Time-Consistent Public Good Provision in Heterogeneous Agent Economies

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PRELIMINARY*

Abstract

This paper analyzes optimal public-good provision in heterogeneous-agent economies with incomplete markets and no commitment. Relative to the Samuelson (1954) benchmark, time-consistent policy in such a setup introduces a dynamic wedge because public spending affects capital accumulation and, through it, equilibrium wages and interest rates. In an Aiyagari-style environment, these general-equilibrium price effects redistribute resources across households with different labor productivities and asset positions, so public spending also becomes an indirect redistributive and insurance instrument. Optimal policy can therefore feature either over- or under-provision relative to Samuelson, and both uninsurable income risk and wealth inequality systematically shape the size of government. The analysis combines cross-country empirical evidence, closed-form results in a tractable model, and a novel computational method for Markov-perfect policy in a rich heterogeneous-agent economy.

Keywords: Heterogeneous Agents, Optimal Policy, Time Consistency, Public Good Provision, Taxation.

JEL codes: E52, D52, E21.

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1 Introduction

Countries differ markedly in the provision of public goods and the fiscal instruments used to finance them. Such heterogeneity generated a vast literature on the determinants of the size of government. This lively line of research can be seen as an extension of the seminal characterization of optimal public-good provision by Samuelson (1954), which considers a static environment and shows that a utilitarian government equates the marginal rate of substitution between private and public goods to their marginal rate of transformation. This result no longer holds when public goods are financed with distortionary taxes or when allocations are not constrained efficient (see the literature review below).

In this paper, we study how household heterogeneity and market incompleteness affect the optimal size of government in Bewley–Huggett–Aiyagari economies. These models are now workhorses of quantitative macroeconomics.[1]

Recent contributions on optimal policy in these settings study Ramsey plans. While these analyses are useful for understanding the normative properties of heterogeneous-agent models, the assumption of commitment is arguably unrealistic for actual policy decisions.

We, instead, study the problem of a benevolent government who provides a public good financed by lump-sum taxes and, importantly, faces a lack of commitment. We therefore characterize time-consistent policy, that is, a Markov-perfect equilibrium in a neoclassical growth model with heterogeneous agents (Aiyagari, 1994). In each period, the government must choose the level of public-good provision —a choice which affects the entire wealth distribution next period— anticipating that future governments may revise this provision in response to changes in that distribution. We show that this environment generates new mechanisms linking risk and inequality to the provision of public goods.

To clarify these mechanisms, we present our results as deviations from the standard Samuelson rule. To start, the absence of commitment generates a wedge relative to the static Samuelson benchmark, as each government has an incentive to manipulate capital accumulation through public-good provision. This wedge is a reformulation of the one identified by Klein et al. (2008) in the standard neoclassical growth model with representative agent.

In the heterogeneous-agent version, two key additional wedges arise. First, precautionary saving leads to capital over-accumulation, which represents an initial distortion. Second, public-good provision affects capital accumulation and therefore equilibrium

[1] See, for example, the surveys by Heathcote et al. (2009); Quadrini and Ríos-Rull (2015); Krueger et al. (2016); Ragot (2018); Auclert et al. (2025). Cherrier et al. (2023) offer an historical account of the evolution of this literature.

wages and interest rates. As a consequence, a utilitarian government can adjust public-good provision at the margin to influence capital accumulation and general-equilibrium prices in order to improve welfare. The direction of this effect is ambiguous and depends on the distribution of labor and capital income across households, as emphasized by Dávila et al. (2012). Public goods may be either over-provided or under-provided relative to this dynamic complete market benchmark. Each government anticipates that higher public-good provision reduces private wealth next period through the negative wealth effect and therefore reduces capital accumulation, even when taxes are non-distortionary. The government, however, may wish to increase capital accumulation to raise wages and lower interest rates when poorer households rely primarily on labor income rather than capital income. In this case, risk and inequality reduce public good provision, over and above what the Samuelson rule would predict.

To identify and quantify these mechanisms, we perform this analysis in two environments. First, we consider a tractable model. We assume that the economy consists of two islands across which agents move according to the realization of their idiosyncratic shocks, with full risk sharing within each island. This representation nests the deterministic-transition economy of Woodford (1990) as a special case. We derive an explicit expression for the generalized Euler equation (GEE), in the spirit of Klein et al. (2008), in which the effect of inequality on public-good provision, and therefore the deviation from the Samuelson rule, can be made explicit.

This GEE admits a closed-form characterization, which allows us to provide analytical conditions under which the public good is overprovided or underprovided relative to the Samuelson benchmark, closely related to the analysis of Dávila et al. (2012). We show that an increase in inequality or a rise in uninsurable risk systematically reduce public-good provision.

Second, we perform the analysis in a general Bewley–Huggett–Aiyagari model. This environment is substantially more challenging, both theoretically and computationally. We develop a general algorithm to compute time-consistent policies and Markov-perfect equilibria in this class of economies. Our method relies on a finite state-space approximation of the incomplete-markets economy, which allows us to derive and solve the government's first-order conditions without commitment.

Following LeGrand and Ragot (2025), we show that the allocation of a suitably defined "island" representation of the economy can approximate arbitrarily well the equilibrium allocation of the full Bewley–Huggett–Aiyagari model. Using this representation, we derive the first-order optimality condition characterizing the Markov-perfect equilibrium and relate it to the generalized Euler equation in representative-agent economies derived by Klein et al. (2008). This condition provides a tractable way to obtain the optimal time-consistent policy rule, defined as a mapping from the entire wealth distribution to the optimal level of public-good provision.

The paper also contains a cross-country empirical analysis which confirms that higher wealth inequality is associated with lower public good provision, after controlling for other key determinants of spending on public goods.

Literature review. This paper sits at the intersection of the literature on optimal public-good provision and the literature on optimal policy in heterogeneous-agent models.

The classic benchmark in the first branch is Samuelson (1954), where a utilitarian government in a static setting equates the average marginal rate of substitution between private and public goods to their marginal rate of transformation. Sandmo (1998) studies the problem in a static model with heterogeneous agents and a limited set of tax instruments, and highlights a redistributive motive. Kaplow (2004) extends this analysis to the Mirrleesian benchmark. Here, the tax system is flexible enough to implement the constrained efficient allocation, so the Samuelson rule for public good spending is restored. In an effort to keep our analysis interesting to policymakers, we focus on a limited policy toolkit, so public-good provision carries redistributive content.

Our paper also relates to the literature on optimal policy design in heterogeneous-agent models. Foundational analyses include Aiyagari (1995) and Aiyagari and McGrattan (1998), which derive normative implications of precautionary saving in such environments. More recently, Dávila et al. (2012) characterizes constrained efficiency in heterogeneous-agent models. A large quantitative literature solves Ramsey policy with commitment, including LeGrand and Ragot (2022), Bhandari et al. (2021), Dyrda and Pedroni (2023), McKay and Wolf (2022), Açikgöz et al. (2022), and Auclert et al. (2024). These papers implicitly assume the existence of a commitment device and thus feature is no policy feedback channel across periods.

The no-commitment problem is harder because policy must satisfy a Markov-perfect equilibrium condition. In complete-markets settings, Cohen and Michel (1988) and Klein et al. (2008) derive the time-consistent first-order condition as a generalized Euler equation. In incomplete-markets environments, results remain scarce: Dávila and Schaab (2023) study optimal policy in continuous time with an infinitesimal commitment horizon, while Jang (2026) solves for time-consistent rules with the Reiter (2010) algorithm. We contribute by using the finite state-space representation introduced in LeGrand and Ragot (2022) and refined in LeGrand and Ragot (2025) to derive and solve the MPE first-order condition and show how it differs from the complete-markets generalized Euler equation.

The rest of the paper proceeds as follows. Section 2 presents cross-country empirical evidence supporting theoretical results. Section 3 analyzes the simple, tractable model. Section 4 develops the general model and the government's first-order condition. Finally, Section 5 reports quantitative and computational results. All derivations are

collected in Appendix.

2 Empirical Motivation

To motivate our theoretical analysis, we investigate the relationship between wealth inequality and the size of government using a panel of 38 OECD countries over the period 1980–2024. The data combine information on wealth inequality from the World Inequality Database, data on spending on public goods from the World Bank, data on various economic indicators from the Global Macro Database, and data on political indicators from the Database of Political Institutions assembled by the Inter-American Development Bank.

Our baseline specification relates government spending as a share of GDP to measures of wealth inequality, controlling for standard macroeconomic and demographic factors, as well as country and year fixed effects:

$$\frac{G_{it}}{Y_{it}} = \beta_1 \text{Gini}_{it} + \beta_2 \log(\text{GDP per capita}_{it}) + \gamma' X_{it} + \alpha_i + \alpha_t. \tag{1}$$

The vector X_{it} includes the population shares of young (0–14) and elderly (65+) individuals, the unemployment rate, and trade openness. Standard errors are clustered at the country level.

Table 1 reports the main results. Across specifications, we find a robust negative relationship between wealth inequality and government spending. In the baseline specification, a higher wealth Gini is associated with a lower government spending-to-GDP ratio, with a coefficient of -0.074, statistically significant at conventional levels. The magnitude is economically meaningful: a 10-point increase in the wealth Gini is associated with a reduction in government spending of about 0.7 percentage points of GDP.

This negative association remains stable when additional controls are introduced (column 2), when observations are weighted by population (column 3), and when country-specific trends are included (column 4). The robustness of this result across these specifications suggests that it is not driven by omitted macroeconomic dynamics or differential country trends.

Column (5) allows for non-linear effects by including a quadratic term in the wealth Gini. The estimates point to a convex relationship, with a negative linear term and a positive quadratic term. This suggests that the marginal effect of inequality on government size becomes less negative - and potentially reverses - at very high levels of inequality.

Column (6) replaces the Gini coefficient with the wealth share held by the top 10 percent. The coefficient is again negative and highly significant, reinforcing the baseline

	Government Spending / GDP					
	(1) Baseline	(2) Controls	(3) Weighted	(4) Trends	(5) Quadratic	(6) Top 10%
Wealth Gini	-0.0740*	-0.0819**	-0.0828**	-0.0828*	-0.6268***	-0.1291***
	[-1.77]	[-2.23]	[-2.17]	[-1.84]	[-2.93]	[-4.45]
Log GDP per capita	-0.0505***	-0.0345**	-0.0321**	-0.0417**	-0.0354**	-0.0351**
	[-3.30]	[-2.24]	[-2.15]	[-2.53]	[-2.32]	[-2.28]
Population share 65+	0.0028**	0.0026*	0.0027**	-0.0004	0.0028**	0.0029**
	[2.17]	[2.01]	[2.25]	[-0.23]	[2.21]	[2.39]
Population share 0-14	-0.0021**	-0.0014*	-0.0015**	0.0009	-0.0014*	-0.0014**
	[-2.61]	[-1.95]	[-2.09]	[0.68]	[-1.90]	[-2.05]
Unemployment rate		0.0013***	0.0013***		0.0012**	0.0012***
		[2.79]	[2.91]		[2.61]	[2.82]
Trade openness		-0.0133	-0.0147		-0.0133	-0.0134
		[-0.92]	[-1.02]		[-0.91]	[-0.93]
Wealth Gini squared					0.3580**	
					[2.50]	
Country & Year fixed effects	√	√	√	√	√	√
Country-specific trends				√		
SE clustered by	Country	Country	Country	Country	Country	Country
Weights	None	None	Population	None	None	None
Observations	1,570	1,566	1,566	1,570	1,566	1,574
R^2	0.864	0.874	0.879	0.930	0.875	0.875
Within R^2	0.195	0.234	0.241	0.584	0.243	0.240

Table 1: Cross-country regression of G/Y on measures of wealth inequality and various additional control variables.

finding that greater concentration of wealth is associated with smaller governments.

The estimated coefficients on control variables are consistent with standard interpretations. Government spending is decreasing in GDP per capita, suggesting that richer countries tend to have lower spending shares once fixed effects are accounted for. Demographic structure matters: a larger elderly population is associated with higher public spending, while a larger share of young individuals is associated with lower spending. The unemployment rate is positively correlated with government size, reflecting the role of automatic stabilizers. Trade openness, by contrast, does not appear to have a statistically significant effect in our sample. Although not shown in the table, when we control for various political indicators

3 A Tractable Model

This section sets up a two-island economy with incomplete markets and a borrowing constraint, and derives the optimal public-spending rule under both *time-consistent* (Markov-perfect) policy and *commitment* (Ramsey policy).

The two island economy is an environment where there is full risk-sharing within each island, but where agents move across islands depending of the realization of idiosyncratic risk. This environment captures precautionary-saving and heterogeneity

in a tractable way, and embeds the the case of a deterministic transition of Woodford (1990), as shown below.[2] Section 3.1 presents this two-island setup and provides baseline definitions. Section 3.2 derives the Markov-perfect equilibrium (MPE) corresponding to the policy implemented by a time-consistent government. Conversely, Section 3.3 assumes the existence of a commitment device for the government. It solves for the Ramsey program and compares MPE and Ramsey allocations.

3.1 Economy and private-sector equilibrium

3.1.1 Demographics and income risk

Time is discrete and indexed by $t = 0, 1, 2, \ldots$. The economy is populated by a measure one of families. Each family has a continuum of individual members (agents). Individuals are infinitely lived.

The economy is composed of two islands, $i \in \{R, P\}$, each of measure $1/2$ in every period. Island productivities are $(1 + \Delta)$ in R (the "rich" island) and $(1 - \Delta)$ in P (the "poor" island). Individuals move stochastically across islands according to the Markov transition matrix

$$\begin{pmatrix} 1-\pi & \pi \\ \pi & 1-\pi \end{pmatrix}, \qquad \pi \in [0, 1],$$

where π is the switching probability. The unique stationary distribution is $(1/2, 1/2)$ for $\pi \in (0, 1)$. This Markov chain determines individual income risk.

3.1.2 Risk sharing

A family planner chooses consumption allocations for each family member. In addition, at each date, the family planner chooses how much income to save in capital on each island. At the beginning of the period, the family planner can move capital across islands only proportionately to the share of its family members moving, i.e. when a share π of agents move from an island to the other, the same share of capital saved on that island goes with them to the other. This is the extent to which resources can be transferred across islands (states) within a period which allows for some self-insurance. A borrowing constraint, set to 0, prevents agents in island P from ending the period with negative assets. As a result, there is perfect risk-sharing across islands, but only imperfect risk-sharing between island, even within members of the same family. Importantly, each family planner takes prices as given because it is measure zero.

Throughout this section, we follow a guess-and-verify strategy and consider the case where constraint binds for agents on the P-island, such that P-agents choose zero

[2] The island environment is studied in Challe et al. (2017), Bilbiie and Ragot (2021), and Bilbiie (2025) among many others. See also Heathcote et al. (2017) for another strategy to obtain tractable environment.

end-of-period assets.

3.1.3 Technology and Factor Prices

Aggregate output is produced using capital and labor: $Y_t = AK_{t-1}^{\alpha}L^{1-\alpha}, \alpha \in (0,1)$, with full depreciation between $t-1$ and t. Labor supply will be normalized to $L = 1$. Hence, output is:

$$Y_t = AK_{t-1}^{\alpha}.$$

Let k_t denote end-of-period per-capita assets chosen by agents located in island R at date t (the other island holds zero assets at date t in the credit-constrained equilibrium). For normalization purposes, we set $A = 2^{\alpha-1}$ and write aggregate capital as $K_{t-1} = \frac{1}{2}k_{t-1}$, so that the aggregate production function becomes:

$$Y_t = \frac{1}{2}k_{t-1}^{\alpha}. \tag{2}$$

Factor prices are:

$$w_t = (1-\alpha)Y_t = \frac{1}{2}(1-\alpha)k_{t-1}^{\alpha}, \qquad 1 + r_t = \alpha A K_{t-1}^{\alpha-1} = \alpha k_{t-1}^{\alpha-1}. \tag{3}$$

Labor income in island i is therefore $(1 \pm \Delta)w_t$, and total effective labor is $L = \frac{1}{2}(1+\Delta) + \frac{1}{2}(1-\Delta) = 1$.

3.1.4 Government and preferences

A benevolent government chooses public spending $g_t \geq 0$ each period and finances it with a lump-sum tax of g_t per capita. Per-period utility for an individual is

$$U(c_t^i, g_t) := u(c_t^i) + \frac{1}{2}v(g_t),$$

where u and v are increasing and concave; $\beta \in (0,1)$ is the discount factor. Up to a constant factor, utilitarian social welfare equals

$$\mathcal{W}_t := \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left[u(c_{t+j}^R) + u(c_{t+j}^P) + v(g_{t+j}) \right], \tag{4}$$

since each island has mass $1/2$.[3]

[3] In this version of the paper we only analyze utilitarian welfare, but our results are immediately generalizable to the case of unequal social welfare weights.

3.1.5 Budget Constraints

At date $t-1$, agents in island R end the period with asset holdings k_{t-1}, which becomes the economy's state at date t. At date t, a fraction $(1-\pi)$ of those agents remain in island R, so island R receives per-capita capital income $(1+r_t)(1-\pi)k_{t-1}$, while island P receives $(1+r_t)\pi k_{t-1}$. We denote island-R (island-P) consumption by c^R (c^P).

Define the *reduced-form income shares*

$$\theta_R(1-\pi,\Delta) := \frac{1}{2}(1-\alpha)(1+\Delta)+\alpha(1-\pi), \qquad \theta_P(1-\pi,\Delta) := \frac{1}{2}(1-\alpha)(1-\Delta)+\alpha\pi, \tag{5}$$

which satisfy $\theta_R+\theta_P=1$. When no confusion arises, we suppress the arguments $(1-\pi,\Delta)$ and write (θ_R,θ_P). Then the budget constraints when the borrowing constraint binds for island P are:

$$c_t^R = \theta_R\, k_{t-1}^{\alpha} - k_t - g_t, \tag{6}$$

$$c_t^P = \theta_P\, k_{t-1}^{\alpha} - g_t. \tag{7}$$

3.1.6 Within-period timing

Fix a date $t \geq 0$. The predetermined state is k_{t-1}, the end-of-period assets chosen at date $t-1$ by agents located in island R (in the constrained case, agents in island P carry zero assets). A period unfolds as follows:

1. *Mobility.* Agents draw next-period locations according to the transition matrix in Section 3.1.2, so that a fraction $(1-\pi)$ of last period's island-R agents remain in R (and a fraction π move to P), and vice versa for island-P agents.

2. *Production.* Output is produced with aggregate capital $K_{t-1} = \frac{1}{2}k_{t-1}$ and labor $L=1$, generating output Y_t and factor prices $(w_t, 1+r_t)$. Under full depreciation, K_{t-1} is entirely consumed in production at date t.

3. *Policy.* The government chooses public spending g_t and finances it with a lump-sum tax of g_t per capita.

4. *Risk Sharing and Saving.* Within each island, agents share risk so that contemporaneous consumption is common within each island. Agents currently in island R choose next period assets k_t, while agents in island P optimally choose zero end-of-period assets.

5. *Transition.* The economy moves to date $t+1$ with state k_t, which becomes the next period capital input $K_t = \frac{1}{2}k_t$.

3.1.7 Euler equation and implementability

In the constrained case, only island-R agents choose k_t. Their Euler equation is:

$$u_c(c_t^R) = \beta(1+r_{t+1})\Big((1-\pi)\,u_c(c_{t+1}^R) + \pi u_c(c_{t+1}^P)\Big), \tag{8}$$

where $1 + r_{t+1} = \alpha k_t^{\alpha-1}$ from (3). An equilibrium must also satisfy the Kuhn–Tucker condition that island-P agents prefer not to borrow, which we relegate to Appendix A.6.

For later use, we define the residual of the Euler equation of R agents.

$$\eta_t := u_c(c_t^R) - \beta\alpha k_t^{\alpha-1}\Big((1-\pi)\,u_c(c_{t+1}^R) + \pi u_c(c_{t+1}^P)\Big). \tag{9}$$

3.2 Markov-perfect (time-consistent) policy

3.2.1 Definition and Bellman representation

The previous formulation allows us to follow closely the strategy of Klein et al. (2008), which we will sometimes shorten as KKR in what follows. To simplify the exposition, we consider an equilibrium where the credit constraints are binding for the poor, which we later verify exists.[4] An equilibrium consists of a value function $V(k)$, a government policy function $g = \Psi(k)$ and a savings function for island R, $k' = h'(k)$ solving

$$V(k) = \max_{g,k'}\Big\{u(c^R(k,k',g)) + u(c^P(k,g)) + v(g) + \beta V(k')\Big\} \tag{10}$$

subject to the Euler equation

$$u_c((c^R(k,k',g))) = \beta\alpha k'^{\alpha-1}\Big((1-\pi)\,u_c(c^R(k',h(k'),\Psi(k))) + \pi u_c(c^P(k',\Psi(k')))\Big).$$

and

$$V(k) = u(c^R(k,h(k),\Psi(k))) + u(c^P(k,\Psi(k))) + v(\Psi(k)) + \beta V(k')$$

The previous expressions are the same as Klein et al. (2008), adapted to two distinct agents instead of a representative one.

To derive the conditions defining time-consistent policy rules, we first define the intertemporal wedge, when it is assumed that agents follow equilibrium rules in the future as

$$\eta(k,g,k') := u_c(c^R(k,k',g)) - \beta\alpha(k')^{\alpha-1}\Big((1-\pi)\,u_c(c^R(k',h(k'),\Psi(k'))) + \pi u_c(c^P(k',\Psi(k')))\Big).$$

[4] In what follows, we use primes to denote next-state values, evaluated at the next state and policy (e.g., $x' := x(k',g')$).

Define implicitly the function $k' = \mathcal{H}(k, g)$, as

$$\eta(k, g, \mathcal{H}(k, g)) = 0$$

This function is defined in the neighborhood of a steady state, where credit constraints are binding for the poor. This function is the reaction of the current savings decision of agents to a current deviation of public spending, assuming that the government and agents will follow equilibrium rules in the future.

3.2.2 A generalized Euler equation for the Markov-perfect policy

Using the implicit function theorem, we can denote partial derivatives of $\mathcal{H}$ by

$$\mathcal{H}_k := \frac{\partial \mathcal{H}(k,g)}{\partial k}, \qquad \mathcal{H}_g := \frac{\partial \mathcal{H}(k,g)}{\partial g},$$

and similarly at next period's state (k', g') by $\mathcal{H}'_k, \mathcal{H}'_g$. Define next-period consumption derivatives (holding g' fixed, as in an envelope derivative)

$$\begin{aligned} C^{P'}_{k'} &:= \frac{\partial c^P(k', g')}{\partial k'} = \alpha\theta_P(1-\pi, \Delta)\,(k')^{\alpha-1} \\ C^{R'}_{k'} &:= \frac{\partial c^R(k', k'', g')}{\partial k'} = \alpha\theta_R(1-\pi, \Delta)\,(k')^{\alpha-1}. \end{aligned}$$

Proposition 1 (Markov-perfect generalized Euler equation)**.** *Suppose V is differentiable and the Markov problem admits an interior solution. Then any interior Markov-perfect policy satisfies, at every state k:*

$$\underbrace{v_g(g) - \Big(u_c(c^R) + u_c(c^P)\Big)}_{\text{(i) static Samuelson residual}} = \underbrace{\beta \frac{\mathcal{H}_g}{\mathcal{H}'_g}\mathcal{H}'_k\Big(v_g(g') - \Big(u_c(c^{R,\prime}) + u_c(c^{P,\prime})\Big)\Big)}_{\text{(ii) policy feedback}} + \underbrace{\mathcal{H}_g\Big(u_c(c^R) - \beta u_c(c^{R,\prime})C^{R'}_{k'} - \beta u_c(c^{P,\prime})C^{P'}_{k'}\Big)}_{\text{(iii) intertemporal (capital / price) channel (IC)}}. \tag{11}$$

All objects are evaluated on the Markov equilibrium path induced by Ψ and $\mathcal{H}$.

The equation (11) is the expression of the Generalized Euler Equation of Klein et al. (2008) in our two-island environment.[5] Term (i) is the *static Samuelson residual* and is common to Ramsey and MPE: if the next state k' were held fixed, the government's first-order condition would set $v_g(g) = u_c(c^R) + u_c(c^P)$. Term (ii) is specific to the MPE: today's g affects the next state k', hence tomorrow's policy choice $g' = \Psi(k')$, and therefore tomorrow's static residual; the coefficient $\beta(\mathcal{H}_g/\mathcal{H}'_g)\mathcal{H}'_k$ arises from

[5]This reduced one-state GEE is derived for $\pi > 0$. For $\pi = 0$, credit constraints are slack and the correct MPE system has two states (k_R, k_P).

differentiating through the continuation policy.

Term (iii) denoted Intertemporal Price Channel, or IC in short, is common to Ramsey and MPE, as shown below. It captures the effect of g on welfare through the induced change in private saving $k' = \mathcal{H}(k, g)$. The derivatives $C^{R'}_{k'}$ and $C^{P'}_{k'}$ are envelope derivatives holding g' fixed, so the effect of k' on g' enters separately via term (ii). Section 3.2.3 further decomposes term (iii). Appendix A.2 derives (11).

3.2.3 Decomposing the intertemporal channel: a distributional component

The "capital / price" term (iii) in (11) can be decomposed to isolate a redistribution component operating through how capital affects factor prices.

Let $f_k(k') := \alpha(k')^{\alpha-1}$ be the marginal product of capital after normalizing using equation (2). Using equation (11) and $\theta_R + \theta_P = 1$,

$$C^{R'}_{k'} = f_k(k') - C^{P'}_{k'}. \tag{12}$$

Substituting (12) into the bracket in term (iii) yields the following identity.

Remark 1 (Two equivalent expressions for term (iii)). Defining the term IC (i.e. term (iii) in 11), we have:

$$IC = \beta\left(-\mathcal{H}_g\right)\left(-f_{kk}(k')\right)k' \sum_{i=P,R} u_c(c^i)\left(\frac{z^i}{z} - \frac{k^i}{k}\right). \tag{13}$$

where z^i/z is the labor share of agent $i = P, R$, k^i/k is the capital share of agents $i = P, R$, and $f_{kk}(k) = -\alpha(1-\alpha)k^{\alpha-2}$. Appendix A.2.5 reports the algebra to derive (13).

Equation (13) presents a decomposition in the spirit of Dávila et al. (2012) to highlight why the term ICc an be either positive or negative. One can show that the term IC depends on the difference between the labor share and the capita share of the rich and the poor, weighted by their respective marginal utilities. When the term $\sum_{i=P,R} u_c(c^i)\left(\frac{z^i}{z} - \frac{k^i}{k}\right)$ is positive, then under-providing g increases k'. This increases the real wage and decreases the real interest rate. It redistributes the "right way" when the welfare effect on the labor share dominates. The term is negative when a marginal increase in the interest rate and a decrease in the real wage increases welfare.

Remark 2. In the representative-agent limit, set $c := c^R = c^P$ and $c' := c^{R,\prime} = c^{P,\prime}$. Then

the redistribution wedge is zero and (11) collapses to:

$$\underbrace{v_g(g) - 2u_c(c)}_{\text{(i) static Samuelson residual}} = \underbrace{\beta \frac{\mathcal{H}_g}{\mathcal{H}'_g} \mathcal{H}'_k \Big(v_g(g') - 2u_c(c') \Big)}_{\text{(ii) policy feedback}} + \underbrace{\mathcal{H}_g \Big(u_c(c) - \beta u_c(c') f_k(k') \Big)}_{\text{(iii) intertemporal (capital / price) channel}} . \tag{14}$$

With $\gamma := v_g(g) - 2u_c(c)$ and $\nu := -u_c(c) + \beta u_c(c') f_k(k')$, (14) is

$$\gamma = \beta \frac{\mathcal{H}_g}{\mathcal{H}'_g} \mathcal{H}'_k \gamma' - \mathcal{H}_g \nu,$$

which is the KKR generalized Euler form. A compact mapping to their equation (4) is provided in Appendix A.2.6.

3.2.4 An analytical expression in the log-utility case

The remaining results in this section assume log utility, $u(c^i) = \log c^i$ and $v(g) = \log g$, which delivers an analytical expression for the planner's policy.

Assumption 1. $u(c) = \log c \quad$ *and* $\quad v(g) = \log g$.

Within this section, equilibrium characterization follows a guess-and-verify sequence: conjecture a functional form, derive the Euler relation between $(\mathcal{H}, \Psi)$, substitute into the Markov generalized Euler equation to obtain a closed-form equation for Ψ, and then select the admissible root to sign comparative statics.

Conjecture 1 (Scale-free Markov policies)**.** *In a Markov steady state, policies and saving are proportional to output:*

$$g = \Psi k^\alpha, \qquad k' = \alpha\beta\Xi k^\alpha,$$

for constant scalars $\Psi > 0$ *and* $\Xi > 0$.

Conjecture 1 is the guess; Propositions 2–8 and Lemma 2 verify existence of an admissible steady-state candidate and uniqueness of the admissible root.

Proposition 2 (Euler equation in the log case)**.** *Under Assumption 1 and Conjecture 1, the Euler condition implies the linear constraint*

$$0 = (\Xi - (1 - \pi))(\theta_P - \Psi) - \pi(\theta_R - \Psi - \alpha\beta\Xi), \tag{15}$$

which is equivalent to

$$\Xi = \frac{(1 - \pi)(\theta_P - \Psi) + \pi(\theta_R - \Psi)}{(\theta_P - \Psi) + \pi\alpha\beta}. \tag{16}$$

In the log case, the Euler equation residual becomes linear, as marginal utilities are proportional to the reciprocals of consumption. Equation (15) therefore pins down the steady-state saving parameter Ξ as a function of the spending share Ψ via (16). Credit-constrained equilibrium existence requires $\Psi < \theta_P$ (so that $c^P > 0$) and $\theta_R - \Psi - \alpha\beta\Xi > 0$ (so that $c^R > 0$ under Conjecture 1).

Proposition 3 (Closed-form Markov spending share)**.** *Under Assumption 1 and Conjecture 1, the Markov steady-state spending share Ψ is characterized, within the scale-free class, as the unique admissible root in $(0, \theta_P)$ of a quadratic equation*

$$12\Psi^2 + b\Psi + c = 0, \tag{17}$$

with coefficients b and c reported in Appendix A.3.

Lemma 1 (Upper bound for the admissible MPE share (scale-free class))**.** *Under Assumption 1 and Conjecture 1, the admissible MPE root satisfies*

$$0 < \Psi < \frac{\theta_P}{2}. \tag{18}$$

Equation (18) is proved in Appendix A.4.

Proposition 3 is obtained by substituting the conjecture into the Markov generalized Euler equation and then eliminating Ξ using (16). The restriction $\Psi \in (0, \theta_P/2)$ is not a normalization: it enforces strictly positive poor consumption when credit constraints bind. The second root of (17) typically violates either feasibility (e.g., $c^P \leq 0$) or the inequalities ensuring the existence of the equilibrium with binding credit constraints on island P. Appendix A.4 shows that in the constrained case there is a unique admissible root and provides implicit-derivative formulas for comparative statics. A useful preview is that both comparative statics share a common mechanism: $(1 - \pi, \Delta)$ enter the quadratic only through a common factor, and the admissible root is decreasing in that index in the constrained case. This common-index structure drives $\Psi_\Delta < 0$ and $\Psi_{(1-\pi)} < 0$ in Proposition 4.

Proposition 4 (Comparative statics of the Markov spending share)**.** *Under Assumption 1 and Conjecture 1, let Ψ denote the admissible Markov steady-state spending share characterized in Proposition 3. Define the dispersion threshold:*

$$\bar{\Delta} := \frac{\alpha(1+\beta)}{1-\alpha}.$$

Suppose $\Delta > \bar{\Delta}$ which implies that the constrained equilibrium exists for all $\pi \in (0, 1]$ (Appendix A.6). Then, for every $\pi \in (0, 1)$:

$$\Psi_\Delta < 0 \qquad \textit{and} \qquad \Psi_{(1-\pi)} < 0.$$

Appendix A.4 contains the proof. Proposition 4 states two results. First, when income inequality is sufficiently large, $\Delta > \bar{\Delta}$, the planner cannot undo credit constraints and the equilibrium with binding credit constraints for the P agents exists. Second, higher dispersion or lower mobility both tilt the credit-constrained economy toward tighter poor-consumption feasibility, so the admissible Markov rule lowers the spending share. Because both effects work through the same index, the sign logic is parallel across the two derivatives.

Proposition 5 (Dispersion monotonicity of Markov capital)**.** *Under Assumption 1, Conjecture 1, and the conditions of Proposition 4, for every $\pi \in (0,1)$ in the credit-constrained economy,*

$$K_\Delta > 0.$$

Proposition 5 states that with $\pi > 0$, higher dispersion raises steady-state capital. The mechanism is that a higher risk reinforces the precautionary saving motive, which leads R agents to save more. This then maps into higher capital and $K_\Delta > 0$. Since output in this economy evolves as capital, the effect of inequality on the *level* public spending of share. Indeed, as in shown Appendix Proposition 8, this effect can be decomposed into: a *policy-share channel*, making public spending share decrease with inequality, and a *capital-base channel*, making capital and output rising with inequality. Overall, the effect is ambiguous and the comparative statics for the level of public spending G need not coincide with share comparative statics (for G/Y).

Persistence (π) has an ambiguous effect on capital and also on the level of public spending.

3.3 Ramsey plan with commitment

The Ramsey program assumes the existence of a commitment device for the planner. The Ramsey planner at date $t = 0$ chooses an entire plan $\{g_t, k_t\}_{t\geq 0}$ to maximize social welfare (4) subject to feasibility and implementability. In the credit-constrained economy, consumption is given by equations (6) and (7), and implementability is given by the rich Euler equation (8).

Define the date-t Euler residual:

$$\eta_t := u_c(c_t^R) - \beta\alpha k_t^{\alpha-1}\Big((1-\pi)\,u_c(c_{t+1}^R) + \pi\,u_c(c_{t+1}^P)\Big), \qquad \eta_t = 0\ \forall t \geq 0, \tag{19}$$

which corresponds to the sequential version of (9): under Ramsey it becomes a forward-looking constraint with multipliers $\{\lambda_t\}$. Using the discounted-multiplier convention,

the first-order condition with respect to g_t is

$$0 = \underbrace{v_g(g_t) - \left(u_c(c_t^R) + u_c(c_t^P)\right)}_{\text{static Samuelson residual}} - \lambda_t \, \eta_{g,t} - \lambda_{t-1} \, \alpha k_{t-1}^{\alpha-1} \Big((1-\pi) u_{cc}(c_t^R) + \pi u_{cc}(c_t^P) \Big), \tag{20}$$

while the first-order condition with respect to k_t is

$$\begin{aligned} 0 = & -u_c(c_t^R) + \beta \Big(u_c(c_{t+1}^R) \, C_{k,t+1}^R + u_c(c_{t+1}^P) \, C_{k,t+1}^P \Big) - \lambda_t \, \eta_{k,t} - \lambda_{t-1} \, \alpha k_{t-1}^{\alpha-1} \, (1-\pi) \, u_{cc}(c_t^R) \\ & - \beta \, \lambda_{t+1} \, u_{cc}(c_{t+1}^R) \, C_{k,t+1}^R. \end{aligned} \tag{21}$$

Equations (20)–(21) together with (19) characterize Ramsey allocations (plus feasibility and transversality).

3.3.1 Comparing Ramsey and Markov-perfect policies

Both problems can be written in the same reduced-form economy with state k_{t-1}, controls (g_t, k_t), and are subject to a zero Euler residual (19). The key difference is *how* the government internalizes the effect of current choices on future policy: the Markov-perfect condition (11) features policy feedback, while the Ramsey conditions feature history and commitment terms through the multiplier sequence $\{\lambda_t\}$.

3.4 Non-separable utility function

We now consider a general non-separable utility function $u(c, g)$ to analyze the effect of the complementarity in public and private consumption. For the sake of generality, we consider the general case. However, for some components of in-kind transfer, such as health and eduction, the literature mostly favors the substitutable case. Following the same step as before, one can write the Samuelson gap as

Proposition 6 (Markov-perfect generalized Euler equation)**.** *Suppose V is differentiable and the Markov problem admits an interior solution. Then any interior Markov-perfect policy satisfies, at every state k:*

$$\begin{aligned} \underbrace{u_g(c^R, g) + u_g(c^P, g) - \Big(u_c(c^R, g) + u_c(c^P, g)\Big)}_{\text{(i) static Samuelson residual}} = & \underbrace{\beta \frac{\mathcal{H}_g}{\mathcal{H}_g'} \mathcal{H}_k' \Big(u_g(c^{R\prime}, g') + u_g(c^{P\prime}, g') - \Big(u_c(c^{R\prime}, g') + u_c(c^{P\prime}, g')\Big) \Big)}_{\text{(ii) policy feedback}} \\ & + \underbrace{\mathcal{H}_g \Big(u_c(c^R, g) - \beta u_c(c^{R,\prime}, g') C_{k'}^{R'} - \beta u_c(c^{P,\prime}, g') C_{k'}^{P'} \Big)}_{\text{(iii) intertemporal (capital / price) channel (IC)}}. \end{aligned} \tag{22}$$

All objects are evaluated on the Markov equilibrium path induced by Ψ and $\mathcal{H}$.

At the steady-state, one can write the Samuelson gap as

$$\underbrace{u_g(c^R,g)+u_g(c^P,g)-\Big(u_c(c^R,g)+u_c(c^P,g)\Big)}_{\text{(i) static Samuelson residual}}=\frac{-\mathcal{H}_g}{1-\beta\mathcal{H}_k}\frac{1}{4}\beta\left(-f_{kk}(k)k\right)\left(\frac{k^R}{k}-\frac{z^R}{z}\right)\Big)\Big(u_c(c^P,g)-u_c(c^R,g)\Big)$$

We now have a new effect. The provision of the public good can affet the difference in the marginal utility of private consumption, and thus on the welfare cost of the financing of the public good. If an increase in the public good contributes to a decrease in the difference in marginal utility, ie. $u_c(c^P,g)-u_c(c^R,g)$ for $c^P<c^R$ is decreasing in g, then this reduces the financing cost of the public good and tends to increase the provision of public good. Note that the difference in marginal utility is decreasing with g if $u_{ccg}<0$, which is a condition on the third derivative.

4 The general problem

Throughout this subsection, we adopt a recursive notation and drop the time subscript.

4.1 Economy and recursive household program

Time is discrete, households face idiosyncratic risk and an exogenous borrowing constraint, preferences are separable and discounted, production has constant returns to scale, and the government uses lump sum taxes to finance public spending according to the balanced budget $T=G$. Aggregate feasibility is enforced through market clearing. We denote by Γ the current distribution of households across wealth and idiosyncratic states. The policy rule maps Γ into current public spending.

Idiosyncratic productivity y follows a Markov chain with transition matrix Π. Throughout this section, π denotes a scalar switching probability, while Π denotes transition matrices. Period utility is $u(c)+\theta v(G)$ with discount factor $\beta\in(0,1)$. Competitive prices r and w are pinned down by f_K and f_L, and market clearing requires:

$$K=\int k\,d\Gamma(k,y),\qquad \bar{L}=\int y\,d\Gamma(k,y).$$

Given Γ, these conditions pin down prices and disposable resources for the current household distribution.

Fix a Markov government rule Ψ and denote by H_Ψ the induced aggregate law of

motion. A household with individual state (k, y) solves:

$$\begin{aligned} V_\Psi(k, y, \Gamma) &= \max_{c,k'} u(c) + \theta v(\Psi(\Gamma)) + \beta \mathbb{E} V_\Psi(k', y', \Gamma') \\ c + k' &= w(\Gamma) y + (1 + r(\Gamma)) k - \Psi(\Gamma) \\ k' &\geq 0 \\ \Gamma' &= H_\Psi(\Gamma) \end{aligned}$$

The constraint $k' \geq 0$ prevents agents from borrowing. The law of motion of the distribution $\Gamma' = H_\Psi(\Gamma)$ enforces consistency between individual decisions, prices, and the policy rule.

This decomposition will be used repeatedly below: private optimality is characterized for a given rule Ψ, and equilibrium policy is then imposed through a fixed-point condition on that same rule.

4.2 Markov-perfect policy rule

To define time consistency under lack of commitment, fix a continuation rule Ψ and consider a one-period deviation: at the current state Γ the government sets temporary public spending g, while future governments return to the rule Ψ. We denote this one-step deviation by g. With balanced budget ($T = g$), the associated household value is:

$$\begin{aligned} v_\Psi(k, y, \Gamma, g) &= \max_{c,k'} u(c) + v(g) + \beta \mathbb{E} V_\Psi(k', y', \Gamma'), \\ c + k' &= w(\Gamma) y + (1 + r(\Gamma)) k - g, \\ k' &\geq 0, \\ \Gamma' &= h_\Psi(\Gamma, g). \end{aligned}$$

Here, $h_\Psi(\Gamma, g)$ denotes the one-step transition map for aggregate states: given current state Γ and current deviation g, it returns next period's distribution when private agents anticipate continuation rule Ψ thereafter. By construction, the baseline law of motion is recovered when the one-step deviation equals the continuation rule: $H_\Psi(\Gamma) = h_\Psi(\Gamma, \Psi(\Gamma))$.

The planner is utilitarian, so its one-step objective is the sum with equal weights of household continuation values under the current distribution Γ. Formally, the integration variable is the individual state (k, y), and integration is taken with respect to the measure Γ. Therefore, the government chooses the best current deviation as:

$$g_\Psi(\Gamma) = \arg\max_g \int v_\Psi(k, y, \Gamma, g) d\Gamma(k, y).$$

A Markov-perfect equilibrium is a fixed point of this one-step best response. For all admissible aggregate states Γ in the state space of feasible distributions, the policy rule must satisfy

$$g_\Psi(\Gamma) = \Psi(\Gamma).$$

This equality is the core MPE consistency condition: current policy is optimal given the continuation rule Ψ, and that continuation rule is exactly the one used in equilibrium at every future state. The finite-state implementation of this recursive definition is presented in the next subsection.

4.3 Finite state-space representation

We now build a finite-state approximation based on truncation. The economic primitives (risk, preferences, government budget rule, production, and market clearing) were introduced above in this section; here we focus on state reduction and induced island dynamics.

Truncation. We consider a *truncation length* $N \geq 0$. All agents with the same idiosyncratic history over the last N periods are grouped together on the same island. For instance, an agent i with the date-t idiosyncratic history

$$y^{i,t} := (\ldots, y_{-N+k}, \ldots, y_{-N-1}, y_{-N}, y_{-N+1}, \ldots, y_{-1}, y_0)$$

will belong to the island characterized by the *truncated history* $y^N := (y_{-N+1}, \ldots, y_{-1}, y_0)$, regardless of the prior history $(\ldots, y_{-N+k}, \ldots, y_{-N-1}, y_{-N})$. For the sake of simplicity, we will call truncated history both the vector y^N and the island (i.e., the group of agents) characterized by y^N. The set of vectors $(y^N)_{y^N \in \mathcal{Y}^N}$ implies a well-defined partition of the set of truncated histories: any full idiosyncratic history is mapped to a unique truncation history. The transition matrix between truncated histories can be defined from π and is denoted by Π. For instance, the probability $\Pi_{y^N \tilde{y}^N}$ to switch from y^N in the current period to $\tilde{y}^N$ is equal to the probability to switch from the current state y_0 of y^N to the current state $\tilde{y}_0$ of $\tilde{y}^N$ – provided that $\tilde{y}^N$ is a possible continuation of y^N. Formally:

$$\Pi_{y^N \tilde{y}^N} = \begin{cases} 0 & \text{if } (y_{-N+2}, \ldots, y_{-1}, y_0) \neq (\tilde{y}_{-N+1}, \ldots, \tilde{y}_{-1}) \\ \pi_{y_0 \tilde{y}_0} & \text{otherwise.} \end{cases}$$

In the remainder of this section, and for the sake of generality, we will denote by i a truncated history (or an island) and by $\mathcal{I}$ the set of islands, whose cardinality is $I := |\mathcal{I}|$. The Markov condition for Π can be written as: $\Pi_{ij} \geq 0$ and $\sum_{j \in \mathcal{I}} \Pi_{ij} = 1$.

Each island i is characterized by a current productivity $y_{i,0}$ and a demand shifter

ξ_i, such that the common utility on the island over an individual consumption pair (c_i, G) is $\xi_i u(c_i) + \theta v(G)$. The private consumption level c_i is individual and common to all agents belonging to i. The aggregate consumption on island i is thus $S_i c_i$, where S_i is the size of island i. We assume the vector of sizes $\boldsymbol{S} := (S_i)_i$to be stationary and to verify for all i: $S_i = \sum_\ell \Pi_{\ell i} S_\ell$, and we normalize $\sum_i S_i = 1$, or using matrix notation ():

$$\Pi^\top \boldsymbol{S} = \boldsymbol{S},$$
$$\mathbf{1}_I^\top \boldsymbol{S} = 1,$$

where $\mathbf{1}_I^\top$ is a row vector of 1 with length I, and $\boldsymbol{M}^\top$ denoting the transpose of any square matrix $\boldsymbol{M}$. We will also denote by bold symbols vectors and matrices. Without loss of generality we will assume that all islands have a nonzero mass: $S_i > 0$ for all i.

Agents can save in capital shares paying off the gross interest rate R. An agent i, who saved $a_{i,t-1}$ in date $t-1$, receives the saving payoff $R_t a_{i,t-1}$ at date t. We denote by $a_{i,t}$ the end-of-period per-capita wealth on island i at date t. However, the beginning-of-period wealth on i is made of the wealth of agents arriving from all possible islands ℓ. We assume that wealth flows are identical to agent flows. This amounts to assuming that agents move from one island to another with the per-capita wealth of their origin island. If the beginning-of-period wealth is denoted $\tilde{a}_{i,t}$, the wealth flows across islands implies the following relationship between island wealth levels $(S_i \tilde{a}_{i,t})_i$:

$$S_i \tilde{a}_{i,t} = \sum_\ell \Pi_{\ell i} S_\ell a_{\ell,t-1}.$$

Defining the matrix $(\Pi^a_{\ell i})_{\ell i} = \left(\frac{\Pi_{\ell i} S_\ell}{S_i}\right)_{\ell i}$, we also have with matrix notation:

$$\tilde{\boldsymbol{a}}_t = (\Pi^a)^\top \boldsymbol{a}_{t-1},$$
$$\Pi^a = \operatorname{diag}(\boldsymbol{S})\, \Pi \operatorname{diag}(\boldsymbol{S})^{-1}.$$

The matrix Π^a verifies three properties that are used repeatedly below to characterize aggregation and transition dynamics. First, Π^a is the transpose of a stochastic matrix: it verifies $\Pi^a_{\ell i} \geq 0$ and $\sum_\ell \Pi^a_{\ell i} = 1$ for all i, but in general $\sum_i \Pi^a_{\ell i} \neq 1$. Second, $(\Pi^a)^\top$ is the Markov matrix for wealth "pushed forward" by the agent flow. Third, pooling preserves aggregate wealth exactly: $\sum_i S_i \tilde{a}_{i,t} = \sum_i S_i a_{i,t}$ at all dates t.

Aggregate welfare. An allocation in the truncated economy consists of private consumption paths $(c_{i,t})_{t\geq 0}^{i\in\mathcal{I}}$ and a common public consumption path $(G_t)_{t\geq 0}$. We will denote such an allocation $\boldsymbol{A}$ and the set of allocations by $\mathcal{A}$.

The government is assumed to rank allocations using a utilitarian social welfare function. The social welfare function is denoted W and maps any allocation $\boldsymbol{A} \in \mathcal{A}$ to a

real value, as follows:

$$W(\boldsymbol{A}) = \sum_{t=0}^{\infty} \beta^t \sum_{i \in \mathcal{I}} S_i \left(\xi_i u(c_{i,t}) + \theta v(G_t) \right),$$

where the $(\xi_i)_{i \in \mathcal{I}}$ are preference shifters chosen for the island allocation to equal the island-aggregation of the individual allocation.

Market clearing. The financial market clearing implies that the date-t capital equals the sum of date-t end-of-period wealth levels, or equivalently the sum of next-period (date $t+1$) beginning-of-period wealth levels:

$$K_t = \sum_i S_i a_{i,t} = \sum_i S_i \tilde{a}_{i,t+1}.$$

The latter relationship follows from the definition of beginning of period wealth $(\tilde{a}_{i,t})_i$ and the properties of matrix Π^a.

The labor market clearing condition is stationary and defines the constant aggregate labor supply $\bar{L}$ in efficiency units:

$$\bar{L} = \sum_i S_i y_{i,0}$$

4.3.1 Recursive representation in the island representation

Assume that there is a family head maximizing the welfare of all agents. This family head chooses the savings for each island – with a per-island credit constraint. The family head also takes as given the transitions across islands, and the prices R and w. These prices depend on aggregate capital but this general equilibrium effect is ignored by the family head. The family head is also prevented from implementing transfers across islands. The transitions across islands impose that agents transfer from one island to another according to the law of motion of histories (and the transition probabilities $(\Pi^a_{\ell i})_{\ell i}$) and arrive on their beginning-of-period island with a wealth equal to the per-capita wealth of their origin island. Given current transfer T and current public spending G, the family-head program consists in maximizing family intertemporal welfare by choosing the end-of-period wealth vector, denoted $\boldsymbol{a}'$, assuming that the

wealth in the end of the previous period was $\boldsymbol{a}$. Formally:

$$V(\boldsymbol{a}) = \max_{\boldsymbol{a}'=(a'_i)_i} \sum_i S_i \left(\xi_i u(c_i) + \theta v(G)\right) + \beta V(\boldsymbol{a}'), \tag{23}$$

$$\text{s.t. } c_i + a'_i = w y_{i,0} + R\tilde{a}_i - T, \tag{24}$$

$$\tilde{a}_i = \sum_\ell \Pi^a_{\ell i} a_\ell, \tag{25}$$

$$a'_i \geq 0. \tag{26}$$

Hence, the FOC for a'_i is:

$$S_i \xi_i u'(c_i) = \beta \frac{\partial V(\boldsymbol{a}')}{\partial a'_i} + S_i \mu_i,$$

where μ_i is the Lagrange multiplier on the constraint $a'_i \geq 0$ (normalized by S_i to simplify the algebra). Since the envelope condition yields $\frac{\partial V(\boldsymbol{a})}{\partial a_i} = \sum_j S_j R \xi_j u'(c_j) \frac{\partial \tilde{a}_j}{\partial a_i}$ and $\frac{\partial \tilde{a}_j}{\partial a_i} = \Pi^a_{ij}$, we find:

$$\xi_i u'(c_i) = \beta \sum_j \frac{\Pi^a_{ij} S_j}{S_i} \xi_j R' u'(c'_j) + \mu_i.$$

As $\Pi^a_{ij} = \frac{\Pi_{ij} S_i}{S_j}$, we obtain a standard Euler equation:

$$\xi_i u'(c_i) = \beta \sum_j \Pi_{ij} \xi_j R' u'(c'_j) + \mu_i. \tag{27}$$

When the length of the truncation tends toward ∞, we can show that $\xi_i \to 1$ for all i. We thus have a consistent representation, which can approximate the Aiyagari economy, when the number of islands increases.

4.3.2 Markov-perfect equilibrium

We adapt the framework of Klein et al. (2008), applied to the island representation.

Some notation. With matrix notation, financial market clearing implies that capital is equal to the scalar product of the vector of island sizes by the vector of asset holdings:

$$K = \boldsymbol{S}^\top \boldsymbol{a}.$$

The budget constraint (24) becomes after substituting the expressions of G, w, R, and $\tilde{a}_i$:

$$c_i = f_L(\boldsymbol{S}^\top \boldsymbol{a}, \bar{L}) y_{i,0} + \left(f_K(\boldsymbol{S}^\top \boldsymbol{a}, \bar{L}) + 1 - \delta\right) \sum_\ell \Pi^a_{\ell i} a_\ell - G - a'_i,$$

or with matrix notation:

$$\boldsymbol{c} = f_L(\boldsymbol{S}^\top \boldsymbol{a}, \bar{L})\boldsymbol{y}_0 + \left(f_K(\boldsymbol{S}^\top \boldsymbol{a}, \bar{L}) + 1 - \delta\right)(\Pi^a)^\top \boldsymbol{a} - G\,\mathbf{1}_I - \boldsymbol{a}'.$$

We define the function $\mathcal{C} : (\mathbb{R}^+)^I \times (\mathbb{R}^+)^I \times \mathbb{R}^+ \to (\mathbb{R}^+)^I$ mapping past and current savings choices and public spending into current consumption as:

$$\mathcal{C}(\boldsymbol{a}, \boldsymbol{a}', G) := f_L(\boldsymbol{S}^\top \boldsymbol{a}, \bar{L})\boldsymbol{y}_0 + \left(f_K(\boldsymbol{S}^\top \boldsymbol{a}, \bar{L}) + 1 - \delta\right)(\Pi^a)^\top \boldsymbol{a} - G\,\mathbf{1}_I - \boldsymbol{a}'.$$

We also denote:

$$\boldsymbol{\xi} u(\boldsymbol{c}) := (\xi_i u(c_i))_{i\in\mathcal{I}},$$
$$\boldsymbol{\xi} u'(\boldsymbol{c}) := (\xi_i u'(c_i))_{i\in\mathcal{I}},$$

such that the Euler equation (27) becomes with matrix notation:

$$\boldsymbol{\xi} u'(\boldsymbol{c}) = \beta R' \boldsymbol{\Pi} \boldsymbol{\xi} u'(\boldsymbol{c}') + \boldsymbol{\mu}.$$

Equilibrium. Under the balanced-budget constraint $G = T$, the equilibrium is pinned down by four objects that play distinct roles: continuation values, policy, private responses, and Lagrange multipliers. First, the continuation value is a function $V : \boldsymbol{a} \in (\mathbb{R}^+)^I \mapsto V(\boldsymbol{a}) \in \mathbb{R}$ that summarizes expected discounted welfare from each aggregate state. Second, the policy object is a Markov rule $\Psi : (\mathbb{R}^+)^I \to \mathbb{R}^+$ mapping current wealth distribution into current public spending: $G = \Psi(\boldsymbol{a})$. Third, households' private response is encoded by a saving function $\boldsymbol{h} : (\mathbb{R}^+)^I \to (\mathbb{R}^+)^I$ that determines next-period island savings $\boldsymbol{a}' = \boldsymbol{h}(\boldsymbol{a})$. Fourth, a Lagrange multiplier mapping $\boldsymbol{\mu} : (\mathbb{R}^+)^I \to (\mathbb{R}^+)^I$ tracks the borrowing constraints island by island and separates constrained from unconstrained responses.

Given the current state $\boldsymbol{a}$, the policy-saving pair $(\Psi(\boldsymbol{a}), \boldsymbol{h}(\boldsymbol{a}))$ is defined by

$$\boldsymbol{a}', G = \arg\max_{\boldsymbol{a}', G} \left\{ \boldsymbol{S}^\top \boldsymbol{\xi} u\left(\mathcal{C}(\boldsymbol{a}, \boldsymbol{a}', G)\right) + v(G) + \beta V(\boldsymbol{a}') \right\}, \tag{28}$$

subject to

$$\boldsymbol{\xi} u'\left(\mathcal{C}(\boldsymbol{a}, \boldsymbol{a}', G)\right) = \beta \left(f_K(\boldsymbol{S}^\top \boldsymbol{a}', \bar{L}) + 1 - \delta\right) \boldsymbol{\Pi} \boldsymbol{\xi} u'\left(\mathcal{C}(\boldsymbol{a}', \boldsymbol{h}(\boldsymbol{a}'), \Psi(\boldsymbol{a}'))\right) + \boldsymbol{\mu}(\boldsymbol{a}, G),$$

and

$$V(\boldsymbol{a}) = \boldsymbol{S}^\top \boldsymbol{\xi} u\left(\mathcal{C}(\boldsymbol{a}, \boldsymbol{a}', \Psi(\boldsymbol{a}))\right) + v(\Psi(\boldsymbol{a})) + \beta V(\boldsymbol{h}(\boldsymbol{a})).$$

This can be interpreted as follows. In the current period, the state is the cross-island asset vector $\boldsymbol{a}$ (or, equivalently, the capital $K = \boldsymbol{S}^\top \boldsymbol{a}$ together with the distribution across histories). A benevolent government that cannot commit chooses current public spending $G = \Psi(\boldsymbol{a})$ taking as given that future governments will apply the same

Markov rule Ψ. Households then choose next-period savings $\boldsymbol{a}'$ taking the current prices (w, R) and the current policy G as given and anticipating future policy through Ψ. Hence, for any current $(\boldsymbol{a}, G)$ the household best response $\boldsymbol{a}' = \mathcal{H}(\boldsymbol{a}, G)$ is characterized by the Euler equation and the budget constraint.

Writing the equilibrium as an "argmax over $(\boldsymbol{a}', G)$ subject to the Euler equation" is therefore a reformulation of the government's problem: instead of explicitly substituting the response function $\mathcal{H}(\boldsymbol{a}, G)$ into the government objective (a nested fixed-point), we treat $\boldsymbol{a}'$ as an allocation variable and impose the Euler equation as a constraint guaranteeing that $\boldsymbol{a}'$ coincides with households' optimal saving given G and the anticipated continuation rule. In this sense, the government is not assumed to directly control $\boldsymbol{a}'$: the chosen $(\boldsymbol{a}', G)$ can be decentralized as a competitive household choice at the given prices and policy rule.

4.4 FOCs of the Markov-perfect equilibrium

Throughout this subsection we assume that the set of islands on which the borrowing constraint binds is *known* and *locally constant* (e.g. in a neighborhood of the steady state). Let the set of indices be partitioned as

$$\mathcal{I} = \mathcal{U} \cup \mathcal{B}, \qquad \mathcal{U} \cap \mathcal{B} = \varnothing,$$

where $\mathcal{U}$ denotes the set of *unconstrained* islands and $\mathcal{B}$ ($\mathcal{C}$ taken for consumption) the set of *credit-constrained* (binding) islands.

4.4.1 Core equilibrium system

Define the *Euler gap* (also equal to the multiplier at the optimum)

$$\boldsymbol{\eta}(\boldsymbol{a}, G, \boldsymbol{a}') := \boldsymbol{\xi} u'\left(\mathcal{C}(\boldsymbol{a}, \boldsymbol{a}', G)\right) - \beta\left(f_K(\boldsymbol{S}^\top \boldsymbol{a}', \bar{L}) + 1 - \delta\right) \boldsymbol{\Pi}\, \boldsymbol{\xi} u'\left(\mathcal{C}\left(\boldsymbol{a}', \boldsymbol{h}(\boldsymbol{a}'), \Psi(\boldsymbol{a}')\right)\right). \tag{29}$$

The Kuhn–Tucker conditions for the constraint $\boldsymbol{a}' \geq \mathbf{0}$ are:

$$\boldsymbol{a}' \geq \mathbf{0},\ \boldsymbol{\eta}(\boldsymbol{a}, G, \boldsymbol{a}') \geq \mathbf{0},\ \boldsymbol{a}' \odot \boldsymbol{\eta}(\boldsymbol{a}, G, \boldsymbol{a}') = \mathbf{0},$$

where is the $\odot$ Hadamard product. Hence:

$$i \in \mathcal{U} \Longrightarrow a_i' > 0 \Longrightarrow \boldsymbol{\eta}_i(\boldsymbol{a}, G, \boldsymbol{a}') = 0,$$
$$i \in \mathcal{B} \Longrightarrow a_i' = 0 \Longrightarrow \boldsymbol{\eta}_i(\boldsymbol{a}, G, \boldsymbol{a}') \geq 0.$$

Implicit saving response The optimal saving response $\tilde{\mathcal{H}}(\boldsymbol{a}, G)$ is defined implicitly by the Euler system on unconstrained islands and by binding constraints on constrained

islands:

$$\begin{cases} \eta_i\left(\boldsymbol{a}, G, \tilde{\mathcal{H}}(\boldsymbol{a}, G)\right) = 0, & \forall i \in \mathcal{U}, \\ \left(\tilde{\mathcal{H}}(\boldsymbol{a}, G)\right)_i = 0, & \forall i \in \mathcal{B}. \end{cases} \tag{30}$$

We use the reduced-form derivatives $\tilde{\mathcal{H}}_2$ and $\boldsymbol{J}_{\tilde{\mathcal{H}},1}$ in the first-order condition and envelope equation below. The full Jacobian derivations and matrix identities are moved to Appendix A.1.

FOC on G. The FOC on G is:

$$\underbrace{\left(\boldsymbol{S} \odot \xi u'(\boldsymbol{c})\right)^\top \left(\mathbf{1}_I + \tilde{\mathcal{H}}_2(\boldsymbol{a}, G)\right)}_{F(G,\boldsymbol{a})} = \underbrace{v'(G) + \beta \nabla V(\boldsymbol{a}')^\top \tilde{\mathcal{H}}_2(\boldsymbol{a}, G)}_{K(G,\boldsymbol{a})}. \tag{31}$$

Envelope condition. The envelope condition can be written as:

$$\nabla V(\boldsymbol{a}) = \left[\boldsymbol{J}_{\mathcal{C},1}(\boldsymbol{a}, \boldsymbol{a}', G) - \boldsymbol{J}_{\tilde{\mathcal{H}},1}(\boldsymbol{a}, G)\right]^\top \left(\boldsymbol{S} \odot \xi \boldsymbol{u}'(\boldsymbol{c})\right) + \beta \boldsymbol{J}_{\tilde{\mathcal{H}},1}^\top(\boldsymbol{a}, G) \nabla V(\boldsymbol{a}')$$

To keep the main text focused on the equilibrium conditions, we move the full matrix identities for $\boldsymbol{J}_{\mathcal{C},1}$, $\tilde{\mathcal{H}}_2$, and $\boldsymbol{J}_{\tilde{\mathcal{H}},1}$ to Appendix A.1.

Steady-state condition. At the steady state, combining the FOC on G and the envelope condition yields:

$$\begin{aligned} 0 = &-\left(\boldsymbol{S} \odot \xi u'(\boldsymbol{c})\right)^\top \left(\mathbf{1}_I + \tilde{\mathcal{H}}_2\right) + v'(G) \\ &+ \beta \left(\boldsymbol{S} \odot \xi u'(\boldsymbol{c})\right)^\top \left(\boldsymbol{J}_{\mathcal{C},1} - \boldsymbol{J}_{\tilde{\mathcal{H}},1}\right) \left(\boldsymbol{I}_I - \beta \boldsymbol{J}_{\tilde{\mathcal{H}},1}\right)^{-1} \tilde{\mathcal{H}}_2. \end{aligned} \tag{32}$$

Define the (weighted) static Samuelson residual as:

$$\mathcal{S} := v'(G) - \left(\boldsymbol{S} \odot \xi u'(\boldsymbol{c})\right)^\top \mathbf{1}_I,$$

which is an extension of the Samuelson residual we use in the tractable model of Section 3. It has the same interpretation as in the tractable model and in particular it would be zero in a static setting for islands. Rearranging (32) gives:

$$\mathcal{S} = \left(\boldsymbol{S} \odot \xi u'(\boldsymbol{c})\right)^\top \left(\boldsymbol{I}_I - \beta \boldsymbol{J}_{\mathcal{C},1}\right) \left(\boldsymbol{I}_I - \beta \boldsymbol{J}_{\tilde{\mathcal{H}},1}\right)^{-1} \tilde{\mathcal{H}}_2, \tag{33}$$

which uses $\tilde{\mathcal{H}}_2 = \left(\boldsymbol{I}_I - \beta \boldsymbol{J}_{\tilde{\mathcal{H}},1}\right)\left(\boldsymbol{I}_I - \beta \boldsymbol{J}_{\tilde{\mathcal{H}},1}\right)^{-1} \tilde{\mathcal{H}}_2$ and $\boldsymbol{I}_I - \beta \boldsymbol{J}_{\mathcal{C},1} = \left(\boldsymbol{I}_I - \beta \boldsymbol{J}_{\tilde{\mathcal{H}},1}\right) - \beta\left(\boldsymbol{J}_{\mathcal{C},1} - \boldsymbol{J}_{\tilde{\mathcal{H}},1}\right)$.

Constrained-set decomposition $(\mathcal{U}, \mathcal{B})$. We still consider as given a partition $(\mathcal{U}, \mathcal{B})$: islands in $\mathcal{U}$ are unconstrained, while those on $\mathcal{B}$ are constrained. We denote by

$U := |\mathcal{U}|$ and $B := |\mathcal{B}|$ the cardinality of the two island sets, that verify $U + B = I$. Under a fixed active set,

$$\tilde{\mathcal{H}}_2 = \begin{pmatrix} -(\tilde{\boldsymbol{J}}^{UU}_{\eta,3})^{-1}\mathbf{1}_U \\ \mathbf{0}_B \end{pmatrix},$$

and

$$\left(\boldsymbol{I}_I - \beta \boldsymbol{J}_{\tilde{\mathcal{H}},1}\right)^{-1}\tilde{\mathcal{H}}_2 = \begin{pmatrix} \left(\boldsymbol{I}_U - \beta \boldsymbol{J}^{UU}_{\tilde{\mathcal{H}},1}\right)^{-1}\left[-(\tilde{\boldsymbol{J}}^{UU}_{\eta,3})^{-1}\mathbf{1}_U\right] \\ \mathbf{0}_B \end{pmatrix}.$$

Substituting into (33) yields

$$\begin{aligned}\mathcal{S} = &\underbrace{(\boldsymbol{S}\odot\boldsymbol{\xi}u'(\boldsymbol{c}))^{\top}_{\mathcal{U}}\left(\boldsymbol{I}_U - \beta \boldsymbol{J}^{UU}_{\mathcal{C},1}\right)\left(\boldsymbol{I}_U - \beta \boldsymbol{J}^{UU}_{\tilde{\mathcal{H}},1}\right)^{-1}\left[-(\tilde{\boldsymbol{J}}^{UU}_{\eta,3})^{-1}\mathbf{1}_U\right]}_{\text{unconstrained intertemporal channel}} \\ &- \underbrace{\beta(\boldsymbol{S}\odot\boldsymbol{\xi}u'(\boldsymbol{c}))^{\top}_{\mathcal{B}}\boldsymbol{J}^{BU}_{\mathcal{C},1}\left(\boldsymbol{I}_U - \beta \boldsymbol{J}^{UU}_{\tilde{\mathcal{H}},1}\right)^{-1}\left[-(\tilde{\boldsymbol{J}}^{UU}_{\eta,3})^{-1}\mathbf{1}_U\right]}_{\text{consumption response on } \mathcal{B} \text{ due to } \mathcal{U}\text{-saving}}. \end{aligned} \tag{34}$$

The second term is weighted by marginal utility on constrained islands, but is driven by the sensitivity of their consumption to unconstrained saving through $\boldsymbol{J}^{BU}_{\mathcal{C},1}$. We distinguish two cases, depending on the cardinal of $\mathcal{U}$.

4.4.2 Steady-state interpretation when $|\mathcal{U}| > 1$

Under a fixed active set, (34) becomes

$$\mathcal{S} = \boldsymbol{T}_U^{\top}\boldsymbol{\Omega}_U, \tag{35}$$

$$\text{with: } \boldsymbol{T}_U^{\top} := (\boldsymbol{S}\odot\boldsymbol{\xi}u'(\boldsymbol{c}))^{\top}_{\mathcal{U}}\left(\boldsymbol{I}_U - \beta \boldsymbol{J}^{UU}_{\mathcal{C},1}\right) - \beta(\boldsymbol{S}\odot\boldsymbol{\xi}u'(\boldsymbol{c}))^{\top}_{\mathcal{B}}\boldsymbol{J}^{BU}_{\mathcal{C},1}, \tag{36}$$

$$\boldsymbol{\Omega}_U := \left(\boldsymbol{I}_U - \beta \boldsymbol{J}^{UU}_{\tilde{\mathcal{H}},1}\right)^{-1}\left[-(\tilde{\boldsymbol{J}}^{UU}_{\eta,3})^{-1}\mathbf{1}_U\right]. \tag{37}$$

We have the following decomposition for $\boldsymbol{T}_U$ and $\boldsymbol{\Omega}_U$ (which are both vectors in $\mathbb{R}^{|\mathcal{U}|}$) and

Proposition 7. *For each island $r \in \mathcal{U}$,*

$$(\boldsymbol{T}_U^{\top})_r = \beta S_r K f_{KK}\sum_{n\in\mathcal{I}} S_n\left(\xi_r u'(c_r) - \xi_n u'(c_n)\right)\left(\frac{\left((\Pi^a)^{\top}\boldsymbol{a}\right)_n}{K} - \frac{y_{0,n}}{\bar{L}}\right), \tag{38}$$

while $\boldsymbol{\Omega}_U = (\boldsymbol{I}_U - \beta\frac{\partial \boldsymbol{a}'_U}{\partial \boldsymbol{a}_U^{\top}})^{-1}\frac{\partial \boldsymbol{a}'_U}{\partial G}$.

The proof and supporting algebra for Proposition 7 are moved to Appendix Section A.1, together with the intermediate Jacobian identities.

The decomposition (38) extends the decomposition (13) of the two-island case. The MPE equilibrium crucially relies on the two same forces: the consumption inequality across islands and the difference between labor capital shares on each island.

Regarding Ω_U, we have: $\Omega_U = (\boldsymbol{I}_U - \beta \frac{\partial \boldsymbol{a}'_{\mathcal{U}}}{\partial \boldsymbol{a}_{\mathcal{U}}^\top})^{-1} \frac{\partial \boldsymbol{a}'_{\mathcal{U}}}{\partial G}$, which, when the inverse exists, can be written as:

$$\Omega_U = \sum_{m=0}^{\infty} \beta^m \left(\frac{\partial \boldsymbol{a}'_{\mathcal{U}}}{\partial \boldsymbol{a}_{\mathcal{U}}^\top}\right)^m \left[\frac{\partial \boldsymbol{a}'_{\mathcal{U}}}{\partial G}\right].$$

This clarifies the interpretation: The $m = 0$ term is the *direct* effect of today's G on next-period unconstrained savings, while the $m \geq 1$ terms collects feedback rounds through endogenous law-of-motion effects on future savings.

4.5 The algorithm: A "Krusell and Smith"-type of approximation

We solve for a Markov rule Ψ with a Krusell-Smith-type fixed-point loop on policy coefficients. To solve for savings rules on each island, we follow a method in the spirit of Klein et al. (2008). Each specification uses the same structure: guess coefficients, solve the island equilibrium, evaluate government optimality residuals, update coefficients, and iterate to convergence.

4.5.1 Constant rule: $\Psi(\boldsymbol{a}) = G$

The constant-rule case is the baseline fixed-point problem in which policy does not react to the state vector beyond its steady-state level. It is useful both as an economically interpretable benchmark and as a numerical initializer for richer specifications. the algorithm can be summarized as follows.

1. Guess a scalar level G for the constant rule $\Psi(\boldsymbol{a}) = G$.
2. Given G, solve the steady state $\boldsymbol{a}^*$ and the implied household response $\boldsymbol{h}(\boldsymbol{a}^*) = \boldsymbol{a}^*$.
3. Evaluate the government residual in (33) at the candidate steady state.
4. Update G with a one-dimensional root-finding step until the residual is below tolerance.

4.5.2 Linear rule: $\Psi(\boldsymbol{a}) = \Psi_0 + \alpha f(\boldsymbol{a})$

The linear rule introduces first-order state dependence through a low-dimensional statistic $f(\boldsymbol{a})$. This specification is the main local approximation: it captures directional policy responses while keeping the coefficient update problem tractable and stable.

1. Choose a basis statistic $f(\boldsymbol{a})$ (baseline: $f(\boldsymbol{a}) = \boldsymbol{S}^\top \boldsymbol{a}$) and initialize (Ψ_0, α_Ψ) in $\Psi(\boldsymbol{a}) = \Psi_0 + \alpha_\Psi f(\boldsymbol{a})$.

2. Given (Ψ_0, α_Ψ), solve the induced equilibrium objects. To solve for the household savings rule for island i, we conjecture it is a polynomial of degree 1 in *each island's* assets. Thus, each of the I islands will have $I+1$ unknown coefficients to solve for. To solve for the $I^2 + I$ coefficients we do the following: for islands that are constrained, all coefficients on the rule are set to 0, for islands that are unconstrained we require that the gradient vector of the the Euler equation residual (equation 29) for island i is equal to 0. This is the multidimensional analog of the method of Klein et al. (2008). We use automatic differentiation to compute the Jacobian of equation 29 efficiently.

3. By computing equation 31 in the steady state, obtain the mapping $F(G, \boldsymbol{a}) - K(G, \boldsymbol{a})$.

4. By taking the gradient of equation 31 with respect to $\boldsymbol{a}$, obtain the gradient-implied condition
$$\nabla G(\boldsymbol{a}) = -\frac{\nabla_{\boldsymbol{a}} F(G(\boldsymbol{a}), \boldsymbol{a}) - \nabla_{\boldsymbol{a}} K(G(\boldsymbol{a}), \boldsymbol{a})}{F_G - K_G}. \tag{39}$$
used to update the slope α_Ψ

5. Project the updated gradient on the basis direction and update (Ψ_0, α); iterate until coefficient and residual convergence.

4.5.3 Polynomial rule

The polynomial rule extends the linear approximation by adding curvature terms. In practice, one should progressively increase the degree of the polynomial rule, starting at $m = 2$ and ending when the equilibrium value of government spending (or other statistics of interest) does not change when increasing the degree.

1. Choose degree M and parameterize
$$G(\boldsymbol{a}) = \Psi_0 + \sum_{k=1}^{M} \alpha_k (\boldsymbol{S}^\top \boldsymbol{a}^k).$$
Build the basis-gradient matrix $U = [u_1, \ldots, u_M]$ with $u_k = k(\boldsymbol{S} \odot \boldsymbol{a}^{k-1})$.

2. Given coefficients, solve for the household savings rule. Since only the gradient of the planner's GEE is necessary, follow the same methods from the previous subsection to find the household savings rule– the household savings rule on each island is still linear.

3. Given the savings rule, compute the residual $F(G, \boldsymbol{a}) - K(G, \boldsymbol{a})$ from equation 31

and then by taking its gradient with respect to $\boldsymbol{a}$ obtain

$$\nabla G(\boldsymbol{a}) = -\frac{\nabla_{\boldsymbol{a}} F(G(\boldsymbol{a}), \boldsymbol{a}) - \nabla_{\boldsymbol{a}} K(G(\boldsymbol{a}), \boldsymbol{a})}{F_G - K_G}, \tag{40}$$

Then project the gradient update on the basis and update $(\Psi_0, \alpha_1, \ldots, \alpha_M)$ until both coefficient updates and government optimality residuals are below tolerance.

5 Results

We present and compare results for both the numerical model of Section 3 and the numerical model of Section 4. We show results for fixed levels of the transition probability between low and high income states, π, and vary the dispersion of the shock Δ. First, we provide our parameterization below.

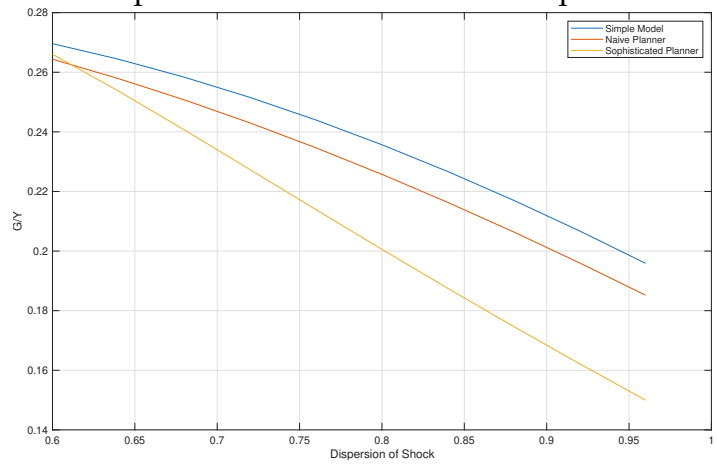
Parameter	Value
α	.2
β	.75
δ	1
γ	1
π	varying, see below
Δ	varying, see below

First, we fix a value of the transition probability π. Then, we consider all numerically permissible values of the dispersion of the shock Δ, i.e. those values of Δ for which the credit constraint for the poor, with income $1 - \Delta$ binds. For these values, we solve for the MPE of the simple model, the numerical model with a naive planner that follows a static Samuelson rule, and the sophisticated planner who solves the problem in equation 28.

To obtain these equilibria, we first solve the MPE of the simple model and obtain its level of government spending G. We then solve and discretize the true Aiyagari model with the parameters in the above table (setting the fixed level of government spending equal to G from the simple model), obtaining an island representation. We then follow the algorithms from section 3.5. Starting with order 0 (the constant rule), we solve for the MPE of the island economy with progressively higher orders of the rule Ψ. We increase the order of our conjectured rule Ψ until the equilibrium level of G for order n, G_n, does not change as we increase the order: $G_n = G_{n-1}$.

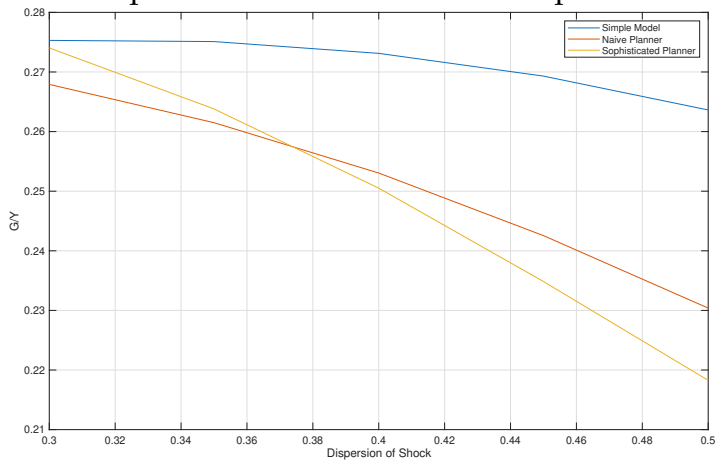
First, we set $\pi = 1$, recovering the Woodford limit. When $\pi = 1$, we must have exactly 2 islands in the numerical model, corresponding to the rich and poor agent of the simple model. Figure 1 compares the MPE value of G/Y in the simple model, for the

Figure 1: Comparison of Numerical and Simple Model $\pi = 1$



naive planner in the numerical model, and the sophisticated planner in the numerical model. In all three specifications G/Y decreases with Δ. Since the public good is financed by both rich and poor agents, as Δ increases, it becomes more costly to provide G as the consumption of the agents with a low shock realization falls. The distance between the naive and sophisticated planner's solutions represents the Samuelson Gap, as the naive planner strictly adheres to a Samuelson rule when choosing consumption. In this case, as the dispersion of the shock, or the level of inequality, increases, the Samuelson gap increases as well.

Figure 2: Comparison of Numerical and Simple Model $\pi = .7$



We now set $\pi = .7$, representing a departure from the Woodford limit. The island representation of this model will now have many islands. Figure 2 compares the MPE value of G/Y in the same 3 scenarios as in Figure 1. Again, as we increase the dispersion of the shock, the ratio G/Y falls.

6 Concluding Remarks

Going forward, our analysis of the tractable model will incorporate endogenous labor supply and distortionary taxation. We also plan to extend our numerical solution of the general model, which is currently based on a linear rule for G, to higher-order rules.

Both the analysis of the tractable model and the numerical algorithm we proposed can be used more generally for the analysis of Markov-perfect fiscal and monetary policy in rich dynamic heterogeneous-agent incomplete-markets models with production.

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Appendix

A Proofs for the tractable model

A.1 General-problem derivatives and decomposition algebra

This subsection collects technical derivations moved from Section 4, including Jacobian formulas and the algebra behind Proposition 7.

Using $\mathcal{C}(\boldsymbol{a}, \boldsymbol{a}', G)$ from Section 4, the Jacobian with respect to current assets is

$$\boldsymbol{J}_{\mathcal{C},1}(\boldsymbol{a}, \boldsymbol{a}', G) = \Big(f_{LK}(\boldsymbol{S}^\top \boldsymbol{a}, \bar{L})\boldsymbol{y}_0 + f_{KK}(S^\top \boldsymbol{a}, \bar{L})(\Pi^a)^\top \boldsymbol{a}\Big)\, \boldsymbol{S}^\top + \Big(f_K(\boldsymbol{S}^\top \boldsymbol{a}, \bar{L}) + 1 - \delta\Big)\,(\Pi^a)^\top. \tag{41}$$

Under a fixed active set, the derivative of the implicit saving function with respect to current public spending is

$$\tilde{\mathcal{H}}_2(\boldsymbol{a}, G) = -\boldsymbol{J}^{-1}_{\tilde{\eta},3}(\boldsymbol{a}, G, \boldsymbol{a}')\tilde{\boldsymbol{\eta}}_2(\boldsymbol{a}, G, \boldsymbol{a}') \tag{42}$$

$$= -\boldsymbol{J}^{-1}_{\tilde{\eta},3}(\boldsymbol{a}, G, \boldsymbol{a}')\boldsymbol{P}_{\mathcal{U}}\mathbf{1}_I. \tag{43}$$

For the decomposition in Proposition 7, the key intermediate identity is

$$\boldsymbol{S}_{\mathcal{U}}^\top \boldsymbol{J}^{UU}_{\mathcal{C},1} + \boldsymbol{S}_{\mathcal{B}}^\top \boldsymbol{J}^{BU}_{\mathcal{C},1} = (f_K + 1 - \delta)\,\boldsymbol{S}_{\mathcal{U}}^\top, \tag{44}$$

which follows from stationarity of island masses and CRS. Substituting this identity in (36) yields (38).

A.2 Derivation of the Markov generalized Euler equation

This appendix derives Proposition 1. The derivation is standard but easy to lose track of because the control g affects continuation values both *directly* (through current utility) and *indirectly* through the private saving response $k' = \mathcal{H}(k, g)$ and the dependence of tomorrow's optimal policy on the next state.

A.2.1 Step 0: a convenient notational device

For a given (k, g, k'), define the *current-period payoff* (holding k' fixed)

$$\tilde{U}(k, g, k') := u\big(c^R(k, k', g)\big) + u\big(c^P(k, g)\big) + v(g),$$

where c^R and c^P are given in (6)–(7). In the Markov problem, the next state is pinned down by the private sector, $k' = \mathcal{H}(k, g)$, so that

$$V(k) = \max_g\ \tilde{U}\big(k, g, \mathcal{H}(k, g)\big) + \beta V\big(\mathcal{H}(k, g)\big).$$

A.2.2 Step 1: the current-period first-order condition

Differentiate the Bellman objective w.r.t. g (an interior optimum is assumed):

$$0 = \tilde{U}_g(k, g, k') + \Big(\tilde{U}_{k'}(k, g, k') + \beta V_k(k')\Big)\mathcal{H}_g(k, g), \qquad k' = \mathcal{H}(k, g). \tag{45}$$

Using (6)–(7), the partial derivatives are

$$\tilde{U}_g(k, g, k') = -u_c(c^R) - u_c(c^P) + v_g(g), \qquad \tilde{U}_{k'}(k, g, k') = -u_c(c^R).$$

A.2.3 Step 2: envelope at the next state

Apply the envelope theorem to the Bellman equation at k':

$$V_k(k') = \tilde{U}_k(k', g', k'') + \Big(\tilde{U}_{k'}(k', g', k'') + \beta V_k(k'')\Big)\mathcal{H}'_k, \tag{46}$$

where $g' = \Psi(k')$ is tomorrow's optimal policy, $k'' = \mathcal{H}(k', g')$, and $\mathcal{H}'_k := \partial\mathcal{H}(k', g')/\partial k'$. The partial derivative $\tilde{U}_k(k', g', k'')$ is exactly the "next-period marginal value of capital":

$$\tilde{U}_k(k', g', k'') = u_c(c^{R,\prime})\, C^{R'}_{k'} + u_c(c^{P,\prime})\, C^{P'}_{k'},$$

with $C^{R'}_{k'}$ and $C^{P'}_{k'}$ defined in (11).

A.2.4 Step 3: eliminate continuation terms with the next-state FOC

Define the (auxiliary) marginal value of the next state,

$$W(k) := \tilde{U}_{k'}(k, g, k') + \beta V_k(k'), \qquad k' = \mathcal{H}(k, g).$$

Then the current-period first-order condition (45) is $0 = \tilde{U}_g + W\mathcal{H}_g$.

At the next state (k', g'), the same condition implies

$$W(k') = -\frac{\tilde{U}_g(k', g', k'')}{\mathcal{H}'_g}, \tag{47}$$

where $\mathcal{H}'_g := \partial\mathcal{H}(k', g')/\partial g'$. Using (46) and the definition of W,

$$W(k) = -u_c(c^R) + \beta V_k(k') = -u_c(c^R) + \beta\tilde{U}_k(k', g', k'') + \beta W(k')\,\mathcal{H}'_k.$$

Substitute (47) and then plug the resulting $W(k)$ into $0 = \tilde{U}_g + W\mathcal{H}_g$. After rearranging, this delivers exactly (11) in Proposition 1:

$$-\frac{\tilde{U}_g}{\mathcal{H}_g} = -u_c(c^R) + \beta\tilde{U}_k(k', g', k'') - \beta\frac{1}{\mathcal{H}'_g}\mathcal{H}'_k\tilde{U}_g(k', g', k'').$$

$$0 = \tilde{U}_g - \beta \frac{\mathcal{H}_g}{\mathcal{H}'_g} \mathcal{H}'_k \, \tilde{U}_g(k', g', k'') - \mathcal{H}_g \Big(u_c(c^R) - \beta \tilde{U}_k(k', g', k'') \Big).$$

Replacing $\tilde{U}_g$ and $\tilde{U}_k$ by their expressions

$$\tilde{U}_g(k, g, k') = -u_c(c^R) - u_c(c^P) + v_g(g), \qquad \tilde{U}_k(k', g', k'') = u_c(c^{R,\prime}) \, C^{R'}_{k'} + u_c(c^{P,\prime}) \, C^{P'}_{k'}.$$

above yields (11):

$$\begin{aligned} 0 = &\Big(-u_c(c^R) - u_c(c^P) + v_g(g) \Big) - \beta \frac{\mathcal{H}_g}{\mathcal{H}'_g} \mathcal{H}'_k \Big(-u_c(c^{R,\prime}) - u_c(c^{P,\prime}) + v_g(g') \Big) \\ &- \mathcal{H}_g \Big(u_c(c^R) - \beta \Big(u_c(c^{R,\prime}) \, C^{R'}_{k'} + u_c(c^{P,\prime}) \, C^{P'}_{k'} \Big) \Big). \end{aligned}$$

A.2.5 Algebra for term (iii) decomposition

$$IC = \mathcal{H}_g \Big(u_c(c^R) - \beta u_c(c^{R,\prime}) f_k(k') \Big) - \beta \mathcal{H}_g \Big(u_c(c^{P,\prime}) - u_c(c^{R,\prime}) \Big) C^{P'}_{k'}.$$

Using $\eta = 0$ on the Markov path,

$$u_c(c^R) = \beta f_k(k') \Big((1 - \pi) u_c(c^{R,\prime}) + \pi u_c(c^{P,\prime}) \Big),$$

so

$$u_c(c^R) - \beta u_c(c^{R,\prime}) f_k(k') = \beta \pi f_k(k') \Big(u_c(c^{P,\prime}) - u_c(c^{R,\prime}) \Big).$$

Substituting gives

$$IC = \beta \mathcal{H}_g \Big(f_k(k') \pi - C^{P'}_{k'} \Big) \Big(u_c(c^{P,\prime}) - u_c(c^{R,\prime}) \Big), \tag{48}$$

which is (48). Since $C^{P'}_{k'} = \alpha \theta_P (k')^{\alpha - 1}$, $f_k(k') = \alpha (k')^{\alpha - 1}$, and $\theta_R = 1 - \theta_P$,

$$f_k(k') \pi - C^{P'}_{k'} = \alpha (k')^{\alpha - 1} \big(\pi - \theta_P \big) = \alpha (k')^{\alpha - 1} \big(\theta_R \pi - (1 - \pi) \theta_P \big).$$

Finally, using (5),

$$\theta_R \pi - (1 - \pi) \theta_P = \frac{1 - \alpha}{2} \big(1 + \Delta - 2(1 - \pi) \big),$$

which yields (13).

A.2.6 Compact map to KKR notation

In the representative-agent limit, let

$$\gamma := v_g(g) - 2u_c(c), \qquad \nu := -u_c(c) + \beta u_c(c') f_k(k').$$

Equation (14) is

$$\gamma = \beta \frac{\mathcal{H}_g}{\mathcal{H}'_g} \mathcal{H}'_k \gamma' - \mathcal{H}_g \nu.$$

Expanding γ and γ' gives

$$0 = v_g(g) - (2 + \mathcal{H}_g)u_c(c) + \beta\mathcal{H}_g\left[u_c(c')f_k(k') - \frac{\mathcal{H}'_k}{\mathcal{H}'_g}\left(v_g(g') - 2u_c(c')\right)\right],$$

which is algebraically equivalent to the KKR equation-(4) representation up to normalization and the known sign correction in the printed version.

A.3 Quadratic characterization of Ψ in the log case

This appendix collects the derivation of Proposition 3.

A.3.1 Coefficients

Under Assumption 1 and Conjecture 1, the quadratic in (17) has coefficients

$$b(1-\pi,\Delta) = 6\alpha^2\beta(2(1-\pi)-1) + 6\alpha\beta(1-\alpha)\Delta + 2\alpha\beta - 8, \tag{49}$$

$$c(1-\pi,\Delta) = (\alpha\beta-1)\left(m(1-\pi,\Delta)^2 - 1\right), \qquad m(1-\pi,\Delta) := \alpha(1-2(1-\pi)) - (1-\alpha)\Delta. \tag{50}$$

A.3.2 How the quadratic arises

Starting from the general Markov condition (11) under log preferences, use the scale-free structure of Conjecture 1 to express all variables in terms of (Ψ, Ξ). Eliminate Ξ using (16) and clear denominators to obtain (17).

A.4 Admissible root and comparative statics for Ψ

This appendix provides the complete proof of admissible-root selection and the sign claims in Proposition 4.

A.4.1 Step 1: reduce the quadratic to a one-index representation

Let $q := \alpha\beta \in (0,1)$ and $\zeta := 1 - \pi$. Using (49)–(50),

$$b(\zeta,\Delta) = 2q - 8 - 6q\, m(\zeta,\Delta), \qquad c(\zeta,\Delta) = (q-1)\big(m(\zeta,\Delta)^2 - 1\big),$$

with

$$m(\zeta,\Delta) := \alpha(1-2\zeta) - (1-\alpha)\Delta.$$

Hence

$$F(\Psi;\zeta,\Delta) = \widetilde{F}(\Psi,m) := 12\Psi^2 + \big(2q - 8 - 6qm\big)\Psi + (q-1)(m^2-1). \tag{51}$$

So (ζ, Δ) affects F only through m.

A.4.2 Step 2: admissible root is unique and equals the smaller root

From $g = \Psi k^{\alpha}$, admissibility requires $\Psi > 0$. From $\hat{c}^{P} > 0$, admissibility also requires

$$\Psi < \theta_P(\zeta, \Delta) = \frac{1}{2}(1 + m).$$

In Proposition 4, Appendix A.6 gives $m < -\alpha\beta = -q$. Admissibility implies $\theta_P > 0$, hence $m > -1$. Therefore $m \in (-1, -q)$.

Evaluate (51) at the interval endpoints:

$$\widetilde{F}(0, m) = (1 - q)(1 - m^2) > 0,$$

and, with $\theta_P = (1 + m)/2$,

$$\widetilde{F}(\theta_P, m) = 2(1 - q)m(1 + m) < 0. \tag{52}$$

Evaluating at half of the admissible upper bound gives

$$\widetilde{F}\left(\frac{\theta_P}{2}, m\right) = -(1 + 2q)\,\theta_P^2 < 0. \tag{53}$$

Since $\widetilde{F}(\cdot, m)$ is strictly convex in Ψ (coefficient $12 > 0$), (52) implies exactly one root in $(0, \theta_P)$. Therefore the admissible root is unique and equals the smaller quadratic root:

$$\Psi(\zeta, \Delta) = \Psi_{-}(\zeta, \Delta) = \frac{-b(\zeta, \Delta) - \sqrt{b(\zeta, \Delta)^2 - 48c(\zeta, \Delta)}}{24}.$$

Because $\widetilde{F}(0, m) > 0$, (53), and the admissible root is the smaller root, we also obtain

$$0 < \Psi(\zeta, \Delta) < \frac{\theta_P(\zeta, \Delta)}{2},$$

which is (18).

A.4.3 Step 3: derivatives and their signs

Implicitly differentiating $F(\Psi(\zeta, \Delta); \zeta, \Delta) = 0$ gives

$$\Psi_{\Delta} = -\frac{b_{\Delta}\Psi + c_{\Delta}}{24\Psi + b}, \qquad \Psi_{\zeta} = -\frac{b_{\zeta}\Psi + c_{\zeta}}{24\Psi + b}.$$

At the smaller root, $24\Psi + b = -\sqrt{b^2 - 48c} < 0$. Hence the sign of each derivative is the sign of its numerator.

Compute derivatives:

$$b_\Delta = 6\alpha\beta(1-\alpha), \qquad b_\zeta = 12\alpha^2\beta,$$

$$c_\Delta = 2(1-\alpha)(1-\alpha\beta)m, \qquad c_\zeta = 4\alpha(1-\alpha\beta)m.$$

Therefore

$$b_\Delta\Psi + c_\Delta = 2(1-\alpha)\Big(3\alpha\beta\Psi + (1-\alpha\beta)m\Big),$$

$$b_\zeta\Psi + c_\zeta = 4\alpha\Big(3\alpha\beta\Psi + (1-\alpha\beta)m\Big).$$

So both signs are pinned down by

$$B := 3\alpha\beta\Psi + (1-\alpha\beta)m.$$

To sign B, evaluate (51) at

$$\Psi^\star := \frac{1-\alpha\beta}{3}.$$

Direct substitution gives

$$\widetilde{F}(\Psi^\star, m) = -(1-\alpha\beta)\left[(m+\alpha\beta)^2 + \frac{(1-\alpha\beta)(1+3\alpha\beta)}{3}\right] < 0. \tag{54}$$

Because $\widetilde{F}(0,m) > 0$ and Ψ is the smaller root, (54) implies $\Psi < \Psi^\star = (1-\alpha\beta)/3$. Using also $m < -\alpha\beta$,

$$B = 3\alpha\beta\Psi + (1-\alpha\beta)m < \alpha\beta(1-\alpha\beta) + (1-\alpha\beta)(-\alpha\beta) = 0.$$

Hence $B < 0$, so both numerators are strictly negative, and therefore

$$\Psi_\Delta(\zeta,\Delta) < 0, \qquad \Psi_\zeta(\zeta,\Delta) < 0.$$

Since $\zeta = 1-\pi$, this is exactly

$$\Psi_\Delta(1-\pi,\Delta) < 0, \qquad \Psi_{(1-\pi)}(1-\pi,\Delta) < 0,$$

as claimed in Proposition 4.

A.5 Comparative static for K and G

A.5.1 Proposition

Proposition 8 (From spending shares to levels in Markov steady state)**.** *Under Assumption 1, Conjecture 1, and the conditions of Proposition 4, define steady-state levels*

$$K(1-\pi,\Delta) := \frac{1}{2}\Big(\alpha\beta\,\Xi(1-\pi,\Delta)\Big)^{\frac{1}{1-\alpha}}, \qquad G(1-\pi,\Delta) := \Psi(1-\pi,\Delta)\Big(\alpha\beta\,\Xi(1-\pi,\Delta)\Big)^{\frac{\alpha}{1-\alpha}},$$

where $\Xi(1-\pi,\Delta)$ *is pinned down by* (16)*. Then for each* $x \in \{(1-\pi),\Delta\}$,

$$\frac{\partial \log K}{\partial x} = \frac{1}{1-\alpha}\frac{\Xi_x}{\Xi}, \tag{55}$$

$$\frac{\partial \log G}{\partial x} = \frac{\Psi_x}{\Psi} + \frac{\alpha}{1-\alpha}\frac{\Xi_x}{\Xi}, \tag{56}$$

$$\frac{\partial}{\partial x}\left(\frac{G}{Y}\right) = 2\Psi_x. \tag{57}$$

Hence $\operatorname{sgn}(K_x) = \operatorname{sgn}(\Xi_x)$*, while*

$$\operatorname{sgn}(G_x) = \operatorname{sgn}\left(\frac{\Psi_x}{\Psi} + \frac{\alpha}{1-\alpha}\frac{\Xi_x}{\Xi}\right).$$

A.5.2 Proof

The goal is to map spending-share derivatives into level derivatives and then make sign conditions explicit through quotient differentiation of Ξ. Write

$$\Xi(1-\pi,\Delta) = \frac{N(1-\pi,\Delta)}{D(1-\pi,\Delta)},$$

with

$$N := (1-\pi)\big(\theta_P - \Psi\big) + \pi\big(\theta_R - \Psi\big), \qquad D := \big(\theta_P - \Psi\big) + \pi\alpha\beta.$$

By (16), $\Xi = N/D$. Under Conjecture 1, steady-state allocations satisfy

$$k = (\alpha\beta\Xi)^{\frac{1}{1-\alpha}}, \qquad K = \frac{1}{2}k, \qquad G = \Psi k^{\alpha}, \qquad Y = \frac{1}{2}k^{\alpha}.$$

Therefore

$$\log K = \log\frac{1}{2} + \frac{1}{1-\alpha}\big(\log(\alpha\beta) + \log\Xi\big),$$

and differentiation with respect to $x \in \{(1-\pi),\Delta\}$ gives

$$\frac{\partial \log K}{\partial x} = \frac{1}{1-\alpha}\frac{\Xi_x}{\Xi},$$

which is (55).

Next,

$$\log G = \log\Psi + \frac{\alpha}{1-\alpha}\big(\log(\alpha\beta) + \log\Xi\big),$$

so

$$\frac{\partial \log G}{\partial x} = \frac{\Psi_x}{\Psi} + \frac{\alpha}{1-\alpha}\frac{\Xi_x}{\Xi},$$

which is (56).

For output shares, since $G/Y = 2\Psi$ under $Y = \frac{1}{2}k^{\alpha}$,

$$\frac{\partial}{\partial x}\left(\frac{G}{Y}\right) = 2\Psi_x,$$

which is (57).

Finally, because $K > 0$ and $G > 0$:

$$K_x = K\frac{\partial \log K}{\partial x}, \qquad G_x = G\frac{\partial \log G}{\partial x},$$

so the signs of K_x and G_x are exactly those stated in the proposition.

Lemma 2 (Sign characterization for $\Xi_{(1-\pi)}$ and Ξ_Δ)**.** *Under the assumptions of Proposition 8, define*

$$\mathcal{D}(1-\pi, \Delta) := (\theta_P - \Psi) + \pi\alpha\beta > 0.$$

Then

$$\Xi_\Delta = \frac{\frac{1-\alpha}{2}\big(\Xi + 1 - 2(1-\pi)\big) + (\Xi - 1)\Psi_\Delta}{\mathcal{D}}, \tag{58}$$

$$\Xi_{(1-\pi)} = \frac{\alpha(1+\beta)\Xi + 2\alpha(1-2(1-\pi)) - (1-\alpha)\Delta + (\Xi-1)\Psi_{(1-\pi)}}{\mathcal{D}}. \tag{59}$$

Hence signs are determined by the numerators in (58)–(59). With $\Xi > 1$, $\Psi_\Delta < 0$ and $\Psi_{(1-\pi)} < 0$, this is equivalent to

$$\Xi_\Delta > 0 \iff |\Psi_\Delta| < \frac{(1-\alpha)\big(\Xi + 1 - 2(1-\pi)\big)}{2(\Xi - 1)},$$

and

$$\Xi_{(1-\pi)} > 0 \iff |\Psi_{(1-\pi)}| < \frac{\alpha(1+\beta)\Xi + 2\alpha(1-2(1-\pi)) - (1-\alpha)\Delta}{\Xi - 1}.$$

Proof. Differentiate N and D term by term. Quotient differentiation gives

$$\Xi_x = \frac{N_x D - N D_x}{D^2},$$

where

$$N_\Delta = \frac{1-\alpha}{2}(1 - 2(1-\pi)) - \Psi_\Delta, \qquad D_\Delta = -\frac{1-\alpha}{2} - \Psi_\Delta,$$

and

$$\begin{aligned} N_{(1-\pi)} &= (\theta_P - \theta_R) + (1-\pi)(\theta_{P,(1-\pi)} - \Psi_{(1-\pi)}) \\ &\quad + \pi(\theta_{R,(1-\pi)} - \Psi_{(1-\pi)}), \\ D_{(1-\pi)} &= \theta_{P,(1-\pi)} - \Psi_{(1-\pi)} - \alpha\beta, \end{aligned}$$

with $\theta_{R,(1-\pi)} = \alpha$ and $\theta_{P,(1-\pi)} = -\alpha$. Using $N = \Xi D$, write

$$\Xi_x = \frac{N_x - \Xi D_x}{D}.$$

For $x = \Delta$,

$$\Xi_\Delta = \frac{\frac{1-\alpha}{2}(1-2(1-\pi)) - \Psi_\Delta - \Xi\left(-\frac{1-\alpha}{2} - \Psi_\Delta\right)}{D} = \frac{\frac{1-\alpha}{2}\big(\Xi + 1 - 2(1-\pi)\big) + (\Xi - 1)\Psi_\Delta}{D},$$

which is (58). For $x = (1-\pi)$,

$$N_{(1-\pi)} = 2\alpha(1-2(1-\pi)) - (1-\alpha)\Delta - \Psi_{(1-\pi)}, \qquad D_{(1-\pi)} = -\alpha(1+\beta) - \Psi_{(1-\pi)},$$

so

$$\Xi_{(1-\pi)} = \frac{N_{(1-\pi)} - \Xi D_{(1-\pi)}}{D} = \frac{\alpha(1+\beta)\Xi + 2\alpha(1-2(1-\pi)) - (1-\alpha)\Delta + (\Xi-1)\Psi_{(1-\pi)}}{D},$$

which is (59). Since $\mathcal{D} = D > 0$, signs are governed by numerators, giving the stated threshold conditions. □

Lemma 3 (Slope bound for the admissible root in the common index)**.** *Under Assumption 1, Conjecture 1, and the credit-constraints conditions in Appendix A.4, let $q := \alpha\beta \in (0,1)$ and $m := \alpha(1-2(1-\pi)) - (1-\alpha)\Delta$. Then the admissible root satisfies*

$$0 < \Psi_m < \frac{1}{2}, \qquad \textit{and therefore} \qquad |\Psi_\Delta| < \frac{1-\alpha}{2}.$$

Proof. In Appendix A.4, the reduced quadratic is

$$\widetilde{F}(\Psi, m) = 12\Psi^2 + \big(2q - 8 - 6qm\big)\Psi + (q-1)(m^2-1) = 0,$$

so implicit differentiation gives

$$\Psi_m = -\frac{\widetilde{F}_m}{\widetilde{F}_\Psi} = \frac{2B}{24\Psi + b}, \qquad B := 3q\Psi + (1-q)m, \qquad b := 2q - 8 - 6qm.$$

From Appendix A.4, $B < 0$ and at the smaller root $24\Psi + b = -\sqrt{b^2 - 48c} < 0$. Hence $\Psi_m > 0$.

To show $\Psi_m < 1/2$, note that with denominator $24\Psi + b < 0$,

$$\Psi_m < \frac{1}{2} \iff 4B - (24\Psi + b) > 0.$$

Direct algebra yields

$$4B - (24\Psi + b) = -12(2-q)\Psi + (4+2q)m + 2(4-q).$$

Using $\Psi < \theta_P/2 = (1+m)/4$ from Lemma 1,

$$4B - (24\Psi + b) > 2 + q + (-2 + 5q)m.$$

Because $q \in (0,1)$ and $m \in (-1,1)$, the right-hand side is strictly positive: if $q \geq 2/5$,

its minimum over $m \in [-1,1]$ is at $m = -1$ and equals $4(1-q) > 0$; if $q < 2/5$, its minimum is at $m = 1$ and equals $6q > 0$. Therefore $\Psi_m < 1/2$.

Finally, $m_\Delta = -(1-\alpha)$, so $\Psi_\Delta = -(1-\alpha)\Psi_m$ and thus $|\Psi_\Delta| = (1-\alpha)\Psi_m < (1-\alpha)/2$. □

Proof of Proposition 5. By (55), $\operatorname{sgn}(K_\Delta) = \operatorname{sgn}(\Xi_\Delta)$. From (58),

$$\Xi_\Delta = \frac{\frac{1-\alpha}{2}\big(\Xi + 1 - 2(1-\pi)\big) + (\Xi - 1)\Psi_\Delta}{\mathcal{D}}, \qquad \mathcal{D} > 0.$$

Using Lemma 3,

$$|\Psi_\Delta| < \frac{1-\alpha}{2},$$

so the numerator is strictly bounded below by

$$\begin{aligned}\frac{1-\alpha}{2}\big(\Xi + 1 - 2(1-\pi)\big) - (\Xi - 1)|\Psi_\Delta| &> \frac{1-\alpha}{2}\big(\Xi + 1 - 2(1-\pi)\big) - \frac{1-\alpha}{2}(\Xi - 1) \\ &= (1-\alpha)\pi > 0.\end{aligned}$$

Hence $\Xi_\Delta > 0$, therefore $K_\Delta > 0$.

Numerical illustrations for $\Xi_{(1-\pi)}$ sign ambiguity. Using the admissible root Ψ_- and implicit derivatives from Appendix A.4, one calibrations already show opposite mobility-derivative signs. For $(\alpha, \beta, \Delta) = (0.30, 0.96, 0)$, $\Xi_{(1-\pi)} \approx 0.399 > 0$ for $(1-\pi) = 0.73$, while $\Xi_{(1-\pi)} \approx -0.199 < 0$ for $(1-\pi) = 0.81$.

A.6 Credit-constraint condition in the log case

This appendix records the condition under which the credit constraint equilibrium exists.

Criterion and sufficient condition. In this credit-constrained equilibrium, island-P agents would like to borrow at equilibrium prices and policy. Under log preferences and the scale-free steady state, the borrowing constraint binds if and only if

$$\Xi > 1 \iff \hat{c}^R > \hat{c}^P, \tag{60}$$

using (15)–(16).

To isolate the sign of the borrowing wedge, rewrite $\Xi - 1$ as a product of a mobility factor π and a price/income term. From (16),

$$\Xi - 1 = \frac{(1-\pi)(\theta_P - \Psi) + \pi(\theta_R - \Psi) - \big[(\theta_P - \Psi) + \pi\alpha\beta\big]}{(\theta_P - \Psi) + \pi\alpha\beta} = \frac{\pi\big(\theta_R - \theta_P - \alpha\beta\big)}{(\theta_P - \Psi) + \pi\alpha\beta}.$$

On the admissible set, $\Psi < \theta_P$ so the denominator is strictly positive. Also

$$\theta_R - \theta_P = (1-\alpha)\Delta + \alpha(2(1-\pi) - 1),$$

hence

$$\theta_R - \theta_P - \alpha\beta = (1-\alpha)\Delta + \alpha(2(1-\pi) - 1 - \beta) \geq (1-\alpha)\Delta - \alpha(1+\beta).$$

Therefore, if

$$\Delta > \bar{\Delta} := \frac{\alpha(1+\beta)}{1-\alpha},$$

then $\theta_R - \theta_P - \alpha\beta > 0$ for every $\pi \in (0,1]$, so

$$\Xi - 1 > 0 \qquad \forall\, \pi \in (0,1].$$

By (60), the credit-constrained equilibrium exists for all $\pi > 0$ under this condition.

A.7 Proof that $\mathcal{H}_g < 0$ in our two-island economy

We show that holding continuation objects (k'', g') fixed (as in the envelope derivatives used in the generalized Euler equation), higher current public spending crowds out private saving: $k' = \mathcal{H}(k, g)$ is decreasing in g. The argument has two steps: first show $\eta_g > 0$ and $\eta_{k'} > 0$ for the Euler residual, then apply the implicit-function theorem.

We recall that

$$\eta(k, g, k'; k'', g') := u_c(c^R) - \beta\alpha(k')^{\alpha-1}\Big((1-\pi)\, u_c(c^{R,\prime}) + \pi\, u_c(c^{P,\prime})\Big).$$

Differentiating η with respect to g while holding (k'', g') fixed, and using $\partial c^R/\partial g = -1$, gives

$$\eta_g = u_{cc}(c^R)\frac{\partial c^R}{\partial g} = -u_{cc}(c^R) > 0.$$

Next, $\partial c^R/\partial k' = -1$, $\partial c^{R,\prime}/\partial k' = \alpha\theta_R(1-\pi, \Delta)(k')^{\alpha-1}$, and $\partial c^{P,\prime}/\partial k' = \alpha\theta_P(1-\pi, \Delta)(k')^{\alpha-1}$. Let

$$\bar{u}_c := (1-\pi)\, u_c(c^{R,\prime}) + \pi\, u_c(c^{P,\prime}) > 0.$$

Then:

$$\begin{aligned}\frac{\partial}{\partial k'}\Big(\alpha(k')^{\alpha-1}\bar{u}_c\Big) &= \alpha(\alpha-1)(k')^{\alpha-2}\bar{u}_c + \alpha(k')^{\alpha-1}\frac{\partial \bar{u}_c}{\partial k'} \\ &= \alpha(\alpha-1)(k')^{\alpha-2}\bar{u}_c + \alpha^2(k')^{2\alpha-2}\Big((1-\pi)\theta_R u_{cc}(c^{R,\prime}) + \pi\theta_P u_{cc}(c^{P,\prime})\Big).\end{aligned}$$

Because $\alpha \in (0,1)$ implies $\alpha(\alpha-1) < 0$, because $\bar{u}_c > 0$, and because $u_{cc} < 0$, the

right-hand side is strictly negative. Therefore

$$\eta_{k'} = u_{cc}(c^R)\frac{\partial c^R}{\partial k'} - \beta\frac{\partial}{\partial k'}\Big(\alpha(k')^{\alpha-1}\bar{u}_c\Big) = -u_{cc}(c^R) - \beta\cdot(\text{negative}) > 0.$$

Finally, by the implicit-function theorem for the saving response $k' = \mathcal{H}(k,g)$ defined by $\eta(k,g,\mathcal{H}(k,g);k'',g') = 0$, we obtain $\mathcal{H}_g = -\eta_g/\eta_{k'} < 0$ and thus $\eta_g/\eta_{k'} > 0$.

A.8 Ramsey derivations (general case)

This appendix collects derivations underlying the Ramsey first-order conditions (20)–(21) and the pseudo-GEE in Sections 3.3

A.8.1 Ramsey Lagrangian and first-order conditions

We derive the Ramsey first-order conditions (20)–(21) from the Lagrangian (61):

$$\mathcal{L} = \sum_{t=0}^{\infty}\beta^t\Big[u(c_t^R) + u(c_t^P) + v(g_t) + \lambda_t\eta_t\Big]. \tag{61}$$

The implementability constraint is

$$\eta_t = u_c(c_t^R) - \beta\alpha k_t^{\alpha-1}\Big((1-\pi)u_c(c_{t+1}^R) + \pi u_c(c_{t+1}^P)\Big),$$

and the relevant envelope derivatives (holding g_{t+1} fixed) are

$$\frac{\partial c_{t+1}^R}{\partial k_t} = C_{k,t+1}^R = \alpha\theta_R k_t^{\alpha-1}, \qquad \frac{\partial c_{t+1}^P}{\partial k_t} = C_{k,t+1}^P = \alpha\theta_P k_t^{\alpha-1}.$$

Holding (k_{t+1}, g_{t+1}) fixed, the derivative of the Euler residual with respect to k_t is

$$\begin{aligned}\eta_{k,t} &= u_{cc}(c_t^R)\frac{\partial c_t^R}{\partial k_t} - \beta\,\frac{\partial}{\partial k_t}\Big(\alpha k_t^{\alpha-1}[(1-\pi)u_c(c_{t+1}^R) + \pi u_c(c_{t+1}^P)]\Big)\\ &= -u_{cc}(c_t^R) - \beta\,\alpha(\alpha-1)k_t^{\alpha-2}\Big((1-\pi)u_c(c_{t+1}^R) + \pi u_c(c_{t+1}^P)\Big)\\ &\quad - \beta\,\alpha^2 k_t^{2\alpha-2}\Big((1-\pi)\theta_R u_{cc}(c_{t+1}^R) + \pi\theta_P u_{cc}(c_{t+1}^P)\Big).\end{aligned} \tag{62}$$

We also use $\partial c_t^R/\partial g_t = \partial c_t^P/\partial g_t = -1$ and $\partial c_t^R/\partial k_t = -1$.

FOC for g_t. Differentiating (61) with respect to g_t and dividing by β^t yields three contributions: (i) the direct utility term, (ii) the current implementability term $\lambda_t\eta_t$, and (iii) the lagged implementability term $\lambda_{t-1}\eta_{t-1}$ (because c_t^R and c_t^P enter η_{t-1}). Using $\eta_{g,t} = -u_{cc}(c_t^R)$ and

$$\frac{\partial\eta_{t-1}}{\partial g_t} = \beta\alpha k_{t-1}^{\alpha-1}\Big((1-\pi)u_{cc}(c_t^R) + \pi u_{cc}(c_t^P)\Big),$$

we obtain (20).

FOC for k_t. Differentiating with respect to k_t and dividing by β^t yields: (i) the current utility term, (ii) the next-period utility term via c^R_{t+1} and c^P_{t+1}, (iii) the current term $\lambda_t \eta_t$, (iv) the lagged term $\lambda_{t-1}\eta_{t-1}$, and (v) the lead term $\lambda_{t+1}\eta_{t+1}$. Using $\eta_{k,t}$ from (62) (holding (k_{t+1}, g_{t+1}) fixed),

$$\frac{\partial \eta_{t-1}}{\partial k_t} = \beta\alpha k_{t-1}^{\alpha-1}(1-\pi)u_{cc}(c^R_t), \qquad \frac{\partial \eta_{t+1}}{\partial k_t} = u_{cc}(c^R_{t+1})\, C^R_{k,t+1},$$

and the utility derivative

$$\beta\Big(u_c(c^R_{t+1})\, C^R_{k,t+1} + u_c(c^P_{t+1})\, C^P_{k,t+1}\Big),$$

we obtain (21).