_2 are the corresponding Lagrange multipliers. By Equations (7), (10), (9), (13), and Itô's lemma, the stochastic welfare weight follows the dynamics

$$d\lambda(t) = -\nu(t)(\kappa(t) - \nu(t))\lambda(t)dt - \nu(t)\lambda(t)dw(t). \tag{14}$$

Note that $\nu(t)$, the difference between the market prices of risk faced by the two agents, measures the volatility of the stochastic weight.

The first order condition of agent 1 is

$$e^{-\rho t} u_1'(c_1^*(t)) = \psi_1 \pi_1(t), \tag{15}$$

where

$$d\pi_1(t) = -r(t)\pi_1(t)dt - \kappa(t)\pi_1(t)dw(t), \tag{16}$$

Equations (12), (15), and (16) together yield

$$de^{-\rho t} u_c(t) = -r(t)e^{-\rho t} u_c(t)dt - \kappa(t)e^{-\rho t} u_c(t)dw(t). \tag{17}$$

On the other hand, by Itô's lemma,

$$de^{-\rho t}u_c(t) = e^{-\rho t}[(\mathcal{D}u_c(t) - \rho u_c(t))dt + (u_{cc}(t)\sigma_\delta(t) - u_{c\lambda}(t)\nu(t)\lambda(t))dw(t)], \tag{18}$$

where $\mathcal{D}u_c(t)$ denotes the drift of $u_c(t)$. Matching the drift and diffusion terms of Equation (17) and Equation (18) respectively, we obtain

$$r(t) = \rho - \frac{\mathcal{D}u_c(t)}{u_c(t)} \tag{19}$$

and

$$\nu(t) = \frac{u_{cc}(t)\sigma_\delta(t) + u_c(t)\kappa(t)}{u_{c\lambda}(t)\lambda(t)}. \tag{20}$$

Let

$$A_i(t) = -\frac{u_i''(c_i(t))}{u_i'(c_i(t))}, P_i(t) = -\frac{u_i'''(c_i(t))}{u_i''(c_i(t))}, i = 1, 2$$
$$A(t) = -\frac{u_{cc}(\delta(t), \lambda(t))}{u_c(\delta(t), \lambda(t))}, P(t) = -\frac{u_{ccc}(\delta(t), \lambda(t))}{u_{cc}(\delta(t), \lambda(t))}.$$

denote the absolute risk aversion and absolute prudence coefficient at time t for agent 1, 2, and the representative agent respectively. The following lemma formalizes the above discussion.

Lemma 1 *In the* ***buy and hold economy****, if equilibrium exists, the interest rate and the market price of risk are given respectively by*

$$r(t) = \rho + A(t)\mu_\delta(t) - \frac{1}{2}P_1(t)A(t)A_1(t)^{-2}\kappa(t)^2 - \frac{1}{2}P_2(t)A(t)A_2(t)^{-2}\kappa_\nu(t)^2 \tag{21}$$

and

$$\kappa(t) = A(t)[\sigma_\delta(t) + A_2(t)^{-1}\nu(t)]. \tag{22}$$

Equation (22) implies the equity premium is given by

$$\mu(t) - r(t) = A(t)\sigma(t)\sigma_\delta(t) + A(t)A_2(t)^{-1}\sigma(t)\nu(t). \tag{23}$$

The optimal consumption policies c_1 and c_2 follow the processes

$$dc_1(t) = [A_1(t)^{-1}(r(t) - \rho) + \frac{1}{2}P_1(t)A_1(t)^{-2}\kappa(t)^2]dt + A_1(t)^{-1}\kappa(t)dw(t). \tag{24}$$

$$dc_2(t) = [A_2(t)^{-1}(r(t) - \rho) + \frac{1}{2}P_2(t)A_2(t)^{-2}\kappa_\nu(t)^2]dt + A_2(t)^{-1}\kappa_\nu(t)dw(t). \tag{25}$$

Remark 1 *Equation (23) shows that the equity premium consists of two terms. The first term is the traditional Consumption CAPM term of covariation between stock return and consumption growth. In addition, there is a second term related to the covariation between stock return and the stochastic weight λ. (Recall from Equation (14) that ν measures how volatile λ is.) In a complete market, the weight λ is deterministic. The only source of randomness in the representative agent's marginal utility is from the fluctuation of aggregate consumption. With an incomplete market, λ is stochastic and its fluctuation results in an additional source of randomness in the representative agent's marginal utility (See Equation (12)). Whether the equity premium is higher or lower than that predicted by the Consumption CAPM depends critically on the sign of the shadow process ν.*

Remark 2 *Similarly, Equation (22) shows that whether the Sharpe ratio in the buy and hold economy is higher or lower than that in the complete market economy (which equals $A(t)\sigma_\delta(t)$) depends on the sign of the shadow process ν.*

C. Agent 1's Wealth and Stock Return Volatility

We look for an equilibrium in which the state variables (δ, λ, η) follow a joint Markov process. In such an equilibrium, agent 1's wealth W_1 and the stock price S must be

deterministic functions of the state variables and time: $W_1(t) = J(\delta(t), \lambda(t), \eta(t), t)$ and $S(t) = F(\delta(t), \lambda(t), \eta(t), t)$ for some functions J and F. J and F are assumed to be continuously differentiable with respect to t and η, and twice continuously differentiable with respect to δ and λ.[17] Agent 1's wealth is defined by

$$J(\delta(t), \lambda(t), \eta(t), t) = \pi_1(t)^{-1} \mathrm{E}_t \left[\int_t^T \pi_1(s) c_1(s) ds \right]. \tag{26}$$

The stock return volatility $\sigma(t)$ is then related to $J(t)$ and $F(t)$ by the following lemma:

Lemma 2 *If there exists an equilibrium, the stock return volatility is given by*

$$\sigma(t) = \frac{F_\delta(t)\sigma_\delta(t) - F_\lambda(t)\nu(t)\lambda(t)}{F(t)}, \tag{27}$$

where the shadow process ν satisfies

$$\nu(t) = \frac{[(1-\eta(t))F_\delta(t) - J_\delta(t)]\sigma_\delta(t)}{[(1-\eta(t))F_\lambda(t) - J_\lambda(t)]\lambda(t)}. \tag{28}$$

From now on, to keep notations at a minimum, we suppress the time subscripts whenever possible.

D. Equilibrium Consumption Allocation

Equation (13) and the CRRA assumption on agents' preferences yield

$$\lambda = \frac{c_1^{-\gamma_1}}{c_2^{-\gamma_2}}. \tag{29}$$

Combining Equation (29) with the resource constraint

$$c_1 + c_2 = \delta, \tag{30}$$

it follows that both c_1 and c_2 are functions of δ and λ only. In particular, if we write $c_1 = h(\delta, \lambda)$, the following lemma holds:

[17] We use F_δ and $F_{\delta\delta}$ to denote $\frac{\partial F}{\partial \delta}$ and $\frac{\partial^2 F}{\partial \delta^2}$ respectively.

Lemma 3 *Assuming equilibrium exists, the consumption allocations are characterized by*

$$c_1 = h(\delta, \lambda) \text{ and } c_2 = \delta - h(\delta, \lambda),$$

where

$$h : (0, +\infty) \times (0, +\infty) \to (0, \delta) \text{ and } h(\delta, \lambda) + \lambda^{\frac{1}{\gamma_2}} h(\delta, \lambda)^{\frac{\gamma_1}{\gamma_2}} = \delta. \tag{31}$$

Remark 3 *Equation (31) can be easily solved numerically. Notice that function*

$$f(x) = x + \lambda^{\frac{1}{\gamma_2}} x^{\frac{\gamma_1}{\gamma_2}}$$

increases monotonically on $[0, \delta]$*. Moreover, since* $f(0) = 0$ *and* $f(\delta) > \delta$ *(because* $\lambda > 0$*),* x *exists and is unique.*

E. Agents' Stock, Bond Investments and Total Wealth

Each agent's stock and bond investments and total wealth in equilibrium are given in the following lemma:

Lemma 4 *If equilibrium exists, the amounts invested in the bond by agents are given respectively by*

$$\begin{aligned} \alpha_1(t) &= J(t) - (1 - \eta(t))F(t), \\ \alpha_2(t) &= (1 - \eta(t))F(t) - J(t). \end{aligned}$$

The amounts invested in the stock by agents are given respectively by

$$\begin{aligned} \theta_1(t) &= (1 - \eta(t))F(t), \\ \theta_2(t) &= \eta(t)F(t). \end{aligned}$$

Agents' total wealth is given by

$$\begin{aligned} W_1(t) &= J(t), \\ W_2(t) &= F(t) - J(t). \end{aligned} \tag{32}$$

The proof is by definition and straight forward, therefore omitted for brevity.

F. PDEs for Agent 1's Optimal Wealth and the Stock Price

The previous four lemmas express the equilibrium interest rate, market price of risk, equity premium, stock return volatility, consumption allocations, agents' trading strategies and total wealth all in terms of the state variables, partial derivatives of agent 1's optimal wealth, and partial derivatives of the stock price. In this section, agent 1's optimal wealth and the stock price are presented as solutions of two coupled partial differential equations.

Theorem 1 *If there exists an equilibrium, agent 1's optimal wealth J solves the following PDE,*

$$\mathcal{D}J - rJ - \kappa\sigma(1-\eta)F + c_1 = 0, \tag{33}$$

with terminal condition

$$J(\delta, \lambda, \eta, T) = 0,$$

where

$$\mathcal{D}J = J_\delta \mu_\delta - J_\lambda(\kappa - \nu)\nu\lambda + J_\eta x F^{-1} + J_t + \frac{1}{2}[J_{\delta\delta}\sigma_\delta^2 + J_{\lambda\lambda}\nu^2\lambda^2] - J_{\delta\lambda}\sigma_\delta\nu\lambda,$$

κ is given by Equation (22) and ν is given by Equation (28).

Similarly, the stock price is also given by a PDE:

Theorem 2 *Assuming equilibrium exists, the stock price F solves the following PDE,*

$$\mathcal{D}F - rF - \kappa\sigma F + \delta = 0, \tag{34}$$

with terminal condition

$$F(\delta, \lambda, \eta, T) = 0,$$

where

$$\mathcal{D}F = F_\delta \mu_\delta - F_\lambda(\kappa - \nu)\nu\lambda + F_\eta x F^{-1} + F_t + \frac{1}{2}[F_{\delta\delta}\sigma_\delta^2 + F_{\lambda\lambda}\nu^2\lambda^2] - F_{\delta\lambda}\sigma_\delta\nu\lambda.$$

By now, we have expressed all terms in the two PDEs in terms of only the state variables — (δ, λ, η), t, J, F, J and F's first-order partial derivatives. The plan is to solve for J and F numerically. All the other variables in the model, κ, ν, μ, σ, r, can be expressed in terms of only the state variables — (δ, λ, η), time t, J, F, J's and F's first-order partial derivatives. Therefore, once J and F are known, the rest can be easily computed. The system of coupled second order PDEs on J and F is then solved numerically using the numerical methodology described in the appendix.[18] Next, we verify that the equilibrium characterized by the four lemmas and two theorems indeed meet other equilibrium requirements.

G. Optimality of Agent 2's Stock Investment under Constraint

First, we show analytically that θ_2 is optimal for agent 2 under the buy and hold constraint.

[18] We note that our system of coupled PDEs does not fall into a class whose existence of solutions has been formally shown. In fact, the solution of very few continuous-time equilibrium models with frictions have been formally proven for existence. Existence has been shown for the Basak and Cuoco (1998) model, which is a special case of our model when both the initial holding and contribution rate of the stock is set to zero for the constrained agent. Although it is far from a proof of existence, we have conducted extensive numerical analysis of our model for a wide-range of parameters with no evidence of non-existence. Sections IV and V include a subset of this analysis.

It follows from Equation (66) and the martingale representation theorem that there exists a process φ_2 with $\int_0^T |\varphi_2(t)|^2 dt < \infty$ a.s. such that

$$\pi_2(t)W_2(t) + \int_0^t \pi_2(s)[c_2(s) - \Delta(\nu(s))]ds = W_2(0) + \int_0^t \varphi_2(s)dw(s). \tag{35}$$

Compare the diffusion terms on both sides of Equation (35) and use

$$W_2(t) = F - W_1(t) = F - J, \tag{36}$$

we get

$$\varphi_2 = \pi_2 \left[\sigma F - \sigma_J + \kappa_\nu W_2\right] = \pi_2 \left[\sigma\theta_2 + \kappa_\nu W_2\right],$$

where the second equality follows from Equation (60). Solving for θ_2:

$$\theta_2 \;=\; \sigma^{-1}(\pi_2^{-1}\varphi_2 - \kappa_\nu W_2),$$

which is the optimality condition we am looking for.

H. Trading Strategies and the Financing of Agents' Consumption Plans

In this section, we verify that agent 1's trading strategy (α_1, θ_1) finances her consumption plan c_1.

Equations (60) and (33) imply

$$\begin{aligned} dJ &= (rJ - \kappa\sigma\theta_1)dt - c_1 dt + \sigma\theta_1 dw(t) \\ &= [(J - \theta_1)r + \theta_1\mu]dt - c_1 dt + \sigma\theta_1 dw(t) \end{aligned} \tag{37}$$

Since

$$\alpha_1 = J - \theta_1,$$

Equation (37) shows that c_1 is indeed financed by (α_1, θ_1). In addition, it is easy to see that the representative agent's consumption is financed by aggregate dividend. Therefore, since we have proved that all markets clear, agent 2's trading strategy must finance c_2.

I. Expected Consumption Growth Rates and Volatilities

Agents' consumption processes satisfy:

$$dc_i^*(t) = \mu_{c_i^*}(t)c_i^*(t)dt + \sigma_{c_i^*}(t)c_i^*(t)dw(t),\ i = 1, 2$$

where Equations (1), (24), (45), and Itô's Lemma imply

$$\begin{aligned}
\mu_{c_1^*}(t) &= \bar{\mu} + (\bar{\sigma} + \kappa)(2\kappa + \nu), \\
\mu_{c_2^*}(t) &= \bar{\mu} - \frac{(\bar{\sigma} + \kappa)(2\kappa + \nu)}{\lambda}, \\
\sigma_{c_1^*}(t) &= -\kappa,
\end{aligned}$$

and

$$\sigma_{c_2^*}(t) = \bar{\sigma} + \frac{(\bar{\sigma} + \kappa)}{\lambda}. \tag{38}$$

To facilitate comparisons across economies, we also reiterate the corresponding results for the complete market economy and limited participation economy. In the complete market economy,

$$\begin{aligned}
\mu_{c_1^*}(t) &= \mu_{c_2^*}(t) = \bar{\mu}, \\
\sigma_{c_1^*}(t) &= \sigma_{c_2^*}(t) = \bar{\sigma}.
\end{aligned}$$

While in the limited participation economy,

$$\begin{aligned}
\mu_{c_1^*}(t) &= \bar{\mu} + (\lambda + \lambda^2)\bar{\sigma}^2, \\
\mu_{c_2^*}(t) &= \bar{\mu} - (1 + \lambda)\bar{\sigma}^2, \\
\sigma_{c_1^*}(t) &= (1 + \lambda)\bar{\sigma},
\end{aligned}$$

and

$$\sigma_{c_2^*}(t) = 0.$$

III. The Limited Participation Economy and the Complete Market Economy

For comparison purposes, the next two corollaries collects the results on the equilibrium characterizations of 1) an otherwise identical complete market economy, where both agents can trade the bond and the stock dynamically, and 2) a limited stock market participation economy where the non-stockholder can only trade the bond. The third corollary compares the interest rate in a complete market economy with that in a buy and hold economy.

Corollary 1 *In the* ***limited participation economy***[19] *, the equilibrium is characterized by*

$$r(t) = \rho + A(t)\mu_\delta(t) - \frac{1}{2}A(t)P_1(t)\sigma_\delta(t)^2,$$

$$\kappa(t) = A_1(t)\sigma_\delta(t), \tag{39}$$

$$\nu(t) = \kappa(t) = A_1(t)\sigma_\delta(t), \tag{40}$$

$$\sigma(t) = \frac{F_\delta(t)\sigma_\delta(t) - F_\lambda(t)\nu(t)\lambda(t)}{F(t)},$$

$$\mu(t) - r(t) = A_1(t)\sigma(t)\sigma_\delta(t).$$

[19]In the limited participation economy of Basak and Cuoco (1998), the non-stock holder is restricted to having log preference to obtain closed form solution. The limited participation model considered here allows both agents to have arbitrary CRRA coefficients. Moreover, we solve for the equilibrium volatility endogenously.

The stock price $F_{LP}(\delta, \lambda, t)$ satisfies the following PDE:

$$\mathcal{D}F - rF - \kappa\sigma F + \delta = 0, \tag{41}$$

with terminal condition

$$F(\delta, \lambda, T) = 0,$$

where

$$\mathcal{D}F \;=\; F_\delta \mu_\delta + F_t + \frac{1}{2}[F_{\delta\delta}\sigma_\delta^2 + F_{\lambda\lambda}\nu^2\lambda^2] - F_{\delta\lambda}\sigma_\delta\nu\lambda.$$

Agents' optimal consumption processes are

$$dc_1(t) = A(t)[A_1(t)^{-1}\mu_\delta(t) + \frac{1}{2}P_1(t)A_2(t)^{-1}\sigma_\delta(t)^2]dt + \sigma_\delta(t)dw(t),$$

and

$$dc_2(t) = A(t)A_2(t)^{-1}[\mu_\delta(t) - \frac{1}{2}P_1(t)\sigma_\delta(t)^2]dt.$$

Corollary 2 *In the* ***complete market economy***[20]*, the equilibrium is characterized by*

$$r(t) = \rho + A(t)\mu_\delta(t) - \frac{1}{2}A(t)P(t)\sigma_\delta(t)^2,$$

$$\kappa(t) = A(t)\sigma_\delta(t),$$

$$\nu(t) = 0,$$

$$\sigma(t) = \frac{F_\delta(t)\sigma_\delta(t)}{F(t)},$$

[20]The complete market economy set-up is identical to that of Basak and Cuoco (1998). Again, we numerically solve for the equilibrium volatility endogenously.

$$\mu(t) - r(t) = A(t)\sigma(t)\sigma_\delta(t).$$

The stock price $F_{CM}(\delta, \lambda, t)$ *satisfies the following PDE:*

$$\mathcal{D}F - rF - \kappa\sigma F + \delta = 0, \tag{42}$$

with terminal condition

$$F(\delta, \lambda, T) = 0,$$

where

$$\mathcal{D}F = F_\delta \mu_\delta + F_t + \frac{1}{2} F_{\delta\delta}\sigma_\delta^2.$$

Agents' optimal consumption processes are

$$\begin{aligned} dc_i(t) &= [A(t)A_i(t)^{-1}\mu_\delta(t) - \frac{1}{2}A(t)A_i(t)^{-1}P(t)\sigma_\delta(t)^2 + \frac{1}{2}A(t)^2 A_i(t)^{-2}P_i(t)\sigma_\delta(t)^2]dt \\ &\quad + A(t)A_i(t)^{-1}\sigma_\delta(t)dw(t), \quad i = 1, 2. \end{aligned}$$

The next corollary compares the interest rate in the complete market economy with that in the buy and hold economy.

Corollary 3 *If agents have homogenous constant relative risk aversion coefficients, for a given pair of* (δ, λ)*, the interest rate in the buy and hold economy is bounded from above by the interest rate in the complete market economy. They are equal if and only if* $\nu(\delta, \lambda, \eta, t) = 0$.

IV. Effects of Buy and Hold Investors on the Equilibrium (the Case of Logarithmic Preferences)

To study the impact of buy and hold investors, in this section we specialize our economy to both agents having logarithmic preferences and the dividend process following geometric

Brownian motion (GBM). (In the calibration section, we will allow both agents to have relative risk aversion different from one.). In particular, we compare the buy and hold (BH) economy with the complete market (CM) economy and the limited participation (LP) economy. The latter two are studied in detail by Basak and Cuoco (1998), so that we directly invoke their results as benchmarks for comparison. The BH model is also calibrated using their parameters, which are taken to match the data in Mehra and Prescott (1985). In particular, the mean and volatility for consumption growth rate are set to $\bar{\mu} = 0.0183$ and $\bar{\sigma} = 0.0357$, respectively. The time preference parameter ρ is set to 0.001. In addition, we assume the buy and hold investor's contribution rate into the stock $x(t)$ is given by $\bar{x}\delta(t)$, where $\bar{x}$ is a constant. Table 1 indicates that the ratio between average contribution given participation and per capita consumption for the whole population seems quite stable over the years. The constant $\bar{x}$ is set to 12% to match the data. Department of Labor data indicates that retirement contribution is typically 20% as large as per capita consumption (See Table 1). In addition, Exhibit 8 of Ameriks and Zeldes (2000) shows that on average, about 60% of contribution flow goes to stocks. Therefore, $\bar{x} \approx 20\% \times 60\% = 12\%$. Note that $x(t)$ is exogenous. The purpose of the proportional assumption is to get an idea of the relative size of the contribution rate for model calibration. We are not taking a stand on the functional form of the contribution rate. The model can easily accommodate more sophisticated functional forms for $x(t)$. More importantly, most of the results are driven by the "holding" rather than the "buying" of the stock by the restricted agent as the last section on intuitions will demonstrate. Given the relative small size of the contribution rate, the "buying" part is insignificant relative to the "holding" part in delivering our results.

A. Equilibrium in the Log Case with GBM Dividend

Specializing the results in Section III, we obtain the following characterization of asset price dynamics and optimal consumption allocations.

Corollary 4 *When both agents in the* ***buy and hold economy*** *have* ***logarithmic*** *preferences and aggregate dividend follows GBM:*

$$d\delta(t) = \bar{\mu}\delta(t)dt + \bar{\sigma}\delta(t)dw(t), \tag{43}$$

if there exists an equilibrium, the consumption allocations are given by

$$c_1(t) = \frac{\delta(t)}{1+\lambda(t)}, c_2(t) = \frac{\lambda(t)\delta(t)}{1+\lambda(t)}. \tag{44}$$

The weighting process λ follows

$$d\lambda(t) = \frac{(1+\lambda(t))(\bar{\sigma}-\kappa(t))[(1+\lambda(t))\bar{\sigma}-\kappa(t)]}{\lambda(t)}dt + (1+\lambda(t))(\bar{\sigma}-\kappa(t))dw(t). \tag{45}$$

The interest rate and market price of risk are characterized respectively by

$$r(t) = \rho + \bar{\mu} - \bar{\sigma}^2 - \frac{(\bar{\sigma}-\kappa(t))^2}{\lambda(t)} \tag{46}$$

and

$$\kappa(t) = \bar{\sigma} + \frac{\nu(t)\lambda(t)}{1+\lambda(t)}. \tag{47}$$

The equity premium is given by

$$\mu(t) - r(t) = \sigma(t)\bar{\sigma} + \sigma(t)\frac{\nu(t)\lambda(t)}{1+\lambda(t)} = cov_t(\frac{dS}{S}, \frac{d\delta}{\delta}) - \frac{1}{1+\lambda(t)}cov_t(\frac{dS}{S}, d\lambda). \tag{48}$$

Agents' optimal consumptions follow

$$dc_i^*(t) = \mu_{c_i^*}(t)c_i^*(t)dt + \sigma_{c_i^*}(t)c_i^*(t)dw(t),\ i = 1, 2$$

where

$$\mu_{c_1^*}(t) = \bar{\mu} + (\bar{\sigma} - \kappa)(-2\kappa + \nu),$$
$$\mu_{c_2^*}(t) = \bar{\mu} - \frac{(\bar{\sigma} - \kappa)(-2\kappa + \nu)}{\lambda},$$
$$\sigma_{c_1^*}(t) = \kappa,$$
$$\sigma_{c_2^*}(t) = \bar{\sigma} + \frac{(\bar{\sigma} - \kappa)}{\lambda}.$$

To facilitate comparisons across economies, we also re-iterate the corresponding results for the complete market economy and limited participation economy. In the ***complete market economy****,*

$$\mu_{c_1^*}(t) = \mu_{c_2^*}(t) = \bar{\mu},$$
$$\sigma_{c_1^*}(t) = \sigma_{c_2^*}(t) = \bar{\sigma}.$$

While in the ***limited participation economy****,*

$$\mu_{c_1^*}(t) = \bar{\mu} + (\lambda + \lambda^2)\bar{\sigma}^2,$$
$$\mu_{c_2^*}(t) = \bar{\mu} - (1 + \lambda)\bar{\sigma}^2,$$
$$\sigma_{c_1^*}(t) = (1 + \lambda)\bar{\sigma},$$
$$\sigma_{c_2^*}(t) = 0.$$

B. Effects on Equilibrium Asset Price Dynamics (the Log Case)

Basak and Cuoco (1998) show that with logarithmic preferences, in both the complete market economy and limited participation economy, the expected return of the stock $\mu_S^{CM/LP}$ equals $\rho + \bar{\mu}$, and the volatility of the stock $\sigma_S^{CM/LP}$ equals $\bar{\sigma}$. In other words, restricting agent 2 from investing in the stock has no impact on either the expected return or the volatility of the stock.

Figure 1(a) shows that stock return volatility in the buy and hold economy can be either higher or lower than that in the complete market economy or the limited participation economy. When the buy and hold investor's stock holding η is held constant at small values, increasing her consumption share λ leads to higher stock return volatility. But with log utility, the effect is small.

The effect of the buy and hold investor on expected stock return is similar to that on stock return volatility, as shown in Figure 1(b). Expected stock return in the buy and hold economy can be either higher or lower than that in the complete market economy or the limited participation economy. Increasing the buy and hold investor's consumption share λ leads to higher expected stock return. Again, the change is small with log utility, .

Figure 1(c) compares the Sharpe ratio in the buy and hold economy with those in the complete market and the limited participation economy. Note that with log preference, agent 1 (the dynamic asset allocator)'s fraction of wealth invested in the stock is $\sigma^{-1}\kappa$. As η, the fraction of stock held by agent 2 (the buy and hold investor), increases from 0 to 1, agent 1 optimally chooses to hold less stock. Since the change in σ is small, the Sharpe ratio, given by κ, must decrease to induce her to invest less in stock. As η reaches 1, the Sharpe ratio declines to zero and agent 1 optimally choose to hold zero amount in the stock. In addition, holding η constant, the Sharpe ratio increases as the buy and hold investor's consumption share increases. The same intuition also applies to the equity premium, which is obtained by multiplying the Sharpe ratio by the stock return volatility. Figure 1(d) plots a similar picture for the equity premium as that for the Sharpe ratio.

Figure 2(a) shows that the interest rate in the buy and hold economy r^{BH} is uniformly lower than that in the complete market economy, given by $r^{CM} = \rho + \bar{\mu} - \bar{\sigma}^2$. This is obviously seen from Equation (46). In fact, Corollary 3 states that r^{BH} is bounded from above by r^{CM}

as long as agents have the same relative risk aversion coefficient. In the complete market economy with homogenous agents, there is no borrowing or lending between agents. In the corresponding economy with buy and hold investors, as η decreases, Sharpe ratio increases, inducing agent 1 to invest more in the stock by borrowing, the interest rate decreases. By substituting Equation (47) into Equation (46), we see that r^{BH} is quadratic in ν. Moreover, r^{BH} attains its maximum at $\nu = 0$. The maximum value is equal to the interest rate in the corresponding complete market economy. Figure 2(b) indicates that ν is approximately linear in η given a particular consumption share λ. Therefore r appears close to quadratic in η. In addition, when η, the fraction of the stock held by the buy and hold investor is zero, r^{BH} is very close to $r^{LP} = \rho + \bar{\mu} - (1 + \lambda(t))\bar{\sigma}^2$, the interest rate in the corresponding limited participation economy. Moreover, holding η at fixed small values, the higher the λ (consumption share for the buy and hold investor), the lower the interest rate.

C. Effects on Expected Consumption Growth and Volatility

Figures 3(a) and 3(b) show that in the economy with buy and hold investors, the higher the consumption share claimed by the buy and hold investor (represented by λ), the higher the unrestricted agent's expected consumption growth rate and volatility. Moreover, the unrestricted agent's expected consumption growth rate and volatility will exceed those in the complete market case if she holds more stock than she would hold in the corresponding complete market economy. When $\eta = 0$, the unrestricted agent holds all of the stock and her expected consumption growth rate and volatility are close to those in the limited participation economy.

Figures 3(c) and 3(d) show that the buy and hold investor's expected consumption growth rate and volatility start off close to the non-stock holder case in the limited participation

economy at $\eta = 0$ and gradually increase as she holds more shares. Moreover, her expected consumption growth rate and volatility will exceed those in the complete market case if she holds more stock than she would hold had she faced a complete market.

V. The Effects of Buy and Hold Investors on the Equilibrium (the CRRA Model)

In this section we allow both agents to have constant relative risk aversion coefficients different from one. We demonstrate that the qualitative features of the model discussed in the previous section carries over to the more general model. A simple calibration with relative risk aversion coefficients $\gamma_1 = 1$ and $\gamma_2 = 3$ suggests that the model with buy and hold investors can match the data better than both the complete market model and the limited participation model.

First we look at the complete market case. If we calibrate the complete market model such that the real interest rate is reasonable at below 3% as shown in Figure 4(a), but then the Sharpe ratio is too low (Figure 4(b)). If we increase agents' relative risk aversion coefficients to increase the Sharpe ratio, the interest rate becomes too high. The model cannot generate a low interest rate and a high Sharpe ratio at the same time. This is the well known equity premium/risk free rate puzzle. The reason behind this puzzle is obvious from Corollary 2. Both the interest rate and the Sharpe ratio respond positively to an increase in the representative agent's risk aversion. Increasing the Sharpe ratio inevitably increases the interest rate.

Next, we examine the limited participation model. Figures 5(a) and 5(b) show that the limited participation model can produce a low interest rate and a high Sharpe ratio at

λ around 4. As shown in Corollary 1, in the limited participation model, the Sharpe ratio responds only to agent 1's risk aversion. Therefore, we can use agent 1's relative risk aversion coefficient to match the Sharpe ratio and agent 2's relative risk aversion coefficient to match the risk free rate. However, as indicated by Figures 5(c) and 6(a), at λ around 4, the stock return volatility σ is too low while the volatility for the risk free rate is too high compared with those in the data. Consequently, the expected return is also low for the stock (Figure 5(d)).

Finally, Figures 7(a) and 7(b) show that the model with buy and hold investors at λ around 4 and η around 0.3 can generate both a low interest rate and a high Sharpe ratio.[21] In addition, it can produce stock return volatility more than twice that in the limited participation model and a higher expected return (Figures 7(c) and 7(d)). Note that the stock in this paper is a claim on aggregate consumption, rather than the market portfolio, which is the claim on aggregate dividend and represents *levered* equity. Given that aggregate dividend is about three times as volatile as aggregate consumption in the data, the 6% annual volatility for the claim on aggregate consumption in our model would translate into about 18% annual volatility for the market portfolio. See also Bansal and Yaron (2004). We decompose the stock return volatility into its two components:

$$\sigma = \frac{\partial \log F}{\partial \log \delta}\bar{\sigma} - \frac{\partial \log F}{\partial \log \lambda}\nu.$$

The first component is caused by the dividend (consumption) risk. The second component is related to the risk of randomly shifting wealth represented by the weighting process. The higher stock return volatility is mainly a result of the higher sensitivity of the stock price with respect to dividend risk, given by $\frac{\partial \log F}{\partial \log \delta}$, in the buy and hold economy (Figure 8(a)) than that in the corresponding limited participation economy at λ around 4 (Figure 8(b)).

[21] $\lambda = 4$ corresponds to a consumption share of 85% claimed by the buy and hold investor. η around 0.3 seems realistic given the fraction of stocks held by defined contribution plans.

To keep things simple, we consider the case where the contribution rate is zero, i.e., $x = 0$. If there is a negative shock to the dividend, in the absence of buy and hold constraint, the less risk averse agent 1 would sell some shares of the stock to the more risk averse agent 2, and the stock price drops to S_1 where agent 1 no longer wants to sell. With the buy and hold constraint, selling is impossible. Therefore, to reach equilibrium, the stock price has to drop further than S_1 such that agent 1 does not want to sell at such a low price. If there is a positive shock to the dividend, in the absence of buy and hold constraint, the less risk averse agent 1 would buy some shares of the stock from the more risk averse agent 2, and the stock price rises to S_2 where agent 1 no longer wants to buy. With the buy and hold constraint, buying is impossible. Therefore, to reach equilibrium, the stock price has to rise further than S_2 such that agent 1 does not want to buy at such a high price. The pressure to trade after a dividend shock becomes more severe as the buy and hold investor holds more shares of the stock, leading to the higher sensitivity of the stock price with respect to dividend risk.

Moreover, as shown in Figure 6(b), interest rate volatility in the buy and hold economy at λ around 4 and η around 0.3 is kept at 3%, which is within the reasonable range. The interest rate sensitivities to both dividend risk and the risk to randomly shifting wealth represented by the weighting process are comparable in both types of economies. Similarly we decompose the interest rate volatility into its two components:

$$\sigma_r = \frac{\partial \log r}{\partial \log \delta}\bar{\sigma} - \frac{\partial \log r}{\partial \log \lambda}\nu.$$

The lower interest rate volatility in the buy and hold economy compared to that in the limited participation economy is mostly due to the fact that ν, the volatility of the weighting process λ is higher in the corresponding limited participation economy at λ around 4 than in the buy and hold economy (Figures 9(a) and 9(b)).[22] This is because in the limited participation

[22] $\frac{\partial \log r}{\partial \log \lambda}$ is negative at λ around 4 and η around 0.3.

economy, a positive (negative) dividend shock increases (decreases) only the value of the stock owned by the stockholder (the non-stockholder owns no stock by definition), while in the buy and hold economy, such shock increases (decreases) the values of stock holdings of *both* agents, therefore having a smaller effect on the welfare weight process in the latter economy.

VI. Conclusion

This paper presents a continuous-time economy populated by dynamic asset allocators and buy and hold investors. A buy and hold investor effectively faces a portfolio constraint and therefore has state price density different from that of the dynamic asset allocators. The equilibrium is solved through the construction of a representative investor with stochastic weights assigned to the two types of agents. In equilibrium, the fraction of the stock held by the buy and hold investor emerges as an additional state variable to capture the information of the historical dividend and stock price. We characterize all equilibrium quantities as functions of the state variables. The main results of this paper are then presented by comparing the buy and hold economy with an otherwise identical complete market economy and a limited participation economy. A simple calibration of our model shows that unlike the complete market economy, the buy and hold economy can simultaneously produce a low interest rate and a high Sharpe ratio. In addition, the buy and hold economy can deliver stock return volatility more than twice that in the limited participation economy, because the stock price can potentially be more sensitive to dividend shocks in the buy and hold economy. Moreover, the buy and hold economy achieves higher stock return volatility while keeping interest rate volatility at reasonably low levels at the same time. This is due to the smaller impact of dividend shocks on the welfare weight in the buy and hold economy than

that in the limited participation economy as both types of agents hold stocks in the former.

VII. Appendices

A. Proofs

A.1. Proof of Lemma 1.

Define $f_1(\cdot)$ and $f_2(\cdot)$ to be the inverse function of $u_1'(\cdot)$ and $u_2'(\cdot)$ respectively. Then the identity

$$c_1 = f_1(u_1'(c_1)) \tag{49}$$

holds. Taking partial derivatives of Equation (49) with respect to c_1 yields

$$f_1'(u_1'(c_1)) = u_1''(c_1)^{-1} = -A_1^{-1}u_c(\delta,\lambda)^{-1}, \tag{50}$$

Taking partial derivatives of Equation (50) with respect to c_1 leads to

$$f_1''(u_1'(c_1)) = -u_1'''(c_1)u_1''(c_1)^{-3} = P_1A_1^{-2}u_c(\delta,\lambda)^{-2}. \tag{51}$$

Substituting Equation (12) into Equation (49) yields

$$c_1 = f_1(u_c(\delta,\lambda)). \tag{52}$$

Applying Itô's lemma to Equation (52) and using Equations (17), (50), and (51) lead to Equation (24).

The derivation of c_2's dynamics is essentially the same, except that

$$c_2 = f_2(\lambda^{-1}u_c(\delta,\lambda))$$

and Equation (14) is used.

Adding the drift terms of c_1 and c_2 then equating the sum to μ_δ yield the expression for the interest rate in Equation (21).

By definition,

$$f_1(u_c(\delta,\lambda)) + (f_2(\lambda^{-1}u_c(\delta,\lambda)) = \delta. \tag{53}$$

Differentiating both sides of Equation (53) with respect to δ and using Equation (50) and

$$f_2'(u_2'(c_2)) = u_2''(c_2)^{-1} \tag{54}$$

yield

$$u_2''(c_2)^{-1} = \lambda(u_1(c_1)'')^{-1}u_{cc}(\delta,\lambda)^{-1} \tag{55}$$

Differentiating both sides of Equation (53) with respect to λ and using Equation (55) lead to

$$u_{c\lambda}(\delta,\lambda) = \lambda^{-1}AA_2^{-1}u_c(\delta,\lambda). \tag{56}$$

Substituting Equation (56) into Equation (20) and re-arranging terms yield Equation (22). Equation (23) follows from Equation (22).

A.2. Proof of Lemma 2.

It follows from Equation (26) and the martingale representation theorem that there exists a process φ_1 with $\int_0^T |\varphi_1(t)|^2 dt < \infty$ a.s. such that

$$\pi_1(t)J(t) + \int_0^t \pi_1(s)c_1(s)ds = J(0) + \int_0^t \varphi_1(s)dw(s) \tag{57}$$

is a martingale. Matching the diffusion terms on both sides of Equation (57) we get

$$\sigma_J = \pi_1^{-1}\varphi_1 + \kappa J, \tag{58}$$

where σ_J denotes the diffusion of J. On the other hand, standard argument such as that in Cuoco (1997) suggests that agent 1's optimal investment in the stock is given by

$$\theta_1 = \sigma^{-1}(\pi_1^{-1}\varphi_1 + \kappa J). \tag{59}$$

Comparing Equation (58) and Equation (59) yields

$$\sigma_J = \sigma\theta_1 = \sigma(1-\eta)F. \tag{60}$$

Alternatively, σ_J can be written as

$$\sigma_J = J_\delta\sigma_\delta - J_\lambda\nu\lambda. \tag{61}$$

Putting equations (60) and (61) together produces

$$\sigma = \frac{J_\delta\sigma_\delta - J_\lambda\nu\lambda}{(1-\eta)F}. \tag{62}$$

Similarly, the diffusion of the stock price F, is given by

$$\sigma F = F_\delta\sigma_\delta - F_\lambda\nu\lambda. \tag{63}$$

Solving for σ from Equation (63) and equating the result to the expression for σ in Equation (62) leads to the following equation

$$J_\delta\sigma_\delta - J_\lambda\nu\lambda = (1-\eta)(F_\delta\sigma_\delta - F_\lambda\nu\lambda). \tag{64}$$

Solving ν from Equation (64), we obtain Equation (28).

A.3. Proof of Theorem 1.

The PDE for J is obtained by using the fact that the martingale in Equation (57) must have a zero drift.

A.4. Proof of Theorem 2.

Define W_2, the optimal wealth process[23] for agent 2 by

$$W_2(t) = \pi_2(t)^{-1} \mathrm{E}_t \left[\int_t^T \pi_2(s)[c_2(s) - \Delta_s(\nu(s))] ds \right], \tag{65}$$

where the function $\Delta_t(\nu) = \theta_2(t)\sigma(t)\nu(t)$ is given in Equation (8). It then follows from Equations (32) and (65) that the martingale

$$\pi_2(t)\left[F(t) - J(t)\right] + \int_0^t \pi_2(s)\left[c_2(s) - \theta_2(s)\sigma(s)\nu(s)\right] ds \tag{66}$$

must have a zero drift term, i.e.,

$$\mathcal{D}F - \mathcal{D}J - r(F - J) - (\kappa - \nu)\sigma\theta_2 + c_2 - \theta_2\sigma\nu = 0. \tag{67}$$

Substituting Equation (33) into Equation (67), we obtain the PDE for stock price F.

A.5. Proof of Corollary 1.

It follows from Basak and Cuoco (1998) that in limited participation economy,

$$\kappa_\nu(t) = 0, \;\; \nu(t) = -\kappa(t). \tag{68}$$

Substituting Equation (68) into Equation (22) yields Equation (39) and Equation (40). Further similar substitutions yield the remaining results.

A.6. Proof of Corollary 2.

In the complete market economy,

$$\kappa_\nu(t) = \kappa(t), \;\; \nu(t) = 0. \tag{69}$$

Substituting Equation (69) into Lemma 1 to 3, Theorem 1 and 2 yield the desired results.

[23] Cuoco (1997) proves that the optimal wealth of constrained agent are financed by associated trading strategies that satisfy agent's portfolio constraint.

A.7. Proof of Corollary 3.

Substituting Equation (22) into Equation (21) yields an expression for r^{BH} as a quadratic polynomial in ν with a negative coefficient for the quadratic term. The constant term of the polynomial is equal to r^{CM}. It is easy to see that the first order term vanishes if $\gamma_1 = \gamma_2$ and the polynomial achieves its minimum at $\nu = 0$.

A.8. Proof of Corollary 4.

Solving Lemma 3 with $\gamma_1 = \gamma_2 = 1$ directly yields Equation (44). The rest follows from substituting Equation (44) into Lemma 1 and Lemma 2.

B. Numerical Method

The two coupled PDEs in Equation (33) and Equation (34) are solved using the explicit finite difference method. This is a terminal value problem. Other finite difference schemes, such as the implicit method and the Crank-Nicolson method cannot be applied here because implicit methods require solving a system of linear difference equations. The PDEs we need to solve transform into a system of non-linear difference equations after discretization. To insure stability of the numerical solution, the time-spacing needs to be taken much smaller relative to the space-spacings. Heuristically speaking, if one starts with a stable solution, then increasing the number of points along the space dimensions by a factor of n requires increasing the number of points along the time dimension by a factor of n^2 to retain stability. Detailed discussion of stability conditions for explicit finite difference can be found in Press et. al. (1993).

First, Equation (34) can be rewritten as

$$F_t = -\mathcal{L}F(\delta, \lambda, \eta, t), \tag{70}$$

where $\mathcal{L}$ is an operator on F that satisfies

$$\mathcal{L}F(\delta, \lambda, \eta, t) =$$
$$F_\delta \mu_\delta - F_\lambda(\kappa - \nu)\nu\lambda + F_\eta \bar{x}\delta F^{-1} + \frac{1}{2}[F_{\delta\delta}\sigma_\delta^2 + F_{\lambda\lambda}\nu^2\lambda^2] - F_{\delta\lambda}\sigma_\delta\nu\lambda + \delta - rF + \kappa\sigma F.$$

Then we represent function $F(\delta, \lambda, \eta, t)$ by its values at the discrete set of points:

$$\delta(k) = k\Delta\delta, k = 0, 1, ..., K$$
$$\lambda(l) = l\Delta\lambda, l = 0, 1, ..., L$$
$$\eta(m) = m\Delta\eta, m = 0, 1, ..., M$$
$$t(n) = n\Delta t, n = 0, 1, ..., N$$

where the Δs are the grid spacings along each dimension. We discretize Equation (70) by using forward differencing along the time dimension and central differencing along the spatial dimension:

$$F_t(\delta, \lambda, \eta, t) = \frac{F(\delta, \lambda, \eta, t + \Delta t) - F(\delta, \lambda, \eta, t)}{\Delta t},$$
$$F_\delta(\delta, \lambda, \eta, t) = \frac{F(\delta + \Delta\delta, \lambda, \eta, t) - F(\delta - \Delta\delta, \lambda, \eta, t)}{2\Delta\delta},$$
$$F_{\delta\delta}(\delta, \lambda, \eta, t) = \frac{F(\delta + \Delta\delta, \lambda, \eta, t) - 2F(\delta, \lambda, \eta, t) + F(\delta - \Delta\delta, \lambda, \eta, t)}{(\Delta\delta)^2},$$
$$F_{\delta\lambda}(\delta, \lambda, \eta, t) = \frac{F_\delta(\delta, \lambda + \Delta\lambda, \eta, t) - F_\delta(\delta, \lambda - \Delta\lambda, \eta, t)}{2\Delta\lambda},$$
$$...$$

This way, we obtain a method of calculating F through iterating backward, starting at time T:

$$F(\delta, \lambda, \eta, T) = 0,$$
$$F(\delta, \lambda, \eta, t - \Delta t) = F(\delta, \lambda, \eta, t) + \Delta t\mathcal{L}F(\delta, \lambda, \eta, t),\ t = T, T - \Delta t, ..., \Delta t.$$

Additional boundary conditions:

$$F(0, \lambda, \eta, t) = 0,$$

$$F(+\infty, \lambda, \eta, t) = +\infty.$$

Once the value for F is known, its first- and second-order partial derivatives can be computed as shown above. The same calculation is carried out simultaneously for J. Since we have expressed all the equilibrium quantities, e.g. κ, ν, μ, σ, r, in terms of only the state variables — (δ, λ, η), t, J, F, J and F's first- and second-order partial derivatives, they are also readily obtained. Although we have only demonstrated the computation strategy for the buy and hold economy, similar methodology can be applied to solving equilibriums in the limited participation economy and complete market economy.

References

[1] Allen, F. and D. Gale, 1994, Limited Market Participation and Volatility of Asset Prices, *American Economic Review*, 84, 933-955.

[2] Ameriks, J. and S. Zeldes, 2004, How Do Household Portfolio Shares Vary with Age? working paper, Columbia University.

[3] Aiyagari, S. R., and M. Gertler, 1991, Asset Returns with Transaction Costs and Uninsured Individual Risk, *Journal of Monetary Economics*, 27, 311-331.

[4] Bansal, R. and A. Yaron, 2004, Risks For the Long Run: A Potential Resolution of Asset Pricing Puzzles, *Journal of Finance*, 59, 1481-1509.

[5] Basak, S. 1995, A General Equilibrium Model of Portfolio Insurance, *Review of Financial Studies*, 8, 1059-1090.

[6] Basak, S., 2000, A Model of Dynamic Equilibrium Asset Pricing with Heterogeneous Beliefs and Extraneous Risk, *Journal of Economic Dynamics and Control*, 24, 63-95.

[7] Basak, S. and D. Cuoco, 1998, An Equilibrium Model with Restricted Stock Market Participation, *Review of Financial Studies*, 11, 309-341.

[8] Cao, H., T. Wang, and H. Zhang, 2005, Model Uncertainty, Limited Market Participation, and Asset Prices, *Review of Financial Studies*, 18, 1219-1258.

[9] Choi, J., D. Laibson, B. Madrian, and A. Metrick, 2002, Defined Contribution Pensions: Plan Rules, Participant Choices, and the Path of Least Resistance, *Tax Policy and the Economy, MIT Press.*

[10] Cottle, S., and W.T. Whitman, 1950, Focmula Plans and the Institutional Investor, *Harvard Business Review*, 28, 84-96.

[11] Cox, J. and C.-f. Huang, 1989, Optimal Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process, *Journal of Economic Theory*, 49, 33-83.

[12] Cuoco, D., 1997, Optimal Consumption and Equilibrium Prices with Portfolio Constraints and Stochastic Income, *Journal of Economic Theory*, 72, 33-73.

[13] Cuoco, D. and H. He, 1994a, Dynamic Aggregation and Computation of Equilibria in Finite Dimensional Economies with Incomplete Financial Markets, working paper, University of Pennsylvania.

[14] Cuoco, D. and H. He, 1994b, Dynamic Equilibrium in Infinite Dimensional Economies with Incomplete Financial Markets, working paper, University of Pennsylvania.

[15] Cvitanic, J. and I. Karatzas, 1992, Convex Duality in Constrained Portfolio Optimization, *The Annals of Applied Probability*, 2, 767-818.

[16] Gormley, T., H. Liu, and G. Zhou, 2010, Limited Participation and Consumption-Saving Puzzles: A Simple Explanation and the Role of Insurance, *Journal of Financial Economics* 96, 331-344.

[17] Heaton, J., and D. J. Lucas, 1996, Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing, *Journal of Political Economy*, 104, 443-487.

[18] He, H., and N. Pearson, 1991, Consumption and Portfolio Policies with Incomplete Market and Short- Sale Constraint: The Infinite Dimensional Case, *Journal of Economic Theory*, 54, 259-304.

[19] Hull, J., and A. White, 1990, Valuing Derivative Securities Using the Explicit Finite Difference Method, *Journal of Financial and Quantitative Analysis*, 25, 87-100.

[20] Linnainmaa, J., 2005, Learning and Stock Market Participation, working paper, UCLA.

[21] Lucas R., 1978, Asset Prices in an Exchange Economy, *Econometrica*, 46, 1429-1445.

[22] Mankiw, N. and S. Zeldes, 1991, The Consumption of Stockholders and Non-Stockholders, *Journal of Financial Economics*, 29, 97-112.

[23] Mehra, R. and E. Prescott, 1985, The Equity Premium: A Puzzle, *Journal of Monetary Economics*, 15, 145-161.

[24] Merton, R. 1971, Optimal Consumption and Portfolio Rules in a Continuous Time Model, *Journal of Economic Theory*, 3, 373-413.

[25] Poterba, J., S. Venti and D. Wise, 2001, Pre-retirement Cashouts and Foregone Retirement Saving: Implications for 401(k) Asset Accumulation, D. Wise, ed., *Themes in the Economics of Aging*, 23-56.

[26] Press, W., B. Flannery, S. Teukolsky, and W. Vetterling. 1993, *Numerical Recipes in C*, (2nd ed.) Cambridge: Cambridge University Press.

[27] Saito, M., 1996, Limited Participation and Asset Pricing, working paper, University of British Columbia.

[28] Samuelson, P., 1994, The Long-Term Case for Equities, *Journal of Portfolio Management*, 21, 15-24.

[29] Samuelson, W. and R. Zeckhauser, 1998, Status Quo Bias in Decision Making, *The Journal of Risk and Uncertainty*, 1(1), 7-59.

[30] Schwert, G., 1989, Why Does Stock Market Volatility Change Over Time, *Journal of Finance*, 44, 1115-1153.

[31] Shapiro, A., 2002, The Investor Recognition Hypothesis in a Dynamic General Equilibrium: Theory and Evidence, *Review of Financial Studies*, 15(1), 97-141.

[32] Shiller, R., 1981, Do Stock Prices Move Too Much to Be Justified by Subsequent Changes in Dividends? *American Economic Review*, 71, 421-436.

[33] Storesletten, K., C. Telmer, A. Yaron, 2007, Asset Pricing with Idiosyncratic Risk and Overlapping Generations, Review of Economic Dynamics, 10(4), 519-548.

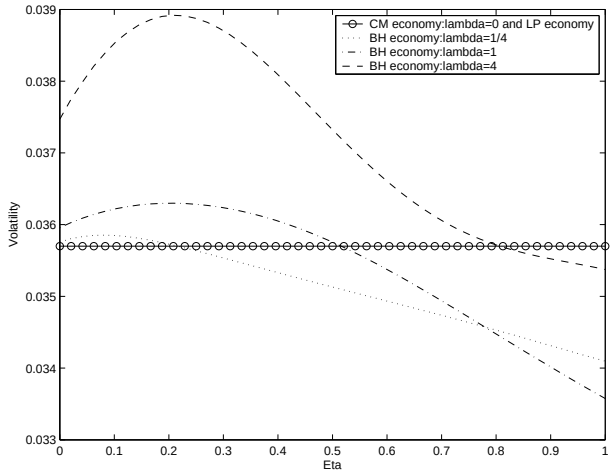
[34] Statman, M., 1995, Behavioral Framework for Dollar-Cost Averaging, *Journal of Portfolio Management*, 22, 70-78.

[35] Weil, P., 1989, The Equity Premium Puzzle and the Risk free Rate Puzzle, *Journal of Monetary Economics*, 24, 401-421.

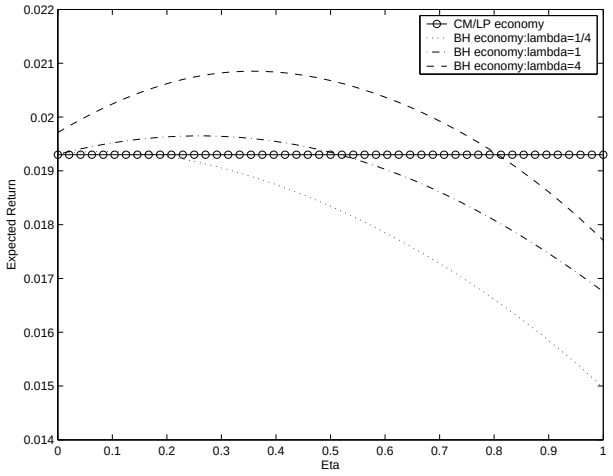
Table 1: **401(k) Contribution as a Percentage of Personal Consumption (1989-1998)**

The table displays the relationship between the average 401(k) plan contribution and the corresponding personal consumption data for that year. Consumption data is from NIPA (National Income and Product Accounts Tables), CPI is obtained from CRSP, and 401(k) contribution data is from the Department of Labor. The first column is real personal consumption of non-durable goods and services measured in 1998 dollars, the second column is the average annual personal contribution into 401(k) accounts given participation, also measured in 1998 dollars. The last column indicates the ratio between contribution and consumption.

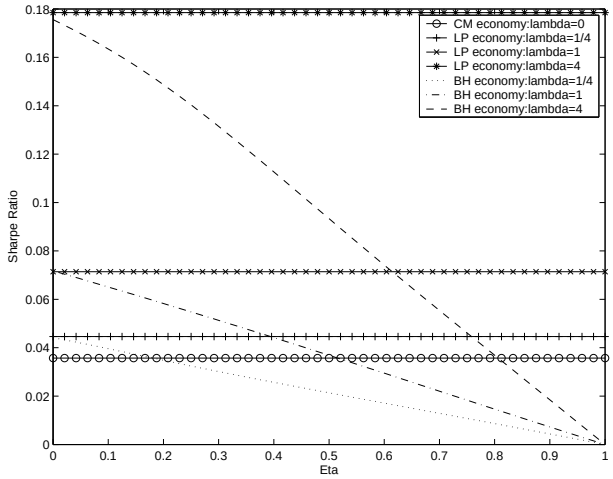
Year	Consumption	Contribution	Contribution/Consumption
1989	16,439	3,455	0.21
1990	16,484	3,070	0.19
1991	16,553	3,202	0.19
1992	16,824	3,317	0.20
1993	17,038	3,368	0.20
1994	17,284	3,296	0.19
1995	17,555	3,326	0.19
1996	17,721	3,484	0.20
1997	18,205	3,471	0.19
1998	18,709	3,628	0.19



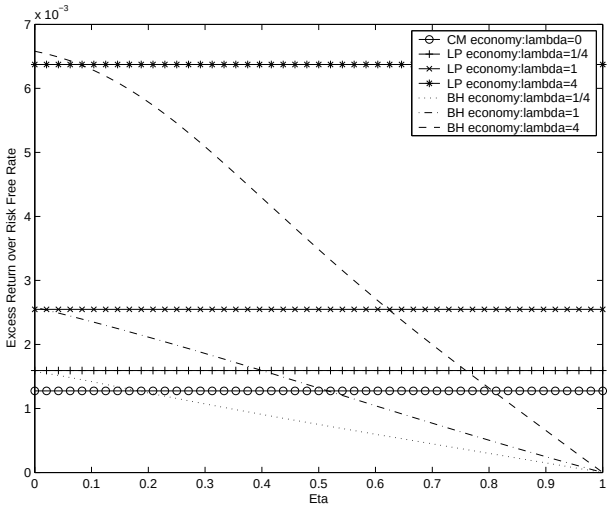
(a) Stock return volatility



(b) Expected return

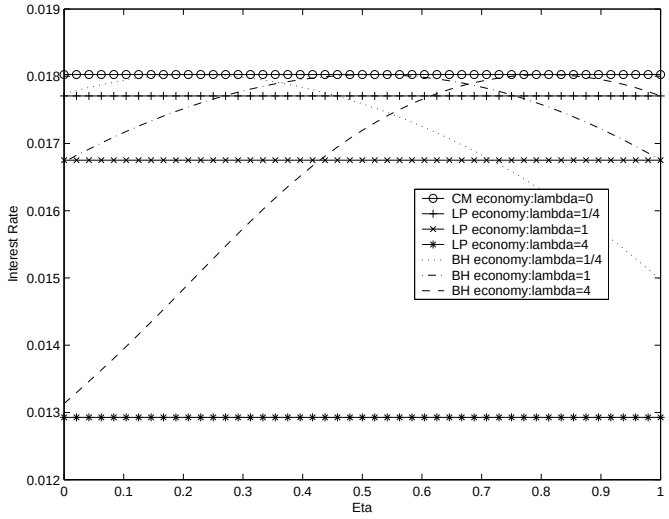


(c) Sharpe ratio

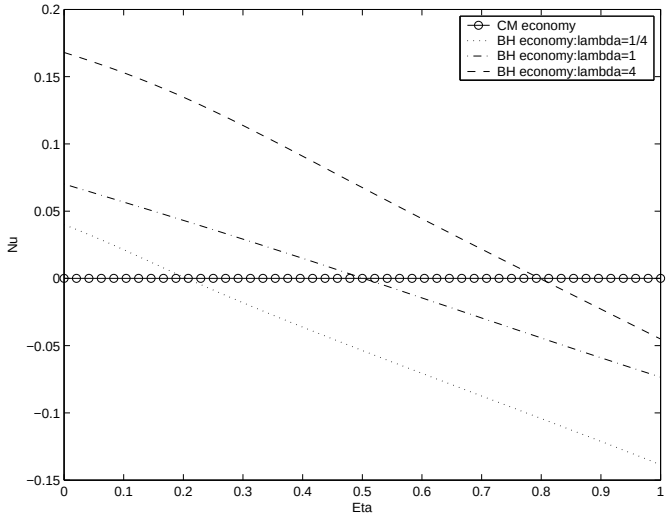


(d) Equity premium

Figure 1: **Stock return volatility, expected return, Sharpe ratio and equity premium in the buy and hold economy vs that in the complete market/limited participation economy (the log case).** The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. In Panels (a) and (b), the circled line is the case for complete market economy and limited participation economy. In Panels (c) and (d), the line with '$*$' is for limited participation economy with $\lambda = 4$. The line with '$\times$' is for limited participation economy with $\lambda = 1$. The line with '|' is for limited participation economy with $\lambda = 1/4$. The dashed line is for buy and hold economy with $\lambda = 4$. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. Parameter values of the economy: agent's relative risk aversion γ_1=γ_2=1, impatience parameter ρ=0.001, aggregate consumption volatility $\bar{\sigma}$=0.0357, expected aggregate consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

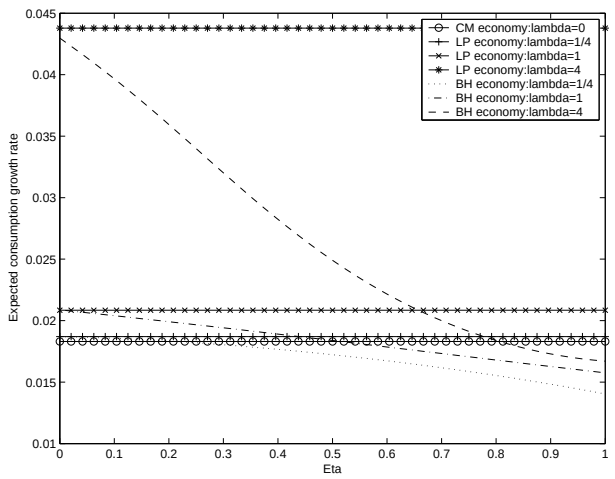


(a) Interest rate

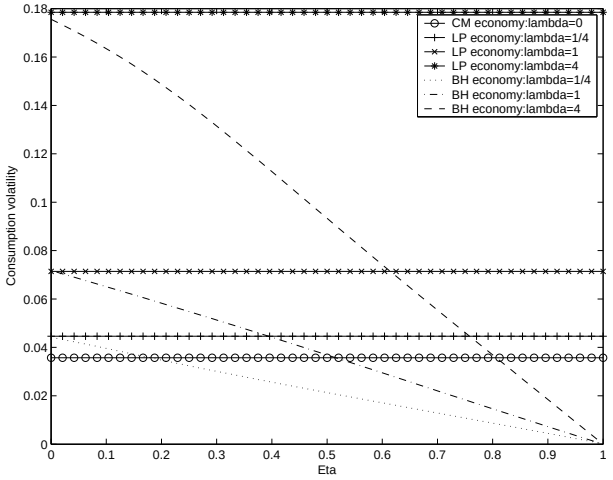


(b) Shadow process ν

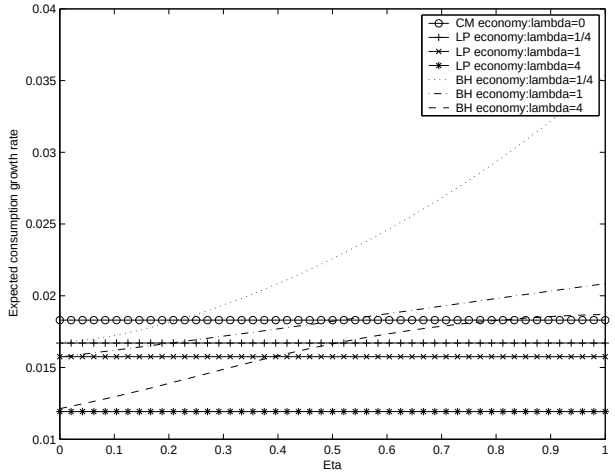
Figure 2: **Interest rate and shadow process ν in the buy and hold economy vs that in the complete market/limited participation economy (the log case).** The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The circled line is the case for complete market economy. The line with '$*$' is for limited participation economy with $\lambda = 4$. The line with '$\times$' is for limited participation economy with $\lambda = 1$. The line with '|' is for limited participation economy with $\lambda = 1/4$. The dashed line is for buy and hold economy with $\lambda = 4$. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. Parameter values of the economy: agent's relative risk aversion γ_1=γ_2=1, impatience parameter ρ=0.001, aggregate consumption volatility $\bar{\sigma}$=0.0357, expected aggregate consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.



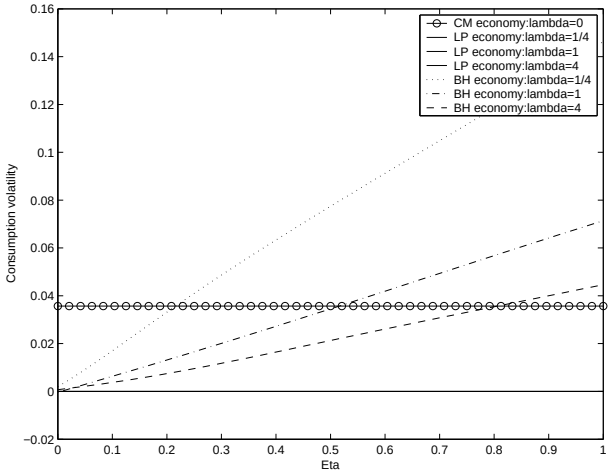
(a) Unrestricted agent's expected consumption growth rate



(b) Unrestricted agent's consumption volatility

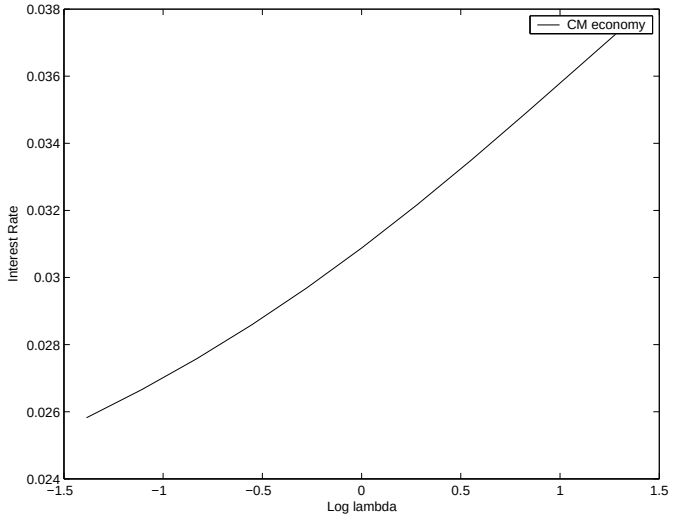


(c) Restricted agent's expected consumption growth rate

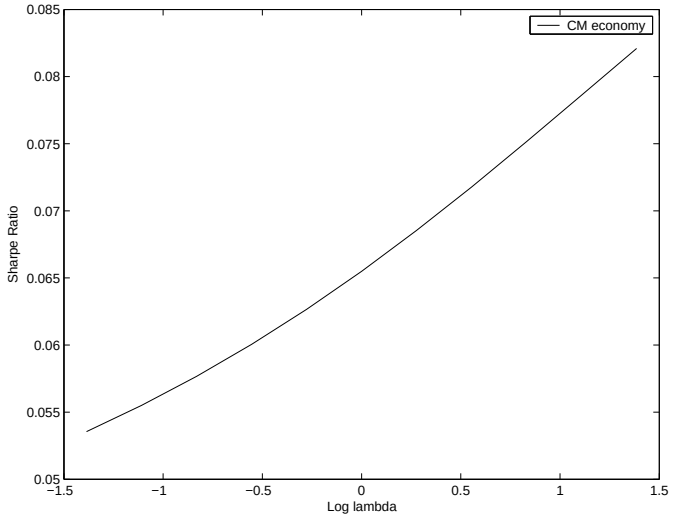


(d) Restricted agent's consumption volatility

Figure 3: **Unrestricted agent's and restricted agent's expected consumption growth rate and volatility in the buy and hold economy vs that in the complete market/limited participation economy (the log case).** The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The circled line is the case for complete market economy. The line with '$*$' is for limited participation economy with $\lambda = 4$. The line with '$\times$' is for limited participation economy with $\lambda = 1$. The line with '|' is for limited participation economy with $\lambda = 1/4$. The dashed line is for buy and hold economy with $\lambda = 4$. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. Parameter values of the economy: agent's relative risk aversion γ_1=γ_2=1, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

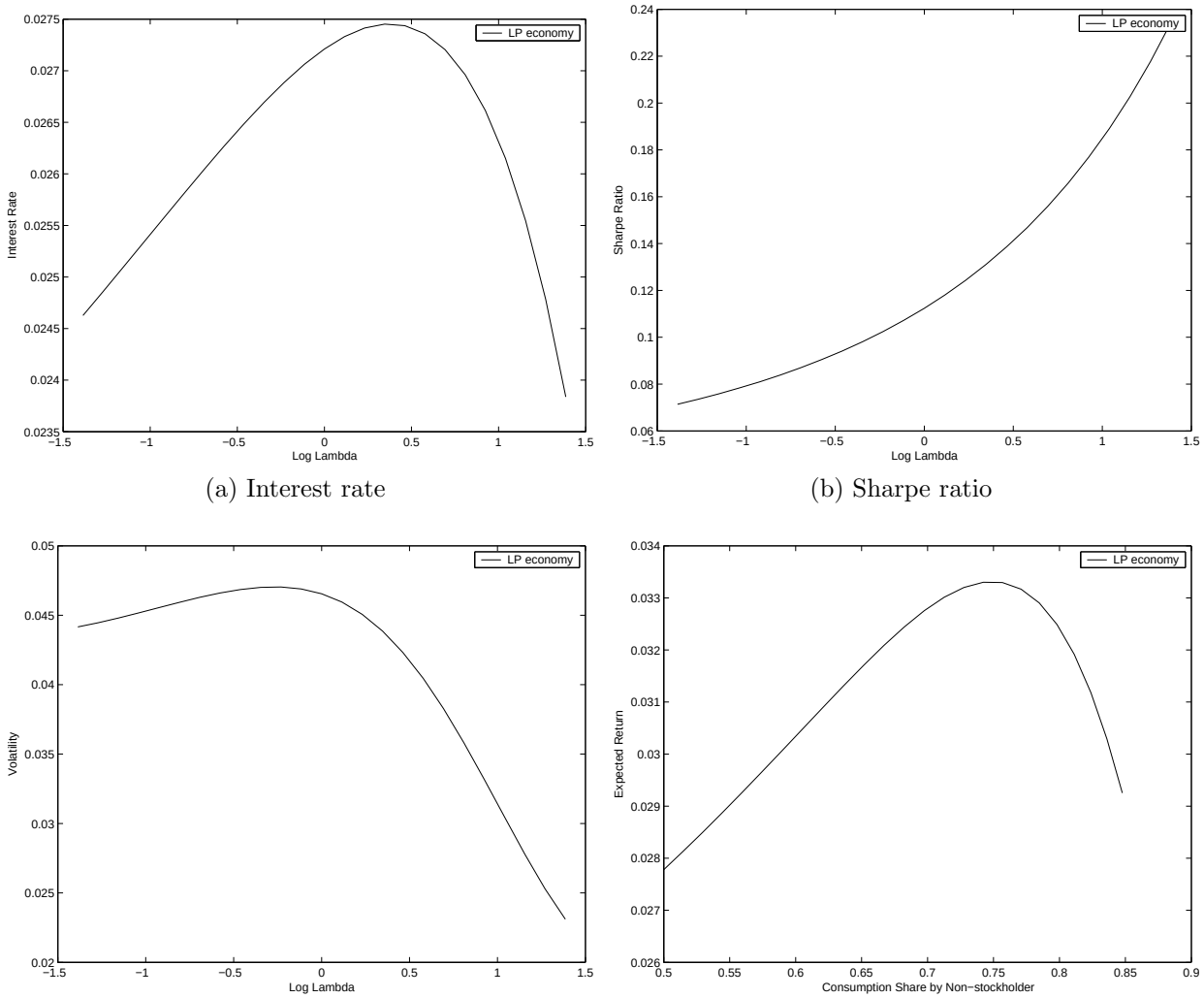


(a) Interest rate



(b) Sharpe ratio

Figure 4: **Interest rate and sharpe ratio in the complete market economy.** The x-axis corresponds to log λ. The corresponding range for λ is [1/4,4]. Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter $\rho = 0.001$, aggregate consumption volatility $\bar{\sigma}$=0.0357, expected aggregate consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.



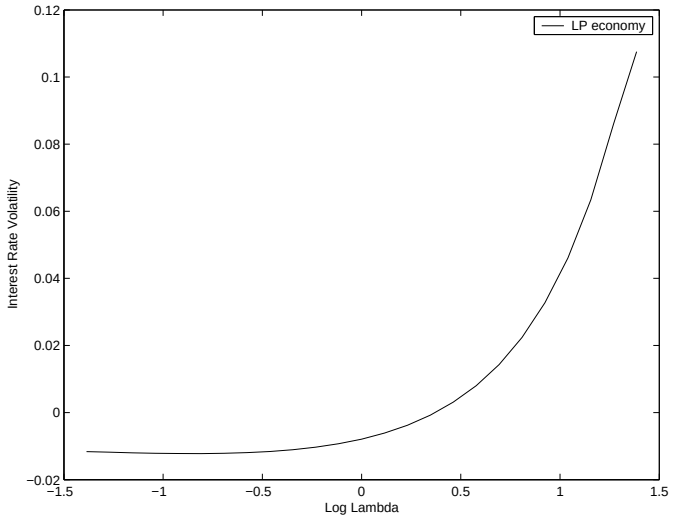
(a) Interest rate

(b) Sharpe ratio

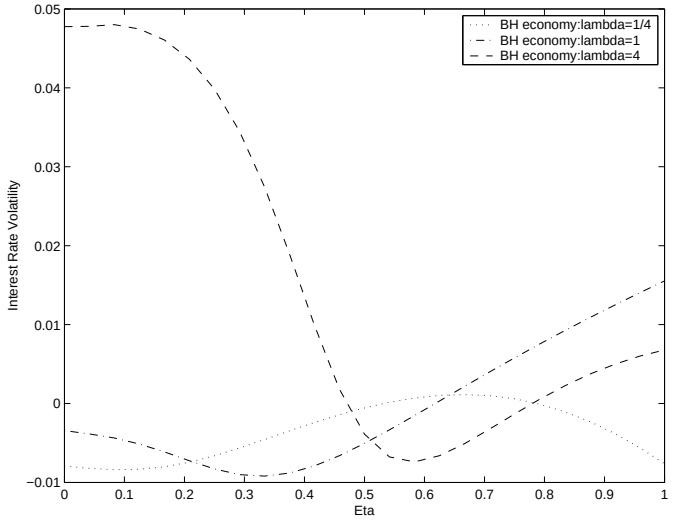
(c) Stock return volatility

(d) Expected return

Figure 5: **Interest rate, Sharpe ratio, stock return volatility and expected return in the limited participation economy.** The x-axis corresponds to log λ. The corresponding range for λ is [1/4,4]. Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter $\rho = 0.001$, aggregate consumption volatility $\bar{\sigma}$=0.0357, expected aggregate consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

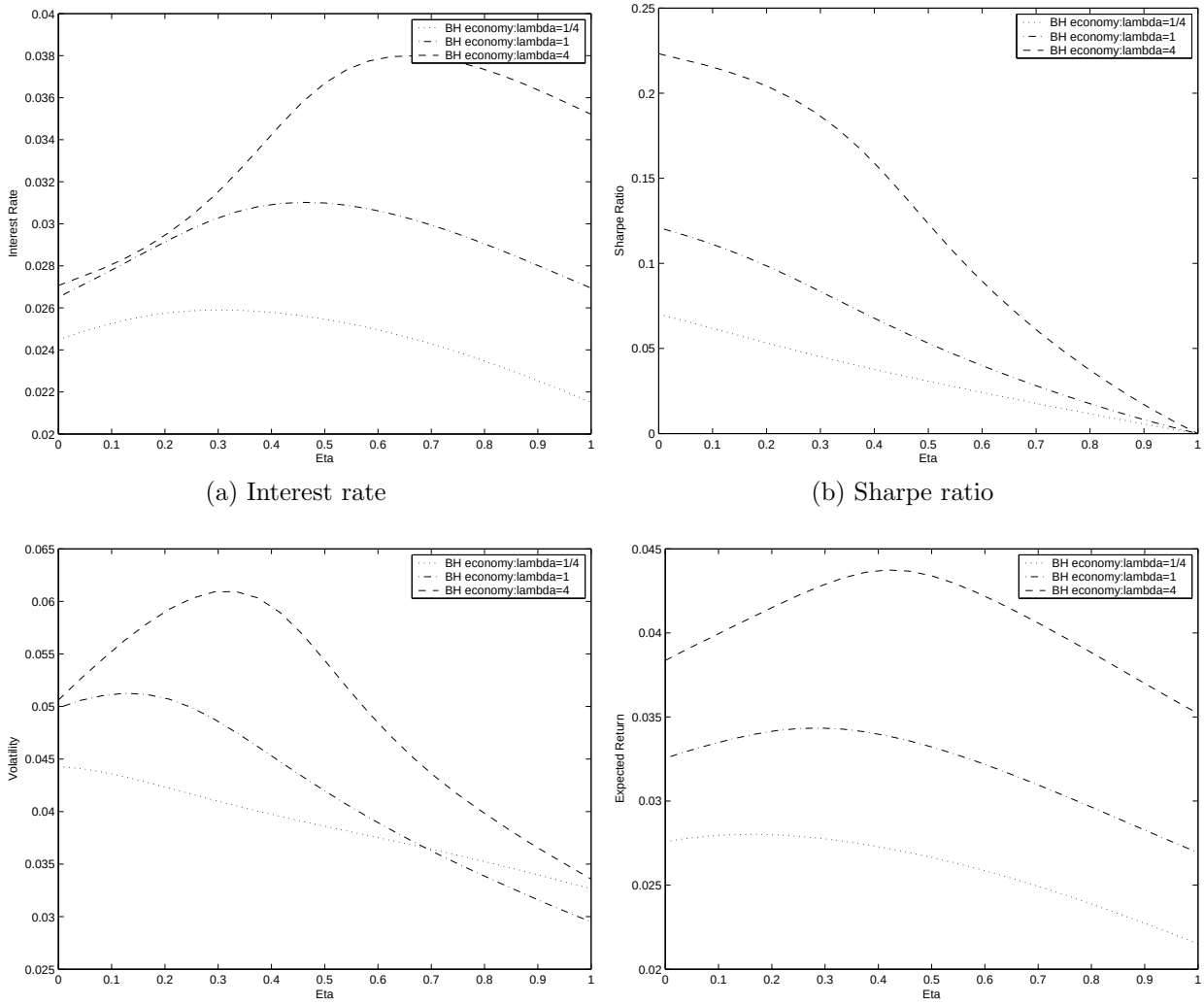


(a) Interest rate volatility in the limited participation economy



(b) Interest rate volatility in the buy and hold economy

Figure 6: **Interest rate volatility in the limited participation economy vs that in the buy and hold economy.** (Note that a negative value simply means the interest rate is counter-cyclical in that region.) In Panel (a), The x-axis corresponds to log λ. The corresponding range for λ is [1/4,4]. In Panel (b), the x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The dashed line is for buy and hold economy with $\lambda = 4$. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter ρ=0.001, aggregate consumption volatility $\bar{\sigma}$=0.0357, expected aggregate consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.



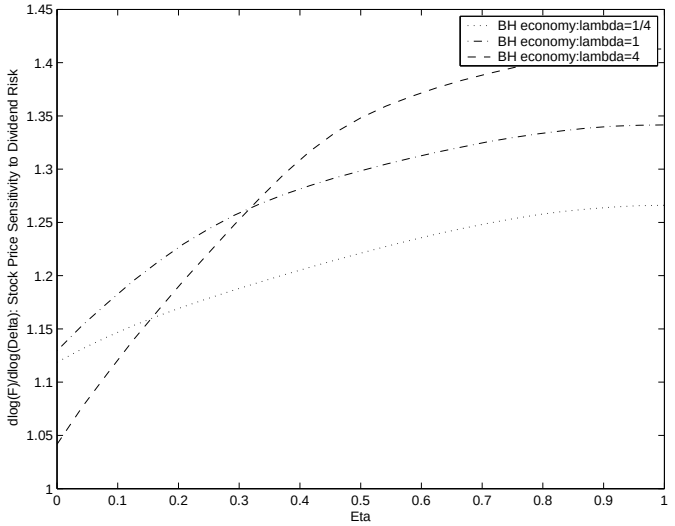
(a) Interest rate

(b) Sharpe ratio

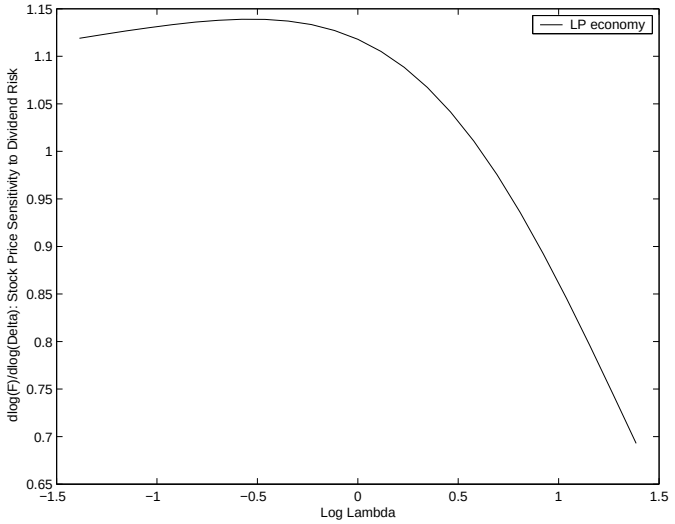
(c) Stock return volatility

(d) Expected return

Figure 7: **Interest rate, Sharpe ratio, stock return volatility and expected return in the buy and hold economy.** The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The dashed line is for buy and hold economy with $\lambda = 4$. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter ρ=0.001, aggregate consumption volatility $\bar{\sigma}$=0.0357, expected aggregate consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

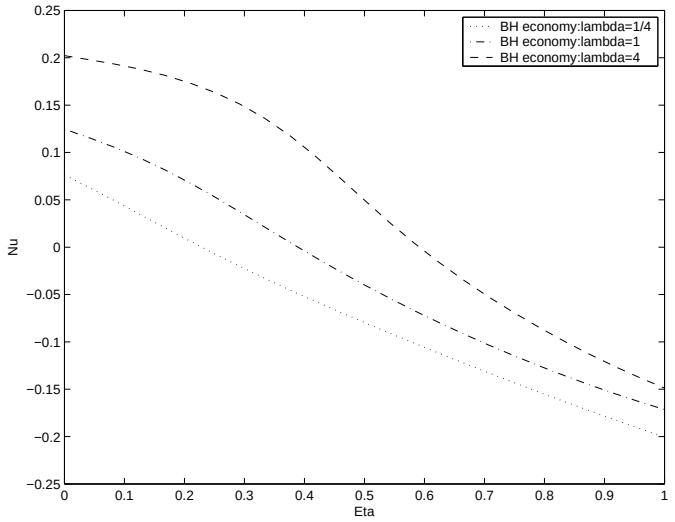


(a) Stock price sensitivity to dividend risk in the buy and hold economy

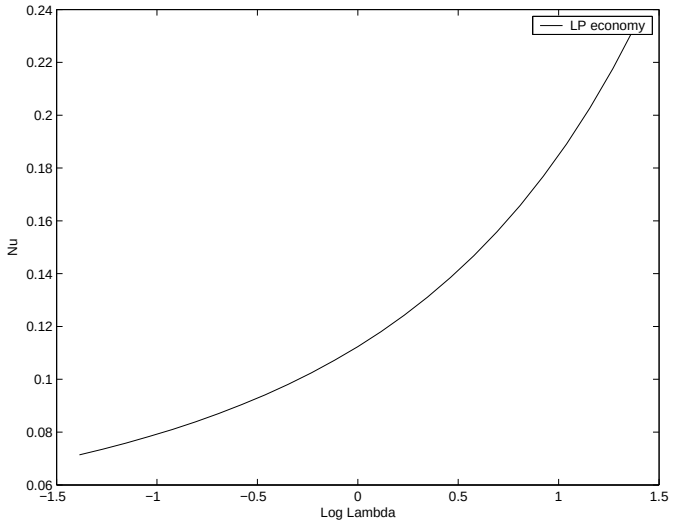


(b) Stock price sensitivity to dividend risk in the limited participation economy

Figure 8: **Stock price sensitivity to dividend risk in the buy and hold economy vs that in the limited participation economy.** In Panel a, the x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The dashed line is for buy and hold economy with $\lambda = 4$. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. In Panel b, the x-axis corresponds to log λ. The corresponding range for λ is [1/4,4]. Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter ρ=0.001, aggregate consumption volatility $\bar{\sigma}$=0.0357, expected aggregate consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.



(a) Shadow process ν (volatility of the stochastic weight) in the buy and hold economy.



(b) Shadow process ν (volatility of the stochastic weight) in the limited participation economy

Figure 9: **Shadow process ν (volatility of the stochastic weight) in the buy and hold economy vs that in the limited participation economy.** In Panel a, the x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The dashed line is for buy and hold economy with $\lambda = 4$. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. In Panel b, the x-axis corresponds to log λ. The corresponding range for λ is [1/4,4]. Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter ρ=0.001, aggregate consumption volatility $\bar{\sigma}$=0.0357, expected aggregate consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.