Monopsony, Search Frictions, and Firm Dynamics *

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Abstract

We develop a tractable model of firm dynamics with random search frictions in which firms set wage contracts for workers—a strict generalization of Hopenhayn and Rogerson (1993) and Burdett and Mortensen (1998). We provide an analytical characterization of equilibrium wage contracts as a function only of the distribution of marginal surplus. When calibrated to the US data, the model replicates empirical regularities on how firm wages vary by size, growth, and age. Finally, we explore the efficiency properties of the equilibrium both analytically and quantitatively. We find that the equilibrium firm growth is too slow at the early stage of the life cycle but too fast at the later stage.

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1 Introduction

2 Model

We focus on the steady state of the economy and drop time subscript wnenever there is no ambiguity.

Production Technology. There is a single homogenous goods. Workers can be either unemployed or employed by a firm. When workers are unemployed, they produce b units of goods at home. A firm with productivity z employing n workers produce $y(z,n)$ units of goods. We assume $y(z,n)$ is strictly increasing in n and z, and weakly concave in n. Note that $y(z,n)$ can potentially include a fixed cost of operation, so that $y(0,z) < 0$.

Firm Dempgraphics. Firm's productivity z evolves stochastically with the following stochastic process that is a discrete-time analogue of the diffusion process in continuous time:

$$z_{t+dt} = z_t + \mu(z_t)\,dt + \sigma(z_t)\,\sqrt{dt}\,\varepsilon_{t+dt}, \tag{1}$$

where $\mu(z)$ is the drift, $\sigma(z) \geq 0$ governs the volatility, and $\varepsilon_{t+dt} \sim \mathcal{N}(0,1)$. Firm's employment n evolves beacuse of four reasons. First, workers exogenous separate into unemployment at rate $s > 0$. Second, employed workers receive outside offers at rate $\lambda^E \geq 0$ and may quit to move to the other firms. Third, firms can hire workers by posting vacancies subject to vacancy costs. Vacancy costs of a firm with employment n take the form $c(v)n$, where v denotes the vacancy rate (number of vacancies divided by current employment). We assume the vacancy cost takes the isoelastic form:

$$c(v) = \bar{c}^{v}\frac{v^{1+\nu}}{1+\nu}. \tag{2}$$

We furthermore assume that the elasticity parameter satisfies $\nu \geq 1$ throughout the paper, for the reason that will become clear later. Finally, firms can fire workers at no cost. We let $\varsigma \in [0,1]$ denote the fraction of workers fired.

There is free entry. A potential firm can pay fixed cost c^e to become an operating firm. After paying the fixed cost, firms draw initial employment size, n_0, and initial productivity, z_0, from a joint distribution function $\psi(n_0, z_0)$. Firms exit at an exogenous rate $\varkappa > 0$. Firms also can exit endogenously. We denote the total mass of firms in the economy as m and the mass of entrants per unit of time as m_0.

Preferences and Worker Demographics. All agents discount the future with the discount rate $r > 0$ and are risk-neutral. We normalize the total mass of workers to be one. Let $u \in [0, 1]$ denote the fraction of unemployed workers.

Matching. An unemployed worker has a search efficiency of 1 and meets a firm at rate $\lambda^U > 0$. An employed worker has a search efficiency $\zeta > 0$ and meets another firm at rate $\lambda^E = \zeta\lambda^U > 0$. A vacancy meets a worker at rate $\lambda^F > 0$. All these meeting rates are endgenously determined by a constant-returns-to-scale matching function $M(\tilde{u}, \tilde{v})$, where $\tilde{u} \equiv u + \zeta(1 - u)$ is the total efficeincy of search by workers and $\tilde{v}$ denotes the total number of vacancy postings. The equilibrium meeting rates are given by

$$\lambda^U = M(\tilde{u}, \tilde{v})/\tilde{u}, \quad \lambda^E = \zeta M(\tilde{u}, \tilde{v})/\tilde{u}, \quad \lambda^F = M(\tilde{u}, \tilde{v})/\tilde{v}. \tag{3}$$

They can be expressed as a function of market tightness $\theta = \tilde{v}/\tilde{u}$.

Timing. The timing within a period is as follows. First, the idiosyncratic productivity shocks are realized, and firms produce with the workers and pay wages. Second, firms exit with arrival rate $\varkappa$. Third, the exogenous separation takes place. Fourth, firms hire workers and entrants enter also hire workers to acheive the initial firm size At the same time, unemployed and employed workers contact their potential employers. Finally, firms fire workers.

Cotracts. Firms offer wage contracts to workers. As in the tradition of Burdett and Mortensen (1998), we impose equal-treatment constraint within a firm. That is, firms have to offer the same wage w to all the workers within a firm, and whenever firms fire, firms have to randomize which workers to fire. This assumption rules out outside offer matching.

Firms have full commitment to the contracts. Workers cannot commit. The cotract specifies wages, hiring and firing policies for all possible histories of productivity shocks. As in Spear and Srivastava (1987), firm's past commitment can be summarized by the promised utility to the workers. Let $\mathcal{W}$ be worker's utility and U be unemployment value. In what follows, we write the recursive contracts in terms of worker utility in excess of unemployment value:

$$W = \mathcal{W} - U. \tag{4}$$

Note that the value of unemployment corresponds to $W = 0$.

We denote $F(W)$ as the vacancy-weighted distribution of continuation value across firms before the separation takes place. Likewise, we denote $G(W)$ as the employment-weighted distribution of continuation value across firms before the separation takes place. We will later formally

define $F(W)$ and $G(W)$ as equilibrium objects.

At time t, given the past commitment W_t as well as other state variables (z, n), firms offer a combination of real wage w_t, and continuation values (in excess of unemployment value) for each realization of productivity shocks z_{t+dt}. Formally, they are given by

$$w_t(n, W_t), \quad W_{t+dt}(z_{t+dt}; n, W_t). \tag{5}$$

The contracts have to satisfy the promise-keeping constraints.

$$W_t = w_t dt + e^{-rdt}\left[e^{-\lambda^E dt}V_{t+dt} + (1 - e^{-\lambda^E dt})\int \max\{V_{t+dt}, \tilde{V}\}dF(\tilde{V})\right] - (1 - e^{-rdt})U, \tag{6}$$

where

$$V_{t+dt} \equiv e^{-(s+\varkappa)dt}(1 - \varsigma_t)(1 - \chi_t)\mathbb{E}[W_{t+dt}(z_{t+dt}; n, W_t)]$$

is the expected continuation value at the current firm, where χ is an indicator function that takes one if the firm exits. [1] The promise-keeping constraint states that the firms have to deliver the past commitment W_t either through the current wage payments w_t or through the continuation value. The continuation value derives either from staying in the current firm or moving to the new firm.

Given the contracts, the evolution of firm-level employment is given by

$$n_{t+dt} = e^{-sdt}(1 - \varsigma_t)\left[\left(1 - (1 - e^{-\lambda^E dt})(1 - F(V_{t+dt}))\right)n_t + (1 - e^{\lambda^F dt})G(V_{t+dt})vn_t\right] \tag{7}$$

where $(1 - e^{\lambda^F dt})G(W_t)$ is the probability that a vacancy successfully hire a worker, and $(1 - e^{-\lambda^E dt})(1 - F(V_{t+dt})$ is the probability that a worker is poached.

2.1 Firm's Problem

Given the state variables (z, n, W), firm's problem is to choose vacancy rates v, the wage w, how many workers to fire, how much utility to promise W_{t+dt}, in order to maximize present discounted value of profits. Let $J(z, n, W)$ denote the firm's value function. The Bellman equation is

$$J(z_t, n_t, W_t) = \max_{w,v,\varsigma,\chi,n_{t+dt},W_{t+dt}} y(z_t, n_t)dt - wn_t dt - c(v)n_t dt \tag{8}$$

$$+ e^{-(r+\varkappa)dt}(1 - \chi)\mathbb{E}J(z_{t+dt}, n_{t+dt}, W_{t+dt}(z_{t+dt})) \tag{9}$$

$$\text{s.t.} \qquad (6) \text{ and } (7). \tag{10}$$

[1] Without fixed costs $y(z, 0) = 0$, $chi_t = 1$ is equivalent to $\varsigma_t = 1$.

This original problem is quite complex. The Bellman equation features three state variables, z, n, W, and the control variables are high-dimensional, promised utility in each possible realization of productivity shocks, $W_{t+dt}(z_{t+dt})$.

We overcome this problem by showing that the firm's problem can be represented as a joint surplus of firms and workers. To this end, we guess and verify that the firm's value function takes the following form:

$$J(z, n, W) = S(z, n) - Wn. \tag{11}$$

Since the term $S(z, n)$ is the sum of firm's value and the workers' value (in excess of unemployment value), it can be interpreted as the total surplus of the firm and workers inside the firm. Importantly, the surplus function $S(z, n)$ is independent from W, which derives from the the fact that utilities are transferrable.

The following writes the firm's problem in terms of the surplus function $S(z, n)$:

Problem 1. *The surplus function $S(z, n)$ solves the following Bellman equation:*

$$\begin{aligned} S(z_t, n_t) = \max_{v,\varsigma,\chi,n_{t+dt},V_{t+dt}} & \; y(z_t, n_t)dt - c(v)n_t dt - (1 - e^{-rdt})Un_t \\ & - e^{-rdt}V_{t+dt}(1 - e^{\lambda^F dt})G(V_{t+dt})vn_t + e^{-rdt}(1 - e^{-\lambda^E dt}) \int_{V_{t+dt}} \tilde{V} dF(\tilde{V}) n_t \\ & + e^{-(r+\varkappa)dt}(1 - \chi)\mathbb{E}S(z_{t+dt}, n_{t+dt}) \\ \textit{s.t.} \qquad n_{t+dt} = & \; e^{-sdt}(1 - \varsigma)\left[\left(e^{-\lambda^E dt} + (1 - e^{-\lambda^E dt})(1 - F(V_{t+dt}))\right) + (1 - e^{\lambda^F dt})G(V_{t+dt})v\right] n_t, \end{aligned}$$

The first line is the net flow surplus of the firms and workers, which is the production minus the vacancy cost and the flow unemployment value. The first term in the second line captures the flow value that accrues to the new hires. From the viewpoint of the incumbent workers, raising the promised utility has a cost of giving more resources to the new hires. The second terms in the second line captures the additional value whenever workers are poached by the other firms offering higher continuation value. We let $v(z, n)$, $\varsigma(z, n)$, $n_{t+dt}(z, n)$, $V_{t+dt}(z, n)$, $\chi(z, n)$ be the policy functions associated with the above Bellman equation, where $\chi(z, n)$ is an indicator function that takes one if the firm exits.

There are several pieces to unpack in the above problem. First, the promised utility W drops from the state variable. This is because, under the transferrable utility, the promised utility is irrelevant for the firm's decisions. Second, the Bellman equation only determines the expected continuation value for the workers, V_{t+dt}, but how the continuation value is split across the re-

alization of the states, $W_{t+dt}(z_{t+dt})$, is indeterminate. This comes from the risk-neutrality of the workers. Wokrers are indifferent between being offered V_{t+dt} with certainty and being offered $W_{t+dt}(z_{t+dt})$ that varies across the states, as long as the mean is the same. This indeterminancy naturally translates into the indeterminancy of the wages, which is determined residually from the promise-keeping constraint (6) given the stochastic process of promised utilities. It is important to note that there is no indeterminancy in terms of the allocations, such as hiring, firing, and job-ladder.

However, we show in Appendix A.3 that the wages are uniquely determined as long as workers have arbitrarily small risk aversion. Specifically, let $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ be the utility function of a worker with risk aversion $\gamma > 0$. While the split of $W_t(z_t)$ across z_t are indeterminate with $\gamma = 0$, when $\gamma \to 0$, they are uniquely determined so as to satisfy the following condition:

$$\tilde{w}\left(W_t(z_t^1), V_{t+dt}(z_t^1, n_t)\right) = \tilde{w}\left(W_t(z_t^2), V_{t+dt}(z_t^2, n_t)\right), \tag{12}$$

for any z_t^1 and z_t^2, where $\tilde{w}(W, V)$ is the wage implied by the promise-keeping constraint (6) given the promised utility W and the continuation value V:

$$\tilde{w}(W, V)dt \equiv W - e^{-rdt}\left[e^{-\lambda^E dt}V + (1 - e^{-\lambda^E dt})\int \max\{V, \tilde{V}\}dF(\tilde{V})\right] - (1 - e^{-rdt})U, \tag{13}$$

Therefore, the wages are uniquely determined so as to minimize the fluctuations across the realizations of productivity shocks. This result is intuitive. While workers do not care about the fluctuations in wages whenever they are exactly risk-neutral, they would like to smooth the fluctuations the moment they are arbitrarily risk-averse. For this reason, in what follows, we select the path of wages such that (12) holds.

2.2 Aggregation and Equilibrium Definition

Entrants draw the initial productivity z_0 and employment n_0 from a joint distribution function $\Psi(z_0, n_0)$. We assume that the entrants are endowed with as many vacancies as needed to achieve the initial level of employment. This implies that the initial vacancy is given by

$$v^0(z_0, n_0) = \frac{n_0}{\lambda^F G(W)}. \tag{14}$$

The entrants choose the initial promised utility to maximize the entry value: $\max_{W \geq 0} J(z_0, n_0, W) = \max_{W \geq 0} S(z_0, n_0, W) - W n_0$. This clearly implies that the entrants choose $W = 0$ and hire only from unemployed workers.

The free-entry condition is

$$\int S_0(z_0, n_0) d\Psi(z^0, n^0) = c^e, \tag{15}$$

where $S_0(z_0, n_0)$ is the value of entry defined as

$$S_0(z_0, n_0) = \max_{\varsigma, n_{t+dt}} e^{-(r+\varkappa)dt} \max\{\mathbb{E}S(z_{t+dt}, n_{t+dt}), 0\} \tag{16}$$

$$\text{s.t.} \qquad n_{t+dt} = e^{-sdt}(1-\varsigma)n_0. \tag{17}$$

Let $\varsigma_0(z_0, n_0), n_{t+dt,0}(z_0, n_0)$ and $\chi_0(z_0, n_0)$ be the policy functions associated with the above problem, where $\chi_0(z_0, n_0)$ is an indicator function that takes one if the firm exits.

Let $g(\mathcal{Z}, \mathcal{N})$ denote the steady state measure of firms with productivty $z \in \mathcal{Z}$ and employment $n \in \mathcal{N}$ at the beginning of the period. It satisfies the following steady state law of motion:

$$\begin{aligned} g(\mathcal{Z}, \mathcal{N}) = e^{-\varkappa dt} &\int (1 - \chi(z, n)) \mathbb{I}\left[n_{t+dt}(z, n) \in \mathcal{N}\right] \pi(\mathcal{Z}|z) dg(z, n) \\ &+ m^0 e^{-\varkappa dt} \int (1 - \chi_0(z_0, n_0)) \mathbb{I}\left[n_{t+dt,0}(z_0, n_0) \in \mathcal{N}\right] \pi(\mathcal{Z}|z_0) d\Psi(z_0, n_0), \end{aligned} \tag{18}$$

where m^0 is the mass of entrants.

Each vacancy meets randomly meets a worker. With probability $p^u \equiv \frac{u}{u+\zeta(1-u)}$ vacancies meet unemployed workers who always accept the offer. With probability $1 - p_u$ the worker is employed and accepts it whenever the offered value is higher than the value that they are currently offered. Therefore the probability that a vacancy at a firm offering V hires a worker is given by

$$G(V) = \frac{1}{\tilde{u}} \left[u + \zeta \frac{\int_{V_{t+dt}(z,n) \leq V} n dg(z, n)}{\int n dg(z, n)} \right]. \tag{19}$$

The aggregate vacancy posting is given by

$$\tilde{v} = \int v(z, n) n dg(z, n) + m^0 \int v^0(z_0, n_0) d\Psi(z_0, n_0). \tag{20}$$

The vacancy-weighted distribuution of continuation value is given by

$$F(V) = \frac{1}{\tilde{v}} \left[\int_{V_{t+dt}(z,n) \leq V} v(z, n) n dg(z, n) + m^0 \int v^0(n_0/z_0) d\Psi(z_0, n_0) \right], \tag{21}$$

where we have imposed that the entrants post the continuation value of $V = 0$.

The value of unemployment, U, solves

$$U = bdt + e^{-rdt}\left[U + (1 - e^{-\lambda^U dt})\int V dF(V)\right]. \tag{22}$$

The steady state unemployment satisfies

$$u = 1 - \int n dg(z,n) \tag{23}$$

We define the steady state equilibrium of this economy as follows. A steady stateequilibrium consists of the value and policy functions of incumbent firms $\{S(z,n), v(z,n), n_{t+dt}(z,n), V_{t+dt}(z,n), \chi(z,n)\}$, of entrants $\{S_0(z_0,n_0), n_{t+dt,0}(z_0,n_0), \chi_0(z_0,n_0)\}$, the two distributions of continuouation value $G(V)$ and $F(V)$, the distribution of firms $\{g(z,n), m^0\}$, the meeting rates $\{\lambda^U, \lambda^E, \lambda^F\}$, the unemployment value U, and the aggregate vacancies and unemployment, $\tilde{v}$ and u such that:

1. Given $\{G(V), F(V), \lambda^E, U\}$, firms solve Problem 1.
2. The steady state firm distribution $\{g(z,n), m^0\}$ solve (18).
3. The two distributions of continuouation value $G(V)$ and $F(V)$ satisfy (19) and (21).
4. The unemployment value U satisfies (22).
5. The aggregate vacancies and unemployment satisfy (20) and (23).
6. The meeting rates are given by (3).

3 Continuous Time Limit

While all our analysis will carry through in the discrete time setup, taking the continuous time limit will provide particuarly transparent and clean characterization of the equilibrium. In the limit of $dt \to 0$, the stochastic process of z_t described by (1), is governed by the following diffusion process:

$$dz = \mu(z)dt + \sigma(z)dW_t, \tag{24}$$

where dW_t is an increment of a standard Brownian motion.

The following problem is the continuous time limit of Problem 1:

Problem 2. *In the cotinuous time limit, the surplus function $S(z,n)$ solves the HJB equation:*

$$\rho S = \max_{v,V} y(z,n) - c(v)n - rUn - V\lambda^F G(V)vn + \lambda^E \int_V \tilde{V} dF(\tilde{V})n + \mathcal{L}(z,n,v,V)S$$

where $\rho \equiv r + \varkappa$, $\mathcal{L}(z, n, v, V)$ *is the generator defined for an arbitrary function* $S(z, n)$*:*

$$\begin{aligned}\mathcal{L}(z,n,v,V)S(z,n) = \mu(z)\partial_z S(z,n) + \frac{\sigma^2(z)}{2}\partial_{zz}S(z,n) \\ + \left[\lambda^F G(V)v - s - \lambda^E(1-F(V))\right] n\partial_n S(z,n),\end{aligned} \tag{25}$$

with the following boundary conditions:

$$S(z,n) \geq 0, \quad S_n(z,n) \geq 0. \tag{26}$$

The first-order condition with respect to V is given by

$$\left[\lambda^F g(V)v + \lambda^E f(V)\right](\partial_n S(z,n) - V) = \lambda^F G(V)v. \tag{27}$$

This left-hand side is the marginal value of offering higher promised utility. Firms can additinally higher at an additional rate $\lambda^F g(V)v$ of workers and prevent workers from being poached at an addtional rate of $\lambda^E f(n)$. Each additional worker has a marginal value of $(\partial_n S - V)$. The right hand side is the cost of offering higher promised utility, which accurues to new hires.

The first-order condition with respect to v is

$$\lambda^F G(V)(\partial_n S(z,n) - V) = c'(v). \tag{28}$$

The left-hand side is the marginal benefit of a vacancy, and the right-hand side is the marginal cost of vacancy. The boundary condition for firing is

$$\partial_n S(z, n^*) = 0, \tag{29}$$

which says that the firms fire workers until the marginal value of a worker is weakly positive.

As one can see from the optimality conditions (27) and (28), firm's optimal promised utility and vacancy decisions are solely summarized by $S_n \equiv \partial_n S(z,n)$, which we refer to as the marginal surplus. For this reason, we will write the policy functions in terms of the marginal surplus: $V(S_n(z,n)) \equiv V(z,n)$ and $v(S_n(z,n)) \equiv v(z,n)$. Condition (29) suggests that the marginal suprlus is bounded from below by zero.

Since the marginal benefit of increasing promised utility – the lefth-hand side of (27) – is strictly increasing in the marginal surplus, it is tempting to think that the firms with higher marginal surplus offers higher promised utility. While this observation is obviously correct whenever the vacancy is exogenous, the endogenous vacancy introduces additional complication. With endogenous vacancy, firms with higher marginal surplus has two ways of increasing its work-

force: offering higher utility or posting more vacancies. If the vacancy creation is sufficiently elastic, firms would primaly use vacancy postings as a way to increase its workforce. This increase in new hires, in turn, disourage firms from offering higher utility because the cost of doing so increases with the size of new hires – the right-hand side of (27).

We show, however, that as long as the elasticity of vacancy creation satisfies $\nu \geq 1$, the promised utility offer is strictly increasing in the marginal surplus.

Proposition 1. *Assume $\nu \geq 1$. Promised utility offer $V(S_n)$ is strictly increasing in S_n, and consequently, workers at firm (z,n) accept the job offer from firm (z',n') if and only if $S_n(z,n) \geq S_n(z',n')$.*

The monotonicy of promised utility implies that the job-ladder is ranked according to the marginal surplus. At any point in time, we can define two rankings of the job-ladder at the firm-level, deferring in terms of how the ranking is weighted. The first is the ranking of the marginal suprlus weighted by the mass of search efficiency of workers:

$$\hat{G}(S_n) \equiv \frac{1}{\tilde{u}}\left[u + \zeta \hat{H}(S_n)\right], \tag{30}$$

where $\hat{H}(S_n) \equiv \int_{S_n(z,n)\leq S_n} ndg(z,n)$ denotes the mass of workers at a firm with marginal surpus below S_n. The second is the vacancy-weighted ranking of the marginal surplus

$$\hat{F}(S_n) \equiv \frac{1}{\tilde{v}}\left[\int_{S_n(z,n)\leq S_n} v(z,n)ndg(z,n) + m^0 \int v^0(n_0/z_0)d\Psi(z_0,n_0)\right], \tag{31}$$

The monotonicity of promised utiliy offer implies

$$G(V(S_n)) = \hat{G}(S_n), \quad F(V(S_n)) = \hat{F}(S_n). \tag{32}$$

Exploiting the rank-preserving property, we have a tight analytical characterization of the equilibrium policies as a function of the distribution of the marignal surplus.

Proposition 2. *Assume $\nu \geq 1$. The promised utility each firm with marginal surplus S_n offers is given by*

$$V(S_n) = \frac{1}{\left(u + \zeta \hat{H}(S_n)\right)^2} \int_0^{S_n} \left\{(u + \zeta H(\tilde{S}_n))2\zeta h(\tilde{S}_n)\right\} \tilde{S}_n d\tilde{S}_n \tag{33}$$

$$\equiv \mathbb{E}_{\hat{G}^2}\left[\tilde{S}_n | \tilde{S}_n \leq S_n\right], \tag{34}$$

where $\mathbb{E}_{\hat{G}^2_\zeta}[\tilde{S}_n|\tilde{S}_n \leq S_n]$ is the weighted distribution of marginal surplus conditional on being below

S_n. The weighting function is given by the following cumulative density function:

$$\hat{G}^2(\tilde{S}_n|\tilde{S}_n \leq S_n) \equiv \frac{(u+\zeta\hat{H}(\tilde{S}_n))^2}{\Big(u+\zeta\hat{H}(S_n)\Big)^2}, \tag{35}$$

which is the quadratic of the search intensity-weighted cumulative mass of workers below $\tilde{S}_n$ conditional on being below S_n. Vacancy policy function is given by

$$v(S_n) = c'^{-1}\Big(\lambda^F G(V(S_n))(S_n - V(S_n)))\Big). \tag{36}$$

While wage posting model is often considered as challenging to solve due to the infinite-dimensional fixed point problem, we obtain a closed-form solution to the firm's policies conditional on the distribution of the marginal surplus.

The distribution of the marginal surplus can be obtained from the steady state distribution, which satisfies the following Kolmogorov forward equation:

$$0 = \mathcal{L}^\dagger\left(z, n, v(S_n(z,n)), V(S_n(z,n))\right) g(z,n) - \varkappa g(z,n) + m^0\psi(z,n), \tag{37}$$

whenever $S_n(z,n) > 0$ and $S(z,n) > 0$, where $\mathcal{L}^\dagger$ is the adjoint operator of $\mathcal{L}$.

The free-entry condition in the continuous time limit is

$$\int S(z,n)d\Psi(z,n) = c^e. \tag{38}$$

The value of unemployment in continuous time is

$$rU = b + \lambda^U \int V dF(V). \tag{39}$$

Finally, the wage is given by

$$w = (r+s+\varkappa)W - \lambda^E \int_W (V-W)dF(V) - rU. \tag{40}$$

The rest of the equilibrium conditions remain unchanged.

Building on these characterizations, we can rewrite the HJB equation as

$$\begin{aligned}\rho S = y(z,n) - c(v(S_n))n - bn - \lambda^U \mathbb{E}_{\hat{F}}[V(\tilde{S}_n)]n - \lambda^F \hat{G}(S_n)V(S_n)v(S_n)n \\ + \lambda^E(1-\hat{F}(S_n))\mathbb{E}_{\hat{F}}[V(\tilde{S}_n)|\tilde{S}_n \geq S_n]n + \mathcal{L}(z,n,v(S_n),V(S_n))S\end{aligned}. \tag{41}$$

4 Planning Problem

We now explore the efficiency of the equilibrium. We consider the planning problem of a social planner who maximizes the expected discounted consumption of the economy.

Problem 3. *The constrained planner solves*

$$\max_{\{g_t, v_t, m_t^0, dn_t, \varsigma_t^e, \varsigma_t^f, x_{t,zn,z'n'}, x_{t,u,zn}\}} \int_0^\infty e^{-rt} \left\{ \int \left(y(z,n) - c(v_t(z,n))n \right) dg_t + (1 - \int n dg_t)b - m_t^0 c^e \right\} dt$$

subject to

$$\begin{aligned} \partial_t g_t(z,n) &= -\partial_n \left[dn_t(z,n) g_t(z,n) \right] - \partial_z [\mu(z) g_t(z,n)] + \frac{1}{2} \partial_{zz} [(\sigma(z))^2 g_t(z,n)] \\ &\quad - (\varkappa + \varsigma_t^e) g_t(z,n) + m_t^0 \psi(z,n) \\ dn_t(z,n) &= \left[\lambda_t^F G(z,n) v(z,n) - s - \lambda_t^E (1 - F(z,n)) \right] n - \varsigma_t^f \end{aligned}$$

where

$$\begin{aligned} G(z,n) &\equiv \frac{1}{\tilde{u}} \left[x_{t,u,zn} \left(1 - \int n dg_t \right) + \zeta \int x_{t,z'n',zn} n' dg_t(z',n') \right] \\ 1 - F(z,n) &\equiv \frac{1}{\tilde{v}} \int x_{t,zn,z'n'} v(z',n') n' dg_t(z',n') \\ \lambda^F &\equiv \frac{M(\tilde{u},\tilde{v})}{\tilde{v}}, \quad \lambda^E \equiv \zeta \frac{M(\tilde{u},\tilde{u})}{\tilde{u}}, \\ \tilde{u} &\equiv 1 - \int n dg_t + \zeta \int n dg_t, \quad \tilde{v} \equiv \int v(z,n) n dg(z,n) + m^0 \int v^0(z_0,n_0) d\Psi(z_0,n_0), \end{aligned}$$

and $x_{t,zn,z'n'} \in [0,1]$, $v_t(z,n) \geq 0$, $\varsigma_t^e \geq 0$ *and* $\varsigma_t^f \geq 0$.

Let $S_t^p(z,n)$ denote the co-state variable (Lagrange multiplier) associated with the law of motion of $g_t(z,n)$. This variable captures the social surplus of the firm with productivity z and employment size n. We also define

$$\eta^U \equiv -\frac{\partial \ln \lambda^U}{\partial \ln \tilde{u}} = 1 - \frac{\partial M(\tilde{u},\tilde{v})}{\partial \ln \tilde{u}} \tag{42}$$

$$\eta^F \equiv -\frac{\partial \ln \lambda^F}{\partial \ln \tilde{v}} = 1 - \frac{\partial M(\tilde{u},\tilde{v})}{\partial \ln \tilde{v}} \tag{43}$$

as the steady-state elasticity of congestion from additional worker search and vacancy posting,

respectively. We also define the social gains from a meeting in the steady state as

$$\Omega \equiv \int G(z',n') \left(S_n^p(z',n') - \mathbb{E}_{\hat{G}}[\tilde{S}_n | \tilde{S}_n \leq S_n^p(z',n')] \right) \frac{1}{\tilde{v}} v(z',n') n' dg_t(z',n') \tag{44}$$

The following proposition characterizes the planner's solution.

Proposition 3. *In the steady state, the efficient allocation has the following properties. The social surplus of a firm solves*

$$\begin{aligned} \rho S_t^p(z,n) = y(z,n) - c(v(z,n))n - bn + dn_t(z,n) S_n^p(z,n) + \mu(z) S_z^p(z,n) + \frac{1}{2}(\sigma(z))^2 S_{zz}^p(z,n) \\ - \lambda^U \left\{ \mathbb{E}_{\hat{F}}[\tilde{S}_n] - \eta^U \Omega \right\} n + \lambda^E \left\{ (1 - \hat{F}(S_n^p)) \mathbb{E}_{\hat{F}}[\tilde{S}_n | \tilde{S}_n \geq S_n^p] - \eta^U \Omega \right\} n \\ - \lambda^F \left\{ \hat{G}(S_n^p) \mathbb{E}_{\hat{G}}[\tilde{S}_n | \tilde{S}_n \leq S_n^p] + \eta^F \Omega \right\} v(z,n) n, \end{aligned}$$

with the boundary conditions $S^p(z,n) \geq 0$ and $S_n^p(z,n) \geq 0$. Firms exit if $S^p(z,n) \leq 0$ and fire workers if $S_n^p(z,n) \leq 0$. The vacancy rate is

$$v(z,n) = c'^{-1} \left(\lambda^F \left\{ G(S_n^p) \left(S_n^p(z,n) - \mathbb{E}_{\hat{G}}[\tilde{S}_n | \tilde{S}_n \leq S_n^p] \right) - \eta^F \Omega \right\} \right).$$

The unemployed workers accept the job as long as $S_n^p(z,n) \geq 0$, and the employed workers at firm (z',n') accept the job offer from firm (z,n) if and only if $S_n^p(z,n) \geq S_n^p(z',n')$. The entry satisfies

$$\int S^p(z,n) d\Psi(z,n) = c^e.$$

5 Quantitative Analysis

5.1 Steady State Computational Algorithm

5.2 Calibration

We make the following parameteric assumptions. First, we parametrize the production function as

$$y(z,n) = z^{1-\alpha} n^\alpha - c_f, \tag{45}$$

where $c_f \geq 0$ is the fixed cost of operation, and $\alpha \in (0,1)$ is the returns to scale of production. We set $\alpha = 0.85$, a standard value of returns to scale in the literature. Second, the productivity process is assumed to be a geometric Brownian motion: $\mu(z) = \mu z$ and $\sigma(z) = \sigma z$. Third, we

Paramter	Value	Description	Target
A. Externally assigned parameters			
r	0.0125	Steady state interest rate	Annual 5% interest rate
α	0.85	Returns to scale	Standard
$\varkappa$	0.0225	Exit rate	Annual 9% exit rate
n^0	5.4	Entrants initial employment	BDS 2019
ν	3.45	Vacancy cost elasticity	Bilal, Engbom, Mongey and Violante (2022)
η	0.5	Matching efficiency	Petrongolo and Pissarides (2001)
$\bar{c}^v$	-	Vacancy cost level	Normalization
B. Ineternally calibrated parameters			
μ	0.015	Productivity growth drift	Emp. share 500+ firms 50%
σ	0.18	Productivity growth variance	std($\Delta \ln(n)$) of 0.41
c^e	10.6	Entry cost	Market tightness 1
z^0	2500	Initial productivity	Average firm size 23
c_f	1.0	Fixed cost	Exit rate of young firms
s	0.02	Exogenous separation rate	Monthly EU rate 2%
ζ	0.10	On-the-job search efficiency	Monthly EE rate 2%
A	1.0	Matching efficiency	UE rate
z^0	1.0	Entrants initial productivity	Normalization

Table 1: Parameter Values

assume the Cobb-Douglas matching function:

$$M(\tilde{u}, \tilde{v}) = A\tilde{u}^{\eta}\tilde{v}^{1-\eta}, \tag{46}$$

where A is the matching efficiency parameter, and η is the elasticity of matching function with respect to the job search by workers. We set $\eta = 0.5$, as suggested by Petrongolo and Pissarides (2001).

We simply set $b = 0$ to reduce the number of free parameters, but we explore robustness to alternative parameterization in Appendix XXX.

6 Transition Dynamics

7 Conclusion

Parameter	Value	Description	Target
A. Externally assigned parameters			
r	0.05	Steady state interest rate	Annual 5% interest rate
α	0.85	Returns to scale	Standard
$\varkappa$	0.0	Exogenous exit rate	Largest firm exit rates
n^0	6	Entrants initial employment	BDS 2023
κ	3.5	Vacancy cost elasticity	Bilal et al. (2022)
η	0.5	Matching efficiency	Petrongolo and Pissarides (2001)
$\bar{c}^v$	100.0	Vacancy cost level	Normalization
ζ	0.2	On-the-job search efficiency	-
B. Internally calibrated parameters			
μ	-0.001	Productivity growth drift	Emp. share 10,000+ firms 30%
σ	0.6	Productivity growth variance	std($\Delta \ln(n)$) of 0.41
c_f	0.1	Fixed cost	Exit rate 9%
s	0.25	Exogenous separation rate	Unemployment rate 5%
A	4.8	Matching efficiency	UE rate

Table 2: Parameter Values

Moments	Target	Model
Firm size Pareto tail	1.05	
Average firm size	23	23
Standard deviation of firm growth		
Unemployment rate	0.05	0.05
EU rate		
EE rate		
Entrants productivity gap		
Profit share or Std($\log w$)		

Table 3: Targeted Moments

Moment	Model	Data
Average firm size	22.952	24.0
Unemployment rate	0.047	0.05
Emp. share firms >10,000	0.379	0.3
Emp. share firms >250	0.629	0.57
Exit rate	0.106	0.075
Entrant size	6	6.0
Std. firm employment growth	0.477	0.41

Table 4: Targeted Moments

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A Derivations and Proofs

A.1 Derivation of Problem 1

The original Bellman equation is

$$\begin{aligned}
J(z_t, n_t, W_t) = &\max_{w,v,\varsigma,n_{t+dt},W_{t+dt},\chi} y(z_t, n_t)dt - wn_t dt - c(v)n_t dt \\
&\quad + e^{-(r+\varkappa)dt}(1-\chi)\mathbb{E}J(z_{t+dt}, n_{t+dt}, W_{t+dt}(z_{t+dt})) \\
\text{s.t.} \quad W_t = &\, w_t dt + e^{-rdt}\left[e^{-\lambda^E dt}V_{t+dt} + (1-e^{-\lambda^E dt})\int \max\{V_{t+dt}, \tilde{V}\}dF(\tilde{V})\right] - (1-e^{-rdt})U \\
n_{t+dt} = &\, e^{-sdt}(1-\varsigma)\left[\left(e^{-\lambda^E dt} + (1-e^{-\lambda^E dt})(1-F(V_{t+dt}))\right)n_t + (1-e^{\lambda^F dt})G(V_{t+dt})vn_t\right]
\end{aligned}$$

where $V_{t+dt} = e^{-(s+\varkappa)dt}(1-\varsigma)(1-\chi)\mathbb{E}[W_{t+dt}(z_{t+dt}; n, W_t)]$. Substituting the promise-keeping constraint to eliminate w, we have

$$\begin{aligned}
J(z_t, n_t, W_t) = &\max_{w,v,\varsigma,n_{t+dt},W_{t+dt},\chi} y(z_t, n_t)dt - c(v)n_t dt - W_t n_t \\
&\quad + e^{-rdt}\left[e^{-\lambda^E dt}V_{t+dt} + (1-e^{-\lambda^E dt})\int \max\{V_{t+dt}, \tilde{V}\}dF(\tilde{V})\right]n_t - (1-e^{-rdt})Un_t \\
&\quad + e^{-(r+\varkappa)dt}(1-\chi)\mathbb{E}J(z_{t+dt}, n_{t+dt}, W_{t+dt}(z_{t+dt})) \\
\text{s.t.} \quad n_{t+dt} = &\, e^{-sdt}(1-\varsigma)\left[\left(e^{-\lambda^E dt} + (1-e^{-\lambda^E dt})(1-F(V_{t+dt}))\right)n_t + (1-e^{\lambda^F dt})G(V_{t+dt})vn_t\right]
\end{aligned}$$

We now guess and verify that the value function is of the form

$$J(z_t, n_t, W_t) = S(z_t, n_t) - W_t n_t.$$

Substituting this into the Bellman equation, we get

$$\begin{aligned}
S(z_t, n_t) = &\max_{w,v,\varsigma,n_{t+dt},W_{t+dt},\chi} y(z_t, n_t)dt - c(v)n_t dt \\
&\quad + e^{-rdt}\left[e^{-\lambda^E dt}V_{t+dt} + (1-e^{-\lambda^E dt})\int \max\{V_{t+dt}, \tilde{V}\}dF(\tilde{V})\right]n_t - (1-e^{-rdt})Un_t \\
&\quad + e^{-(r+\varkappa)dt}(1-\chi)\mathbb{E}S(z_{t+dt}, n_{t+dt}) - e^{-(r+\varkappa)dt}(1-\chi)\mathbb{E}[W_{t+dt}(z_{t+dt})]n_{t+dt} \\
\text{s.t.} \quad n_{t+dt} = &\, e^{-sdt}(1-\varsigma)\left[\left(e^{-\lambda^E dt} + (1-e^{-\lambda^E dt})(1-F(V_{t+dt}))\right)n_t + (1-e^{\lambda^F dt})G(V_{t+dt})vn_t\right]
\end{aligned}$$

Using the employment evolution equation, we can rewrite the above equation as

$$S(z_t, n_t) = \max_{v,\varsigma,n_{t+dt},V_{t+dt},\chi} y(z_t, n_t)dt - c(v)n_t dt$$

$$
\begin{aligned}
&+ e^{-rdt}\left[e^{-\lambda^E dt}V_{t+dt} + (1-e^{-\lambda^E dt})\int \max\{V_{t+dt},\tilde{V}\}dF(\tilde{V})\right]n_t - (1-e^{-rdt})Un_t \\
&+ e^{-(r+\varkappa)dt}(1-\chi)\mathbb{E}S(z_{t+dt},n_{t+dt}) \\
&- e^{-rdt}V_{t+dt}\left[\left(e^{-\lambda^E dt} + (1-e^{-\lambda^E dt})(1-F(V_{t+dt}))\right)n_t + (1-e^{\lambda^F dt})G(V_{t+dt})vn_t\right] \\
\text{s.t.}\quad & n_{t+dt} = e^{-sdt}(1-\varsigma)\left[\left(e^{-\lambda^E dt} + (1-e^{-\lambda^E dt})(1-F(V_{t+dt}))\right)n_t + (1-e^{\lambda^F dt})G(V_{t+dt})vn_t\right],
\end{aligned}
$$

which further simplifies to

$$
\begin{aligned}
S(z_t,n_t) = \max_{v,\varsigma,n_{t+dt},V_{t+dt},\chi}\ & y(z_t,n_t)dt - c(v)n_t dt - (1-e^{-rdt})Un_t - e^{-rdt}V_{t+dt}(1-e^{\lambda^F dt})G(V_{t+dt})vn_t \\
&+ e^{-rdt}(1-e^{-\lambda^E dt})\int_{V_{t+dt}}\tilde{V}dF(\tilde{V})n_t + e^{-(r+\varkappa)dt}(1-\chi)\mathbb{E}S(z_{t+dt},n_{t+dt}) \\
\text{s.t.}\quad & n_{t+dt} = e^{-sdt}(1-\varsigma)\left[\left(e^{-\lambda^E dt} + (1-e^{-\lambda^E dt})(1-F(V_{t+dt}))\right)n_t + (1-e^{\lambda^F dt})G(V_{t+dt})vn_t\right],
\end{aligned}
$$

A.2 Continuous Time Limit

We can rewrite Problem 1 as

$$
\begin{aligned}
S(z_t,n_t) = \max_{v,n_{t+dt},V_{t+dt}}\ & y(z_t,n_t)dt - c(v)n_t dt - (1-e^{-rdt})Un_t \\
&- e^{-rdt}V_{t+dt}(1-e^{\lambda^F dt})G(V_{t+dt})vn_t + e^{-rdt}(1-e^{-\lambda^E dt})\int_{V_{t+dt}}\tilde{V}dF(\tilde{V})n_t \\
&+ e^{-(r+\varkappa)dt}\max\{\mathbb{E}S(z_{t+dt},n_{t+dt}),S^*(z_{t+dt},n_{t+dt})\} \\
\text{s.t.}\quad & n_{t+dt} = e^{-sdt}\left[\left(e^{-\lambda^E dt} + (1-e^{-\lambda^E dt})(1-F(V_{t+dt}))\right)n_t + (1-e^{\lambda^F dt})G(V_{t+dt})vn_t\right],
\end{aligned}
$$

where $S^*(z_t,n_t)$ is the value from firing or exiting:

$$
S^*(z_t,n_t) = \max\left\{\max_{n^*\le n_t}\mathbb{E}S(z_{t+dt},n^*),0\right\} \tag{47}
$$

Now we will consider a continuous time limit of the above equation. We divide into two cases: (i) firms do not fire or exit ($\mathbb{E}S(z_{t+dt},n_{t+dt}) > S^*(z_{t+dt},n_{t+dt})$), and (ii) firms fire or exit ($\mathbb{E}S(z_{t+dt},n_{t+dt}) \le S^*(z_{t+dt},n_{t+dt})$). If firms do not fire or exit, the continuous time limit (i.e., $dt \to 0$) of the Bellman equation is

$$
\begin{aligned}
\rho S(z_t,n_t) = \max_{v,V_t}\ & y(z_t,n_t) - c(v)n_t - rUn_t - V_t\lambda^F G(V_t)vn_t + \lambda^E\int_{V_t}\tilde{V}dF(\tilde{V})n_t \\
&+ \mathcal{L}(z_t,n_t,V_t,v)S(z_t,n_t),
\end{aligned} \tag{48}
$$

where $\mathcal{L}(z_t, n_t, W_t)$ is the generator for (z, n) defined for arbitrary function $f(z, n)$:

$$\begin{aligned}\mathcal{L}(z, n, V, v)f(z, n) = \mu(z)\partial_z f(z, n) + \frac{\sigma^2(z)}{2}\partial_{zz} f(z, n) \\ + \left[\lambda^F G(V)v - s - \lambda^E(1 - F(V))\right] n\partial_n f(z, n).\end{aligned} \tag{49}$$

Now we turn to the case where firms fire or exit. In the continuous time limit, the value from firing or exiting is given by

$$S^*(z_t, n_t) = \max\{\max_{n^* \leq n_t} S(z_t, n^*), 0\}. \tag{50}$$

If $S(z_t, n_t) > S^*(z_t, n_t)$, then firms do not fire or exit and follow the HJB equation in (48). Otherwise, firms fire or exit with $S(z_t, n_t) = S^*(z_t, n_t)$, in which case $S(z, t)$ would be strictly higher than the value from not adjusting, which is the right-hand side of the HJB equation (48). This can be compactly written as the HJB Quasi-Variational Inequality (QVI), where $S \equiv S(z, n)$ solves

$$\begin{aligned}\min\Bigg\{\rho S - \max_{v,V} y(z, n) - c(v)n - rUn - V\lambda^F G(V)vn + \lambda^E \int_V \tilde{V} dF(\tilde{V})n + \mathcal{L}(z, n, V, v)S, \\ S - S^*(z, n)\Bigg\} = 0,\end{aligned} \tag{51}$$

A.3 Risk-Averse Workers

In the baseline model, we assumed that the workers are risk-neutral. We relax this assumption by assuming that the worker flow utility is given

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}. \tag{52}$$

The baseline model is nested as a special case with $\gamma = 0$.

With this modification, the Bellman equation for the firm's problem becomes

$$\begin{aligned}J(z_t, n_t, W_t) = \max_{w,v,\varsigma,n_{t+dt},W_{t+dt},\chi} y(z_t, n_t)dt - wn_t dt - c(v)n_t dt \\ + e^{-(r+\varkappa)dt}(1 - \chi)\mathbb{E}J(z_{t+dt}, n_{t+dt}, W_{t+dt}(z_{t+dt})) \\ \text{s.t.} \quad W_t = u(w)dt + e^{-rdt}\left[e^{-\lambda^E dt}V_{t+dt} + (1 - e^{-\lambda^E dt})\int \max\{V_{t+dt}, \tilde{V}\}dF(\tilde{V})\right] - (1 - e^{-rdt})U \\ n_{t+dt} = e^{-sdt}(1 - \varsigma)\left[\left(e^{-\lambda^E dt} + (1 - e^{-\lambda^E dt})(1 - F(V_{t+dt}))\right)n_t + (1 - e^{\lambda^F dt})G(V_{t+dt})vn_t\right]\end{aligned}$$

and $V_{t+dt} = e^{-(s+\varkappa)dt}(1-\varsigma)(1-\chi)\mathbb{E}[W_{t+dt}(z_{t+dt})]$. We can subsitute the promise-keeping constraint into the Bellman equation to eliminate w:

$$\begin{aligned} J(z_t, n_t, W_t) = &\max_{w,v,\varsigma,n_{t+dt},W_{t+dt},\chi} y(z_t, n_t)dt - \tilde{w}(W_t, V_{t+dt})n_t dt - c(v)n_t dt \\ &+ e^{-(r+\varkappa)dt}(1-\chi)\mathbb{E}J(z_{t+dt}, n_{t+dt}, W_{t+dt}(z_{t+dt})) \\ \text{s.t.} \qquad n_{t+dt} = &e^{-sdt}(1-\varsigma)\left[\left(e^{-\lambda^E dt} + (1-e^{-\lambda^E dt})(1-F(V_{t+dt}))\right)n_t + (1-e^{\lambda^F dt})G(V_{t+dt})vn_t\right], \end{aligned}$$

where

$$\begin{aligned} &\tilde{w}(W_t, V_{t+dt}) \\ &\quad \equiv u^{-1}\left(\frac{1}{dt}\left(W_t - e^{-rdt}\left[e^{-\lambda^E dt}V_{t+dt} + (1-e^{-\lambda^E dt})\int \max\{V_{t+dt}, \tilde{V}\}dF(\tilde{V})\right] + (1-e^{-rdt})U\right)\right). \end{aligned}$$

The first-order optimality condition with respect to $W_{t+dt}(z_{t+dt})$ implies that for any z^1_{t+dt} and z^2_{t+dt}, we must have

$$\partial_W J(z^1_{t+dt}, n_{t+dt}, W_{t+dt}(z^1_{t+dt})) = \partial_W J(z^2_{t+dt}, n_{t+dt}, W_{t+dt}(z^2_{t+dt})). \tag{53}$$

By the envelope condition, we have

$$\partial_W J(z_t, n_t, W_t) = \frac{1}{u'(\tilde{w}(W_t, V_{t+dt}(z_t, n_t, W_t)))}n_t, \tag{54}$$

where $V_{t+dt}(z_t, n_t, W_t)$ is the policy function. Therefore, as long as workers are risk-averse, $W_{t+dt}(z^1_t; n_t, W_t)$ must satisfy that for any z^1_t and z^2_t,

$$\frac{1}{u'\left(\tilde{w}(W_{t+dt}(z^1_t), V_{t+dt}(z^1_{t+dt}, n_{t+dt}, W_{t+dt}(z^1_{t+dt})))\right)}n_{t+dt} \tag{55}$$

$$= \frac{1}{u'\left(\tilde{w}(W_{t+dt}(z^2_t), V_{t+dt}(z^2_{t+dt}, n_{t+dt}, W_{t+dt}(z^2_{t+dt})))\right)}n_{t+dt}. \tag{56}$$

This implies that

$$\tilde{w}(W_{t+dt}(z^1_{t+dt}), V_{t+dt}(z^1_{t+dt}, n_{t+dt}, W_{t+dt}(z^1_{t+dt}))) = \tilde{w}(W_{t+dt}(z^2_{t+dt}), V_{t+dt}(z^2_{t+dt}, n_{t+dt}, W_{t+dt}(z^2_{t+dt}))).$$

Note that this condition must hold for any $\gamma > 0$, and hence in the limit of $\gamma \to 0$.

A.4 Monotonicity of Promised Utility Offer

We will look for an equilibrium in which $F(V)$ and $G(V)$ have support $[0, \bar{V}]$, and the distributions are smooth for any $V > 0$. In such an equilibrium, almost all firms will offer V that is an interior opitmum $V \in [0, \bar{V}]$. The first-order condition with respect to v is

$$\lambda^F G(V) \left(S_n - V\right) = c'(v). \tag{57}$$

Since $c'(v) = \bar{c}^v v^\nu$, we can explictly solve for v as

$$v = \left(\frac{\lambda^F G(V)(S_n - V)}{\bar{c}^v}\right)^{\frac{1}{\nu}} \tag{58}$$

$$\equiv \hat{v}(S_n, V) \tag{59}$$

The first-order condition with respect to V is given by

$$\left[\lambda^F g(V)\hat{v}(S_n, V) + \lambda^E f(V)\right](S_n - V) = \lambda^F G(V)\hat{v}(S_n, V), \tag{60}$$

which implicity defines the policy function as a function of the marginal suprlus, $V = V(S_n)$. The associated second-order condition is given by

$$\begin{aligned} &\left[\lambda^F g'(V)\hat{v}(S_n, V) + \lambda^E f'(V) + \lambda^F g(V)\frac{\partial \hat{v}(S_n, V)}{\partial V}\right](S_n - V) \\ &- \left[\lambda^F g(V)\hat{v}(S_n, V) + \lambda^E f(V)\right] - \lambda^F g(V)\hat{v}(S_n, V) - \lambda^F G(V)\frac{\partial \hat{v}(S_n, V)}{\partial V} \qquad (61) \\ &\equiv \mathcal{G}(S_n, V) < 0, \qquad (62) \end{aligned}$$

Totally differentiating (60) with respect to S_n and V, we get

$$\left[\lambda^F g(V)\hat{v}(S_n, V) + \lambda^E f(V)\right] d(S_n) - \lambda^F G(V)\frac{\partial \hat{v}(S_n, V)}{\partial (S_n)} d(S_n) \tag{63}$$

$$+\lambda^F g(V)(S_n - V)\frac{\partial \hat{v}(S_n, V)}{\partial S_n} dS_n + \mathcal{G}(S_n, V) dV = 0, \tag{64}$$

which we can rewrite as

$$\frac{dV}{d(\partial_n S)} = -\frac{\lambda^E f(V)\left(1 - \frac{1}{\nu}\right) + \lambda^F g(V)\hat{v}(\partial_n S, V)}{\mathcal{G}(\partial_n S, V)} > 0, \tag{65}$$

where the last inequality follows from the second-order condition $\mathcal{G}(S_n, V) < 0$ as well as our assumption that $\nu \geq 1$. This shows that the optimal promised utility offer is strictly increasing

in the marginal surplus.

A.5 Equilibrium Promised Utility Offer

The monotonicity of promised utiliy offer implies

$$G(V(S_n)) = \hat{G}(S_n), \quad F(V(S_n)) = \hat{F}(S_n). \tag{66}$$

Differentiating both sides with respect to S_n, we get

$$g(V(S_n))V'(S_n) = \hat{g}(S_n), \quad f(V(S_n))V'(S_n) = \hat{f}(S_n), \tag{67}$$

where $\hat{g}(S_n) \equiv G'(S_n)$ and $\hat{f}(S_n) \equiv \hat{F}'(S_n)$. Recall the first-order condition for promised utility offer (60) is

$$\left[\lambda^F g(V(S_n))\hat{v}(S_n, V(S_n)) + \lambda^E f(V(S_n))\right](S_n - V(S_n)) = \lambda^F G(V(S_n))\hat{v}(S_n, V(S_n)), \tag{68}$$

Substituting (67) into (68), we have

$$\left[\lambda^F \frac{\hat{g}(S_n)}{V'(S_n)}\hat{v}(S_n, V(S_n)) + \lambda^E \frac{\hat{f}(S_n)}{V'(S_n)}\right](S_n - V(S_n)) = \lambda^F G(V(S_n))\hat{v}(S_n, V(S_n)), \tag{69}$$

Note that

$$\hat{f}(S_n) = \frac{1}{\tilde{v}}\hat{v}(S_n, V(S_n))h(S_n), \tag{70}$$

where $h(S_n) \equiv \hat{H}'(S_n)$ denote the density of workers at a firm with marginal surpus S_n. We can rewrite (69) as

$$\left[\lambda^F \hat{g}(S_n) + \lambda^E \frac{1}{\tilde{v}}h(S_n)\right](S_n - V(S_n)) = \lambda^F G(V(S_n))V'(S_n), \tag{71}$$

Note that

$$\frac{\lambda^E}{\lambda^F} = \frac{\zeta M(\tilde{u}, \tilde{v})/\tilde{u}}{M(\tilde{u}, \tilde{v})/\tilde{v}} \tag{72}$$

$$= \zeta \frac{\tilde{v}}{\tilde{u}}. \tag{73}$$

Substituting this into (71), we get

$$\left[\hat{g}(S_n) + \zeta \frac{1}{\tilde{u}} h(S_n)\right] (S_n - V(S_n)) = G(V(S_n))V'(S_n), \tag{74}$$

Also note that

$$\hat{g}(S_n) = (1 - p^u)\frac{h(S_n)}{1-u} \tag{75}$$

$$= \zeta \frac{1}{\tilde{u}} h(S_n), \tag{76}$$

and

$$G(V(S_n)) = \hat{G}(S_n) \tag{77}$$

$$= p^u + \zeta \frac{1}{\tilde{u}} \hat{H}(S_n) \tag{78}$$

Substituting (76) and (78) into (74),

$$2\zeta \frac{1}{\tilde{u}} h(S_n) (S_n - V(S_n)) = \left(p^u + \zeta \frac{1}{\tilde{u}} \hat{H}(S_n)\right) V'(S_n), \tag{79}$$

which we can rewrite as

$$2\zeta h(S_n) (S_n - V(S_n)) = \left(u + \zeta \hat{H}(S_n)\right) V'(S_n). \tag{80}$$

Now we solve the above ODE for $V(S_n)$. Rearranging the terms,

$$V'(S_n) + \frac{2\zeta h(S_n)}{u + \zeta \hat{H}(S_n)} V(S_n) = \frac{2\zeta h(S_n)}{u + \zeta \hat{H}(S_n)} S_n \tag{81}$$

$$\Leftrightarrow \quad \left(u + \zeta \hat{H}(S_n)\right)^2 V'(S_n) + 2\zeta h(S_n)(u + \zeta \hat{H}(S_n))V(S_n) = (u + \zeta \hat{H}(S_n))2\zeta h(S_n) S_n \tag{82}$$

$$\Leftrightarrow \quad \frac{d}{dS_n}\left[\left(u + \zeta \hat{H}(S_n)\right)^2 V'(S_n)\right] = (u + \zeta \hat{H}(S_n))2\zeta h(S_n) S_n \tag{83}$$

$$\tag{84}$$

With the boundary condition $V(0) = 0$, we can integrate the above equation to obtain

$$\left(u + \zeta \hat{H}(S_n)\right)^2 V(S_n) = \int_0^{S_n} \left\{(u + \zeta \hat{H}(\tilde{S}_n))2\zeta h(\tilde{S}_n)\tilde{S}_n\right\} d\tilde{S}_n \tag{85}$$

$$\Leftrightarrow \quad V(S_n) = \frac{1}{\left(u + \zeta \hat{H}(S_n)\right)^2} \int_0^{S_n} \left\{(u + \zeta \hat{H}(\tilde{S}_n))2\zeta h(\tilde{S}_n)\right\} \tilde{S}_n d\tilde{S}_n \tag{86}$$

One can view

$$\hat{G}^2(X|X \leq S_n) \equiv \frac{1}{\left(u + \zeta \hat{H}(S_n)\right)^2} \int_0^{\tilde{S}_n} \left\{(u + \zeta \hat{H}(\hat{S}_n))2\zeta h(\hat{S}_n)\right\} d\hat{S}_n \tag{87}$$

$$= \frac{\left(u + \zeta \hat{H}(\tilde{S}_n)\right)^2}{\left(u + \zeta \hat{H}(S_n)\right)^2} \tag{88}$$

as the distribution of the marginal surplus connditional on the marginal surplus being less than S_n. Let $\hat{g}^2(X|X \leq S_n)$ be the associated probability density function. Therefore one can rewrite $V(S_n)$ as

$$V(S_n) = \frac{1}{\left(u + \zeta \hat{H}(S_n)\right)^2} \int_0^{S_n} \left\{(u + \zeta \hat{H}(\tilde{S}_n))2\zeta h(\tilde{S}_n)\right\} \tilde{S}_n d\tilde{S}_n \tag{89}$$

$$= \int_0^{S_n} \hat{g}^2(\tilde{S}_n|\tilde{S}_n \leq S_n)\tilde{S}_n d\tilde{S}_n \tag{90}$$

$$\equiv \mathbb{E}_{H^2_\zeta}\left[\tilde{S}_n|\tilde{S}_n \leq S_n\right]. \tag{91}$$

Altarnatively, one can use integration by parts to rewrite $V(S_n)$ as

$$V(S_n) = \int_0^{S_n} \hat{g}^2(\tilde{S}_n|\tilde{S}_n \leq S_n)\tilde{S}_n d\tilde{S}_n \tag{92}$$

$$= S_n - \int_0^{S_n} \hat{G}^2(\tilde{S}_n|\tilde{S}_n \leq S_n) d\tilde{S}_n \tag{93}$$

The vacancy policy function is given by

$$v(S_n) = \hat{v}(S_n, V(S_n)) \tag{94}$$

$$= \left(\frac{\lambda^F G(V(S_n))(S_n - V(S_n))}{\bar{c}^v}\right)^{\frac{1}{\nu}} \tag{95}$$

Now we verify that the second-order condition, (62), is satisfied. Further differenciating (67),

$$\hat{f}'(S_n) = f'(V(S_n))V'(S_n)^2 + f(V(S_n))V''(S_n) \tag{96}$$

$$\hat{g}'(S_n) = g'(V(S_n))V'(S_n)^2 + g(V(S_n))V''(S_n) \tag{97}$$

Rearranging (69), the first-order condition with respect to V is

$$\left[\lambda^F \hat{g}(S_n)v(S_n) + \lambda^E \hat{f}(S_n)\right](S_n - V(S_n)) = \lambda^F \hat{G}(S_n)v(S_n)V'(S_n), \tag{98}$$

Taking the derivative w.r.t. S_n,

$$\left[\lambda^F \hat{g}'(S_n)v(S_n) + \lambda^F \hat{g}(S_n)v'(S_n) + \lambda^E \hat{f}'(S_n)\right](S_n - V(S_n)) \tag{99}$$

$$+ \left[\lambda^F \hat{g}(S_n)v(S_n) + \lambda^E \hat{f}(S_n)\right](1 - V'(S_n)) \tag{100}$$

$$= \lambda^F \hat{g}(S_n)v(S_n)V'(S_n) + \lambda^F \hat{G}(S_n)v'(S_n)V'(S_n) + \lambda^F \hat{G}(S_n)v(S_n)V''(S_n). \tag{101}$$

We can solve for $V''(S_n)$ as

$$V''(S_n) = \frac{1}{\lambda^F \hat{G}(S_n)v(S_n)} \times \left\{ \left[\lambda^F \hat{g}'(S_n)v(S_n) + \lambda^F \hat{g}(S_n)v'(S_n) + \lambda^E \hat{f}'(S_n)\right](S_n - V(S_n)) \right.$$
$$\left. + \left[\lambda^F \hat{g}(S_n)v(S_n) + \lambda^E \hat{f}(S_n)\right](1 - V'(S_n)) - \lambda^F \hat{g}(S_n)v(S_n)V'(S_n) - \lambda^F \hat{G}(S_n)v'(S_n)V'(S_n) \right\} \tag{102}$$

Now consider the following expression:

$$\begin{aligned}
&\lambda^F g'(V(S_n))\hat{v}(S_n, V(S_n)) + \lambda^E f'(V(S_n)) \\
&= \frac{1}{V'(S_n)^2}\left[\lambda^F \hat{v}(S_n, V(S_n))(\hat{g}'(S_n) - g(V(S_n))V''(S_n)) + \lambda^E(\hat{f}'(S_n) - f(V(S_n))V''(S_n))\right] \\
&= \frac{1}{V'(S_n)^2}\left[\lambda^F \hat{v}(S_n, V(S_n))\hat{g}'(S_n) + \lambda^E \hat{f}'(S_n) - V''(S_n)\left(\lambda^F \hat{v}(S_n, V(S_n))g(V(S_n)) + \lambda^E f(V(S_n))\right)\right],
\end{aligned}$$

where the first equality uses (96) and (97). Substituting (102) into the above expression, we get

$$\begin{aligned}
&\lambda^F g'(V(S_n))\hat{v}(S_n, V(S_n)) + \lambda^E f'(V(S_n)) \\
=&\frac{1}{V'(S_n)^2}\left[\lambda^F \hat{v}(S_n, V(S_n))\hat{g}'(S_n) + \lambda^E \hat{f}'(S_n)\right. \\
&\quad \underbrace{- \frac{(S_n - V(S_n))\left(\lambda^F \hat{v}(S_n, V(S_n))g(V(S_n)) + \lambda^E f(V(S_n))\right)}{\lambda^F \hat{G}(S_n)v(S_n)}}_{=1 \text{ by FOC}} \\
&\quad \left. \times \left[\lambda^F \hat{g}'(S_n)v(S_n) + \lambda^F \hat{g}(S_n)v'(S_n) + \lambda^E \hat{f}'(S_n)\right]\right]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\left(\lambda^F \hat{v}(S_n, V(S_n)) g(V(S_n)) + \lambda^E f(V(S_n))\right)}{\lambda^F \hat{G}(S_n) v(S_n)} \Bigg\{ \left[\lambda^F \hat{g}(S_n) v(S_n) + \lambda^E \hat{f}(S_n)\right] (1 - V'(S_n)) \\
&- \lambda^F \hat{g}(S_n) v(S_n) V'(S_n) - \lambda^F \hat{G}(S_n) v'(S_n) V'(S_n) \Bigg\} \Bigg] \\
=&\frac{1}{V'(S_n)^2} \Bigg[- \lambda^F \hat{g}(S_n) v'(S_n) \\
&- \frac{\left(\lambda^F \hat{v}(S_n, V(S_n)) g(V(S_n)) + \lambda^E f(V(S_n))\right)}{\lambda^F \hat{G}(S_n) v(S_n)} \Bigg\{ \left[\lambda^F \hat{g}(S_n) v(S_n) + \lambda^E \hat{f}(S_n)\right] (1 - V'(S_n)) \\
&- \lambda^F \hat{g}(S_n) v(S_n) V'(S_n) - \lambda^F \hat{G}(S_n) v'(S_n) V'(S_n) \Bigg\} \Bigg] \qquad (103)
\end{aligned}
$$

Now we check the second-order condition, (62). Substituting (103) into the second-order condition, we have

$$
\begin{aligned}
\mathcal{G}(S_n, V) = &\left[\lambda^F g'(V) \hat{v}(S_n, V) + \lambda^E f'(V)\right] (S_n - V) \\
&+ \lambda^F g(V) \frac{\partial \hat{v}(S_n, V)}{\partial V} (S_n - V) \\
&\quad - \left[\lambda^F g(V) \hat{v}(S_n, V) + \lambda^E f(V)\right] \\
&- \lambda^F g(V) \hat{v}(S_n, V) - \lambda^F G(V) \frac{\partial \hat{v}(S_n, V)}{\partial V} \\
=&\frac{1}{V'(S_n)^2} \Bigg[- \lambda^F \hat{g}(S_n) v'(S_n) (S_n - V) \\
&- \underbrace{\frac{\left(\lambda^F \hat{v}(S_n, V(S_n)) g(V(S_n)) + \lambda^E f(V(S_n))\right) (S_n - V)}{\lambda^F \hat{G}(S_n) v(S_n)}}_{=1} \\
&\times \Bigg\{ \left[\lambda^F \hat{g}(S_n) v(S_n) + \lambda^E \hat{f}(S_n)\right] (1 - V'(S_n)) \\
&\quad - \lambda^F \hat{g}(S_n) v(S_n) V'(S_n) - \lambda^F \hat{G}(S_n) v'(S_n) V'(S_n) \Bigg\} \Bigg] \\
&+ \lambda^F g(V) \frac{\partial \hat{v}(S_n, V)}{\partial V} (S_n - V) \\
&- \left[\lambda^F g(V) \hat{v}(S_n, V) + \lambda^E f(V)\right] \\
&- \lambda^F g(V) \hat{v}(S_n, V) - \lambda^F G(V) \frac{\partial \hat{v}(S_n, V)}{\partial V} \\
=&- \frac{1}{V'(S_n)^2} \lambda^F \hat{g}(S_n) v'(S_n) (S_n - V)
\end{aligned}
$$

$$- \frac{1}{V'(S_n)^2}\left\{\left[\lambda^F \hat{g}(S_n)v(S_n) + \lambda^E \hat{f}(S_n)\right] - \lambda^F \hat{g}(S_n)v(S_n)V'(S_n) - \lambda^F \hat{G}(S_n)v'(S_n)V'(S_n)\right\}\Bigg]$$
$$+ \lambda^F g(V)\frac{\partial \hat{v}(S_n, V)}{\partial V}(S_n - V) - \lambda^F g(V)\hat{v}(S_n, V) - \lambda^F G(V)\frac{\partial \hat{v}(S_n, V)}{\partial V}$$

Note that the rank-preserving property implies

$$\lambda^F \hat{G}(S_n)\hat{v}(S_n) = \lambda^F G(V(S_n))\hat{v}(S_n, V(S_n)) \tag{104}$$

Differentiating the above equation with respect to S_n, we have

$$\lambda^F \hat{g}(S_n)\hat{v}(S_n)\frac{1}{V'(S_n)} + \frac{1}{V'(S_n)}\lambda^F \hat{G}(S_n)\hat{v}'(S_n) \tag{105}$$
$$= \lambda^F g(V)v(S_n) + \lambda^F G(V(S_n))\frac{\partial \hat{v}(S_n, V(S_n))}{S_n}\frac{1}{V'(S_n)} + \lambda^F G(V(S_n))\frac{\partial \hat{v}(S_n, V(S_n))}{V} \tag{106}$$

Using this equation, we can simplify the second-order condition as

$$\mathcal{G}(S_n, V) = -\frac{1}{V'(S_n)^2}\lambda^F \hat{g}(S_n)v'(S_n)(S_n - V)$$
$$- \frac{1}{V'(S_n)^2}\left[\lambda^F \hat{g}(S_n)v(S_n) + \lambda^E \hat{f}(S_n)\right] + \lambda^F G(V(S_n))\frac{1}{V'(S_n)}\frac{\partial \hat{v}(S_n, V(S_n))}{\partial S_n}$$
$$+ \lambda^F g(V)\frac{\partial \hat{v}(S_n, V)}{\partial V}(S_n - V)$$

We can rewrite the last term as

$$\lambda^F g(V)\frac{\partial \hat{v}(S_n, V)}{\partial V}(S_n - V) = \lambda^F \frac{\hat{g}(S_n)}{V'(S_n)}\frac{\partial \hat{v}(S_n, V)}{\partial V}(S_n - V)$$
$$= \lambda^F \frac{\hat{g}(S_n)}{V'(S_n)^2}\left[v'(S_n) - \frac{\partial \hat{v}(S_n, V)}{\partial S_n}\right](S_n - V).$$

Plugging this back into the second-order condition, we have

$$\mathcal{G}(S_n, V) = -\frac{1}{V'(S_n)^2}\left[\lambda^F \hat{g}(S_n)v(S_n) + \lambda^E \hat{f}(S_n)\right] + \lambda^F G(V(S_n))\frac{\partial \hat{v}(S_n, V(S_n))}{S_n}\frac{1}{V'(S_n)}$$
$$- \lambda^F \frac{\hat{g}(S_n)}{V'(S_n)^2}\frac{\partial \hat{v}(S_n, V)}{\partial S_n}(S_n - V)$$

Note that

$$\frac{\partial \hat{v}(S_n, V(S_n))}{S_n} = v(S_n)\frac{1}{\nu}\frac{1}{S_n - V(S_n)}.$$

Therefore, we have

$$\begin{aligned}
\mathcal{G}(S_n, V) &= -\frac{1}{V'(S_n)^2}\left[\lambda^F \hat{g}(S_n)v(S_n) + \lambda^E \hat{f}(S_n)\right] + \frac{\lambda^F G(V(S_n))v(S_n)}{S_n - V(S_n)}\frac{1}{\nu}\frac{1}{V'(S_n)} \\
&\quad - \lambda^F \frac{\hat{g}(S_n)}{V'(S_n)^2}\frac{\partial \hat{v}(S_n, V)}{\partial S_n}(S_n - V) \\
&= -\frac{1}{V'(S_n)^2}\left[\lambda^F \hat{g}(S_n)v(S_n) + \lambda^E \hat{f}(S_n)\right] + \left[\lambda^F g(V(S_n))v(S_n) + \lambda^E f(V(S_n))\right]\frac{1}{\nu}\frac{1}{V'(S_n)} \\
&\quad - \lambda^F \frac{\hat{g}(S_n)}{V'(S_n)^2}\frac{\partial \hat{v}(S_n, V)}{\partial S_n}(S_n - V) \\
&= -\frac{1}{V'(S_n)^2}\left[\lambda^F \hat{g}(S_n)v(S_n) + \lambda^E \hat{f}(S_n)\right]\left(1 - \frac{1}{\nu}\right) \\
&\quad - \lambda^F \frac{\hat{g}(S_n)}{V'(S_n)^2}\frac{\partial \hat{v}(S_n, V)}{\partial S_n}(S_n - V) \\
&< 0,
\end{aligned}$$

where the last inequality follows from the fact that $\nu > 1$. This verifies the second-order condition, (62).

A.6 Planning Problem

Consider the Halmitonian of the planning problem 3. We let $S_t^p(z, n)$ denote the co-state variable associated with the law of motion of the distribution. The current value function is given by

$$\begin{aligned}
\mathcal{H} &= \int \left(y(z, n) - c(v_t(z, n))n\right) dg_t + (1 - \textstyle\int n dg_t)b - m_t^0 c^e \\
&\quad + \int\int S^p(z, n)\Big[-\partial_n\left[dn_t(z, n)g_t(z, n)\right] - \partial_z[\mu(z)g_t(z, n)] + \frac{1}{2}\partial_{zz}[(\sigma(z))^2 g_t(z, n)] \\
&\quad - \varkappa g_t(z, n) + m_t^0 \psi(z, n)\Big] dz dn \\
&= \int \left(y(z, n) - c(v_t(z, n))n\right) dg_t + (1 - \textstyle\int n dg_t)b - m_t^0 c^e \\
&\quad + \int\int \Big[S_{t,n}^p(z, n) dn_t(z, n) g_t(z, n) + S_{t,z}^p(z, n)\mu(z)g_t(z, n) + \frac{1}{2}S_{t,zz}^p(z, n)(\sigma(z))^2 g_t(z, n) \\
&\quad - \varkappa S_t^p(z, n)g_t(z, n) + S_t^p(z, n)m_t^0 \psi(z, n)\Big] dz dn,
\end{aligned}$$

where the second line uses integration by parts,

$$dn_t(z, n) = \left[\lambda_t^F G(z, n)v(z, n) - s - \lambda_t^E(1 - F(z, n))\right] n dt - \varsigma_t$$

and

$$G(z,n) \equiv \frac{1}{\tilde{u}}\left[x_{t,u,zn}\left(1-\int n dg_t\right)+\zeta\int x_{t,z'n',zn}n'dg_t(z',n')\right]$$
$$1-F(z,n) \equiv \frac{1}{\tilde{v}}\int x_{t,zn,z'n'}v(z',n')n'dg_t(z',n')$$
$$\lambda_t^F \equiv \frac{M(\tilde{u},\tilde{v})}{\tilde{v}}, \quad \lambda_t^E \equiv \zeta\frac{M(\tilde{u},\tilde{u})}{\tilde{u}}, \quad \tilde{u} \equiv 1-\int n dg_t+\zeta\int n dg_t, \quad \tilde{v} \equiv \int v_t(z,n)n dg_t$$

and $x_{t,zn,z'n'} \in [0,1]$, $v_t(z,n) \geq 0$, and $\varsigma_t \geq 0$.

The FOC w.r.t. $x_{t,u,zn}$ implies

$$x_{t,u,zn} = \begin{cases} 1 & \text{if } S_{t,n}^p(z,n) \geq 0 \\ 0 & \text{otherwise} \end{cases}. \tag{107}$$

Likewise, the derivative of the Hamiltonian w.r.t. $x_{t,z'n',zn}$ is

$$\frac{\partial \mathcal{H}}{\partial x_{t,z'n',zn}} = \left[\lambda_t^F\frac{\zeta}{\tilde{u}}v(z,n)n'ndg_t(z',n')dg_t(z,n)S_n^p(z,n)\right. \tag{108}$$
$$\left.\lambda_t^E\frac{1}{\tilde{v}}v(z,n)nn'dg_t(z,n)dg_t(z',n')S^p\right] \tag{109}$$
$$= \zeta\frac{M(\tilde{u},\tilde{v})}{\tilde{v}\tilde{u}}\left[S_{t,n}^p(z,n)-S_{t,n}^p(z,n)\right], \tag{110}$$

which implies

$$x_{t,z'n',zn} = \begin{cases} 1 & \text{if } S_{t,n}^p(z,n) \geq S_{t,n}^p(z',n') \\ 0 & \text{otherwise} \end{cases}. \tag{111}$$

The optimality condition w.r.t. ς_t is

$$S_{t,n}^p(z,n) \geq 0.$$

The FOC w.r.t. $v(z,n)$ is

$$\begin{aligned}
&\lambda_t^F G(z,n) S^p(z,n) - \frac{\lambda_t^E}{\tilde{v}} \int x_{t,z'n',zn} S^p_{t,n}(z',n') n' dg_t(z',n') \\
&\quad + \frac{\partial \ln \lambda_t^F}{\partial \tilde{v}} \int \lambda_t^F G(z,n) v(z,n) n S^p_{t,n}(z,n) dg_t(z,n) \\
&\quad - \frac{\partial \ln \lambda_t^F}{\partial \tilde{v}} \int \lambda_t^E (1 - F(z,n)) n S^p_{t,n}(z,n) dg_t(z,n) = c'(v(z,n)).
\end{aligned} \tag{112}$$

We seek to rewrite the left-hand side of the above expression as

$$\frac{\partial \ln \lambda_t^F}{\partial \tilde{v}} \int \lambda_t^F G(z,n) v(z,n) n S^p_{t,n}(z,n) dg_t(z,n) \tag{113}$$

$$- \frac{\partial \ln \lambda^F}{\partial \tilde{v}} \int \lambda^E (1 - F(z,n)) n S^p_{t,n}(z,n) dg_t(z,n) \tag{114}$$

$$= \frac{\partial \ln \lambda_t^F}{\partial \ln \tilde{v}} \lambda^{F_t} \int G(z',n') \left(S^p_n(z',n') - \mathbb{E}_{\hat{G}}[\tilde{S}_n | \tilde{S}_n \leq S^p_n(z',n')] \right) \frac{1}{\tilde{v}} v(z',n') n' dg_t(z',n'), \tag{115}$$

where

$$\begin{aligned}
\mathbb{E}_{\hat{G}}[\tilde{S}_n | \tilde{S}_n \leq S^p_n(z,n)] &\equiv \frac{1}{\tilde{u}} \left[\zeta \int x_{t,z'n',zn} S_n(z',n') n' dg_t(z',n') \right] \\
&= \frac{1}{\tilde{u}} \left[\zeta \int_{z',n': S^p_n(z',n') \leq S_n(z,n)} S_n(z',n') n' dg_t(z',n') \right]
\end{aligned}$$

is the weighted average of the marignal surplus conditional on the marginal surplus being less than $S^p_n(z,n)$, and the weight is given by the search intensity weighted employment distribution. Note that the marginal surplus of unemployed worker is zeo. The second line in the above expression imposes the optiamlity conditions for $x_{t,z'n',zn}$ in (111).

Define

$$\Omega_t \equiv \int G(z',n') \left(S^p_n(z',n') - \mathbb{E}_{\hat{G}}[\tilde{S}_n | \tilde{S}_n \leq S^p_n(z',n')] \right) \frac{1}{\tilde{v}} v(z',n') n' dg_t(z',n')$$

as the average gains in the marginal surplus from a meeting. Using this definition, we can rewrite (112) as

$$\lambda^F \left\{ G(z,n) \left(S^p(z,n) - \mathbb{E}_{\hat{G}}[\tilde{S}_n | \tilde{S}_n \leq S^p_n(z,n)] \right) - \eta_t^F \Omega_t \right\} = c'(v(z,n)),$$

where

$$\eta_t^F \equiv - \frac{\partial \ln \lambda_t^F}{\partial \ln \tilde{v}}$$

is the congestion effect from additional vacancy posting.

The optimality condition w.r.t. $S^p_{t,n}(z,n)$, evaluated at the steady state, is

$$
\begin{aligned}
(r+\varkappa)S^p_n(z,n) &= y(z,n) - c(v(z,n))n - bn \\
&\quad + dn_t(z,n)S_n(z,n) + \mu(z)S_z(z,n) + \frac{1}{2}(\sigma(z))^2 S_{zz}(z,n) \\
&\quad - \frac{1}{\tilde{u}}\int \lambda^F_t v(z',n') x_{u,z'n'} S_n(z',n') n' dg_t(z',n') n \\
&\quad + \frac{\zeta}{\tilde{u}}\int \lambda^F_t v(z',n') x_{zn,z'n'} S_n(z',n') n' dg_t(z',n') n \\
&\quad - \lambda^E_t \frac{1}{\tilde{v}} v(z,n) \int x_{z'n'n,zn} n' S_n(z',n') dg_t(z',n') n \\
&\quad - (1-\zeta)\frac{\partial \ln \lambda^F_t}{\partial \tilde{u}} \int \lambda^F G(z',n') v(z',n') n' S^p_{t,n}(z',n') dg_t(z',n') n \\
&\quad + (1-\zeta)\frac{\partial \ln \lambda^E_t}{\partial \tilde{u}} \int \lambda^E (1-F(z',n')) n' S^p_{t,n}(z',n') dg_t(z',n') n \\
&\quad + v(z,n) n \frac{\partial \ln \lambda^F_t}{\partial \tilde{v}} \int \lambda^F G(z,n) v(z,n) n S^p_{t,n}(z,n) dg_t(z,n) \\
&\quad - v(z,n) n \frac{\partial \ln \lambda^E_t}{\partial \tilde{v}} \int \lambda^E (1-F(z,n)) n S^p_{t,n}(z,n) dg_t(z,n) \\
&= y(z,n) - c(v(z,n))n - bn \\
&\quad + dn_t(z,n)S_n(z,n) + \mu(z)S_z(z,n) + \frac{1}{2}(\sigma(z))^2 S_{zz}(z,n) \\
&\quad - \lambda^U_t \int \frac{1}{\tilde{v}} v(z',n') x_{u,z'n'} S_n(z',n') n' dg_t(z',n') n \\
&\quad + \lambda^E_t \int \frac{1}{\tilde{v}} v(z',n') x_{zn,z'n'} S_n(z',n') n' dg_t(z',n') n \\
&\quad - \lambda^F_t v(z,n) \int \frac{1}{\tilde{u}} \zeta x_{z'n'n,zn} n' S_n(z',n') dg_t(z',n') n \\
&\quad - (1-\zeta)\frac{\partial \ln \lambda^U_t}{\partial \ln \tilde{u}} \lambda^U_t \Omega_t n \\
&\quad + \frac{\partial \ln \lambda^F_t}{\partial \ln \tilde{v}} \lambda^{F_t} \Omega_t v(z,n) n \\
&= y(z,n) - c(v(z,n))n - bn \\
&\quad + dn_t(z,n)S_n(z,n) + \mu(z)S_z(z,n) + \frac{1}{2}(\sigma(z))^2 S_{zz}(z,n) \\
&\quad - \lambda^U_t \left\{ \mathbb{E}_{\hat{F}}[\tilde{S}_n] - \eta^U_t \Omega_t \right\} n \\
&\quad + \lambda^E_t \left\{ (1-F(S^p_n)) \mathbb{E}_{\hat{F}}[\tilde{S}_n | \tilde{S}_n \geq S^p_n] - \eta^U_t \Omega_t \right\} n \\
&\quad - \lambda^F_t \left\{ G(S^p_n) \mathbb{E}_{\hat{G}}[\tilde{S}_n | \tilde{S}_n \leq S^p_n] + \eta^F_t \Omega_t \right\} v(z,n) n
\end{aligned}
$$

B Computation Appendix

Algorithm 1 Calibration Algorithm

Set θ^0
Fix tolerances $\varepsilon_{FG} > 0$
Set iteration counter $k \leftarrow 0$
Guess $(F^{(0)}, G^{(0)})$ given $\theta^{(k)}$
Set inner iteration counter $\ell \leftarrow 0$
while $\|(F^{(\ell+1)}, G^{(\ell+1)}) - (F^{(\ell)}, G^{(\ell)})\| > \varepsilon_{FG}$ **do**
    Solve the HJB equation
    Set $m^0 = 1$ and guess z^0
    while The average firm size is not at the target **do**
        Solve the KFE equation
        Update z^0
    end while
    Obtain m^0 to match the market tightness at the target θ^0
    Update $F^{(\ell+1)} = \mathcal{F}\big(F^{(\ell)}, G^{(\ell)}; \theta^{(k)}\big)$
    Update $G^{(\ell+1)} = \mathcal{G}\big(F^{(\ell)}, G^{(\ell)}; \theta^{(k)}\big)$
    $\ell \leftarrow \ell + 1$
end while

Algorithm 2 Steady State Counterfactual Algorithm

Guess $\theta^{(0)}$
Fix tolerances $\varepsilon_\theta, \varepsilon_{FG} > 0$
Set iteration counter $k \leftarrow 0$
while $\| \int S(z,n) d\psi - c^e \| > \varepsilon_\theta$ **do**
    Guess $(F^{(0)}, G^{(0)})$ given $\theta^{(k)}$
    Set inner iteration counter $\ell \leftarrow 0$
    while $\|(F^{(\ell+1)}, G^{(\ell+1)}) - (F^{(\ell)}, G^{(\ell)})\| > \varepsilon_{FG}$ **do**
        Solve the HJB equation
        Solve the KFE equation with $m^0 = 1$
        Obtain m^0 to match $\theta^{(k)}$
        Update $F^{(\ell+1)} = \mathcal{F}\big(F^{(\ell)}, G^{(\ell)}; \theta^{(k)}\big)$
        Update $G^{(\ell+1)} = \mathcal{G}\big(F^{(\ell)}, G^{(\ell)}; \theta^{(k)}\big)$
        $\ell \leftarrow \ell + 1$
    end while
    Update $\theta^{(k+1)}$
    $k \leftarrow k + 1$
end while