Wealth Inequality and Safe Asset Demand*

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Abstract

This paper studies how wealth inequality shapes the demand for safe assets in heterogeneous-agent economies. We show that asset bubbles can arise when agents face a sufficiently high probability of falling into extreme poverty. In such cases, an asset insulated from idiosyncratic risk appears infinitely large relative to agents' wealth *ex-post*, which makes it highly valuable *ex-ante*. This mechanism differs fundamentally from classic rational bubbles, and generates bubbles even under stationary wealth distributions without aggregate uncertainty and growth. Using this insight, we revisit the pricing of long-lived assets in standard heterogeneous-agent economies, and show that prior analyses may be incomplete due to overlooked possibility of transversality condition violation.

Keywords: Asset Bubble, Heterogeneous Agents, Wealth Inequality

JEL Codes: E43, G12

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1 Introduction

When households face idiosyncratic risk, precautionary saving generates strong demand for safe assets and depresses the risk-free interest rate. At the same time, such risk generates a non-degenerate wealth distribution and potentially substantial wealth inequality. While both precautionary saving and wealth inequality have been widely studied, less is known about *how* the shape of the wealth distribution influences the demand for safe assets, and *when* it lowers the discount rate enough to sustain asset bubbles. Clarifying this link is essential for understanding the valuation of long-lived assets such as government debt, and for evaluating fiscal sustainability in heterogeneous-agent economies.

This paper studies the relationship between wealth inequality and safe asset demand. We characterize conditions on the wealth distribution under which the present discounted value of a long-lived asset approaches infinity in the distant future, violating the transversality condition (TVC). In such cases, the asset price exceeds the present discounted value of cash flows, i.e., $P > PV(CF)$, permanently deviating from its fundamentals.

We find that the TVC is closely linked to the wealth distribution, which is in turn shaped by the nature of the idiosyncratic risk. We uncover a novel mechanism for TVC violation in heterogeneous-agent economies, which is distinct from the classic rational bubble: in our setting, TVC violation requires the wealth of a significant fraction of agents to become sufficiently low relative to the average wealth in the economy. When these agents evaluate an asset that is unaffected by their idiosyncratic risk, they face the possibility that the asset will become infinitely large relative to their own wealth *ex-post*. Anticipating this outcome *ex-ante*, they place exceptionally high value on such assets, leading to TVC violation even in the absence of aggregate uncertainty or economic growth.

Our analysis offers a new perspective on how wealth inequality interacts with asset prices. Through the lens of our characterization, we revisit some workhorse heterogeneous-agent models like Constantinides and Duffie (1996), and show that prior analyses may be incomplete due to the overlooked possibility of TVC violation.

Analysis of Permanent Idiosyncratic Risk. We begin with a standard setting in which households face uninsurable idiosyncratic risk that is *permanent* and follows a random walk for each household. As the wealth distribution evolves, the vast majority of households become extremely poor, while a vanishing share of households accumulates nearly all the wealth. Over time, this divergence leads to an exploding degree of wealth inequality.

To understand the implications for asset pricing, consider how a household in this economy prices a long-lived aggregate asset such as government debt. The value of this asset depends on its cash flow and discount rate. In standard representative-agent models, asset cash flows and household consumption are cointegrated, and their covariance capturing risk premium is the primary determinant of valuation. In contrast, with permanent idiosyncratic risk, the household anticipates its wealth to decline far below the population average in almost all states of the world. As a result, the asset's cash flow will become arbitrarily larger than the household's wealth.

This divergence in scale between the household and the asset is the key determinant of asset valuation in this environment: while the household's marginal utility growth correctly prices the asset's return *period by period*, the asset's long-run discounted value tends to infinity *in the limit*, leading to TVC violation for aggregate assets.[1] In other words, the precondition for asset bubbles to arise is that the household anticipates its wealth share to become negligible in the long run with a sufficiently high probability, which requires a sufficiently large amount of wealth inequality.

Importantly, not all permanent idiosyncratic risk leads to bubbles. When the magnitude of risk is sufficiently small, although inequality still grows without bound and individual wealth shares still vanish, this process occurs gradually and is outweighed by the subjective discount factor. In this case, the asset's terminal value remains finite and converges to zero in the limit. Therefore, evaluating asset bubbles requires a *quantitative* comparison between the degree of wealth inequality and the strength of discounting.

[1] In contrast, each household's TVC for its own consumption stream still holds, as the consumption stream is discounted at a higher rate due to its exposure to the household's idiosyncratic risk. As a result, the household perceives finite wealth, so that it will not demand more consumption goods than available in the economy.

Analysis of Transitory Idiosyncratic Risk. The above analysis suggests that TVC violation arises when permanent idiosyncratic risk is sufficiently strong, which causes wealth inequality to diverge over time. A natural question is whether TVC violation can also occur when idiosyncratic risk is transitory and the wealth distribution is stationary—a setting that may be more realistic for studying asset prices.

To address this question, we consider two mechanisms to generate a stationary wealth distribution under idiosyncratic risk: wealth reset and death. In the first case, an idiosyncratic redistribution shock resets a randomly selected fraction of households' wealth to the average wealth level every period. This shock generates a stationary wealth distribution in a tractable way, while preserving the total amount of resources in the economy. To our surprise, even in this setting, TVC violation can still occur. As before, the key lies in understanding the properties of the wealth distribution.

Specifically, when the intensity of the wealth reset shock is low, each household's wealth can deviate from the population average for an extended period before returning to the average. While the wealth distribution remains stationary, if the left tail of the wealth distribution is sufficiently heavy, the household anticipates large enough dispersion in its marginal utility growth, which depresses the discount rate and causes bubbles to emerge.

Conversely, when the intensity of the wealth reset shock is sufficiently high, the wealth distribution becomes more concentrated around its mean. In this case, households expect their wealth levels to be comparable to the size of the economy in most states, raising the discount rate and restoring the TVC.

These results show that TVC violation for aggregate assets can occur when the wealth distribution is stationary but sufficiently left-skewed. Quantitatively, when we calibrate our model to U.S. post-war data, we find that the level of wealth inequality as measured by the Gini coefficient is potentially consistent with such a case, making asset bubbles an empirically relevant possibility. That said, the Gini coefficient only offers a partial assessment. Evaluating whether the U.S. wealth distribution has the shape required to generate bubbles calls for a more detailed empirical analysis, with particular attention to the left tail of the distribution, which has received less scrutiny

compared to the right tail that describes the very rich.

In the second case, we introduce death to our model. We assume that households die at a constant rate, and new households are born to maintain a constant population size. As a result of this mortality risk, households become more impatient, consuming a larger fraction of their wealth and discounting future cash flows more heavily.

We consider two variants. First, if each newborn household inherits the wealth of a specific dying household, the wealth distribution diverges as in the permanent risk case. If the magnitude of the idiosyncratic risk is sufficiently large relative to the subjective discount rate, households are concerned about their vanishing wealth shares in scenarios where they do survive, leading to TVC violation. The only difference is that the subjective discount rate needs to be adjusted upwards to account for impatience due to the mortality risk.

Second, if each newborn household starts with the average wealth, the wealth distribution is stationary like the previous case with wealth reset. TVC violation may still occur if wealth inequality is sufficiently large, with death playing a role similar to wealth reset. Therefore, mortality risk affects TVC outcomes, but does not overturn our core intuition.

Implications for Asset Bubbles. Our findings contribute to the literature on asset bubbles by identifying and characterizing a novel class of asset bubbles. Santos and Woodford (1997) develop a general characterization of asset bubbles, with the main result showing "nonexistence of pricing bubbles in an equilibrium with the property that the economy's aggregate endowment has a finite value." Consistent with their result, our bubble arises in an economy in which the aggregate endowment has infinite value. On the other hand, we deviate from their focus on assets with positive cash flows. Our analysis allows for cases with potentially negative cash flows, such as a government running persistent deficits. Relaxing this assumption enables us to uncover a new class of bubbles distinct from the examples studied in Santos and Woodford (1997). Below, we offer a brief discussion of the classic bubble types, and how they differ from our bubble.

First, consider classic rational bubbles in OLG models. These bubbles arise when agents

use assets for intertemporal consumption smoothing, which gives value to otherwise intrinsically worthless assets. A canonical example of such a bubble is fiat money.

Second, Hirano and Toda (2025) show that asset bubbles necessarily emerge in models featuring multiple assets or sectors with unbalanced growth. These bubbles typically apply to settings like land in a growing economy or sectors with divergent growth rates.

Our bubble comprises a third, distinct type. The key driving force is the dispersion in the agents' marginal utility growth, which is shaped by the idiosyncratic risk they face. This form of bubble is relevant for studying assets whose cash flows serve as good hedges against idiosyncratic risk, such as government debt.

There is an interesting parallel between the second and our types of bubbles. The former relies on divergent growth rates across *sectors*, whereas ours depends on dispersion in *wealth accumulation* across agents. That said, our case does not require permanent divergence in wealth, as bubbles can arise even under a stationary wealth distribution.

Despite these similarities, our bubble is fundamentally different from the other two types in both mechanism and implications for asset pricing. Internet Appendix E presents stylized models for each bubble type to illustrate the following key differences:

(1) Cash Flow/Price Ratio: In the first two types of bubbles, this ratio is either zero or vanishing over time. In fact, when cash flows are non-negative, bubbles can arise only if the cash flow/price ratio is zero or vanishing (Montrucchio, 2004; Hirano and Toda, 2025). For instance, money (Type 1 bubble) has no cash flow, and rents diminish relative to land prices (Type 2 bubble) over time. In contrast, our bubble exhibits a non-zero, non-vanishing cash flow/price ratio, which is made possible by *negative* cash flows. For example, the government can run a persistent deficit with a stationary deficit/debt ratio.

(2) Transversality Condition: All three bubble types feature a positive present discounted value of the asset price in distant future, violating the standard TVC. However, this value is finite in Types 1 and 2 bubbles, but infinite in our bubble. Specifically, in the first type of bubble, since the asset has no cash flow, its value equals the current price of the bubble asset, which is finite. In the second

type, its value measures the difference between the current asset price and the present value of cash flows, both of which are finite. In contrast, our bubble has an infinitely positive value, which is balanced by an infinitely negative present value of cash flows to result in a finite asset price.

(3) Short-Sale Constraint: The first two bubble types require short-sale constraints on the bubble asset, so that agents cannot profit from selling the overvalued asset to finance higher consumption (Kocherlakota, 1992). For instance, OLG models inherently impose such constraints since agents must liquidate positions at death. Our bubble, however, does not rely on such constraints. In our framework, agents can freely trade the bubble asset like government debt without violating equilibrium conditions.

In sum, by providing a characterization of the conditions under which this new type of bubble emerges, our analysis broadens the scope of asset pricing behaviors that can be rationalized by a bubbly economy, and offers a clear framework for future research to assess the empirical relevance of such bubbles.

Implications for Asset Valuation. Our results highlight the left tail of the wealth distribution as an important determinant of asset valuation. Viewed through the lens of our framework, the model proposed by Constantinides and Duffie (1996) corresponds to a case with permanent idiosyncratic risk, which leads to a diverging wealth distribution. When such risk is sufficiently large, the marginal utility growth in the lower tail becomes so steep that aggregate assets are infinitely valuable. In this case, the no-trade equilibrium analyzed by Constantinides and Duffie (1996) does not exist. Moreover, Constantinides and Duffie (1996) also consider a model variant which yields a stationary wealth distribution in their appendix. Our framework implies that, even in this case, TVC violation can still occur if the wealth distribution is sufficiently left-skewed. This, too, precludes the existence of no-trade equilibrium.

These findings suggest that, while heterogeneous-agent economies may offer a promising avenue to generate realistic risk premia, they also introduce a new challenge for the TVC to hold. Ignoring the possibility of TVC violation may risk an incomplete analysis of asset valuation. Con-

versely, our results clarify what is needed for the TVC to hold—agents need to have a finite expected growth rate in marginal utility, which requires mechanisms to induce mean-reversion in idiosyncratic risk and limit the left tail of the wealth distribution. Since these mechanisms are also central to determining the finite-period risk-return trade-off, the risk premium question needs to be jointly considered with the infinite-horizon TVC.

Our characterization of the TVC also extends to Bewley-type models like Krueger and Lustig (2010), which have occasionally binding borrowing constraints. These borrowing constraints introduce an important caveat, as the standard present value equation becomes an inequality: $P > PV(CF) + TVC$. Asset prices can exceed discounted cash flows, i.e., $P > PV(CF)$, even when TVC is not violated.

This inequality is a different mechanism which elevates asset value above discounted cash flows, which can arise even in models without idiosyncratic risk (Townsend, 1980; Woodford, 1990). Even in these cases, our results still offer a valid characterization of TVC violation, which remains a sufficient, though no longer necessary condition for $P > PV(CF)$. For example, Krueger and Lustig (2010) consider a model in which households have a guaranteed labor income which generates a bounded wealth distribution. Since no households' wealth will become infinitesimally small relative to the population average, the TVC always holds. In workhorse HANK models like McKay, Nakamura, and Steinsson (2016); Kaplan, Moll, and Violante (2018), the wealth distribution is also bounded from below, which guarantees the TVC to hold.

Literature Review. Our paper contributes to a large literature on heterogeneous-agent asset pricing (Grossman and Shiller, 1982; Constantinides and Duffie, 1996; Angeletos, 2007; Toda, 2014; Kogan, Papanikolaou, and Stoffman, 2020; Panageas, 2020; Schmidt, 2024; Gomez, 2025; Marin and Singh, 2025). While much of this literature focuses on the idiosyncratic risk's implications for risk premia, we investigate its implications for the TVC. We show that wealth inequality, which is also the key ingredient to generate realistic risk premia, can lead to asset bubbles.

Our analysis parallels the literature on asset bubbles in overlapping-generation (OLG) mod-

els (Samuelson, 1958; Diamond, 1965; Blanchard and Watson, 1982), as well as in more general frameworks (Santos and Woodford, 1997; Hirano and Toda, 2024). Just as Tirole (1985) characterizes conditions under which bubbly equilibrium exists in OLG economies, we characterize conditions under which TVC fails in heterogeneous-agent economies. To our surprise, bubbles can arise even when wealth distribution is stationary. Our results can be applied to any aggregate assets including government debt, which has been studied by Blanchard (2019); Abel and Panageas (2022); Gârleanu and Panageas (2023); Panageas (2024) in OLG models, and by Jiang et al. (2020); Reis (2021); Mian et al. (2022); Chen et al. (2022); Kocherlakota (2025) in models with long-lived agents.

Moreover, Lucas (1992); Atkeson and Lucas (1992) study resource allocation under idiosyncratic shocks. Lucas (1992) notes that permanent idiosyncratic risk generates a diverging wealth distribution, which lowers the equilibrium interest rate. Our analysis of the permanent-shock case builds on this insight. Our contribution is to derive the valuation implications of this divergence, and characterize exactly when and how the discount rate is lower than the aggregate growth rate. We also show that this divergence in wealth distribution is neither sufficient nor necessary to generate TVC violation. What matters instead is sufficient dispersion in marginal utility growth, which can happen even when the wealth distribution is stationary.

Finally, our analysis complements the *dynamic trading perspective* in Brunnermeier et al. (2022), which prices the cash flows each individual receives from the bond market. These cash flows, due to individuals' trading and rebalancing, are not the same as the aggregate cash flows paid out by the government. This perspective leads to a well-defined TVC *for individuals' wealth and asset holdings*, which offers important insights about the value of government debt.

Our results do not contradict with their results. Instead, we focus on the *buy-and-hold perspective*, which prices the issuer's *aggregate cash flows*. While Brunnermeier et al. (2020, 2022) show that the issuer's TVC may fail, its precise behavior remains an open question. We clarify the precise conditions under which this issuer-level TVC fails, which is central to the present value approach commonly used in financial valuation, and offers a rigorous foundation to interpret the

buy-and-hold valuation exercise in Jiang et al. (2024b).

Our paper proceeds as follows. Section 2 analyzes the baseline model with permanent idiosyncratic risk. Section 3 introduces wealth reset and mortality risk to generate transitory idiosyncratic risk, and discusses the model's empirical relevance. Section 4 discusses our results' implications for workhorse heterogeneous-agent models. Section 5 concludes.

2 Transversality Condition under Permanent Idiosyncratic Risk

2.1 Model Set-Up

We study a standard heterogeneous-agent economy with idiosyncratic risk, which nests a broad class of models including Constantinides and Duffie (1996); Brunnermeier et al. (2022). This general framework enables us to derive results that are applicable across a wide range of heterogeneous-agent settings.

We begin by outlining the key equilibrium conditions of the model. The economy consists of a continuum of infinitely lived households indexed by $i \in [0, 1]$, each maximizing expected lifetime utility:

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log(c_t^i) dt \right].$$

For expositional simplicity, we assume log preference. We allow for more general risk aversion in our calibration and present details in Internet Appendix A.1.

In equilibrium, individual consumption is subject to both aggregate and idiosyncratic shocks:

$$\frac{dc_t^i}{c_t^i} = g_t dt + \sigma_t dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^i, \tag{1}$$

where g_t denotes the common growth rate, σ_t captures the volatility of the aggregate shock Z_t, and $\tilde{\sigma}_t$ captures the volatility of the idiosyncratic shock $\tilde{Z}_t^i$.

This consumption process naturally arises from numerous heterogeneous-agent settings. In Internet Appendix A.1, we follow Brunnermeier et al. (2022) to derive it as the endogenous outcome of a model with capital accumulation and government debt. Alternatively, it can be obtained from Constantinides and Duffie (1996), which we discuss in Section 4. Our analysis focuses on the implications of this consumption process for asset valuation, which does not rely on specific model details.

Under these assumptions, the marginal utility can be expressed as

$$m_t^i = e^{-\rho t}(c_t^i)^{-1}.$$

Substituting in the law of motion for consumption yields the following dynamics of marginal utility:

$$\frac{dm_t^i}{m_t^i} = -r_t^f dt - \sigma_t dZ_t - \tilde{\sigma}_t d\tilde{Z}_t^i, \tag{2}$$

which implies that the risk-free rate equals

$$r_t^f = \rho + g_t - \sigma_t^2 - \tilde{\sigma}_t^2. \tag{3}$$

The risk-free rate r_t^f reflects the expected growth rate g_t and the variances of aggregate and idiosyncratic risks. Higher volatility strengthens the precautionary saving motive, exerting a downward pressure on the risk-free rate.

2.2 Transversality Condition for Individual Wealth

Let n_t^i denote household wealth, which is the market value of its asset holdings. Let $r_t^{w,i}$ denote the return on the household's wealth portfolio. In the microfounded model presented in Appendix A.1, this return is endogenously determined by the household's optimal portfolio choice over financial

assets such as equities and bonds. Here, we consider the more general version:

$$dn_t^i = -c_t^i dt + n_t^i dr_t^{w,i}.$$

Iterating forward this equation, we obtain that the household's wealth is equal to the present discounted value of its future consumption plus a transversality term:

$$m_t^i n_t^i = \mathbb{E}_t \left[\int_0^\infty m_{t+j}^i c_{t+j}^i dj \right] + \lim_{T \to \infty} \mathbb{E}_t \left[m_T^i n_T^i \right]. \tag{4}$$

The transversality condition (TVC) for individual wealth is

$$\lim_{T \to \infty} \mathbb{E}_t \left[m_T^i n_T^i \right] = 0. \tag{5}$$

When this condition holds, wealth can be expressed as the present discounted value of future consumption.

In our setting, this individual-level TVC holds for all households. Given the log preference, the envelope condition implies that a constant consumption-to-wealth ratio $c_t^i / n_t^i = \rho$. The pricing kernel can be expressed as

$$m_t^i = e^{-\rho t} \rho^{-1} (n_t^i)^{-1}. \tag{6}$$

From (2) and (6), we obtain

$$\frac{d(m_t^i n_t^i)}{m_t^i n_t^i} = -[\underbrace{r_t^f}_{\text{risk-free rate}} + \underbrace{\sigma_t^2}_{\text{risk premium for aggregate risk}} + \underbrace{\tilde{\sigma}_t^2}_{\text{risk premium for idiosyncratic risk}} - \underbrace{g_t}_{\text{growth rate}}]dt, \tag{7}$$

which expresses the dynamics of the transversality term $m_t^i n_t^i$ in terms of the discount rate and the

growth rate. Plugging in the risk-free rate in (3) yields

$$(r_t^f + \sigma_t^2 + \tilde{\sigma}_t^2) - g_t = \rho > 0. \tag{8}$$

Thus, the appropriate discount rate for individual wealth, which accounts for both aggregate and idiosyncratic risk premia, exceeds the growth rate: r is always above g. This condition guarantees that the discounted individual wealth $m_t^i n_t^i$ converges to zero in the limit.

2.3 Transversality Condition for Aggregate Wealth

We now turn to the TVC associated with the aggregate wealth, defined as $\bar{n}_t = \int_0^1 n_t^i di$. Its law of motion is given by

$$\frac{d\bar{n}_t}{\bar{n}_t} = g_t dt + \sigma_t dZ_t,$$

which, compared to any individual household's wealth, averages out the idiosyncratic risk.

Consider a claim that pays out the aggregate wealth $\bar{n}_T$ at some distant future date T. Its time-0 valuation from the perspective of household i is $\mathbb{E}_0\left[m_T^i \bar{n}_T\right]$, which captures how much the household is willing to pay in exchange for receiving the aggregate wealth in the distant future.

The corresponding aggregate TVC is given by

$$\lim_{T\to\infty} \mathbb{E}_0\left[m_T^i \bar{n}_T\right] = 0. \tag{9}$$

Whether this TVC holds has important implications for asset valuation. Consider, for example, how households price government debt. Similar to the individual household's budget constraint (4), the government faces the following standard intertemporal budget condition:

$$b_t = \mathbb{E}_t\left[\int_0^\infty m_{t+j}^i s_{t+j} dj\right] + \lim_{T\to\infty} \mathbb{E}_t\left[m_T^i b_T\right], \tag{10}$$

where b_t denotes the market value of government debt and s_t denotes the primary surplus. Let $\vartheta_t = b_t/\bar{n}_t$ denote the government debt-to-aggregate wealth ratio. The government issuer's transversality condition is given by

$$\lim_{T\to\infty} \mathbb{E}_t \left[m_T^i \vartheta_T \bar{n}_T \right].$$

Since $\vartheta_t \in (0,1)$, $0 < \mathbb{E}_t \left[m_T^i \vartheta_T \bar{n}_T \right] < \mathbb{E}_t \left[m_T^i \bar{n}_T \right]$ for all time T. So, the TVC (9) for aggregate wealth is a sufficient condition for the TVC for government debt to hold.[2] More generally, for any asset whose cash flow is cointegrated with the size of the economy, the TVC (9) for aggregate wealth provides a key condition for ruling out explosive valuation paths.

Equivalently, we can analyze this TVC from the perspective of the average household. We define the equal-weighted average marginal utility growth as $\bar{m}_t = \int_0^1 (m_t^i/m_0^i) di$, which evolves according to

$$\frac{d\bar{m}_t}{\bar{m}_t} = -r_t^f dt - \sigma_t dZ_t.$$

Compared with the individual household's stochastic discount factor (SDF) m_t^i/m_0^i, the average SDF $\bar{m}_t$ has the same drift, but does not load on any idiosyncratic risk. Since the aggregate wealth $\bar{n}_T$ is also free of idiosyncratic risk, the two TVCs are identical:

$$\lim_{T\to\infty} \mathbb{E}_0 \left[m_T^i \bar{n}_T \right] = 0 \quad \Leftrightarrow \quad \lim_{T\to\infty} \mathbb{E}_0 \left[\bar{m}_T \bar{n}_T \right] = 0.$$

For analytical convenience, we focus on the TVC evaluated using the average SDF $\bar{m}_t$.

Unlike the TVC (5) for individual wealth n_t^i, the TVC (9) for aggregate wealth $\bar{n}_t$ does not necessarily hold. We provide two complementary perspectives to characterize this condition, one based on the degree of inequality in the wealth distribution and the other based on the comparison between the discount rate (r) and the growth rate (g).

[2] As a special case, if the government maintains a constant debt/wealth ratio ϑ_t, then, Eq. (9) is also a necessary condition.

We begin with the wealth distribution perspective. Let $\eta_t^i = n_t^i/\bar{n}_t$ denote the wealth share of individual i. By construction, the wealth share has an average of 1: $\int_0^1 \eta_t^i di = 1$. The following proposition characterizes the relationship between the aggregate TVC and the distribution of the wealth share.

Proposition 1. *Assuming all households start with equal wealth shares, i.e., $\eta_0^i = 1$, the transversality condition for aggregate wealth can be expressed as*

$$\lim_{T\to\infty} \mathbb{E}_0[\bar{m}_T \bar{n}_T] = \lim_{T\to\infty} e^{-\rho T} \bar{n}_0 \mathbb{E}_0 \left[\int_0^1 (\eta_T^i)^{-1} di \right]. \tag{11}$$

The proof is presented in Appendix A.2. Eq. (11) is the central equation of our paper: whether the aggregate TVC holds or not depends on the distribution of the wealth share as summarized by $\int_0^1 (\eta_T^i)^{-1} di$, which is a measure of inequality: a more unequal wealth distribution implies that more households have lower wealth shares η_T^i and hence higher marginal utilities $(\eta_T^i)^{-1}$. If inequality grows without bound, most households' wealth shares η_T^i will become infinitely small and their marginal utilities $(\eta_T^i)^{-1}$ will become infinitely large. In this case, the right-hand side of Eq. (11) may diverge to infinity. On the other hand, if wealth share are bounded from below, then, the integral remains finite and the TVC always holds.

In our setting, the idiosyncratic risk is permanent. Each household's wealth share follows a geometric random walk:

$$\frac{d\eta_t^i}{\eta_t^i} = \tilde{\sigma}_t d\tilde{Z}_t^i.$$

As time passes, almost all households become arbitrarily poor, while wealth becomes concentrated in a vanishing measure of rich households. As a result, the wealth distribution becomes increasingly skewed, leading to an exploding amount of inequality.

2.4 Calibrated Example

We illustrate this point by simulating the model under a simple calibration. To generate a realistic magnitude of risk premia, we assume that agents have CRRA utility with a risk aversion coefficient of $\gamma = 2$. We also shut down aggregate risk and aggregate growth by setting $\sigma_t = 0$ and $g_t = 0$, allowing us to isolate the role of idiosyncratic risk. Appendix B presents detailed derivations of this extension, and Appendix Table A.1 reports the parameter values used.[3]

Our calibration also relies on the following assumptions, which simplify our analysis but are not essential to our results. The $r - g$ perspective developed below characterizes the TVC under more general assumptions.

Assumption 1. *(a) The idiosyncratic risk has a constant volatility:* $\tilde{\sigma}_t = \tilde{\sigma}$.

(b) The economy starts with a steady-state value of the portfolio share of equity $\vartheta_t = \vartheta$ *so that it remains constant over time in the absence of aggregate risk.*

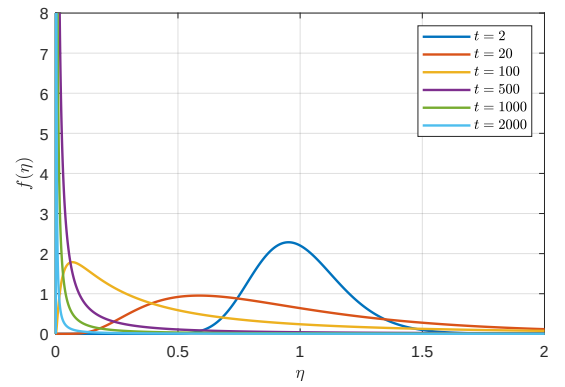
(c) All households start with equal wealth shares: $\eta_0^i = 1$.

Figure 1 illustrates the evolution of the wealth distribution under these assumptions. Panel (a) plots the cross-sectional distribution of wealth shares η_t^i for different time t. As time passes, while the average wealth share remains equal to 1, its distribution becomes increasingly skewed. The median and the mode of the wealth distribution shift to the left, indicating that most households' wealth share η_t^i approaches zero, while a vanishing share of households accumulate a disproportionate share of wealth. Panel (b) plots the cumulative distribution of the wealth share, highlighting the same pattern of increasing concentration.

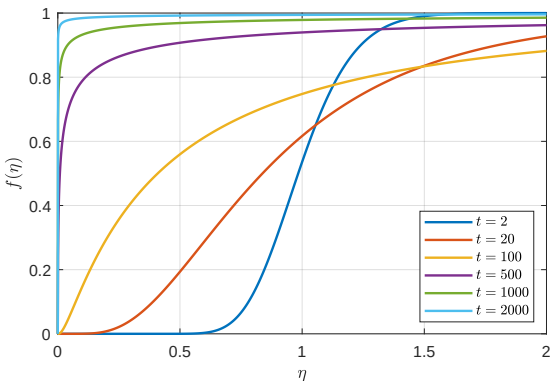
Panel (c) plots the Gini coefficient of the wealth distribution, a standard measure of inequality. As the distribution becomes increasingly dispersed, the Gini coefficient approaches one, indicating extreme concentration of wealth.

As almost all households become negligibly small relative to the aggregate wealth $\bar{n}_T$, they

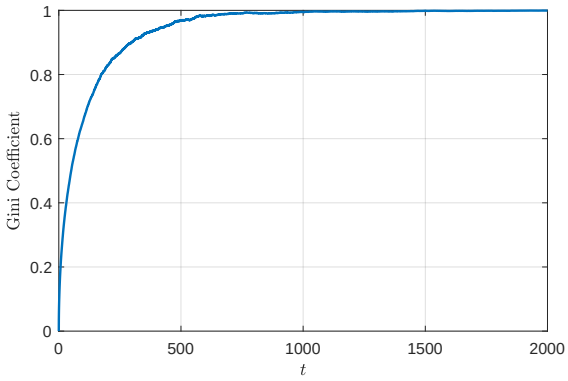
[3] While this assumption is not necessary, it implies a constant aggregate wealth $\bar{n}_T$ which further simplifies Eq. (11), so that we can focus on the distribution term. A non-zero growth rate generates a time-varying $\bar{n}_T^{1-\gamma}$ term. While this term does not affect the distribution of household wealth, it influences the growth rate of marginal utility in the same way as the time discount factor ρ.



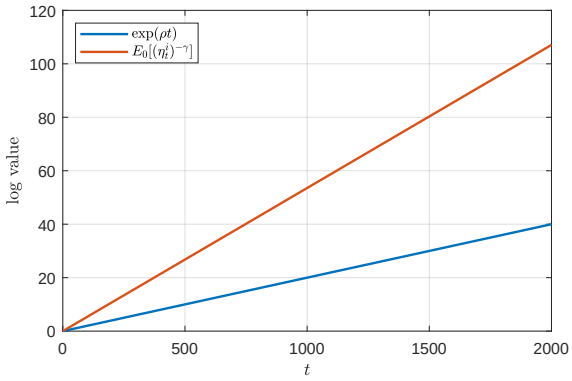
(a) Distribution of Wealth Share.



(b) Cumulative Distribution of Wealth Share.



(c) Gini Coefficient.



(d) TVC for Aggregate Wealth.

Figure 1: Dynamics with Permanent Idiosyncratic Risk.

assign an infinite valuation to claims on aggregate wealth, or, more generally, to any asset whose payoff scales with the size of the economy. With a general risk aversion coefficient γ, the aggregate TVC is proportional to

$$\lim_{T\to\infty} \mathbb{E}_0\left[m_T^i \bar{n}_T\right] \propto \lim_{T\to\infty} \exp(-\rho T)\cdot \mathbb{E}_0\left[(\eta_T^i)^{-\gamma}\right].$$

On the right-hand side, $\exp(-\rho T)$ represents the subjective discount factor, while $(\eta_T^i)^{-\gamma}$ captures the marginal utility growth. If the marginal utility grows to infinity faster than the subjective discount factor approaching zero, this transversality term diverges to infinity.

Panel (d) plots these two competing forces. Both the inequality term $\int_0^1 (\eta_T^i)^{-\gamma} di$ and the inverse subjective discount factor $\exp(\rho T)$ approach infinity in the limit. Under our calibration,

the magnitude of the idiosyncratic risk is sufficiently high, so that the rise in inequality—reflected in the inequality term—dominates the effect of discounting. This results in an infinite value of expected marginal utility growth in the limit.

It is worth noting that this result is not driven by a force which, as a household becomes poorer, increases the magnitude of its idiosyncratic risk and strengthens the precautionary saving motive. In our setting, poor and rich households hold the same portfolio, and their difference lies in the size of their wealth, which determines how large any aggregate asset appears to them. The TVC is violated when all but a vanishing share of the households become infinitesimally small relative to the size of the aggregate asset, which is different from a mechanism driven by increasing risk premium.

2.5 The r Minus g Perspective

We can also interpret our results through the lens of the familiar "r minus g" condition. Note that the discounted aggregate wealth has the following dynamics:

$$\frac{d(\bar{m}_t\bar{n}_t)}{\bar{m}_t\bar{n}_t} = [-\underbrace{r_t^f}_{\text{risk-free rate}} + \underbrace{g_t}_{\text{growth rate}} - \underbrace{\sigma_t^2}_{\text{agg. risk premium}}]dt.$$

In contrast to the discounted individual wealth in (7), the discounted aggregate wealth does not contain the risk premium for idiosyncratic risk, because it is not exposed to such risk. The relevant $r - g$ condition becomes

$$(r_t^f + \sigma_t^2) - g_t = \rho - \tilde{\sigma}_t^2,$$

which, compared to the condition (8) for individual wealth, is no longer guaranteed to be positive. This condition governs the aggregate TVC, which holds if and only if

$$\lim_{T\to\infty} \mathbb{E}_0\left[\bar{m}_T\bar{n}_T\right] = \lim_{T\to\infty} \bar{n}_0\mathbb{E}_0\left[\exp\left(-\int_0^T [\rho - \tilde{\sigma}_t^2]\, dt\right)\right] = 0. \tag{12}$$

Thus, the aggregate TVC is violated when the magnitude of the idiosyncratic risk $\tilde{\sigma}_t^2$ is sufficiently large relative to the subjective discount rate ρ. In this case, because inequality explodes fast enough, households lower their discount rate on the aggregate wealth below its growth rate, leading to $r < g$.

Conversely, if the variance of the permanent idiosyncratic risk is sufficiently low, i.e., $\tilde{\sigma}_t^2 < \rho$, the discount rate r exceeds the growth rate g, and the aggregate TVC holds: $\lim_{T\to\infty} \mathbb{E}_0 \left[\bar{m}_T \bar{n}_T\right] = 0$. This conclusion is consistent with the wealth distribution perspective above: in this case, although wealth inequality continues to rise due to the permanent idiosyncratic risk, it does so slowly enough that the growth in marginal utility is dominated by time discounting.

This discussion shows that the TVC for aggregate wealth is violated only when wealth inequality explodes sufficiently quickly, which drives the discount rate below the growth rate, i.e., $r < g$. What $r - g$ captures in this setting is how fast permanent idiosyncratic risk drives the wealth inequality to explode.

2.6 What Does It Mean to Violate the TVC for Aggregate Wealth?

When the TVC for aggregate wealth is violated, the government intertemporal budget condition, reproduced below,

$$b_t = \mathbb{E}_t \left[\int_0^\infty m_{t+j}^i s_{t+j} dj \right] + \lim_{T\to\infty} \mathbb{E}_t \left[m_T^i b_T \right],$$

has an infinite TVC term $\lim_{T\to\infty} \mathbb{E}_t \left[m_T^i b_T\right] = \infty$.

In an economy like Brunnermeier et al. (2022), the asset value b_t can remain well-defined and finite, because the government runs deficits in enough states of the world to generate an infinitely negative present value of surpluses: $\mathbb{E}_t \left[\int_0^\infty m_{t+j}^i s_{t+j} dj\right] = -\infty$ that is just enough to offset the infinitely negative TVC term. In this case, the asset bubble benefits the asset issuer, allowing it to run deficits in perpetuity. In contrast, when the TVC holds and there is no asset bubble, the government issuer must run surpluses to back up a positive value of debt.

In an economy like Constantinides and Duffie (1996), an infinite TVC term implies that the equilibrium commonly studied in that literature does not exist. One-period Euler equations and the assumption $\mathbb{E}_0\left[m_T^i\right] = 0$ are not enough to guarantee the agents; optimality. We will discuss the implications of our results for this class of models in Section 4.

3 Transversality Condition under Transitory Idiosyncratic Risk

So far, our analysis has focused on the case of *permanent* idiosyncratic risk, and shows it can lead to TVC violation for aggregate assets. As shown in Proposition 1, the key mechanism operates through the left tail of the wealth distribution: permanent idiosyncratic shocks generate an increasingly heavy left tail, causing expected marginal utility growth to diverge over time.

In practice, however, it is more realistic to consider environments where the wealth distribution remains stationary, which requires idiosyncratic risk to be *transitory*. In this section, we study the implications of transitory idiosyncratic risk for the TVC. Surprisingly, TVC violations can still arise. In fact, standard mechanisms that produce stationary wealth distributions can generate sufficiently heavy left tails. As a result, expected marginal utility can still diverge, despite the absence of permanent idiosyncratic shocks.

3.1 Model Set-Up

We introduce a simple modification to our baseline model to make the idiosyncratic risk transitory. Specifically, each household's wealth n_t^i is reset at random intervals governed by a Poisson process with intensity ζ. Let $\tilde{J}_t^i$ denote this jump process, which is i.i.d. across households.[4] For simplicity, we assume each household's wealth is always reset to the average level $\bar{n}_t$ upon the shock, which preserves the total amount of wealth in this economy and can thus be interpreted as a form of redistribution. As we discuss below, the key results are robust to more general specifications of

[4]Modeling mean reversion in this way provides a tractable mechanism for generating a stationary wealth distribution. Similarly, Irie (2024) proposes a setting where agents face Poisson shocks that switch their types. By contrast, proportional wealth taxation leads to substantially more complex dynamics, even under log utility.

post-reset wealth distribution.

In this setting, individual consumption evolves according to

$$\frac{dc_t^i}{c_t^i} = g_t dt + \sigma_t dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^i + \frac{\bar{n}_t - n_t^i}{n_t^i} d\tilde{J}_t^i,$$

which endogenously arises from the household's optimization problem in the presence of wealth reset shocks. We present detailed derivations of this model variant in Appendix A.1.

This consumption process differs from (1) in the baseline model with no wealth reset in three ways. First, exposure to the redistribution risk induces mean reversion in the wealth share n_t^i: when the jump $d\tilde{J}_t^i$ occurs, wealth is reset to the average level $\bar{n}_t$.

Second, the household internalizes the redistribution risk in its consumption decision. Since a realization of the reset shock erases accumulated wealth, the household becomes effectively more impatient and consumes a higher share of its wealth: the consumption-wealth ratio $c_t^i/n_t^i = \rho + \zeta$ is increasing in ζ. Since the household consumes more and saves less, the growth rate in this economy can be different from the baseline model under the same calibration.

Third, the risk-free rate is given by

$$r_t^f = \rho + \zeta + g_t - \sigma_t^2 - \tilde{\sigma}_t^2.$$

Compared to (3) in the baseline model with permanent idiosyncratic risk, this expression includes the jump intensity ζ, which captures the additional impatience induced by the wealth reset mechanism.

Throughout this section, we focus on the parameter region in which the magnitude of idiosyncratic risk is sufficiently large such that the TVC fails in the absence of any jump risk. This restriction is without loss of generality for our purpose: if the TVC holds in the absence of jump risk, it will continue to hold for any level of jump intensity, since the reset mechanism can only dampen inequality and compress the dispersion in marginal utility growth.

3.2 Transversality Condition for Aggregate Wealth

Next, we revisit the TVC for aggregate assets in the presence of wealth reset. As in Section 2, the TVC can be characterized using either an individual household's SDF or the average SDF. We find it convenient to work with the average SDF, defined as $\bar{m}_t = \int_0^1 (m_t^i/m_0^i)di$. Its law of motion is given by

$$d\bar{m}_t = -[\rho + g_t - \sigma_t^2 - \tilde{\sigma}^2]\bar{m}_t dt - \sigma_t \bar{m}_t dZ_t + \int_0^1 e^{-\rho t}\hat{\rho}^{-1}\left(\bar{n}_t^{-1} - (n_{t-}^i)^{-1}\right)\frac{1}{m_0^i}d\tilde{J}_t^i di.$$

On the right-hand side, the drift and the dZ_t terms are standard as in the baseline model, whereas the last term captures the effect of the wealth reset shock. While this shock is purely redistributive and leaves the aggregate wealth $\bar{n}_t$ unchanged, it affects the average marginal utility $\bar{m}_t$ since individual marginal utility is a non-linear function of wealth.

Our analysis applies to any aggregate assets whose cash flows are unaffected by idiosyncratic risk. Here, we take the government debt as a concrete example. Let b_t denote the aggregate market value of government debt in real terms. We obtain the following result which generalizes the government budget constraint (10):

Proposition 2. *In the presence of wealth reset, the intertemporal government budget condition is given by*

$$b_0 = \mathbb{E}_0\left[\int_0^T \bar{m}_t s_t dt\right] + \mathbb{E}_0\left[\bar{m}_T b_T\right] - \zeta\mathbb{E}_t\left[\int_0^T e^{-\rho t}\left(\frac{\bar{n}_t}{\bar{n}_0}\right)^{-1} b_t dt\right].$$

The proof is presented in Appendix A.5. This expression has a natural interpretation from the perspective of the average household: government debt is valued based on (i) future primary surpluses (the first term on the right-hand side), (ii) the standard transversality term (the second term), and (iii) an additional term capturing the effect of wealth reset.

To see why, note that when a reset occurs, a household's wealth is replaced by the average wealth level regardless of its prior holdings. From the household's perspective, this effectively

wipes out its asset position. The final term captures this expected loss, which equals the jump intensity ζ times the present value of asset holdings. This loss of value reduces the valuation b_0 of government debt relative to the standard present value of surpluses plus terminal value.

Since the market value of government debt is bounded by the aggregate wealth, i.e., $b_t < \bar{n}_t$, the last term is finite. Proposition 2 implies

$$b_0 \leq \mathbb{E}_0 \left[\int_0^T \bar{m}_t s_t dt \right] + \mathbb{E}_0 \left[\bar{m}_T \bar{n}_T \right],$$

which shows that the discounted aggregate wealth $\bar{m}_t \bar{n}_t$ again arises as the key object governing the TVC for aggregate assets. In what follows, we characterize its behavior using the wealth distribution and the $r - g$ perspectives as in the baseline case.

We begin with the wealth distribution perspective. For simplicity, we maintain the assumption that households begin with identical wealth shares: $\eta_0^i = 1$. Then, the TVC (11) in Proposition 1, reproduced below, also holds in this setting:

$$\lim_{T \to \infty} \mathbb{E}_0[\bar{m}_T \bar{n}_T] = \lim_{T \to \infty} e^{-\rho T} \bar{n}_0 \mathbb{E}_0 \left[\int_0^1 (\eta_T^i)^{-1} di \right]. \tag{13}$$

As in the baseline case with permanent idiosyncratic risk, whether the TVC holds for the aggregate wealth depends on how the integral of inverse wealth share $\int_0^1 (\eta_T^i)^{-1} di$ evolves over time and compares to the decay of time discount $e^{-\rho T}$. The key difference is that transitory idiosyncratic risk induces mean-reverting wealth dynamics, whereas permanent idiosyncratic risk generates ever-increasing dispersion.

3.3 Calibrated Example

To illustrate these results, we again consider the simple calibration with a general risk aversion coefficient γ under Assumption 1. The following proposition characterizes the stationary wealth distribution.

Proposition 3. *Starting from any initial distribution, the wealth distribution converges to the distribution with density function f that satisfies*

$$f(\eta) = \begin{cases} C\eta^{-\frac{3}{2}+\frac{\sqrt{1+\frac{8\zeta}{\tilde{\sigma}^2}}}{2}}, & 0 < \eta < 1, \\ C\eta^{-\frac{3}{2}-\frac{\sqrt{1+\frac{8\zeta}{\tilde{\sigma}^2}}}{2}}, & \eta \geq 1, \end{cases} \tag{14}$$

where C is a constant that ensures that the distribution integrates to one.

The proof is presented in the Appendix A.6. Even though this distribution is obtained by introducing wealth reset to an expanding log-normal process, the limiting stationary distribution is not log-normal—an important feature that we will revisit shortly.[5]

The stationary density has two distinct regimes depending on the relative strength of mean reversion and idiosyncratic risk. If

$$\zeta > \frac{\gamma(1+\gamma)}{2}\tilde{\sigma}^2, \tag{15}$$

i.e., the jump intensity is sufficiently high relative to the magnitude of the idiosyncratic risk, then, the left tail of the wealth distribution is light enough to ensure a finite transversality term:

$$\lim_{T\to\infty} \mathbb{E}_0 \left[\int_0^1 (\eta_T^i)^{-\gamma} di \right] < \infty.$$

In this case, the TVC (13) for aggregate wealth holds.

Conversely, when (15) fails, i.e., $\zeta \leq \frac{\gamma(1+\gamma)}{2}\tilde{\sigma}^2$, the jump intensity is overshadowed by the magnitude of the idiosyncratic risk. Although the limiting wealth distribution remains stationary, its left tail is sufficiently heavy to generate an infinite $\lim_{T\to\infty} \mathbb{E}_0 \left[\int_0^1 (\eta_T^i)^{-\gamma} di \right]$, which, as we show below, may lead to TVC violation.

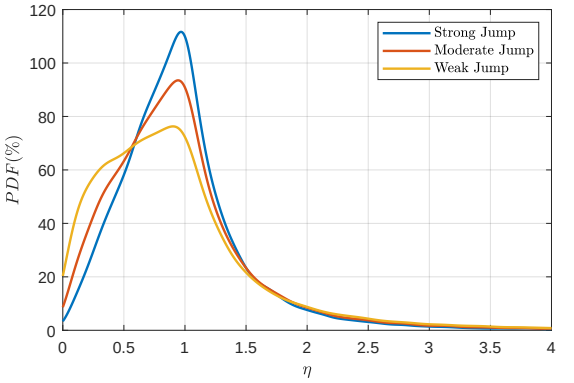
We consider three calibrations to illustrate these cases: a case with a high jump intensity $\zeta = 5.4\%$, a case with an intermediate jump intensity $\zeta = 3.8\%$, and a case with a low jump

[5]This feature is consistent with the more general cases studied by Reed (2001); Toda (2014). The "power law on both tails" is a common feature in models with a constant probability of birth and death.

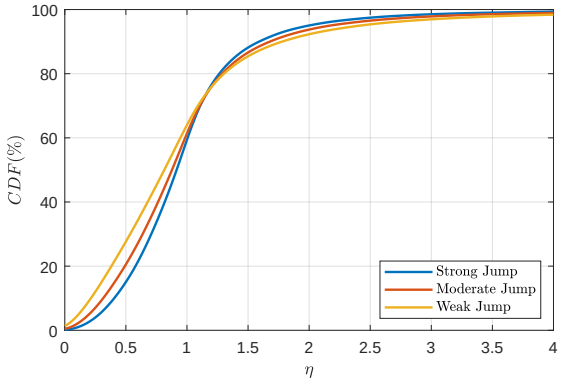
intensity $\zeta = 2.5\%$. The expected marginal utility growth $\lim_{T\to\infty} \mathbb{E}_0 \left[\int_0^1 (\eta_T^i)^{-\gamma} di\right]$ is finite in the first case, and infinite in the second and third cases. All other variables are identical to the previous baseline case. If we set the jump intensity ζ_t to zero, we recover the case with permanent idiosyncratic risk in Figure 1.

In all three cases, households start with identical wealth shares at $t = 0$. The wealth distribution quickly converges to the stationary distribution given by (14). Figure 2 illustrates the properties of these three cases.

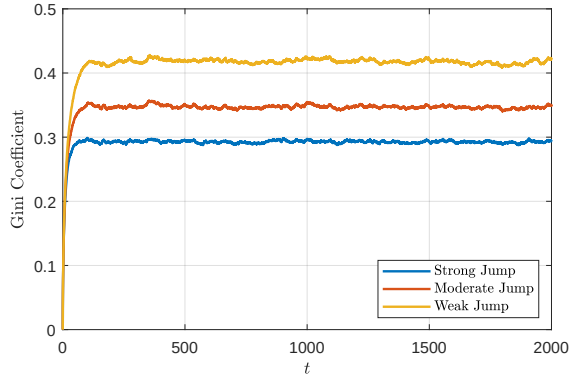
Panel (a) plots the limiting stationary distribution of wealth share η_t^i. The case with a higher jump intensity has a greater mass around the mean and a thinner left tail, while the case with a lower jump intensity has a heavier left tail. In all three cases, the left tail of the wealth distribution



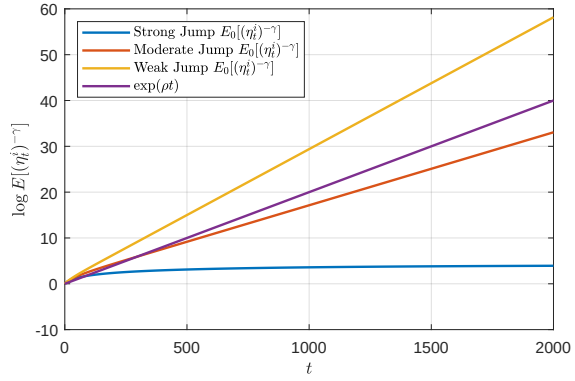
(a) Distribution of Wealth Share.



(b) Cumulative Distribution of Wealth Share.



(c) Gini Coefficient.



(d) TVC for Aggregate Wealth.

Figure 2: Dynamics with Transitory Idiosyncratic Risk

declines with a power function. While it is not visually obvious, this tail is heavier than the one implied by log-normal distribution which guarantees a finite $\mathbb{E}_0\left[\int_0^1(\eta_T^i)^{-\gamma}di\right]$.

Panel (b) plots the cumulative distribution of the wealth share η_t^i. Both panels show that there is no qualitative difference between the distributions obtained in these three cases, as they belong to the same family of distributions with slightly different jump intensity ζ.

Panel (c) plots the evolution of the Gini coefficient. Consistent with the convergent behavior of the wealth distribution, the Gini coefficient also stabilizes quickly and takes similar values across the three cases, ranging from 0.3 to 0.4. Thus, standard inequality measures do not distinguish sharply between these cases.

Finally, panel (d) reports the average path of marginal utility growth in Eq. (13), which determines the TVC for aggregate wealth. We break down this path into two components: the integral of inverse wealth share $\mathbb{E}_0\left[\int_0^1(\eta_T^i)^{-\gamma}di\right]$ and the subjective discount factor $\exp(\rho T)$. The expected marginal utility growth is given by the ratio between these two components. In contrast to the previous panels, this decomposition reveals substantial differences across the three cases.

With high jump intensity (blue), the integral of inverse wealth share $\mathbb{E}_0\left[\int_0^1(\eta_T^i)^{-\gamma}di\right]$ converges to a finite value. By contrast, with lower (yellow) or intermediate (red) jump intensity, this integral diverges to infinity, reflecting a sufficiently heavy left tail and large downside risk in individual consumption paths. This leads to an infinite expected value of the marginal utility growth in the limit.

Comparing these values of expected marginal utility growth to the subjective discount factor $\exp(\rho T)$ yields further differentiation. With a high jump intensity (blue), the expected value of marginal utility growth is finite while the subjective discount factor grows exponentially, which guarantees a finite TVC term. With intermediate jump intensity (red), while the expected marginal utility growth diverges to infinity, it does so at a slower rate than the subjective discount factor. As a result, the TVC still holds. With a low jump intensity (yellow), the expected marginal utility growth diverges to infinity at a faster rate than the subjective discount factor, leading to a violation of the TVC for aggregate wealth.

Table 1(a) summarizes these different cases. With high jump intensity, the TVC holds and $\mathbb{E}_0\left[\int_0^1(\eta_T^i)^{-\gamma}di\right]$ converges to a finite value. With low jump intensity, the TVC is violated and $\mathbb{E}_0\left[\int_0^1(\eta_T^i)^{-\gamma}di\right]$ diverges to infinity. With an intermediate jump intensity, while $\mathbb{E}_0\left[\int_0^1(\eta_T^i)^{-\gamma}di\right]$ diverges to infinity, the TVC still holds.

These results underscore a key insight: despite the visual similarities of the wealth distribution in these three cases, small differences in the thickness of the left tail have first-order implications for asset valuation. According to Eq. (13), the TVC for aggregate wealth is violated only when the left tail is heavy enough to make the expected marginal utility growth diverge faster than the rate of time discounting.

Table 1(b) reports the corresponding results for the model without wealth reset, which we studied in the previous section. In these cases, idiosyncratic risk is permanent and $\mathbb{E}_0\left[\int_0^1(\eta_T^i)^{-\gamma}di\right]$ always diverges to infinity. However, the TVC still holds if the idiosyncratic risk is sufficiently small, so that time discounting dominates the growth in marginal utility.

Table 1: Summary of the TVC for Aggregate Wealth.

(a) With Wealth Reset, Transitory Idiosyncratic Risk				
Jump Intensity	Parameter Region	TVC holds?	$\mathbb{E}[\int(\eta_\infty^i)^{-\gamma}di]=\infty$?	$r>g$?
High	$\hat{c}>\gamma\tilde{\sigma}^2+\rho+(\gamma-1)g$	Yes	No	Yes
Moderate	$\gamma\tilde{\sigma}^2<\hat{c}\leq\bar{\gamma}\bar{\sigma}^2+\rho+(\gamma-1)g$	Yes	Yes	Yes
Low	$0<\hat{c}\leq\gamma\tilde{\sigma}^2$	No	Yes	No
(b) Without Wealth Reset, Permanent Idiosyncratic Risk				
Idiosyncratic Volatility	Parameter Region	TVC holds?	$\mathbb{E}[\int(\eta_\infty^i)^{-\gamma}di]=\infty$?	$r>g$?
Low	$\hat{c}>\gamma\tilde{\sigma}^2$	Yes	Yes	Yes
High	$\hat{c}\leq\gamma\tilde{\sigma}^2$	No	Yes	No

Notes: $\hat{c}$ is the consumption-wealth ratio for the general case with wealth reset and power utility. Under log utility, $\hat{c}=\rho+\zeta$.

3.4 The r Minus g Perspective

Next, we turn to the $r-g$ perspective, which allows us to obtain more general results characterizing the dynamics of the TVC term $\bar{m}_t\bar{n}_t$ without relying on Assumption 1 and the specific wealth

distribution.

Proposition 4. *The TVC term for aggregate wealth has the following dynamics:*

$$d[\bar{m}_t \bar{n}_t] = -\left[\rho + \zeta - \tilde{\sigma}^2\right] \bar{m}_t \bar{n}_t dt + \zeta e^{-\rho t} \bar{n}_0 dt.$$

If $\tilde{\sigma}_t = \tilde{\sigma}$ is constant, the aggregate TVC holds if and only if

$$\rho + \zeta > \tilde{\sigma}^2. \tag{16}$$

The proof is presented in Appendix A.7. This proposition shows that the sign of $\rho + \zeta - \tilde{\sigma}^2$ plays a crucial role in determining the TVC for aggregate wealth. As

$$(r_t^f + \sigma^2) - g_t = \rho + \zeta - \tilde{\sigma}^2,$$

this proposition shows that the relevant condition for the TVC is again "$r - g > 0$." To satisfy the TVC for aggregate wealth, we need either a small enough volatility $\tilde{\sigma}_t$ of the idiosyncratic risk, or a sufficiently high jump risk ζ so that the households' wealth frequently reverts to the mean, or a sufficiently high subjective discount rate ρ.

Table 1(a) also reports the $r - g$ condition in the last column. Eq. (16) is satisfied when the jump intensity is high or moderate, which ensures that $r > g$ and the TVC holds. In contrast, with a low jump intensity, the marginal utility growth explodes fast enough to dominate the subjective discount rate in (13), which leads to TVC violation.

The results we obtain from the $r - g$ perspective are more general than the previous results from the wealth distribution perspective, which assumes a specific post-reset wealth distribution. In Internet Appendix A.8, we impose an arbitrary wealth distribution for households after experiencing the wealth reset, including a log-normal one. In this case, while the stationary wealth distribution no longer takes the form of Eq. (14), Proposition 4 from the $r - g$ perspective still holds. In particular, with a low enough jump intensity ζ, the TVC for aggregate wealth is still vio-

lated, which implies that the left tail of the stationary wealth distribution has to be heavier than the log-normal distribution. This can happen even when the stationary wealth distribution is obtained from introducing wealth reset to an expanding log-normal wealth distribution.

3.5 Comparison to U.S. Inequality Data

The results in this section show that a stationary wealth distribution does not guarantee the TVC for aggregate assets. TVC violation can still occur when the left tail of the stationary wealth distribution is heavy enough, so that the households expect to experience a very high marginal utility upon negative realizations of idiosyncratic shock. In this case, the integral of inverse wealth share, which measures the degree of inequality in the wealth distribution, can diverge to infinity just like the case with permanent idiosyncratic risk.

A natural question is whether the wealth distribution implied by cases with TVC violation is consistent with the data. If the wealth distribution needs to be incredibly skewed to generate TVC violation, then TVC violation is just a theoretical curiosity.

Figure 3 compares our model variants with the U.S. data. We use the Gini coefficient based on income data to measure the degree of inequality in the U.S.[6] From 1963 to 2022, the Gini coefficient has remained stable, fluctuating around 0.4. In the baseline model with permanent idiosyncratic risk, if we start with the same degree of inequality, the wealth distribution will diverge over time. Under our calibration, which is motivated by the empirical moments considered in Brunnermeier et al. (2022), we expect the Gini coefficient to rise to 0.6 around 2020 and approach 0.7 around 2050.

In contrast, models with transitory idiosyncratic risk imply a stable Gini coefficient. We again consider the three cases with different values of jump intensity ζ. Both the case with moderate jump intensity and the case with low jump intensity yield a Gini coefficient around 0.4, even though the

[6]The wealth in our model encompasses not only financial wealth but also human capital. Greenwald et al. (2021)'s estimates show that the Gini for financial wealth is approximately 0.74 in the U.S., whereas the adjusted total wealth Gini is 0.41, which aligns closely with the figures used here. 0.4 is also close to the Gini index for consumption, which is relevant in marginal utility dispersion.

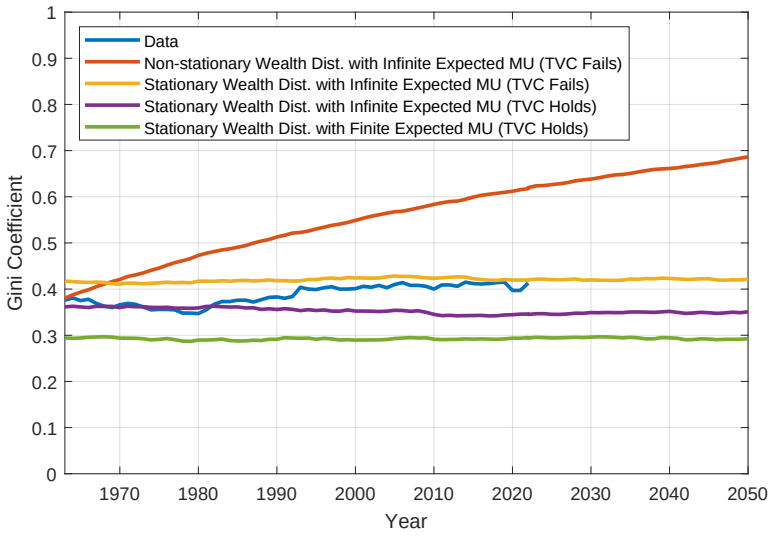


Figure 3: Gini Coefficient of Consumption Inequality.

TVC holds in the former case and fails in the latter.

Thus, we obtain a possibility result: we do not need an extremely skewed wealth distribution to generate TVC violation, as long as households have reasonably high risk aversion ($\gamma = 2$ in our calibration). In this case, the market value of long-lived assets may exceed the discounted present value of its cash flows.

That said, two caveats must be noted. First, the Gini coefficient is not the only relevant metric for evaluating the degree of inequality in the wealth distribution. TVC violation relies on households expecting sufficiently large dispersions in marginal utility growth, which makes it important to examine household beliefs about wealth dynamics. We leave this analysis to future work.

Second, and relatedly, TVC violation in our setting relies on households' marginal utility being unbounded—that is, the Inada condition plays an important role. If, for example, we assume that the households' marginal utility remains finite when their consumption approaches zero, the TVC holds even when the wealth distribution has similar statistics.

3.6 Summary

Our analysis of permanent and transitory idiosyncratic risk yields several insights. First, a non-stationary wealth distribution may be able to generate a violation of TVC for aggregate assets, but it depends on the rate of divergence. The TVC can still hold if the wealth distribution diverges more slowly than the subjective discount factor. Only when the divergence is sufficiently rapid is the TVC violated.

Second, a stationary wealth distribution is not sufficient to ensure the aggregate TVC holds. The left tail is critical: if it is sufficiently heavy, the TVC can still be violated. In such cases, households anticipate their wealth shrinking to a negligible fraction of the economy for long enough time, and therefore perceive aggregate assets to be infinitely valuable.

Third, the condition $r - g > 0$ remains both necessary and sufficient for the TVC to hold. In our setting, this gap reflects the relative magnitude of the idiosyncratic risk and the subjective discount factor.

Fourth, the degree of inequality in post-war U.S. data does not rule out TVC violations. With a plausible level of risk aversion, the left-tail risk required for such violations may be consistent with the observed Gini coefficient. To evaluate the possibility of asset bubbles, we need to look further into the left tail of the wealth distribution.

3.7 Transversality Condition under Stochastic Death

Our characterizations of the TVC continue to hold in a model where households face a constant probability of death, denoted by ζ. For concreteness, let us first consider the following argument in discrete time. Let c_t^i denote the household's consumption at time t, conditional on survival, and assume utility is zero in all periods after death. Consider a long-lived asset with price p_t and dividend d_t at time t. The household alive at time t has the following first-order condition:

$$u'(c_t^i)p_t = (1-\zeta)\mathbb{E}_t\left[\delta u'(c_{t+1}^i)(p_{t+1}+d_{t+1})\right] + \zeta \times 0,$$

reflecting the fact that the asset is valuable only insofar as the household survives in the next period to consume the payoff.

Iterating this equation forward, we obtain

$$u'(c_t^i)p_t = \mathbb{E}_t\left[\sum_{k=1}^{\infty}((1-\zeta)\delta)^k u'(c_{t+k}^i)d_{t+k}\right] + \lim_{k\to\infty}\mathbb{E}_t\left[((1-\zeta)\delta)^k u'(c_{t+k}^i)p_{t+k}\right],$$

which is very similar to the case without death, except that the subjective discount rate δ is augmented with the survival probability $(1-\zeta)$. Intuitively, the risk of death increases the household's effective discount rate, making them behave more impatiently.

Aside from this adjustment, the valuation of the asset continues to depend on the distribution of marginal utility growth $u'(c_{t+k}^i)/u'(c_t^i)$ conditional on survival. If the wealth and consumption distributions are sufficiently left-skewed, the TVC can still fail. Thus, even in the presence of mortality risk, our main result holds: the validity of the aggregate TVC hinges on the thickness of the left tail of the wealth distribution.

Internet Appendix A.9 formalizes this argument in a continuous-time setting that extends our baseline model. Households face a constant death rate ζ, while a fraction ζ of new households are born and receive the wealth of the deceased households. This ensures that aggregate wealth and financial assets are preserved. We consider two plausible cases.

First, suppose each new household directly inherits the wealth of a deceased household. In this case, the model is equivalent to the baseline setting without jump risk, as discussed in Section 2. The wealth distribution has an explosive amount of inequality, and the aggregate TVC may be violated if idiosyncratic risk is sufficiently large. The only difference is that the subjective discount rate ρ needs to be augmented by the death rate ζ, mirroring the discrete-time example. The necessary and sufficient condition for the aggregate TVC to hold is

$$\rho + \zeta - \tilde{\sigma}_t^2 > 0, \tag{17}$$

which is identical to the condition (16) in the previous case with wealth reset.

Second, suppose the wealth of deceased households is pooled and redistributed equally among the newborns. This setup corresponds to the model with wealth reset shocks described in Section 3. The wealth distribution becomes stationary, but the aggregate TVC remains sensitive to the left tail of the wealth distribution. As before, the death rate affects the consumption-wealth ratio $\hat{c}$, and the TVC condition is again given by Eq. (17). This condition ensures that the left tail of the stationary wealth distribution is not too heavy. Conversely, if this condition fails, TVC violation occurs even when the wealth distribution is stationary.

This discussion shows that introducing stochastic death is not a simple fix to TVC violation for long-lived assets. Whether the TVC holds or not continues to hinge on the degree of wealth inequality, or, more precisely, the balance between the effective subjective discount factor and the magnitude of idiosyncratic risk. While death increases impatience and thus makes TVC violations less likely to happen, it does not fundamentally alter the mechanism by which left-tail risk generates such violations.

4 Implications for Theories of Heterogeneous-Agent Economies

Our results identify wealth inequality as a central determinant of TVC in heterogeneous-agent economies. In this section, we apply this insight to workhorse models in finance and macroeconomics, and examine its implications for asset valuation.

4.1 Constantinides-Duffie

Constantinides and Duffie (1996) study an economy with heterogeneous households facing idiosyncratic risk, which admits a no-trade equilibrium. In their framework, while households can freely trade financial assets such as equities and bonds, their labor income is subject to permanent idiosyncratic shocks. These shocks generate a gap between individual and aggregate consumption

growth. In equilibrium, the consumption growth of household i is given by

$$\frac{c_{t+1}^i}{c_t^i} = \frac{c_{t+1}}{c_t} \exp\left(-\frac{1}{2}y_{t+1}^2 + y_{t+1}\varepsilon_{t+1}^i\right),$$

where c_{t+1}/c_t is the aggregate consumption growth, and ε_{t+1}^i is an i.i.d. idiosyncratic shock. Additional model details are provided in Appendix C.

Let $\eta_t^i = c_t^i/c_t$ denote the household's consumption share. Assuming CRRA utility, the household's one-period pricing kernel is

$$m_{t,t+1}^i = e^{-\rho}\left(\frac{c_{t+1}^i}{c_t^i}\right)^{-\gamma} = e^{-\rho}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\left(\frac{\eta_{t+1}^i}{\eta_t^i}\right)^{-\gamma}.$$

Relative to the representative-agent benchmark, the additional term $\left(\frac{\eta_{t+1}^i}{\eta_t^i}\right)^{-\gamma}$ introduces extra variation in the pricing kernel, which can help generate more realistic risk premia. A large subsequent literature builds on this mechanism by specifying idiosyncratic risk processes that better match key asset pricing moments.

In this setting, the idiosyncratic shocks are permanent. The consumption share η_t^i follows a random walk, leading to an exploding degree of inequality in the limit. This setting closely parallels our baseline model in Section 2, in which permanent idiosyncratic shocks drive most households' consumption shares to zero in the limit. Consequently, if the magnitude of permanent idiosyncratic risk is sufficiently large, the TVC may fail. This possibility is overlooked in prior analysis, which generally typically the TVC as an assumption.

Asset Pricing Example. To illustrate this point, we develop an example with sufficiently large permanent idiosyncratic risk. Consider a long-lived asset, such as equity, that pays a non-negative dividend stream d_t. We would like to construct a case where the one-period Euler equation holds,

$$\mathbb{E}_t[m_{t,t+1}^i r_{t+1}] = 1,$$

yet the TVC fails, so that the discounted value of the dividend stream approaches infinity:

$$\sum_{t=1}^{\infty} \mathbb{E}_0 \left[m_{0,t}^i d_t \right] = \infty.$$

Suppose this asset is the claim to the aggregate consumption. The argument readily extends to assets whose cash flows are not subject to idiosyncratic risk. For simplicity, assume constant aggregate consumption growth $g_t = g > 0$, constant idiosyncratic risk volatility $y_t = y > 0$, and parameter restrictions $\rho < y^2 < \rho + g$. Assume households have log utility: $\gamma = 1$. The pricing kernel from time 0 to t is

$$m_{0,t}^i = \exp(-\rho t) \left(\frac{c_t^i}{c_0^i} \right)^{-1} = \exp(-\rho t) \left(\frac{c_t}{c_0} \right)^{-1} \exp\left(\frac{t}{2} y^2 - y \sum_{k=1}^{t} \varepsilon_k^i \right).$$

The present value of the consumption strip at horizon t is

$$\mathbb{E}_0[m_{0,t}^i c_t] = \mathbb{E}_0 \left[\exp(-\rho t) \left(\frac{c_t}{c_0} \right)^{-1} \exp\left(\frac{t}{2} y^2 - y \sum_{k=1}^{t} \varepsilon_k^i \right) c_t \right] = c_0 . \exp((y^2 - \rho)t).$$

When the idiosyncratic risk is sufficiently large such that $y^2 > \rho$, the present value of the consumption strip grows exponentially, leading to an infinite sum of discounted cash flows $\sum_{t=1}^{\infty} \mathbb{E}_0[m_{0,t}^i c_t]$. Therefore, the consumption claim will be infinitely valuable, and the no-trade equilibrium breaks down.[7]

This example illustrates that the one-period risk-return relationship and the TVC contain distinct information about asset valuation. Correctly pricing one-period returns alone is not sufficient to ensure well-defined asset prices in heterogeneous-agent models. A complete characterization requires verifying that the TVC holds in the presence of persistent heterogeneity and idiosyncratic risk.

[7] Constantinides and Duffie (1996) impose the condition $\lim_{T \to \infty} \mathbb{E}[m_{0,T}] = 0$ (their Eq. (5)). This condition is not sufficient to rule out TVC violation. In our example, the pricing kernel satisfies

$$\lim_{T \to \infty} \mathbb{E}_0[m_{0,T}^i] = \lim_{T \to \infty} \exp((-\rho - g + y^2)t) = 0,$$

so the entire term structure of risk-free rates is positive, yet the TVC still fails.

Transitory Idiosyncratic Risk. In their appendix, Constantinides and Duffie (1996) also explore a model variant that incorporates wealth reset, which resembles our model variant with a stationary wealth distribution. As we show in Section 3, the TVC can still fail even in this case, if the left tail of the stationary wealth distribution is sufficiently heavy. Therefore, the TVC cannot be taken for granted in heterogeneous-agent economies.

4.2 Brunnermeier-Merkel-Sannikov

Our results offer a complementary view to Brunnermeier et al. (2022) on the intertemporal government budget constraint. As discussed in the literature review, their *dynamic trading perspective* focuses on individual wealth dynamics and time-varying Treasury holdings. Since the TVC for individual wealth always holds, their approach provides insights into how households value government debt. Our analysis does not contradict this view.

In contrast, we focus on the *buy-and-hold perspective*, which evaluates the aggregate government budget from the issuer's stand point. As Brunnermeier et al. (2022) point out, the TVC for aggregate government debt portfolio is not guaranteed to hold because it does not stem from an individual's optimization problem. Nevertheless, standard asset pricing approaches evaluate debt based on aggregate cash flows, making it essential to understand whether and when the TVC holds at the aggregate level.

Our results provide a general framework to consider this issue, as well as concrete examples in which the TVC fails. In particular, we show that the validity of the aggregate TVC hinges on the shape of the wealth distribution. When the left tail is sufficiently heavy, the TVC for aggregate assets can fail, and asset prices may diverge from the present value of expected cash flows.

4.3 Aiyagari-Bewley-Huggett Models

Compared to Brunnermeier et al. (2022); Constantinides and Duffie (1996) and the settings analyzed in our paper, classic Aiyagari-Bewley-Huggett models introduce a borrowing constraint,

which restricts each household's asset holding b_t^i from falling below a certain level q_t^i:

$$b_t^i \geq q_t^i.$$

When $q_t^i = 0$, this is a no-short-selling constraint. More generally, $q_t^i > 0$ can reflect minimum asset holding requirements as in financial repression.

When the borrowing constraint binds, households would prefer to reallocate consumption from the future to the present, but they are unable to do so through borrowing. In the discrete-time framework typically used in this literature, the Euler equation becomes an inequality:

$$m_t^i \geq \mathbb{E}_t \left[m_{t+1}^i (1+r) \right],$$

with strict inequality when the constraint is binding.

As a result, the government's intertemporal budget condition (10) also becomes an inequality:

$$b_t \geq \lim_{T\to\infty} \mathbb{E}_t \left[\sum_{k=1}^{T} \frac{m_{t+k}^i}{m_t^i} s_{t+k} \right] + \lim_{T\to\infty} \mathbb{E}_t \left[\frac{m_T^i}{m_t^i} b_T \right]. \tag{18}$$

The proof is presented in the Internet Appendix D.1. This expression implies that the market value of government debt b_t can exceed the present value of expected future surpluses for two distinct reasons. First, the TVC fails, resulting in an infinite $\lim_{T\to\infty} \mathbb{E}_t \left[\frac{m_T^i}{m_t^i} b_T \right]$ term.

Second, the inequality in the Euler equation creates slack even if the TVC holds: the present value of surpluses is only a lower bound on the value of government debt. In this sense, TVC violation for long-lived assets provide a sufficient, though not necessary, condition for $P > PV(CF)$.[8]

Our characterizations of the TVC still apply to these settings. Consider Krueger and Lustig (2010) for example, who develop a heterogeneous-agent model in which a fraction of the aggregate endowment is distributed as equity dividends, while the remainder is allocated as labor income. On

[8]Belief distortion can also lead to $P > PV(CF)$ under econometricians' information set, even when the TVC holds from the households' perspective. Jiang et al. (2024a) provides a concrete example in the context of government debt valuation.

the equity side, all households hold the market index, albeit in different proportions. On the labor income side, households face idiosyncratic labor income risk governed by a Markov process.

In this environment, the distribution of the individual household's marginal utility growth, i.e., m_T^i/m_t^i, remains central to determining whether the TVC for aggregate assets holds. If labor income is bounded below by a strictly positive amount, then household consumption shares are likewise bounded from below. This, in turn, implies that marginal utility is bounded above, ruling out the possibility of explosive marginal utility growth. As we detail in the Internet Appendix D.2, this ensures that households do not expect to become infinitely poor relative to the population average, ruling out the possibility of TVC violation.

4.4 Townsend and Woodford

Townsend (1980) provides another illustration of how borrowing constraints affect asset valuation. We consider a simplified version of Woodford (1990) to highlight this mechanism.

The economy consists of two types of infinitely-lived households. Type A receives endowment e_1 in even periods and e_2 in odd periods, while type B receives endowment e_2 in even periods and e_1 in odd periods, with $e_1 > e_2 > 0$. This staggered income pattern generates a motive for intertemporal trade between the two types.

Households can invest in government debt with total supply d, which is financed by lump-sum taxes τ equally split between the two household types. The interest rate is r. Households face a borrowing constraint: $b_{i,t} \geq 0$, for $i \in \{A, B\}$. Each household's budget constraint is

$$c_{i,t} + \tau/2 + b_{i,t} = e_{i,t} + (1+r)b_{i,t-1},$$

where the tax equals the interest on the government debt, i.e., $\tau = rd$.

Households maximize lifetime utility $\sum_{t=0}^{\infty} \rho^t u(c_{i,t})$. We assume that the endowment level e_2 is sufficiently lower than e_1 so that the borrowing constraint is binding in the low-endowment period. By market clearing, the other household must hold all the government debt.

For example, in period 1, household A holds no government, whereas household B holds all the government debt:

$$c_{A,1} = e_2 + (1+r)d - \tau/2 \stackrel{\text{def}}{=} \underline{c},$$
$$c_{B,1} = e_1 - d - \tau/2 \stackrel{\text{def}}{=} \bar{c}.$$

Now, we are in a position to evaluate the government debt. Since the borrowing constraint always binds for the low-endowment type, Eq. (18) applies. For household A, we have

$$d > \lim_{T\to\infty} \mathbb{E}_0 \left[\sum_{t=1}^{T} \rho^t \frac{u'(c_{A,t})}{u'(c_{A,0})} \tau \right] + \lim_{T\to\infty} \mathbb{E}_0 \left[\rho^T \frac{u'(c_{A,T})}{u'(c_{A,0})} d \right] = \lim_{T\to\infty} \mathbb{E}_0 \left[\sum_{t=1}^{T} \rho^t \frac{u'(c_{A,t})}{u'(c_{A,0})} \tau \right].$$

The proof is presented in the Internet Appendix D.3. Because marginal utility is bounded, the transversality term vanishes. This example shows that even when the TVC holds, borrowing constraints can increase the value of government debt above the present value of surpluses.

4.5 HANK Models

HANK models also feature heterogeneous agents facing idiosyncratic risk. As a result, the TVC for long-lived assets also depends on the wealth distribution. For example, in McKay, Nakamura, and Steinsson (2016), households receive equal shares of firm dividends and face borrowing constraints. The present value relationship again becomes an inequality as in Eq. (18). Yet, as household wealth is bounded from below by the dividend-output ratio, which limits the degree of wealth inequality, the TVC holds in this model.

Similarly, in Kaplan, Moll, and Violante (2018), households receive lump-sum transfers and face borrowing constraints. Transfers minus borrowing costs ensure a lower bound on household income, which is positive under the paper's calibration. Thus, despite an extremely skewed financial asset distribution, the TVC for aggregate assets holds due to bounded left tails of the wealth distribution.

5 Conclusion

This paper characterizes the conditions under which asset bubbles arise and TVC fails for long-lived assets in heterogeneous-agent economies. We find that the degree of wealth inequality is the key to understanding this question. Both non-stationary wealth distributions with explosive inequality and stationary distributions with heavy left tails can lead to TVC violations. The r minus g condition remains a useful guide for interpreting the TVC in heterogeneous-agent models.

This new type of asset bubbles has important implications for asset valuation in heterogeneous-agent models. For example, the TVC may fail in settings considered by Constantinides and Duffie (1996) and Brunnermeier et al. (2022), where the wealth distribution explodes over time. In contrast, the TVC holds in settings considered by Krueger and Lustig (2010) and McKay et al. (2016); Kaplan et al. (2018), when the wealth distribution is bounded from below. These results offer a new perspective on the valuation of long-lived assets in heterogeneous-agent models.

References

Andrew B Abel and Stavros Panageas. Running primary deficits forever in a dynamically efficient economy: Feasibility and optimality. Technical report, National Bureau of Economic Research, 2022.

S Rao Aiyagari. Uninsured idiosyncratic risk and aggregate saving. The Quarterly Journal of Economics, 109(3):659–684, 1994.

George-Marios Angeletos. Uninsured idiosyncratic investment risk and aggregate saving. Review of Economic dynamics, 10(1):1–30, 2007.

Andrew Atkeson and Robert E Lucas. On efficient distribution with private information. The Review of Economic Studies, 59(3):427–453, 1992.

Olivier Blanchard. Public debt and low interest rates. American Economic Review, 109(4):1197–1229, 2019.

Olivier Blanchard and Mark Watson. Bubbles, rational expectations and financial markets, 1982.

Markus K Brunnermeier, Sebastian A Merkel, and Yuliy Sannikov. The fiscal theory of price level with a bubble. Technical report, National Bureau of Economic Research, 2020.

Markus K Brunnermeier, Sebastian A Merkel, and Yuliy Sannikov. Debt as safe asset. Technical report, National Bureau of Economic Research, 2022.

Zefeng Chen, Zhengyang Jiang, Hanno Lustig, Stijn Van Nieuwerburgh, and Mindy Z Xiaolan. Exorbitant privilege gained and lost: Fiscal implications. Technical report, National Bureau of Economic Research, 2022.

George M Constantinides and Darrell Duffie. Asset pricing with heterogeneous consumers. Journal of Political economy, 104(2):219–240, 1996.

Peter Diamond. National debt in a neoclassical growth model. American economic review, 55(5): 1126–1150, 1965.

Nicolae Gârleanu and Stavros Panageas. Heterogeneity and asset prices: An intergenerational approach. Journal of Political Economy, 131(4):839–876, 2023.

Matthieu Gomez. Wealth inequality and asset prices. Review of Economic Studies, page rdaf008, 2025.

Daniel L Greenwald, Matteo Leombroni, Hanno Lustig, and Stijn Van Nieuwerburgh. Financial and total wealth inequality with declining interest rates. Technical report, National Bureau of Economic Research, 2021.

Sanford J Grossman and Robert J Shiller. Consumption correlatedness and risk measurement in economies with non-traded assets and heterogeneous information. Journal of financial Economics, 10(2):195–210, 1982.

Tomohiro Hirano and Alexis Akira Toda. Bubble economics. Journal of Mathematical Economics, 111:102944, 2024.

Tomohiro Hirano and Alexis Akira Toda. Bubble necessity theorem. Journal of Political Economy, 133(1):111–145, 2025.

Magnus Irie. Innovations in entrepreneurial finance and top wealth inequality. Technical report, Working paper, 2024.

Zhengyang Jiang, Hanno Lustig, Stijn Van Nieuwerburgh, and Mindy Z Xiaolan. Manufacturing risk-free government debt. Technical report, National Bureau of Economic Research, 2020.

Zhengyang Jiang, Hanno Lustig, Stijn Van Nieuwerburgh, and Mindy Z Xiaolan. What drives variation in the us debt-to-output ratio? the dogs that did not bark. The Journal of Finance, 2024a.

Zhengyang Jiang, Hanno Lustig, Stijn Van Nieuwerburgh, and Mindy Z Xiaolan. The us public debt valuation puzzle. Econometrica, 92(4):1309–1347, 2024b.

Greg Kaplan, Benjamin Moll, and Giovanni L Violante. Monetary policy according to hank. American Economic Review, 108(3):697–743, 2018.

Narayana R Kocherlakota. Bubbles and constraints on debt accumulation. Journal of Economic theory, 57(1):245–256, 1992.

Narayana R Kocherlakota. Public debt bubbles, liquidity, and risk: Policy assessments based on the zero-beta interest rate. Technical report, National Bureau of Economic Research, 2025.

Leonid Kogan, Dimitris Papanikolaou, and Noah Stoffman. Left behind: Creative destruction, inequality, and the stock market. Journal of Political Economy, 128(3):855–906, 2020.

Dirk Krueger and Hanno Lustig. When is market incompleteness irrelevant for the price of aggregate risk (and when is it not)? Journal of Economic Theory, 145(1):1–41, 2010.

Robert E Lucas. On efficiency and distribution. The Economic Journal, 102(411):233–247, 1992.

Emile A Marin and Sanjay R Singh. Incomplete markets and exchange rates. Technical report, 2025.

Alisdair McKay, Emi Nakamura, and Jón Steinsson. The power of forward guidance revisited. American Economic Review, 106(10):3133–3158, 2016.

Atif R Mian, Ludwig Straub, and Amir Sufi. A goldilocks theory of fiscal deficits. Technical report, National Bureau of Economic Research, 2022.

Luigi Montrucchio. Cass transversality condition and sequential asset bubbles. Economic Theory, 24:645–663, 2004.

Stavros Panageas. The implications of heterogeneity and inequality for asset pricing. Foundations and Trends® in Finance, 12(3):199–275, 2020.

Stavros Panageas. Growth-indexed securities. 2024.

William J Reed. The pareto, zipf and other power laws. Economics letters, 74(1):15–19, 2001.

Ricardo Reis. The constraint on public debt when r< g but g< m. CEPR Discussion Papers, 15950, 2021.

Paul A Samuelson. An exact consumption-loan model of interest with or without the social contrivance of money. Journal of political economy, 66(6):467–482, 1958.

Manuel S Santos and Michael Woodford. Rational asset pricing bubbles. Econometrica: Journal of the Econometric Society, pages 19–57, 1997.

Lawrence Schmidt. Climbing and falling off the ladder: Asset pricing implications of labor market event risk. Available at SSRN 2471342, 2024.

Jean Tirole. Asset bubbles and overlapping generations. Econometrica: Journal of the Econometric Society, pages 1499–1528, 1985.

Alexis Akira Toda. Incomplete market dynamics and cross-sectional distributions. Journal of Economic Theory, 154:310–348, 2014.

Robert M Townsend. Models of money with spatially separated agents. Models of monetary economies, pages 265–303, 1980.

Michael Woodford. Public debt as private liquidity. The American Economic Review, 80(2): 382–388, 1990.

Appendix

Contents

A Model Details **44**

A.1 Model Derivation 44

A.2 Proof for Proposition 1 56

A.3 Transversality Conditions with Permanent Idiosyncratic Risk 57

A.4 Transversality Conditions with Transitory Idiosyncratic Risk 59

A.5 Proof for Proposition 2 61

A.6 Proof for Proposition 3 63

A.7 Proof for Proposition 4 66

A.8 Arbitrary Wealth Distribution After Wealth Reset 67

A.9 Model with Stochastic Death 69

A.10 Calibration 73

B Model with General CRRA Utility **74**

B.1 Model Derivation 74

B.2 Transversality Conditions with Transitory Idiosyncratic Risk 77

C TVC in the Constantinides-Duffie Economy **81**

D Model Extension with Borrowing Constraints **83**

D.1 Intertemporal Government Budget Condition 83

D.2 Consumption Lower Bound in Bewley Model 85

D.3 Townsend and Woodford Model 87

E Comparing Different Types of Asset Bubbles **88**

E.1 Classic Rational Bubbles 89

E.2 Unbalanced Growth 91

A Model Details

In this Appendix, we provide derivation details and additional explanations for models in Section 3 and 3.7. Note that the permanent risk model in Section 2 can be viewed as a special case of the transitory risk model in Section 3 with wealth reset intensity $\zeta = 0$.

Throughout the proof, for any jump-diffusion process $X_t = X_0 + \int_0^t \mu_{s-}^X ds + \int_0^t \sigma_{s-}^X dZ_s + \sum_{k=1}^{N(t)} \Delta X_k$, where Z_t is a standard Brownian motion, $N(t)$ is a counting process we use $\mathbb{E}_t[dX_t]/dt$ to denote μ_{t-}^X and $\langle dX_t, dX_t\rangle/dt$ to denote $(\sigma_{t-}^X)^2$.

A.1 Model Derivation

A.1.1 Set-up

Households. Household's lifetime utility is given by

$$V_0^i = \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^i dt\right].$$

The capital evolution follows

$$\frac{d\tilde{k}^i}{\tilde{k}^i} = \left(\Phi(\iota_t^i) - \delta\right) dt + \tilde{\nu}_t d\tilde{Z}_t^i + d\Delta_t^{k,i},$$

where $d\Delta_t^{k,i}$ is the physical capital transactions between households, and

$$\Phi(\iota) = \frac{1}{\phi}\log(1+\phi\iota).$$

The idiosyncratic risk processes $\tilde{Z}_t^i$ are independent across households. Besides, households are subject to wealth reset shock with intensity ζ. In the permanent shock case, $\zeta = 0$. Formally, let $\tilde{J}_t^i$ be a Poisson process with intensity ζ, the wealth dynamic for household i is given by

$$\frac{dn_t^i}{n_{t-}^i} = -\frac{c_t^i}{n_{t-}^i}dt + dr_t^B + \theta_t^{K,i}\left(dr_t^{K,i}(\iota_t^i) - dr_t^B\right) + \theta^{E,i}\left(dr_t^{E,i} - dr_t^B\right) + \theta_t^{\bar{E},i}\left(dr_t^{\bar{E},i} - dr_t^B\right) + \frac{\bar{n}_t - n_{t-}^i}{n_{t-}^i} d\tilde{J}_t^i,$$

where $\tilde{J}_t^i$ and $\tilde{J}_t^j$ are stochastically independent if $i \neq j$. Households face a skin-in-the-game constraint when issuing equity, i.e.,

$$-\theta_t^{E,i} \leq (1-\bar{\chi})\theta_t^{K,i}.$$

Government. The nominal growth of government debt follows

$$dB_t = \mu_t^B B_t dt.$$

Government budget constraint is given by

$$i_t B_t + P_t \mathfrak{g} K_t = \mu_t^B B_t + P_t \tau_t a_t K_t,$$

where nominal rate i_t, spending and tax rate $\mathfrak{g}$, productivity a_t are exogenous processes. Assume those processes, together with $\tilde{\nu}_t$, are jointly Markov and are driven by some Brownian motion Z_t. This creates aggregate risk in the economy. Tax rate τ_t is determined endogenous. With q_t^B defined below, the budget constraint can be restated as

$$\check{\mu}_t^B q_t^B + s_t = 0, \tag{A.1}$$

where $\check{\mu}_t^B := \mu_t^B - i_t$ is the bond issuance (net of interest payment) rate. $s_t := \tau_t a_t - \mathfrak{g}$ is the real surplus-capital ratio.

A.1.2 Symmetric Equilibrium

Aggregation. Under symmetric equilibrium,

$$\forall i \in [0,1], \quad (\theta_t^{K,i}, \theta_t^{E,i}, \theta_t^{\bar{E},i}, \iota_t^i, \hat{c}_t^i) = (\theta_t^K, \theta_t^E, \theta_t^{\bar{E}}, \iota_t, \hat{c}_t),$$

where $\hat{c}_t^i = c_t^i / n_t^i$ is the consumption-wealth ratio.

Under symmetric equilibrium, the returns adjust such that no one trades physical capital with others, i.e., $\Delta_t^{k,i} = 0$. Aggregate capital evolves according to

$$dK_t = (\Phi(\iota_t) - \delta) K_t dt.$$

Define the real price of physical capital as q_t^K and the real bond-capital ratio as $q_t^B = \frac{B_t/P_t}{K_t}$. The state variable ϑ_t is defined as the share of wealth in government bond, i.e.,

$$\vartheta_t = \frac{q_t^B}{q_t^B + q_t^K}. \tag{A.2}$$

Assume that q_t^K, q_t^B and ϑ_t all follow geometric Brownian motion with drift terms $\mu_t^{q,K}$, $\mu_t^{q,B}$ and μ_t^{ϑ} and volatility terms $\sigma_t^{q,K}$, $\sigma_t^{q,B}$ and σ_t^{ϑ}.

The difference of volatility $\sigma_t^{q,B} - \sigma_t^{q,K}$ is $\frac{\sigma_t^{\vartheta}}{1-\vartheta_t}$. Indeed, by definition,

$$\vartheta_t = \frac{q_t^B}{q_t^B + q_t^K}.$$

By Itô's lemma,

$$\vartheta_t \sigma_t^{\vartheta} = \frac{q_t^B}{q_t^B + q_t^K} \left(\sigma_t^{q,B} - \frac{\sigma_t^{q,B} q_t^B + \sigma_t^{q,K} q_t^K}{q_t^B + q_t^K} \right) = \frac{q_t^B}{q_t^B + q_t^K} \frac{(\sigma_t^{q,B} - \sigma_t^{q,K}) q_t^K}{q_t^B + q_t^K},$$

i.e.,

$$\sigma_t^{\vartheta} = \sigma_t^{q,B} - \frac{\sigma_t^{q,B} q_t^B + \sigma_t^{q,K} q_t^K}{q_t^B + q_t^K} = (\sigma_t^{q,B} - \sigma_t^{q,K})(1 - \vartheta_t). \tag{A.3}$$

The difference of drift $\mu_t^{q,B} - \mu_t^{q,K}$ can be derived similarly. Indeed, by Itô's lemma,

$$\begin{aligned}
\mu_t^{\vartheta} \vartheta_t &= \frac{q_t^B}{q_t^B + q_t^K} \left(\mu_t^{q,B} - \frac{q_t^B \mu_t^{q,B} + q_t^K \mu_t^{q,K}}{q_t^B + q_t^K} + \left(\frac{q_t^B \sigma_t^{q,B} + q_t^K \sigma_t^{q,K}}{q_t^B + q_t^K} \right)^2 - \left(\frac{q_t^B \sigma_t^{q,B} + q_t^K \sigma_t^{q,K}}{q_t^B + q_t^K} \right) \sigma_t^{q,B} \right) \\
&= \frac{q_t^B}{q_t^B + q_t^K} (1 - \vartheta_t) \left((\mu_t^{q,B} - \mu_t^{q,K}) - \left(\frac{q_t^B \sigma_t^{q,B} + q_t^K \sigma_t^{q,K}}{q_t^B + q_t^K} \right) (\sigma_t^{q,B} - \sigma_t^{q,K}) \right),
\end{aligned}$$

i.e.,

$$\frac{\mu_t^{\vartheta}}{1-\vartheta_t} = \mu_t^{q,B} - \mu_t^{q,K} + (\sigma_t^{q,B} - \sigma_t^{\vartheta})(\sigma_t^{q,K} - \sigma_t^{q,B}). \tag{A.4}$$

Return Processes. Here, we express the return process with the drift terms of q_t^B, q_t^K and ϑ_t.

Inflation:

$$\frac{dP_t}{P_t} = \frac{d(B_t/(q_t^B K_t))}{B_t/(q_t^B K_t)} = \left(\mu_t^B - \mu_t^{q,B} - \Phi(\iota_t) + \delta + (\sigma_t^{q,B})^2\right) dt - \sigma_t^{q,B} dZ_t$$

Bond return:

$$dr_t^B = i_t dt + \frac{d(1/P_t)}{1/P_t} = i_t dt + \frac{d(q_t^B K_t/B_t)}{q_t^B K_t/B_t} = \left(\Phi(\iota_t) - \delta - \breve{\mu}_t^B + \mu_t^{q,B}\right) dt + \sigma_t^{q,B} dZ_t. \tag{A.5}$$

Capital return:

$$\begin{aligned} dr_t^K &= \frac{(1-\tau_t)a_t - \iota_t^i}{q_t^K} dt + \frac{d(q_t^K \tilde{k}_t^i)}{q_t^K \tilde{k}_t^i} \\ &= \left(\frac{(1-\tau_t)a_t - \iota_t^i}{q_t^K} + \mu_t^{q,K} + (\Phi(\iota_t^i) - \delta)\right) dt + \sigma_t^{q,K} dZ_t + \tilde{\nu}_t d\tilde{Z}_t^i. \end{aligned} \tag{A.6}$$

Equity has the same risk loading as capital, but the expected return may be different when skin-in-the-game constraint is binding. Idiosyncratic risk loadings average out for aggregate equity.

Market Clearing. Goods market clearing:

$$\hat{c}_t(q_t^B + q_t^K) + \mathfrak{g} + \iota_t = a_t. \tag{A.7}$$

Asset market clearing:

$$\theta_t^K = \frac{q_t^K K_t}{(q_t^B + q_t^K)K_t} = 1 - \vartheta_t, \tag{A.8}$$

$$\theta^E = \theta^{\bar{E}}. \tag{A.9}$$

A.1.3 Solution

Households Problem. We use dynamic programming to solve the households' portfolio choice problem. Guess that the value function takes the form $V(n_t^i, v_t) = v_t + \frac{1}{\rho+\zeta}\log n_t^i$, where v_t depends on aggregate level variables only. Heuristically,

$$\begin{aligned} V(n_t^i, t) &= \max_{c_t^i, \iota_t^i, \theta_t^i} \mathbb{E}_t \left[\frac{(c_t^i)^{1-\gamma}}{1-\gamma} dt + e^{-\rho dt}(1_{\text{jump}} V(n_{t+dt}^i, t+dt) + 1_{\text{nojump}} V(n_{t+dt}^i, t+dt)) \right] \\ &= \max_{c_t^i, \iota_t^i, \theta_t^i} \mathbb{E}_t \left[\frac{(c_t^i)^{1-\gamma}}{1-\gamma} dt + (1-\rho dt)(\zeta dt V(\bar{n}_t, t+dt) + (1-\zeta dt) V(n_{t+dt}^i, t+dt)) \right] \end{aligned}$$

The Hamilton-Jacobi-Bellman equation is

$$\begin{aligned} &(\rho+\zeta) V(n_{t-}^i, v_t) \\ &= \max_{c_t^i, \iota_t^i, \theta_t^i} \mathbb{E}_t \left[\log c_t^i dt + \zeta V(\bar{n}_t, v_t) dt + V_t dt + V_n(n_{t-}^i, v_t) dn_t^i + \frac{1}{2} V_{nn}(n_{t-}^i, v_t) \langle dn_t^i, dn_t^i \rangle \right] / dt \\ &\Rightarrow \rho v_t + \log n_{t-}^i - \frac{\zeta}{\rho+\zeta} \log \bar{n}_t \\ &= \max_{c_t^i, \iota_t^i, \theta_t^i} \mathbb{E}_t \left\{ \log c_t^i dt + \dot{v}_t dt + \frac{1}{\rho+\zeta} \frac{dn_t^i}{n_{t-}^i} - \frac{1}{2} \frac{1}{\rho+\zeta} \left\langle \frac{dn_t^i}{n_{t-}^i}, \frac{dn_t^i}{n_{t-}^i} \right\rangle \right\} / dt, \end{aligned}$$

where we use $\mathbb{E}_t[dn_t^i]/dt$ to denote the drift term of n_t^i, as we have emphasized at the beginning of the appendix. For simplicity, we denote $\hat{\rho} := \rho + \zeta$. The HJB equation can be restated as

$$
\begin{aligned}
0 = \max_{c_t^i, \iota_t^i, \theta_t^i} \Bigg\{ & -(\hat{\rho} - \zeta) v_t - \log n_{t-}^i + \frac{\zeta}{\hat{\rho}} \log \bar{n}_t + \dot{v}_t + \log c_t^i \qquad \text{(A.10)} \\
& - \frac{1}{\hat{\rho}} \frac{c_t^i}{n_t^i} + \frac{1}{\hat{\rho}} (1 - \theta_t^{K,i} - \theta_t^{E,i} - \theta_t^{\bar{E},i}) \left(\Phi(\iota_t) - \delta - \check{\mu}_t^B + \mu_t^{q,B} \right) \\
& + \frac{1}{\hat{\rho}} \theta_t^{K,i} \left(\frac{(1 - \tau_t) a_t - \iota_t^i}{q_t^K} + \mu_t^{q,K} + (\Phi(\iota_t^i) - \delta) \right) + \frac{1}{\hat{\rho}} \theta_t^{E,i} \frac{\mathbb{E}_t[dr_t^{E,i}]}{dt} + \frac{1}{\hat{\rho}} \theta_t^{\bar{E},i} \frac{\mathbb{E}_t[dr_t^{\bar{E}}]}{dt} \\
& - \frac{1}{2\hat{\rho}} \left((1 - \theta_t^{K,i} - \theta_t^{E,i} - \theta_t^{\bar{E},i}) \sigma_t^{q,B} + (\theta_t^{K,i} + \theta_t^{E,i} + \theta_t^{\bar{E},i}) \sigma_t^{q,K} \right)^2 - \frac{1}{2\hat{\rho}} (\theta_t^{K,i} + \theta_t^{E,i})^2 \tilde{\nu}_t^2 \Bigg\}. \qquad \text{(A.11)}
\end{aligned}
$$

s.t.

$$
-\theta_t^{E,i} \leq (1 - \bar{\chi}) \theta_t^{K,i}
$$

The lagrangian is hence given by

$$
\begin{aligned}
\mathcal{L}_t^i = & -(\hat{\rho} - \zeta) v_t - \log n_{t-}^i + \frac{\zeta}{\hat{\rho}} \log \bar{n}_t + \dot{v}_t + \log c_t^i \\
& - \frac{1}{\hat{\rho}} \frac{c_t^i}{n_t^i} + \frac{1}{\hat{\rho}} (1 - \theta_t^{K,i} - \theta_t^{E,i} - \theta_t^{\bar{E},i}) \left(\Phi(\iota_t) - \delta - \check{\mu}_t^B + \mu_t^{q,B} \right) \\
& + \frac{1}{\hat{\rho}} \theta_t^{K,i} \left(\frac{(1 - \tau_t) a_t - \iota_t^i}{q_t^K} + \mu_t^{q,K} + (\Phi(\iota_t^i) - \delta) \right) + \frac{1}{\hat{\rho}} \theta_t^{E,i} \frac{\mathbb{E}_t[dr_t^{E,i}]}{dt} + \frac{1}{\hat{\rho}} \theta_t^{\bar{E},i} \frac{\mathbb{E}_t[dr_t^{\bar{E}}]}{dt} \\
& - \frac{1}{2\hat{\rho}} \left((1 - \theta_t^{K,i} - \theta_t^{E,i} - \theta_t^{\bar{E},i}) \sigma_t^{q,B} + (\theta_t^{K,i} + \theta_t^{E,i} + \theta_t^{\bar{E},i}) \sigma_t^{q,K} \right)^2 - \frac{1}{2\hat{\rho}} (\theta_t^{K,i} + \theta_t^{E,i})^2 \tilde{\nu}_t^2 \\
& + \lambda_t^i (\theta_t^{E,i} + (1 - \bar{\chi}) \theta_t^{K,i})
\end{aligned}
$$

The first order conditions w.r.t. c_t^i, ι_t^i, $\theta_t^{K,i}$, $\theta_t^{E,i}$ and $\theta_t^{\bar{E},i}$ are

$$-\frac{1}{\hat{\rho} n_{t-}^i} + \frac{1}{c_t^i} = 0, \tag{A.12}$$

$$-\frac{1}{q_t^K} + \frac{1}{1+\phi\iota_t^i} = 0, \tag{A.13}$$

$$\frac{\mathbb{E}_t[dr_t^{K,i}]}{dt} - \frac{\mathbb{E}_t[dr_t^B]}{dt} = \left(\sigma_t^{q,B} - (\theta_t^{K,i} + \theta_t^{E,i} + \theta_t^{\bar{E},i})\frac{\sigma_t^\vartheta}{1-\vartheta_t}\right)(\sigma_t^{q,K} - \sigma_t^{q,B}) + (\theta_t^{K,i} + \theta_t^{E,i})\tilde{\nu}_t^2 - (1-\bar{\chi})\lambda_t^i, \tag{A.14}$$

$$\frac{\mathbb{E}_t[dr_t^{E,i}]}{dt} - \frac{\mathbb{E}_t[dr_t^B]}{dt} = \left(\sigma_t^{q,B} - (\theta_t^{K,i} + \theta_t^{E,i} + \theta_t^{\bar{E},i})\frac{\sigma_t^\vartheta}{1-\vartheta_t}\right)(\sigma_t^{q,K} - \sigma_t^{q,B}) + (\theta_t^{K,i} + \theta_t^{E,i})\tilde{\nu}_t^2 - \lambda_t^i, \tag{A.15}$$

$$\frac{\mathbb{E}_t[dr_t^{\bar{E}}]}{dt} - \frac{\mathbb{E}_t[dr_t^B]}{dt} = \left(\sigma_t^{q,B} - (\theta_t^{K,i} + \theta_t^{E,i} + \theta_t^{\bar{E},i})\frac{\sigma_t^\vartheta}{1-\vartheta_t}\right)(\sigma_t^{q,K} - \sigma_t^{q,B}). \tag{A.16}$$

Note that we have plugged in $\sigma_t^{q,B} - \sigma_t^{q,K} = \frac{\sigma_t^\vartheta}{1-\vartheta_t}$.

Characterizing Consumption, Investment q_t^B and q_t^K. The first order condition Eq. (A.12) implies constant consumption-wealth ratio

$$c_t^i = \hat{\rho} n_{t-}^i. \tag{A.17}$$

FOC (A.13) reproduces the Tobin's q condition, and confirms that every household has the same investment ratio

$$q_t^K = 1 + \phi\iota_t. \tag{A.18}$$

Now we can (jointly) pin down ϑ_t, q_t^B and q_t^K. Recall goods market clearing condition

$$C_t + \mathfrak{g}K_t + \iota_t K_t = a_t K_t.$$

Plug in $C_t = \hat{\rho}\bar{n}_t = \hat{\rho} q_t^K K_t/(1-\vartheta_t)$ to obtain

$$\frac{\hat{\rho}}{1-\vartheta_t} + \frac{\hat{\rho}\phi}{1-\vartheta_t}\iota_t + \mathfrak{g} + \iota_t = a_t,$$

i.e.,

$$\iota_t = \frac{(a_t - \mathfrak{g})(1-\vartheta_t) - \hat{\rho}}{1 + \hat{\rho}\phi - \vartheta_t}. \tag{A.19}$$

Hence, we can express price of capital q_t^K as

$$q_t^K = \frac{1 - \vartheta_t + \phi(a_t - \mathfrak{g})(1-\vartheta_t)}{1 + \hat{\rho}\phi - \vartheta_t} = (1-\vartheta_t)\frac{1 + \phi(a_t - \mathfrak{g})}{1 + \hat{\rho}\phi - \vartheta_t}.$$

Since $q_t^B/q_t^K = \vartheta_t/(1-\vartheta_t)$,

$$q_t^B = \vartheta_t \frac{1 + \phi(a_t - \mathfrak{g})}{1 + \hat{\rho}\phi - \vartheta_t}. \tag{A.20}$$

Portfolio Weights Under symmetric equilibrium,

$$\theta_t^{K,i} = \theta_t^K, \quad \theta_t^{E,i} = \theta_t^E, \quad \theta_t^{\bar{E},i} = \theta_t^{\bar{E}}.$$

Asset Market clearing implies

$$1 - \theta_t^K - \theta_t^E - \theta_t^{\bar{E}} = \vartheta_t$$

and

$$\theta_t^E + \theta_t^{\bar{E}} = 0.$$

When the skin-in-the-game constraint is binding,

$$\theta_t^E + (1-\bar{\chi})\theta_t^K = 0. \tag{A.21}$$

The equations above jointly solve

$$\begin{aligned}
\theta_t^K &= 1 - \vartheta_t, \\
\theta_t^E &= -(1-\bar{\chi})(1-\vartheta_t), \\
\theta_t^{\bar{E}} &= (1-\bar{\chi})(1-\vartheta_t).
\end{aligned}$$

Return Processes Under symmetric equilibrium, the outside equity has the same expected return as individual equity, i.e.,

$$\mathbb{E}_t[dr_t^{E,i}] = \mathbb{E}_t[dr_t^{\bar{E},i}], \tag{A.22}$$

with which we can pin down the lagrangian multiplier from Eq. (A.15) and (A.16)

$$\lambda_t^i = (\theta_t^{K,i} + \theta_t^{E,i})\tilde{\nu}_t^2 = \bar{\chi}(1-\vartheta_t)\tilde{\nu}_t^2.$$

Note that $\theta_t^{E,i} + \theta_t^{K,i} \geq \chi\theta_t^{K,i}$, hence λ_t^i is always positive and the skin-in-the-game constraint is always binding.

The expected return of equity and outside equity in excess of real bond return can be written as

$$\frac{\mathbb{E}_t[dr_t^{\bar{E}}]}{dt} - \frac{\mathbb{E}_t[dr_t^B]}{dt} = \left(\sigma_t^{q,B} - \theta_t^K \frac{\sigma_t^\vartheta}{1-\vartheta_t}\right)(\sigma_t^{q,K} - \sigma_t^{q,B}) = \left(\sigma_t^{q,B} - \sigma_t^\vartheta\right)(\sigma_t^{q,K} - \sigma_t^{q,B}).$$

The expected return of capital in excess of equity is

$$\frac{\mathbb{E}_t[dr_t^{K,i}]}{dt} - \frac{\mathbb{E}_t[dr_t^{E,i}]}{dt} = \chi\lambda_t^i = (1-\vartheta_t)\bar{\chi}^2\tilde{\nu}_t^2.$$

A.1.4 Equilibrium: Discipline Drift of the State Variable

We discipline μ_t^ϑ as Proposition 1. By Eq. (A.6) and (A.5), the expected return of capital in excess of real bond return is

$$\frac{\mathbb{E}_t[dr_t^{K,i}]}{dt} - \frac{\mathbb{E}_t[dr_t^{B}]}{dt} = \frac{(1-\tau_t)a_t - \iota_t}{q_t^K} + \mu_t^{q,K} + \check{\mu}_t^B - \mu_t^{q,B}.$$

Plugging Eq. (A.1) and (A.4) yields

$$\frac{\mathbb{E}_t[dr_t^{K,i}]}{dt} - \frac{\mathbb{E}_t[dr_t^{B}]}{dt} = \frac{\check{\mu}_t^B - \mu_t^\vartheta}{1-\vartheta_t} + (\sigma_t^{q,B} - \sigma_t^\vartheta)(\sigma_t^{q,K} - \sigma_t^{q,B}) + \frac{a_t - g - \iota_t}{q_t^K}.$$

By Eq. (A.14) and (A.19),

$$(\sigma_t^{q,B} - \sigma_t^\vartheta)(\sigma_t^{q,K} - \sigma_t^{q,B}) + (1-\vartheta_t)\bar{\chi}^2\tilde{\nu}_t^2 = \frac{\check{\mu}_t^B - \mu_t^\vartheta}{1-\vartheta_t} + (\sigma_t^{q,B} - \sigma_t^\vartheta)(\sigma_t^{q,K} - \sigma_t^{q,B}) + \frac{a_t - g - \iota_t}{q_t^K}$$

$$\Rightarrow \mu_t^\vartheta = \check{\mu}_t^B + \hat{\rho} - (1-\vartheta_t)^2\bar{\chi}^2\tilde{\nu}_t^2. \tag{A.23}$$

Specifically, in the steady state, we have

$$1 - \vartheta^{SS} = \frac{\sqrt{\check{\mu}^B + \rho + \zeta}}{\bar{\chi}\tilde{\sigma}}.$$

A.1.5 Macro Synthesis

Before proceeding, we briefly summarize the equation system here. We suppress the individual superscript for simplicity.

There are 4 exogenous variables:

$$(\mu_t^B, i_t, a_t, \tilde{\nu}_t),$$

5 endogenous variables:

$$\left(\hat{c}_t, \iota_t, q_t^B, q_t^K, \tau_t\right),$$

and an endogenous state variable ϑ_t.

We have 5 equations: 2 households' first-order conditions (A.12) and (A.13):

$$\hat{c}_t = \hat{\rho} = \rho + \zeta,$$
$$q_t^K = 1 + \phi\iota_t,$$

1 goods market clearing condition:

$$\hat{c}_t \frac{q_t^K}{1-\vartheta_t} + g + \iota_t = a_t,$$

1 wealth share definition:

$$\vartheta_t = \frac{q_t^B}{q_t^B + q_t^K},$$

and 1 government budget constraint:

$$(\mu_t^B - i_t) q_t^B + \tau_t a_t = \mathfrak{g}.$$

Given the state variable ϑ_t, these 5 equations allow us to solve for the 5 endogenous variables.

Besides, the dynamics of state variable ϑ_t are subject to

$$\mu_t^\vartheta = \check{\mu}_t^B + \hat{\rho} - (1-\vartheta_t)^2\bar{\chi}^2\tilde{\nu}_t^2.$$

In this equilibrium, the expected returns are

$$\begin{aligned}
\frac{\mathbb{E}[dr_t^B]}{dt} &= \Phi(\iota_t) - \delta - \check{\mu}_t^B + \mu_t^{q,B},\\
\frac{\mathbb{E}[dr_t^K]}{dt} &= \Phi(\iota_t) - \delta - \check{\mu}_t^B + \mu_t^{q,B} + (\vartheta_t\sigma_t^{q,B} + (1-\vartheta_t)\sigma_t^{q,K})(\sigma_t^{q,K} - \sigma_t^{q,B}) + (1-\vartheta_t)\bar{\chi}^2\tilde{\nu}_t^2,\\
\frac{\mathbb{E}[dr_t^E]}{dt} &= \Phi(\iota_t) - \delta - \check{\mu}_t^B + \mu_t^{q,B} + (\vartheta_t\sigma_t^{q,B} + (1-\vartheta_t)\sigma_t^{q,K})(\sigma_t^{q,K} - \sigma_t^{q,B}),\\
\frac{\mathbb{E}[dr_t^{\bar{E}}]}{dt} &= \Phi(\iota_t) - \delta - \check{\mu}_t^B + \mu_t^{q,B} + (\vartheta_t\sigma_t^{q,B} + (1-\vartheta_t)\sigma_t^{q,K})(\sigma_t^{q,K} - \sigma_t^{q,B}).
\end{aligned}$$

So far, we have written every aggregate level variable as function of ϑ_t and pinned down its drift term μ_t^ϑ. We can then pin down the volatility term σ_t^ϑ by specifying the exogenous processes a_t, $\tilde{\nu}_t$, $\check{\mu}_t^B = \mu_t^B - i_t$ and apply Itô's lemma. Then we can pin down $\sigma_t^{q,B}$, $\sigma_t^{q,K}$, $\mu_t^{q,B}$ and $\mu_t^{q,K}$ and solve all the expected returns explicitly.

A.1.6 Verifying Value Function

I verify the value function by plugging endogenous variables into the HJB Equation (A.11). Indeed,

$$\begin{aligned}
0 = &-(\hat{\rho}-\zeta)v_t - \log n_{t-}^i + \frac{\zeta}{\hat{\rho}}\log\bar{n}_t + \dot{v}_t + (\log\rho + \log n_{t-}^i) - 1 + \frac{1}{\rho}\left(\Phi(\iota_t) - \delta - \check{\mu}_t^B + \mu_t^{q,B}\right)\\
&+ \frac{1}{\rho}(1-\vartheta_t)\left[(\sigma_t^{q,B} - \sigma_t^\vartheta)(\sigma_t^{q,K} - \sigma_t^{q,B}) + (1-\vartheta_t)\bar{\chi}^2\tilde{\nu}_t^2\right]\\
&- \frac{1}{2\rho}\left(\sigma_t^{q,B} - \sigma_t^\vartheta\right)^2 - \frac{1}{2\rho}(1-\vartheta_t)^2\bar{\chi}^2\tilde{\nu}_t^2,
\end{aligned}$$

where the two $\log n_{t-}^i$ terms cancel out. The equation above is an linear ODE for v_t. Note that $\vartheta_t, \sigma_t^\vartheta, \mu_t^\vartheta$ and $\bar{n}_t$ depend on aggregate shock only, so does v_t.

A.1.7 SDF and Wealth Processes

Household i's marginal utility is

$$m_t^i = e^{-\rho t}\frac{1}{c_t^i} = e^{-\rho t}(\rho+\zeta)^{-1}\frac{1}{n_t^i}.$$

Plug in the expected return processes to obtain her wealth process as

$$dn_t^i = g_t n_{t-}^i dt + \sigma_t n_{t-}^i dZ_t + \bar{\chi}(1-\vartheta_t)\tilde{\nu}_t n_{t-}^i d\tilde{Z}_t^i + (\bar{n}_t - n_{t-}^i)d\tilde{J}_t^i,$$

which implies

$$\frac{dm_t^i}{m_{t-}^i} = -[\rho + g_t - \sigma_t^2 - \tilde{\sigma}_t^2]dt - \sigma_t dZ_t - \tilde{\sigma}_t d\tilde{Z}_t^i + \frac{1}{m_{t-}^i}\left(e^{-\rho t}\frac{1}{(\rho+\zeta)\bar{n}_t} - e^{-\rho t}\frac{1}{(\rho+\zeta)\bar{n}_{t-}^i}\right)d\tilde{J}_t^i$$

where

$$\begin{aligned}\sigma_t &= \sigma_t^{q,B} - \sigma_t^{\vartheta} = (1-\vartheta_t)\sigma_t^{q,K} + \vartheta_t\sigma_t^{q,B},\\ \tilde{\sigma}_t &= \bar{\chi}(1-\vartheta_t)\tilde{\nu}_t.\end{aligned}$$

which is consistent with the investors' portfolio which is ϑ_t in bonds and $1-\vartheta_t$ in capital. Moreover, the idiosyncratic risk loading is $\bar{\chi}$ times the equity allocation $(1-\vartheta_t)$.

A.2 Proof for Proposition 1

In the permanent shock case, $\zeta = 0$, hence

$$m_t^i = e^{-\rho t}\rho^{-1}(n_t^i)^{-1}.$$

We obtain

$$\begin{aligned}\lim_{T\to\infty}\mathbb{E}_0\left[\bar{m}_T\bar{n}_T\right] &= \lim_{T\to\infty}\mathbb{E}_0\left[\left(\int_0^1 e^{-\rho T}\left(\frac{n_T^i}{n_0^i}\right)^{-1}di\right)\bar{n}_T\right]\\ &= \lim_{T\to\infty}\mathbb{E}_0\left[e^{-\rho T}\bar{n}_0\left(\int_0^1\left(\frac{\eta_T^i}{\eta_0^i}\right)^{-1}di\right)\right].\end{aligned}$$

Normalizing $\eta_0^i = 1$ and $\bar{n}_0 = 1$ yields

$$\lim_{T\to\infty}\mathbb{E}_0\left[\bar{m}_T\bar{n}_T\right] = \lim_{T\to\infty}\mathbb{E}_0\left[e^{-\rho T}\left(\int_0^1\left(\eta_T^i\right)^{-1}di\right)\right].$$

A.3 Transversality Conditions with Permanent Idiosyncratic Risk

A.3.1 Individual Level TVC

The r Minus g Perspective. Individual TVC is satisfied iff $\mathbb{E}_t[d(m_t^i n_t^i)] < 0$, i.e.,

$$-\underbrace{\left[\rho + g_t - \sigma_t^2 - \tilde{\sigma}_t^2\right]}_{\text{drift of } m_t^i \text{ (risk free rate)}} + \underbrace{g_t}_{\text{drift of } n_t^i} - \underbrace{\left[\sigma_t^2 + \tilde{\sigma}_t^2\right]}_{\text{cross variation (risk premium)}} < 0,$$

i.e., individual TVC is satisfied iff

$$\rho > 0. \tag{A.24}$$

As a result, individual TVC always hold. The intuition here is that the growth rate of individual wealth equals the return of individual wealth portfolio minus consumption rate. As long as consumption rate is positive, the growth rate of individual wealth is always smaller than the discount rate.

A.3.2 Aggregate Level TVC

The r Minus g Perspective. The aggregate TVC is $\lim_{T\to\infty} \mathbb{E}_0[m_T^i \bar{n}_T] = 0$. Note that

$$\begin{aligned}\frac{dm_t^i \bar{n}_t}{m_t^i \bar{n}_t} &= \{-[\rho + g_t - \sigma_t^2 - \tilde{\sigma}^2] + g_t - \sigma_t^2\}dt + (\sigma_t - \sigma_t)dZ_t - \tilde{\sigma}_t d\tilde{Z}_t^i \\ &= -[\rho - \tilde{\sigma}_t^2]dt - \tilde{\sigma}_t d\tilde{Z}_t^i,\end{aligned}$$

Hence

$$\begin{aligned}d\log(m_t^i \bar{n}_t) &= \frac{dm_t^i \bar{n}_t}{m_t^i \bar{n}_t} - \frac{1}{2}\frac{\langle dm_t^i \bar{n}_t, dm_t^i \bar{n}_t\rangle}{(m_t^i \bar{n}_t)^2} \\ &= -[\rho - \frac{1}{2}\tilde{\sigma}_t^2]dt - \tilde{\sigma}_t d\tilde{Z}_t^i,\end{aligned}$$

which implies

$$m_T^i \bar{n}_T = m_0^i \bar{n}_0 \exp\left(\int_0^T -[\rho - \tilde{\sigma}_t^2]dt\right) \exp\left(\int_0^T -\frac{1}{2}\tilde{\sigma}_t^2 dt - \int_0^T \tilde{\sigma}_t d\tilde{Z}_t^i\right).$$

For simplicity, let idiosyncratic risk be constant over time, i.e., $\tilde{\sigma}_t = \tilde{\sigma}$. Then $\exp\left(\int_0^T -\frac{1}{2}\tilde{\sigma}_t^2 dt - \int_0^T \tilde{\sigma}_t d\tilde{Z}_t^i\right)$ is a martingale, and

$$\mathbb{E}_0[m_T^i \bar{n}_T] = m_0^i \bar{n}_0 \exp\left(\int_0^T -[\rho - \tilde{\sigma}^2]dt\right) = m_0^i \bar{n}_0 \exp\left(-[\rho - \tilde{\sigma}^2]T\right).$$

The aggregate transversality condition holds if and only if

$$\rho > \tilde{\sigma}^2.$$

The Wealth Distribution Perspective. The transversality condition is given by

$$\lim_{T\to\infty} \mathbb{E}[m_T^i \bar{n}_T] = \lim_{T\to\infty} \rho^{-1}\mathbb{E}\left[e^{-\rho T}(\eta_T^i)^{-1}\right] = \lim_{T\to\infty} (\rho^{-1}e^{-\rho T})\mathbb{E}\left[(\eta_T^i)^{-1}\right].$$

Recall that

$$\frac{d\eta_t^i}{\eta_t^i} = \tilde{\sigma} d\tilde{Z}_t^i,$$

by Itô's lemma,

$$d(\eta_t^i)^{-1} = \tilde{\sigma}^2(\eta_t^i)^{-1}dt - (\eta_t^i)^{-1}d\tilde{Z}_t^i.$$

Following the same method above, we have

$$\mathbb{E}[(\eta_T^i)^{-1}] = (\eta_0^i)^{-1} \exp\left(\tilde{\sigma}^2 T\right).$$

Hence,

$$\lim_{T\to\infty} \mathbb{E}[m_T^i \bar{n}_T] = \lim_{T\to\infty} \rho^{-1}(\eta_0^i)^{-1} \exp\left((-\rho + \tilde{\sigma}^2)T\right).$$

Again, the aggregate transversality condition holds if and only if

$$\rho > \tilde{\sigma}^2.$$

A.4 Transversality Conditions with Transitory Idiosyncratic Risk

A.4.1 Aggregate Level TVC

We show the wealth distribution perspective here only, and leave the r minus g perspective in Appendix A.7 which also serves as the proof for Proposition 4.

The Wealth Distribution Perspective. Consider the wealth reset shock with intensity ζ. The wealth dynamics become

$$d\eta_t^i = \tilde{\sigma}_t \eta_{t-}^i d\tilde{Z}_t^i + (1 - \eta_{t-}^i) d\tilde{J}_t^i.$$

By Itô's lemma,

$$\begin{aligned} d(\eta_t^i)^{-1} = &\left[\tilde{\sigma}_t^2 (\eta_{t-}^i)^{-1} + \zeta(1 - (\eta_{t-}^i)^{-1})\right] dt \\ &- \tilde{\sigma}_t (\eta_{t-}^i)^{-1} d\tilde{Z}_t^i + (1 - (\eta_{t-}^i)^{-1})(d\tilde{J}_t^i - \zeta dt), \end{aligned}$$

i.e.,

$$\begin{aligned} (\eta_t^i)^{-1} = (\eta_0^i)^{-1} &+ \int_0^t \left[\tilde{\sigma}_s^2 (\eta_{s-}^i)^{-1} + \zeta(1 - (\eta_{s-}^i)^{-1})\right] ds \\ &+ \int_0^t -\tilde{\sigma}_s (\eta_{s-}^i)^{-1} d\tilde{Z}_s^i + \int_0^t (1 - (\eta_{s-}^i)^{-1})(d\tilde{J}_s^i - \zeta ds). \end{aligned}$$

Let $\tilde{\sigma}_t = \tilde{\sigma}$, i.e. constant over time. Assume $\int_0^t \tilde{\sigma}^2 (\eta_s^i)^{-2} ds < \infty$ and $\int_0^t (1 - (\eta_{s-}^i)^{-1})^2 ds < \infty$ for every t, then the last two terms are both martingales with zero conditional expectation. Take expectation on both sides yields

$$\mathbb{E}_0[(\eta_t^i)^{-1}] = (\eta_0^i)^{-1} + \mathbb{E}_0 \left[\int_0^t \left[\tilde{\sigma}^2 (\eta_{s-}^i)^{-1} + \zeta(1 - (\eta_{s-}^i)^{-1})\right] ds\right]$$

Note that jumps happen at most countable times from time 0 to t, hence $\int_0^t ((\eta_{s-}^i)^{-1} - (\eta_s^i)^{-1}) ds = 0$, and

$$\begin{aligned} \mathbb{E}_0[(\eta_t^i)^{-1}] &= (\eta_0^i)^{-1} + \mathbb{E}_0 \left[\int_0^t \left[\tilde{\sigma}^2 (\eta_s^i)^{-1} + \zeta(1 - (\eta_s^i)^{-1})\right] ds\right] \\ &= (\eta_0^i)^{-1} + \int_0^t \left[\tilde{\sigma}^2 \mathbb{E}_0[(\eta_s^i)^{-1}] + \zeta(1 - \mathbb{E}_0[(\eta_s^i)^{-1}])\right] ds, \end{aligned}$$

differentiating both sides yields an ODE w.r.t. $\mathbb{E}[(\eta_t^i)^{-\gamma}]$, i.e.,

$$d\mathbb{E}_0[(\eta_t^i)^{-1}] = \left[\tilde{\sigma}^2\mathbb{E}_0[(\eta_t^i)^{-\gamma}] + \zeta(1 - \mathbb{E}_0[(\eta_t^i)^{-1}])\right] dt.$$

The solution to the ODE is

$$\mathbb{E}_0\left[(\eta_t^i)^{-1}\right] = \left(\frac{\zeta}{\tilde{\sigma}^2 - \zeta} + (\eta_0^i)^{-1}\right) e^{[\tilde{\sigma}^2 - \zeta]t} - \frac{\zeta}{\tilde{\sigma}^2 - \zeta}.$$

The aggregate TVC condition can be stated as

$$\begin{aligned}\lim_{T\to\infty} \mathbb{E}[m_T^i \bar{n}_T] &= \lim_{T\to\infty} (\rho + \zeta)^{-1} e^{-\rho T} \mathbb{E}\left[(\eta_T^i)^{-1}\right] \\ &= \lim_{T\to\infty} (\rho + \zeta)^{-1} \left(\frac{\zeta}{\tilde{\sigma}^2 - \zeta} + (\eta_0^i)^{-1}\right) e^{[\tilde{\sigma}^2 - \zeta - \rho]T} - \frac{\zeta}{\tilde{\sigma}^2 - \zeta} \exp(-\rho T).\end{aligned}$$

Hence, aggregate TVC holds if and only if

$$\rho + \zeta > \tilde{\sigma}^2.$$

A.5 Proof for Proposition 2

For expositional purposes, we prove this in the same environment as Appendix A.1. Denote the nominal government bond outstanding as B_t and real government bond as b_t. We obtain

$$dB_t = i_t B_t dt - P_t s_t dt.$$

Hence,

$$
\begin{aligned}
d(m_t^i b_t) &= b_t dm_t^i + m_{t-}^i/P_t dB_t + m_{t-}^i B_t d(1/P_t) + B_t dm_t^i d(1/P_t) \\
&= b_t \left(-(r_t^f - \zeta) m_{t-}^i dt - \sigma_t m_{t-}^i dZ_t - \tilde{\sigma}_t m_{t-}^i d\tilde{Z}_t^i + \left(e^{-\rho t}(\rho+\zeta)^{-1} \bar{n}_t^{-1} - m_{t-}^i \right) d\tilde{J}_t^i \right) \\
&+ i_t b_t m_{t-}^i dt - s_t m_t^i dt + m_{t-}^i b_t (dr_t^B - i_t dt) - \sigma_t m_{t-}^i b_t \sigma_t^{q,B} \\
&= -m_{t-}^i s_t dt + m_{t-}^i \zeta b_t dt \\
&- m_{t-}^i b_t (\sigma_t - \sigma_t^{q,B}) dZ_t - m_{t-}^i b_t \tilde{\sigma}_t d\tilde{Z}_t^i + b_t \left(e^{-\rho t}(\rho+\zeta)^{-1} \bar{n}_t^{-1} - m_{t-}^i \right) d\tilde{J}_t^i \\
&= -m_{t-}^i s_t dt + \zeta e^{-\rho t}(\rho+\zeta)^{-1} \bar{n}_t^{-1} b_t dt - m_{t-}^i b_t (\sigma_t - \sigma_t^{q,B}) dZ_t - m_{t-}^i b_t \tilde{\sigma}_t d\tilde{Z}_t^i \\
&+ b_t \left(e^{-\rho t}(\rho+\zeta)^{-1} \bar{n}_t^{-1} - m_{t-}^i \right) d(\tilde{J}_t^i - \zeta t),
\end{aligned}
$$

where we plugged in the definition of bond return, i.e.,

$$
dr_t^B = i_t dt + \frac{d(1/P_t)}{1/P_t} = i_t dt + \frac{d(q_t^B K_t/B_t)}{q_t^B K_t/B_t} = (r_t^f + \sigma_t \sigma_t^{q,B}) dt + \sigma_t^{q,B} dZ_t.
$$

Integrate $d(m_t^i b_t)$ from $t = 0$ to T and take expectation to obtain[9]

$$
\mathbb{E}_0[m_T^i b_t] - m_0^i b_0 = -\mathbb{E}_0 \left[\int_0^T m_t^i s_t \, dt \right] + \zeta \mathbb{E}_0 \left[\int_0^T e^{-\rho t}(\rho+\zeta)^{-1} \bar{n}_t^{-1} \, dt \right].
$$

Divide both sides by m_0^i to obtain

$$
b_0 = \mathbb{E}_0 \left[\int_0^T \frac{m_t^i}{m_0^i} s_t \, dt \right] - \zeta \mathbb{E}_0 \left[\int_0^T e^{-\rho t} \left(\frac{\bar{n}_T}{\bar{n}_0} \right)^{-1} b_t \, dt \right] + \mathbb{E}_0 \left[\frac{m_T^i}{m_0^i} b_T \right].
$$

[9]Note that there are at most countably many jumps between time 0 and T, and we can replace $t-$ by t in the intergrals, as shown in our previous proofs. Heuristically, $\int_0^T m_t^i s_t \, dt = \int_0^T m_{t-}^i s_t \, dt + \int_0^T (m_t^i - m_{t-}^i) s_t \, dt = \int_0^T m_{t-}^i s_t \, dt$.

To derive the bond valuation equation from the aggregate SDF, we start by dividing $d(m_t^i B_t/P_t)$ by m_0^i, i.e.,

$$d[(m_t^i/m_0^i)b_t] = -\frac{m_{t-}^i}{m_0^i}s_t dt + \zeta e^{-\rho t}\left(\frac{\bar{n}_t}{\bar{n}_0}\right)^{-1} b_t dt \\ - \frac{m_{t-}^i}{m_0^i}b_t(\sigma_t - \sigma_t^{q,B})\sigma_t dZ_t - \frac{m_{t-}^i}{m_0^i}b_t\tilde{\sigma}_t d\tilde{Z}_t^i + b_t\left(e^{-\rho t}\left(\frac{\bar{n}_t}{\bar{n}_0}\right)^{-1} - \frac{m_{t-}^i}{m_0^i}\right)d(\tilde{J}_t^i - \zeta t).$$

Intergrate by i to obtain

$$d[\bar{m}_t b_t] = -\bar{m}_t s_t dt + \zeta e^{-\rho t}\left(\frac{\bar{n}_t}{\bar{n}_0}\right)^{-1} b_t dt - \bar{m}_t b_t(\sigma_t - \sigma_t^{q,B})\sigma_t dZ_t.$$

Integrate over time and take expectation to obtain

$$\mathbb{E}_0[\bar{m}_T b_T] - b_0 = -\mathbb{E}_0\left[\int_0^T \bar{m}_t s_t dt\right] + \mathbb{E}_t\left[\int_0^T \zeta e^{-\rho t}\left(\frac{\bar{n}_t}{\bar{n}_0}\right)^{-1} b_t dt\right],$$

i.e.,

$$b_0 = \mathbb{E}_0\left[\int_0^T \bar{m}_t s_t dt\right] - \mathbb{E}_t\left[\int_0^T \zeta e^{-\rho t}\left(\frac{\bar{n}_t}{\bar{n}_0}\right)^{-1} b_t dt\right] + \mathbb{E}_0\left[\bar{m}_T b_T\right].$$

A.6 Proof for Proposition 3

We use the same proof strategy in Toda (2014). Recall that the individual wealth satisfies

$$\frac{dn_t^i}{n_t^i} = g_t dt + \sigma_t dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^i + \frac{\bar{n}_t - n_t^i}{n_t^i}d\tilde{J}_t^i.$$

For aggregate wealth,

$$\frac{d\bar{n}_t}{\bar{n}_t} = g_t dt + \sigma_t dZ_t.$$

Hence,

$$\frac{d\eta_t^i}{\eta_t^i} = \frac{d(n_t^i/\bar{n}_t)}{n_t^i/\bar{n}_t} = \tilde{\sigma}_t d\tilde{Z}_t^i + \frac{1-\eta_t^i}{\eta_t^i} d\tilde{J}_t^i. \tag{A.25}$$

In the absence of wealth reset, the log wealth share follows

$$d\log \eta_t^i = -\frac{1}{2}\tilde{\sigma}_t^2 dt + \tilde{\sigma}_t d\tilde{Z}_t^i. \tag{A.26}$$

Below, we use generation s to denote the collection of agents whose last wealth reset shock happened at time s. Specifically, generation 0 denotes the collection of agents who have never had any reset shock. We also use x to denote log wealth share, i.e., $x_t^i = \log \eta_t^i$.

At time t, denote the (within-generation) probability density function for the log wealth share of generation 0 as $g(x,t,0)$. Under Assumption 1, $g(x,t,0)$ is subject to the following Kolmogorov Forward Equation

$$\frac{\partial g(x,t,0)}{\partial t} = \frac{\tilde{\sigma}^2}{2}\left(\frac{\partial}{\partial x} g(x,t,0) + \frac{\partial^2}{\partial x^2} g(x,t,0)\right),$$

Note that $g(x,t,0)$ is always a valid pdf at any time t, i.e.,

$$\int_0^\infty g(x,t,0)\, dx = 1, \quad \forall t > 0.$$

Denote the p.d.f. of the initial wealth distribution as $f(x,0)$. Since g is the p.d.f. of the initial log

wealth share, we have[10]

$$\frac{1}{\eta}g(\log\eta,0,0)=f(\eta,0).$$

Similarly, for every generation s such that $0<s<t$, we can find the within-generation p.d.f. of the log wealth share $g(x,t,s)$ by plugging the initial condition $\delta(x-1)$, which integrates to 1 but only has mass at $x=1$. By doing this, we obtain that

$$g(x,t,s)=\frac{1}{\tilde{\sigma}\sqrt{2\pi(t-s)}}e^{-\frac{(x+\tilde{\sigma}^2(t-s)/2)^2}{2\tilde{\sigma}^2(t-s)}}$$

Note that the initial mass of generation s is ζds and only $\zeta e^{-\zeta(t-s)}ds$ are left at time t. Hence, the distribution of log wealth share of all agents is a mixture of the within-generation p.d.f.s, i.e.,

$$\begin{aligned}
g(x,t) &= e^{-\zeta t}g(x,t,0)+\int_0^t \zeta e^{-\zeta(t-s)}g(x,t,s)ds \\
&= e^{-\zeta t}g(x,t,0) \\
&+ Ce^{-\frac{1}{2}x}\left[e^{-\frac{\sqrt{1+\frac{8\zeta}{\tilde{\sigma}^2}}|x|}{2}}\Phi\left(-\frac{|x|}{\tilde{\sigma}\sqrt{t}}+\frac{\sqrt{8\zeta+\tilde{\sigma}^2}}{2}\sqrt{t}\right)\right] \\
&- Ce^{-\frac{1}{2}x}\left[e^{-\frac{\sqrt{1+\frac{8\zeta}{\tilde{\sigma}^2}}|x|}{2}}\Phi\left(-\frac{|x|}{\tilde{\sigma}\sqrt{t}}-\frac{\sqrt{8\zeta+\tilde{\sigma}^2}}{2}\sqrt{t}\right)\right]
\end{aligned}$$

where $C=\frac{2\zeta}{\tilde{\sigma}\sqrt{8\zeta+\tilde{\sigma}^2}}$. As $t\to\infty$, the first and third term shrink to zero pointwise, while the second

[10]Note that

$$\begin{aligned}
f(\eta,0) &= \frac{d}{d\eta}\mathbb{P}(\eta_0^i\le\eta) \\
&= \frac{d}{d\eta}\mathbb{P}(\log\eta_0^i\le\log\eta) \\
&= \frac{d}{d\eta}\int_{-\infty}^{\log\eta} g(x,0,0)dx \\
&= \frac{d}{d\log\eta}\int_{-\infty}^{\log\eta} g(x,0,0)dx\frac{d\log\eta}{d\eta} \\
&= \frac{1}{\eta}g(\log\eta,0,0).
\end{aligned}$$

term converges to 1. Hence, $g(x,t)$ converges pointwise to

$$g(x,\infty) = Ce^{-\frac{1}{2}x - \frac{\sqrt{1+\frac{8\zeta}{\tilde{\sigma}^2}}|x|}{2}}.$$

Given the p.d.f. for log wealth share g, we obtain the limit wealth distribution as

$$\begin{aligned} f(\eta,\infty) &= \frac{1}{\eta} g(\log\eta,\infty) \\ &= \begin{cases} C\eta^{-\frac{3}{2}+\frac{\sqrt{1+\frac{8\zeta}{\tilde{\sigma}^2}}}{2}}, & 0<\eta<1, \\ C\eta^{-\frac{3}{2}-\frac{\sqrt{1+\frac{8\zeta}{\tilde{\sigma}^2}}}{2}}, & \eta \geq 1. \end{cases} \end{aligned}$$

A.7 Proof for Proposition 4

Note that

$$\begin{aligned} d(m_t^i \bar{n}_t) = &- \left(\rho - \tilde{\sigma}_t^2\right) m_{t-}^i \bar{n}_t dt \\ &+ \zeta[e^{-\rho t}(\rho+\zeta)^{-1} - m_{t-}^i \bar{n}_t]dt \\ &- \tilde{\sigma}_t m_{t-}^i \bar{n}_t d\tilde{Z}_t^i + [e^{-\rho t}(\rho+\zeta)^{-1} - m_{t-}^i \bar{n}_t] d(\tilde{J}_t^i - \zeta t). \end{aligned}$$

Divide by m_0^i and integrate across i to obtain

$$d(\bar{m}_t \bar{n}_t) = -(\rho + \zeta - \tilde{\sigma}_t^2)\bar{m}_t \bar{n}_t dt + \zeta e^{-\rho t}\bar{n}_0 dt.$$

Let $\tilde{\sigma}_t = \tilde{\sigma}$. The ODE admits the following solution

$$\begin{aligned} \bar{m}_t \bar{n}_t = &\ \bar{n}_0 \exp(-(\rho+\zeta-\tilde{\sigma}^2)t)\left(\frac{\zeta}{\tilde{\sigma}^2-\zeta}+1\right) \\ &- \bar{n}_0 e^{-\rho t}\frac{\zeta}{\tilde{\sigma}^2-\zeta}. \end{aligned}$$

As a special case, when $t=0$, we recover our initial condition $\bar{m}_0\bar{n}_0 = \bar{n}_0$.

The aggregate TVC holds if and only if

$$\rho + \zeta - \tilde{\sigma}^2 > 0.$$

A.8 Arbitrary Wealth Distribution After Wealth Reset

We show that the main results are preserved even if we impose arbitrary wealth distribution after wealth resets. Denote the pdf for wealth resets as $h(\tilde{\eta})$. We require that $\int_0^\infty \tilde{\eta}^{-\gamma} h(\tilde{\eta})\, d\tilde{\eta} < \infty$, i.e., the expected marginal utility immediately after the wealth reset is well-defined. The previous specification can be viewed as a special case where $h(\tilde{\eta}) = \delta(\tilde{\eta} - 1)$, where $\delta(.)$ is the dirac function. First, we show that this general case admits the same solution of aggregate variables in the reset-to-average case. Heuristically, household's HJB equation is given by

$$\begin{aligned} V(n_t^i, t) &= \max_{c_t^i, \iota_t^i, \theta_t^i} \mathbb{E}_t \left[\frac{(c_t^i)^{1-\gamma}}{1-\gamma} dt + e^{-\rho dt} (1_{\text{jump}} V(n_{t+dt}^i, t+dt) + 1_{\text{nojump}} V(n_{t+dt}^i, t+dt)) \right] \\ &= \max_{c_t^i, \iota_t^i, \theta_t^i} \mathbb{E}_t \left[\frac{(c_t^i)^{1-\gamma}}{1-\gamma} dt + (1-\rho dt)(\zeta dt \mathbb{E}^\eta [V(h(\tilde{\eta})\bar{n}_t, t+dt)] + (1-\zeta dt) V(n_{t+dt}^i, t+dt)) \right]. \end{aligned}$$

$\mathbb{E}^\eta[.]$ denotes taking expectation w.r.t. $\tilde{\eta}$. The only difference is that the after-jump value function changed from $V(\bar{n}_t, t+dt)$ to $V(h(\tilde{\eta})\bar{n}_t, t+dt)$. Since the assignment of after-jump wealth is independent with agent's pre-jump wealth, the expectation depends on aggregate variables only. As a result, $h(\tilde{\eta})$ only affects v_t in the value function, which does not matter household's consumption or investment choices.

Next, we show that $h(\tilde{\eta})$ does not matter for the transversality condition either. Define the compound Poisson process

$$\tilde{Q}_t^i = \sum_{k=1}^{\tilde{J}_t^i} (\tilde{\eta}_k^i)^{-\gamma},$$

where $\tilde{\eta}_k^i$ is the post-reset wealth share in each reset shock, i.i.d. drawn from the probability density

function $h(\tilde{\eta})$. The compensated compound Poisson process is

$$\tilde{Q}_t^i - \mathbb{E}[(\tilde{\eta})^{-\gamma}]\zeta t,$$

which is a martingale. The dynamics of SDF can be written as

$$\begin{aligned} dm_t^i = &- \left[\rho + \gamma g - \frac{\gamma(\gamma+1)}{2}\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2 + \zeta\right] m_{t-}^i dt \\ &+ \zeta e^{-\rho t}\hat{c}^{-\gamma}\bar{n}_t^{-\gamma}\mathbb{E}[(\tilde{\eta})^{-\gamma}]dt \\ &- \gamma\bar{\chi}(1-\vartheta)\tilde{\sigma}m_{t-}^i d\tilde{Z}_t^i \\ &+ e^{-\rho t}\hat{c}^{-\gamma}\bar{n}_t^{-\gamma}d(\tilde{Q}_t^i - \mathbb{E}[(\tilde{\eta})^{-\gamma}]\zeta t) \\ &- m_{t-}^i(d\tilde{J}_t^i - \zeta t) \end{aligned}$$

Hence,

$$\begin{aligned} dm_t^i\bar{n}_t = &- \left[\rho + (\gamma-1)g - \frac{\gamma(\gamma+1)}{2}\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2 + \zeta\right] m_{t-}^i\bar{n}_t dt \\ &+ \zeta e^{-\rho t}\hat{c}^{-\gamma}\bar{n}_t^{1-\gamma}\mathbb{E}[(\tilde{\eta})^{-\gamma}]dt \\ &- \gamma\bar{\chi}(1-\vartheta)\tilde{\sigma}m_{t-}^i\bar{n}_t d\tilde{Z}_t^i \\ &+ e^{-\rho t}\hat{c}^{-\gamma}\bar{n}_t^{-\gamma}d(\tilde{Q}_t^i - \mathbb{E}[(\tilde{\eta})^{-\gamma}]\zeta t) \\ &- m_{t-}^i(d\tilde{J}_t^i - \zeta t). \end{aligned}$$

Take expectation to obtain

$$\begin{aligned} d\mathbb{E}\left[m_t^i n_t^i\right] = &- \left[\rho + (\gamma-1)g - \frac{\gamma(\gamma+1)}{2}\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2 + \zeta\right]\mathbb{E}[m_{t-}^i\bar{n}_t]dt \\ &+ \zeta e^{-\rho t}\hat{c}^{-\gamma}\mathbb{E}[(\tilde{\eta})^{-\gamma}]\bar{n}_t^{1-\gamma}dt. \end{aligned}$$

Note that $[\rho + (\gamma-1)g - \frac{\gamma(\gamma+1)}{2}\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2 + \zeta = r^f - g = \hat{c} - \gamma\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2$. The solution is

$$e^{(r^f-g)t}\mathbb{E}[m_t^i\bar{n}_t] - m_0^i\bar{n}_0 = \frac{\zeta\hat{c}^{-\gamma}\mathbb{E}[(\tilde{\eta})^{-\gamma}]\bar{n}_0^{1-\gamma}}{\zeta - \frac{\gamma(\gamma+1)}{2}\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2}(e^{[\zeta - \frac{\gamma(\gamma+1)}{2}\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2]t} - 1),$$

i.e.,

$$\begin{aligned}\mathbb{E}[m_t^i\bar{n}_t] = &\left(m_0^i\bar{n}_0 - \frac{\zeta\hat{c}^{-\gamma}\mathbb{E}[(\tilde{\eta})^{-\gamma}]\bar{n}_0^{1-\gamma}}{\zeta - \frac{\gamma(\gamma+1)}{2}\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2}\right)\exp(-(\hat{c} - \gamma\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2)t) \\ &+ \frac{\zeta\hat{c}^{-\gamma}\mathbb{E}[(\tilde{\eta})^{-\gamma}]\bar{n}_0^{1-\gamma}}{\zeta - \frac{\gamma(\gamma+1)}{2}\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2}\exp(-(\rho + (\gamma-1)g)t).\end{aligned}$$

For all $h(\tilde{\eta})$ such that $\mathbb{E}[(\tilde{\eta})^{-\gamma}]$ exists, aggregate TVC depends on $\hat{c} - \gamma\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2$, i.e., $r^f - g$. In the first part we show that $h(\tilde{\eta})$ does not matter for any aggregate variables. Hence, $h(\tilde{\eta})$ does not matter for aggregate TVC.

A.9 Model with Stochastic Death

We model the stochastic death as follows. Define $\{d_t^i\}$ as a Poisson process with intensity ζ. Death events are independent across agents, i.e., $\{d_t^i\}$ and $\{d_t^j\}$ are stochastically independent for $i \neq j$. Let $d_0^i = 0$. The stochastic death time τ^i is defined as

$$\tau^i := \inf\{t > 0 \mid d_t^i \geq 1\}$$

The death process is hence defined by

$$death_t^i = d^i_{\{t\wedge\tau^i\}},$$

i.e., when agent i is alive, $death_t^i = 0$. Otherwise, $death_t^i = 1$.

Consider household's problem. Heuristically,

$$\begin{aligned} V(n_t^i, t) &= \max_{c_t^i, \iota_t^i, \theta_t^i} \mathbb{E}_t \left[\frac{(c_t^i)^{1-\gamma}}{1-\gamma} dt + e^{-\rho dt} (1_{\text{death}} V(n_{t+dt}^i, t+dt) + 1_{\text{nodeath}} V(n_{t+dt}^i, t+dt)) \right] \\ &= \max_{c_t^i, \iota_t^i, \theta_t^i} \mathbb{E}_t \left[\frac{(c_t^i)^{1-\gamma}}{1-\gamma} dt + (1-\rho dt)(\zeta dt \times 0 + (1-\zeta dt) V(n_{t+dt}^i, t+dt)) \right] \end{aligned}$$

The Hamilton-Jacobi-Bellman equation is

$$\begin{aligned} &(\rho+\zeta) V(n_t^i, t) = \max_{c_t^i, \iota_t^i, \theta_t^i} \mathbb{E}_t \left[\frac{(c_t^i)^{1-\gamma}}{1-\gamma} dt + \zeta \times 0 + V_t dt + V_n(n_t^i, t) dn_t^i + \frac{1}{2} V_{nn}(n_t^i, t) \langle dn_t^i, dn_t^i \rangle \right] / dt \\ &\rho v_t + \omega_t \frac{(n_t^i)^{1-\gamma}}{1-\gamma} - 0 \\ &= \max_{c_t^i, \iota_t^i, \theta_t^i} \mathbb{E} \left\{ \frac{(c_t^i)^{1-\gamma}}{1-\gamma} dt + \frac{\dot{w}_t}{\rho+\zeta} \frac{(n_t^i)^{1-\gamma}}{1-\gamma} dt + \dot{v}_t dt + \omega_t \frac{(n_t^i)^{1-\gamma}}{\rho+\zeta} \frac{dn_t^i}{n_t^i} - \omega_t \frac{\gamma}{2} \frac{(n_t^i)^{1-\gamma}}{\rho+\zeta} \left\langle \frac{dn_t^i}{n_t^i}, \frac{dn_t^i}{n_t^i} \right\rangle \right\} / dt. \end{aligned}$$

Note that the HJB is essetianlly the same as the wealth reset model, except that the utility after death is zero while the utility after wealth reset is $V(\bar{n}_t, t)$. The first order conditions are the same, though the solution for v_t is different. Hence, we obtain the same equilibrium in the wealth reset model.

Below, we discuss household's decision and the relevant transversality conditions in a different perspective. Let n_t^i denote the wealth of agent i at time t. For $t > \tau^i$, i.e., after the agent's death, we assume that n_t^i denotes the wealth following the agent's consumption and investment plan if it were still alive. We denote $\check{m}_t^i = e^{-\rho t} \hat{c}^{-\gamma} (n_t^i)^{-\gamma}$ as the hypothetical marginal utility if the agent were still alive.

We denote m_t^i as the agent's actual marginal utility, which is zero after death:

$$m_t^i := \begin{cases} \check{m}_t^i = e^{-\rho t} \hat{c}^{-\gamma} (n_t^i)^{-\gamma}, & death_t^i = 0, \\ 0, & death_t^i = 1. \end{cases}$$

We obtain

$$dm_t^i = \begin{cases} -\left[\rho + \gamma g_t - \frac{\gamma(\gamma+1)}{2}\bar{\chi}^2(1-\vartheta_t)^2\tilde{\sigma}_t^2\right] m_t^i dt - \gamma\bar{\chi}(1-\vartheta_t)\tilde{\sigma}_t m_t^i d\tilde{Z}_t^i, & t < \tau^i, \\ -m_{\tau^i-}^i, & t = \tau, \\ 0 & t > \tau^i. \end{cases}$$

Aggregate TVC. For long-lived asset, e.g. government bond at $t < \tau^i$,

$$\begin{aligned} d(m_t^i b_t) &= b_t dm_t^i + m_t^i/P_t dB_t + m_t^i B_t d(1/P_t) + B_t dm_t^i d(1/P_t) \\ &= b_t\left(-(r_t^f - \zeta)m_t^i dt - \bar{\chi}(1-\vartheta_t)\tilde{\sigma}_t m_t^i d\tilde{Z}_t^i\right) \\ &+ i_t b_t m_t^i dt - s_t m_t^i dt + m_t^i b_t(dr_t^B - i_t dt) \\ &= -m_t^i s_t dt + m_t^i \zeta b_t dt - m_t^i b_t \bar{\chi}(1-\vartheta_t)\tilde{\sigma}_t d\tilde{Z}_t^i. \end{aligned}$$

At $t = \tau^i$, $d(m_t^i b_t) = -m_{\tau^i-}^i b_{\tau^i}$. At $t \geq \tau^i$, $d(m_t^i b_t) = 0$. Integrate from $t = 0$ to $t = T$ and take expectation. Recall that τ^i is independent of n_t^i and b_t, and

$$\mathbb{E}_0\left[1_{\{\tau^i \leq u\}}\right] = 1 - e^{-\zeta u}.$$

Hence,

$$\begin{aligned} m_0^i b_0 &= \mathbb{E}_0\left[\int_0^T -d(m_t^i b_t) + m_T^i b_T\right] \\ &= (1-e^{-\zeta T})\mathbb{E}_0\left[\int_0^T -d(m_t^i b_t) + m_T^i b_T \middle| \tau^i \leq T\right] + e^{-\zeta T}\mathbb{E}_0\left[\int_0^T -d(m_t^i b_t) + m_T^i b_T \middle| \tau^i > T\right]. \end{aligned}$$

The first term is computed as

$$\begin{aligned} &\mathbb{E}_0\left[\int_0^T -d(m_t^i b_t) + m_T^i b_T \middle| \tau^i \leq T\right] \\ =&\mathbb{E}_0\left[\int_0^{\tau^i} \check{m}_t^i s_t dt \middle| \tau^i \leq T\right] - \mathbb{E}_0\left[\int_0^{\tau^i} \zeta \check{m}_t^i b_t dt \middle| \tau^i \leq T\right] + \mathbb{E}_0\left[\check{m}_{\tau^i-}^i b_{\tau i} \middle| \tau^i \leq T\right], \end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}_0\left[\int_0^{\tau^i} \check{m}_t^i s_t dt \middle| \tau^i \le T\right] &= \mathbb{E}_0\left[\mathbb{E}_0\left[\int_0^{\tau^i} \check{m}_t^i s_t\, dt \middle| \tau^i\right] \middle| \tau^i \le T\right] \\
&= \frac{1}{1-e^{-\zeta T}} \int_0^T \zeta e^{-\zeta u} \mathbb{E}_0\left[\int_0^u \check{m}_t^i s_t\, dt\right] du \\
&= \frac{1}{1-e^{-\zeta T}} \mathbb{E}_0\left[\int_0^T \left(\int_0^u \zeta e^{-\zeta u} \check{m}_t^i s_t\, dt\right) du\right] \\
&= \frac{1}{1-e^{-\zeta T}} \mathbb{E}_0\left[\int_0^T \left(\int_t^T \zeta e^{-\zeta u} \check{m}_t^i s_t\, du\right) dt\right] \\
&= \frac{1}{1-e^{-\zeta T}} \mathbb{E}_0\left[\int_0^T (e^{-\zeta t} - e^{-\zeta T}) \check{m}_t^i s_t dt\right],
\end{aligned}$$

and similarly

$$\begin{aligned}
\mathbb{E}_0\left[\int_0^{\tau^i} \zeta \check{m}_t^i b_t dt \middle| \tau^i \le T\right] &= \frac{1}{1-e^{-\zeta T}} \mathbb{E}_0\left[\int_0^T \zeta (e^{-\zeta t} - e^{-\zeta T}) \check{m}_t^i b_t dt\right], \\
\mathbb{E}_0\left[\check{m}_{\tau^i -}^i b_{\tau i} | \tau^i \le T\right] &= \frac{1}{1-e^{-\zeta T}} \mathbb{E}_0\left[\int_0^T \zeta e^{-\zeta u} \check{m}_u^i b_u du\right].
\end{aligned}$$

The second term is computed as

$$\begin{aligned}
&\mathbb{E}_0\left[\int_0^T -d(m_t^i b_t) + m_T^i b_T \middle| \tau^i > T\right] \\
=&\mathbb{E}_0\left[\int_0^T \check{m}_t^i s_t dt\right] - \mathbb{E}_0\left[\int_0^T \zeta \check{m}_t^i b_t dt\right] + \mathbb{E}_0\left[\check{m}_T^i b_T\right].
\end{aligned}$$

Finally,

$$\begin{aligned}
m_0^i b_0 &= \mathbb{E}_0\left[\int_0^T -d(m_t^i b_t) + m_T^i b_T\right] \\
&= \mathbb{E}_0\left[\int_0^T (e^{-\zeta t} - e^{-\zeta T}) \check{m}_t^i s_t dt\right] + \mathbb{E}_0\left[\int_0^T \zeta e^{-\zeta T} \check{m}_t^i b_t dt\right] \\
&+ e^{-\zeta T} \mathbb{E}_0\left[\int_0^T \check{m}_t^i s_t dt\right] - e^{-\zeta T} \mathbb{E}_0\left[\int_0^T \zeta \check{m}_t^i b_t dt\right] + e^{-\zeta T} \mathbb{E}_0\left[\check{m}_T^i b_T\right] \\
&= \mathbb{E}_0\left[\int_0^T (e^{-\zeta t} \check{m}_t^i) s_t dt\right] + \mathbb{E}_0\left[(e^{-\zeta T} \check{m}_T^i) b_T\right].
\end{aligned}$$

Recall that $\check{m}_t^i$ is the marginal utility if the agent were alive at time t. The TVC term for this aggregate asset is the discounted asset value $m_T^i b_T$ times the probability of surviving until T, which is $e^{-\zeta T}$. The aggregate TVC holds if and only if $\hat{c} - \gamma\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2 > 0$.

Individual TVC. Similarly, for individual wealth, we obtain

$$\begin{aligned} d(m_t^i n_t^i) &= n_t^i dm_t^i + m_t^i dn_t^i + dm_t^i dn_t^i \\ &= n_t^i(-(r_t^f - \zeta)m_t^i dt - \bar{\chi}(1-\vartheta_t)\tilde{\sigma}_t m_t^i d\tilde{Z}_t^i) \\ &+ m_t^i(-c_t^i dt + (r_t^f + \bar{\chi}^2(1-\vartheta_t)^2\tilde{\sigma}_t^2)n_t^i dt + \bar{\chi}(1-\vartheta_t)\tilde{\sigma}_t n_t^i d\tilde{Z}_t^i) - \bar{\chi}^2(1-\vartheta_t)^2\tilde{\sigma}_t^2 m_t^i n_t^i dt \\ &= -c_t^i m_t^i dt + \zeta m_t^i n_t^i dt \end{aligned}$$

for $t < \tau^i$, i.e., if the agent were still alive. Following the same steps above, we obtain

$$m_0^i n_0^i = \mathbb{E}_0\left[\int_0^T (e^{-\zeta t}\check{m}_t^i)c_t^i dt\right] + \mathbb{E}_0\left[(e^{-\zeta T}\check{m}_T^i)\check{n}_T^i\right].$$

The TVC for individual household's wealth is

$$\lim_{T\to\infty} \mathbb{E}_0\left[(e^{-\zeta T}\check{m}_T^i\check{n}_T^i)\right] = 0.$$

Again, the analysis is the same as that in A.1 without wealth reset shock, but the subjective discount rate is augmented by ζ.

A.10 Calibration

Table A.1: Parameter Values.

Variable	Notation	Value	Notes
Baseline Model with Permanent Idiosyncratic Risk			
risk aversion	γ	2	
subjective discount rate	ρ	0.02	
capital adjustment cost	ϕ	8.1	
Depreciation rate	δ	0.055	
Government spending	$\mathfrak{g}$	0.15	
real growth rate	g	0%	Targeted by choosing a
idiosyncratic risk variance	$\tilde{\sigma}$	0.54	From Brunnermeier et al. (2022)
share of undiversifiable risk	$\bar{\chi}$	0.3	From Brunnermeier et al. (2022)
share of equity	ϑ	0.18	From Brunnermeier et al. (2022)
Model Variant with Transitory Idiosyncratic Risk			
weak wealth reset intensity	ζ	2.5%	
moderate wealth reset intensity	ζ	3.8%	
strong wealth reset intensity	ζ	5.4%	

B Model with General CRRA Utility

In this section, we consider a model variant where households have constant relative risk aversion γ. As shown in the log utility case, aggregate risk does not matter for our core intuition. Hence, for tractability, we assume there is no aggregate risk, which implies constant consumption-wealth ratio $\hat{c} := c_t^i/n_t^i$. The derivation is similar to the log case. We provide a proof sketch as follows.

B.1 Model Derivation

B.1.1 Household's Problem

Similar to the log case, guess that the value function takes the form $V(n_t^i, t) = \frac{\omega_t}{\rho+\zeta}\frac{(n_t^i)^{1-\gamma}}{1-\gamma} + v_t$, where ω_t and v_t depends on investment opportunity in aggregate level. The Hamilton-Jacobi-

Bellman equation is

$$(\rho+\zeta)V(n_t^i,t)=\max_{c_t^i,\iota_t^i,\theta_t^i}\mathbb{E}_t\left[\frac{(c_t^i)^{1-\gamma}}{1-\gamma}dt+\zeta V(\bar{n}_t,t)dt+V_t(n_t^i,t)dt+V_n(n_t^i,t)dn_t^i+\frac{1}{2}V_{nn}(n_t^i,t)\langle dn_t^i,dn_t^i\rangle\right]/dt$$

$$\rho v_t+\omega_t\frac{(n_t^i)^{1-\gamma}}{1-\gamma}-\frac{\zeta}{\rho+\zeta}\omega_t\frac{(\bar{n}_t)^{1-\gamma}}{1-\gamma}$$

$$=\max_{c_t^i,\iota_t^i,\theta_t^i}\mathbb{E}_t\left\{\frac{(c_t^i)^{1-\gamma}}{1-\gamma}dt+\frac{\dot{w}_t}{\rho+\zeta}\frac{(n_t^i)^{1-\gamma}}{1-\gamma}dt+\dot{v}_tdt+\omega_t\frac{(n_t^i)^{1-\gamma}}{\rho+\zeta}\frac{dn_t^i}{n_t^i}-\omega_t\frac{\gamma}{2}\frac{(n_t^i)^{1-\gamma}}{\rho+\zeta}\left\langle\frac{dn_t^i}{n_t^i},\frac{dn_t^i}{n_t^i}\right\rangle\right\}/dt.$$

For simplicity, we denote $\hat{\rho}:=\rho+\zeta$. The first order conditions w.r.t. c_t^i, ι_t^i, $\theta_t^{K,i}$, $\theta_t^{E,i}$ and $\theta_t^{\bar{E},i}$ are

$$-\omega_t\frac{(n_t^i)^{1-\gamma}}{\hat{\rho}n_t^i}+(c_t^i)^{-\gamma}=0, \tag{B.1}$$

$$-\frac{1}{q_t^K}+\frac{1}{1+\phi\iota_t^i}=0, \tag{B.2}$$

$$\frac{\mathbb{E}_t[dr_t^{K,i}]}{dt}-\frac{\mathbb{E}_t[dr_t^B]}{dt}=\gamma(\theta_t^{K,i}+\theta_t^{E,i})\tilde{\nu}_t^2-(1-\bar{\chi})\lambda_t^i, \tag{B.3}$$

$$\frac{\mathbb{E}_t[dr_t^{E,i}]}{dt}-\frac{\mathbb{E}_t[dr_t^B]}{dt}=\gamma(\theta_t^{K,i}+\theta_t^{E,i})\tilde{\nu}_t^2-\lambda_t^i, \tag{B.4}$$

$$\frac{\mathbb{E}_t[dr_t^{\bar{E}}]}{dt}-\frac{\mathbb{E}_t[dr_t^B]}{dt}=0. \tag{B.5}$$

B.1.2 Verifying Value Function

We can solve for the symmetric equilibrium and verify the value function following the same steps in the log case. Specifically, plugging endogenous variables into the HJB Equation, he coefficient of $(n_t^i)^{1-\gamma}$ should be zero, i.e.,

$$\begin{aligned}0=&-\omega_t\frac{1}{1-\gamma}+\frac{1}{\hat{\rho}}\dot{\omega}_t\frac{1}{1-\gamma}+\left(\frac{\omega_t}{\hat{\rho}}\right)^{-\frac{1-\gamma}{\gamma}}\frac{1}{1-\gamma}-\omega_t\frac{1}{\hat{\rho}}\left(\frac{\omega_t}{\hat{\rho}}\right)^{-\frac{1}{\gamma}}\\&+\omega_t\frac{1}{\hat{\rho}}\vartheta_t\mathbb{E}[r_t^B]/dt+\omega_t\frac{1}{\hat{\rho}}(1-\vartheta_t)\mathbb{E}[r_t^{K,i}]/dt-\frac{\gamma}{2}\omega_t\frac{1}{\hat{\rho}}\bar{\chi}^2(1-\vartheta_t)^2\tilde{\nu}_t^2,\\0=&-\rho v_t+\dot{v}_t+\frac{\zeta}{\hat{\rho}}\omega_t\frac{(\bar{n}_t)^{1-\gamma}}{1-\gamma}.\end{aligned}$$

which are ODEs for ω_t and v_t. Note that $\vartheta_t,\mu_t^\vartheta$ and $\bar{n}_t$ depend on aggregate level variables only, so do ω_t and v_t. The value function is verified. Besides, in the absence of aggregate shocks,

$\omega_t = \omega$ is constant if the economy starts at the steady state. Plug in $\mathbb{E}[r_t^B]/dt = r^f$, $\mathbb{E}[r_t^{K,i}]/dt = r^f + \gamma\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2$ and $\omega_t = \hat{\rho}\hat{c}^{-\gamma}$ to get

$$0 = -\hat{\rho}\hat{c}^{-\gamma}\frac{1}{1-\gamma} + \hat{c}^{1-\gamma}\frac{1}{1-\gamma} - \hat{c}^{1-\gamma} + \hat{c}^{-\gamma}r^f + \hat{c}^{-\gamma}\gamma\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2 - \hat{c}^{-\gamma}\frac{\gamma}{2}\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2,$$

i.e.,

$$r^f = \frac{-\gamma\hat{c} + \hat{\rho}}{1-\gamma} - \frac{\gamma}{2}\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2, \tag{B.6}$$

The risk free rate r^f equals $\mathbb{E}_t[r_t^B]$ and $\mathbb{E}_t[r_t^E]$ in the absence of aggregate risk, which can be derived as

$$r^f = \Phi(\iota) - \delta + \hat{c} - \gamma\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2.$$

$\hat{c}$ can be solved numerically by

$$\hat{c} - (\gamma-1)\Phi(\iota) = \hat{\rho} - (\gamma-1)\delta - \frac{\gamma(\gamma-1)}{2}\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2 \tag{B.7}$$

where

$$\iota = \frac{(a-\mathfrak{g})(1-\vartheta) - \hat{c}}{1 + \hat{c}\phi - \vartheta}$$
$$\Phi(\iota) = \frac{1}{\phi}\log(1+\phi\iota).$$

Since

$$\frac{d\iota}{d\hat{c}} = \frac{-(1+\hat{c}\phi-\vartheta) - ((a-\mathfrak{g})(1-\vartheta) - \hat{c})\phi}{(1+\hat{c}\phi-\vartheta)^2} < 0,$$

ι and $\Phi(\iota)$ are decreasing in $\hat{c}$. The LHS is monotonically increasing in $\hat{c}$. There is a unique positive solution of $\hat{c}$ if

$$\frac{\gamma}{2}\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2 - \frac{\hat{\rho}}{1-\gamma} + \delta < \Phi(a - \mathfrak{g}).$$

Note that

$$\frac{d\hat{c}}{d\hat{\rho}} = \left(1 - (\gamma-1)\Phi'(\iota)\frac{d\iota}{dc}\right)^{-1} > 0.$$

Consumption rate increases when subjective discount rate ρ or wealth reset shock intensity ζ goes up.

B.1.3 SDF and Wealth Process

We focus on the steady state where consumption-wealth ratio is constant ($\hat{c}$) in the absence of aggregate risks. Again, denote $\tilde{\sigma} = \bar{\chi}(1-\vartheta)\tilde{\nu}$. We have

$$dn_t^i = g_t n_{t-}^i dt + \gamma\tilde{\sigma} n_{t-}^i d\tilde{Z}_t^i + (\bar{n}_t - n_{t-}^i)d\tilde{J}_t^i,$$

which implies

$$\frac{dm_t^i}{m_{t-}^i} = -\left[\rho + \gamma g_t - \frac{\gamma(\gamma+1)}{2}\tilde{\sigma}_t^2\right]dt - \gamma\tilde{\sigma}d\tilde{Z}_t^i + \frac{1}{m_{t-}^i}\left(e^{-\rho t}\hat{c}^{-\gamma}\bar{n}_t^{-\gamma} - e^{-\rho t}\hat{c}^{-\gamma}(n_{t-}^i)^{-\gamma}\right)d\tilde{J}_t^i.$$

B.2 Transversality Conditions with Transitory Idiosyncratic Risk

The Wealth Distribution Perspective. Consider the wealth reset shock with intensity ζ. The wealth dynamics become

$$d\eta_t^i = \tilde{\sigma}\eta_{t-}^i d\tilde{Z}_t^i + (1 - \eta_{t-}^i)d\tilde{J}_t^i.$$

By Itô's lemma,

$$d(\eta_t^i)^{-\gamma} = \left[\frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2(\eta_{t-}^i)^{-\gamma} + \zeta(1-(\eta_{t-}^i)^{-\gamma})\right]dt$$
$$-\gamma\tilde{\sigma}(\eta_{t-}^i)^{-\gamma}d\tilde{Z}_t^i + (1-(\eta_{t-}^i)^{-\gamma})(d\tilde{J}_t^i - \zeta dt).$$

Take expectation on both sides yields an ODE w.r.t. $\mathbb{E}[(\eta_t^i)^{-\gamma}]$, i.e.,

$$d\mathbb{E}[(\eta_t^i)^{-\gamma}] = \left[\frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2\mathbb{E}[(\eta_t^i)^{-\gamma}] + \zeta(1-\mathbb{E}[(\eta_t^i)^{-\gamma})]\right]dt.$$

The solution to the ODE is

$$\mathbb{E}_0\left[(\eta_t^i)^{-\gamma}\right] = \left(\frac{\zeta}{\frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2-\zeta} + (\eta_0^i)^{-\gamma}\right)e^{[\frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2-\zeta]t} - \frac{\zeta}{\frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2-\zeta}.$$

Recall that growth rate of aggregate wealth and risk-free rate jointly satisfy

$$r^f = \rho + \zeta + \gamma g - \frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2, \tag{B.8}$$
$$g = -\hat{c} + r^f + \gamma\tilde{\sigma}^2, \tag{B.9}$$

where the first equation is from household's Euler equation (note that household does not effectively get the payoff from the risk free asset when wealth reset event happens, hence the risk-free rate does not equal the negative drift rate of the SDF), and the second equation is from the fact that household's wealth portfolio is tradable to herself, of which the total return is r^f plus idiosyncratic risk premium $\gamma\tilde{\sigma}^2$. The simultaneous equations jointly solve for

$$g = \frac{-\hat{c}+\rho+\zeta}{1-\gamma} + \frac{\gamma}{2}\bar{\chi}^2(1-\vartheta)^2\tilde{\sigma}^2,$$

The aggregate TVC condition can be stated as

$$\begin{aligned}
\lim_{T\to\infty} \mathbb{E}[m_T^i \bar{n}_T] &= \lim_{T\to\infty} (\hat{c}^{-\gamma} e^{-\rho T} \bar{n}_T^{1-\gamma}) \mathbb{E}\left[(\eta_T^i)^{-\gamma}\right] \\
&= \lim_{T\to\infty} \hat{c}^{-\gamma} \bar{n}_0^{1-\gamma} e^{-\rho T} \exp\left(\left(-\hat{c} + \rho + \zeta - \frac{\gamma(\gamma-1)}{2}\tilde{\sigma}^2\right) T\right) \\
&\times \exp\left(\left(\frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2 - \zeta\right) T\right) \left(\frac{\zeta}{\frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2 - \zeta} + (\eta_0^i)^{-\gamma}\right) \\
&- \lim_{T\to\infty} \hat{c}^{-\gamma} \bar{n}_0^{1-\gamma} e^{-\rho T} \bar{n}_T^{1-\gamma} \frac{\zeta}{\frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2 - \zeta} \\
&= \lim_{T\to\infty} \hat{c}^{-\gamma} \bar{n}_0^{1-\gamma} \exp(-(\hat{c} - \gamma\tilde{\sigma}^2) T) \left(\frac{\zeta}{\frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2 - \zeta} + (\eta_0^i)^{-\gamma}\right) \\
&- \lim_{T\to\infty} \hat{c}^{-\gamma} \bar{n}_0^{1-\gamma} e^{-\rho T} \bar{n}_T^{1-\gamma} \frac{\zeta}{\frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2 - \zeta}.
\end{aligned}$$

Hence, the aggregate TVC holds if and only if

$$\hat{c} > \gamma\tilde{\sigma}^2$$

and

$$-\rho + (1-\gamma)g < 0.$$

The first condition states that the wealth reset intensity should be big enough such that the marginal utility dispersion does not explode too fast, hence risk free rate will not be lower than the growth rate. The second condition is imposed as we are evaluating long-lived assets with the realized marginal utility process, we want to rule out the case where the whole economy shrinks and marginal utilities explode on average. In other word, we assume transversality condition hold had the economy been populated by one representative agent instead of heterogeneous agents. In this way, we shed light on the effect of heterogeneity and idiosyncratic risk.

The r minus g Perspective Note that Eq. (B.9) implies

$$r^f - g = \hat{c} - \gamma\tilde{\sigma}^2.$$

Note that

$$\begin{aligned} d(m_t^i \bar{n}_t) = &- \left[\rho - \gamma g + \frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2 + g\right] m_{t-}^i \bar{n}_t dt \\ &+ \zeta[e^{-\rho t}\hat{c}^{-\gamma}\bar{n}_t^{1-\gamma} - m_{t-}^i \bar{n}_t]dt \\ &- \gamma\tilde{\sigma} m_{t-}^i \bar{n}_t d\tilde{Z}_t^i + [e^{-\rho t}\hat{c}^{-\gamma}\bar{n}_t^{1-\gamma} - m_{t-}^i \bar{n}_t]d(\tilde{J}_t^i - \zeta t). \end{aligned}$$

Plug in the expression of g and $r^f - g = \hat{c} - \gamma\tilde{\sigma}^2$ to get

$$\begin{aligned} d(m_t^i \bar{n}_t) = &- \left[r^f - g\right] m_{t-}^i \bar{n}_t dt + \zeta e^{-\rho t}\hat{c}^{-\gamma}\bar{n}_t^{1-\gamma} dt \\ &- \gamma\bar{\chi}(1-\vartheta)\tilde{\sigma} m_{t-}^i \bar{n}_t d\tilde{Z}_t^i + [e^{-\rho t}\hat{c}^{-\gamma}\bar{n}_t^{1-\gamma} - m_{t-}^i \bar{n}_t]d(\tilde{J}_t^i - \zeta t). \end{aligned}$$

The drift term reflects the discounted growth rate. There is an additional drift adjustment term as wealth jump creates a discrepancy between the SDF and marginal utility (in other word, the wealth portfolio does not pays n_{t-}^i when jump event happens). Divide by m_0^i and integrate across i to obtain

$$d(\bar{m}_t \bar{n}_t) = -(r^f - g)\bar{m}_t \bar{n}_t dt + \zeta e^{-\rho t}\left(\frac{\bar{n}_t}{\bar{n}_0}\right)^{1-\gamma} \bar{n}_0 dt.$$

The ODE admits the following solution[11]

$$\bar{m}_t\bar{n}_t = \bar{n}_0 \exp(-(r^f - g)t)\left(\frac{\zeta}{\frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2 - \zeta} + 1\right) - \bar{n}_0 e^{(-\rho-(\gamma-1)g)t}\frac{\zeta}{\frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2 - \zeta}.$$

As a special case, when $t = 0$, we recover our initial condition $\bar{m}_0\bar{n}_0 = \bar{n}_0$. Given $-\rho-(\gamma-1)g < 0$, the aggregate TVC holds if and only if

$$r^f > g, \quad \text{i.e., } \hat{c} > \gamma\tilde{\sigma}^2.$$

C TVC in the Constantinides-Duffie Economy

The Economy. There is a continuum of households indexed by $i \in [0, 1]$. Time is discrete. There is a single nondurable consumption good which serves as the numeraire. Households maximize lifetime expected utility with CRRA γ. The aggregate dividend is exogenous, denoted as d_t. The aggregate labor income process is exogenous, denoted as ℓ_t. The market clearing condition implies that the aggregate consumption process is $c_t = \ell_t + d_t$.

Uninsured Idiosyncratic Labor Income Risk. Consumer i has labor income ℓ_t^i defined by

$$\ell_t^i = \delta_t^i c_t - d_t,$$

[11] To solve the ODE, note that

$$\begin{aligned} d(\exp((r^f - g)t)\bar{m}_t\bar{n}_t) &= \exp((r^f - g)t)d(\bar{m}_t\bar{n}_t) + (r^f - g)\exp((r^f - g)t)(\bar{m}_t\bar{n}_t)dt \\ &= \zeta \exp((-\rho + (1-\gamma)g + r^f - g)t)\bar{n}_0 dt, \end{aligned}$$

where according to Eq. (B.8),

$$-\rho + (1-\gamma)g + r^f - g = \zeta - \frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2.$$

Integrate both sides to get

$$\exp((r^f - g)t)\bar{m}_t\bar{n}_t = \bar{m}_0\bar{n}_0 + \zeta\bar{n}_0\frac{1}{\zeta - \frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2}[\exp((\zeta - \frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2)t) - 1],$$

note that $\zeta - \frac{\gamma(\gamma+1)}{2}\tilde{\sigma}^2 - (r^f - g) = -\rho + (1-\gamma)g$ and $\bar{m}_0 = 1$. Move $\exp(r^f - g)$ to the RHS to get the solution.

where

$$\delta_t^i = \exp\left[\sum_{s=1}^{t}\left(\varepsilon_s^i y_s - \frac{y_s^2}{2}\right)\right].$$

The shocks ε_t^i are standard normal and i.i.d. across periods and individuals.

Equilibrium. We consider the no-trade equilibrium where each household holds one unit of equity. Household i's consumption is $c_t^i = \delta_t^i c_t$. Denote $\mathcal{F}_t$ as the information set available to consumers. With household i's SDF given by

$$\begin{aligned} m_{t,t+1}^i &= e^{-\rho}\left(\frac{c_{t+1}^i}{c_t^i}\right)^{-\gamma} \\ &= e^{-\rho}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\exp\left(-\gamma\left(\varepsilon_{t+1}^i y_{t+1} - \frac{y_{t+1}^2}{2}\right)\right) \end{aligned}$$

The Euler equation for households i is

$$\begin{aligned} p_t &= \mathbb{E}\left[e^{-\rho}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\exp\left(-\gamma\left(\varepsilon_{i,t+1}y_{t+1} - \frac{y_{t+1}^2}{2}\right)\right)(p_{t+1}+d_{t+1})\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[e^{-\rho}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\exp\left(-\gamma\left(\varepsilon_{i,t+1}y_{t+1} - \frac{y_{t+1}^2}{2}\right)\right)(p_{t+1}+d_{t+1})\middle|\mathcal{F}_t\cup\{y_{t+1}\}\right]\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[e^{-\rho}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\exp\left(\frac{\gamma(\gamma+1)}{2}y_{t+1}^2\right)(p_{t+1}+d_{t+1})\middle|\mathcal{F}_t\right]. \end{aligned}$$

It is verified that no-trade equilibrium is valid, as every household's valuation on the aggregate equity aligns.

The risk-free rate is given by

$$\exp(-r_t) = \mathbb{E}_t\left[e^{-\rho}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\exp\left(\frac{\gamma(\gamma+1)}{2}y_{t+1}^2\right)\right]$$

We consider a special case with constant growth rate $c_{t+1}/c_t = \exp(g)$ and constant intensity of

idiosyncratic risk $y_t = y$, which implies

$$r = \rho + \gamma g - \frac{\gamma(\gamma+1)}{2} y^2.$$

Pricing Long-Lived Asset with Positive Dividend. Assume that the aggregate dividend is positive and grows at the same speed as consumption. If y^2 is large enough, such that $\rho + \gamma g - \frac{\gamma(\gamma+1)}{2} y^2 < g$, i.e., $r < g$, there is no no-trade equilibrium. Indeed, suppose there were a no-trade equilibrium. The present value of dividend flow would be

$$PV_t(d) = \sum_{k=1}^{\infty} d_t \exp(gk - rk) = \infty.$$

The equity would be infinitely valuable. That is, $\forall\ p_t < \infty$, agent can buy and hold an arbitrarily small amount of the asset to obtain an (infinitely) larger life-time utility. Hence, there exists no well-defined price process and no no-trade equilibrium exists.

D Model Extension with Borrowing Constraints

In this appendix, we provide more details for the heterogeneous-agent models with borrowing constraints, which we discussed in Section 4.3.

D.1 Intertemporal Government Budget Condition

First, we show that with borrowing constraint, the Euler equation becomes an inequality:

$$\mathbb{E}_t \left[\frac{m^i_{t+1}}{m^i_t} \exp(r_t) \right] \leq 1.$$

Suppose the household maximizes lifetime utility:

$$\max \sum_{t=0}^{\infty} \mathbb{E}_0 \left[\exp(-\rho t) u(c_t) \right],$$

such that

$$y_t + b_{t-1}\exp(r_{t-1}) \geq c_t + b_t,$$
$$b_t \geq 0.$$

The Lagrangian is

$$\mathcal{L}_t = \sum_{t=0}^{\infty} \mathbb{E}_0 \left[\exp(-\rho t)u(c_t) + \zeta_t(y_t + b_{t-1}\exp(r_{t-1}) - c_t - b_t) + \mu_t b_t\right].$$

The first-order conditions are

$$\exp(-\rho t)u'(c_t) = \zeta_t,$$
$$\zeta_t - \mu_t = \mathbb{E}_t[\zeta_{t+1}\exp(r_t)],$$

which implies

$$\mathbb{E}_t\left[\frac{\exp(-\rho)u'(c_{t+1})}{u'(c_t)}\exp(r_t)\right] = 1 - \frac{\mu_t}{u'(c_t)} \leq 1.$$

with $\mu_t > 0$ if the borrowing constraint is binding.

Next, We evaluate government bond in the presence of borrowing constraint. Recall that

$$b_t \exp(r_t) = s_{t+1} + b_{t+1}.$$

Multiplying both sides by m^i_{t+1}/m^i_t and take expectation to obtain

$$b_t \geq b_t \mathbb{E}_t\left[\frac{m^i_{t+1}}{m^i_t}\exp(r_t)\right] = \mathbb{E}_t\left[\frac{m^i_{t+1}}{m^i_t}s_{t+1}\right] + \mathbb{E}_t\left[\frac{m^i_{t+1}}{m^i_t}b_{t+1}\right].$$

The inequality follows $\mathbb{E}_t\left[\frac{m^i_{t+1}}{m^i_t}\exp(r_t)\right] \leq 1$, which we proved above. Similarly,

$$b_{t+1} \geq \mathbb{E}_{t+1}\left[\frac{m^i_{t+2}}{m^i_{t+1}}s_{t+2}\right] + \mathbb{E}_{t+1}\left[\frac{m^i_{t+2}}{m^i_{t+1}}b_{t+2}\right],$$

which implies

$$b_t \geq \sum_{k=1}^{2}\mathbb{E}_t\left[\frac{m^i_{t+k}}{m^i_t}s_{t+k}\right] + \mathbb{E}_t\left[\frac{m^i_{t+2}}{m^i_t}b_{t+2}\right].$$

In this way, we can iterate forward to obtain

$$b_t \geq \sum_{k=1}^{\infty}\mathbb{E}_t\left[\frac{m^i_{t+k}}{m^i_t}s_{t+k}\right] + \lim_{k\to\infty}\mathbb{E}_t\left[\frac{m^i_{t+k}}{m^i_t}b_{t+k}\right].$$

D.2 Consumption Lower Bound in Bewley Model

Here we feature a simple model without aggregate growth or aggregate uncertainty but with uninsured idiosyncratic risk. There is a continuum of agents on $[0,1]$. Agent i get endowment y^i_t each period, with $y^i_t \geq \underline{y} > 0$. Aggregate (after-tax) endowment is $y = \int_0^1 y^i_t\, di$. Agents can invest into risk-free bonds subject to borrowing constraints, i.e., $b^i_t \geq 0$. The supply of bond is fixed, $b = \int_0^1 b^i_t\, di$. Under steady state, interest rate is constant r. Agents maximize lifetime utility

$$\max\sum_{t=0}^{\infty}\mathbb{E}_0\left[\exp(-\rho t)u(c^i_t)\right].$$

We assume that the utility function satisfies the Inada condition, i.e., $\lim_{c\to 0}u'(c) = \infty$ and $u'(c)$ decreases in c.

In this setting, we show that consumption has a lower bound $\underline{c} > 0$. Denote $z^i_t = b^i_{t-1}\exp(r) + y^i_t$ as the resource available at time t. $z^i_t \geq \underline{y}$. The value function is

$$V(z^i_t, y^i_t) = \max_{c^i_t}\{u(c^i_t) + \exp(-\rho)\mathbb{E}_t[V(z^i_{t+1}, y^i_{t+1})]\}.$$

If y_t^i is i.i.d., the expectation term on the right-hand side does not depend on y_t^i, and we can rewrite value function as $V(z_t^i)$, i.e., a function of available resources. In the case, Aiyagari (1994) proves that the consumption has a positive lower bound, at which household consumes all available resources. The Euler equation is

$$u'(c_t^i) \geq \exp(r-\rho)\mathbb{E}_t[V_z(z_{t+1}^i, y_{t+1}^i)], \tag{D.1}$$

with equality obtained when $b_t^i > 0$.

Next, we show that the consumption has a positive lower bound. Assume that the marginal value of wealth $V_z(z_t^i, y_t^i)$ is decreasing in wealth z_t^i for any given income y_t^i. Note that

$$V_z(z_{t+1}^i, y_{t+1}^i) \leq V_z(\underline{y}, y_{t+1}^i) \leq \overline{V}_z := \sup_{y_{t+1}^i} V_z(\underline{y}, y_{t+1}^i).$$

The first inequality follows that $z_{t+1}^i \geq \underline{y}$ no matter how much agent saves or consumes at time t. The second inequality assumes that given a positive level of resource available ($\underline{y}$), the marginal gain of extra resource $V_z(\underline{y}, y_{t+1}^i)$ is always finite across every possible y_{t+1}^i. Two special cases are (1) i.i.d. idiosyncratic shock case in Aiyagari (1994), where y_{t+1}^i is i.i.d., and (2) finite space Markov chain in Krueger and Lustig (2010), where y_{t+1}^i has finite numbers of value, and the upper bound of $V_z(\underline{y}, y_{t+1}^i)$ exists.

Now suppose to the contrary that c_t^i does not have a positive lower bound. On the one hand, there exists $c^* > 0$, such that

$$u'(c_t^i) > \exp(r-\rho)\overline{V}_z \geq \exp(r-\rho)\mathbb{E}_t[V_z(z_{t+1}^i, y_{t+1}^i)], \quad \forall c_t^i < c^*.$$

On the other hand, since c_t^i does not have a positive lower bound. There must be some states where $c_t^i < \underline{y}$. Note that since $c_t^i + b_t^i \geq \underline{y}$, $c_t^i < \underline{y}$ must imply $b_t^i > 0$ and equality in the Euler equation

(D.1), i.e.,

$$u'(c_t^i) = \exp(r-\rho)\mathbb{E}_t[V_z(z_{t+1}^i, y_{t+1}^i)].$$

By contradiction, c_t^i must have a positive lower bound $\underline{c}$. Moreover, if $\underline{c} < \underline{y}$, since the agent must be saving, the Euler equation holds. Assume the inverse of marginal utility u'^{-1} exists, we obtain

$$\underline{c} \geq u'^{-1}(\bar{V}_z).$$

Hence, $\underline{c} \geq \min\{u'^{-1}(\bar{V}_z), \underline{y}\}$. Following the same steps, it is straightforward to extend the result to an economy with constant growth rate $\exp(g)$. The consumption over aggregate endowment share is no smaller than $\underline{c}/y$.

Finally, we show that the TVC holds for aggregate debt with CRRA $\gamma > 1$ and positive growth rate. Denote the aggregate endowment as $y_t = y\exp(gt)$ and aggregate debt as $b_t = b\exp(gt)$ at time t. Note that

$$\lim_{T\to\infty}[e^{-\rho T}y_T^{-\gamma}b_T] = \lim_{T\to\infty}\exp((-\rho+(1-\gamma)g)T) = 0,$$

i.e., the TVC holds if the economy were populated by a hypothetical representative agent. With bounded left tail in consumption distribution, the TVC also holds with heterogeneous agents. Indeed,

$$\lim_{T\to\infty}\mathbb{E}_0[e^{-\rho T}(c_T^i)^{-\gamma}b_T] = \lim_{T\to\infty}\mathbb{E}_0[e^{-\rho T}y_T^{-\gamma}b_T]\mathbb{E}_0\left[\left(\frac{c_T^i}{y_T}\right)^{-\gamma}\right] \leq \lim_{T\to\infty}\mathbb{E}_0[e^{-\rho T}y_T^{-\gamma}b_T]\underline{c}^{-\gamma} = 0.$$

D.3 Townsend and Woodford Model

The government budget constraint is

$$d(1+r) = \tau + d,$$

which implies

$$d = \frac{\tau}{r}$$

for $r > 0$ cases. Since household purchase government bonds only at good times, we obtain the following Euler equations:

$$\rho \frac{u'(\underline{c})}{u'(\bar{c})} = (1+r)^{-1},$$
$$\rho \frac{u'(\bar{c})}{u'(\underline{c})} \leq (1+r)^{-1},$$

which imply

$$\mathbb{E}_0 \left[\sum_{t=1}^{T} \rho^t \frac{u'(c_{A,t})}{u'(c_{A,0})} \tau \right] \leq \mathbb{E}_0 \left[\sum_{t=1}^{T} (1+r)^{-t} \tau \right] = \frac{\tau}{r}.$$

We have $\mathbb{E}_0 \left[\sum_{t=1}^{T} \rho^t \frac{u'(c_{A,t})}{u'(c_{A,0})} \tau \right] < d$ if borrowing constraint binds infinitely often and $\rho \frac{u'(\bar{c})}{u'(\underline{c})} \leq (1+r)^{-1}$. In the meantime, the transversality condition holds.

$$0 \leq \lim_{T \to \infty} \mathbb{E}_0 \left[\rho^T \frac{u'(c_{A,T})}{u'(c_{A,0})} d \right] \leq \lim_{T \to \infty} \mathbb{E}_0 \left[\rho^T \frac{u'(\underline{c})}{u'(\bar{c})} d \right] = \lim_{T \to \infty} \rho^{T-1} (1+r)^{-1} d = 0$$

E Comparing Different Types of Asset Bubbles

In this section we provide two simple models to illustrate the first and second types of bubbles, which follow Hirano and Toda (2024). For simplicity, we focus on deterministic economies. Denote q_t as the price of the Arrow-Debreu security that delivers one unit of consumption at time t. Denote the price and cash flow as p_t and d_t. No arbitrage implies

$$q_t p_t = q_{t+1}(p_{t+1} + d_{t+1}).$$

Iterate forward to obtain

$$q_t p_t = \sum_{k=1}^{\infty} q_{t+k} d_{t+k} + \lim_{T \to \infty} q_T p_T.$$

We define the first term as fundamental value, and the second term (the TVC term) as the bubble term.

E.1 Classic Rational Bubbles

Consider a model of classic rational bubble in Hirano and Toda (2024), which is based on Samuelson (1958). In period t, a new agent is born and lives for two periods. Its utility function is given by

$$u(c_t(t), c_{t+1}(t)) = \log c_t(t) + \delta \log c_{t+1}(t).$$

We use (t) to denote the agent's cohort, which is based on when it is born. In any given period, there is a young agent who is just born and an old agent who was born in the previous period.

Each agent receives an endowment of a when young and b when old. There is also an intrinsically useless asset (i.e., cash) that has a unit supply and is initially owned by the old agent in period 0. Let p_t denote the asset's price and let $x_t(t)$ denote the amount of the asset that is purchase by the young agent. The agent's budget constraints are given by

$$\begin{aligned} c_t(t) + p_t x_t(t) &= a, \\ c_{t+1}(t) &= b + p_{t+1} x_t(t). \end{aligned}$$

We are interested in competitive equilibria in which all agents maximize their utility functions subject to their budget constraints, and the following asset and goods market clearing conditions

hold:

$$a + b = c_t(t) + c_t(t-1),$$
$$x_t(t) + 0 = 1.$$

The young agent's Euler equation with respect to the asset is given by

$$\frac{1}{c_t(t)} p_t = \delta \frac{1}{c_{t+1}(t)} p_{t+1}.$$

Unlike earlier models we studied, there are multiple equilibria in this economy. First, there is a *fundamental equilibrium* in which the asset price is zero, i.e., $p_t = 0$ for all t. The asset is fundamentally worthless, and it does not affect agents' autarky consumption patterns.

In addition, there may exist multiple *bubbly equilibria* in which the asset price is positive, i.e., $p_t > 0$ for all t. In this case, the Euler equation requires

$$\frac{1}{p_{t+1}} = \frac{a\delta}{b} \frac{1}{p_t} - \frac{\delta + 1}{b},$$

If $a\delta \leq b$, the fundamental equilibrium is the only equilibrium. Otherwise, given $a\delta > b$, there exist a continuum of bubbly equilibria indexed by initial asset price $0 < p_0 \leq \frac{a\delta - b}{\delta + 1}$, such that

$$\frac{1}{p_t} = \left(\frac{a\delta}{b}\right)^t \left(\frac{1}{p_0} - \frac{\delta + 1}{a\delta - b}\right) + \frac{\delta + 1}{a\delta - b}.$$

The bubble is stationary if $p_0 = \frac{a\delta - b}{1 + \delta}$.

The bubble asset offers a saving vehicle that allows the agents to smooth their consumption across periods. The young agent can purchase the bubble asset from the old agent, with the expectation that it can sell the asset to the next generation when it becomes old. In this way, the agent delays some of its consumption until it becomes old. Note that since the bubble is intrinsically worthless, the fundamental value is zero. The (finite) value of TVC term equals the price of the

bubble asset.

E.2 Unbalanced Growth

Consider another type of bubble of which the cash flow/price ratio is vanishing from Hirano and Toda (2025). The model features similar OLG settings. In period t, a new agent is born and lives for two periods. The agent has a unit supply of labor when young and zero when old.

There is only one type of goods and two production sectors in this economy. The two sectors are indexed by $i = 1, 2$. The first sector produces goods using labor ℓ, while the second sector produces goods y using labor ℓ and land x. The production functions are given by

$$f_{1,t}(h) = \exp(g_1 t)\ell,$$
$$f_{2,t}(h) = \exp(g_2 t)\ell^{\alpha} x^{1-\alpha},$$

where the productivity growth in the two sectors satisfies $g_1 > g_2$. Let w_t denote the wage rate, and h_t denote the rental rate of land. In equilibrium, the rental rate of land is

$$h_t = (1-\alpha)\alpha^{\alpha/(1-\alpha)} \exp\left(\left(g_2 + \frac{\alpha}{(1-\alpha)}(g_2 - g_1)\right) t\right),$$

and the land price is

$$p_t = \frac{\delta}{1+\delta} \exp(g_1 t).$$

The cash flow/price ratio is

$$\frac{h_t}{p_t} = \delta^{-1}(1+\delta)(1-\alpha)\alpha^{\alpha/(1-\alpha)} \exp\left(\frac{g_2 - g_1}{1-\alpha} t\right),$$

which decays geometrically, hence

$$\sum_{t=1}^{\infty} \frac{h_t}{p_t} = \delta^{-1}(1+\delta)(1-\alpha)\alpha^{\alpha/(1-\alpha)} \frac{\exp\left(\frac{g_2-g_1}{1-\alpha}\right)}{1-\exp\left(\frac{g_2-g_1}{1-\alpha}\right)} < \infty$$

By the bubble characterization lemma (Hirano and Toda, 2025; Montrucchio, 2004), the land price exhibits a bubble.

Over time, the cash flow of the land is going to be a vanishing share of its price, generating a classical behavior of asset bubbles. Intuitively, think of the two sectors as tech and traditional industries. The tech industry is growing faster, and its growth ignites a fast growth rate of wage. A fraction of wage income becomes saving and flows into the land, the only saving vehicle in the economy. Since the land supply is fixed, the price have to go up. The gap between the land price and its fundamental value, i.e., the present value of rents, can be viewed as its premium for being the store of value. As the rent vanishes relative to land price and the tech industry overwhelm the traditional sector, land loses its function as production factor and becomes a pure bubble in the limit.

In this example, there is a unique bubbly equilibrium, and no fundamental equilibrium with no asset bubble. More generally, Hirano and Toda (2025) show that asset bubbles have to arise when two sectors' growth rates diverge while the counterfactual long-run autarky interest rate is below dividend growth, which rules out fundamental equilibria with no asset bubble.