Quantifying Knowledge Spillovers Using Firm and Product Dynamics

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Abstract

Intertemporal knowledge spillovers are a common rationale for government support of innovation, yet estimates of their magnitude remain scant. Since many innovations are not patented, I use data on product and firm dynamics, interpreted through the lens of a new growth model, to quantify these spillovers. In the model, stronger spillovers accelerate improvements across successive generations of new products and thereby increase the rate at which incumbent products lose market share and exit. At the same time, idiosyncratic product volatility generates turnover even absent spillovers. Exit by age then allows separate identification of these competing drivers of turnover. Using data on all U.S. nonfarm private employer businesses, I estimate that spillovers drive an 11 to 28 percentage point wedge between social and private returns to innovation under laissez-faire.

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Intertemporal knowledge spillovers are a common rationale for government support of innovation: if the cumulative stock of knowledge is an input into the production of new ideas, then an individual inventor might not fully capture the social value of their innovation (Arrow, 1962). Quantifying how large these spillovers are is therefore a key ingredient for informed innovation policy (Atkeson and Burstein, 2019; Akcigit, Hanley and Stantcheva, 2022).

Yet, evidence on these spillovers' magnitude "is quite thin" (Bryan and Williams, 2021, p. 290). The challenge is that knowledge flows are inherently difficult to measure. As Griliches notes, "in this desert of data, patent statistics loom up as a mirage of wonderful plenitude and objectivity" (Griliches, 1990, p. 287). Consequently, existing approaches to estimate these spillovers rely on patents.[1] However, many innovations are not patented, including the polio vaccine, the World Wide Web, and Linux. Patent data also skew systematically toward manufacturing: the sector accounts for 10% of GDP (BEA, 2025) and TFP growth (BLS, 2025) but 64% of patenting (NCSES, 2023).

In this paper, I develop a new model of growth and firm dynamics, then use it to quantify these spillovers. Rather than patents, I use firm-level data covering the universe of U.S. private nonfarm employer businesses. I find sizable spillovers: along the laissez-faire balanced growth path (BGP), they drive an 11 to 28 percentage points (pp) wedge between social and private rates of return to innovation and lead to underprovision of innovation by a factor of 2 to 3.4 relative to the first best.

To build intuition, consider today's inventors creating new products. Compared to their predecessors, they have access to a stock of knowledge made larger by recent innovations. How much does this larger knowledge stock improve what they invent? In search for an answer, I turn to dynamics in product markets. The rationale is simple: if better products render worse ones obsolete, then, all else equal, stronger spillovers—by accelerating improvements across successive generations of new products—accelerate the rate at which incumbent products lose market share and exit.

Using this intuition to quantify spillovers requires a model for three reasons. First, the thought experiment takes as given how much knowledge is available to inventors, but this stock is accumulated endogenously through innovation. Accordingly, I need a model endogenizing this process and formalizing how, under laissez-faire, the resulting spillovers lead to a wedge between social and private returns to innovation. Second, the thought experiment abstracts from the possibility that high turnover could reflect high volatility of idiosyncratic shocks rather than strong spillovers. The model must therefore allow separately identifying these competing forces. Third, the thought experiment presumes access to product data, which are only available for a few sectors. As a result, the theory must bridge this data gap, showing how to quantify the model with more widely available firm data.

To meet these three requirements, I develop a new model of growth and firm dynamics. Both new and incumbent firms do R&D to increase the likelihood of creating a new product (variety). The quality of each such new product is drawn from an entry distribution, then evolves as a geometric

[1] By using patent citations directly or by using patenting activity to measure technological proximity between firms.

Brownian motion. Knowledge spillovers are the source of improvement in this entry distribution over time. Specifically, product innovations increase the aggregate stock of knowledge. In turn, a higher stock of knowledge leads to a first-order stochastic dominance improvement in this entry distribution (Kortum, 1997). While these innovations allow new firms to enter and incumbent ones to expand, the endogenous shutdown of existing products, à la Hopenhayn (1992), acts as a countervailing force, culminating in a firm's exit upon the shutdown of its final product.

The model delivers on all three requirements. First, it provides an intuitive characterization of the wedge between social and private rates of returns to innovation created by knowledge spillovers under laissez-faire. Along the BGP, this wedge is the product of: (i) the spillover elasticity, governing how much a higher stock of knowledge improves the entry distribution; and (ii) the pace of knowledge accumulation, which, in my semi-endogenous growth model, is the population growth rate.

Second, the model formalizes how dynamics of product exit are informative about this wedge, even with idiosyncratic shocks as another driver of product turnover. The key is that a firm shuts down a product when its market share becomes too small to justify incurring the labor-denominated overhead cost. Stronger spillovers catalyze this process: by speeding improvements across generations of products, they quicken the substitution toward newer ones, accelerating the gradual erosion of an incumbent product's market share. It is this gradual component of product exit, due to downward drift toward the exit threshold, that is informative about spillovers.

Third, the model overcomes the scarcity of product data by providing a way to infer this gradual component of product exit from widely available data on firm exit by age. This is possible because product dynamics aggregate to firm dynamics: a firm starts with a single product, attempts to grow its portfolio via R&D, and exits upon shutdown of its final product. As a result, the model's profile of firm exit by age is governed by three sufficient statistics: (i) the extent of product exit due to downward drift (the component informative about spillovers), (ii) the extent of product exit due to shocks, and (iii) the endogenous rate at which an incumbent firm adds a product to its portfolio per existing product. These are separately identified because they leave distinct signatures on the firm exit hazard over its life cycle: (i) a higher product exit due to downward drift raises firm exit at all ages, (ii) a higher product exit due to shocks increases exit among young firms but, due to selection, decreases it among older ones, and (iii) a higher rate of product addition has little effect on young single-product firms but lowers the exit for mature firms by increasing their number of products.

Before using the model to quantify spillovers, I provide supporting evidence for the key model features underpinning the sufficient statistic result just described. Specifically, I document five facts using the NielsenIQ Retail Scanner dataset, which provides barcode-level data for consumer packaged goods. First, products with higher sales are less likely to exit. Second, the product exit rate is lower among firms with more products. Taken together, these first two facts corroborate negative selection into product exit. Third, the rate of new product introductions per existing product shows

no systematic variation with the firm's number of products—consistent with my model's constant product addition rate per existing product. Fourth, prior to exit, a product's sales decline gradually. Fifth, this gradual decline is driven by a collapse in quantity sold while relative price falls only modestly, a pattern consistent with a negative residual demand shock to the product. These last two facts provide support for gradual erosion of a product's appeal preceding its exit, as in my model.

To quantify the model, I use firm-level data covering the universe of U.S. private nonfarm employer businesses. Targeting the profile of firm exit at ages 1 through 19, I estimate that the extent of product exit due to downward drift toward the exit threshold is 16 pp. The model requires such a large estimate to match the non-trivial exit rate observed among older firms: a firm exits upon shutting down its final product, and while such product exits could result from the accumulation of bad idiosyncratic shocks, firms that survive to old age have positively selected products, making the sequence of shocks needed to push their products to the exit threshold extremely unlikely. This establishes the need for downward drift in a product's quality relative to the exit threshold, which results from this threshold itself rising over time. Improvements in the entry distribution are then necessary to match the fact that, despite this rising exit threshold, there has been sustained entry of new firms with the average number of workers per new firm remarkably stable over time. Without such spillovers, new firms would enter progressively further behind the average incumbent and hire fewer and fewer workers. Quantitatively, these spillovers drive a 16 pp wedge between social and private rates of return to innovation along the laissez-faire BGP.

I bolster confidence in my results through a series of exercises. Quantitatively, the model fits several untargeted moments well, including features of the firm size distribution, average firm employment by age, aggregate productivity growth, and the share of aggregate growth attributable to incumbent firms. Qualitatively, I leverage cross-sector variation to provide supporting evidence. First, I document a positive correlation across sectors between firm exit rates and productivity growth, a necessary condition for firm dynamics to be informative about the magnitude of spillovers. Second, when I re-estimate the model sector-by-sector, the estimated product exit due to downward drift tends to be larger in sectors where narrative evidence points to an important role for spillovers.

Finally, I assess the sensitivity of my results to relaxing some assumptions I make in my baseline quantification. First, I entertain the possibility that, in addition to the rising exit threshold, drift in an incumbent product's *relative* quality may also be driven by drift in the *level* of its quality. Any positive such drift, reflecting for example innovation on incumbent products or learning by doing, makes my baseline results biased downward. Intuitively, if incumbent products are themselves improving over time, rationalizing the same rate of product exit due to gradual creative destruction by newer products requires the entry distribution to be improving even more rapidly. In contrast, my baseline results are biased upward if the level of a product's quality drifts down over time, which could for instance reflect taste for novelty. Entertaining a range of values for this drift, informed by

existing estimates in the literature and different calibration strategies, I still find a sizable wedge between social and private returns to innovation, ranging from 11 to 28 pp. Second, I entertain the possibility that improvements in the entry distribution could potentially be due to growth in human capital. Specifically, I develop an extension with exogenous such human capital growth, and show that even in this case, the wedge of interest remains substantial at 12 pp.

The rest of the paper proceeds as follows. After a brief review of the literature, Section 1 lays out the model, derives the wedge between social and private rates of return to innovation under laissez-faire, and formalizes how dynamics of product exit are informative about this wedge. Section 2 then characterizes the model's *firm* dynamics, a necessary step to quantify this wedge using firm data. Section 3 uses product level data from the consumer packaged goods sector to corroborate my modeling of product dynamics. Section 4 quantifies the model using firm-level data covering the universe of U.S. private nonfarm employer businesses. Finally, Section 5 concludes.

Related Literature. My theoretical contribution is a growth model with multiproduct firms, selection into product exit, and idiosyncratic product-level demand shocks. As in Romer (1990), growth results from the creation of new products. Unlike standard expanding-variety models, my model also features quality growth and new products rendering incumbents obsolete. This relates my work to the large Schumpeterian growth literature, building on Aghion and Howitt (1992) and Grossman and Helpman (1991), and reviewed in Aghion, Akcigit and Howitt (2014). In contrast to standard quality ladder models, this obsolescence unfolds gradually here, because old and new products are imperfect substitutes—as in Caballero and Jaffe (1993), with the caveat that products eventually exit in my setting because of the labor denominated overhead cost (Hopenhayn, 1992).

My modeling of multiproduct firms builds on Klette and Kortum (2004) and Luttmer (2011): a firm is a collection of products, each of which gives "birth" to a new product at a constant Poisson rate in equilibrium.[2] Motivated by product-level evidence I document, a key departure I take is modeling product exit as an optimal shutdown decision by the firm, rather than a Poisson shock. Argente, Lee and Moreira (2024) and Berlingieri, De Ridder, Lashkari and Rigo (2025) similarly use product-level data to discipline growth models with multiproduct firms, but do not focus on how the hazard rate of exit varies across products. Acemoglu, Akcigit, Alp, Bloom and Kerr (2018) features similar endogenous product shutdown in a setting with multiproduct firms, while also entertaining heterogeneity in firm types and skilled versus unskilled labor, but in a quality ladder model with a fixed measure of products. In contrast, I allow for growth in the number of products and firms to match a positive firm net entry rate, use the Kortum (1997) entry distribution rather than quality ladders to nest the null of no intertemporal knowledge spillovers, and entertain idiosyncratic product-level Brownian shocks to generate product turnover unrelated to innovation.

[2] In Section 4, I explain that, despite this feature, my model is still consistent with the arrival rate of a new product per worker employed declining with firm size, in line with evidence documented in Akcigit and Kerr (2018).

My modeling of product dynamics in a setting with growth builds on Luttmer (2007). A first notable difference is that I allow for multiproduct firms. In addition to the evidence on their ubiquity (Bernard, Redding and Schott, 2010; Broda and Weinstein, 2010), this leads to lower volatility of firm growth among larger firms (Arkolakis, 2016). The second is that Luttmer (2007) models entrants drawing from the incumbent distribution while I follow Kortum (1997) in having them draw from a potentially endogenously shifting distribution.[3] In addition to allowing me to nest the null of no intertemporal knowledge spillovers, this leads to our models yielding diverging comparative statics: in my model, higher volatility of shocks does not increase growth, product entry and exit *rates* are not affected by the cost of entry, and the resulting growth rate is semi-endogenous.

The link from spillovers to dynamics in product markets is not unique to my model. Akcigit and Ates (2023) uses a step-by-step innovation model—à la Aghion, Harris, Howitt and Vickers (2001)—with Bertrand duopoly in each sector to argue that declining knowledge diffusion from frontier to laggard firms can quantitatively explain a large share of the recent slowdown in business dynamism in the U.S. (Decker, Haltiwanger, Jarmin and Miranda, 2016; Akcigit and Ates, 2021).

In using the quantified model to shed light on the magnitude of intertemporal knowledge spillovers, I also relate to a large body of innovation research, reviewed in Bryan and Williams (2021). Closest in spirit is Jones and Summers (2021), which argues that, through the lens of endogenous growth theory, the growth rate of TFP divided by the share of GDP spent on R&D provides a measure of the *average* social rate of return to R&D (in a cost-benefit sense). In contrast, I focus on the gap between *marginal* social and private rates of returns arising due to spillovers and do not need to take a stance on the share of resources spent on R&D. Such macro approaches complement existing approaches to quantifying spillovers which typically rely on patents. Following Jaffe, Trajtenberg and Henderson (1993), many have used patent citations as a paper trail for knowledge flows. However, Jaffe, Trajtenberg and Fogarty (2000) uses survey evidence on the familiarity of inventors with patents they cite to conclude that patent citations are at best a noisy signal of spillovers. In fact, among U.S. patents granted between 2001 and 2003, Alcácer, Gittelman and Sampat (2009) finds that examiners added, on average, 63% of citations and all citations in 40% of the cases. An alternative approach consists of using patenting activity to define a measure of technological proximity between firms and then looking at the effect of a firm's R&D on R&D by its "technological neighbors" (Jaffe, 1986; Zacchia, 2019). Controlling for proximity in product markets and using changes in R&D tax credits as instruments, Bloom, Schankerman and Van Reenen (2013) estimates large spillovers among patenting publicly listed firms, leading to a gap between social and private marginal products of R&D of 20 to 45 p.p. Arqué-Castells and Spulber (2022) shows this gap remains large (30 p.p.) even when accounting for voluntary technology transfers between firms. My approach is complementary to these and motivated by the literature acknowledging the limitations

[3]Tying entry and incumbent distributions also features in Bilal et al. (2021) and Ottonello and Winberry (2024).

of patent and R&D data. First, the propensity to patent an invention varies across industries (Levin, Klevorick, Nelson and Winter, 1987). Second, even within sectors where patenting is common, larger firms are more likely to patent an innovation (Cohen, Nelson and Walsh, 2000; Mezzanotti and Simcoe, 2023; Argente, Baslandze, Hanley and Moreira, 2023). Third, patents capture a small share of the knowledge spilling out of research conducted in universities (Agrawal and Henderson, 2002). Fourth, firms may strategically relabel expenses as R&D (Chen, Liu, Suárez Serrato and Xu, 2021), with these incentives changing as the tax code evolves (Cowx, Lester and Nessa, 2024).

1 Model

This section lays out the model and derives two key results. The first characterizes the wedge between social and private returns to innovation created by knowledge spillovers under laissez-faire. The second formalizes how the dynamics of product exit are informative about the magnitude of this wedge. Appendix A keeps track of all symbols I introduce when setting up and solving the model.

1.1 Economic Environment

Time is continuous and population N_t grows at rate $\eta > 0$:

$$\dot{N}_t = \eta N_t \ . \tag{1}$$

Each individual inelastically supplies a unit of labor and household preferences are given by

$$U_0 = \int_0^\infty e^{-\rho t} N_t \frac{c_t^{1-\gamma} - 1}{1-\gamma} dt \ ,$$

where consumption c_t is a CES aggregate over a continuum of imperfectly substitutable products

$$c_t = \left[\int_{p \in \Omega_t} \left(Q_{pt} c_{pt} \right)^{\frac{\sigma-1}{\sigma}} dp \right]^{\frac{\sigma}{\sigma-1}} \ .$$

Here, Q_{pt} is the quality of product p at t, c_{pt} its quantity consumed, $\sigma > 1$ the elasticity of substitution, and Ω_t the set of products at t, which evolves over time through creation of new products and exit of existing ones. Each product is supplied by one, potentially multiproduct, firm and its production requires incurring a fixed overhead cost $\mathcal{F}$, denoted in labor units. If unpaid, the product is irreversibly shut down. Otherwise, the firm produces it using production labor L_{pt}:

$$Y_{pt} = A L_{pt} \ .$$

For a product to be supplied, the underlying blueprint must have already been developed. Such innovations are carried out by entering as well as incumbent firms.

All firms begin as single product firms. Motivated by the findings of Klenow and Li (2025),

there is a labor-denominated firm entry cost. Specifically, an individual attempting to set up a new firm—by developing the blueprint for a new product—succeeds at Poisson rate ε. Subsequently, this single product firm can expand its portfolio through R&D, but these investments run into diminishing returns as hiring I_p R&D workers leads to the arrival of a new blueprint at Poisson rate:

$$\iota\left(I_p\right) = \vartheta \frac{I_p^{1-\delta}}{1-\delta} \quad \text{where} \quad \delta \in (0,1)\ ;\ \vartheta > 0. \tag{2}$$

Once the firm becomes multiproduct, it operates as a collection of product lines. As with production, innovation is organized around existing blueprints: for each product p in its portfolio, a firm chooses how many R&D workers I_p to hire. Using the superposition property of the Poisson process, a firm f currently producing the set of products $\{p_1, \ldots, p_n\}$ succeeds in adding a product to its portfolio with Poisson arrival rate

$$\sum_{p \in \{p_1, \ldots, p_n\}} \iota\left(I_p\right) \ .$$

Equivalently, one could write down, as in Klette and Kortum (2004), a firm-level idea production function with constant returns to scale in firm-wide R&D labor and number of products. The product-level formulation allows me to defer the introduction of firm level variables until Section 2.

Knowledge spillovers, which I seek to quantify, stem from the cumulative stock of innovation being an input into the production of new ideas. With innovation taking the form of product development, the cumulative stock of innovation is the mass of product blueprints already developed, which I denote by K_t. Its law of motion is:

$$\dot{K}_t = \varepsilon S_t + \int_{p \in \Omega_t} \iota\left(I_{pt}\right) dp \ . \tag{3}$$

The first term reflects the flow of new blueprints developed by entrants, as S_t is the mass of individuals working as startup entrepreneurs and ε is the Poisson rate an entrepreneur succeeds. The second term reflects the flow of new blueprints developed by incumbents.

Whether developed by an entrant or incumbent, the quality of a new product at t is drawn from an entry distribution with complementary cumulative distribution function (CCDF):

$$\overline{F}_t^E(Q) = \Pr\left(\text{Draw}_t > Q\right) = \begin{cases} K_t^{\theta} Q^{-\alpha} & \text{if} \quad Q > K_t^{\frac{\theta}{\alpha}} \\ 1 & \text{otherwise} \end{cases} \qquad \text{with} \quad \alpha > 0 \ \text{ and } \theta \geq 0.$$

After this initial draw, the product's quality Q_{pt} evolves as a geometric Brownian motion with increments that are independent across the continuum of products:

$$d \ln Q_{pt} = \beta dt + \nu dB_{pt} \ .$$

In this specification of the entry distribution, due to Kortum (1997), α is the time-invariant Pareto

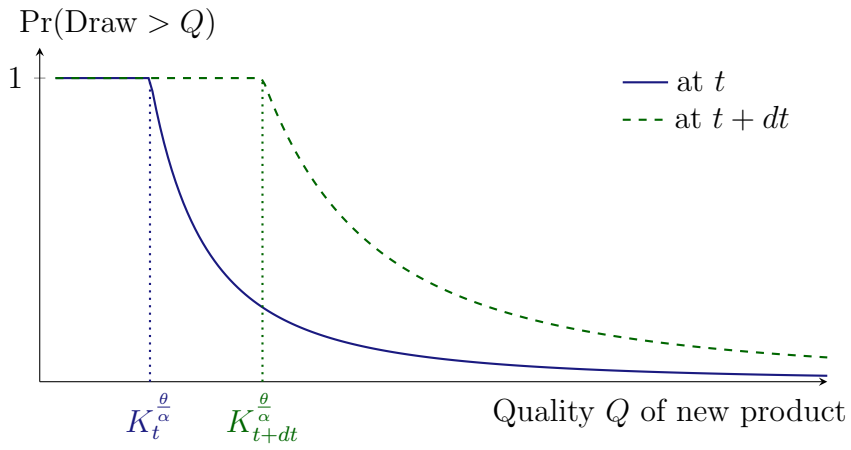


Figure 1: Improvement in entry distribution due to spillovers when $\theta > 0$

shape of the entry distribution, with a higher value for α corresponding to a thinner tail. θ is the spillover elasticity: it governs how strongly a higher cumulative stock of innovation improves—in a first-order stochastic dominance sense—the distribution of quality for new products. For example, when there are no spillovers, $\theta = 0$ and the entry distribution is time-invariant. In contrast, when $\theta > 0$, the flow of new blueprints today increases K_t, which in turn improves the entry distribution inventors draw from in the future, as shown in Figure 1.

In terms of resource constraints, each product's aggregate consumption cannot exceed production:

$$N_t c_{pt} \leq Y_{pt} \ .$$

Each individual inelastically supplies a unit of labor and can work for an incumbent as a production, overhead, or R&D worker. Alternatively, the individual can choose to be a startup entrepreneur and attempt to create a new firm. The resource constraint on labor is thus given by:

$$\int_{p \in \Omega_t} \left(L_{pt} + \mathcal{F} + I_{pt} \right) dp + S_t = N_t \ . \tag{4}$$

Table 1 summarizes the model's economic environment. The allocation decision is how to divide labor among its competing uses, which governs how the set of products Ω_t consumed by households evolves over time. To appreciate the economic trade-offs involved, consider the benefit of a marginal unit of labor in each of its uses. Allocating more labor to production increases the output of products currently supplied. Allocating more labor to overhead expands the range of products that remain active. Because of the love of variety effect, both of these margins enhance current consumption. In contrast, allocating more labor to R&D or entry expands the stock of blueprints, thereby increasing the number and quality of products available for future consumption.

Discussion. I focus on product quality, rather than process efficiency, due to the findings of Argente et al. (2024) and Fact 5 in Section 3. This only affects *how* new products displace incumbents, but

Table 1: Economic Environment

Population	$\dot{N}_t = \eta N_t$	$\eta > 0$
Preferences	$U_0 = \int_0^\infty e^{-\rho t} N_t \frac{c_t^{1-\gamma}-1}{1-\gamma} dt$	$\rho > \eta$
Consumption	$c_t = \left[\int_{p\in\Omega_t} \left(Q_{pt} c_{pt}\right)^{\frac{\sigma-1}{\sigma}} dp\right]^{\frac{\sigma}{\sigma-1}}$	$\sigma > 1$
Production	Overhead $\mathcal{F}$ (exit if unpaid)	$\mathcal{F} > 0$
	$Y_{pt} = A L_{pt}$	
Flow of new blueprints	$\dot{K}_t = \varepsilon S_t + \int_{p\in\Omega_t} \frac{\vartheta}{1-\delta} I_{pt}^{1-\delta} dp$	$\vartheta > 0\,;\, 0 < \delta < 1$
Entry distribution	$\overline{F}_t^E(Q) = K_t^\theta Q^{-\alpha}$	$\theta \geq 0\,;\, \alpha > 0$
Set of products	Ω_t evolves through entry & exit	
Quality evolution	$d\ln Q_{pt} = \beta dt + \nu dB_{pt}$	$\nu \geq 0$
Resource constraints	$N_t c_{pt} = Y_{pt}$	
	$S_t + \int_{p\in\Omega_t} \left(L_{pt} + \mathcal{F} + I_{pt}\right) dp = N_t$	

Notes: The scarce resource to allocate in this economy is labor. The decision of how to divide this resource among its competing uses in production (L), overhead ($\mathcal{F}$), incumbent innovation (I), and entry (S) governs how the the set of products Ω_t consumed evolves over time.

not the pace of this process, which, as will become clear, is what matters for my quantitative exercise. My use of the Kortum (1997) entry distribution—rather than quality ladders (Acemoglu et al., 2018) or the incumbent distribution (Luttmer, 2007)—allows me to nest the null of no intertemporal knowledge spillovers ($\theta = 0$). In that case, my model is a one-country Melitz (2003) with product and firm turnover due to the Brownian shocks and growth due to expanding varieties. Finally, my modeling of spillovers as affecting the distribution of quality conditional on getting a draw, rather than the arrival rate of a draw, aligns with evidence on entry costs from Klenow and Li (2025).

1.2 Decision Problems

To pin down how labor is allocated across its competing uses, I consider the following market structure. Each differentiated product is supplied by a single, potentially multiproduct, monopolistically competitive firm. There are no barriers to firm entry and the frictionless labor market is perfectly competitive. The consumption bundle serves as the numeraire so its price is normalized to unity. In terms of assets, in addition to the blueprints which entitle their owners to a stream of dividends, there is a risk-free bond in zero-net supply.

I characterize equilibrium labor allocations and the evolution of the product set Ω_t without explicitly tracking firms. This is possible because a firm's problem is separable across its products.

On the technology side, there are no production or innovation synergies across a firm's portfolio. On the demand side, firms are atomistic and the elasticity of substitution is constant across products, so there are no cannibalization effects. While well defined, boundaries of the firm only matter for firm-level outcomes, which I return to in Section 2.

Household's problem. Given paths of interest rates r_t, wages w_t, and product prices P_{pt}, the household chooses consumption of products p to maximize utility subject to their budget constraint:

$$\max_{\{c_{pt},a_t\}} \int_0^\infty e^{-\rho t} N_t \frac{c_t^{1-\gamma}-1}{1-\gamma} dt \qquad \text{subject to} \qquad c_t = \left[\int_{p\in\Omega_t} \left(Q_{pt}c_{pt}\right)^{\frac{\sigma-1}{\sigma}} dp\right]^{\frac{\sigma}{\sigma-1}}$$
$$\dot{a}_t = (r_t - \eta)\, a_t + w_t - \int_{p\in\Omega_t} P_{pt}c_{pt}dp \ . \tag{5}$$

Incumbent's problem. For each product in its portfolio, a firm solves an optimal stopping time problem in which it chooses pricing, production, and R&D to maximize the expected present discounted value (PDV) of dividends from that product. These consist of profits from operating the product and an option value of adding a new product to the firm's portfolio through R&D, and accrue until the firm chooses to irreversibly shut down production with this product.

$$V_t(Q_{pt}) = \max_{\substack{T \\ \{P_{pt},I_{pt},L_{pt}\}}} \mathbb{E}_t \left\{ \int_t^T e^{-\int_t^\tau r_s ds} \left[\underbrace{P_{p\tau}Y_{p\tau} - w_\tau\left(\mathcal{F} + L_{p\tau}\right)}_{\text{operating profit}} + \overbrace{\underbrace{\frac{\vartheta}{1-\delta}}_{\substack{\text{arrival rate of} \\ \text{new product}}} \underbrace{\int V_\tau(Q)\, dF_t^E(Q)}_{\substack{\text{expected value} \\ \text{of new product}}} - w_\tau I_{p\tau}}^{\text{option value}} \right] d\tau \right\}$$
$$\text{subject to, for all } \ t \le \tau < T, \ \begin{cases} Y_{p\tau} = AL_{p\tau} \\ Y_{p\tau} = Q_{p\tau}^{\sigma-1} P_{p\tau}^{-\sigma} N_\tau c_\tau \\ d\ln Q_{p\tau} = \beta d\tau + \nu dB_{p\tau} \end{cases} \tag{6}$$

The value of a firm is then obtained by summing the value of products in its portfolio. For endogenous product exit to occur, the option value of R&D expansion must not exceed the overhead cost. In essence, the effective fixed cost is the overhead net of the option value, and it must be strictly positive for some products to be optimally shut down. Assumption 1 below provides a condition on parameters that guarantees this holds in equilibrium. As in Hopenhayn (1992), endogenous exit is then characterized by a quality threshold, denoted $\underline{Q}_t$, below which firms stop supplying a product.

Firm Entry. Since there are no barriers to entry, individuals must be indifferent between employment and entrepreneurship in equilibrium. With the wage as the outside option, this requires:

$$S_t \left(w_t - \varepsilon \int V_t(Q) dF_t^E(Q) \right) = 0 \ . \tag{7}$$

Along an interior equilibrium ($S_t > 0$), the expected payoff from attempting entry equals the wage.

1.3 Equilibrium

The model's dynamics center on the evolution of the product set Ω_t. Since products differ in quality, this set can be represented by a distribution over product qualities. With stationarity in mind, it is helpful to work with a product's log-quality relative to the exit threshold as its state variable:

$$q_{pt} = \ln\left(\frac{Q_{pt}}{\underline{Q}_t}\right) .$$

I denote the corresponding cross-sectional measure by $m(q,t)$. Using Ito's lemma,

$$dq_{pt} = \left(\beta - g_{Q_t}\right) dt + \nu dB_{pt} , \tag{8}$$

where g_{Q_t} is the instantaneous growth rate of $\underline{Q}_t$ at t. It follows that $m(0,t) = 0$ and for $q > 0$ the law of motion for $m(q,t)$ is given by a Kolmogorov Forward Equation (KFE):

$$\dot{m}(q,t) = \underbrace{-\left(\beta - g_{Q_t}\right)\frac{\partial m(q,t)}{\partial q}}_{\text{Drift}} + \underbrace{\frac{\nu^2}{2}\frac{\partial^2 m(q,t)}{\partial q^2}}_{\text{Diffusion}} + \underbrace{\dot{K}_t K_t^{\theta}\underline{Q}_t^{-\alpha}\alpha e^{-\alpha q}\,\mathbb{1}_{\left\{q \geq \frac{\theta}{\alpha}\ln K_t - \ln \underline{Q}_t\right\}}}_{\text{Entry}} . \tag{9}$$

Here, the measure of products entering with log-relative quality q is simply the flow of new blueprints $\dot{K}_t$ multiplied by the density of draws at that relative quality.

Definition 1. Given initial population N_0, stock of existing blueprints K_0, and distribution of product log-relative qualities $m(q,0)$, an equilibrium consists of time paths for N_t, K_t, $m(q,t)$, $\underline{Q}_t$, prices $\{r_t, w_t, P_{pt}, V_t(Q_{pt})\}$ and allocations $\{c_t, a_t, c_{pt}, Y_{pt}, L_{pt}, I_{pt}, S_t\}$ such that for all t:

1. c_t, a_t, and c_{pt} solve the household's problem (5)
2. L_{pt}, I_{pt}, P_{pt}, $V_t\left(Q_{pt}\right)$, and $\underline{Q}_t$ solve the incumbent's problem (6)
3. S_t satisfies the free entry condition (7)
4. Y_{pt} satisfies the market clearing condition for product p, $Y_{pt} = N_t\, c_{pt}$
5. w_t clears the labor market (4)
6. r_t clears the asset market, $N_t a_t = \int V_t\left(\underline{Q}_t e^q\right) m(q,t) dq$
7. N_t, K_t, and $m(q,t)$ evolve respectively according to equations (1), (3), and (9).

Solving. In equilibrium, consumption per capita and the wage rate are then given by

$$c_t = A M_t^{\frac{1}{\sigma-1}} \overline{Q}_t \frac{L_t}{N_t} \qquad \text{and} \qquad w_t = \frac{\sigma - 1}{\sigma} A M_t^{\frac{1}{\sigma-1}} \overline{Q}_t , \tag{10}$$

where L_t is production labor, M_t the measure of products supplied, and $\overline{Q}_t$ their average quality:

$$M_t \equiv \int_0^\infty m(q,t)dq \text{ and } \overline{Q}_t \equiv \left(\frac{1}{M_t}\int_{p\in\Omega_t} Q_{pt}^{\sigma-1}dp\right)^{\frac{1}{\sigma-1}} = \underline{Q}_t\left(\frac{1}{M_t}\int_0^\infty e^{(\sigma-1)q}m(q,t)dq\right)^{\frac{1}{\sigma-1}} .$$

Equation 10 highlights the two potential sources of productivity growth: expanding varieties, which contributes to growth due to love of variety, and increasing average quality of products consumed.

Moving on to the model's dynamics, note that the innovation technology in (2) satisfies an Inada condition at 0, so that an incumbent always finds it optimal to do some R&D. Along an interior equilibrium ($S_t > 0$), this level of incumbent innovation per product is constant:

$$\left.\begin{aligned} \text{Incumbent's FOC:}\quad & w_t = \vartheta I_{pt}^{-\delta}\int V_t(Q)\,\mathrm{d}F_t^E(Q) \\ \text{Free entry condition:}\quad & w_t = \varepsilon\int V_t(Q)\,\mathrm{d}F_t^E(Q)\end{aligned}\right\} \implies I_{pt} = I \equiv \left(\frac{\vartheta}{\varepsilon}\right)^{1/\delta} . \tag{11}$$

Thus the flow of new blueprints at t is simply

$$\dot{K}_t = \varepsilon S_t + (1-\delta)^{-1}\varepsilon I M_t . \tag{12}$$

The scaling by $(1-\delta)^{-1} > 1$ reflects that, while the marginal R&D worker has to be as productive as a startup entrepreneur in equilibrium, the infra-marginal ones will be more productive. Plugging the equilibrium level of incumbent R&D back into the firm's objective function yields an option value from R&D expansion equal to Ow_t where $O \equiv \delta(1-\delta)^{-1}(\vartheta/\varepsilon)^{1/\delta}$.

Assumption 1. Endogenous product exit requires $\mathcal{F} > \delta(1-\delta)^{-1}(\vartheta/\varepsilon)^{1/\delta}$.

Finally, denoting by $V(q,t) \equiv V_t(\underline{Q}_t e^q)$ the value at t of a product with log-relative quality q, this value function satisfies the following Hamilton–Jacobi–Bellman (HJB) equation:

$$V(0,t) = \frac{\partial V(0,t)}{\partial q} = 0\ ;\ \forall q > 0,\ r_t V(q,t) = w_t\left[\frac{1}{\sigma-1}\frac{L_t}{M_t}\left(\frac{\underline{Q}_t}{\overline{Q}_t}\right)^{\sigma-1} e^{(\sigma-1)q} - (\mathcal{F}-O)\right] + \dot{V}(q,t) + \left(\beta - g_{Q_t}\right)\frac{\partial V(q,t)}{\partial q} + \frac{\nu^2}{2}\frac{\partial^2 V(q,t)}{\partial q^2} . \tag{13}$$

1.4 Balanced Growth Path

Definition 2. A balanced growth path (BGP) is an allocation with

1. stationary labor allocations: $\dot{L}_t = \eta L_t$, $\dot{M}_t = \eta M_t$, and $\dot{S}_t = \eta S_t$;
2. growth in the quality threshold $\underline{Q}_t$ at constant rate $g_Q \geq 0$;
3. a stationary distribution of relative qualities: $m(q,t) = f_p(q)M_t$, with f_p a probability density.

Proposition 1. Along a BGP,

$$\frac{\dot{K}_t}{K_t} = \eta \quad , \quad g_Q = \frac{\theta}{\alpha}\eta \quad \text{and} \quad g \equiv \frac{\dot{c}_t}{c_t} = \underbrace{\frac{\eta}{\sigma - 1}}_{\text{Variety}} + \underbrace{\frac{\theta}{\alpha}\eta}_{\text{Quality}} .$$

Proof. All derivations and proofs are in Appendix B. □

Assumption 2. Finite lifetime utility and firm value require

$$\rho > \eta + (1-\gamma)g \quad , \quad (\sigma - 1)\left(\beta - g_Q\right) + \frac{1}{2}(\sigma - 1)^2 \nu^2 < \eta \quad \text{and} \quad \alpha > \sigma - 1 .$$

To build intuition for the rate of quality growth g_Q, notice first that it does not depend on the drift β and volatility ν. While these affect growth of an individual incumbent product, they do not contribute to aggregate growth along a path with a stationary distribution of relative quality. Intuitively, the reason is that the quality gains experienced by a product are "lost" once it exits.

It is also noteworthy that when $\theta = 0$, there is no quality growth along the BGP. In this case, the entry distribution is time invariant, and along a stationary equilibrium the distribution of incumbent quality (in levels) is itself time-invariant, as in Hopenhayn (1992) and Melitz (2003). This underscores that the driver of quality growth in this economy is the improvement in the entry distribution. When $\theta > 0$, this improvement happens as a result of the endogenous accumulation of knowledge (blueprints). In this case, as shown in Figure 1, exponential growth in the stock of blueprints K_t at rate η shifts the entry distribution to the right at rate $\frac{\theta}{\alpha}\eta$.

Sustained quality growth thus requires a growing population and positive spillovers. The first is common in semi-endogenous growth models (Jones, 1995, 2022): if new ideas drive growth and individuals come up with ideas, then the economy's growth rate is tied to the rate of population growth. That $\theta > 0$ is necessary to sustain quality growth in this model stands in contrast to Kortum (1997). Key to understanding the difference is that, while these two semi-endogenous growth models share the same entry distribution, the mass of products in Kortum (1997) is exogenously fixed as new products perfectly substitute for older ones. As a result, what matters for the rate of quality growth in that setup is the maximum draw. Even if the distribution is time invariant, Kortum (1997) shows that exponential growth in the number of draws (due to population growth) from a Pareto distribution leads to exponential growth in the maximum draw. In contrast, what matters for the rate of quality growth in my setting is the average draw from the entry distribution, and for that to grow over time, the entry distribution has to shift out, which requires $\theta > 0$. While indicative about the *presence* of spillovers ($\theta > 0$), quality growth alone is not informative about their *magnitude*. Rapid quality growth can reflect strong spillovers (high θ) or a thick-tailed entry distribution (low α). In addition to this identification challenge, quality growth is notoriously challenging to measure, see for example Bils (2009). This is why, to quantify the model, I leverage its predictions regarding product and firm dynamics.

Assumption 3. By Proposition 1, the exit threshold $\underline{Q}_t$ and the minimum draw from the entry distribution $K_t^{\theta/\alpha}$ grow at the same rate. I focus on the case $\underline{Q}_t \geq K_t^{\theta/\alpha}$, which obtains if the PDV of the effective fixed cost is not too small relative to the entry cost (see Appendix B4).

Proposition 2. The stationary distribution of product log-relative quality is

$$f_p(q) = \frac{\alpha\zeta}{\zeta - \alpha}\left(e^{-\alpha q} - e^{-\zeta q}\right) \quad \text{where} \quad \zeta \equiv \frac{g_Q - \beta + \sqrt{(g_Q - \beta)^2 + 2\eta\nu^2}}{\nu^2},$$

and the stationary product entry and exit rates are respectively

$$\frac{E_t}{M_t} = \eta + \frac{\nu^2}{2}\alpha\zeta \quad ; \quad \frac{D_t}{M_t} = \frac{\nu^2}{2}\alpha\zeta = \frac{1}{2}\alpha\left(g_Q - \beta\right) + \frac{1}{2}\sqrt{\left(\alpha\left(g_Q - \beta\right)\right)^2 + 2\eta\alpha^2\nu^2}.$$

It follows that the incumbent quality distribution (in levels) has a Pareto tail with index $\min\{\alpha, \zeta\}$. While α is inherited from the entry distribution, ζ is the Luttmer (2007) tail that arises endogenously as a result of the geometric Brownian motion. This Pareto tail reflects the positive selection of products that have accumulated favorable Brownian shocks over time. As such, it is unsurprising that higher volatility ν makes this tail thicker (smaller ζ). In contrast, faster population growth (η) or higher growth in entrant's quality relative to incumbents ($g_Q - \beta$) make this endogenous tail thinner (larger ζ). Intuitively, both of these forces increase the share of products that are young and which haven't had enough time to accumulate as many favorable Brownian shocks.

Note that, as in Hopenhayn (1992) and the version of Luttmer (2007) with exogenous growth, stationary product entry and exit rates do not depend on the cost of entry (ε, ϑ, and δ). Intuitively, "cheaper" entry makes innovation more appealing at all times, equally raising the flow of entry and the measure of products, as the latter reflects cumulative past entry.

Proposition 3. Along the BGP, the value of a product with log-relative quality q is

$$V(q,t) = w_t\mathcal{V}(q) \quad \text{where} \quad \mathcal{V}(q) \equiv \frac{\mathcal{F} - O}{r - g}\left[\frac{\xi}{\xi + \sigma - 1}e^{(\sigma-1)q} + \frac{\sigma - 1}{\xi + \sigma - 1}e^{-\xi q} - 1\right]$$
$$\xi \equiv \frac{\beta - g_Q + \sqrt{(\beta - g_Q)^2 + 2\nu^2(r - g)}}{\nu^2},$$

and average production labor per product is

$$\frac{L_t}{M_t} = (\mathcal{F} - O)\,\frac{\alpha\zeta(\sigma - 1)}{(\alpha - (\sigma - 1))(\zeta - (\sigma - 1))}\,\frac{\xi}{\xi + \sigma - 1}\,\frac{r - [g + (\sigma - 1)(\beta - g_Q) + \frac{\nu^2}{2}(\sigma - 1)^2]}{r - g}.$$

With both L_t/M_t and I_t pinned down, one more equation is needed to fully characterize the stationary labor allocations. It follows from the free entry condition:

$$\varepsilon K_t^\theta \underline{Q}_t^{-\alpha}\int_0^\infty V(q,t)\alpha e^{-\alpha q}dq = w_t \overset{\textit{Proposition 3}}{\Longrightarrow} \varepsilon K_t^\theta \underline{Q}_t^{-\alpha}\int_0^\infty \mathcal{V}(q)\alpha e^{-\alpha q}dq = 1. \tag{14}$$

Intuitively, ε is the arrival rate of a new blueprint, $K_t^\theta \underline{Q}_t^{-\alpha}$ is the probability that a new blueprint leads to entry (quality draw is above threshold), and the integral gives the expected value of a new

product conditional on entry. As I show in Appendix B, this can be rewritten as

$$\frac{S_t}{M_t} + \frac{I}{1-\delta} = \left(\eta + \frac{\nu^2}{2}\alpha\zeta\right)\int_0^\infty \mathcal{V}(q)\alpha e^{-\alpha q}dq \ ;$$

which makes clear that the free entry condition pins down the level of innovation along the BGP.

1.5 Efficiency

Having characterized the stationary allocations arising in equilibrium, I now turn to assessing their efficiency. To do so, I solve for the first best, delegating the setup and solution to Appendix B. Given the model's semi-endogenous nature, the growth rates are as in Proposition 1 and the stationary distribution of product qualities as in Proposition 2, so any inefficiency has a level effect.

Proposition 4. If $\theta = 0$, the equilibrium BGP is efficient. In contrast, if $\theta > 0$, then

$$\left(\frac{L_t}{M_t}\right)^{\mathbf{DE}} = \left(\frac{L_t}{M_t}\right)^{\mathbf{FB}} \quad \text{and} \quad I^{\mathbf{DE}} = I^{\mathbf{FB}} \quad \text{but} \quad S_t^{\mathbf{DE}} < S_t^{\mathbf{FB}}\ ,\ M_t^{\mathbf{DE}} > M_t^{\mathbf{FB}}\ ,\ \text{and}\ L_t^{\mathbf{DE}} > L_t^{\mathbf{FB}}\ .$$

The divergence is due to differing free entry conditions, which I reproduce below:

$$\textbf{Equilibrium}: \quad 1 = \varepsilon K_t^\theta \underline{Q}_t^{-\alpha}\int_0^\infty \mathcal{V}(q)\alpha e^{-\alpha q}dq$$

$$\textbf{First Best}: \quad 1 = \left(1 + \frac{\theta\eta}{\rho + (\gamma-1)g}\right)\varepsilon K_t^\theta \underline{Q}_t^{-\alpha}\int_0^\infty \mathcal{V}(q)\alpha e^{-\alpha q}dq$$

Given $\eta > 0$, these conditions coincide if and only $\theta = 0$. Otherwise with $\theta > 0$, for a given K_t and $\underline{Q}_t$, the planner—internalizing intertemporal knowledge spillovers—values entry more. In fact, the additional term in the planner's free entry condition has a natural Pigouvian interpretation. The externality consists of any individual failing to internalize that their innovation raises the aggregate stock of knowledge K_t. Along the BGP, this stock grows at rate η, which improves the distribution future entrants draw from at rate $\theta\eta$. The denominator corresponds to taking the present discounted value (it is simply $r - g$), as the benefits of a higher stock of knowledge last forever. As a result of this externality, there is underprovision of innovation in the laissez-faire equilibrium.[4]

Unlike the textbook quality ladder model, spillovers are the only source of inefficiency here. While my model features business stealing as a negative externality, it also features uninternalized consumer surplus as a positive externality since new products imperfectly substitute for incumbents. With CES, the two exactly offset, as in Melitz (2003) (Dhingra and Morrow, 2019).

Governments support innovation through a wide array of policies (Bloom, Van Reenen and Williams, 2019). With that in mind, I focus on the wedge knowledge spillovers create between social and private rates of return under laissez-faire—rather than, for example, the optimal subsidy

[4]Along an interior BGP, this underprovision shows up entirely along the entry of new firm margin, and the per-product incumbent innovation is efficient. I discuss the underlying intuition in detail at end of Appendix B5.

rate. The definition of the social rate of return I use, due to Jones and Williams (1998), involves a variational argument around the BGP. Suppose the economy sacrifices some consumption at t in order to do devote more resources to R&D, then "eats the proceeds" of this investment at $t + dt$ by doing sufficiently less R&D to be back to the initial BGP by $t + 2dt$. The social rate of return is the rate of return on this variation, in terms of higher consumption at $t + dt$. The derivation in Appendix B.3 shows that doing this variation around the laissez-faire BGP yields:

$$\boxed{\text{Social Rate of Return to R\&D along Equilibrium BGP} = r + \theta\eta}$$

In contrast, by no-arbitrage, the private rate of return is r, so that $\theta\eta$ is the gap between social and privates rates of return to innovation along the laissez-faire BGP.

1.6 Knowledge Spillovers and Product Dynamics

Proposition 5. The hazard rate of a product exiting at age a is

$$d_p(a) \equiv \frac{\ell(a)}{1 - \int_0^a \ell(\tau)d\tau},$$

where $\ell(a)$, the density of a product's lifespan (i.e. age at exit), is given by:

$$\ell(a) = \exp\left(a\left(\frac{(\alpha\nu)^2}{2} - \alpha\left(g_Q - \beta\right)\right)\right)\left(\frac{\alpha\nu}{\sqrt{a}}\phi\left(z_a\right) - \left((\alpha\nu)^2 - \alpha\left(g_Q - \beta\right)\right)\Phi\left(-z_a\right)\right)$$

$$\text{with} \quad z_a \equiv \frac{\sqrt{a}}{\alpha\nu}\left((\alpha\nu)^2 - \alpha\left(g_Q - \beta\right)\right), \ \phi(.) \text{ and } \Phi(.) \text{ standard normal PDF and CDF.}$$

Proposition 5 shows that $\alpha\left(g_Q - \beta\right) = \theta\eta - \alpha\beta$ and $\alpha\nu$ are sufficient statistics for the profile of product exit by age. These capture the two sources of influx at the exit threshold, due to drift $\beta - g_Q$ and diffusion ν in relative quality q, scaled up by α, which governs how much density there is close to the exit threshold. Intuitively, $\theta\eta - \alpha\beta$ is the extent of product exit due to downward drift toward the exit threshold, while $\alpha\nu$ governs the extent of product exit due to idiosyncratic shocks.

By targeting the profile of product exit by age, these two statistics can be separately identified. The reason is that a higher $\theta\eta - \alpha\beta$ increases product exit rate at all ages, whereas a higher $\alpha\nu$ increases the product exit rate at young ages but decreases it at older ones. This non-monotonic effect of volatility on the exit rate by age is a consequence of selection: when volatility is higher, products that survive to old age are even more positively selected. It is the first of these two statistics that is informative about the wedge $\theta\eta$, and coincides precisely with it when there is no drift in the level of quality of an incumbent product ($\beta = 0$). The challenge this approach runs into is that product data are not widely available, so that my ultimate quantitative exercise has to rely on firm data. This requires characterizing the model's firm dynamics, which I turn to next.

Before doing so, it is important to emphasize that the downward drift which results in the gradual

component of product exit is not driven by growth in the wage (which happens at rate g rather g_Q). For example, when $\theta = 0$, there is still wage growth due to the expanding measure of varieties, but this has no bearing on product turnover. The reason is that, holding quantity sold Y_{pt} constant, wage growth increases expenditures on overhead but also operating revenues since $P_{pt} = \frac{\sigma}{\sigma-1}\frac{w_t}{A}$. This illustrates that the downward drift and the resulting gradual component of product exit are instead driven by incumbent products gradually losing market share to newer ones.

2 Model's Firm Dynamics

Through the lens of the model in Section 1, dynamics of product exit are informative about the magnitude of knowledge spillovers. The challenge, however, is that comprehensive datasets track the exit of firms rather than products. This section provides the necessary bridge, formalizing how, within the model, underlying product dynamics aggregate to observable firm dynamics along a BGP.

2.1 Firm's Life Cycle

A new firm f starts with a single product p. Its initial log-relative quality q_{pt} is drawn from $\text{Exp}(\alpha)$, then evolves according to (8) until it hits zero (p exits). While produced, p has I R&D workers associated with it (11). This R&D generates a new blueprint at Poisson rate $\iota(I)$, which leads to product entry if the draw is above the threshold $\underline{Q}_t$, so with probability $\overline{F}_t^E(\underline{Q}_t) = K_t^\theta \underline{Q}_t^{-\alpha}$ (constant along BGP, by Proposition 1). As a result, while produced, p "gives birth" to a new product at rate:

$$x \equiv \iota(I)\,\overline{F}_t^E(\underline{Q}_t) = \frac{\varepsilon}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{1/\delta} K_t^\theta \underline{Q}_t^{-\alpha} = \frac{1}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{1/\delta} \frac{(r-g)(\alpha+\xi)(\alpha-(\sigma-1))}{(\mathcal{F}-O)(\sigma-1)}\,. \tag{15}$$

The log-relative quality of this new product is again drawn from $\text{Exp}(\alpha)$, then evolves according to Equation 8 with independent Brownian increments across products; while active, this product in turn gives birth to a new one at Poisson rate x.

Accordingly, the number of products firm f produces at ages $a \geq 0$, $\{n_f(a)\}_{a\geq 0}$, is a branching process. At age 0, f starts with a single product, so $n_f(0) = 1$. Firm f exits once all its products have become obsolete, so its age at exit, or lifespan, is the smallest a such that $n_f(a) = 0$:

$$a_f = \inf\left\{a \geq 0 \mid n_f(a) = 0\right\}\,.$$

This is a random variable for two reasons: product births arrive as a Poisson process, and each product's lifespan is itself random. Denoting by Γ the cumulative distribution function (CDF) of a_f, the hazard rate of firm f exiting at an age a is:

$$d_f(a) \equiv \frac{\Gamma'(a)}{1-\Gamma(a)}\,.$$

Assumption 4. To guarantee that products and firms exit in finite time almost surely, I assume:

$$\beta \leq g_Q \quad \text{and} \quad 0 \leq x \leq \alpha \left(g_Q - \beta\right) \ .$$

Proposition 6. The CDF of a firm's lifespan satisfies $\Gamma(0) = 0$ and, for all $a > 0$,

$$\Gamma(a) = \int_0^a \ell(\tau) \exp\left(\int_0^\tau x \left[\Gamma(a-s) - 1\right] ds\right) d\tau \ ,$$

where $\ell(.)$ is the density of a product's lifespan, given in Proposition 5.

In conjunction with Proposition 5, this highlights that the model's profile of firm exit by age is governed by three sufficient statistics: the product addition rate per existing product x, the extent of product exit due to downward drift toward the exit threshold $\alpha(g_Q - \beta) = \theta\eta - \alpha\beta$, and the extent of product exit due to shocks $\alpha\nu$. Note that, computationally, solving for Γ only requires a single forward iteration on a fine age grid, as $\Gamma(a)$ only depends on $\{\Gamma(\tau) : \tau < a\}$ and $\Gamma(0) = 0$.

2.2 Evolution of the Firm Size Distribution

Having shed light on the life cycle of a single firm, I now turn to characterizing the evolution of the distribution of firms along the equilibrium BGP. Unlike Klette and Kortum (2004) and Luttmer (2011), only keeping track of a firm's number of products is insufficient. This is because a firm with n products becomes a firm with $n-1$ products when it endogenously shuts down production of one of its products, which happens when this product's log-relative quality hits zero. Accordingly, I need to keep track of a firm's portfolio of products. In this vein, define:

$$\mu_{nt}\left(q_1, \ldots, q_n\right) \equiv \text{ measure at } t \text{ of } n\text{-product firms with portfolio } q_1, \ldots, q_n \ .$$

Since μ_{nt} is the measure of n-product firms and a product exits once its relative quality hits zero, the support of μ_{nt} is the positive orthant $(0,\infty)^n$, and μ_{nt} vanishes on the boundary of this support. To avoid cumbersome notation, I denote by $\mathbf{q}$ the vector $(q_1, \ldots, q_n)$, with its dimension to be inferred from the function it is an argument for. Also, given $\mathbf{q} \in (0,\infty)^n$, $\mathbf{q}^{\setminus i} \in (0,\infty)^{n-1}$ is the vector $\mathbf{q}$ with its i^{th} entry removed and $\mathbf{q}^{i \leftarrow 0}$ the $(n+1)$ dimensional vector obtained by inserting a 0 into $\mathbf{q}$ at the i^{th} index. The evolution of the measure of firms with n products and portfolio $\mathbf{q}$ is then characterized by a system of coupled partial differential equations (PDEs) in $\{\mu_{nt}(\mathbf{q})\}_n$:

$$\begin{aligned}
\forall q > 0, \ \dot{\mu}_{1t}(q) \ &= \left(g_Q - \beta\right)\mu'_{1t}(q) \ + \ \frac{\nu^2}{2}\mu''_{1t}(q) && \text{Drift and diffusion} \\
&- \ x \ \mu_{1t}(q) && \text{Single product firm adds a product} \\
&+ \frac{\nu^2}{2}\partial_1\mu_{2t}(0,q) + \frac{\nu^2}{2}\partial_2\mu_{2t}(q,0) && \text{2-product firm shuts down a product} \\
&+ E_t^f \ \alpha \ e^{-\alpha q} && \text{Entry of new firm}
\end{aligned}$$

and $\forall n > 1,\ \forall \mathbf{q} \in (0,\infty)^n$

$$\begin{aligned}
\dot{\mu}_{nt}(\mathbf{q}) \;=\; & (g_Q - \beta) \sum_{i=1}^{n} \partial_i \mu_{nt}(\mathbf{q}) + \frac{\nu^2}{2} \sum_{i=1}^{n} \partial_i^2 \mu_{nt}(\mathbf{q}) && \text{Drift and diffusion} \\
& - n\, x\, \mu_{nt}(\mathbf{q}) && n\text{-product firm adds product} \\
& + \frac{\nu^2}{2} \sum_{i=1}^{n+1} \partial_i \mu_{(n+1)t}\left(\mathbf{q}^{i\leftarrow 0}\right) && (n+1)\text{-product firm loses a product} \\
& + (n-1)\, x\, \frac{1}{n} \sum_{i=1}^{n} \alpha e^{-\alpha q_i} \mu_{(n-1)t}\left(\mathbf{q}^{\setminus i}\right) && (n-1)\text{-product firm adds a product}
\end{aligned}$$

The first line in each of these two equations reflects the contributions of drift and diffusion in the log-relative quality q of a product, while the second records outflow resulting from the addition of a product to the firm's portfolio. The third lines capture inflow due to $(n+1)$-product firms shutting down one of their products. In fact, the instantaneous flow of $(n+1)$-product firms who shut down production of their i^{th} product *and* upon doing so become n-product firms with portfolio $\mathbf{q}$ is $\frac{\nu^2}{2}\partial_i \mu_{(n+1)t}\left(\mathbf{q}^{i\leftarrow 0}\right)$. To clarify notation, the second term is the partial derivative of $\mu_{(n+1)t}$ with respect to its i^{th} argument, evaluated at $\left(\mathbf{q}^{i\leftarrow 0}\right)$.The summation then reflects that any of the $n+1$ products could have been the one the firm chose to shut down.

The fourth line in the $n = 1$ case corresponds to the entry of new firms, which leads to inflow at q given by the total flow of entering firms E_t^f multiplied by the density of draws at that relative quality. For $n > 1$, it records inflow due to $n-1$-product firms becoming n-product firms, where any of the n products could be the newly added one.[5] Suppose for a second that the new product is $i = 1$. The measure of $(n-1)$-product firms with product qualities $(q_2, \ldots, q_n)$ is precisely $\mu_{(n-1)t}(\mathbf{q}^{\setminus 1})$. Such a firm adds a product to its portfolio at rate $(n-1)x$, and if successful, the relative quality of this new product is independent of the firm's portfolio and drawn from the density $\alpha e^{-\alpha q_1}$. The term on the fourth line of the law of motion for $n > 1$ is simply obtained by averaging such contributions over $i = 1, \ldots, n$ as each of these is equally likely to be the new product.

2.3 Stationary Firm Size Distribution

Along the BGP, the measure of products M_t grows at the rate of population growth η. A stationary firm size distribution then requires the measure of firms to grow at rate η as well.

Definition 3. Let Ψ_{nt} be the share of products at t held by firms with n products and $f_{nt}(\mathbf{q})$ the probability density function on $(0,\infty)^n$ of product qualities among n-product firms. Formally,

$$\Psi_{nt} \equiv \frac{1}{M_t}\, n \int_{\mathbf{q}\in(0,\infty)^n} \mu_{nt}(\mathbf{q})\; d\mathbf{q} \quad \text{and} \quad f_{nt}(\mathbf{q}) \equiv \frac{n\, \mu_{nt}(\mathbf{q})}{\Psi_{nt}\, M_t}\,.$$

[5] The firm's state variable is its *set* of products, as reflected by the PDE system satisfying permutation symmetry.

A stationary firm size distribution then corresponds to Ψ_{nt} and $f_{nt}(\mathbf{q})$ being time invariant:

$$\forall t, \Psi_{nt} = \Psi_n \text{ and } f_{nt}(\mathbf{q}) = f_n(\mathbf{q}) .$$

Proposition 7. A stationary firm size distribution consists of

- a sequence of non-negative numbers $\{\Psi_n\}_{n\geq 1}$ with $\sum_{n=1}^{\infty} \Psi_n = 1$,
- a sequence of functions $\{f_n\}_{n\geq 1}$, with f_n a probability density function on $(0,\infty)^n$,

such that $\{\Psi_n\}$ and $\{f_n\}$ jointly satisfy the following system of coupled PDEs:

$$(\eta+x) f_1(q) = \left(g_Q - \beta\right) f_1'(q) + \frac{\nu^2}{2} f_1''(q) + \frac{\eta + \frac{\nu^2}{2}\alpha\zeta - x}{\Psi_1} \alpha e^{-\alpha q} + \frac{\nu^2}{2}\frac{\Psi_2}{2\Psi_1}\left[\partial_1 f_2(0,q) + \partial_2 f_2(q,0)\right] ;$$

with the boundary condition $f_1(0) = 0$; and for $n > 1$,

$$\begin{aligned}\left(\frac{\eta}{n} + x\right) f_n(\mathbf{q}) \;=\; & \frac{1}{n}\sum_{i=1}^{n}\left(\left(g_Q - \beta\right)\partial_i f_n(\mathbf{q}) + \frac{\nu^2}{2}\partial_i^2 f_n(\mathbf{q})\right) \\ & + \; x\,\frac{\Psi_{n-1}}{\Psi_n}\frac{1}{n}\sum_{i=1}^{n}\alpha e^{-\alpha q_i} f_{n-1}\left(\mathbf{q}^{\setminus i}\right) + \frac{\nu^2}{2}\frac{\Psi_{n+1}}{\Psi_n}\frac{1}{n+1}\sum_{i=1}^{n+1}\partial_i f_{n+1}\left(\mathbf{q}^{i\to 0}\right)\end{aligned}$$

with the boundary condition: $\forall\ 1 \leq i \leq n,\ q_i = 0 \implies f_n(\mathbf{q}) = 0$.

These coupled PDEs are the analogue of the difference equations characterizing the stationary firm size distribution in Klette and Kortum (2004) and Luttmer (2011). To make the parallel even clearer, the following definition will prove helpful.

Definition 4. The average *product* exit rate among n-product firms is λ_n, with:

$$\lambda_1 \equiv \frac{\nu^2}{2} f_1'(0) \quad \text{and for } n > 1,\ \lambda_n \equiv \frac{\nu^2}{2}\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbf{q}\in(0,\infty)^{n-1}} \partial_i f_n\left(\mathbf{q}^{i\to 0}\right)\, d\mathbf{q} .$$

Proposition 8. Given a stationary firm size distribution $\{\Psi_n, f_n\}$ and the corresponding $\{\lambda_n\}$,

$$\begin{cases} \eta\Psi_1 = -\left(x + \lambda_1\right)\Psi_1 + \lambda_2\Psi_2 & + \eta + \frac{\nu^2}{2}\alpha\zeta - x \\ \frac{\eta}{n}\Psi_n = -\left(x + \lambda_n\right)\Psi_n + \lambda_{n+1}\Psi_{n+1} + x\Psi_{n-1} & \text{for } n > 1 \end{cases}$$

The recurrence relation in Proposition 8 is a balance condition similar to the one in Luttmer (2011). The difference is that, in his setup, the product exit rate λ does not depend on n, as product exit is a Poisson process. In contrast, here, the average product exit rate among n-product firms depends on the endogenous shape of f_n. It is perhaps clearer to appreciate the balance interpretation when the recurrence is rewritten in terms of Φ_n, the share of *firms* with n products:

$$\Phi_n \equiv \frac{\frac{\Psi_n}{n}}{\sum_{k=1}^{\infty}\frac{\Psi_k}{k}} \implies -\overbrace{\underbrace{nx\Phi_n}_{n\to n+1} - \underbrace{n\lambda_n\Phi_n}_{n\to n-1}}^{\text{Outflow}} + \overbrace{\underbrace{(n-1)x\Phi_{n-1}}_{n-1\to n} + \underbrace{(n+1)\lambda_{n+1}\Phi_{n+1}}_{n+1\to n}}^{\text{Inflow}} - \overbrace{\eta\Phi_n}^{\text{Dilution}} = 0$$

Outflows from n correspond to a n-product firm adding or shutting down a product. The former happens with Poisson arrival rate nx. While the latter is not Poisson, the resulting outflow mass is $n\lambda_n\Phi_n$ since λ_n is the average product exit rate among n-product firms. Inflows from $n-1$ to n correspond to a $(n-1)$-product firm adding a product, which happens at Poisson rate $(n-1)x$. Inflows from $n+1$ to n correspond to $(n+1)$-product firm shutting down production of one of its products, since the average product exit rate among such firms is λ_{n+1}, the resulting inflow mass is $(n+1)\lambda_{n+1}\Phi_{n+1}$. Finally, growth in the total measure of firms at rate η dilutes these shares.

3 Empirical Evidence on Product Dynamics

My quantitative exercise in Section 4 leverages Proposition 6 to pin down some of the model's parameters by targeting firm exit by age. Before doing so, I provide supporting evidence for the key model features underpinning this result: that product exit features negative selection and unfolds gradually, and that among incumbent firms the product addition rate per existing product is constant.

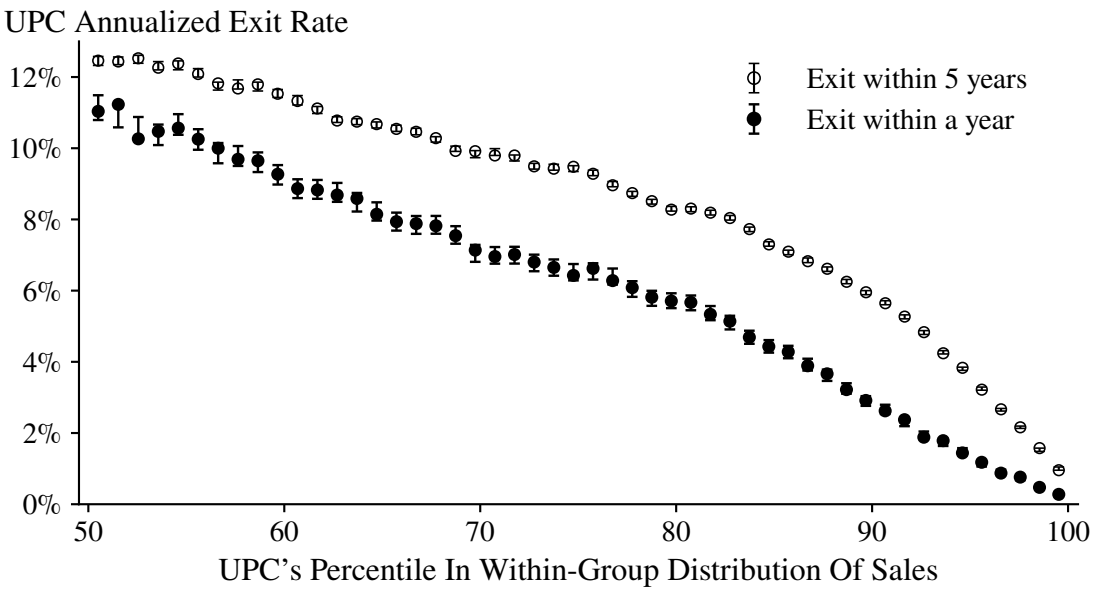
Providing such supporting evidence requires product-level data. For this purpose, I use the NielsenIQ retail scanner dataset, which provides high-frequency data at the UPC (universal product code, or barcode) level. Its coverage spans consumer packaged goods such as groceries and cosmetics. Different UPCs are partitioned into 104 product categories (groups), including coffee, vitamins, and laundry supplies. The analysis draws on data from 2006 to 2019, covering a balanced panel of 25,400 retail stores. Each retailer provides weekly sales and price data at the UPC level. The below results are obtained by aggregating these data across retailers. This yields 6 million UPC-year observations and 1.2 million unique UPCs. Using the GS1 database, I link UPCs to the manufacturing firm and obtain 40,600 unique firms in my sample. Appendix D provides further details on sample construction and variable definitions, with summary statistics in Appendix E.

Fact 1: UPCs with more sales are less likely to exit. Figure 2 shows the annualized hazard rate of UPC exit as a function of a UPC's percentile in the sales distribution for the corresponding group-year. To minimize concerns about measurement error, it focuses on UPCs with above median sales (which account for 98% of sales). It clearly illustrates that, within a group, UPC exit declines with UPC sales.[6] This is true even when looking at the hazard rate of exiting within 5 years, and when zooming in on the right tail of the sales distribution within a group.[7] This pattern corroborates the evidence in Broda and Weinstein (2010), which documents higher exit among smaller UPCs using the NielsenIQ Homescan panel. Showing the same relationship in retailer scanner data, where exit reflects disappearance from store shelves rather than zero purchases by a household sample,

[6] See Appendix E for the corresponding regression results.

[7] The latter is why, unlike Acemoglu et al. (2018), I do not add to the model a Poisson shock of creative destruction.

Figure 2: UPC Sales and Hazard Rate of Exit



Notes: Cattaneo, Crump, Farrell and Feng (2024) binscatter with 50 bins, corresponding to the 50th through 99th percentile of the distribution of sales within a group in year t. "Exit within a year" refers to year $t+1$ being the UPC's last year of sales. "Exit within 5 years" refers to UPC's last year of sales being $t+6$ or earlier. Vertical bars are 95% pointwise confidence intervals. Underlying number of UPC-year observations is 2.73M (so roughly 54,600 per bin).

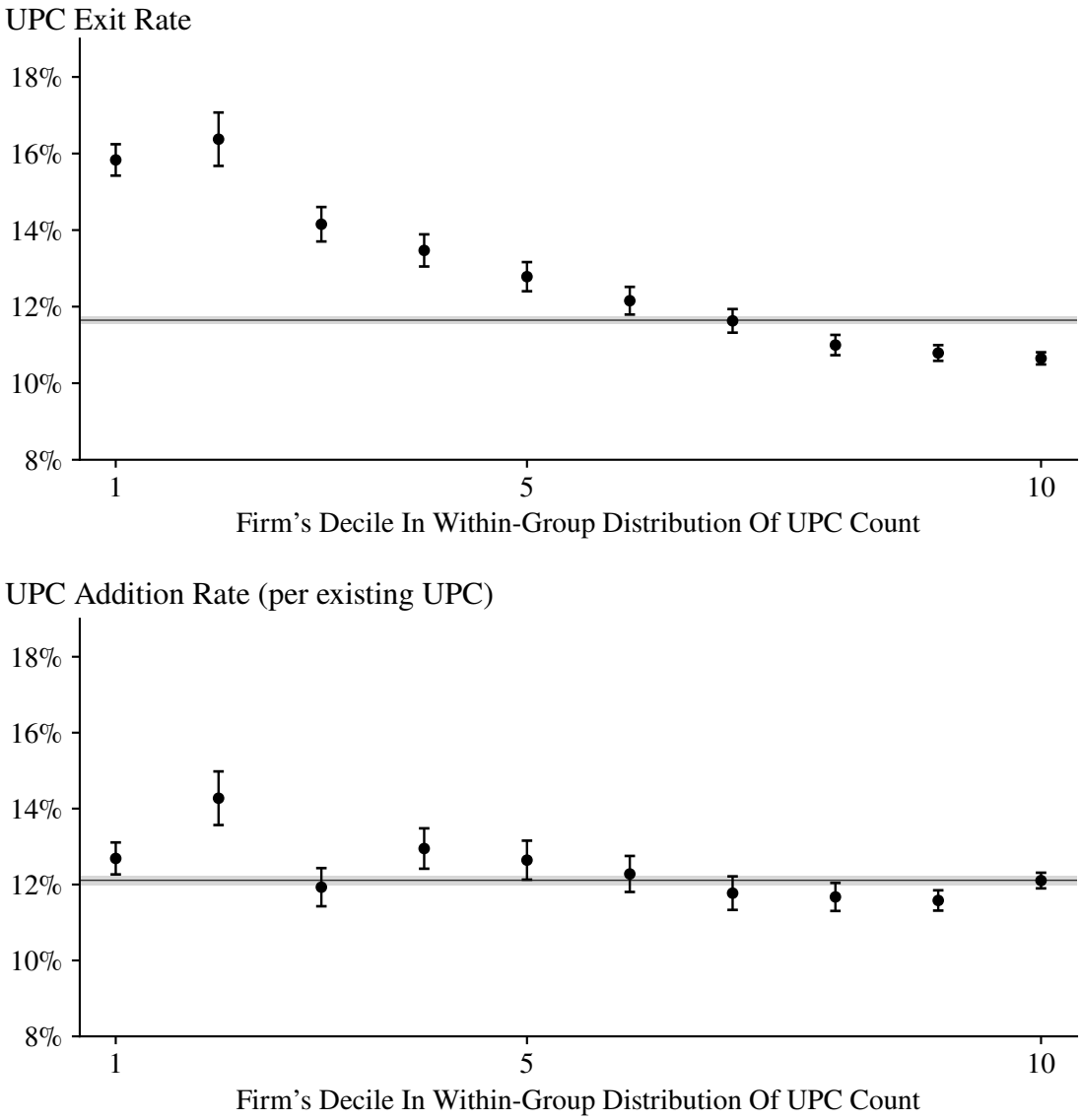
lends further credibility to this relationship.[8] Bernard, Redding and Schott (2010) reach similar conclusions at a coarser product definition (SIC codes in U.S. manufacturing).

Fact 2: UPC exit rate is lower among firms with more UPCs. The top panel of Figure 3 displays another facet of selection into product exit. It shows a firm's UPC exit rate as a function of its number of products. The former is defined as the number of exiting UPCs divided by the number of UPCs in the firm's portfolio (averaged between last and current year). So this is the empirical counterpart of the relationship between λ_n and n. The figure illustrates that, across firms in a group-year, the hazard rate of UPC exit is lower for firms with more UPCs (λ_n decreasing in n). This pattern is inconsistent with models where product exit is a Poisson process. In Appendix F2, I show that my model can generate this pattern as a consequence of negative selection into product exit.

Fact 3: Similar rates of UPC addition per existing UPC across multiproduct firms. The bottom panel of Figure 3 shows that, in contrast to the UPC exit rate, the UPC addition rate does not seem to be systematically related to the firm's UPC count. The addition rate is defined as the firm's

[8]Consistent with entry and exit being measured with non-negligible error in the Homescan panel, the entry and exit rates reported here and in Appendix E are substantially lower than those reported in Broda and Weinstein (2010).

Figure 3: UPC Exit and Addition Rates Across Multiproduct Firms

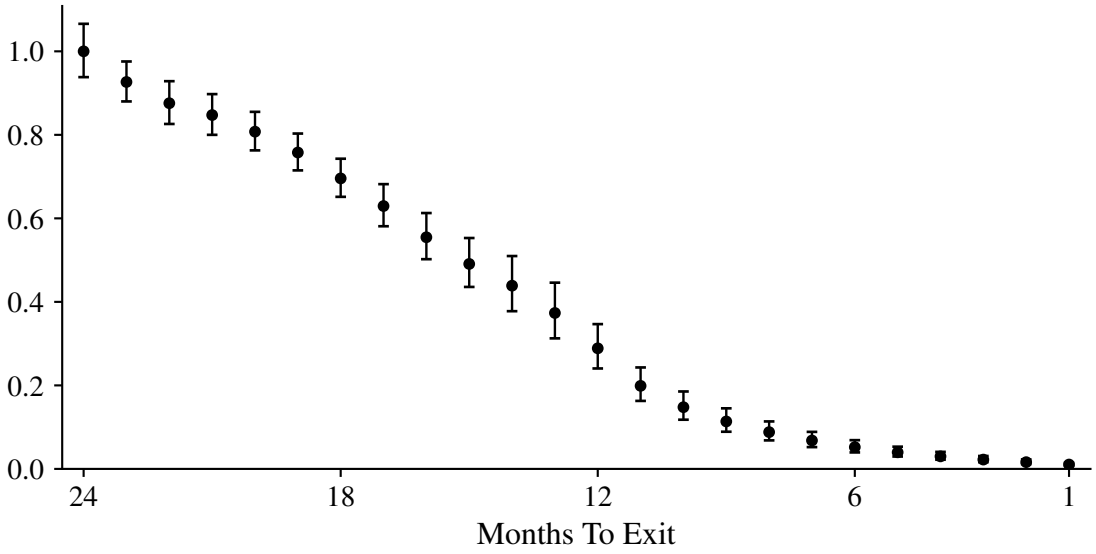


Notes: Cattaneo et al. (2024) binscatter with 10 bins, corresponding to deciles of distribution of UPC count among multiproduct firms within a group-year. Exiting (entering) UPCs are those for which the current year is the last (first) year of sales. Rates obtained through division by firm's UPC count in corresponding group, averaged between previous and current year. Vertical bars are 95% pointwise confidence intervals. Horizontal line is inverse-weighted mean, with 95% confidence interval around it. Underlying number of firm-year observations is 264,809. Single product firms are not shown because they account for 42% of firms—among them, UPC exit rate is 16.6% and add rate is 5.2%.

number of new UPCs divided by the number of UPCs in the firm's portfolio (averaged between last and current year). So this is the empirical counterpart of the relationship between x and n in the model. The lack of a systematic relationship provides empirical support for the rate of new product

introductions per existing product being constant.

Figure 4: Evolution of UPC Sales prior to Exit



Notes: Path of $\exp(\beta_m)$ from Equation 16, normalized such that $\exp(\beta_{24}) = 1$. Underlying regression has 54M UPC-month observations with $R^2 = 0.66$. Vertical bars correspond to 95% confidence intervals, based on standard errors clustered at the group level.

Fact 4: Prior to exit, UPC's sales decline gradually. Besides the granularity, another advantage of the NielsenIQ Retail Scanner dataset is its high frequency, which allows me to track dynamics prior to exit. Specifically, I estimate:

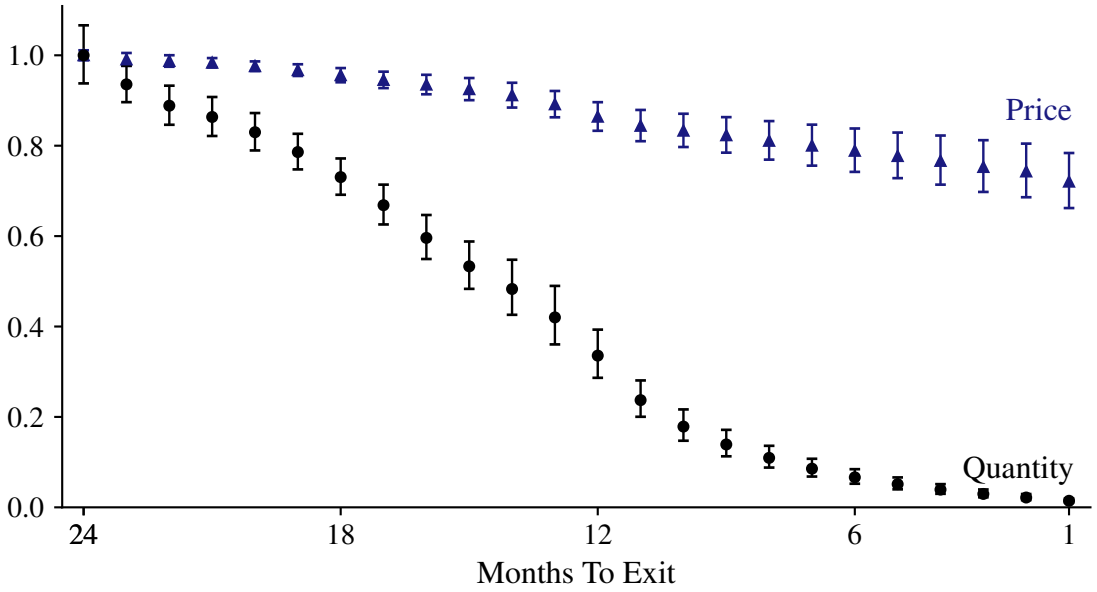
$$\log \text{Sales}_{pt} = \gamma_p + \sum_{m=1}^{24} \beta_m D_{pt}^m + \gamma_{gt} + \varepsilon_{pt} \; ; \tag{16}$$

where p is a UPC, g its group, t a month, γ_p a UPC fixed effect, γ_{gt} a group-month fixed effect, and D_{pt}^m a dummy variable equal to 1 m months prior to the UPC's exit. The path of $\exp(\beta_m)$ then represents the evolution of sales in the two years leading up to exit. Figure 4 plots the resulting path for $\exp(\beta_m)$, normalizing to 1 sales two years prior to exit. The figure reveals that, in the lead up to exit, a UPC experiences a gradual decline in sales.[9]

Fact 5: Pre-exit price–quantity patterns are consistent with a negative residual demand shock. Figure 5 tracks a UPC's price and quantity in the lead up to its exit. While the product's relative price falls modestly, the quantity sold collapses. These patterns align with the product experiencing a gradual negative residual demand shock in the lead up to exit: despite the UPC becoming relatively cheaper, its consumption is decreasing.

[9] In Appendix E, I show a gradual decline in number of retailers selling the UPC as well as sales per store.

Figure 5: Evolution of UPC Price and Quantity prior to Exit



Notes: Price (quantity) curve corresponds to path of $\exp(\beta_m)$ from Equation 16 but with price (quantity) as dependent variable, normalized such that $\exp(\beta_{24}) = 1$; underlying regression has 54M UPC-month observations with $R^2 = 0.86$ ($R^2 = 0.68$). Vertical bars correspond to 95% confidence intervals, based on standard errors clustered at the group level.

Taken together, these facts corroborate my modeling of product dynamics. Fact 3 supports the constant product addition rate per existing product, a feature my model inherits from Klette and Kortum (2004) and Luttmer (2011). At the same time, the remaining facts ground the key departure I take relative to these models and Kortum (1997): perfect substitution between old and new products and Poisson product exit do not generate the observed patterns of negative selection and gradual exit.

4 Quantitative Results

4.1 Calibration

To quantify the model, I proceed in two steps. First, by targeting data on firm exit by age, I pin down the three sufficient statistics characterizing the model's profile of firm exit by age. Second, I use a combination of external calibration and additional targets to pin down other relevant parameters.

The empirical moments I use in the first step are firm exit rates at ages 1 through 19, as reported in Sterk, Sedláček and Pugsley (2021) using data from the U.S. Census Longitudinal Business Database (LBD) for the years 1979 to 2012. The coverage of the LBD is the universe of U.S. nonfarm private employer firms. As emphasized by Hopenhayn, Neira and Singhania (2022) and

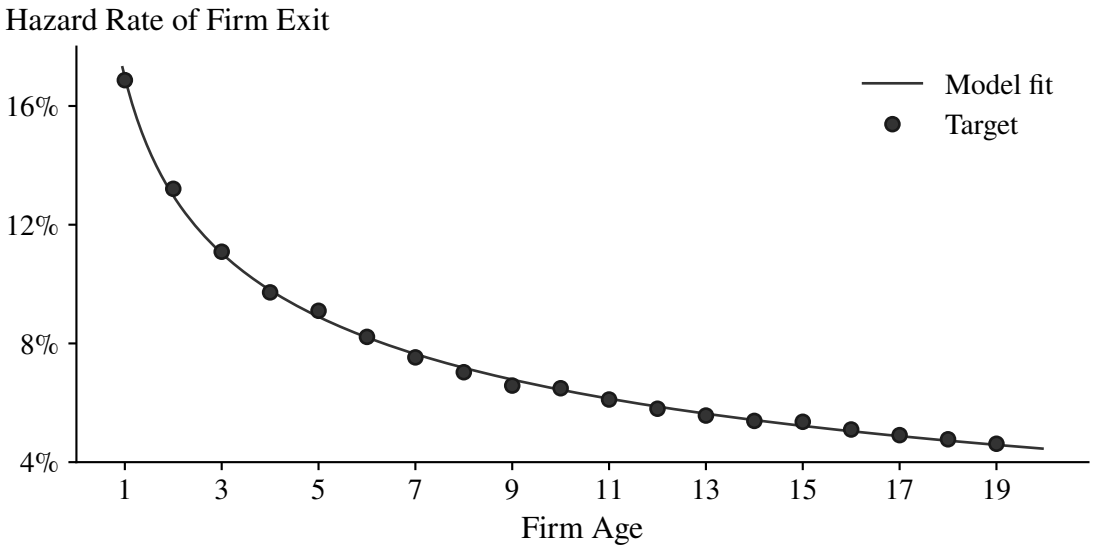
Karahan, Pugsley and Şahin (2024), these age-specific firm exit rates have been remarkably stable in the U.S. in recent decades, making them great targets for the estimation of my stationary model. To estimate the three statistics governing firm exit by age, I minimize the equally weighted sum of squared deviations between these 19 empirical moments and the model-implied ones, which are obtained by numerically solving the integral equation in Proposition 6. Table 2 reports the resulting estimates and Figure 6 shows the model's excellent fit to the 19 targeted moments.

Table 2: Estimation Results

	Symbol	Estimate
Product exit due to drift	$\theta\eta - \alpha\beta$	0.158
Relative volatility of shocks	$\alpha\nu$	0.296
Product addition rate (per existing product)	x	0.127

Notes: Estimates obtained by minimizing equally weighted sum of squared deviations between empirical and model's firm exit rate at ages 1 through 19.

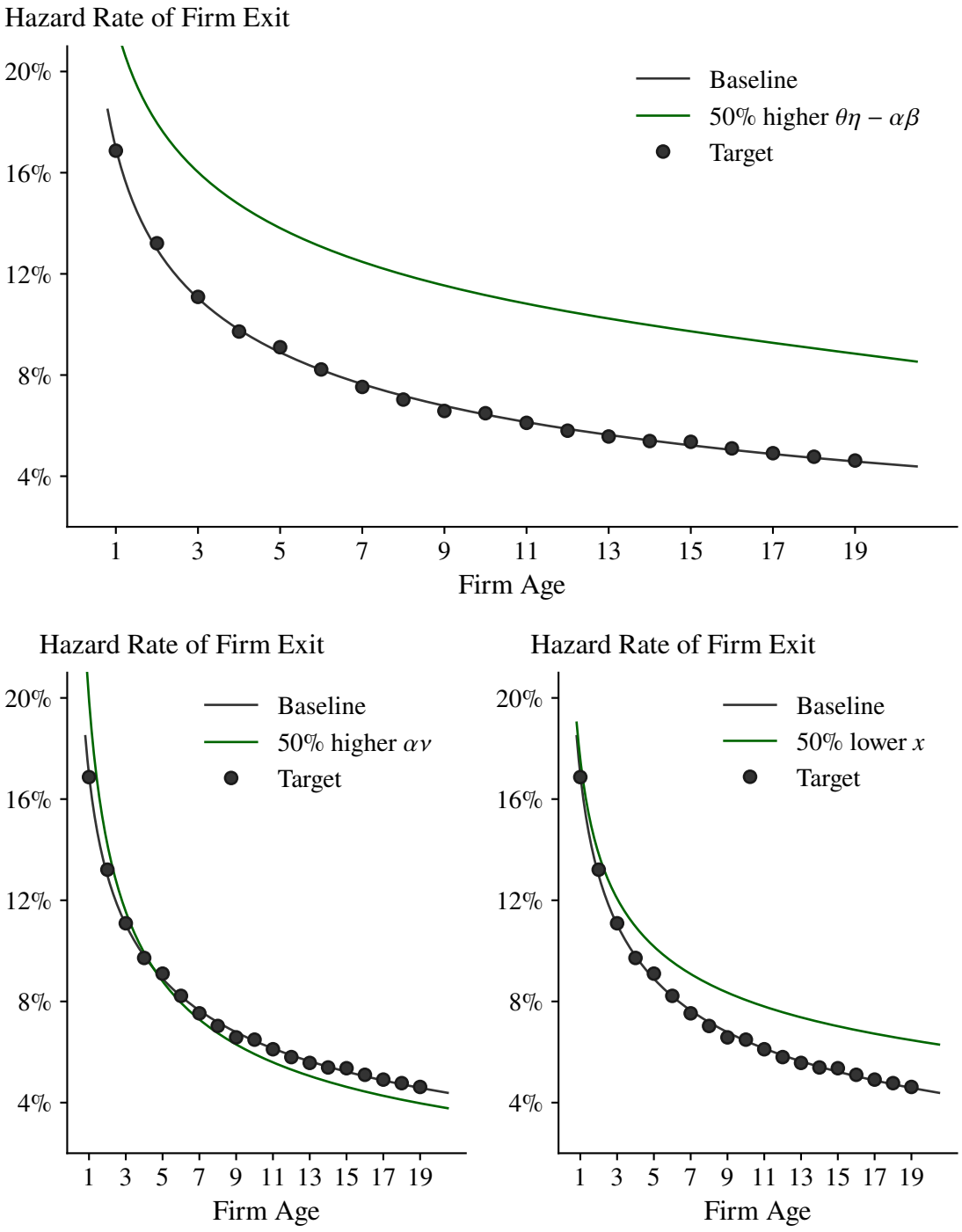
Figure 6: Model Fit to Targeted Profile of Firm Exit by Age



Notes: The 19 target moments are firm exit rates at ages 1 through 19, reported in Sterk, Sedláček and Pugsley (2021) and calculated from LBD. Model counterpart obtained by solving integral equation in Proposition 6. RMSE is 0.11 pp.

Figure 7 builds intuition for the sensitivity of the moments with respect to the three statistics, locally around the estimated values, by showing the effect of changing each statistic while holding

Figure 7: Effect of each of the three sufficient statistics on firm exit by age



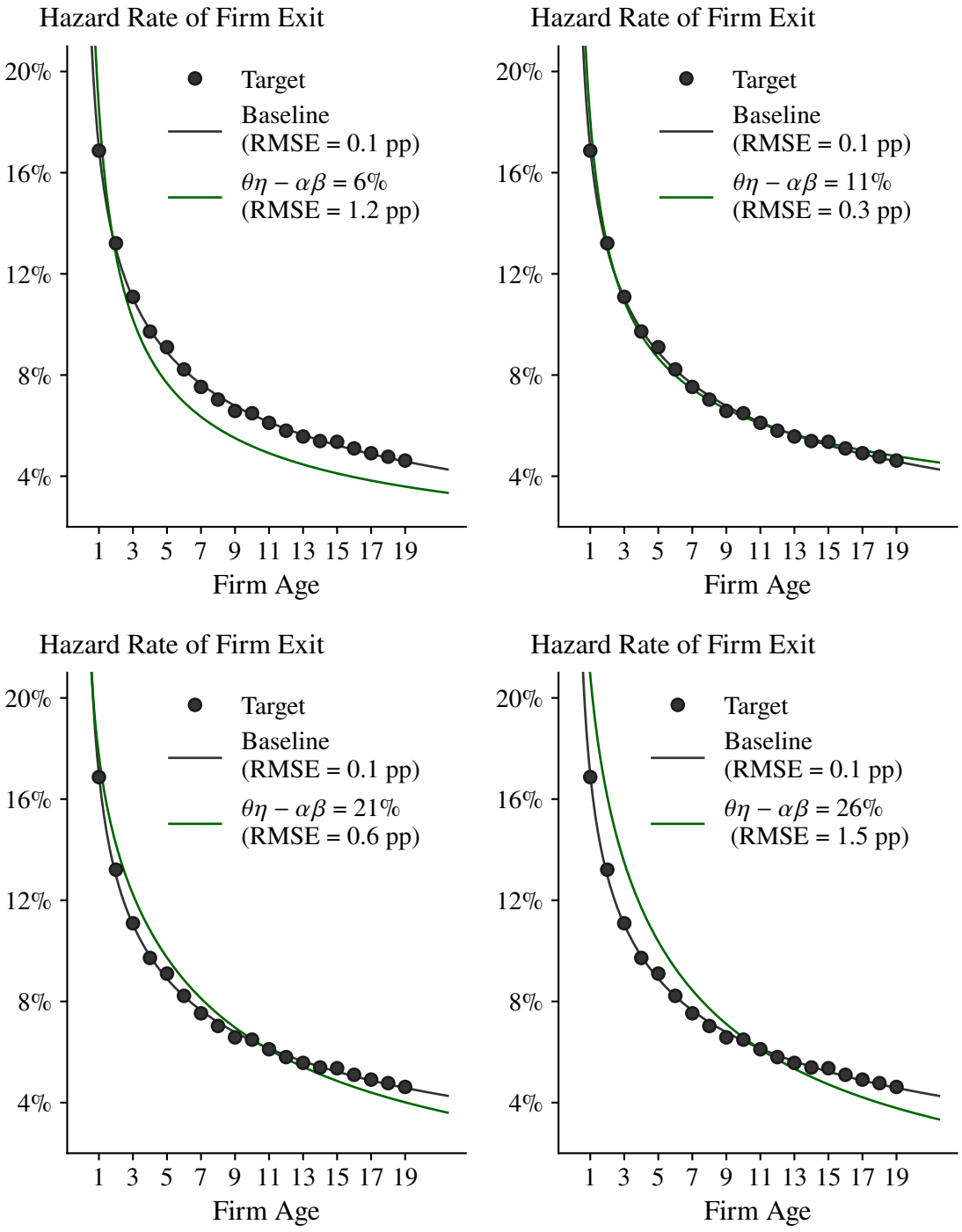
Notes: Each panel shows the effect of changing one of the three estimated statistics, holding the other two fixed. $\theta\eta - \alpha\beta$ is product exit due to downward drift toward the exit threshold, $\alpha\nu$ the relative volatility of shocks, and x the product addition rate per existing product.

the other two constant. The top panel shows that more *product* exit due to downward drift toward the exit threshold (higher $\theta\eta - \alpha\beta$) raises the *firm* exit rate at all ages. The bottom left panel shows that, in contrast, raising relative volatility (higher $\alpha\nu$) raises the exit rate among young firms but lowers it among older firms. The latter reflects that surviving firms have even more positively selected products when volatility is higher. Finally, the bottom right panel shows that a lower product addition rate (x) primarily increases the exit rate for mature firms: while new firms are still single product, this comparative static decreases the number of products older firms have.[10]

To demonstrate how strongly the moments constrain the estimate of the product exit due to drift, $\theta\eta - \alpha\beta$, I re-estimate restricted versions of the model and assess their fit. Specifically, I exogenously set $\theta\eta - \alpha\beta$ to a value different than the baseline estimate of 15.8% and re-estimate the other two statistics by targeting the same firm exit profile. Figure 8 plots the results when $\theta\eta - \alpha\beta$ is exogenously set to 6% (top left panel), 11% (top right panel), 21% (bottom left panel) and 26% (bottom right panel). Unsurprisingly, all restricted models have a worse statistical fit, with a larger root mean squared error (RMSE) than the baseline unrestricted model. However, it is worth noting that the fit is still good when $\theta\eta - \alpha\beta$ is exogenously set to 11% (top right panel). Given Figure 7, this is not surprising: one way to "compensate" for the lower $\theta\eta - \alpha\beta$ is with a higher relative volatility $\alpha\nu$ (lifts firm exit among younger firms) and a lower product addition rate x (lifts firm exit among older firms). Consistent with this intuition, the estimates I get for this restricted model are $\alpha\nu = 0.41$ (higher than baseline value of 0.296) and $x = 4.1\%$ (lower than baseline value of 12.7%). This though highlights the superiority of the baseline estimate: while this restricted model can still achieve a good statistical fit to the targeted moments, it does so by implying a counterfactually low product addition rate of 4.1%, far lower than the 12% observed in my NielsenIQ sample (see Figure 3), which is line with the baseline estimate for x (12.7%). The reason the fit to targeted moments gets much worse in the $\theta\eta - \alpha\beta = 6\%$ case (top left panel) is that the product addition rate x is bounded below by 0, so that the above "compensation" argument can only go so far (and this constraint binds, as this restricted model yields $x = 0$). The top left panel shows that with such little product exit due to drift, the model has a hard time matching the non-trivial exit rate among older firms: while idiosyncratic shocks are another driver of product exit, firms that survived to become old have positively selected products and the sequence of shocks needed to get them to the exit threshold is unlikely. The bottom two panels complete this sensitivity analysis, showing that setting $\theta\eta - \alpha\beta$ above the baseline estimate (to 21% or 26%) also leads to a substantial deterioration in the model's fit to targeted moments, with the RMSE rising to 0.6 pp and 1.5 pp, respectively. Taken together, these exercises demonstrate that the 15.8% estimate is tightly pinned down, as lower and higher values lead to worse fit to targeted and/or untargeted moments.

[10] Appendix F1 shows results in deviation relative to baseline, and reports sensitivity of model's hazard rate of firm exit at each age with respect to the three statistics and sensitivity of estimated statistics with respect to empirical moments.

Figure 8: Fit of restricted models with $\theta\eta - \alpha\beta$ exogenously set to different values



Notes: Each panel plots the 19 target moments, the baseline fit, and the fit of a restricted model with $\theta\eta - \alpha\beta$ exogenously set but $\alpha\nu$ and x re-estimated. Top left: $\alpha\nu = 0.51$, $x = 0$. Top right: $\alpha\nu = 0.41$, $x = 4.1\%$. Bottom left: $\alpha\nu = 0.21$, $x = 20.5\%$. Bottom right: $\alpha\nu = 0.28$, $x = 25.9\%$.

The second step in my quantification of the model involves using a mix of external calibration and matching additional moments. Specifically, I externally set the elasticity of substitution to $\sigma = 10$, following Atkeson and Burstein (2008). I also externally set the drift in the level of quality for incumbent products to $\beta = 0$ and discuss the the sensitivity of my results to $\beta \neq 0$ in subsection 4.5.

Moving on to parameters calibrated to match additional moments, I set the population growth rate to $\eta = 1\%$ to match the average annual growth rate in the number of private nonfarm employer businesses from 1978 to 2019. Along with $\beta = 0$ and the estimates from Table 2, it follows that $\theta = 15.8$. To calibrate α, note that $\alpha/(\sigma - 1)$ is the Pareto tail index of the distribution of employment across firms. The reason is the number of products per firm has a geometric-like thin tail,[11] so that the Pareto tail in the cross-firm employment distribution arises from the thick tailed distribution of product quality, which by Proposition 2 has tail index $\min\{\alpha, \zeta\}$. The results from Table 2 along with my calibration $\eta = 1\%$ imply $\min\{\alpha, \zeta\} = \alpha$. Since product-level employment is proportional to quality raised to the power $\sigma - 1$, it follows that the Pareto tail of the distribution of employment is $\alpha/(\sigma - 1)$. So, to match the tail of 1.06 in the data (Luttmer, 2010), I set $\alpha = 9.54$. The estimate of $\alpha\nu$ from Table 2 then requires $\nu = 0.03$. Finally, I pin down two additional statistics: $\mathcal{F} + I$, which is the constant component of product-level employment consisting of overhead and R&D labor, and $\underline{L}$, which I define as production employment for a product approaching the exit threshold. Along with the product addition rate per existing product x, these three statistics depend differentially on three (combinations of) structural parameters that have not been calibrated: $\mathcal{F}$, δ and ϑ/ε (see Appendix C3 for details). To pin down $\mathcal{F} + I$ and $\underline{L}$, I jointly target an average of 22 workers per firm and 58% of firms having 1 to 4 employees (averages for private nonfarm businesses between 1978 and 2019).

Table 3: Recap of Parameter Values

	Symbol	Value		Symbol	Value
Incumbent quality drift	β	0	Volatility of Brownian shocks	ν	0.03
Elasticity of substitution	σ	10	Product addition rate	x	0.127
Population growth rate	η	0.01	Fixed employment per product	$\mathcal{F} + I$	0.89
Spillover elasticity	θ	15.8	Min production employment	$\underline{L}$	0.37
Thinness of entry distribution	α	9.54			

Notes: x is the product addition rate per existing product, $\mathcal{F} + I$ the sum of overhead and R&D labor per product, and $\underline{L}$ the production employment for a product approaching the exit threshold. Expressions for these in terms of structural parameters are given in Appendix C.3. See text for details about calibration strategy, and subsection 4.5 for robustness to $\beta \neq 0$.

[11] Like Luttmer (2011), my model could generate a thick tailed distribution of products per firm, precisely when $x > \theta\eta - \alpha\beta$, the latter being the limit of λ_n for large n. But the estimates satisfy $x < \theta\eta - \alpha\beta$ (in line with Assumption 4).

4.2 Model Fit

Table 4: Model Fit to Untargeted Moments

	Model	Data
Productivity growth	1.8%	1.8%
Firm Entry Rate	9.8%	9.9%
Firm Exit Rate	8.8%	8.9%
Incumbent firms' contribution to growth	74.3%	75.2%

Notes: For productivity growth, $g = g_Q + \eta/(\sigma - 1)$ in the model and empirically I use the BLS average labor productivity growth in the U.S. private nonfarm business sector for 1978-2019. Firm entry and exit rates are from the Business Dynamics Statistics (BDS) for 1978-2019 and cover the universe of U.S. private nonfarm employer businesses. Data on the share of growth due to incumbent firms is from Garcia-Macia, Hsieh and Klenow (2019).

Table 4 reports the model's fit to various untargeted moments. First, productivity growth in the model ($g = 1.77\%$)—mostly driven by "vertical" growth ($g_Q = 1.66\%$)—closely aligns with the average labor productivity growth for the U.S. private nonfarm business sector for 1978-2019 (1.84%). Second, despite only targeting the profile of firm exit until age 19, the model closely fits the overall exit rate of firms. With η calibrated to match the net firm entry rate, it is then unsurprising that the model also closely fits the firm entry rate. Finally, I use as an untargeted moment the share of growth due to incumbent firms estimated by Garcia-Macia, Hsieh and Klenow (2019) based largely on the employment growth of surviving incumbent firms. Given that (i) entry of new products is the engine of productivity growth in my model and (ii) at any point in time entering and innovating incumbent firms draw the quality of their new products from the same distribution, that share in my model is simply the share of product entry accounted for by incumbent firms. In my calibration, the product entry rate is 17.1%, and from Table 2, each product of an incumbent firm leads to entry of new product at rate 12.7%. Therefore, the share of growth due to incumbents in the model is 74.3% (12.7/17.1), remarkably close to the estimate of 75.2% from Garcia-Macia et al. (2019).

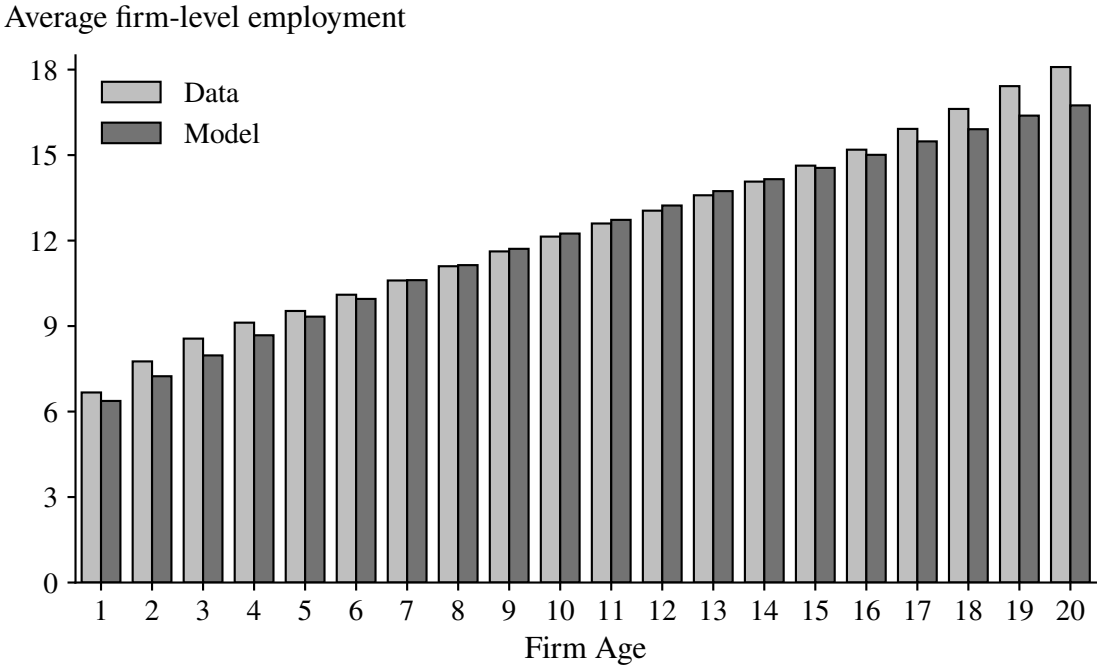
Table 5 compares the model's cross-firm distribution of employment to its empirical counterpart. The calibration only targeted three features of this distribution: the Pareto tail index, the average firm size, and the share of firms in the 1-4 employee bin (58%). Yet, the model is still capable to track the empirical share of firms and the share of employment across different firm size bins. Employment per product plays a more important role than number of products in accounting for the model's cross-sectional heterogeneity in employment per firm: compared to firms with 1-4 employees, firms with 20+ employees have five times more products but employ 54 times more production workers

Table 5: Model Fit to Firm Size Distribution

Firm size bin	Share of firms		Share of employment	
	Model	Data	Model	Data
1 to 4 employees	58.1%	58.1%	8.1%	6.2%
5 to 9 employees	20.5%	19.0%	8.1%	6.2%
10 to 19 employees	11.9%	11.4%	9.6%	7.5%
20 to 99 employees	8.5%	9.5%	17.2%	17.5%
100 to 499 employees	0.8%	1.6%	9.0%	13.8%
500+ employees	0.2%	0.4%	48.0%	48.8%

Notes: BDS data for 1978-2019 covering universe of U.S. private nonfarm employer businesses.

Figure 9: Model Fit to Untargeted Average Firm Size by Firm Age



Notes: Data on firm average employment by firm age as reported in Sterk, Sedláček and Pugsley (2021) and calculated from the LBD. To compute the model counterpart, I simulate a cohort of 5 million firms up until age 20, compute average employment by age, repeat this simulation 1000 times, and report the median of average employment by age across the simulations.

per product (see Appendix F2 for details). This feature of the quantified model is appealing for two reasons. First, it aligns with the findings of Hottman et al. (2016) that the quality (rather than count)

of a firm's products explains most of the variation in size across firms. Second, since the arrival rate of new products scales linearly with the firm's number of products (rather than its workers), the model also generates a declining relationship between arrival rate of a new product per worker and firm's number of workers—in line with the evidence from Akcigit and Kerr (2018).

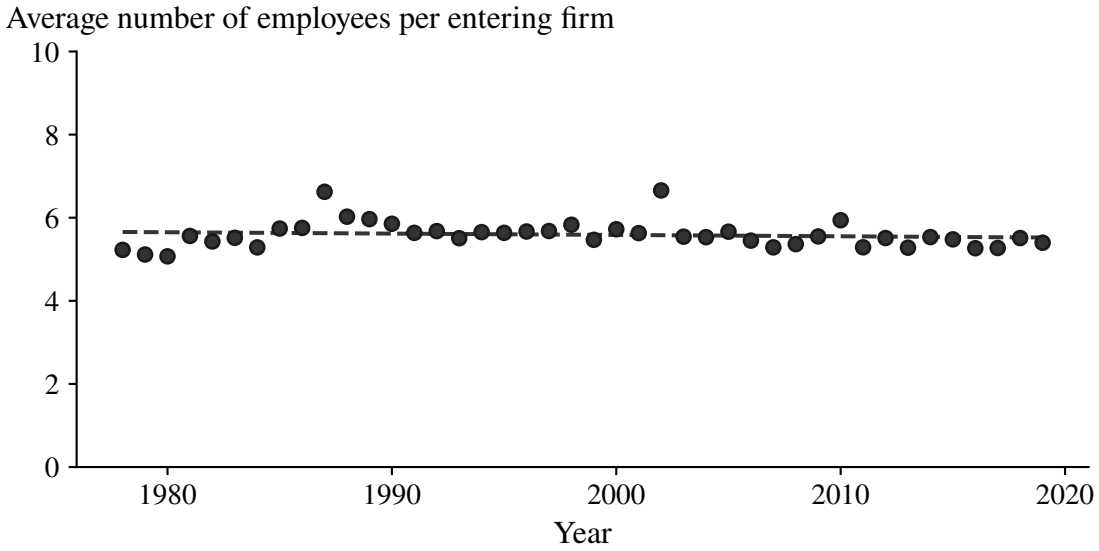
Lastly, despite average firm size by firm age not being targeted, Figure 9 shows that the model closely fits the average number of employees per firm at ages 1 through 20.

4.3 Magnitude of Knowledge Spillovers

My baseline quantitative results imply that knowledge spillovers create a wedge of $\theta\eta = 15.8$ pp between social and private rates of return to innovation along the laissez-faire BGP. Such sizable spillovers are necessary to simultaneously match: (i) exit rates among older firms, and (ii) sustained entry of new firms with stable average number of employees per new firm.

As Figure 8 reveals, matching the first of these requires a large extent of product exit due to downward drift toward the exit threshold ($\theta\eta - \alpha\beta$). In fact, a firm exits upon shutting down its final product, and while such product exits could result from the accumulation of bad idiosyncratic shocks, firms that survive to old age have positively selected products, making the sequence of shocks needed to push their products to the exit threshold unlikely. With $\beta = 0$, this necessary sizable product exit due downward drift is entirely driven by the exit threshold itself rising over time. Intuitively, this force is a substitute to having a Poisson death shock, which many models of firms dynamics and growth introduce to generate exit among older firms and/or products.

Figure 10: Average Size of Entering Firm



Notes: BDS data on average number of employees for firms age 0. Dashed line is a linear time trend.

A natural question then is why this requires knowledge spillovers. In principle, one could imagine a model with firms attempting to innovate by building on internal knowledge stocks they have accumulated. The rise in the exit threshold over time would then be driven by those incumbents that succeed in innovating, and the exit of old firms by those incumbents that fail to do so. A key informative moment in this regard is the second feature of the data mentioned above and depicted in Figure 10: despite this rising exit threshold, there has been sustained entry of new firms with the average number of workers per new firm remarkably stable over time. This requires improvements in the entry distribution over time (so intertemporal knowledge spillovers), because otherwise new firms would enter progressively further behind the average incumbent and hire fewer and fewer workers. Such intertemporal knowledge spillovers are regularly assumed in existing growth theories: in quality ladder models, entrants attempt to improve on the highest rung; and following Luttmer (2007), many have written models where the entry distribution is tied to that of incumbents.

It is important to emphasize that the 15.8 pp gap I identify is the one knowledge spillovers create under laissez-faire. I leave the question of how much of this gap is already internalized by existing policies to future research. This being said, my identification is robust to such policies, as these affect the cost of product entry, but do not affect stationary product dynamics (entry and exit *rates*).

An alternative metric illustrating how large the knowledge spillovers I am backing out are is the resulting extent of innovation underprovision under laissez-faire, defined as:

$$\text{Innovation underprovision under laissez-faire} = \frac{S_t^{\mathbf{FB}} + (1-\delta)^{-1} I_t^{\mathbf{FB}} M_t^{\mathbf{FB}}}{S_t^{\mathbf{DE}} + (1-\delta)^{-1} I_t^{\mathbf{DE}} M_t^{\mathbf{DE}}} .$$

In words, this is the level of innovation along the first-best BGP relative to that along the laissez-faire BGP, taking into account innovation by entrants as well as incumbents, with the $(1-\delta)^{-1}$ accounting for the fact that the infra-marginal R&D worker is more productive than a startup entrepreneur (see Equation 12). Unlike the gap between social and private rates of return, this metric depends on the effective discount rate, $r - g$. In Table 6, I consider a range of values for $r - g$. On the lower end, I set $r - g = 5\%$, based on estimate of r as the average real return to productive capital of around 6.5% from Caballero, Farhi and Gourinchas (2017). On the higher end, I set $r - g = 15\%$, based on the evidence from Gormsen and Huber (2025) on discount rates firms use. Across this range, I find that the social planner would like 2 to 3.4 times more innovation than the laissez-faire economy provides.

Table 6: Extent of innovation underprovision along stationary laissez-faire equilibrium

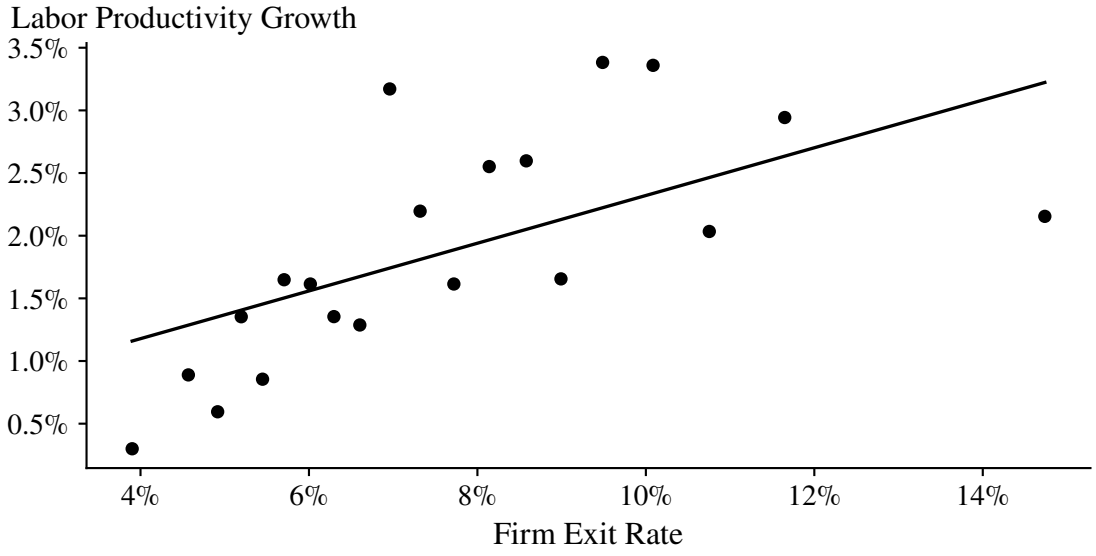
$r - g$	5%	7.5%	10%	12.5%	15%
Innovation Underprovision	3.4	2.7	2.4	2.1	2.0

Notes: Table reports level of innovation along first best BGP relative to that along laissez-faire BGP.

4.4 Validation Exercises

If dynamics in product markets are indeed informative about the magnitude of knowledge spillovers, then I should be able to relate metrics of these dynamics to manifestations or proxies of these spillovers. In this vein, Figure 11 shows that, across industries, there is a positive association between labor productivity growth and the firm exit rate. While not sufficient, such a correlation is necessary to lend credibility to my approach, as stronger spillovers accelerate growth in my model.

Figure 11: Firm exit rate and labor productivity growth across industries



Notes: Cattaneo et al. (2024) binscatter with 20 bins plotting 5-year average labor productivity growth against the 5-year average firm exit rate at the 4-digit industry level. Underlying number of observations is 1120 with 160 4-digit NAICS code industries observed over seven 5-year periods (1988-2022). Labor productivity growth data from BLS; firm exit data from BDS.

As a second exercise, I redo my quantitative exercise for each of 19 2-digit NAICS sector. Specifically, I estimate the three sufficient statistics governing firm exit by age by separately targeting each sector's profile of firm exit by age. If my quantitative analysis plausibly identifies knowledge spillovers, then the estimated product exit due to downward drift should be largest in sectors where narrative evidence points to important spillovers. As the formal setup of the multisector version of the model in Appendix G shows, the sector-level estimate of product exit due to downward drift is informative about spillovers *received*, rather than *generated*, by that sector.

Table 7 reports the estimated product exit due to downward drift ($\theta\eta - \alpha\beta$) for the four sectors in which the procedure yields the highest estimates. The fact that the highest estimate is for Arts, Entertainment, and Recreation is consistent with a taste for novelty being a potential concern, which is why I do robustness to $\beta < 0$ in subsection 4.5. However, several factors suggest that this

Table 7: Sectors with Largest Estimated Product Exit due to Downward Drift

2-digit Sector	Estimated $\theta\eta - \alpha\beta$	Share of firms	Firm Exit Rate
Arts, Entertainment, and Recreation	25.6%	1.7%	8.6%
Mining, Quarrying, and Oil and Gas Extraction	21.6%	0.4%	8.2%
Transportation and Warehousing	18.9%	2.8%	10.8%
Information	18.5%	1.2%	10.2%

Notes: Estimate of $\theta\eta - \alpha\beta$ when targeting sector's firm exit rate at ages 1, 2, 3, 4, 5, 8, 13, and 18. Underlying data are from the BDS for the years 1996-2019.

confounder does not drive my results. First, this is a relatively small sector, accounting for less than 2% of both firms and employment in the U.S. economy. Second, and more importantly, the pattern among the other sectors with a high estimate provides strong evidence supporting my interpretation of the results. As I discuss next, these are technology-intensive industries where narrative evidence points to a central role for exactly the kind of spillovers my model is designed to capture.

The Mining sector provides a prime example, as spillovers played a central role in enabling the shale gas boom. While George Mitchell's company is credited with developing the breakthrough formula combining horizontal drilling with slickwater fracturing—which the industry then adapted with awesome rapidity" (Golden and Wiseman, 2015, p. 960)—cross-industry spillovers were equally crucial. Indeed, this formula built on a "web of technological developments" such as "3D seismic imaging techniques [...] that have benefited from advances in computing and that draw on technology originally developed to track submarines" (Golden and Wiseman, 2015, p. 973). This example also illustrates how my approach to quantifying spillovers complements patent-based ones. In fact, "although it is somewhat surprising and counterintuitive, during the late 1990s and early 2000s, neither Mitchell nor Devon pursued patent protection for their respective innovations in slickwater hydraulic fracturing and horizontal drilling" (Cahoy et al., 2013, p. 291). However, these innovations left a detectable trace in product markets: by lowering natural gas costs and reducing coal demand, they drove coal-dependent firms out of business (Linn and McCormack, 2019).

Turning to Transportation and Warehousing, the major developments the sector has experienced over this period also built on new technologies developed in other sectors. Arguably the most pivotal of these was the Global Positioning System (GPS). Originally developed for military purposes, GPS became widely available for commercial use in the U.S. in the mid 1990s. Its high precision capabilities were critical "to unlocking most of the benefits of telematics", the "field of technology that uses in-vehicle equipment to remotely monitor vehicles" (O'Connor et al., 2019, p. 13-1).

Advanced on-board computers (OBCs) are an example of such equipment adopted by trucking companies and Hubbard (2003) estimates that they increased capacity utilization among adopting trucks by 13%. These gains in fleet management stem from dynamic route optimization as well as improved monitoring of driver behavior (Hubbard, 2000). A more recent transformation in the transportation sector is the rise of ride-sharing platforms like Uber and Lyft. This new business model also leverages a confluence of technologies developed in other sectors, combining the ubiquity of GPS-enabled smartphones and mobile data networks with sophisticated matching algorithms. Cramer and Krueger (2016) document a higher utilization rate among UberX drivers compared to traditional taxi drivers, and argue that the more efficient technology for matching drivers and passengers is a leading contender in explaining this finding.

Finally, the case of the Information sector is perhaps the least surprising, as it encompasses software and digital industries, home to the Open Source Software (OSS) paradigm. In fact, OSS offers a tangible illustration of my model's aggregate stock of knowledge: it is a public stock that firms simultaneously contribute to and benefit from (Gortmaker, 2025).

4.5 Robustness Checks

Drift in quality ($\beta \neq 0$). I obtain my baseline results under the assumption that there is no drift in the level of quality for incumbent products, so that any drift in relative quality is entirely driven by changes in the exit threshold itself. This is likely a conservative benchmark, since any $\beta > 0$, reflecting learning by doing or incumbent innovation, would make my baseline results biased downward, as $\theta\eta = 0.158 + 9.54\beta$. Intuitively, my estimation identifies the rate of product exit resulting from the gradual creative destruction of older products by newer ones. If incumbent products are themselves improving over time ($\beta > 0$), they become a moving target. To rationalize the same rate of product exit due to gradual creative destruction, the quality distribution of new entrants must therefore be advancing even more rapidly to overcome this incumbent quality growth (larger $\theta\eta$). To calibrate β, I estimate the Hottman et al. (2016) demand system on the product-level data used in Section 3. Doing so, I find an average annual reduction of marginal cost of 1.3%, leading to the upper bound of 1.3% on the range of values for β I consider in Table 8.

In contrast, any $\beta < 0$, would make my baseline results biased upward. One, albeit generous, interpretation of such an exogenously negative drift is a taste for novelty: irrespective of the features of new products, demand for an incumbent drifts down as it ages. Bils (2009) finds that ignoring such taste for novelty would overstate quality growth among new products by a third. Given his sample consists of durables in the CPI where taste for novelty is likely more important, I use this to calibrate the lower end of the values I consider for β. Table 8 shows that, across this span of values for β, the wedge $\theta\eta$ created by knowledge spillovers under laissez-faire ranges from 11 to 28.2 pp.

Table 8: Sensitivity of estimated wedge ($\theta\eta$) to drift in the level of incumbent product quality

β	−0.5%	−0.25%	0%	0.65%	1.3%
$\theta\eta$	11%	13.4%	15.8%	22%	28.2%

Notes: Wedge $\theta\eta$ between social and private rates of return to innovation under laissez-faire for different values of β, the exogenous drift in the level of quality of an incumbent product.

Human capital growth. So far, I attribute all improvements in the entry distribution to the accumulation of knowledge. If higher human capital leads to draws from a better distribution, then human capital growth biases my baseline results upward. Appendix G2 extends my model with such (exogenous) human capital growth. Using the BLS labor composition index to calibrate the contribution of human capital to growth, I find that even in this conservative case, the wedge $\theta\eta$ between social and private returns to innovation under laissez-faire remains substantial at 12.4 pp.

Quantification with Product Data. The reason I rely on firm-level data in my quantitative exercise is that comprehensive data at the product level are not available. In Appendix F3, I assess the sensitivity of my results to doing the quantification with product versus firm level data when both are available. Focusing on the product exit due to downward drift—the statistic of interest—the estimate obtained using product-level data is 17%, while that obtained from firm-level data is 16%. This alignment lends credibility to the headline finding by suggesting my methodology successfully recovers the gradual component of product exit from firm data.

5 Conclusion

How large are intertemporal knowledge spillovers? This paper offers a novel and complementary perspective to answering this policy-relevant question. Instead of patents or reported R&D, I use firm-level data covering the universe of U.S. private nonfarm employer firms, viewed through the prism of a novel growth model featuring multiproduct firms, selection into product exit, and idiosyncratic product-level demand shocks. I show that these shocks generate too little exit among older firms relative to what we observe, implying that incumbent products must gradually become obsolete. This selection mechanism means the pool of surviving incumbents continuously improves. Despite facing these improving incumbents, new firms enter with a remarkably stable average size over time. This stability is only possible if the entry distribution also improves over time—a direct manifestation of intertemporal knowledge spillovers. Without such spillovers, entrants would fall progressively further behind incumbents and enter at ever-smaller scales.

References

Acemoglu, Daron, Ufuk Akcigit, Harun Alp, Nicholas Bloom, and William Kerr, "Innovation, Reallocation, and Growth," *American Economic Review*, November 2018, *108* (11), 3450–91.

Aghion, Philippe and Peter Howitt, "A Model of Growth Through Creative Destruction," *Econometrica*, 1992, *60* (2), 323–351.

_ , Christopher Harris, Peter Howitt, and John Vickers, "Competition, Imitation and Growth with Step-by-Step Innovation," *The Review of Economic Studies*, 07 2001, *68* (3), 467–492.

_ , Ufuk Akcigit, and Peter Howitt, "What Do We Learn From Schumpeterian Growth Theory?," in Philippe Aghion and Steven N. Durlauf, eds., *Handbook of Economic Growth*, Vol. 2, Elsevier, 2014, pp. 515–563.

Agrawal, Ajay and Rebecca Henderson, "Putting Patents in Context: Exploring Knowledge Transfer from MIT," *Management Science*, 2002, *48* (1), 44–60.

Akcigit, Ufuk and Sina T. Ates, "Ten Facts on Declining Business Dynamism and Lessons from Endogenous Growth Theory," *American Economic Journal: Macroeconomics*, January 2021, *13* (1), 257–98.

_ and _ , "What Happened to US Business Dynamism?," *Journal of Political Economy*, 2023, *131* (8), 2059–2124.

_ and William R. Kerr, "Growth through Heterogeneous Innovations," *Journal of Political Economy*, 2018, *126* (4), 1374–1443.

_ , Douglas Hanley, and Stefanie Stantcheva, "Optimal Taxation and R&D Policies," *Econometrica*, 2022, *90* (2), 645–684.

Alcácer, Juan, Michelle Gittelman, and Bhaven Sampat, "Applicant and examiner citations in U.S. patents: An overview and analysis," *Research Policy*, 2009, *38* (2), 415–427.

Argente, David, Munseob Lee, and Sara Moreira, "The Life Cycle of Products: Evidence and Implications," *Journal of Political Economy*, 2024, *132* (2), 337–390.

_ , Salomé Baslandze, Douglas Hanley, and Sara Moreira, "Patents to Products: Product Innovation and Firm Dynamics," March 2023. Working paper.

Arkolakis, Costas, "A Unified Theory of Firm Selection and Growth *," *The Quarterly Journal of Economics*, 02 2016, *131* (1), 89–155.

Arqué-Castells, Pere and Daniel F. Spulber, "Measuring the Private and Social Returns to R&D: Unintended Spillovers versus Technology Markets," *Journal of Political Economy*, 2022, *130* (7), 1860–1918.

Arrow, Kenneth J., "Economic Welfare and the Allocation of Resources for Invention," in "The Rate and Direction of Inventive Activity: Economic and Social Factors," Princeton University Press, 1962, pp. 609–626.

Atkeson, Andrew and Ariel Burstein, "Pricing-to-Market, Trade Costs, and International Relative Prices," *American Economic Review*, December 2008, *98* (5), 1998–2031.

_ and _ , "Aggregate Implications of Innovation Policy," *Journal of Political Economy*, 2019, *127* (6), 2625–2683.

BEA, "Value Added by Industry," 2025. Accessed: 2025-09-14.

Berlingieri, Giuseppe, Maarten De Ridder, Danial Lashkari, and Davide Rigo, "Creative Destruction through Innovation Bursts," April 2025. Working paper.

Bernard, Andrew B., Stephen J. Redding, and Peter K. Schott, "Multiple-Product Firms and Product Switching," *American Economic Review*, March 2010, *100* (1), 70–97.

Bilal, Adrien, Niklas Engbom, Simon Mongey, and Giovanni L Violante, "Labor Market Dynamics When Ideas are Harder to Find," Working Paper 29479, National Bureau of Economic Research November 2021.

Bils, Mark, "Do Higher Prices for New Goods Reflect Quality Growth or Inflation?*," *The Quarterly Journal of Economics*, 05 2009, *124* (2), 637–675.

Bloom, Nicholas, John Van Reenen, and Heidi Williams, "A Toolkit of Policies to Promote Innovation," *Journal of Economic Perspectives*, August 2019, *33* (3), 163–84.

_ , Mark Schankerman, and John Van Reenen, "Identifying Technology Spillovers and Product Market Rivalry," *Econometrica*, 2013, *81* (4), 1347–1393.

BLS, "Total factor productivity by major industries," 2025. Accessed: 2025-09-14.

Broda, Christian and David E. Weinstein, "Product Creation and Destruction: Evidence and Price Implications," *American Economic Review*, June 2010, *100* (3), 691–723.

Bryan, Kevin A. and Heidi L. Williams, "Innovation: market failures and public policies," in Kate Ho, Ali Hortaçsu, and Alessandro Lizzeri, eds., *Handbook of Industrial Organization, Volume 5*, Vol. 5 of *Handbook of Industrial Organization*, Elsevier, 2021, pp. 281–388.

Caballero, Ricardo J. and Adam B. Jaffe, "How High Are the Giants' Shoulders: An Empirical Assessment of Knowledge Spillovers and Creative Destruction in a Model of Economic Growth," *NBER Macroeconomics Annual*, 1993, *8*, 15–74.

_ , Emmanuel Farhi, and Pierre-Olivier Gourinchas, "Rents, Technical Change, and Risk Premia Accounting for Secular Trends in Interest Rates, Returns on Capital, Earning Yields, and Factor Shares," *American Economic Review*, May 2017, *107* (5), 614–20.

Cahoy, Daniel R., Joel Gehman, and Zhen Lei, "Fracking Patents: The Emergence of Patents as Information-Containment Tools in Shale Drilling," *Michigan Telecommunications and Technology Law Review*, 2013, *19* (2), 279.

Cattaneo, Matias D., Richard K. Crump, Max H. Farrell, and Yingjie Feng, "On Binscatter," *American Economic Review*, May 2024, *114* (5), 1488–1514.

Chen, Zhao, Zhikuo Liu, Juan Carlos Suárez Serrato, and Daniel Yi Xu, "Notching R&D Investment with Corporate Income Tax Cuts in China," *American Economic Review*, July 2021, *111* (7), 2065–2100.

Cohen, Wesley M, Richard R Nelson, and John P Walsh, "Protecting Their Intellectual Assets: Appropriability Conditions and Why U.S. Manufacturing Firms Patent (or Not)," Working Paper 7552, National Bureau of Economic Research February 2000.

Cowx, Mary, Rebecca Lester, and Michelle L. Nessa, "The Consequences of Limiting the Tax Deductibility of R&D," Research Paper 4998845, Stanford University Graduate School of Business July 2024.

Cramer, Judd and Alan B. Krueger, "Disruptive Change in the Taxi Business: The Case of Uber," *American Economic Review*, May 2016, *106* (5), 177–82.

Decker, Ryan A., John Haltiwanger, Ron S. Jarmin, and Javier Miranda, "Declining Business Dynamism: What We Know and the Way Forward," *American Economic Review*, May 2016, *106* (5), 203–07.

Dhingra, Swati and John Morrow, "Monopolistic Competition and Optimum Product Diversity under Firm Heterogeneity," *Journal of Political Economy*, 2019, *127* (1), 196–232.

Garcia-Macia, Daniel, Chang-Tai Hsieh, and Peter J. Klenow, "How Destructive Is Innovation?," *Econometrica*, 2019, *87* (5), 1507–1541.

Golden, John M. and Hannah J. Wiseman, "The Fracking Revolution: Shale Gas as a Case Study in Innovation Policy," *Emory Law Journal*, 2015, *64* (4), 955.

Gormsen, Niels Joachim and Kilian Huber, "Corporate Discount Rates," *American Economic Review*, June 2025, *115* (6), 2001–49.

Gortmaker, Jeff, "Open Source Software Policy in Industry Equilibrium," January 2025. Working Paper.

Griliches, Zvi, "Patent Statistics as Economic Indicators: A Survey," Working Paper 3301, National Bureau of Economic Research March 1990.

Grossman, Gene M. and Elhanan Helpman, "Quality Ladders in the Theory of Growth," *The Review of Economic Studies*, 01 1991, *58* (1), 43–61.

Hopenhayn, Hugo A., "Entry, Exit, and firm Dynamics in Long Run Equilibrium," *Econometrica*, 1992, *60* (5), 1127–1150.

Hopenhayn, Hugo, Julian Neira, and Rish Singhania, "From Population Growth to Firm Demographics: Implications for Concentration, Entrepreneurship and the Labor Share," *Econometrica*, 2022, *90* (4), 1879–1914.

Hottman, Colin J., Stephen J. Redding, and David E. Weinstein, "Quantifying the Sources of Firm Heterogeneity *," *The Quarterly Journal of Economics*, 03 2016, *131* (3), 1291–1364.

Hubbard, Thomas N., "The Demand for Monitoring Technologies: The Case of Trucking," *The Quarterly Journal of Economics*, 2000, *115* (2), 533–560.

_ , "Information, Decisions, and Productivity: On-Board Computers and Capacity Utilization in Trucking," *American Economic Review*, September 2003, *93* (4), 1328–1353.

Jaffe, Adam B., "Technological Opportunity and Spillovers of R D: Evidence from Firms' Patents, Profits, and Market Value," *The American Economic Review*, 1986, *76* (5), 984–1001.

_ **, Manuel Trajtenberg, and Michael S. Fogarty**, "Knowledge Spillovers and Patent Citations: Evidence from a Survey of Inventors," *The American Economic Review*, 2000, *90* (2), 215–218.

_ **,** _ **, and Rebecca Henderson**, "Geographic Localization of Knowledge Spillovers as Evidenced by Patent Citations," *The Quarterly Journal of Economics*, 1993, *108* (3), 577–598.

Jones, Benjamin F. and Lawrence H. Summers, "A Calculation of the Social Returns to Innovation," in Austan Goolsbee and Benjamin F. Jones, eds., *Innovation and Public Policy*, Chicago: University of Chicago Press, 2021, pp. 13–60.

Jones, Charles I., "R&D-Based Models of Economic Growth," *Journal of Political Economy*, 1995, *103* (4), 759–784.

_ , "The Past and Future of Economic Growth: A Semi-Endogenous Perspective," *Annual Review of Economics*, 2022, *14* (Volume 14, 2022), 125–152.

_ **and John C. Williams**, "Measuring the Social Return to R&D*," *The Quarterly Journal of Economics*, 11 1998, *113* (4), 1119–1135.

Karahan, Fatih, Benjamin Pugsley, and Ayşegül Şahin, "Demographic Origins of the Start-up Deficit," *American Economic Review*, July 2024, *114* (7), 1986–2023.

Klenow, Peter J. and Huiyu Li, "Entry Costs Rise with Growth," *Journal of Political Economy Macroeconomics*, 2025, *3* (1), 43–74.

Klette, Tor Jakob and Samuel Kortum, "Innovating Firms and Aggregate Innovation," *Journal of Political Economy*, 2004, *112* (5), 986–1018.

Kortum, Samuel S., "Research, Patenting, and Technological Change," *Econometrica*, 1997, *65* (6), 1389–1419.

Levin, Richard C., Alvin K. Klevorick, Richard Nelson, and Sidney Winter, "Appropriating the Returns from Industrial Research and Development," *Brookings Papers on Economic Activity*, 1987, *18* (3, Special Issue on Microeconomics), 783–832.

Linn, Joshua and Kristen McCormack, "The roles of energy markets and environmental regulation in reducing coal-fired plant profits and electricity sector emissions," *The RAND Journal of Economics*, 2019, *50* (4), 733–767.

Luttmer, Erzo G. J., "Selection, Growth, and the Size Distribution of Firms," *The Quarterly Journal of Economics*, 08 2007, *122* (3), 1103–1144.

_ , "On the Mechanics of Firm Growth," *The Review of Economic Studies*, 02 2011, *78* (3), 1042–1068.

Luttmer, Erzo G.J., "Models of Growth and Firm Heterogeneity," *Annual Review of Economics*, 2010, *2* (1), 547–576.

Melitz, Marc J., "The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity," *Econometrica*, 2003, *71* (6), 1695–1725.

Mezzanotti, Filippo and Timothy Simcoe, "Innovation and Appropriability: Revisiting the Role of Intellectual Property," Working Paper 31428, National Bureau of Economic Research July 2023.

NCSES, "Business Enterprise Research and Development: 2020," NSF 23-314. Alexandria, VA: National Science Foundation 2023. Available at https://ncses.nsf.gov/pubs/nsf23314/.

O'Connor, A. C., M. P. Gallaher, K. Clark-Sutton, D. Lapidus, Z. T. Oliver, T. J. Scott, D. W. Wood, M. A. Gonzalez, E. G. Brown, and J. Fletcher, "Economic Benefits of the Global Positioning System (GPS)," Technical Report RTI Report Number 0215471, RTI International, Research Triangle Park, NC June 2019. Sponsored by the National Institute of Standards and Technology.

Ottonello, Pablo and Thomas Winberry, "Capital, Ideas, and the Costs of Financial Frictions," Working Paper 32056, National Bureau of Economic Research January 2024.

Romer, Paul M., "Endogenous Technological Change," *Journal of Political Economy*, 1990, *98* (5, Part 2), S71–S102.

Sterk, Vincent, Petr Sedláček, and Benjamin Pugsley, "The Nature of Firm Growth," *American Economic Review*, February 2021, *111* (2), 547–79.

Zacchia, Paolo, "Knowledge Spillovers through Networks of Scientists," *The Review of Economic Studies*, 05 2019, *87* (4), 1989–2018.

Online Appendix for: Quantifying Knowledge Spillovers Using Firm and Product Dynamics

Mohamad Adhami

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A Summary of Notation and Symbols

Table A1: Summary of notation and symbols

η	Population growth rate
ρ	Rate of time preference
γ	Coefficient of relative risk aversion
σ	Elasticity of substitution
β	Drift in incumbent's log quality
ν	Diffusion in incumbent's log quality
θ	Spillover elasticity
α	Thinness of entry distribution
ε	Inverse of firm entry cost
$\mathcal{F}$	Overhead cost
δ	Per-product diminishing returns in incumbent innovation
ϑ	Scale parameter in incumbent innovation
A	Process efficiency
ζ	Luttmer tail index (defined in Proposition 2)
ξ	Parameter in HJB solution (defined in Proposition 3)
I	Equilibrium level of R&D labor per incumbent product (equation 11 in main text)
x	Product addition rate per existing product along BGP (equation 15 in main text)
$\underline{L}$	Production employment at exit threshold (defined in Table C3)
$\theta\eta - \alpha\beta$	Product Exit due to downward drift
$\alpha\nu$	Relative volatility of shocks

Table A2: Summary of notation and symbols (continued)

N_t	Population	K_t	Stock of products created
Q_{pt}	Product's quality	$\underline{Q}_t$	Endogenous exit threshold
q_{pt}	Product's log quality relative to $\underline{Q}_t$	$m(q,t)$	Cross-sectional measure of q_{pt}
r_t	Interest rate	w_t	Wage
P_{pt}	Price of p	$V_t(Q_{pt})$	Product's value, with $V(q,t) = V_t\left(\underline{Q}_t e^q\right)$
c_t	Per capita consumption	a_t	Individual's asset holding
c_{pt}	Per capita consumption of p	Y_{pt}	Aggregate supply of p
L_{pt}	p's production labor	I_{pt}	p's R&D labor (= I in equilibrium)
S_t	Startup entrepreneurs (entry labor)	$\overline{F}_t^E(.)$	CCDF of entry distribution
g_Q	Quality growth rate	g	Consumption per capita growth rate
$\mathcal{V}(q)$	Stationary product value	Ow_t	Option value of product addition
L_t	Aggregate production labor	Ω_t	Set of products supplied at t
M_t	Measure of products	$\overline{Q}_t$	Average quality supplied
$f_p(q)$	Stationary PDF of q_{pt}	E_t^f	Flow of entering firms
E_t/M_t	Product entry rate	D_t/M_t	Product exit rate
$\ell(a)$	PDF of product's lifespan	$d_p(a)$	Product's exit hazard at age a
$\Gamma(a)$	CDF of firm's lifespan	$d_f(a)$	Firm's exit hazard at age a
$\mu_{nt}(\mathbf{q})$	Measure of $n-$prod firms with portfolio $\mathbf{q}$	$f_n(\mathbf{q})$	$= \prod_{i=1}^{n} \varphi_n(q_i)$ PDF among $n-$prod firms
Ψ_n	Share of products held by $n-$prod firms	Φ_n	Share of firms with n products
λ_n	Mean prod exit rate for $n-$prod firm		

B Proofs and derivations of results in main text

B.1 Proof of Proposition 1

Using the KFE, a time invariant $f_p(q)$ requires $K_t^{\theta}\underline{Q}_t^{-\alpha}$ to be constant. But $\underline{Q}_t$ grows at constant rate g_Q, so K_t grows at constant rate, and the latter is η since

$$\dot{K}_t = \varepsilon S_t + \frac{\varepsilon}{1-\delta} I M_t \quad \text{and} \quad \frac{\dot{M}_t}{M_t} = \frac{\dot{S}_t}{S_t} = \eta \ .$$

It follows that $\underline{Q}_t$ grows at rate $\frac{\theta}{\alpha}\eta$. Finally,

$$c_t = A M_t^{\frac{1}{\sigma-1}} \overline{Q}_t \frac{L_t}{N_t} \implies g = \frac{1}{\sigma-1}\eta + g_Q \ ;$$

where I used the fact that along BGP, $\underline{Q}_t$ and $\overline{Q}_t$ grow at the same rate.

B.2 Proof of Proposition 2

Along the BGP, $m(q,t) = M_t f_p(q)$ and M_t grows at rate η, so:

$$\frac{\partial m(q,t)}{\partial t} = \eta\, M_t\, f_p(q) \quad ; \quad \frac{\partial m(q,t)}{\partial q} = M_t\, f_p'(q) \quad ; \quad \frac{\partial^2 m(q,t)}{\partial q^2} = M_t\, f_p''(q)$$

Plugging back into the KFE (equation 9 in main text) yields a second order ODE in $f_p(q)$:

$$\frac{\nu^2}{2} f_p''(q) + \left(g_Q - \beta\right) f_p'(q) - \eta\, f_p(q) = -\frac{E_t}{M_t}\, \alpha\, e^{-\alpha q} \ .$$

The stationary distribution solves this ODE subject to:

$$\int_0^{\infty} f_p(q)\, dq = 1 \quad ; \quad f_p(q) \geq 0 \quad ; \quad f_p(0) = 0$$

The first requirement leads to a zero coefficient on the positive homogeneous root, so:

$$f_p(q) = C_2 e^{-\zeta q} + C_3 e^{-\alpha q} \quad \text{with} \quad C_3 = \frac{-\alpha \frac{E_t}{M_t}}{\frac{\nu^2}{2}\alpha^2 - (g_Q - \beta)\alpha - \eta}$$

and ζ as defined in Proposition 2. The boundary condition $f_p(0) = 0$ yields $C_2 = -C_3$. Hence

$$1 = \int_0^{\infty} f_p(q) dq = \int_0^{\infty} C_3 \left(e^{-\alpha q} - e^{-\zeta q}\right) dq \implies C_3 = \frac{\alpha\zeta}{\zeta - \alpha} \implies f_p(q) = \frac{\alpha\ \zeta}{\zeta - \alpha}\left(e^{-\alpha q} - e^{-\zeta q}\right)$$

Equating the two expressions for C_3 and simplifying yields an expression for the entry rate:

$$\begin{aligned}
\frac{\alpha\zeta}{\zeta-\alpha} \;=\; \frac{-\frac{E_t}{M_t}\,\alpha}{\frac{\nu^2}{2}\alpha^2-(g_Q-\beta)\alpha-\eta} \quad &\Longrightarrow \quad \frac{E_t}{M_t} \;=\; \frac{\zeta}{\alpha-\zeta}\left(\frac{\nu^2}{2}\alpha^2-(g_Q-\beta)\alpha-\eta\right) \\
&\overset{\star}{\Longrightarrow} \quad \frac{E_t}{M_t} \;=\; \frac{\zeta}{\alpha-\zeta}\left(\frac{\nu^2}{2}\alpha^2-(g_Q-\beta)\alpha-\left(\frac{\nu^2}{2}\zeta^2-(g_Q-\beta)\zeta\right)\right) \\
&\Longrightarrow \quad \frac{E_t}{M_t} \;=\; \frac{\nu^2}{2}\zeta\,\alpha \;+\; \frac{\nu^2}{2}\zeta^2-(g_Q-\beta)\zeta \\
&\overset{\star}{\Longrightarrow} \quad \boxed{\frac{E_t}{M_t} \;=\; \frac{\nu^2}{2}\zeta\,\alpha \;+\; \eta}
\end{aligned}$$

where $\star$ follows from $-\zeta$ being a root of the characteristic polynomial. Notice this can be equivalently rewritten as:

$$\frac{E_t}{M_t} \;=\; \eta \;+\; \frac{\nu^2}{2}\,f'(0)\;.$$

Since the second summand is the exit rate, this confirms that the net entry rate is η, consistent with growth rate of M_t.

B.3 Proof of Proposition 3

Plugging the stationary distribution into the definition of $\overline{Q}_t$ yields

$$\overline{Q}_t \equiv \underline{Q}_t\left(\frac{1}{M_t}\int_0^\infty e^{(\sigma-1)q}m(q,t)dq\right)^{\frac{1}{\sigma-1}} \;=\; \underline{Q}_t\left(\int_0^\infty e^{(\sigma-1)q}f_p(q)dq\right)^{\frac{1}{\sigma-1}}$$

Assumption 2 in the main text guarantees that $\alpha > \sigma-1$ and $\zeta > \sigma-1$, so that:

$$\left(\frac{\overline{Q}_t}{\underline{Q}_t}\right)^{\sigma-1} \;=\; \frac{\alpha}{\alpha-(\sigma-1)}\,\frac{\zeta}{\zeta-(\sigma-1)}\;.$$

Plugging back into the HJB yields

$$\begin{aligned}
r_tV(q,t) = w_t\left[\frac{(\alpha-(\sigma-1))(\zeta-(\sigma-1))}{(\sigma-1)\alpha\zeta}\frac{L_t}{M_t}e^{(\sigma-1)q}-(\mathcal{F}-O)\right] \\
+\;\dot{V}(q,t)+\left(\beta-g_{Q_t}\right)\frac{\partial V(q,t)}{\partial q}+\frac{\nu^2}{2}\frac{\partial^2V(q,t)}{\partial q^2}\;.
\end{aligned}$$

Guess that $V(q,t) = w_t\mathcal{V}(q)$, then $\mathcal{V}(q)$ solves the second order ODE with constant coefficients:[1]

$$\frac{\nu^2}{2}\mathcal{V}''(q)+(\beta-g_Q)\mathcal{V}'(q)-(r-g)\mathcal{V}(q) \;=\; -\left[\frac{(\alpha-(\sigma-1))(\zeta-(\sigma-1))}{\alpha\zeta(\sigma-1)}\,\frac{L_t}{M_t}\,e^{(\sigma-1)q}-(\mathcal{F}-O)\right]$$

[1] L_t and M_t both grow at rate η, so their ratio is constant.

subject to:

$$\mathcal{V}(q) \geq 0 \; ; \; \mathcal{V}(0) = 0 \; ; \; \mathcal{V}'(0) = 0 \text{ and } \mathcal{V}(q) \; < \; \frac{\frac{(\alpha-(\sigma-1))(\zeta-(\sigma-1))}{\alpha\zeta(\sigma-1)} \frac{L_t}{M_t} e^{(\sigma-1)q}}{r - \left[g + (\sigma-1)(\beta - g_Q) \; + \; \frac{\nu^2}{2}(\sigma-1)^2\right]} \; .$$

The first constraint follows from the fact that the firm can always choose to shutdown production. The second and third constraints are respectively the value matching and smooth pasting conditions. To understand the fourth condition, note that the flow of dividends, gross of overhead and option value, is the numerator of the right hand side scaled by w_t. Using Ito's lemma, this flow grows at rate $g + (\sigma-1)(\beta - g_Q) \; + \; \frac{\nu^2}{2}(\sigma-1)^2$. So the right hand side of the fourth constraint is the PDV of flow of dividends gross of the overhead and option value. The inequality then follows from $\mathcal{F} > O$. The solution to this ODE is:

$$\mathcal{V}(q) \; = \; C_1 \, e^{zq} \; + \; C_2 \, e^{-\xi q} \; + \; \frac{\frac{(\alpha-(\sigma-1))(\zeta-(\sigma-1))}{\alpha\zeta(\sigma-1)}}{r - \left[g + (\sigma-1)(\beta - g_Q) \; + \; \frac{\nu^2}{2}(\sigma-1)^2\right]} \frac{L_t}{M_t} \, e^{(\sigma-1)q} \; - \; \frac{\mathcal{F} - O}{r - g}$$

$$\text{with} \quad z \equiv \frac{-(\beta - g_Q) + \sqrt{(g_Q - \beta)^2 + 2\nu^2(r-g)}}{\nu^2} \; \text{ and } \; \xi \equiv \frac{\beta - g_Q + \sqrt{(\beta - g_Q)^2 + 2\nu^2(r-g)}}{\nu^2}$$

Since $z > 0$ (while $-\xi < 0$), satisfying the inequality constraints on $\mathcal{V}(q)$ requires $C_1 = 0$. The value matching and smooth pasting conditions are two equations in two unknowns, $\frac{L_t}{M_t}$ and C_2. Solving yields:

$$\mathcal{V}(q) = \frac{\mathcal{F} - O}{r - g} \left[\frac{\xi}{\xi + \sigma - 1} e^{(\sigma-1)q} + \frac{\sigma - 1}{\xi + \sigma - 1} e^{-\xi q} - 1 \right]$$

$$\frac{L_t}{M_t} = (\mathcal{F} - O) \, \frac{\alpha\zeta(\sigma-1)}{(\alpha - (\sigma-1))(\zeta - (\sigma-1))} \frac{\xi}{\xi + \sigma - 1} \frac{r - [g + (\sigma-1)(\beta - g_Q) + \frac{\nu^2}{2}(\sigma-1)^2]}{r - g} \; .$$

B.4 Labor allocations along balanced growth path

One more equation is needed to pin down labor allocations along BGP. It follows from the free entry condition along the BGP. Using a change of variables, relative quality of new products is drawn from the density

$$\frac{K_t^\theta}{\underline{Q}_t^\alpha} \alpha e^{-\alpha q} \quad \text{for} \quad q \; \geq \; \ln\left(\frac{K_t^\theta}{\underline{Q}_t^\alpha}\right)$$

where this pdf is time-invariant along the BGP since $\underline{Q}_t^\alpha$ and K_t^θ both grow at rate $\theta\eta$. Focusing on the case described in assumption 3 of the main text, the free entry condition reads

$$\varepsilon \int_0^\infty w_t \mathcal{V}(q) \frac{K_t^\theta}{\underline{Q}_t^\alpha} \alpha e^{-\alpha q} \, dq \; = \; w_t$$

The integration starts at 0 because for $q \leq 0$, $\mathcal{V}(q) = 0$. Plugging in $\mathcal{V}$ yields:

$$\frac{\underline{Q}_t^{\alpha}}{K_t^{\theta}} \;=\; \varepsilon\, \frac{\mathcal{F}-O}{r-g}\, \frac{(\sigma-1)\xi}{(\alpha+\xi)(\alpha-(\sigma-1))}\,.$$

Accordingly, the case described in assumption 3 obtains if the PDV of the effective fixed cost of operation relative to the entry cost is not small:

$$\boxed{\frac{\frac{\mathcal{F}-O}{r-g}}{\frac{1}{\varepsilon}} \;\geq\; \left(1+\frac{\alpha}{\xi}\right)\left(\frac{\alpha}{\sigma-1}-1\right)} \tag{1}$$

To see how the expression for $\frac{\underline{Q}_t^{\alpha}}{K_t^{\theta}}$ helps us pin down the labor allocations, note that

$$\eta K_t = \dot{K}_t \;=\; \varepsilon S_t \;+\; \frac{\vartheta}{1-\delta} I^{1-\delta} M_t \;\Longrightarrow\; \frac{S_t}{M_t} \;=\; \frac{\eta}{\varepsilon}\frac{K_t}{M_t} \;-\; \frac{I}{1-\delta}$$

where I used the fact that $I = \left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}}$. Now,

$$E_t = \overline{F}_t^{E}\left(\underline{Q}_t\right)\dot{K}_t = \frac{K_t^{\theta}}{\underline{Q}_t^{\alpha}}\eta K_t \;\Longrightarrow\; \frac{K_t}{M_t} = \frac{1}{\eta}\frac{E_t}{M_t}\frac{\underline{Q}_t^{\alpha}}{K_t^{\theta}}$$

Plugging the expression for $\frac{K_t}{M_t}$ back into $\frac{S_t}{M_t}$, I get:

$$\frac{S_t}{M_t} \;=\; \frac{1}{\varepsilon}\,\frac{E_t}{M_t}\,\frac{\underline{Q}_t^{\alpha}}{K_t^{\theta}} \;-\; \frac{I}{1-\delta}\,.$$

Plugging in the expressions for the entry rate and $\frac{\underline{Q}_t^{\alpha}}{K_t^{\theta}}$ yields:

$$\frac{S_t}{M_t} \;=\; \left(\eta \;+\; \frac{\nu^2}{2}\zeta\alpha\right)\,\frac{\mathcal{F}-O}{r-g}\,\frac{(\sigma-1)\xi}{(\alpha+\xi)(\alpha-(\sigma-1))} \;-\; \frac{I}{1-\delta}$$

Adding $\mathcal{F}+I$ on both sides and using $O = \frac{\delta}{1-\delta}I$ yields:

$$\frac{S_t}{M_t} \;+\; I \;+\; \mathcal{F} \;=\; (\mathcal{F}-O)\left[\frac{\eta \;+\; \frac{\nu^2}{2}\zeta\alpha}{r-g}\,\frac{(\sigma-1)\xi}{(\alpha+\xi)(\alpha-(\sigma-1))} \;+\; 1\right].$$

Using the labor resource constraint, one can solve for M_t, S_t, and L_t as functions of N_t and parameters.

B.5 Proof of Proposition 4: Planner's problem

Define $z_{pt} = \ln Q_{pt}$ and denote by $\mu(z,t)$ the measure of products with log-quality z at t. The entry distribution in terms of z then has density

$$\widetilde{f}_t^E(z) = \alpha K_t^\theta e^{-\alpha z} \text{ for } z \geq \frac{\theta}{\alpha} \ln K_t$$

By symmetry, the planner picks $I_{pt} = I_t$ and strict positivity follows from the Inada condition at 0. Moreover, once total production L_t is chosen, standard CES allocations yield

$$L_t = \int_{z>\underline{z}_t} L_t(z)\mu(z,t)dz \implies c_t = \frac{AL_t}{N_t}\left(\int_{z>\underline{z}_t} e^{(\sigma-1)z}\,\mu(z,t)\,dz\right)^{\frac{1}{\sigma-1}}.$$

So the planner's problem features an infinite dimensional state variable (Nuño and Moll, 2018):

$$\max_{\substack{S_t,L_t,I_t\\ d_t}} \int_0^\infty e^{-(\rho-\eta)t} \frac{\left(A\frac{L_t}{N_t}\left(\int_{z>\underline{z}_t} e^{(\sigma-1)z}\,\mu(z,t)\,dz\right)^{\frac{1}{\sigma-1}}\right)^{1-\gamma} - 1}{1-\gamma} dt$$

$$\begin{aligned}
\text{subject to} \quad & \mu\left(\underline{z}_t, t\right) = 0 \\
& \forall z > \underline{z}_t, \quad \dot{\mu}(z,t) = -\beta\frac{\partial\mu(z,t)}{\partial z} + \frac{\nu^2}{2}\frac{\partial^2\mu(z,t)}{\partial z^2} + \dot{K}_t\, K_t^\theta \alpha e^{-\alpha z}\mathbb{1}_{\{z>\frac{\theta}{\alpha}\ln K_t\}} \\
& \dot{\underline{z}}_t = d_t \geq 0 \\
& \dot{K}_t = \varepsilon S_t + \frac{\vartheta}{1-\delta} I_t^{1-\delta}\int_{z>\underline{z}_t}\mu(z,t)dz \\
& N_t = S_t + L_t + \int_{z\geq\underline{z}_t} (\mathcal{F} + I_t)\;\mu(z,t)\,dz
\end{aligned}$$

This is an optimal control problem with S_t, L_t, I_t, and d_t (how much to lift the threshold) as controls. The states are $\underline{z}_t$ (lowest quality still "alive"), K_t and $\{\mu(z,t)\}$. Let ω_t be the Lagrange multiplier on the labor resource constraint, $\Upsilon(z,t)$ be the costate associated with $\mu(z,t)$, χ_t the costate associated with K_t, and Ξ_t the costate associated with $\underline{z}_t$. Then the current-value Hamiltonian is:

$$\begin{aligned}
\mathcal{H} = \frac{c_t^{1-\gamma}-1}{1-\gamma} + \chi_t\left(\varepsilon S_t + \frac{\vartheta}{1-\delta}I_t^{1-\delta}\int_{z>\underline{z}_t}\mu(z,t)dz\right) + \int \Upsilon(z,t)\dot{\mu}(z,t)dz \\
+ \omega_t\left(N_t - S_t - L_t - \int_{z\geq\underline{z}_t}(\mathcal{F}+I_t)\;\mu(z,t)\,dz\right) + \Xi_t d_t
\end{aligned}$$

It is going to be helpful to plug the KFE into $\int \Upsilon(z,t)\dot{\mu}(z,t)dz$ and then integrate by parts to move the derivatives to Υ. To deal with the boundary terms,

Assumption 1. I suppose (and verify later) that:

$$\lim_{z\to\infty} \Upsilon(z,t)\mu(z,t) = 0 \quad \text{and} \quad \lim_{z\to\infty} \Upsilon(z,t)\frac{\partial \mu(z,t)}{\partial z} = 0\,.$$

Using these assumptions, along with $\mu(\underline{z}_t, t) = 0$:

$$\int_{\underline{z}_t}^{\infty} \Upsilon(z,t)\dot{\mu}(z,t)dz = -\frac{\nu^2}{2}\frac{\partial \mu\left(\underline{z}_t,t\right)}{\partial z}\Upsilon\left(\underline{z}_t,t\right) + \int_{\underline{z}_t}^{\infty} \mu(z,t)\left(\beta\frac{\partial \Upsilon(z,t)}{\partial z} + \frac{\nu^2}{2}\frac{\partial^2\Upsilon(z,t)}{\partial z^2}\right)dz$$
$$+ \dot{K}_t K_t^{\theta}\int_{\underline{z}_t}^{\infty} \Upsilon(z,t)\alpha e^{-\alpha z}\mathbb{1}_{\left\{z>\frac{\theta}{\alpha}\ln K_t\right\}}dz$$

$$\implies \mathcal{H} = \frac{\left(A\frac{L_t}{N_t}\left(\int_{z>\underline{z}_t} e^{(\sigma-1)z}\,\mu(z,t)\,dz\right)^{\frac{1}{\sigma-1}}\right)^{1-\gamma} - 1}{1-\gamma} + \chi_t\left(\varepsilon S_t + \frac{\vartheta}{1-\delta}I_t^{1-\delta}\int_{z>\underline{z}_t}\mu(z,t)dz\right)$$
$$- \frac{\nu^2}{2}\Upsilon\left(\underline{z}_t,t\right)\frac{\partial \mu\left(\underline{z}_t,t\right)}{\partial z} + \int_{\underline{z}_t}^{\infty}\mu(z,t)\left(\beta\frac{\partial\Upsilon(z,t)}{\partial z} + \frac{\nu^2}{2}\frac{\partial^2\Upsilon(z,t)}{\partial z^2}\right)dz$$
$$+ \left(\varepsilon S_t + \frac{\vartheta}{1-\delta}I_t^{1-\delta}\int_{z>\underline{z}_t}\mu(z,t)dz\right)K_t^{\theta}\int_{\underline{z}_t}^{\infty}\alpha e^{-\alpha z}\Upsilon(z,t)\mathbb{1}_{\left\{z>\frac{\theta}{\alpha}\ln K_t\right\}}dz$$
$$+ \omega_t\left(N_t - S_t - L_t - \int_{z\geq\underline{z}_t}(\mathcal{F}+I_t)\;\mu(z,t)\;dz\right) + \Xi_t d_t\,.$$

The optimality conditions for the controls L_t, S_t, I_t and d_t are (respectively):

$$\mathcal{H}_{L_t} = 0 \implies \omega_t = \frac{c_t^{1-\gamma}}{L_t} \tag{2}$$
$$\mathcal{H}_{S_t} = 0 \implies \omega_t = \varepsilon\left(\chi_t + K_t^{\theta}\int_{\underline{z}_t}^{\infty}\alpha e^{-\alpha z}\Upsilon(z,t)\mathbb{1}_{\left\{z>\frac{\theta}{\alpha}\ln K_t\right\}}dz\right) \tag{3}$$
$$\mathcal{H}_{I_t} = 0 \implies \omega_t = \vartheta I_t^{-\delta}\left(\chi_t + K_t^{\theta}\int_{\underline{z}_t}^{\infty}\alpha e^{-\alpha z}\Upsilon(z,t)\mathbb{1}_{\left\{z>\frac{\theta}{\alpha}\ln K_t\right\}}dz\right) \tag{4}$$
$$\Xi_t d_t = 0 \implies d_t = 0 \text{ or } \Xi_t = 0$$

$$\underline{z}_t \text{ adjoint } \implies (\rho-\eta)\Xi_t = \dot{\Xi}_t - \frac{\nu^2}{2}\left(\frac{\partial \Upsilon\left(\underline{z}_t,t\right)}{\partial z}\frac{\partial \mu\left(\underline{z}_t,t\right)}{\partial z} + \Upsilon\left(\underline{z}_t,t\right)\frac{\partial^2 \mu\left(\underline{z}_t,t\right)}{\partial z^2}\right)$$
$$-\left(\varepsilon S_t + \frac{\vartheta}{1-\delta}I_t^{1-\delta}\int_{z>\underline{z}_t}\mu(z,t)dz\right)K_t^\theta \alpha e^{-\alpha \underline{z}_t}\Upsilon\left(\underline{z}_t,t\right)\mathbb{1}_{\left\{\underline{z}_t > \frac{\theta}{\alpha}\ln K_t\right\}}dz \tag{5}$$

$$K_t \text{ adjoint } \implies (\rho-\eta)\chi_t = \dot{\chi}_t + \theta\frac{\dot{K}_t}{K_t}\left(K_t^\theta\int_{\underline{z}_t}^\infty \alpha e^{-\alpha z}\Upsilon(z,t)\mathbb{1}_{\left\{z>\frac{\theta}{\alpha}\ln K_t\right\}}dz - \Upsilon\left(\frac{\theta}{\alpha}\ln K_t,t\right)\mathbb{1}_{\left\{\underline{z}_t \le \frac{\theta}{\alpha}\ln K_t\right\}}\right) \tag{6}$$

Finally, adjoint corresponding to $\mu(z,t)$ yields:

$$(\rho-\eta)\Upsilon(z,t) = \dot{\Upsilon}(z,t) + \beta\frac{\partial\Upsilon(z,t)}{\partial z} + \frac{\nu^2}{2}\frac{\partial^2\Upsilon(z,t)}{\partial z^2} + \frac{1}{\sigma-1}\frac{c_t^{1-\gamma}}{M_t}\left(\frac{e^z}{\overline{Q}_t}\right)^{\sigma-1} - \omega_t(\mathcal{F}+I_t)$$
$$+\frac{\vartheta}{1-\delta}I_t^{1-\delta}\left(\chi_t + K_t^\theta\int_{\underline{z}_t}^\infty \alpha e^{-\alpha z}\Upsilon(z,t)\mathbb{1}_{z>\frac{\theta}{\alpha}\ln K_t}dz\right) \tag{7}$$

where I used M_t and $\overline{Q}_t$ as defined in the main text so that

$$\int_{z\ge\underline{z}_t}\mu(z,t)dz = M_t \quad \text{and} \quad \int_{z\ge\underline{z}_t}e^{(\sigma-1)z}\mu(z,t)dz = M_t\overline{Q}_t^{\sigma-1}$$

Finally, the transversality conditions are

$$0 = \lim_{t\to\infty}e^{-(\rho-\eta)t}\Xi_t\underline{z}_t = \lim_{t\to\infty}e^{-(\rho-\eta)t}\chi_t K_t = \lim_{t\to\infty}e^{-(\rho-\eta)t}\Upsilon(z,t)\mu(z,t)$$

$$\text{Combining Equation 3 and Equation 4} \implies I_t = \left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}} \implies \text{as in equilibrium}$$

Plugging Equation 4 and Equation 2 into Equation 7 and using $O \equiv \frac{\delta}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}}$ (as in equilibrium):

$$(\rho-\eta)\Upsilon(z,t) = \dot{\Upsilon}(z,t) + \beta\frac{\partial\Upsilon(z,t)}{\partial z} + \frac{\nu^2}{2}\frac{\partial^2\Upsilon(z,t)}{\partial z^2} + \omega_t\left(\frac{1}{\sigma-1}\frac{L_t}{M_t}\left(\frac{e^z}{\overline{Q}_t}\right)^{\sigma-1} - (\mathcal{F}-O)\right)$$

Change of coordinates

With stationarity in mind, it is going to be helpful to work with relative coordinates: $q = z - \underline{z}_t$. For $q \ge 0$, define the value function $V^{\text{SP}}(q,t)$ and cross-sectional distribution $m(q,t)$ by

$$V^{\text{SP}}(q,t) = \Upsilon(q+\underline{z}_t,t) \quad \text{and} \quad m(q,t) = \mu(q+\underline{z}_t,t)$$

Using the chain rule and $\dot{\underline{z}}_t = d_t$, the planner's HJB becomes:

$$(\rho-\eta)V^{\text{SP}}(q,t) = \dot{V}^{\text{SP}}(q,t) + (\beta - d_t)\frac{\partial V^{\text{SP}}(q,t)}{\partial q} + \frac{\nu^2}{2}\frac{\partial^2 V^{\text{SP}}(q,t)}{\partial q^2} + \omega_t\left[\frac{1}{\sigma-1}\frac{L_t}{M_t}\left(\frac{\underline{Q}_t}{\overline{Q}_t}\right)^{\sigma-1} e^{(\sigma-1)q} - (\mathcal{F}-O)\right]$$

The KFE is as in Equation 9 of the main text. Equation 3, Equation 5, and Equation 6 become:

$$\omega_t = \varepsilon\left(\chi_t + K_t^\theta \underline{Q}_t^{-\alpha}\int_0^\infty \alpha e^{-\alpha q} V^{\text{SP}}(q,t)\mathbb{1}_{\left\{q>\ln \frac{K_t^{\frac{\theta}{\alpha}}}{\underline{Q}_t}\right\}} dq\right), \tag{8}$$

$$(\rho-\eta)\Xi_t = \dot{\Xi}_t - \frac{\nu^2}{2}\left[\frac{\partial V^{\text{SP}}(0,t)}{\partial q}\frac{\partial m(0,t)}{\partial q} + V^{\text{SP}}(0,t)\frac{\partial^2 m(0,t)}{\partial q^2}\right] - \left(\varepsilon S_t + \frac{\vartheta}{1-\delta}I_t^{1-\delta}M_t\right)\alpha K_t^\theta \underline{Q}_t^{-\alpha} V^{\text{SP}}(0,t)\mathbb{1}_{\{\frac{\theta}{\alpha}\ln K_t<0\}}, \tag{9}$$

$$(\rho-\eta)\chi_t = \dot{\chi}_t + \theta\frac{\dot{K}_t}{K_t}\left(K_t^\theta \underline{Q}_t^{-\alpha}\int_0^\infty \alpha e^{-\alpha q} V^{\text{SP}}(q,t)\mathbb{1}_{\left\{q>\ln \frac{K_t^{\frac{\theta}{\alpha}}}{\underline{Q}_t}\right\}} dq - V^{\text{SP}}\left(\ln\frac{K_t^{\frac{\theta}{\alpha}}}{\underline{Q}_t}, t\right)\mathbb{1}_{\left\{\ln \frac{K_t^{\frac{\theta}{\alpha}}}{\underline{Q}_t}>0\right\}}\right) \tag{10}$$

Finally, the transversality conditions become:

$$0 = \lim_{t\to\infty} e^{-(\rho-\eta)t}\Xi_t \underline{z}_t = \lim_{t\to\infty} e^{-(\rho-\eta)t}\chi_t K_t = \lim_{t\to\infty} e^{-(\rho-\eta)t} V^{\text{SP}}(q,t)m(q,t)$$

And Assumption 1 becomes

$$\lim_{q\to\infty} V^{\text{SP}}(q,t)m(q,t) = 0 \quad \text{and} \quad \lim_{q\to\infty} V^{\text{SP}}(q,t)\frac{\partial m(q,t)}{\partial q} = 0\,.$$

Balanced Growth Path

The definition of BGP was given in the main text (Definition 2). Proposition 1 still applies. Here again, I focus on the BGP satisfying Assumption 3 of the main text—as the below will make clear, Equation 1 is sufficient since:

$$\left(\frac{\underline{Q}_t^\alpha}{K_t^\theta}\right)^{\textbf{FB}} \geq \left(\frac{\underline{Q}_t^\alpha}{K_t^\theta}\right)^{\textbf{DE}}.$$

Since the KFE is unchanged, $f_p(q)$ is as in equilibrium. For the HJB, guess and verify $V^{\text{SP}}(q,t) = \omega_t \mathcal{V}(q)$. As I will show, this is not an abuse of notation, as $\mathcal{V}(q)$ will coincide with the function

defined in Proposition 3. To get there, note that

$$\text{Equation 2} \implies \frac{\dot{\omega}_t}{\omega_t} = (1-\gamma)g - \eta\,,$$

so that by plugging back into the HJB, we know $\mathcal{V}^{\text{SP}}(q)$ satisfies the following ODE:

$$(\rho + (\gamma-1)g)\mathcal{V}(q) = (\beta - g_Q)\mathcal{V}'(q) + \frac{\nu^2}{2}\mathcal{V}''(q) + \frac{1}{\sigma-1}\frac{L_t}{M_t}\left(\frac{\underline{Q}_t}{\overline{Q}_t}\right)^{\sigma-1} e^{(\sigma-1)q} - (\mathcal{F} - O)$$

This matches the ODE I solve in subsection B.3 to prove Proposition 3. What remains to be shown is that the boundary conditions are the same. The ones at 0 follow from Equation 9 combined with $\Xi_t = 0$ along the BGP. The latter follows directly from $g_Q > 0$ if $\theta > 0$ and from $\underline{Q}_t > K_t^{\frac{\theta}{\alpha}}$ otherwise. As such, the TVC for Ξ_t is trivially satisfied and:

$$-\frac{\nu^2}{2}\omega_t\left[\mathcal{V}'(0)f_p'(0) + \mathcal{V}(0)f_p''(0)\right] = \eta K_t \alpha K_t^{\theta}\underline{Q}_t^{-\alpha}\mathcal{V}(0)$$

Suppose $\mathcal{V}(0) \neq 0$, then the growth rate of the LHS is $(1-\gamma)g - \eta$ while that of the RHS is η, which is a contradiction. Hence $\mathcal{V}(0) = 0$. Since $f_p''(0) > 0$, it follows that $\mathcal{V}'(0) = 0$. The last boundary condition will follow from Assumption 1, which requires

$$\lim_{q\to\infty}\mathcal{V}(q)f_p(q) = 0 \quad \text{and} \quad \lim_{q\to\infty}\mathcal{V}(q)f_p'(q) = 0$$

In addition to the requirements $\alpha > \sigma - 1$ and $\zeta > \sigma - 1$, these lead to the inequality constraint on $\mathcal{V}(q)$ (see Appendix B.1.2). As a result, we end up with the same ODE and boundary condition so that $\mathcal{V}(q)$ is indeed the same.

Finally, to obtain the planner's analogue of the free entry condition, divide both sides of Equation 8 by ω_t, which yields

$$\frac{1}{\varepsilon} = \frac{\chi_t}{\omega_t} + K_t^{\theta}\underline{Q}_t^{-\alpha}\int_0^{\infty}\alpha e^{-\alpha q}\mathcal{V}(q)dq \implies \frac{\dot{\chi}_t}{\chi_t} = \frac{\dot{\omega}_t}{\omega_t} = (1-\gamma)g - \eta \tag{11}$$

Plugging this growth rate of $\frac{\dot{\chi}_t}{\chi_t}$ into Equation 10,

$$\rho + (\gamma-1)g = \frac{\omega_t}{\chi_t}\theta\eta\frac{K_t^{\theta}}{\underline{Q}_t^{\alpha}}\int_0^{\infty}\alpha e^{-\alpha q}\mathcal{V}(q)dq \implies \frac{\chi_t}{\omega_t} = \frac{\theta\eta}{\rho + (\gamma-1)g}K_t^{\theta}\underline{Q}_t^{-\alpha}\int_0^{\infty}\alpha e^{-\alpha q}\mathcal{V}(q)dq$$

$$\overset{\text{Equation 11}}{\implies} \boxed{1 = \varepsilon\left(1 + \frac{\theta\eta}{\rho + (\gamma-1)g}\right)K_t^{\theta}\underline{Q}_t^{-\alpha}\int_0^{\infty}\alpha e^{-\alpha q}\mathcal{V}(q)dq}$$

This is the planner's analogue of the free entry condition, and it is the only condition that differs across the first best and equilibrium. That the level of incumbent innovation per product (I) is

efficient might seem surprising at first, as this activity generates positive knowledge spillovers – just as much as the entry of new firms. The way to think about it is as follows. Due to spillovers, there is too little aggregate innovation in equilibrium. However, the incumbent's innovation technology satisfies an Inada condition at 0. So, both in equilibrium and the first best, initial units of innovation are carried out by incumbents – until diminishing returns push the marginal product from this technology to ε, at which point the rest of innovation to be done is carried with the linear entry technology (new firms). Along an interior BGP, this point is necessarily reached, so that the underprovision of innovation shows up entirely along the entry of new firms margin.

B.6 Derivation of the social rate of return to R&D

Here I derive the social rate of return to R&D as the return on a variational argument around a BGP (Jones and Williams, 1998). For these purposes, note that the economy is simply given by:

$$\begin{aligned}
Y_t &= L_t \underline{Q}_t A_t^{\frac{1}{\sigma-1}} \qquad \text{where} \quad A_t \equiv A \int_0^\infty e^{(\sigma-1)q} m(q,t) dq \\
\dot{m}(q,t) &= \left(g_Q - \beta\right) \frac{\partial m(q,t)}{\partial q} + \frac{\nu^2}{2} \frac{\partial^2 m(q,t)}{\partial q^2} + \varepsilon R_t K_t^\theta \underline{Q}_t^{-\alpha} \alpha e^{-\alpha q} \\
m(0,t) &= 0 \\
\dot{K}_t &= \varepsilon R_t \\
N_t &= L_t + R_t + (\mathcal{F} - O) M_t \qquad \text{where} \quad R_t \equiv S_t + \frac{1}{1-\delta} I M_t \ .
\end{aligned}$$

Denoting by ∇ deviations from the initial balanced growth path, note that from the law of motion:

$$\forall t, \quad \nabla \dot{m}(q,t) = \mathcal{L} \nabla m(q,t) + \varepsilon \alpha e^{-\alpha q} \left(K_t^\theta \underline{Q}_t^{-\alpha} \nabla R_t + \theta K_t^{\theta-1} \underline{Q}_t^{-\alpha} R_t \nabla K_t - K_t^\theta \alpha \underline{Q}_t^{-\alpha-1} R_t \nabla \underline{Q}_t \right)$$

$$\text{where} \quad \mathcal{L} \equiv -(\beta - g_Q)\partial_q + \frac{\nu^2}{2} \partial_{qq} \ .$$

The specific variational argument of interest is:

1. from t to $t + dt$, the economy does more R&D by reducing L_t and raising R_t;
2. from $t + dt$ to $t + 2dt$, the economy "eats the proceeds" by doing sufficiently less R&D to be back at initial BGP path by $t + 2dt$.

The social rate of return is then defined as the rate of return on this variational argument as $dt \to 0$:

$$\widetilde{r} \equiv \lim_{dt \to 0} \frac{\nabla Y_{t+dt} - \frac{Y_t}{L_t} \nabla R_t}{\frac{Y_t}{L_t} \nabla R_t dt} \ .$$

Intuitively, this is a rate of return because $\frac{Y_t}{L_t} \nabla R_t$ is the amount of output (and hence consumption) that the variational argument sacrifices at t, while ∇Y_{t+dt} is the resulting increase in output at $t + dt$.

Since m is a state variable and the variational argument starts at t, $\nabla m(q,t) = 0$, so

$$m(q,t+dt) = m(q,t) + \dot{m}(q,t)dt \implies \nabla m(q,t+dt) = \varepsilon \underline{Q}_t^{-\alpha} K_t^{\theta} \alpha e^{-\alpha q} \nabla R_t dt$$

In contrast

$$\begin{aligned}
\nabla m(q,t+2dt) &= \nabla m(q,t+dt) + \nabla \dot{m}(q,t+dt)dt \\
&= \nabla m(q,t+dt) + \left(\mathcal{L}\nabla m(q,t+dt) + \varepsilon \underline{Q}_{t+dt}^{-\alpha} \alpha e^{-\alpha q} \left(K_{t+dt}^{\theta} \nabla R_{t+dt} + \theta K_{t+dt}^{\theta-1} \nabla K_{t+dt} R_{t+dt} \right) \right) dt \\
\text{with} \quad \mathcal{L}\nabla \mu(q,t+dt) &= \varepsilon \underline{Q}_t^{-\alpha} K_t^{\theta} \alpha e^{-\alpha q} \left((\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 \right) \nabla R_t dt \\
\dot{K}_{t+dt} = K_t + \varepsilon R_t dt &\implies \nabla K_{t+dt} = \varepsilon \nabla R_t dt
\end{aligned}$$

The variational argument requires $\nabla m(q,t+2dt) = 0$. Solving for ∇R_{t+dt} in terms of ∇R_t yields

$$-\nabla R_{t+dt} = \left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}} \right)^{-\alpha} \left(\frac{K_t}{K_{t+dt}} \right)^{\theta} \nabla R_t + \left[\left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}} \right)^{-\alpha} \left(\frac{K_t}{K_{t+dt}} \right)^{\theta} \left((\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 \right) + \theta\varepsilon \frac{R_{t+dt}}{K_{t+dt}} \right] \nabla R_t dt \ .$$

The increase in output at $t + dt$ is due to higher TFP and higher productional labor:

$$Y_{t+dt} = L_{t+dt} \underline{Q}_{t+dt} A_{t+dt}^{\frac{1}{\sigma-1}} \implies \nabla Y_{t+dt} = \frac{Y_{t+dt}}{L_{t+dt}} \nabla L_{t+dt} + \frac{1}{\sigma - 1} \frac{Y_{t+dt}}{A_{t+dt}} \nabla A_{t+dt}$$

Now

$$\nabla L_{t+dt} + (\mathcal{F} - O)\nabla M_{t+dt} + \nabla R_{t+dt} = 0 \quad \text{and} \quad \nabla M_{t+dt} = K_t^{\theta} \underline{Q}_t^{-\alpha} \varepsilon \nabla R_t dt$$

$$\implies \nabla L_{t+dt} = \frac{\underline{Q}_t^{-\alpha} K_t^{\theta}}{\underline{Q}_{t+dt}^{-\alpha} K_{t+dt}^{\theta}} \nabla R_t + \left[\frac{\underline{Q}_t^{-\alpha} K_t^{\theta}}{\underline{Q}_{t+dt}^{-\alpha} K_{t+dt}^{\theta}} \left((\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 \right) + \theta\varepsilon \frac{R_{t+dt}}{K_{t+dt}} - K_t^{\theta} \underline{Q}_t^{-\alpha} \varepsilon(\mathcal{F} - O) \right] \nabla R_t dt$$

$$\begin{aligned}
\implies \nabla Y_{t+dt} &= \frac{Y_{t+dt}}{L_{t+dt}} \left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}} \right)^{-\alpha} \left(\frac{K_t}{K_{t+dt}} \right)^{\theta} \nabla R_t + \frac{1}{\sigma - 1} \frac{Y_{t+dt}}{A_{t+dt}} \varepsilon \underline{Q}_t^{-\alpha} K_t^{\theta} \frac{\alpha}{\alpha - (\sigma - 1)} \nabla R_t dt \\
&\quad + \frac{Y_{t+dt}}{L_{t+dt}} \left[\left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}} \right)^{-\alpha} \left(\frac{K_t}{K_{t+dt}} \right)^{\theta} \left((\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 \right) + \theta\varepsilon \frac{R_{t+dt}}{K_{t+dt}} - K_t^{\theta} \underline{Q}_t^{-\alpha} \varepsilon(\mathcal{F} - O) \right] \nabla R_t dt \\
\implies \frac{\nabla Y_{t+dt} - \frac{Y_t}{L_t} \nabla R_t}{\frac{Y_t}{L_t} \nabla R_t dt} &= \frac{1}{dt} \left[\frac{\frac{Y_{t+dt}}{L_{t+dt}}}{\frac{Y_t}{L_t}} \left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}} \right)^{-\alpha} \left(\frac{K_t}{K_{t+dt}} \right)^{\theta} - 1 \right] + \frac{1}{\sigma - 1} \frac{\frac{Y_{t+dt}}{A_{t+dt}}}{\frac{Y_t}{L_t}} \varepsilon \underline{Q}_t^{-\alpha} K_t^{\theta} \frac{\alpha}{\alpha - (\sigma - 1)} \\
&\quad + \frac{\frac{Y_{t+dt}}{L_{t+dt}}}{\frac{Y_t}{L_t}} \left[\left(\frac{\underline{Q}_t}{\underline{Q}_{t+dt}} \right)^{-\alpha} \left(\frac{K_t}{K_{t+dt}} \right)^{\theta} \left((\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 \right) + \theta\varepsilon \frac{R_{t+dt}}{K_{t+dt}} - K_t^{\theta} \underline{Q}_t^{-\alpha} \varepsilon(\mathcal{F} - O) \right]
\end{aligned}$$

Define

$$P_{A_t} \equiv \frac{Y_t}{L_t} \underline{Q}_t^{\alpha} K_t^{-\theta}$$

Then taking limits yields

$$\widetilde{r} = \frac{\dot{P}_{A_t}}{P_{A_t}} + (\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + \theta\varepsilon\frac{R_t}{K_t} - \varepsilon(\mathcal{F} - O)K_t^{\theta}\underline{Q}_t^{-\alpha} + \frac{\frac{1}{\sigma-1}\frac{Y_t}{A_t}}{P_{A_t}}\frac{\alpha}{\alpha - (\sigma-1)}\varepsilon$$

$$\text{with} \quad \frac{\dot{P}_{A_t}}{P_{A_t}} = g \ ; \ \dot{K}_t = \varepsilon R_t \ ; \ \frac{\dot{K}_t}{K_t} = \eta \ ; \ A_t = \int_0^{\infty} e^{(\sigma-1)q} M_t \frac{\alpha\zeta}{\zeta - \alpha}(e^{-\alpha q} - e^{-\zeta q})dq$$

$$\implies \widetilde{r} = g + (\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + \theta\eta - \varepsilon(\mathcal{F} - O)K_t^{\theta}\underline{Q}_t^{-\alpha} + \frac{L_t}{M_t}\frac{\zeta - (\sigma-1)}{\zeta(\sigma-1)}\varepsilon K_t^{\theta}\underline{Q}_t^{-\alpha}$$

This gives the social rate of return to R&D as a function of the allocations. Along the decentralization equilibrium, these allocations satisfy

$$\frac{\zeta - (\sigma-1)}{\zeta(\sigma-1)}\frac{L_t}{M_t} = (\mathcal{F} - O)\frac{\alpha}{\alpha - (\sigma-1)}\frac{\xi}{\xi + \sigma - 1}\frac{r - [g + (\sigma-1)(\beta - g_Q) + \frac{\nu^2}{2}(\sigma-1)^2]}{r - g}$$

$$\varepsilon K_t^{\theta}\underline{Q}_t^{-\alpha} = \frac{r-g}{\mathcal{F} - O}\frac{\alpha + \xi}{\xi}\frac{\alpha - (\sigma-1)}{\sigma - 1}$$

$$\implies \widetilde{r}_{\mathbf{DE}} = g + (\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + \theta\eta + (r-g)\left(1 + \frac{\alpha}{\xi}\right)\left(1 - \frac{\alpha}{\sigma - 1}\right)$$
$$+ \frac{\alpha(\alpha + \xi)}{(\sigma-1)(\xi + \sigma - 1)}\left(r - g - (\sigma-1)(\beta - g_Q) - (\sigma-1)^2\frac{\nu^2}{2}\right)$$

$$\text{But from the definition of } \xi \text{ in Proposition 3 :} \quad \frac{\nu^2}{2}\xi^2 - (\beta - g_Q)\xi = r - g$$

Hence

$$\widetilde{r}_{\mathbf{DE}} = g + (\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + \theta\eta + (r-g)\frac{\alpha + \xi}{\xi}\left(1 - \frac{\alpha}{\sigma - 1}\right) + \frac{\alpha(\alpha + \xi)}{\sigma - 1}\left(\frac{\nu^2}{2}(\xi - (\sigma-1)) - (\beta - g_Q)\right)$$

$$= g + (\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + \theta\eta + (r-g)\frac{\alpha + \xi}{\xi}\left(1 - \frac{\alpha}{\sigma - 1}\right) + \frac{\alpha(\alpha + \xi)}{\xi(\sigma - 1)}\left(r - g - \frac{\nu^2}{2}\xi(\sigma-1)\right)$$

$$= \theta\eta + g + (\beta - g_Q)\alpha + \frac{\nu^2}{2}\alpha^2 + (\alpha + \xi)\left(\frac{\nu^2}{2}\xi - (\beta - g_Q) - \frac{\nu^2}{2}\alpha\right)$$

$$= \theta\eta + g + \frac{\nu^2}{2}\xi^2 - (\beta - g_Q)\xi = \theta\eta + g + r - g \implies \boxed{\widetilde{r}_{\mathbf{DE}} = r + \theta\eta}$$

B.7 Proof of Proposition 5

Conditional on an initial draw q, the density of lifespan $\ell(a|q)$ is the density of the first-passage time from above of a drifted Brownian motion (see equation 3.2.13 in Redner (2001)):

$$\ell(a|q) = \frac{q}{\nu\sqrt{2\pi a^3}} \exp\left(-\frac{\left(q + \left(\beta - g_Q\right) a\right)^2}{2\nu^2 a}\right).$$

The unconditional density $\ell(a)$ is then obtained by averaging the conditional density $\ell(a|q)$ over the distribution of initial draws:

$$\ell(a) = \int_0^\infty \alpha e^{-\alpha q} \ell(a|q) dq \ .$$

B.8 Proof of Proposition 6

For the firm to have exited by age A, its initial product must have died at some age $0 \leq a \leq A$. With $\mathcal{L}$ the initial product's lifespan, by the law of total probability:

$$\Gamma(A) = \Pr(f\text{'s lifespan} \leq A) = \int_0^A \ell(a) \Pr(f\text{'s lifespan} \leq A \mid \mathcal{L} = a) \ da \ .$$

If firms were forever single product, the conditional probability in above expression would be 1. With $x > 0$, and conditional on $\mathcal{L} = a$, the initial product may give birth at ages $s \in (0, a)$. By the recursive structure of the branching process, a birth at age s produces a lineage that is obsolete by age A with probability $\Gamma(A - s)$. Since births on $(0, a)$ form a Poisson point process of rate x, splitting this interval into subintervals of length Δ, indexed by i and with midpoints s_i:

$$\begin{aligned}
\Pr(f\text{'s lifespan} \leq A \mid \mathcal{L} = a) &= \lim_{\Delta \downarrow 0} \prod_i (x\Delta)\Gamma(A - s_i) + (1 - x\Delta)1 \\
&= \lim_{\Delta \downarrow 0} \prod_i \exp\left(x\Delta \left[\Gamma(A - s_i) - 1\right]\right) \\
&= \lim_{\Delta \downarrow 0} \exp\left(\sum_i x\Delta \left[\Gamma(A - s_i) - 1\right]\right) \\
&= \exp\left(\int_0^a x \left[\Gamma(A - s) - 1\right] ds\right) \ .
\end{aligned}$$

This leverages that $\{n_f(a)\}_{a \geq 0}$ is a single-type Crump–Mode–Jagers (general age-dependent) branching process; see Crump and Mode (1968, 1969) for foundational theory and Jagers (1975) for applications to population dynamics.

B.9 Proof of Proposition 7

Along a BGP, $\dot{M}_t = \eta M_t$, while Ψ_{nt} and $f_{nt}(\mathbf{q})$ are time invariant. The definition of $f_{nt}(\mathbf{q})$ then implies:

$$\mu_{nt}(\mathbf{q}) = \frac{\Psi_n M_t}{n} f_n(\mathbf{q}) \,, \qquad \dot{\mu}_{nt}(\mathbf{q}) = \frac{\eta}{n} \Psi_n M_t f_n(\mathbf{q}) \,,$$
$$\partial_i \mu_{nt}(\mathbf{q}) = \frac{\Psi_n M_t}{n} \partial_i f_n(\mathbf{q}) \,, \qquad \partial_i^2 \mu_{nt}(\mathbf{q}) = \frac{\Psi_n M_t}{n} \partial_i^2 f_n(\mathbf{q}) \,,$$
$$\partial_i \mu_{(n+1)t}\left(\mathbf{q}^{i\leftarrow 0}\right) = \frac{\Psi_{n+1} M_t}{n+1} \partial_i f_{n+1}\left(\mathbf{q}^{i\leftarrow 0}\right) \,, \qquad \mu_{(n-1)t}(\mathbf{q}^{\setminus i}) = \frac{\Psi_{n-1} M_t}{n-1} f_{n-1}\left(\mathbf{q}^{\setminus i}\right) \,.$$

The boundary conditions on f_n follow immediately from the boundary conditions on μ_{nt}, since $\mu_{nt}(\mathbf{q}) = 0 \iff f_n(\mathbf{q}) = 0$. Now plugging the above into the law of motion for $\mu_{1t}(q)$ yields:

$$\eta \Psi_1 M_t f_1(q) = \left(g_Q - \beta\right) \Psi_1 M_t f_1'(q) + \Psi_1 M_t \frac{\nu^2}{2} f_1''(q) - x \Psi_1 M_t f_1(q)$$
$$+ \frac{\nu^2}{2} \frac{\Psi_2 M_t}{2} \left(\partial_1 f_2(0,q) + \partial_2 f_2(q,0)\right) + E_t^f \; \alpha \; e^{-\alpha q}$$

Dividing both sides by $\Psi_1 M_t$ and rearranging yields

$$(\eta + x) f_1(q) = (g_Q - \beta) f_1'(q) + \frac{\nu^2}{2} f_1''(q) + \frac{E_t^f}{M_t} \frac{1}{\Psi_1} \alpha e^{-\alpha q} + \frac{\nu^2}{2} \frac{\Psi_2}{2\Psi_1} \left(\partial_1 f_2(0,q) + \partial_2 f_2(q,0)\right) \,.$$

By Proposition 2, the product entry rate is

$$\eta + \frac{\nu^2}{2} \alpha \zeta = \frac{E_t}{M_t} = \frac{E_t^f + x M_t}{M_t} \,,$$

where the second equality follows from E_t reflecting entry of products held by new as well as incumbent firms and uses the facts that all new firms are single products while each incumbent product gives birth to a new product at constant Poisson rate x. Combining yields $\frac{E_t^f}{M_t} = \eta + \frac{\nu^2}{2} \alpha \zeta - x$.

$$\implies (\eta + x) f_1(q) = (g_Q - \beta) f_1'(q) + \frac{\nu^2}{2} f_1''(q) + \frac{\eta + \frac{\nu^2}{2} \alpha \zeta - x}{\Psi_1} \alpha e^{-\alpha q} + \frac{\nu^2}{2} \frac{\Psi_2}{2\Psi_1} \left(\partial_1 f_2(0,q) + \partial_2 f_2(q,0)\right) \,.$$

Moving on to the law of motion for $\mu_{nt}(\mathbf{q})$ for $n > 1$, plugging in the above yields

$$\frac{\eta}{n} \Psi_n M_t f_n(\mathbf{q}) = \left(g_Q - \beta\right) \sum_{i=1}^{n} \frac{\Psi_n M_t}{n} \partial_i f_n(\mathbf{q}) + \frac{\nu^2}{2} \sum_{i=1}^{n} \frac{\Psi_n M_t}{n} \partial_i^2 f_n(\mathbf{q}) - x \Psi_n M_t f_n(\mathbf{q})$$
$$+ \frac{\nu^2}{2} \sum_{i=1}^{n+1} \frac{\Psi_{n+1} M_t}{n+1} \partial_i f_{n+1}\left(\mathbf{q}^{i\leftarrow 0}\right) + x \frac{\Psi_{n-1} M_t}{n} \sum_{i=1}^{n} \alpha e^{-\alpha q_i} f_{n-1}\left(\mathbf{q}^{\setminus i}\right)$$

Dividing both sides by $\Psi_n M_t$ and rearranging yields

$$\left(\frac{\eta}{n}+x\right) f_n(\mathbf{q}) \;=\; \frac{1}{n}\sum_{i=1}^{n}\left((g_Q-\beta)\partial_i f_n(\mathbf{q})+\frac{\nu^2}{2}\partial_i^2 f_n(\mathbf{q})\right) + \frac{\nu^2}{2}\frac{\Psi_{n+1}}{\Psi_n}\frac{1}{n+1}\sum_{i=1}^{n+1}\partial_i f_{n+1}\left(\mathbf{q}^{i\leftarrow 0}\right)+x\frac{\Psi_{n-1}}{\Psi_n}\frac{1}{n}\sum_{i=1}^{n}\alpha e^{-\alpha q_i} f_{n-1}\left(\mathbf{q}^{\setminus i}\right)\;.$$

B.10 Proof of Proposition 8

Integrating both sides of the n^{th} PDE over $(0,\infty)^n$ and using linearity of the integral yields:

$$\begin{aligned}\int_{\mathbf{q}\in(0,\infty)^n}(\eta+xn) f_n(\mathbf{q})\,d\mathbf{q} \;=\; &\int_{\mathbf{q}\in(0,\infty)^n}\sum_{j=1}^{n}\left[\left(g_Q-\beta\right)\frac{\partial f_n(\mathbf{q})}{\partial q_j}\;+\;\frac{\nu^2}{2}\frac{\partial^2 f_n(\mathbf{q})}{\partial q_j^2}\right]d\mathbf{q}\\ &+\int_{\mathbf{q}\in(0,\infty)^n} x\,\frac{\Psi_{n-1}}{\Psi_n}\,\sum_{j=1}^{n}\,\alpha e^{-\alpha q_j} f_{n-1}\left(\mathbf{q}^{\setminus j}\right)\,d\mathbf{q}\\ &+\;\frac{\nu^2}{2}\,\frac{n\,\Psi_{n+1}}{\Psi_n}\,\frac{1}{n+1}\,\sum_{j=1}^{n+1}\int_{\mathbf{q}\in(0,\infty)^n}\frac{\partial f_{n+1}\left(\mathbf{q}^{j\to 0}\right)}{\partial q_j}d\mathbf{q}\end{aligned}$$

The goal is to simplify each of four integrals that appear in the expression. Starting with the LHS,

$$f_n \text{ pdf on } (0,\infty)^n \implies \int_{\mathbf{q}\in(0,\infty)^n}(\eta+xn) f_n(\mathbf{q})\,d\mathbf{q} \;=\; \eta+xn\;.$$

On the RHS, three integrals show up:

- Since f_{n-1} is a pdf on $(0,\infty)^{n-1}$ and $\alpha e^{-\alpha q}$ a pdf on $(0,\infty)$,

$$\begin{aligned}&\int_{\mathbf{q}\in(0,\infty)^i} x\frac{\Psi_{n-1}}{\Psi_n}\sum_{j=1}^{n}\alpha e^{-\alpha q_j} f_{i-1}\left(\mathbf{q}^{\setminus j}\right)\,d\mathbf{q}\\ &=\;\sum_{j=1}^{n} x\frac{\Psi_{n-1}}{\Psi_n}\left(\int_0^\infty \alpha e^{-\alpha q_j}\,dq_j\right)\left(\int_{\mathbf{q}^{\setminus j}\in(0,\infty)^{n-1}} f_{n-1}\left(\mathbf{q}^{\setminus j}\right)d\mathbf{q}^{\setminus j}\right)=\;\sum_{j\,=\,1}^{n} x\frac{\Psi_{n-1}}{\Psi_n}\;=\;n\,x\,\frac{\Psi_{n-1}}{\Psi_n}\end{aligned}$$

- Leveraging the definition of λ_n for all n (specifically for $n+1$)

$$\frac{\nu^2}{2}\,\frac{n\,\Psi_{n+1}}{\Psi_n}\,\frac{1}{n+1}\,\sum_{j=1}^{n+1}\int_{\mathbf{q}\in(0,\infty)^n}\frac{\partial f_{n+1}\left(\mathbf{q}^{j\to 0}\right)}{\partial q_j}d\mathbf{q}\;=\;\frac{n\,\Psi_{n+1}}{\Psi_n}\lambda_{n+1}$$

- Finally, to evaluate the integral on the first line of the right hand side, define the continuously differentiable vector field on $(0,\infty)^n$:

$$V(\mathbf{q})\;=\;(V_1(\mathbf{q}),\ldots,V_n(\mathbf{q}))\quad\text{where}\quad V_j(\mathbf{q})\;\equiv\;\left(g_Q-\beta\right)\,f_n(\mathbf{q})\;+\;\frac{\nu^2}{2}\frac{\partial f_n(\mathbf{q})}{\partial q_j}$$

The divergence of this vector field is

$$\text{div } V \;=\; \sum_{j=1}^{n} \left(g_Q - \beta\right) \frac{\partial f_n(\mathbf{q})}{\partial q_j} \;+\; \frac{\nu^2}{2} \frac{\partial^2 f_n(\mathbf{q})}{\partial q_j^2}$$

$$\Longrightarrow \int_{(0,\infty)^n} \left(\left(g_Q - \beta\right) \sum_{j=1}^{n} \frac{\partial f_n(\mathbf{q})}{\partial q_j} \;+\; \frac{\nu^2}{2} \sum_{j=1}^{n} \frac{\partial^2 f_n(\mathbf{q})}{\partial q_j^2} \right) d\mathbf{q} \;=\; \int_{(0,\infty)^n} \text{div } V(\mathbf{q})\; d\mathbf{q}$$

I can now apply the divergence theorem on $(0, R)^i$ and then take limits as $R \to \infty$. This transforms the volume integral of the divergence over the positive orthant into a surface integral on the hyperplanes delimiting the positive orthant:

$$\begin{aligned}
\int_{(0,R)^n} \text{div} V(\mathbf{q}) d\mathbf{q} &= \sum_{j=1}^{n} \left(\int_{\{q_j=0\}} V \cdot (-e_j) dS + \int_{\{q_j=R\}} F \cdot (e_j) dS \right) \\
&= \sum_{j=1}^{n} \left(\int_{\{q_j=0\}} - \left(\left(g_Q - \beta\right) f_n(\mathbf{q}) + \frac{\nu^2}{2} \frac{\partial f_n(\mathbf{q})}{\partial q_j} \right) dS + \int_{\{q_j=R\}} \left(\left(g_Q - \beta\right) f_n(\mathbf{q}) + \frac{\nu^2}{2} \frac{\partial f_n(\mathbf{q})}{\partial q_j} \right) dS \right) \\
&= \sum_{j=1}^{n} \left(\int_{\{q_j=0\}} -\frac{\nu^2}{2} \frac{\partial f_n(\mathbf{q})}{\partial q_j} dS + \int_{\{q_j=R\}} \left(\left(g_Q - \beta\right) f_n(\mathbf{q}) + \frac{\nu^2}{2} \frac{\partial f_n(\mathbf{q})}{\partial q_j} \right) dS \right)
\end{aligned}$$

where the last step uses the boundary condition that f_n vanishes on any of the hyperplanes delimiting the positive orthant. Taking limits as $R \to \infty$, only the first integral within each sum survives, as both f_n and its partial derivatives vanish when any of its entries grows to infinity. And each of these integrals is a surface integral on one of the n hyperplanes delimiting the positive orthant, so the integration is with respect to all variables other than q_j, where q_j itself is zero. Using the notation defined above:

$$\int_{(0,\infty)^n} \text{div } V(\mathbf{q}) \;=\; \sum_{j=1}^{n} -\frac{\nu^2}{2} \int_{\{q_j=0\}} \frac{\partial f_i(\mathbf{q})}{\partial q_j} \;=\; -\,n\,\lambda_n$$

Putting it all together,

$$\eta + xn \;=\; -\,n\,\lambda_n \;+\; n\,x\,\frac{\Psi_{n-1}}{\Psi_n} \;+\; \frac{n\Psi_{n+1}}{\Psi_n}\lambda_{n+1}$$

Multiplying both sides by $\frac{\Psi_n}{n}$, I get:

$$\frac{\eta}{n}\Psi_n \;=\; -\left(x \;+\; \lambda_n\right)\Psi_n \;+x\Psi_{n-1} \;+\; \lambda_{n+1}\Psi_{n+1}$$

C Computational Appendix

C.1 Solving integral equation in Proposition 6

$$\Gamma(A) = \int_0^A \ell(a) \exp\left(\int_0^a x\left[\Gamma(A-s)-1\right] ds\right) da$$

As I highlighted in the main text, this can be computed with a marching forward algorithm, as each $\Gamma(A)$ only depends on lower ages and $\Gamma(0) = 0$. To speed up the process (specifically the inner integral), define $R(A) = \exp\left(-x\int_0^A \Gamma(u)du\right)$, so that $R_0 = 1$ and by the fundamental theorem of calculus $R'(A) = -xR(A)\Gamma(A)$. Now to see how this simplifies the integral equation:

$$\begin{aligned}
\Gamma(A) &= \int_0^A \ell(a) \exp\left(\int_0^a x\left[\Gamma(A-s)-1\right] ds\right) da = \int_0^A \ell(a) \exp\left(x\int_0^a \Gamma(A-s)ds - xa\right) da \\
&= \int_0^A \ell(a) \exp\left(x\int_{A-a}^A \Gamma(u)du - xa\right) da = \int_0^A \ell(a) \exp\left(x\int_0^A \Gamma(u)du - x\int_0^{A-a} \Gamma(u)du - xa\right) da \\
&= \int_0^A \ell(a) \exp\left(x\int_0^A \Gamma(u)du\right) \exp\left(-x\int_0^{A-a} \Gamma(u)du\right) \exp(-xa)\, da = \int_0^A \ell(a)\frac{1}{R(A)} R(A-a)e^{-xa} da \\
&\implies \Gamma(A)R(A) = \int_0^A e^{-xa}\ell(a)R(A-a)da
\end{aligned}$$

Evaluation of the RHS does not depend on values at A, since $\ell(0) = 0$. So I get $\Gamma(A)R(A)$ by simply evaluating the integral. I then get $R(A)$ using the ODE $R'(A) = -xR(A)\Gamma(A)$ and previous value for $R(A)$. I then divide $\Gamma(A)R(A)$ by $R(A)$ to get $\Gamma(A)$.

For the hazard rate, I need $\Gamma'(A)$. Denoting $Y(A) = \Gamma(A)R(A)$, it follows that

$$\Gamma'(A) = \frac{Y'(A)}{R(A)} + x\Gamma(A)^2$$

$$\text{where } \; Y'(A) = e^{-xA}\ell(A) - x\int_0^A e^{-xa}\ell(a)Y(A-a)da$$

Because:

$$\begin{aligned}
Y(A) &= \int_0^A e^{-xa}\ell(a)R(A-a)da \\
Y'(A) &= e^{-xA}\ell(A) + \int_0^A e^{-xa}\ell(a)R'(A-a)da \quad \text{since } \; R(A) = 1 \\
Y'(A) &= e^{-xA}\ell(A) - x\int_0^A e^{-xa}\ell(a)Y(A-a)da \quad \text{using ODE for } \; R \; .
\end{aligned}$$

To see how this helps getting $\Gamma'(A)$:

$$Y(A) = \Gamma(A)R(A) \implies Y'(A) = \Gamma'(A)R(A) + \Gamma(A)R'(A) \implies \Gamma'(A) = \frac{Y'(A)}{R(A)} + \Gamma(A)\frac{R'(A)}{R(A)}$$

Using the ODE for R, this simplifies to

$$\Gamma'(A) = \frac{Y'(A)}{R(A)} + \Gamma(A)^2 x \ .$$

Details for numerical implementation given a set of parameters. To get the hazard rate of firm exit up to age 20, I work on a discrete age grid with 20500 points between 0 and 20.5. I use the above algorithm to "fill" D in a single forward march, using Numpy's builtin numerical integration with the trapezoid rule.

Estimating parameters. I use as objective an equally weighted least squares deviations of model vs empirical hazard rate of firm exit at ages 1 through 19. I use the least squares routine provided by Python's SciPy library.

C.2 Solving for stationary firm size distribution

C.2.1 Dimension reduction

To solve the system from Proposition 7, I start with a special case that then informs the general solution strategy.

Special case with no population growth ($\eta = 0$). Given the semi-endogenous nature of the model, the BGP is now simply a stationary equilibrium. Assumption 2 requires $\beta < 0$, and active firm entry requires $x < -\alpha\beta$. One can guess and verify that a solution to the system is:

$$\forall n \quad f_n(\mathbf{q}) = \prod_{i=1}^{n} f_p(q_i) \quad ; \quad \lambda_n = \lambda \equiv -\alpha\beta > x > 0 \quad ; \quad \Psi_n = \frac{\lambda - x}{x}\left(\frac{x}{\lambda}\right)^n \ , \tag{12}$$

where $f_p(q)$ is the unconditional density of product qualities derived in Proposition 2 (plugging $\eta = g_Q = 0$ into the definition of ζ). In this case, conditioning on the number of products a firm has does not alter the distribution of product qualities. As a result, the average product exit rate does not vary with firm size. The resulting distribution of products per firm, given by Ψ_n, matches that of Klette and Kortum (2004).[2] When $\eta = 0$, whether one models product exit as resulting from a Poisson death shock or negative selection has no bearing on the stationary firm size distribution.

General case ($\eta > 0$). When $\eta > 0$, the sequences in (12) no longer constitute a solution to the system in Proposition 7. The culprit is the $\frac{\eta}{n}$ term on the left hand side of each PDE.

Two points are worth emphasizing to highlight the intuition behind this result. First, as a consequence of volatility and positive selection into surviving, the distribution of quality among incumbent products first order stochastically dominates that among new products. Second, when $\eta > 0$, the size of firm cohorts grows over time. Taken together, since firms that just entered are

[2] See equation (17) of Klette and Kortum (2004). Their nM_n maps to my Ψ_n and their θ to $\frac{\lambda - x}{x}$ here.

necessarily single product, it is unsurprising that the distribution of product quality among single product firms looks different from the distribution of product quality among multiproduct firms.

My strategy to solve the general system is motivated by the fact that $\frac{\eta}{n}$, the term preventing the sequences in (12) from being a solution, vanishes as n grows large. Accordingly, for n large enough, I use the solution obtained under $\eta = 0$ as an approximation for the true solution:

$$\forall n > n_0, \quad f_n(\mathbf{q}) = \prod_{i=1}^{n} f_p(q_i) \quad ; \quad \lambda_n = \alpha\left(g_Q - \beta\right) \quad ; \quad \frac{\Psi_{n+1}}{\Psi_n} = \frac{x}{\alpha(g_Q - \beta)} \, .$$

The next step then consists of solving for $\{f_n, \Psi_n, \lambda_n\}_{n \leq n_0}$. While finite, this system suffers from a severe curse of dimensionality as each f_n is a pdf on $(0, \infty)^n$.

To address this curse of dimensionality, note that f_n is the joint density of n random variables that one expects to be independent. The reason is twofold: (i) the qualities of a firm's current products have no bearing on the quality of a new product it adds to its portfolio; (ii) the Brownian increments are independent across products. In addition, the system inherently satisfies permutation symmetry: a firm's state variable is its *set* of products, so their indexing should be irrelevant. So,

$$f_n(\mathbf{q}) \; = \; \prod_{i=1}^{n} \varphi_n(q_j), \quad \text{with } \varphi_n \text{ pdf on } (0, \infty) \text{ satisfying } \varphi_n(0) = 0 \, . \tag{13}$$

Just like in (12), each joint density f_n is the product of n evaluations of the same marginal density. However, the difference is that now the marginal density potentially depends on the firm's number of products. So, for example, while for a 2-product firm the product with index 1 is not systematically any different than the product with index 2, the typical product held by a 2-product firm is allowed to systematically differ from the typical product of a single product firm.

This observation simplifies the stationary firm size distribution to sequences Ψ_n and $\varphi_n(q)$ satisfying a system of coupled ODEs given in the below lemma.

Lemma 1. Given the ansatz from Equation 13, the stationary firm size distribution consists of sequences Ψ_n and φ_n satisfying the following system of coupled ordinary differential equations:

$$\frac{\nu^2}{2}\varphi_1''(q) + (g_Q - \beta)\varphi_1'(q) - (\eta + x)\varphi_1(q) = -\frac{\eta + \frac{\nu^2}{2}\alpha\zeta - x}{\Psi_1}\alpha e^{-\alpha q} - \frac{\Psi_2}{\Psi_1}\lambda_2\varphi_2(q) \, ,$$

$$\text{and for } n > 1, \quad \frac{\nu^2}{2}\varphi_n''(q) + (g_Q - \beta)\varphi_n'(q) - (\eta + nx + (n-1)\lambda_n)\varphi_n(q)$$
$$= -\frac{x\Psi_{n-1}}{\Psi_n}\left(\alpha e^{-\alpha q} + (n-1)\varphi_{n-1}(q)\right) - n\frac{\Psi_{n+1}}{\Psi_n}\lambda_{n+1}\varphi_{n+1}(q)$$

with $\forall n \geq 1$, $\varphi_n(0) = 0$ and $\lambda_n = \frac{\nu^2}{2}\varphi_n'(0)$.

Proof. For $n > 1$, the n^{th} ODEs is obtained by plugging in the ansatz into the n^{th} PDE then integrating both sides over $(0, \infty)^{n-1}$, so by integrating out all dimensions but one. □

This gets rid of the curse of dimensionality because each $\varphi_n(q)$ is a density on $(0, \infty)$, so the

system can be solved using state of the art ODE solvers. Before passing the system to the solver, I do two further transformations which I now discuss.

C.2.2 A first transformation: rewriting ODE system in terms of identified parameters

Given that the first step of my calibration strategy identifies x, $\alpha(g_Q - \beta) = \theta\eta - \alpha\beta$, and $\alpha\nu$, I start by rewriting the system from Lemma 1 in terms of these parameter combinations and η. To do so, I do the following change of variables

$$s \equiv \alpha q \quad \text{and} \quad \delta_n(s) \equiv \frac{\varphi_n\left(\frac{s}{\alpha}\right)}{\alpha} \implies \varphi_n(q) = \alpha\delta_n(\alpha q)\,,$$

and note that

$$\frac{\nu^2}{2}\alpha\zeta = \frac{1}{2}\alpha\left(g_Q - \beta\right) + \frac{1}{2}\sqrt{\left(\alpha\left(g_Q - \beta\right)\right)^2 + 2\eta\left(\alpha\nu\right)^2}$$

so that this term in the first ODE is taken care of. Plugging into the initial system yields:

$$\forall n \geq 1\,,\;\; \delta_n(0) = 0 \;\;;\;\; \int_0^\infty \delta_n(s)ds = 1 \;\;;\;\; \lambda_n = \frac{(\alpha\nu)^2}{2}\delta_n'(0) \;\;;\;\; \lim_{s\to\infty}\delta(s) = \lim_{s\to\infty}\delta'(s) = 0$$

$$\frac{(\alpha\nu)^2}{2}\delta_1''(s) + \alpha(g_Q - \beta)\delta_1'(s) - (\eta + x)\delta_1(s) = -\frac{\eta + \frac{\nu^2}{2}\alpha\zeta - x}{\Psi_1}e^{-s} - \frac{\Psi_2}{\Psi_1}\lambda_2\delta_2(s)$$

and for $n > 1$

$$\begin{aligned}\frac{(\alpha\nu)^2}{2}\delta_n''(s) + \alpha(g_Q - \beta)\delta_n'(s) - (\eta + nx + (n-1)\lambda_n)\,\delta_n(s) = &- x\frac{\Psi_{n-1}}{\Psi_n}\left(e^{-s} + (n-1)\delta_{n-1}(s)\right)\\ &- n\frac{\Psi_{n+1}}{\Psi_n}\lambda_{n+1}\delta_{n+1}(s)\end{aligned}$$

with the recurrence relation that only depends on identified parameters:

$$\begin{cases}\eta\;\Psi_1 &= -\left(x + \lambda_1\right)\Psi_1 \;+\; \lambda_2\Psi_2 \;+\; \eta \;+\; \frac{\nu^2}{2}\alpha\;\zeta \;-\; x\\ \frac{\eta}{n}\;\Psi_n &= -\left(x + \lambda_n\right)\Psi_n \;+\; \lambda_{n+1}\Psi_{n+1} \;+\; x\Psi_{n-1} \qquad \text{for } n \;>\; 1\end{cases}$$

The limit (fixed point) of the system becomes

$$\delta_\infty(s) = \frac{\frac{\tau}{\alpha}}{\frac{\tau}{\alpha} - 1}\left(e^{-s} - e^{-\frac{\tau}{\alpha}s}\right) \quad \text{where } \frac{\tau}{\alpha} = 2\frac{\alpha(g_Q - \beta)}{(\alpha\nu)^2}$$

along with

$$\lambda_\infty = \alpha(g_Q - \beta) \quad \text{and} \quad \frac{\Psi_{n+1}}{\Psi_n} \;\sim\; \frac{x}{\lambda_\infty} \;<\; 1\,.$$

C.2.3 A second transformation: improving conditioning of the system of ODEs

While the above system is in principle solvable, to guarantee numerical stability I do a second transformation that drastically improves the system's conditioning. In that vein, let:

$$u_n(s) \equiv e^{\frac{\tau}{\alpha}s}\delta_n(s) \quad \text{and} \quad R_n \equiv \frac{\Psi_n}{\Psi_{n+1}}$$

The motivation for working with ratios is that as n grows large, Ψ_n converges to zero, while the ratio of consecutive terms converges to a strictly positive number. Defining the initial condition

$$R_0 \;=\; \frac{1}{x}\frac{\eta + \frac{\nu^2}{2}\alpha\zeta - x}{\Psi_1}\,,$$

The recurrence is then given for all $n \geq 1$ by

$$-\left(\frac{\eta}{n} + x + \lambda_n\right) + \frac{\lambda_{n+1}}{R_n} + xR_{n-1} \;=\; 0\,.$$

Plugging the recurrence into the RHS to avoid having divisions by R_n, the $n = 1$ ODE becomes:

$$\begin{aligned}\frac{(\alpha\nu)^2}{2}u_1''(s) = &-\left[\alpha(g_Q-\beta) - \frac{\tau}{\alpha}(\alpha\nu)^2\right]u_1'(s) + \left[\eta + x + \frac{\tau}{\alpha}\alpha(g_Q-\beta) - \left(\frac{\tau}{\alpha}\right)^2\frac{(\alpha\nu)^2}{2}\right]u_1(s)\\ &- xR_0e^{-\left(1-\frac{\tau}{\alpha}\right)s} + \left[xR_0 - (\eta + x + \lambda_1)\right]u_2(s)\,.\end{aligned}$$

The $n > 1$ ODE becomes:

$$\begin{aligned}\frac{(\alpha\nu)^2}{2}u_n''(s) = &-\left[\alpha(g_Q-\beta) - \frac{\tau}{\alpha}(\alpha\nu)^2\right]u_n'(s) + \left[\eta + nx + (n-1)\lambda_n + \frac{\tau}{\alpha}\alpha(g_Q-\beta) - \left(\frac{\tau}{\alpha}\right)^2\frac{(\alpha\nu)^2}{2}\right]u_n(s)\\ &- xR_{n-1}\left(e^{-\left(1-\frac{\tau}{\alpha}\right)s} + (n-1)u_{n-1}(s)\right) + n\left[xR_{n-1} - \left(\frac{\eta}{n} + x + \lambda_n\right)\right]u_{n+1}(s)\,.\end{aligned}$$

The boundary conditions are with $u_n(0) = 0$; $\lambda_n = \frac{(\alpha\nu)^2}{2}u_n'(0)$ and, for s_max large enough $\delta_n(s_{max}) = 0$ (exponentially decaying tail).

C.2.4 Numerical solution

I transform the system of 2nd order ODEs into a system of 1st order ODEs by introducing the derivatives u' as auxiliary variables. I solve for u_n, u_n', λ_n, and R_n using solve-bvp from Python's Scipy library, which is designed to solve such systems of differential-algebraic equations. The n_0 I choose for truncation purposes is 25 (and checked robustness to increasing n_0 to 40), with a tolerance of 10^{-5} and $s_{max} = 8$.

C.2.5 Verification of solution.

In addition to standard convergence metrics, the theory provides a transparent way of verifying how accurate my solution is. By the law of total probability, aggregating the conditional distributions should yield back the unconditional distribution:

$$f_p(q) = \sum_{n=1}^{\infty} \Psi_n \varphi_n(q) \qquad \text{and} \qquad \frac{D_t}{M_t} = \sum_{n=1}^{\infty} \Psi_n \lambda_n \ .$$

Here $f_p(q)$ is the unconditional distribution of product quality and $\frac{D_t}{M_t}$ is the product exit rate, both characterized in closed form in Proposition 2. Crucially, neither of these identities was used as part of the solution strategy.

So after solving for $u_n(q)$, I obtain $\delta_n(q)$ using

$$\delta_n(s) = e^{-\frac{\tau}{\alpha}s} u_n(s) \ .$$

As discussed in the main text, the theory provides a transparent way to verify the solution:

$$\sum_{n=1}^{\infty} \Psi_n \varphi_n(q) = \frac{\alpha\ \zeta}{\zeta - \alpha} \left(e^{-\alpha q} \ - \ e^{-\zeta q} \right)$$

In terms of what I solved for numerically, this becomes

$$\sum_{n=1}^{\infty} \Psi_n \delta_n(s) = \frac{\zeta/\alpha}{\zeta/\alpha - 1} \left(e^{-s} - e^{-\frac{\zeta}{\alpha}s} \right) \text{ where } \frac{\zeta}{\alpha} = \frac{\alpha(g_Q - \beta) + \sqrt{(\alpha(g_Q - \beta))^2 + 2\eta(\alpha\nu)^2}}{(\alpha\nu)^2} \ .$$

The absolute deviation between the LHS (computed numerically) and the RHS (closed form) has mean 0.0003 and maximum 0.0013.

C.3 Employment distribution along BGP

In equilibrium, production employment for product p is given by

$$L_{pt} = e^{(\sigma-1)q} \cdot \frac{L_t}{M_t} \cdot \frac{\alpha - (\sigma - 1)}{\alpha} \frac{\zeta - (\sigma - 1)}{\zeta} \ .$$

Using the change of variable $s \equiv \alpha q$, the expression for L/M along the BGP and the identities that ξ verifies yields

$$L_{pt} = e^{\frac{\sigma-1}{\alpha}s} (\mathcal{F} - O)^{(\sigma-1)} \left[1 - \frac{\frac{1}{2}(\beta - g_Q)(\sigma - 1) + \frac{1}{2}\sqrt{(\beta - g_Q)^2(\sigma-1)^2 + 2\nu^2(r-g)(\sigma-1)^2}}{r - g} \right] ;$$

which I rewrite as:

$$L_{pt} = \underline{L} e^{\frac{\sigma-1}{\alpha}s} \ ,$$

where $\underline{L}$ is production labor employment at the exit threshold ($s = q = 0$).

Employment at a firm with n products with portfolio $(s_1, \ldots, s_n)$ is then given by:

$$(\mathcal{F} + I)n + \sum_{i=1}^{n} \underline{L} e^{\frac{\sigma-1}{\alpha} s} .$$

Appendix C.2 makes clear that, to solve for the stationary distribution of number of products per firm and conditional distribution of s among firms with n products, one only needs a value for the population growth rate η and values for the statistics estimated by targeting firm exit by age. Once these distributions are characterized, to compute employment, one needs: $\frac{\sigma-1}{\alpha}$, $\mathcal{F} + I$, and $\underline{L}$.

Table C3: Calibrated statistics in terms of structural parameters

Statistic	As function of structural parameters
x	$\frac{1}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}} \frac{\left(\rho+(\gamma-1)\left(\frac{\theta}{\alpha}\eta+\frac{\eta}{\sigma-1}\right)\right)(\alpha-(\sigma-1))}{\left(\mathcal{F}-\frac{\delta}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}}\right)(\sigma-1)} \left(\alpha + \frac{\beta-\frac{\theta}{\alpha}\eta+\sqrt{(\beta-\frac{\theta}{\eta}\alpha)^2+2\nu^2\left(\rho+(\gamma-1)\left(\frac{\theta}{\alpha}\eta+\frac{\eta}{\sigma-1}\right)\right)}}{\nu^2}\right)$
$\mathcal{F} + I$	$\mathcal{F} + \left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}}$
$\underline{L}$	$\left(\mathcal{F} - \frac{\delta}{1-\delta}\left(\frac{\vartheta}{\varepsilon}\right)^{\frac{1}{\delta}}\right)(\sigma-1)\left[1 - \frac{\frac{1}{2}(\beta-\frac{\theta}{\alpha}\eta)(\sigma-1)+\frac{1}{2}\sqrt{(\beta-\frac{\theta}{\eta}\alpha)^2(\sigma-1)^2+2\nu^2\left(\rho+(\gamma-1)\left(\frac{\theta}{\alpha}\eta+\frac{\eta}{\sigma-1}\right)\right)(\sigma-1)^2}}{\rho+(\gamma-1)\left(\frac{\theta}{\alpha}\eta+\frac{\eta}{\sigma-1}\right)}\right]$

It is worth noting that between steps 1 and 2 of my calibration strategy, I am pinning down values for three statistics that are complicated functions of structural parameters. These are reproduced in Table C3. The goal is to highlight that there are enough degrees of freedom (structural parameters) to jointly satisfy the three requirements: for example, given values for other parameters, one can pick values for $\mathcal{F}$, δ and $(\vartheta/\varepsilon)^{1/\delta}$ in order to match the estimated value for x and calibrated values for $\underline{L}$ and $\mathcal{F} + I$.

D Data Appendix

This appendix provides detailed information on the data sources, sample construction, and variable definitions used in the empirical analysis of the paper. Section D.1 describes the product data used to document the facts in Section 3 of the paper. Section D.2 describes the firm data used for the main quantitative estimation in Section 4.

D.1 NielsenIQ Retail Scanner Dataset

Sample Construction. The analysis is restricted to a balanced panel of approximately 25,400 retail stores that are continuously present for the entire 14 year period. This restriction ensures that product exit is not mechanically driven by store closures.

Each UPC in the data is already assigned to one of roughly 120 product group codes. In building my sample, I exclude unclassified products, fresh produce, non-scannable ("magnet") products, control brands, and products classified under "seasonal", "prep food-deli", or groups that get discontinued by NielsenIQ (deferred modules). These exclusions insure that products in my sample can be consistently mapped across different retailers at a point in time as well as over time.

Variable definitions. I obtain a UPC's sales (in $) and volume of sales (quantity sold) by summing across retailers in a given time period. Dividing the former by the latter yields the average unit price. A UPC is defined as entering in year t if it has zero sales in year $t-1$ and positive sales in year t. A UPC is defined as exiting in year t if it has positive sales in year t and zero sales in year $t+1$. A UPC's age in a given year is defined as the current year minus its first year of appearance in the sample. UPCs present in 2006 are considered left-censored

Data cleaning. To guarantee accurate measurement of entry and exit, I drop any UPC that records more than one entry or exit event over the sample period.

To allow for meaningful comparisons of quantity and price within a product group, I harmonize product size units. First, I convert units to a common standard where possible (e.g., pounds and kilograms are converted to ounces; liters and quarts are converted to milliliters). For a small number of products, NielsenIQ provides a secondary size. If a product's primary unit does not match the modal unit of its product group, but its secondary unit does, I use the secondary unit for harmonization. If after these harmonization steps a product group's modal unit of measure accounts for less than 95% of sales, then this group is excluded from my analysis. Similarly, if the group has fewer than 100 products, it is excluded from the sample.

D.2 Publicly Available U.S. Census Data

To quantify spillovers I use publicly available tabulations based on the U.S. Census Longitudinal Business Database (LBD). This is an administrative dataset covering the universe of private nonfarm

employer businesses in the U.S. As it tracks establishments, in tabulations based on this dataset, a firm exits is defined as all its establishments closing.

Estimation across all private nonfarm employer businesses. The firm exit rate at ages 1 through 19 that I use are provided in the replication package of Vincent Sterk, Petr Sedláček and Benjamin Pugsley (2021).

Estimation at the 2-digit sector level. I use the Business Dynamics Statistics, publicly provided by the U.S. Census Bureau. Specifically, I use the State by Firm Age two-way tabulation, which provides, for each 2-digit NAICS sector, the number of firms as well as the number of exiting firms at ages 1, 2, 3, 4, 5, 8, 13, and 18. Since the underlying data source is the LBD (which starts in 1978), I use data from 1996 onward to avoid having any left censored firm in any of my age bins. I define the firm exit rate in a cell as the number of exiting firms divided by the average number of firms between last the previous and current year.

E Additional Results Using NielsenIQ Data

E.1 Summary Statistics

Table E4: Summary Statistics – NielsenIQ Retail Scanner Sample

	Mean	Median	P75	P90	P95
Firm sales (thousands of $)	9,171	87	1,335	34,382	66,022
Firm's market share	10%	8%	17%	23%	27%
Number of UPCs per firm	12	3	8	37	62
UPC sales (thousands of $)	502	14	181	1,026	2,377
UPC's market share	1.2%	0.7%	1.3%	2.4%	3.5%

Notes: NielsenIQ Retail Scanner data with GS1 database used to identify the firm manufacturing a UPC. Each statistic is based on annual data from 2006 to 2019, and computed first at the group-year level, then aggregated to the yearly level weighting a group by its share of sales that year, and finally averaged across the 14 years. So, first three rows correspond to firm-level statistics *within a group*.

Table E5: UPC Entry and Exit Dynamics

Entry Rate	Exit Rate	UPC Age At Exit (in years)	
		Mean	Median
13.6%	12.3%	3.1	2.9

Notes: All statistics are group–year moments, sales-weighted to the year and averaged across years. Last two columns based on exiting UPCs for which age is not left censored.

E.2 Negative Selection into UPC exit (Regression Results)

Table E6: UPC Sales and Hazard Rate of Exit

	Exit_{pt}			
$\log(\text{sales}_{pt-1})$	-1.69	-1.56	-2.70	-2.60
	(0.10)	(0.11)	(0.24)	(0.24)
UPC age FE		✓		✓
Group x Year FE	✓	✓		
Firm x Group x Year FE			✓	✓
Observations	2.58M	2.58M	2.50M	2.50M

Notes: Each column reports results from the OLS regression of a dummy for UPC p's exit at t (multiplied by 100) on p's log annual sales in year $t-1$. Sample consists of UPCs with above-median sales in a group-year. Average of the dependent variable is 7.4%, and distributional moments from UPC sales reported in Table E4. Standard error in parentheses, clustered at the group level. UPC age FE includes a separate dummy for left censored UPCs.

E.3 Gradual Decline in UPS sales

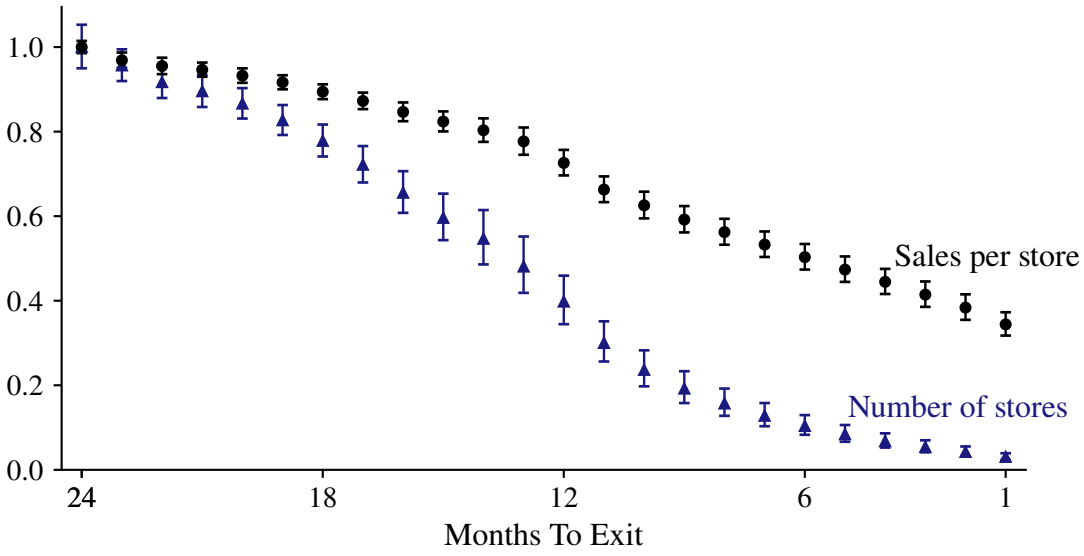
To document the gradual exit process along both the extensive (number of stores at which UPC is sold) and intensive (sales per store) margins, I run the following regressions:

$$\log \text{Sales per store}_{pt} = \gamma_p + \sum_{m=1}^{24} \pi_m D_{pt}^m + \gamma_{gt} + \varepsilon_{pt} \,, \tag{14}$$

$$\log \text{Number of stores}_{pt} = \gamma_p + \sum_{m=1}^{24} \kappa_m D_{pt}^m + \gamma_{gt} + \varepsilon_{pt} \,; \tag{15}$$

where p indexes a UPC, g its group (product category), and t a month, with γ_p a UPC fixed effect, γ_{gt} a group-month fixed effect, and D_{pt}^m a dummy variable equal to 1 m months prior to the UPC's exit. Figure E1 plots $\exp(\pi_m)$ and $\exp(\kappa_m)$, respectively representing the paths of sales per store and number of stores in the two years leading up to the UPC's exit.

Figure E1: Extensive and intensive margins in the lead up to exit

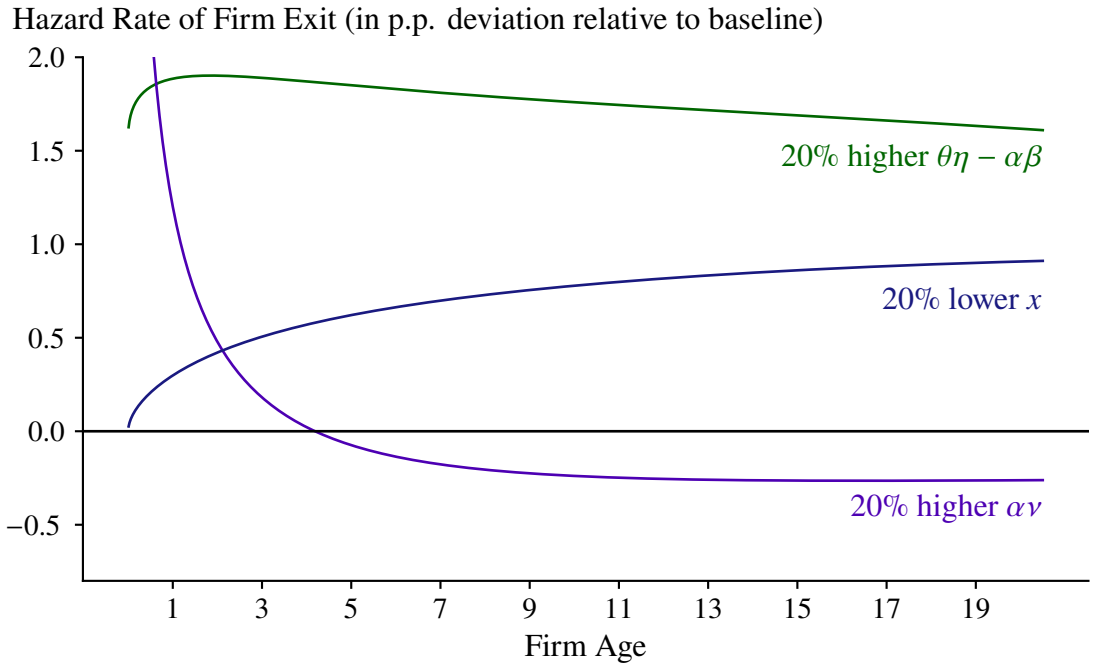


Notes: Sales per store curve corresponds to path of exp (π_m) from Equation 14, normalized such that exp $(\pi_{24}) = 1$; underlying regression has 54M observations. Number of stores curve corresponds to path of exp (κ_m) from Equation 15, normalized such that exp $(\kappa_{24}) = 1$; underlying regression has 54M observations. Vertical bars correspond to 95% confidence intervals, based on SEs clustered at the group level.

F Additional Quantitative Results

F.1 Estimation Targeting Firm Exit by Age: Sensitivity Analysis

Figure F2: Sensitivity of firm exit at different ages with respect to each statistic



Notes: Each curve shows the effect of changing one of the three estimated statistics, while holding the other two fixed. $\theta\eta - \alpha\beta$ is the product exit rate resulting from downward drift toward the exit threshold; $\alpha\nu$ is a measure of the relative volatility of the Brownian shocks; and x is the rate at which incumbent firms add a product to their portfolio.

Table F7: Sensitivity of model-implied hazard rates to the three different statistics

	$\theta\eta - \alpha\beta$	$\alpha\nu$	x
Model's hazard rate of firm exit at age 1	9.16	5.98	-1.78
Model's hazard rate of firm exit at age 2	9.17	2.28	-2.49
Model's hazard rate of firm exit at age 3	9.07	0.75	-2.98
Model's hazard rate of firm exit at age 4	8.95	-0.06	-3.34
Model's hazard rate of firm exit at age 5	8.82	-0.54	-3.63
Model's hazard rate of firm exit at age 6	8.70	-0.84	-3.86
Model's hazard rate of firm exit at age 7	8.58	-1.04	-4.06
Model's hazard rate of firm exit at age 8	8.48	-1.17	-4.22
Model's hazard rate of firm exit at age 9	8.38	-1.25	-4.37
Model's hazard rate of firm exit at age 10	8.29	-1.31	-4.49
Model's hazard rate of firm exit at age 11	8.20	-1.35	-4.60
Model's hazard rate of firm exit at age 12	8.12	-1.37	-4.69
Model's hazard rate of firm exit at age 13	8.05	-1.38	-4.78
Model's hazard rate of firm exit at age 14	7.98	-1.39	-4.86
Model's hazard rate of firm exit at age 15	7.91	-1.39	-4.92
Model's hazard rate of firm exit at age 16	7.84	-1.39	-4.98
Model's hazard rate of firm exit at age 17	7.77	-1.38	-5.04
Model's hazard rate of firm exit at age 18	7.71	-1.37	-5.08
Model's hazard rate of firm exit at age 19	7.65	-1.36	-5.13

Notes: The entry in row i and column j gives the **percentage point** change in the model-implied hazard rate of firm exit at age i resulting from a **100% change** in statistic j. These are local semi-elasticities, evaluated at the estimated values of the three statistics.

Table F8: Sensitivity of estimated statistics to empirical hazard rates

	$\theta\eta - \alpha\beta$	$\alpha\nu$	x
Empirical hazard rate of firm exit at age 1	-5.27	18.26	-14.02
Empirical hazard rate of firm exit at age 2	4.51	0.11	7.76
Empirical hazard rate of firm exit at age 3	6.73	-5.19	12.84
Empirical hazard rate of firm exit at age 4	6.86	-6.77	13.30
Empirical hazard rate of firm exit at age 5	6.23	-6.86	12.05
Empirical hazard rate of firm exit at age 6	5.30	-6.27	10.10
Empirical hazard rate of firm exit at age 7	4.25	-5.37	7.90
Empirical hazard rate of firm exit at age 8	3.18	-4.33	5.63
Empirical hazard rate of firm exit at age 9	2.14	-3.23	3.40
Empirical hazard rate of firm exit at age 10	1.13	-2.14	1.26
Empirical hazard rate of firm exit at age 11	0.18	-1.07	-0.79
Empirical hazard rate of firm exit at age 12	-0.73	-0.03	-2.71
Empirical hazard rate of firm exit at age 13	-1.57	0.95	-4.53
Empirical hazard rate of firm exit at age 14	-2.37	1.89	-6.23
Empirical hazard rate of firm exit at age 15	-3.11	2.77	-7.83
Empirical hazard rate of firm exit at age 16	-3.81	3.61	-9.32
Empirical hazard rate of firm exit at age 17	-4.46	4.39	-10.72
Empirical hazard rate of firm exit at age 18	-5.07	5.13	-12.03
Empirical hazard rate of firm exit at age 19	-5.64	5.82	-13.25

Notes: The entry in row i and column j gives the **percent change** in statistic j resulting from a **1 p.p. change** in the empirical hazard rate of firm exit at age i. These are local semi-elasticities, evaluated at the estimated parameter values (Isaiah Andrews, Matthew Gentzkow and Jesse M. Shapiro, 2017).

F.2 Quantitative Features of the Model

F.2.1 Cross-Sectional Heterogeneity in Employment Per Firm

Table F9: Sources of Firm Size Heterogeneity in the model

	Firm-level Employment	Number of Products	Per-product Production Employment
1 to 4 employees	2.3	1.2	1.1
5 to 9 employees	6.5	2.5	2.3
10 to 19 employees	13.4	4.1	3.8
20+ employees	151.4	6.1	59.3

Notes: Each statistic reported corresponds to an average across firms within that size bin.

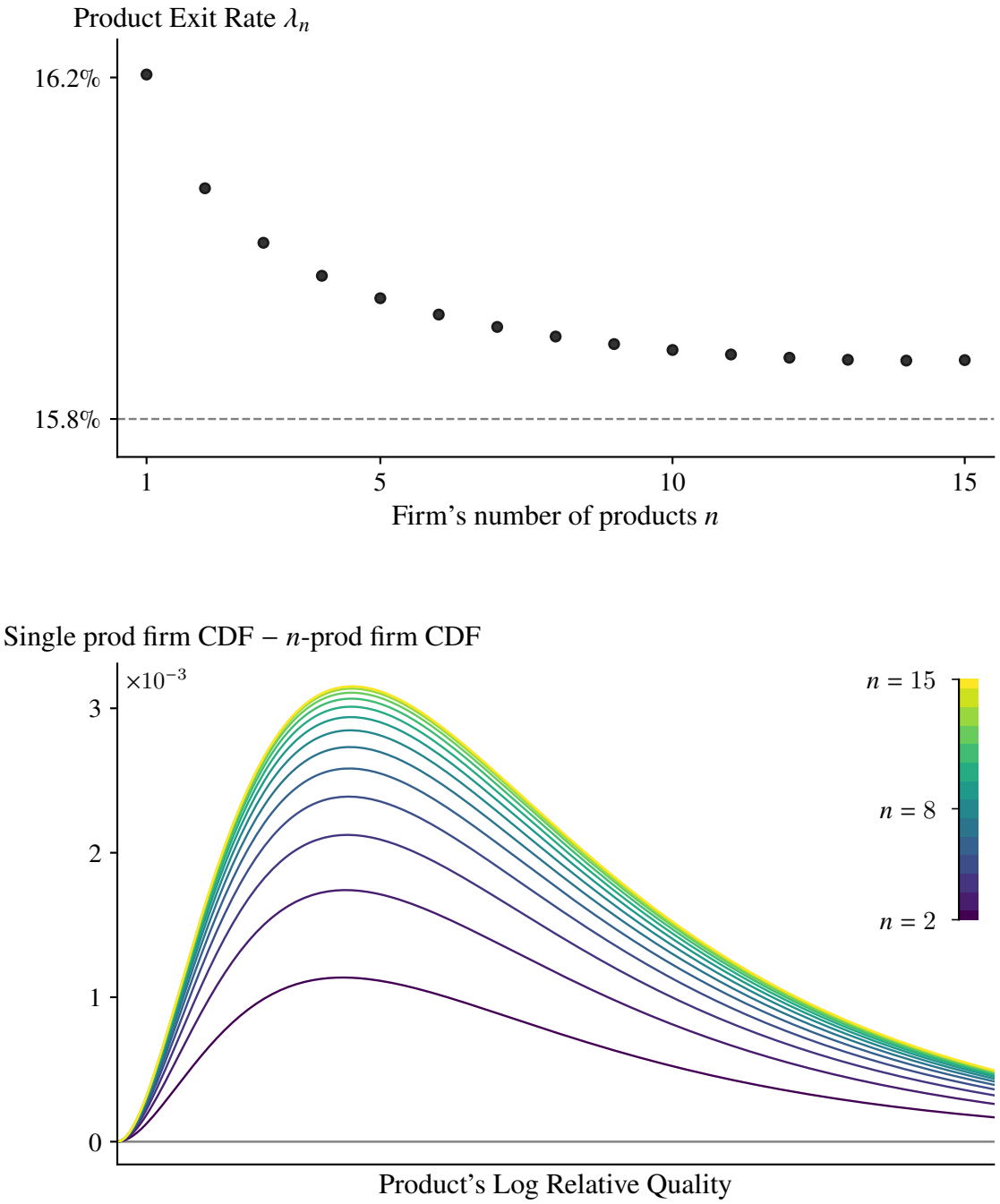
F.2.2 Product exit rate across firms

In Section 3, I showed that the average product exit rate is lower among firms with more products (Fact 2). There, I interpreted the fact as evidence in support of moving away from modeling product exit as a Poisson process. Here, I show how the quantified model endogenously generates this feature as a consequence of negative selection into product exit.

The top panel of Figure F3 illustrates this finding. It plots the model's relationship between a firm's average product exit rate (λ_n) against its number of products (n). The rate is highest for single product firms at 16.2% and declines monotonically as firms grow, slowly approaching a limit of 15.8%. While the range is narrower than the one in Fact 2, it is worth keeping in mind that the underlying samples are different (consumer packaged goods versus the universe of private nonfarm employer businesses).

The bottom panel of Figure F3 shows how the model generates this prediction: firms with *more* products have *better* products. The panel plots the difference between the cumulative distribution function (CDF) of product log-relative quality among single-product firms and that among n-product firms for $n > 1$. The fact that this difference is positive for the different values of $n > 1$ indicates first order stochastic dominance: the product quality distribution for multiproduct firms is better than for single product firms. Since the difference from the single product CDF widens with n, the figure also reveals a ranking of product quality by firm's number of products: the product quality distribution among n-product firms first-order stochastically dominates that among n'-product firms whenever $n > n'$. Finally, the vertical gap between the curves for consecutive values of n shrinks as n increases, showing that the distribution of product quality converges to its large n limit.

Figure F3: Consequences of Selection on Model's Firm Size Distribution

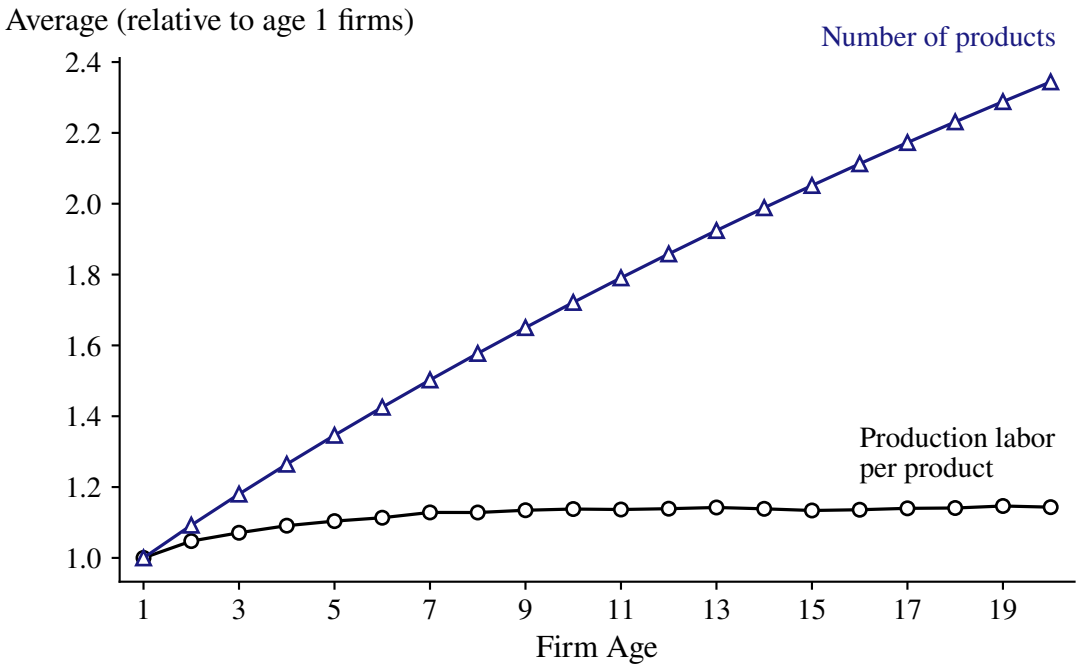


Notes: The top panel shows λ_n, the average product exit rate among n-product firms, as a function of n. The bottom panel shows the CDF of product log-relative quality (q) among single product firms minus the CDF of product product log-relative quality (q) among n-product firms ($n > 1$). A positive gap means the latter ($n > 1$) first order stochastically dominates the former ($n = 1$). Refer to Appendix C for details about computational solution.

This feature of the model's stationary equilibrium is a consequence of positive selection into survival at the product level. Since a firm starts with a single product and gradually expands it portfolio through R&D, ending up with many products requires good draws and/or positive shocks.

F.2.3 Decomposition of Average Firm Employment by Firm Age

Figure F4: Model's firm size by firm age: product count versus employment per product



Notes: To obtain the results, I simulate a cohort of 5 million firms up until age 20, compute average employment and average product count across firms at each age, repeat this simulation 1000 times, and report medians across the simulations.

F.3 Estimation using Product Exit by Age

As I emphasized on a number of occasions, the key insight the paper leverages to quantify spillovers links dynamics of *product* exit to the wedge created by spillovers. The reason I rely on firm-level data to do the quantification is that comprehensive data at the product level are not available.

As a robustness exercise, I assess the sensitivity of my results to doing the quantification with product data instead of firm data. This requires a setting where both product and firm data are available, so that I can compare the results across the two methods. The food manufacturing sector provides such an opportunity. For *firm* exit by age, I use data from the Business Dynamics Statistics for the 3-digit NAICS sector 311 (food manufacturing).[3] For *product* exit by age, I use my NielsenIQ sample and exclude UPCs classified under "Health & Beauty Care", "Non Food Grocery", and "General Merchandise" so that the sample is comparable to food manufacturing.

The quantification with product-level data leverages Propoisition 5. Specifically, by targeting the profile of *product* exit by age, I recover $\theta\eta - \alpha\beta$ and $\alpha\nu$. Identification follows from the former raising product exit at all ages and the latter raising it for young products but lowering it for older ones (due to selection). In contrast, the quantification with firm-level data uses the same approach as in the main text, and identifies the rate of incumbent innovation x in addition to $\theta\eta - \alpha\beta$ and $\alpha\nu$.

Table F10: Results with Product vs Firm Level Data for Food Manufacturing

		Product Data	Firm Data
Product exit due to drift	$\theta\eta - \alpha\beta$	0.170	0.160
		[0.167, 0.173]	
Relative volatility	$\alpha\nu$	0.362	0.484
		[0.342, 0.378]	

Notes: $\theta\eta - \alpha\beta$ is product exit due to downward drift, $\alpha\nu$ governs extent of product exit due to shocks. Product-level results are obtained from an equally weighted GMM targeting the profile of *product* exit at ages 1 through 9, with underlying data from NielsenIQ for sample years 2016, 2017, and 2018. Corresponding 95% confidence intervals obtained using 1000 bootstraps. Firm-level results are obtained from a GMM targeting the profile of *firm* exit at ages 1, 2, 3, 4, 5, 8, 13, and 18 in the food manufacturing sector (NAICS 311), with underlying data from the Business Dynamics Statistics.

Table F10 presents the results from this exercise. Focusing on the spillover product exit due to drift—the statistic informative about the wedge of interest—the estimate obtained using product-level data is 17%, while that obtained from firm-level data is 16%. This alignment lends credibility to the headline finding, suggesting my methodology successfully recovers the gradual component of product exit from firm data.

[3]While I do link NielsenIQ to the GS1 database and can hence observe which products are produced by the same firm, firm age is censored for more than 80% of firms, preventing any meaningful estimation with firm-level data.

F.4 Estimation using Establishment Exit by Age

Given that the primary goal of the LBD is the longitudinal linking of *establishments*, establishment exit is arguably less prone to measurement error (Ron S. Jarmin and Javier Miranda, 2002; Melissa C. Chow, Teresa C. Fort, Christopher Goetz, Nathan Goldschlag, James Lawrence, Elisabeth Ruth Perlman, Martha Stinson and T. Kirk White, 2021). As such, Table F11 reports estimates of $\theta\eta - \alpha\beta$ obtained by targeting the profile of *establishment* exit by age (as well as firm exit by age, as in baseline). The results show that even using estbalishment data, I get a similarly large estimate for this extent of product exit due to downward drift.

Table F11: Results with Establishment vs Firm Level Data

Data	$\theta\eta - \alpha\beta$
Baseline: Firm exit (Vincent Sterk, Petr Sedláček and Benjamin Pugsley, 2021)	15.8%
Establishment exit (BDS)	16.7%
Establishment exit (Vincent Sterk, Petr Sedláček and Benjamin Pugsley, 2021))	17.4%

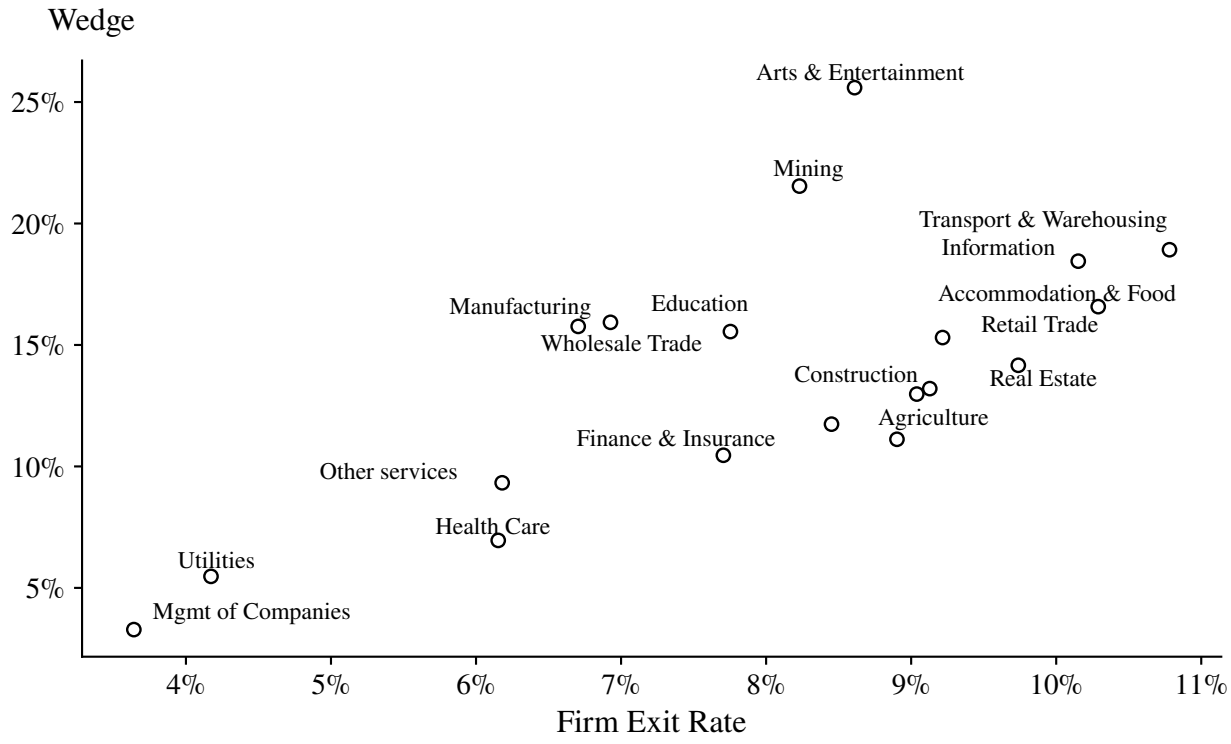
F.5 Sector-level Results

Table F12: Estimation Results Across Sectors

Sector	Firm Exit Rate	Share of firms	Share of emp	Estimated $\theta\eta - \alpha\beta$
Agriculture, Forestry, Fishing, and Hunting	8.9%	0.4%	0.1%	11.1%
Mining, Quarrying, and Oil and Gas Extraction	8.2%	0.3%	0.5%	21.5%
Utilities	4.2%	0.1%	0.6%	5.5%
Construction	9.0%	11.6%	5.4%	13.0%
Manufacturing	6.7%	5.1%	11.6%	15.8%
Wholesale Trade	7.8%	5.7%	5.0%	15.6%
Retail Trade	9.2%	12.2%	13.0%	15.3%
Transportation and Warehousing	10.8%	2.8%	3.7%	18.9%
Information	10.2%	1.2%	2.9%	18.4%
Finance & Insurance	7.7%	4.1%	5.3%	10.5%
Real Estate & Leasing	9.1%	4.5%	1.7%	13.2%
Professional, Scientific, & Tech Services	8.5%	12.2%	6.5%	11.7%
Management of Companies & Enterprises	3.6%	0.5%	2.7%	3.3%
Administrative & Support & Waste Management & Remediation Services	9.7%	5.3%	7.9%	14.2%
Educational Services	6.9%	1.3%	2.6%	15.9%
Health Care and Social Assistance	6.2%	10.8%	14.3%	7.0%
Arts, Entertainment, and Recreation	8.6%	1.7%	1.7%	25.6%
Accommodation and Food Services	10.3%	8.0%	9.8%	16.6%
Other Services (except Public Administration)	6.2%	12.0%	4.6%	9.3%

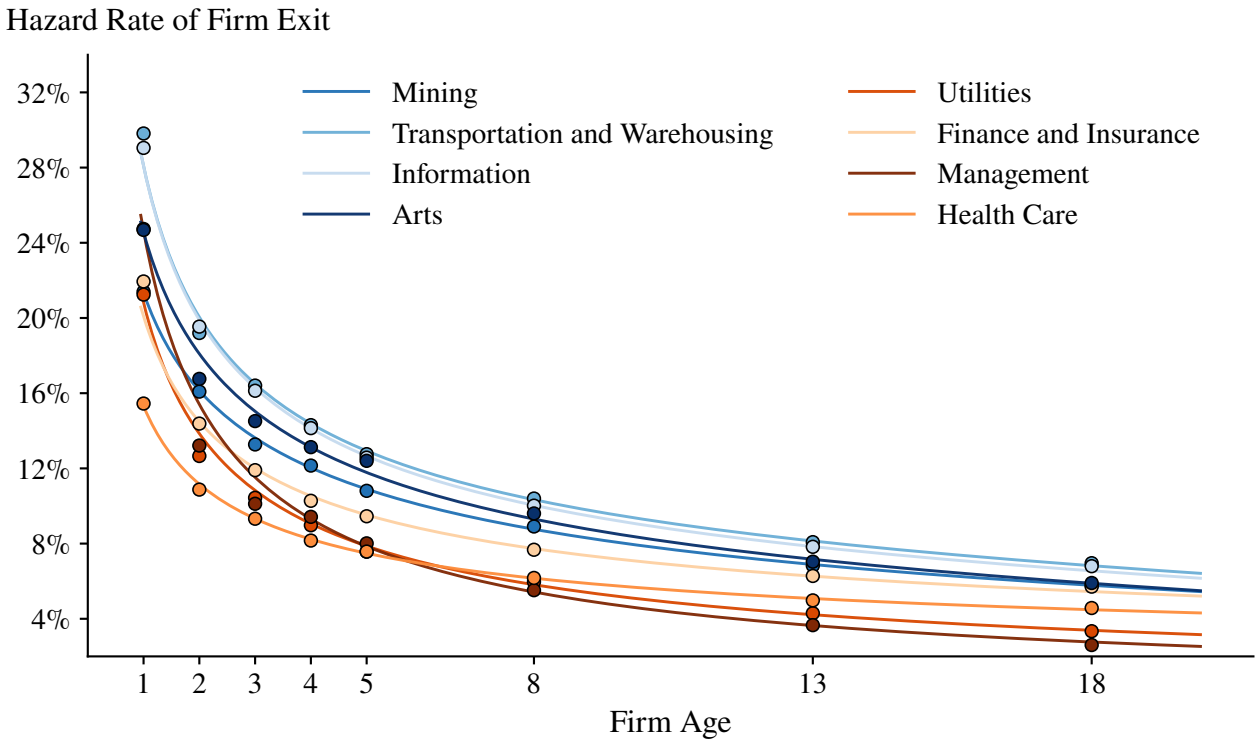
Notes: Underlying data are Business Dynamics Statistics for 1996-2019. The estimation targets the sector's profile of firm exit by age, specifically at ages 1, 2, 3, 4, 5, 8, 13, and 18.

Figure F5: Estimated Wedge and Firm Exit Rate Across Sectors



Notes: Estimated wedge is the point estimate of $\theta\eta - \alpha\beta$ when targeting the sector's firm exit at ages 1, 2, 3, 4, 5, 8, 13, and 18. Pearson correlation coefficient between firm exit rate and estimated wedge is 0.68. Underlying data are BDS for 1996-2019.

Figure F6: Firm Exit by Age in Sectors with Largest and Smallest Estimated Wedges



Notes: Sectors in blue are the four in which I estimate the largest wedge. Sectors in orange are the four in which I estimate the smallest wedge. Dots correspond to empirical moments, each curve is the model's fit when targeting that sector's profile of firm exit by age. The objective puts equal weights on ages 1 through 5, and five times the weight on ages 8, 14, and 18. Underlying data is from Business Dynamics Statistics for the years 1996 to 2019.

G Extensions

G.1 Multi-sector version of the model

The model presented in the main text features a single sector. In this appendix, I present a version of the model with multiple sectors. To avoid cumbersome notation, I shut down volatility ($\nu = 0$) and drift ($\beta = 0$). While these matter for the quantitative aspect of the results, they do not affect the qualitative point I want to make, which is that product turnover or churn in a sector reflects knowledge spillovers *received* by that sector.

Suppose the consumption good is a Cobb-Douglas aggregate of different sectors:

$$c_t = \prod_{s=1}^{S} c_{st}^{\beta_s} \quad \text{where } \beta_s \in (0,1) \text{ and } \sum_{s=1}^{S} \beta_s = 1 \,.$$

As the below will make clear, Cobb Douglas aggregation across sectors is needed to achieve a BGP with constant share of employment in each sector, since different sectors will grow at different rates.

Within each sector, c_{st} is a CES aggregate of a continuum of differentiated varieties, as in the model in the main text:

$$c_{st} = \left[\int_{p \in \Omega_{st}} \left(Q_{pt} c_{pt} \right)^{\frac{\sigma_s - 1}{\sigma_s}} dp \right]^{\frac{\sigma_s}{\sigma_s - 1}} .$$

The set Ω_{st} evolves through entry and exit. Exit results from the presence of a sector-specific fixed overhead cost $\mathcal{F}_s$, denoted in labor units. After payment of this fixed cost, quantity is linear in production labor.

The creation of new products is carried out by incumbent firms in that sector as well as entering firms. Entry of new firms is the result of individuals who choose to be entrepreneurs. If individual i chooses to be an entrepreneur, they draw their idiosyncratic and sector specific entrepreneurship ability from a multivariate Fréchet distribution with cumulative distribution function:

$$F\left(\varepsilon_1, \ldots, \varepsilon_S\right) = \exp\left[- \left(\sum_{s=1}^{S} \varepsilon_s^{-\frac{\widetilde{\lambda}}{1-\kappa}} \right)^{1-\kappa} \right]$$

where$\widetilde{\lambda}$ captures dispersion in entrepreneurship ability between individuals, with lower $\widetilde{\lambda}$ corresponding to more dispersion, and κ measures the correlation between entrepreneurship ability across the different sectors. After observing these draws, the individual chooses toward which sector to direct their entrepreneurship endeavors. If they choose sector s, they generate a new blueprint with Poisson arrival rate $\eta_s \cdot \varepsilon_{is}$ where η_s captures sector-specific features common to all entrepreneurs (ideas are harder to find in some sectors).[4] Denoting by K_{st} the stock of sector s products ever

[4]The individual specific component introduces the diminishing returns necessary to ensure an interior solution for the allocation of entrepreneurship across sectors.

created, its law of motion is then given by:

$$\dot{K}_{st} \;=\; \int_{i\in\mathcal{R}_{st}} \eta_s \cdot \varepsilon_{is} \quad \text{where} \quad \mathcal{R}_{st} \;\equiv\; \text{Sector } s \text{ entrepreneurs} \;.$$

The quality of a new product in sector s at time t is drawn from a sector specific entry distribution with CCDF $\overline{F}^{E}_{st}(Q)$ given by:

$$\overline{F}^{E}_{st}(Q) \;=\; \Pr\left(\text{Draw}_{st} \;>Q\right) \;=\; \begin{cases} 1 & \text{if } Q \;<\; \prod\limits_{j} K_{jt}^{\frac{\theta_{j\to s}}{\alpha_s}} \\ \prod\limits_{j} K_{jt}^{\theta_{j\to s}} Q^{-\alpha_s} & \text{otherwise} \end{cases} \;,$$

where $\theta_{j\to s}$ encodes the strength of knowledge spillovers from sector j to sector s.

A balanced growth path features (i) constant allocation of labor across sectors s and occupations (so that production labor and the measure of products grows at rate η in each sector), (ii) constant growth in the sector-specific exit threshold $\underline{Q}_{st}$, and (iii) stationary distribution of relative quality in each sector.

Along such a balanced growth path,

$$\frac{\dot{K}_{st}}{K_{st}} = \eta \quad \text{and} \quad g = \sum \beta_s \left(\frac{\eta}{\sigma_S - 1} + g_{Q_s} \right) \;,$$

where g_{Q_s}, the growth rate of quality in sector s, is given by:

$$g_{Q_s} = \frac{\Theta_s}{\alpha_s}\eta \quad \text{with} \quad \Theta_s \;\equiv \sum_{j=1}^{S} \theta_{j\to s} \;.$$

The resulting stationary product exit and entry rates are:

$$\frac{D_{st}}{M_{st}} = \Theta_s \eta \quad \text{and} \quad \frac{E_{st}}{M_{st}} = (1+\Theta_s)\,\eta \;.$$

As I mentioned at the top of this subsection, I have set $\nu = 0$ in deriving these results, that's why all turnover is due to spillovers. The point I want to illustrate is that the entry and exit rates in sector s are function of $\sum_j \theta_{j\to s}$, which is a measure of the spillovers *received* by sector s.

G.2 Model incorporating human capital growth

The goal of this subsection is to assess how incorporating human capital growth alters my results. The answer depends on whether human capital growth leads to more draws from the entry distribution or better draws from the entry distribution.

G.2.1 Higher human capital $\implies$ more draws from entry distribution

Suppose that each individual supplies h_t efficiency units of labor per unit of time. The baseline version of the model corresponds to $h_t \equiv 1$ for all t. Here instead I want to entertain exogenous growth in h_t at rate g_h:

$$\dot{h}_t = g_h h_t \ .$$

The only change relative to the environment of the baseline model is that the labor resource constraint is in efficiency units now and reads:

$$\int_{p\in\Omega_t} \left(L_{pt} + \mathcal{F} + I_{pt}\right) dp + S_t = h_t N_t \ .$$

Along the balanced growth path of this model, it follows that

$$\frac{\dot{M}_t}{M_t} = \frac{\dot{S}_t}{S_t} = \frac{\dot{L}_t}{L_t} = \frac{\dot{K}_t}{K_t} = \eta + g_h \quad \text{and} \quad g_Q = \frac{\theta}{\alpha}\left(\eta + g_h\right) \ .$$

Proceeding as I did for the baseline model, one can show that along the equilibrium BGP, the social rate of return to R&D is

$$\widetilde{r}_{\mathbf{DE}} = r + \theta(\eta + g_h) \ ,$$

so that the gap between social and private rate of return is $\widetilde{r} - r = \theta(\eta + g_h) = \theta g_K$. The dynamics of product exit are still informative about this gap as the KFE now leads to the following stationary product entry and exit rates:

$$\frac{E_t}{M_t} = \eta + g_h + \frac{\nu^2}{2}\alpha\widetilde{\zeta} \quad \text{where} \quad \widetilde{\zeta} \equiv \frac{g_Q - \beta + \sqrt{(g_Q-\beta)^2 + 2\nu^2(\eta+g_h)}}{\nu^2} \ ,$$
$$\frac{D_t}{M_t} = \frac{1}{2}\alpha(g_Q - \beta) + \frac{1}{2}\sqrt{(\alpha(g_Q-\beta))^2 + 2\nu^2(\eta+g_h)} = \frac{1}{2}(\theta g_K - \alpha\beta) + \frac{1}{2}\sqrt{(\theta g_K - \alpha\beta)^2 + 2\nu^2(\eta+g_h)} \ .$$

Intuitively the reason is that the SDE for a product's relative quality can still be written as:

$$dq_{pt} = \left(\beta - \frac{\theta}{\alpha}g_K\right)dt + \nu dB_{pt} \ ,$$

So that product exit due to downward drift still exactly maps to the wedge of interest when $\beta = 0$.

The takeaway from this extension is that my baseline approach is robust to having human capital growth that leads to more draws from the entry distribution. The only effect such exogenous growth has is how to "split" the estimated wedge between θ and growth in the knowledge stock (K_t).

G.2.2 Higher human capital $\implies$ better draws from entry distribution

Suppose instead that higher human capital shows up as an ability to draw from a better distribution:

$$\overline{F}^E_t(Q) = h_t^\lambda K_t^\theta Q^{-\alpha} \ .$$

With $\dot{h}_t = g_h h_t$, human capital growth then leads to a shift out in the entry distribution over time. It follows that along the balanced growth path,

$$\frac{\dot{M}_t}{M_t} = \frac{\dot{S}_t}{S_t} = \frac{\dot{L}_t}{L_t} = \frac{\dot{K}_t}{K_t} = \eta \quad \text{and} \quad g_Q = \frac{\theta}{\alpha}\eta + \frac{\lambda}{\alpha} g_h \ .$$

Proceeding as I did for the baseline model, one can show that along the equilibrium BGP, the social rate of return to R&D is still

$$\widetilde{r}_{\mathbf{DE}} = r + \theta\eta \ .$$

It is also still the case that the extent of product exit due to downward drift to the exit threshold is $\alpha(g_Q - \beta)$, which is now equal to $\theta\eta + \lambda g_h - \alpha\beta$. Therefore

$$\theta\eta + \lambda g_h - \alpha\beta = 15.8\% \overset{\beta=0}{\Longrightarrow} \boxed{\theta\eta + \lambda g_h = 15.8\%} \ .$$

To see how to split the 15.8% between $\theta\eta$ and λg_h, note that

$$g_Q = \frac{1}{\alpha}\left(\theta\eta + \lambda g_h\right) \quad \text{and} \quad g = \frac{\eta}{\sigma - 1} + g_Q \ .$$

To pin down $\frac{\lambda}{\alpha} g_h$, I use the labor composition index provided by the BLS to measure the contribution of human capital growth to productivity growth. For the private nonfarm sector between 1978 and 2019, the average annual growth rate of this labor composition index is 0.43%. It follows that $\theta\eta = 12.4\%$.

References

Andrews, Isaiah, Matthew Gentzkow, and Jesse M. Shapiro. 2017. "Measuring the Sensitivity of Parameter Estimates to Estimation Moments*." *The Quarterly Journal of Economics*, 132(4): 1553–1592.

Chow, Melissa C., Teresa C. Fort, Christopher Goetz, Nathan Goldschlag, James Lawrence, Elisabeth Ruth Perlman, Martha Stinson, and T. Kirk White. 2021. "Redesigning the Longitudinal Business Database." National Bureau of Economic Research NBER Working Paper 28839.

Crump, Kenny S., and Charles J. Mode. 1968. "A general age-dependent branching process. I." *Journal of Mathematical Analysis and Applications*, 24(3): 494–508.

Crump, Kenny S., and Charles J. Mode. 1969. "A general age-dependent branching process. II." *Journal of Mathematical Analysis and Applications*, 25(1): 8–17.

Jagers, Peter. 1975. *Branching Processes with Biological Applications. Wiley Series in Probability and Mathematical Statistics*, London; New York; Sydney; Toronto:John Wiley & Sons.

Jarmin, Ron S., and Javier Miranda. 2002. "The Longitudinal Business Database." U.S. Census Bureau, Center for Economic Studies CES Working Paper CES-WP-02-17.

Jones, Charles I., and John C. Williams. 1998. "Measuring the Social Return to R&D*." *The Quarterly Journal of Economics*, 113(4): 1119–1135.

Klette, Tor Jakob, and Samuel Kortum. 2004. "Innovating Firms and Aggregate Innovation." *Journal of Political Economy*, 112(5): 986–1018.

Nuño, Galo, and Benjamin Moll. 2018. "Social optima in economies with heterogeneous agents." *Review of Economic Dynamics*, 28: 150–180.

Redner, Sidney. 2001. *A Guide to First-Passage Processes.* Cambridge University Press.

Sterk, Vincent, Petr Sedláček, and Benjamin Pugsley. 2021. "The Nature of Firm Growth." *American Economic Review*, 111(2): 547–79.