Temperature Shocks and Climate Change: A Conceptual Analysis*

Christian P. Traeger
Department of Economics, University of Oslo
ifo Institute, Munich

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Abstract: This paper addresses the challenge of accurately modeling and estimating climate change damages. Time series approaches rely on weather shocks, while cross-sectional analyses capture climatic differences but suffer from omitted variable bias. Climate is defined as the statistical pattern of weather that persists over time and allows for adaptation, unlike unpredictable weather realizations. To assess econometric approaches, I (i) integrate forward-looking adaptation into a full-fledged integrated assessment model of climate change permitting an analytic solution and (ii) generalize the insights based on a dynamic stochastic envelope argument. I show how a carefully designed time series (or panel) estimation strategy can comprehensively identify the costs of climate change, including the indirect identification of unobserved adaptation costs. The paper also presents the first explicit formula for the social cost of carbon under forward-looking adaptation. This result is not only insightful in its own right but also valuable for clarifying and refining prevailing envelope-theorem arguments in the literature and for emphasizing that adaptation costs are part of the social cost of carbon.

JEL Codes: Q54, H23, D80, D62

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1 Introduction

Can short-run weather shocks be used to estimate the welfare-relevant damages of long-run climate change? This question lies at the center of empirical climate economics. Cross-sectional comparisons across climates appear attractive because they compare locations that have had time to adjust, yet they remain vulnerable to unobserved heterogeneity across places. Within-location time-series variation in weather is cleaner from an identification standpoint, but it is often criticized for capturing only short-run responses to transitory shocks. The underlying difficulty is conceptual. Climate change shifts the distribution of weather over time, whereas weather shocks are realized draws from that distribution. Agents adapt to the former, not to each latter realization. Weather damages are therefore not automatically climate damages.

I study this identification problem in a forward-looking economy with stock adaptation. Many consequential adaptation responses—cooling capital, irrigation, drought-resistant crop varieties, flood protection—are durable investments whose costs and benefits are spread over time. Observed outcomes therefore reflect both realized weather and past adaptation choices. A structural framework is needed to relate observables, such as realized losses and adaptation expenditures, to the marginal welfare objects relevant for policy such as the social cost of carbon (SCC). For that purpose, I embed stock adaptation in a tractable integrated assessment model that links emissions, climate dynamics, adaptation, and welfare in a single framework.

The analytic structure yields a sharp decomposition of damages into a non-adaptable component and an adaptable component, making transparent which parts of the damage process appear in observed outcomes and which enter welfare. Section 2 develops a deterministic benchmark, showing optimal adaptation can render the adaptable component of observed damages locally flat in temperature, even though it continues to reduce output (Proposition 1). Proposition 2 then derives the first explicit formula for the SCC under forward-looking adaptation and shows—contrary to the practice in integrated assessment models (IAMs) and an earlier reading of the envelope theorem—that adaptation costs enter the SCC unambiguously.

This result is a prerequisite for the paper's identification argument. The underlying logic is a follows: once adaptation is chosen optimally, a small increase in temperature does not affect welfare through an induced marginal change in adaptation, because at the optimum the marginal benefit and marginal cost of additional adaptation already balance. This is the sense in which the envelope theorem applies here. A common but misleading inference is that adaptation costs therefore need not be accounted for in inferring marginal welfare effects from observed damage variation. Section 2.3 shows why that conclusion fails. The welfare-relevant object is the partial effect of temperature holding the adaptation state fixed, but in a deterministic environment that object is typically unobserved precisely because agents adapt to temperature. Yet, surprise weather variation is informative because it generates such partial variation. What such a shock reveals is neither post-adaptation damages nor adaptation expenditure itself, but the local slope of realized damages around the optimally adapted

benchmark. That slope combines the marginal damage under optimal adaptation with a markup equal to the marginal adaptation cost required to offset the shock. Moreover, because adaptation capital is persistent, contemporaneous expenditure need not coincide with the marginal welfare value of adaptation relevant for the SCC. Direct observations of adaptation spending therefore do not, in general, resolve the identification problem; surprise weather variation does so indirectly, by revealing the local slope of damages around the adapted state. At the same time, agents typically anticipate at least part of the weather distribution ex ante, so what constitutes a "surprise" must be defined relative to the information set underlying adaptation.

Section 3 introduces uncertainty. Agents choose adaptation before weather is realized and therefore adapt to the distribution of future weather rather than to each realization. Proposition 3 shows that optimal adaptation depends on climate moments, not merely on mean temperature. Proposition 4 characterizes realized damages at a given weather realization. This distinction between climate and weather is central. Because adaptation is chosen ex ante for the prevailing climate distribution, realized weather draws need not coincide with the state for which the existing adaptation stock is optimal. Weather shocks therefore move the economy outside the ex post optimality condition assumed by the deterministic envelope theorem, and it is precisely this deviation that makes realized damages informative about the welfare-relevant marginal effect of temperature. The stochastic model therefore makes precise how climate moments govern optimal adaptation and how realized weather continues to reveal information about damages, including the associated adaptation-cost margin.

This distinction has direct econometric implications. Section 4 shows that empirical work should condition on the relevant climate moments, i.e., on the distributional features of weather to which agents have adapted, and then use the remaining within-unit weather variation to identify marginal damages. Conditional on those climate controls, residual weather realizations are surprises relative to the prevailing adaptation state. The associated weather coefficient recovers the adaptation-inclusive marginal damage relevant for welfare analysis. In the time-varying-climate setting emphasized here, the coefficients on the climate moments can in turn help directly recover the corresponding adaptation-cost component, with weather-climate interactions providing complementary information.

Section 5 develops a general dynamic stochastic framework that nests the ACE analysis and clarifies the broader identification logic. Theorem 1 shows that the welfare effect of climate change can be written in terms of the marginal effect of weather realizations evaluated at the realized state, together with the propagation of climate moments through time. In richer settings, the relevant conditioning set generally extends beyond the temperature statistics to the broader climate and economic states that govern expectations and adaptation. This framework also helps separate two issues that are often conflated in the literature: identifying the local marginal damage object in observed data, and projecting that object beyond the observed state space. The paper addresses both at different levels: it derives the SCC in

a tractable forward-looking setting and, through Theorem 1 and its corollary, clarifies the structure of the broader projection problem. Its main contribution, however, is to clarify the identification problem that underlies any such projection exercise. Section 5.3 extends the logic to accommodate wrong beliefs or delayed updating: when agents adapt to a lagged or misspecified climate state, adaptation is too low, realized damages rise, and so does the SCC.

The paper thus develops a fully dynamic identification strategy for adaptation-inclusive climate damages. Recent state-of-the-art approaches typically infer adaptation from cross-regional heterogeneity in weather responses using quasi-static arguments or base their conceptual analysis on comparisons across long-run steady states. Both are informative, but they are less naturally aligned with the central policy problem, which is to evaluate damages along a transition with ongoing adaptation and changing climate. The framework developed here instead shows how the marginal damage object relevant for welfare analysis and the SCC can be recovered using within-location time-series variation, provided the econometrician conditions on the relevant climate state to which agents have adapted. Properly controlled weather surprises then identify the welfare-relevant marginal object while avoiding the cross-sectional confounding that motivates concern about spatial comparisons.

Literature. The Ricardian approach, introduced by Mendelsohn et al. (1994), has sought to estimate climate impacts by comparing economic outcomes across locations with different climatic conditions while accounting for cross-sectional confounders (Schlenker et al. 2005, Nordhaus 2006). Concerns about omitted variables led Deschênes & Greenstone (2007) to initiate a literature exploiting fixed-effects panel designs and short-run weather fluctuations for identification. The authors note that their estimates may differ from a "preferred long-run effect that allows for all economic substitutions," a limitation still considered a serious shortcoming because short-run shocks cannot capture the long-run consequences of climate change when adaptation is operative (Auffhammer 2018, Hogan & Schlenker 2024). Designs such as long differences (Dell et al. 2012, Burke & Emerick 2016) and spatial first differences (Druckenmiller & Hsiang 2018) incorporate more persistent temperature variation while retaining better identification. In particular, many relevant adaptation costs are difficult to observe directly, and even observed adaptation expenditures need not correspond to the relevant welfare margin.

In the context of flow adaptation, Schlenker (2006), Tulkens & Van Steenberghe (2009), Guo & Costello (2013), Hsiang (2016), Deryugina & Hsiang (2017), and Carleton et al. (2022) discuss how, under certain assumptions, the static envelope theorem can eliminate the need to observe adaptation directly or allow adaptation costs to be inferred indirectly. These applications of the envelope argument typically treat temperature—or its expectation—as the relevant climate state. More generally, evidence from many settings shows that damages depend on volatility and extremes as well as on average conditions (Schlenker & Roberts 2009, Auffhammer & Aroonruengsawat 2011, Barreca et al. 2015, Park et al. 2020, Carleton et al. 2022, Moscona & Sastry 2023). Mérel et al. (2024) emphasize that temperature volatility

is an essential component of climate, as it generally shapes adaptation behavior. They derive conditions under which decisions are certainty-equivalent, implying that uncertainty does not affect adaptation, in order to construct a consistent estimation framework grounded in earlier envelope arguments.[1] More generally, Mérel & Gammans (2021) reinterpret panel estimates as weighted averages of short-run and long-run responses, viewing the short-run component as a source of contamination.[2]

Most papers assume that the econometrician observes maximized output. Yet in practice, output net of adaptation costs is rarely observable. The present paper revisits the interpretation and application of the envelope theorem in this context, and highlights potential issues arising from the direct observation of expenditures in stock adaptation. The paper subsequently extends the envelope theorem along three important dimensions: First, it embeds the envelope argument within a forward-looking dynamic framework. Second, it incorporates stock adaptation—that is, investments in long-lived adaptation measures. Third, it generalizes the approach to a more realistic setting in which climate is represented not simply by temperature or expected temperature, but by the full distribution of weather outcomes.

Kelly et al. (2005) pioneered structural dynamic estimation of agricultural yields under adaptation in a model with learning. Lemoine (2024) is perhaps closest in spirit. He studies whether panel weather variation can recover the effects of moving between stationary climates once agents have adapted, and therefore combines identification with the extrapolation from transient weather shocks to permanent climate change. Because his target is a cross-steady-state comparison, his envelope-style reasoning does not collapse to a directly identified weather coefficient. Relatedly, both Lemoine (2024) and, in a flow-adaptation context, Shrader (2023) emphasize that controlling for weather forecasts or other anticipated weather components mitigates a built-in bias in standard econometric approaches. The present paper instead develops closed-form solutions and a general dynamic envelope argument showing that, once one conditions on the relevant moments of the weather distribution, weather shocks identify the local welfare-relevant marginal object, while the projection to long-run climate costs is a separate step. Rudik et al. (2024) develop a dynamic envelope argument in a structural setting where migration across sectors plays a role analogous to adaptation, showing that migration flows constitute a sufficient statistic for households' expected future welfare across regions. In ongoing work, Costello & Collie (2025) apply a dynamic envelope theorem, treating climate as expected temperature, to estimate the impact of climate change on global fisheries.

The perspective developed here also helps clarify how recent empirical approaches address different margins of adaptation costs. Auffhammer's (2022) CARE approach uses the relationship between temperature and directly observed residential electricity and gas expenditures to project costs of a warming climate in California. Carleton et al. (2022), by contrast, use a

[1] Part of the debate in this literature may stem from the visual appeal of depicting the "long-run response" as the envelope of short-term responses. Under uncertainty, however, this picture is generally misleading: the true envelope is an "invelope", typically lying strictly inside the convex hull of the short-term responses.

[2] The authors define the long-run as a situation without temperature volatility.

static envelope argument to infer adaptation costs indirectly in mortality data, and Hultgren et al. (2025) apply the same logic to agriculture. Viewed through the lens of the present paper, these approaches are complementary. Auffhammer (2022) identifies immediate operating expenditures, a margin largely shielded from envelope-based indirect inference because, once the equipment is in place, households can respond immediately to heat or cold, deterministically changing thermostat settings; the surprise element required for envelope-based identification is absent. By contrast, the approach in Carleton et al. (2022) can, in an ideal setting (see Section 4), recover the stock adaptation margin itself, e.g., the cost of purchasing and installing AC rather than the electricity used once AC is in place. A comprehensive accounting of climate damages would therefore combine these complementary flow and stock margins.

A series of numeric integrated assessment models incorporate adaptation in various ways. For instance, Bosetti et al. (2006) develop a framework in which adaptation emerges endogenously through learning-by-doing and investments in research and development. Tol (2007) analyze adaptation to sea level rise, incorporating both protective measures and displacement responses. De Bruin et al. (2009) extend the DICE model to explicitly represent adaptation as a decision variable. Diaz (2016) and Depsky et al. (2023) quantify the economic impact of sea level rise under scenarios with and without adaptation. Dietz & Lanz (2022) build an integrated assessment model that analyzes a series of adaptation mechanisms, including demographic transitions, technological progress, shifts in land use, and sectoral reallocation. The Climate Impact Lab (2023) develops a spatially detailed IAM that simulates climate damages across multiple sectors, incorporating adaptation into the damage functions. I refer to the excellent recent handbook chapter by Carleton et al. (2024) on adaptation to climate change for a more detailed discussion of several hundred papers on the topic. For broader overviews of empirical climate damage estimation and related measurement issues, see Auffhammer (2018), Hogan & Schlenker (2024), Lemoine et al. (2025), and Hsiang (2025).

The present paper bridges a gap between the structural literature on adaptation modeling, the literature on forward-looking integrated assessment modeling under uncertainty recently reviewed in Jensen & Traeger (2024), and the reduced-form empirical literature on climate damage estimation. In doing so, the paper contributes to resolving a central conceptual identification challenge in climate damage estimation—a problem the most recent literature review has referred to as "a problem from hell" (Lemoine et al. 2025).[3]

2 The Deterministic World

Section 2 develops the production, damage, and adaptation block of ACE, a tractable global IAM. This is the part of the model needed for the present analysis: it determines observed damages, adaptation expenditures, and the cost components that enter welfare accounting.

[3]Lemoine et al. (2025) frame this "problem from hell" as a trilemma among model robustness, climate-change interpretability, and econometric identification.

The remaining ACE structure, linking emissions, the carbon cycle, temperature, and welfare, is summarized in Appendix A.1 and Figure 2. Embedding the analysis in the full global IAM then allows these components to map directly into the SCC and the implied optimal carbon tax.

2.1 Production, Damages, and Adaptation

Gross output is a function of capital $\boldsymbol{K}_t$, labor $\boldsymbol{N}_t$, energy inputs $\boldsymbol{E}_t$, and exogenously evolving technology $\boldsymbol{A}_t$:

$$Y_t = F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t)\,[1 - D(T_t)], \tag{1}$$

where $D(T_t)$ denotes the fraction of output lost due to the atmospheric surface temperature increase T_t relative to 1900. For now, T_t denotes global temperature; Section 4 later reinterprets the same damage block at the regional level. Throughout, bold symbols denote vectors, here, representing sectors and energy sources.

The production function $F(\cdot)$ is assumed to be homogeneous of degree $\kappa \in (0, 1)$ in capital.[4] This specification encompasses the production and abatement structures of common IAMs such as Golosov et al. (2014) and Nordhaus (2017), as well as richer multi-sector models with heterogeneous energy sources and time-varying substitution elasticities.[5]

It is helpful to specify damages in the form

$$D(T_t) = 1 - \exp[-d(T_t)],$$

where $d(T_t)$ represents damages in log-output space:

$$y_t \equiv \log Y_t = \log(F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t)) - d(T_t).$$

From an empirical perspective, $d(T_t)$ is the regression-based damage function. From the IAM perspective, $d(T_t)$ is a convex function of temperature, while the transformation $D(\tau_t) = 1-\exp[-d(T_t)]$ ensures damages cannot exceed total output. A second-order Taylor expansion, $1 - \exp[-x] \approx x - \frac{1}{2}x^2$, shows that to first order $d(T_t)$ can be interpreted as the percentage loss of output, with $D(\cdot)$ being slightly less convex than $d(\cdot)$.

To obtain closed-form solutions, the structural analysis requires a tractable functional form. Traeger (2025) shows that the following specification closely matches the mean damages of

[4] $F(\boldsymbol{A}_t, \boldsymbol{N}_t, \gamma\boldsymbol{K}_t, \boldsymbol{E}_t, i_t) = \gamma^\kappa F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t, i_t)$ for all $\gamma \in \mathbb{R}_+$. I also assume the production optimization problem is well-defined.

[5] Golosov et al. (2014) assume a Cobb–Douglas final goods sector with a CES energy composite of oil, coal, and renewables. DICE's production model (Nordhaus 2017) is a special case when using exogenous business-as-usual emissions. Traeger (2022) calibrates a more detailed version with three final sectors, five primary energy sources, an electricity sector, and sector-specific substitutabilities.

Symbol	Meaning	Symbol	Meaning
Preferences and welfare			
$U(\cdot)$	Period utility.	$V(\cdot)$	Value function.
β	Time preference/utility discount factor.	SCC_t	Social cost of carbon at time t.
Macro quantities and production			
Y_t, y_t	Output and log-output.	Y_t^{net}	Output net of adaptation expenditures.
C_t	Consumption.	I_t	Investment into productive capital.
K_t	Aggregate productive capital stock.	$\boldsymbol{K}_t$	Vector of sectoral capital stocks.
$\boldsymbol{N}_t$	Labor input vector.	$\boldsymbol{E}_t$	Energy input vector.
$\boldsymbol{A}_t$	Productivity / technology shifters.	$F(\cdot)$	Production function.
κ	Capital share / homogeneity parameter.		
Damages and adaptation			
$D(\cdot)$	Fractional climate damages.	$d(\cdot)$	Log-damage index.
ξ_0	Non-adaptable damage semi-elasticity.	ξ_1	Temperature-T-τ scale transformation parameter.
ρ	Scale of adaptable damage component.		
λ	Steepness of adaptable damages.	χ	Effectiveness of adaptation.
i_t, i_t^*	Adaptation investment, optimal level.	η	Adaptation cost coefficient.
A_t	Adaptation capital stock.	δ_A	Depreciation rate of adaptation capital.
Climate, carbon, and temperature			
T_t	Global mean surface temperature (°C).	τ_t	Transformed surface temperature state.
μ_t, $\bar{\tau}_t$	Climatic mean, temperature trend.	$\boldsymbol{\tau}_t$	Vector of temperature states.
$\boldsymbol{\sigma}$	Temperature transition matrix.	σ^{forc}	Part of $\boldsymbol{\sigma}$ related to atmospheric temperature responsiveness.
$\boldsymbol{\Phi}$	Carbon-cycle transition matrix.		
$M_{1,t}$	Atmospheric carbon stock.	M_{pre}	Preindustrial reference carbon level.
$\boldsymbol{R}_t$	Remaining fossil resource stocks.		
Stochastic climate and distributions			
$\mathbb{E}_t[\cdot]$	Conditional expectation.	$\mathrm{Var}_t[\cdot]$	Conditional variance.
$P(T;\boldsymbol{m})$	Temperature density (pdf).	$CDF(T;\boldsymbol{m})$	Cumulative distribution function.
$\boldsymbol{m}_t$	Vector of climate moments.	$\mathrm{CGF}_{\tau_t}(\lambda)$	Cumulant generating function.
κ_n	nth cumulant.		

Table 1: Notation.

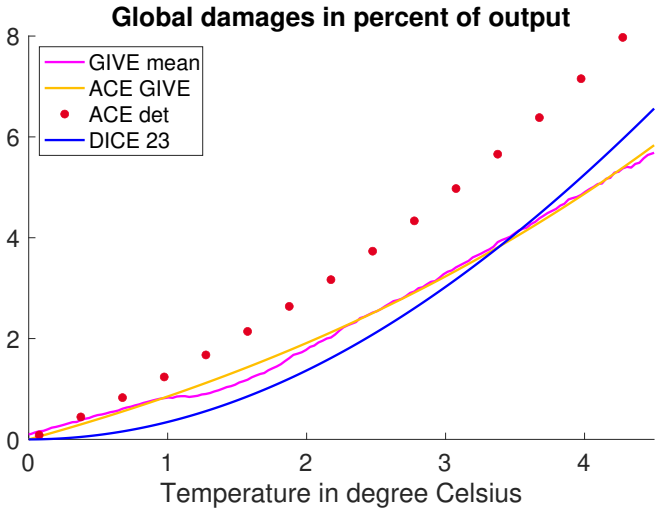


Figure 1: Mean of Rennert et al.'s (2022)'s analysis of global damages (GIVE model, pink solid), and ACE fit using $D(T_t) = 1 - \exp[-d^{na}(T_t)]$ with $\xi_0 = 0.033$. For the numeric illustrations in Section 4.3, I follow Traeger (2025) and augment (non-adaptable) damages to the dotted line to account for missing sectors in the original study ($\xi_0 = 0.049$).

Rennert et al. (2022):

$$d^{\text{na}}(T_t) = \xi_0\big(\underbrace{\exp[\xi_1 T_t]}_{\equiv \tau_t} - 1\big) = \xi_0(\tau_t - 1), \tag{2}$$

where "na" abbreviates *n*o explicit *a*daptation. Rennert et al. (2022), following the National Academies of Sciences & Medicine's (2017) guidance and used in the US SCC update EPA (2023), provide a benchmark estimate based on 10,000 Monte Carlo simulations aggregating country- and regional-level impacts for agriculture, mortality, sea-level rise, and energy. Figure 1 compares their simulated damages with the fitted function $D(T_t) = 1-\exp[-d^{na}(T_t)]$ over the temperature range $[1, 4]$. The figure illustrates that the exponential specification in equation (2) provides a better fit than the commonly used quadratic form.[6]

Equation (2) introduces the transformation $\tau_t = \exp[\xi_1 T_t]$. Beyond notational convenience, the τ-scale is central for solving the forward-looking IAM. As equation (2) shows, expressing damages in τ implies constant marginal losses to log-output; under log-utility these become constant marginal damages in welfare units. This property greatly simplifies the forward-looking analysis and will recur in the SCC expressions.[7]

Most IAMs do not carefully specify whether reduced-form damage functions—which omit

[6]Ever since Nordhaus (1994), quadratic specifications have been standard in IAMs and meta-analyses (Howard & Sterner 2017). The common "one minus one over quadratic" form $D(\tau_t) = 1 - \frac{1}{1+aT_t^2}$ plays a role analogous to $1 - \exp[-d(T_t)]$, limiting damages to not exceed output. The (double-) exponential function proposed here is initially less but eventually more convex than this quadratic "benchmark".

[7]See Traeger (2023) and the Appendix for details. The present analysis focuses on the free damage parameter ξ_0. The parameter ξ_1 also relates to the model's nonlinear temperature dynamics as explained in Traeger (2023); here, I take it as given.

explicit adaptation—represent damages before or after adaptation. The prevailing interpretation is that they reflect damages after adaptation. This study extends the damage structure to include adaptation explicitly. To clarify the relationship between the true damage function, econometrically estimated damages, and implied social costs (including the SCC), it is helpful—if not crucial—to obtain closed-form solutions to the forward-looking problem with adaptation. I am not aware of any previous successful attempt. Maintaining analytic tractability, I extend equation (2) to[8]

$$d(\tau_t, i_t, A_t) = \underbrace{\xi_0(\tau_t - 1)}_{\substack{\text{non-adaptable} \\ \text{component}}} + \underbrace{\rho \overbrace{\exp(\lambda \tau_t)}^{\text{damage}} \overbrace{\exp(-\chi(A_t + i_t))}^{\text{adaptation}}}_{\text{adaptable component}}. \tag{3}$$

The first term preserves the non-adaptable part of equation (2). The second term introduces an explicitly adaptable component, with parameters ρ scaling magnitude, λ the steepness of damages, and χ the effectiveness of adaptation. Adaptation capital A_t and current adaptation investment i_t evolve according to

$$A_{t+1} = (1 - \delta_A)(A_t + i_t), \tag{4}$$

where δ_A is the depreciation rate of adaptation capital.

The joint appearance of $A_t + i_t$ ensures that in the limit of full depreciation ($\delta_A = 1$) the model collapses to the standard flow-adaptation case.[9] Flow adaptation models have dominated the literature, where i_t reduces only current damages in equation (3). More generally, stock adaptation investments reduce damages across multiple periods. The exponential specification further ensures that no amount of adaptation can raise GDP above the no-climate-change baseline.[10] The functional forms are qualitatively reasonable. Most importantly, they permit an analytic solution for optimal adaptation investment, its implications for damage estimation, and the SCC.

Adaptation investment expenses i_t are deducted from output, yielding net output

$$Y_t^{net} = Y_t \exp(-\eta\ i_t),$$

[8]The non-adaptable component is normalized to zero at preindustrial temperatures by the term "-1". Subtracting an additional constant from the second expression would allow for an analogous (or arbitrary) normalization of the adaptable component. For expositional clarity, I suppress this normalization constant, as it does not affect any of the results. I introduce such a normalization in the numerical illustration in Section 4.3. The exponential form in τ_t is highly convex, but the form also permits powerful adaptation, and the results of this combination are didactically particularly helpful.

[9]Thanks to depreciating both A_t and i_t in (4), adaptation capital under $\delta_A = 1$ is always zero at the beginning of a period. Using the common capital accumulation equation $A_{t+1} = (1-\delta_A)A_t + i_t$ would replace the persistence multiplier $(1 - \beta(1 - \delta_A))$ with $\left(1 - \frac{\beta}{1+\beta\delta_A}\right)$ in later results. Unlike the present formulation, this would not collapse to unity under full depreciation, since current investments would also reduce next period's damages.

[10]Although some investments may enhance welfare even at preindustrial temperatures, this assumption distinguishes adaptation investments (aimed at climate impacts) from general investments.

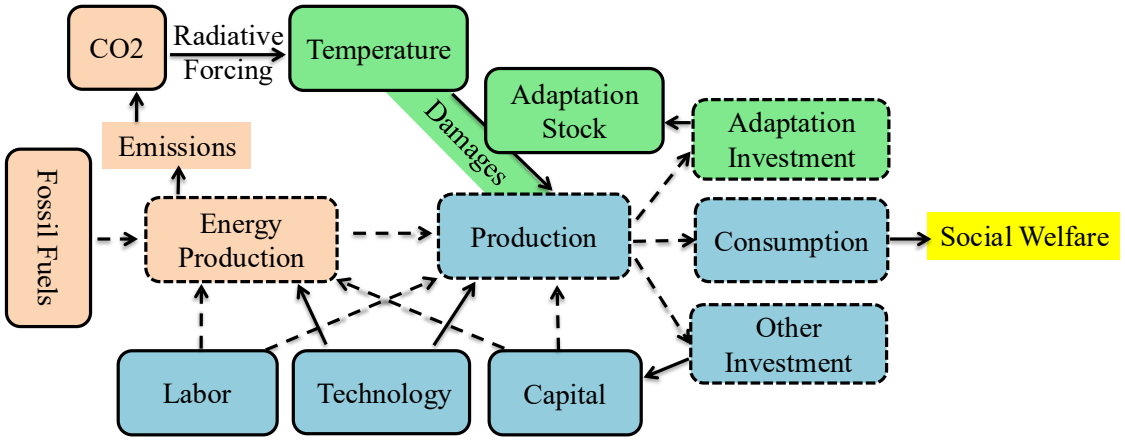


Figure 2: Schematic structure of ACE, representative of most IAMs. Solid boxes denote state variables, dashed boxes flows, and dashed arrows decision variables.

which consumers optimally allocate between consumption and investment.[11] The analysis assumes an interior solution to the adaptation problem with

$$i_t > 0 \ ,$$

which holds for reasonable, though not all, possible parameter values.

Appendix A.1 summarizes the remaining components of the Analytic Climate Economy model, illustrated in Figure 2 (Traeger 2023). The model integrates an updated carbon cycle by Eftekhari et al. (2025) and a nonlinear ocean–atmosphere temperature module, together generating a carbon impulse response consistent with state-of-the-art IAMs (Traeger 2025). It also accounts for potential scarcity of fossil-fuel inputs, which can generate a positive Hotelling rent.

2.2 Results in a Deterministic World

A representative agent maximizes log-utility over an infinite time horizon with time preference discount factor β. The following proposition states the optimal adaptation investment and the resulting shape of the optimally adapted damage function.

Proposition 1 *The optimal investment into adaptation capital is*

$$i_t^* = \frac{1}{\chi}\Big[\log(\rho) - \log\Big(\frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\Big) + \lambda\tau_t\Big] - A_t. \tag{5}$$

[11] The coefficient η converts adaptation spending into consumption equivalents (USD). This share-based approach can be motivated by the fact that also damages are a share of output (equation 1), and the defensive expenditure interacts with damages multiplicatively (equation (3)). Using a time-dependent η is possible, but the present form eases tractability, which is the main motivation for the functional choice.

The resulting optimally adapted damages are

$$d(\tau_t, i_t^*, A_t) = \xi_0(\tau_t - 1) + \frac{\eta}{\chi}\left(1 - \beta(1 - \delta_A)\right). \tag{6}$$

The optimal adaptation investment stated in equation (5) increases exponentially in temperature together with $\tau_t = \exp(\xi_1 T_t)$. This increase is proportional to the ratio $\frac{\lambda}{\chi}$, which reflects the relative steepness of damage escalation (λ) compared to the effectiveness of adaptation investment (χ). Additionally, investment increases logarithmically with the scaling factor of the adaptable damage component, represented by ρ. Conversely, investment decreases logarithmically with the ratio $\frac{\eta}{\chi}$, the cost of adaptation relative to its effectiveness. These costs are further adjusted by a factor that incorporates the persistence of abatement capital, reducing effective costs in proportion to the discounted adaptation capital persistence $\beta(1 - \delta_A)$. Finally, a higher stock of adaptation capital A_t already in place reduces the need for further investment.

Equation (6) implies that the adaptive component of observed damages becomes independent of temperature. To see this, consider an idealized setting with many identical regions that differ only in temperature τ_t. Even if we guess the correct functional form and regress log output on $\tau_t = \exp(\xi_1 T_t)$, equation (6) implies that the estimated relation will be

$$y_t = \ldots - \xi_0 \tau_t = \ldots - \xi_0 \exp(\xi_1 T_t), \tag{7}$$

suppressing observables that are independent of temperature. Even under the best possible controls, cross-sectional regressions recover only the non-adaptable part of damages in this setting.[12]

The simple additive separation of observable damages in equation (7) reflects the specific functional forms chosen to permit analytic tractability: damages decompose into a component that varies with temperature independently of adaptation and a component that no longer varies with temperature once adaptation is in place.[13] The essential point is not the exact form of this separation, but the distinction it highlights: under adaptation, some damages remain unavoidable, while others can be mitigated through adaptive investments. The present example sharpens this distinction by separating residual damages and avoided damages through costly adaptation into two additively separable components. More generally, one can think of realized damages as a convex combination of these two components.

Equation (6) also makes clear that the adaptable component reduces GDP—just not in a way that shows up in temperature-damage regressions. Its effect is proportional to the effective

[12]Section 2.3 discusses in detail when adaptation costs are directly observable and why direct observation does not, in general, resolve the identification problem. Section 4 examines empirical regressions and identification using the stochastic framework yet to be developed.

[13]This result, and the availability of a closed-form solution, also rely on the fact that the problem has a most-rapid-approach-path solution. The combined stock of existing and new adaptation capital, $A_t + i_t$, adjusts directly to its optimal level along any non-stationary trajectory. As a result, A_t does not appear on the right side of equation (7). While this feature is special to the present setup, Section 5 shows that the core insights derived here extend to the general setting, where no closed-form solution is available.

costs of adaptation, $\frac{\eta}{\chi}(1-\beta(1-\delta_A))$: the higher these costs, the lower the achieved adaptation level and the larger the resulting damages. In the end, we are interested in the social cost of a temperature increase and, more specifically, the social cost of the persistent temperature increase triggered by an additional ton of CO_2 emissions. The following proposition clarifies which parameters (or components) we have to estimate in order to calculate the SCC, which specifies the social costs of emitting a ton of carbon along the optimal trajectory and, thus, the optimal carbon tax.

Proposition 2 *The SCC under optimal adaptation is*

$$SCC = \underbrace{\frac{\beta Y_t^{net}}{M_{pre}}}_{\text{cost base}} \Big(\xi_0 + \underline{\eta \frac{\lambda}{\chi}\Big(1-\beta(1-\delta_A)\Big)}\Big) \underbrace{\left[(\mathbf{1}-\beta\mathbf{\Phi})^{-1}\right]_{1,1}}_{\text{carbon dynamics}} \underbrace{\sigma^{forc}\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}}_{\text{temperature dynamics}}, \tag{8}$$

where the underlined contributions are new to the model with adaptable damages.

Proposition 2 extends the base formula in Traeger (2023) by introducing the underlined term. The first term, labeled 'cost base,' relates (discounted) net output Y_t^{net} to the preindustrial carbon concentration M_{pre}. This term sets the scale and defines the units by linking output to CO_2 levels.

The second-to-last term, labeled 'carbon dynamics,' represents a discounted sum over the additional atmospheric CO_2 in all future periods resulting from the release of one extra ton of CO_2 today (Traeger 2023). The magnitude of this multiplier depends on the discount rate, but it is always substantially greater than unity, thereby increasing the SCC—an outcome driven by the mass conservation of carbon. Similarly, the final term, labeled 'temperature dynamics,' reflects the future heat exchange induced by an additional unit of radiative forcing (greenhouse effect).[14] This final temperature-dynamics term is always less than unity because the oceans act as a thermal buffer, delaying warming and, consequently, the associated damages.

Turning to the components of interest, the parameter ξ_0 represents the magnitude of non-adaptable damage. Among the terms inside the bracket, it is the only one that remains in a standard model without adaptation. Here, ξ_0 is the semi-elasticity of output with respect to an exponential increase in temperature. Multiplying ξ_0 by the cost base, we obtain a SCC that would apply in the absence of carbon cycle dynamics, meaning CO_2 would not persist in the atmosphere, and oceanic processes would not delay warming.

The novel underlined term further increases the SCC by accounting for the adaptable damage component, which rises with the cost of building adaptation capital, η. It also increases with the ratio of the steepness of damages to the effectiveness of adaptation capital, $\frac{\lambda}{\chi}$, which reflects the temperature dependence of optimal investment. The subsequent term in brackets

[14]These interpretations derives from expressing the inverted matrices $(\mathbf{1}-\beta\mathbf{\Phi})^{-1}$ and $(1-\beta\boldsymbol{\sigma})^{-1}$ using the Neumann series. The result is a double sum because every increase in atmospheric carbon in the future induces additional radiative forcing, and each additional unit of radiative forcing causes further warming in all subsequent periods, interacting with the oceans, which continue to provide substantial cooling.

adjusts these costs for capital persistence, leading to an interesting and uncommon finding: *this SCC contribution decreases with patience* (i.e., a higher discount factor, β). The reason is that a more patient decision-maker invests more in adaptation capital, thereby mitigating the damage caused by an additional ton of CO_2 released into the atmosphere.

The result underscores that *it's the shape of the damage function that matters for the SCC rather than merely the magnitude of damages.* In particular, the scale parameter ρ of the adaptable damage component in equation (3) does not affect the SCC. While a higher ρ lowers welfare by raising baseline damages, adaptation investment adjusts proportionally to ρ but independently of temperature.[15]

Proposition 2 highlights that the adaptable damage component $\eta\frac{\lambda}{\chi}\big(1-\beta(1-\delta_A)\big)$, which remains unobserved in equation (7), is required to determine the SCC. Even with perfect control for unobserved heterogeneity, cross-sectional variation recovers only the non-adaptable component of damages and therefore misses the additional adaptation costs induced by an extra ton of CO_2. One might wonder why damages that do not vary with temperature nonetheless affect the SCC. The reason is that observed damages reflect outcomes after adaptation: the relevant social costs of such neutralized damages are the adaptation expenditures that offset them, and these costs rise with each additional ton of CO_2 emitted.

Shocks. Turning to a time series, the deterministic setting with perfect adaptation (neutralization of additional damages), would still reveal only equation (7). Yet the supposed weakness of weather-shock regressions can turn into an advantage: unlike slow climatic shifts, short-lived shocks are not matched by adaptation capital, as such investments are typically impractical or undesirable. If adaptation responds only to a temperature trend $\bar{\tau}_t$, then a short-run deviation τ_t implies, via equation (3), a log-GDP loss $y_t = \ldots - \xi_0\tau_t - \rho\exp(\lambda\tau_t)\,\omega_t$ with some constant $\omega_t < 1$. Provided a way to control for the time variation of ω_t, this "surprise variation" promises a handle to uncovers the steepness parameter λ, which is unobservable in long-averages or a cross-section resulting in equation (6). *Paradoxically, the inability to adapt to shocks reveals crucial information about the SCC under adaptation.* Evaluating at the optimal adaptation level gives[16]

$$y_t = \ldots - \xi_0\tau_t - \frac{\eta}{\chi}(1-\beta(1-\delta_A))\exp(\lambda(\tau_t-\bar{\tau}_t)). \tag{9}$$

For a marginal surprise deviation $\Delta\tau \equiv \tau_t - \bar{\tau}_t$, we observe the desired regression coefficient $-\big[\xi_0+\frac{\eta}{\chi}\lambda\big(1-\beta(1-\delta_A)\big)\big]$ at the core of the SCC because $\exp(\lambda(\tau_t-\bar{\tau}_t)) \approx 1+\lambda\Delta\tau$. Section 2.3 shows that this suggestive result is not coincidental but follows from the envelope theorem, and Section 3 develops the stochastic framework distinguishing between shocks and trends, and more generally climate, necessary to formalize this intuitive reasoning.

[15] Welfare is logarithmic, and their multiplicative interaction becomes additively separable.

[16] The constant is $\omega_t = \exp(-\chi(A_t+i_t))$ and, at the optimum, $\omega_t^* = \frac{1}{\rho}\frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\exp(-\lambda\bar{\tau}_t)$.

2.3 The Envelope Theorem and Interpreting Output Variation

Guo & Costello (2013) and subsequent work show that optimal adaptation cancels out in observed output damages, a result attributed to the envelope theorem. Yet interpreting such findings too literally risks obscuring the substantial value and costs of adaptation. This section links the general principles of the envelope theorem to the present findings: first revisiting the static reasoning commonly invoked in this literature (Guo & Costello 2013, Hsiang 2016, Deryugina & Hsiang 2017, Carleton et al. 2022, Mérel et al. 2024), and then extending the analysis to a dynamic setting.

Static envelope theorem. Let net output depend on temperature T_t and adaptation investment i_t: $Y_t^{net}(T_t, i_t, \ldots)$. When investment is chosen optimally, $i_t^* = i_t^*(T_t)$, the total effect of a small temperature change on log output y_t^{net} is

$$\frac{dy_t^{net}}{dT_t} = \frac{\partial y_t^{net}}{\partial T_t} + \underbrace{\frac{\partial y_t^{net}}{\partial i_t}}_{=0} \frac{\partial i_t^*}{\partial T_t} = \frac{\partial y_t^{net}}{\partial T_t}. \tag{10}$$

The second term vanishes because the decision maker optimizes i_t:

$$\max_{i_t} Y_t^{net}(T_t, i_t, \ldots) \quad \Rightarrow \quad \frac{\partial y_t^{net}}{\partial i_t} = 0. \tag{11}$$

This is the envelope theorem at work: the indirect effect of temperature through optimal adaptation cancels, leaving only the direct impact of T_t on net output.

Equation (11) is sometimes interpreted as implying that the marginal adaptation project has no value, e.g. Guo & Costello (2013), and equation (10) is interpreted as suggesting that observed damages alone suffice to measure the social cost of a temperature increase, without explicitly accounting for unobserved adaptation efforts. While these applications of the envelope theorem offer useful insights, they must be treated with caution as they risk obscuring the role and costs of adaptation.

The value of adaptation. Equation (11) should be read as stating that the net economic value of expanding adaptation by one more unit is zero. This is not the same as saying that the marginal value of adaptation itself is zero. An analogy helps clarify: if my marginal valuation for a chocolate bar is \$1, and its store price is also \$1, then the net benefit of buying the bar is zero, since what I pay equals what I gain. Yet the value of the bar remains \$1. Likewise, the value of an adaptation project is strictly positive, even if the net gain from expanding it further is zero.

Formally, using a static version of the production model, equation (11) can be written as

$$\max_{i_t} Y_t^{net}(A_t, \tau_t, i_t, ...) \quad \Rightarrow \quad \underbrace{\chi\rho \exp(\lambda\tau_t) \exp(-\chi(A_t + i_t))}_{\text{marginal benefits}} = \underbrace{\eta}_{\text{marginal costs}}, \tag{12}$$

where costs and benefits are expressed as fractions of output. Either side of equation (12) represents the marginal value of an adaptation project, which is strictly positive.

The interpretation of output: Y_t^{net} vs Y_t. My notation distinguishes between Y_t^{net} and Y_t to stress that the envelope theorem applies only to Y_t^{net}, i.e., output net of adaptation expenses. In practice, however, we observe Y_t rather than Y_t^{net}, since adaptation expenditures are rarely recorded explicitly. For example, measuring GDP at the macroeconomic data we generally do not see how inputs are rerouted for adaptive purposes, and at the micro level we may only observe crop yields but not that a farmer has installed a drip irrigation system.[17]

Equation (7) shows that regressing $\log Y_t$ on temperature identifies only non-adaptable damages and thus understates the social cost in Proposition 2. The reason the envelope theorem applies only to Y_t^{net} is straightforward: the representative agent maximizes consumption, not gross output (and so does the decentralized economy absent market failures). For the same reason, changes in Y_t fall short of measuring the social cost, since resources diverted from consumption or productive investment to adaptation are not properly reflected in gross output.[18] If output net of adaptation investment were observed, then the present model would yield

$$y_t^{net} = \ldots - \xi_0 \tau_t - \eta \frac{\lambda}{\chi} \tau_t = \ldots - \Big(\xi_0 + \eta \frac{\lambda}{\chi}\Big) \tau_t, \tag{13}$$

where I use equation (5) for optimal adaptation investment as a function of temperature (neglecting other dependencies). In a static framework, equation (13) correctly identifies the marginal damages entering the SCC in equation (8).

By contrast, in a setting with adaptation capital, the identification of the SCC fails even if Y_t^{net} is observed. Lemoine (2024) highlights this finding in a long-run steady state, explicitly observing Y_t^{net}. In the present model, the finding follows from comparing equations (13) and (8): *the observed variation of Y_t^{net} with temperature omits the future benefits from current investments into adaptation capital ($1 - \beta(1 - \delta_A)$), overestimating the social cost.*

Using the envelope theorem for the identification of damages. The envelope theorem is usually invoked when adaptation expenditures are not directly observed. The key step is to recognize that $\frac{\partial y_t}{\partial T_t} = \frac{\partial y_t^{net}}{\partial T_t}$: we are isolating the direct effect of temperature on damages while holding adaptation constant. Thus, equation (10) links the observable variation $\frac{\partial y_t}{\partial T_t}$ in total output y_t to the generally unobservable variation in y_t^{net}:

$$\frac{dy_t^{net}}{dT_t} = \frac{\partial y_t^{net}}{\partial T_t} = \frac{\partial y_t}{\partial T_t}.$$

At first sight, this identity seems to offer a way to identify unobserved adaptation costs, at least for flow adaptation. However, observing the partial derivative $\frac{\partial y_t}{\partial T_t}$ is itself a challenge.

[17] In standard national accounts, households' purchases of air-conditioning equipment enter GDP as final consumption. Government construction of a dam is treated as the production of a fixed asset, contributing as gross fixed capital formation; similarly, research and development on drought-resistant crops is treated as the creation of an intellectual property asset and recorded as fixed capital formation, entering GDP. Thus, while the envelope theorem crucially relies on optimizing Y_t minus defensive expenditure (so Y_t^{net}), most of the defensive expenditure is recorded in GDP, making it closest to Y_t.

[18] Several points I make here in the static context have also been emphasized by Mérel et al. (2024).

In a deterministic model, a rational forward-looking agent adapts perfectly to temperature. As a result, the unadapted variation, $\frac{\partial y_t}{\partial T_t} = -\frac{\partial d(\tau_t, i_t, A_t)}{\partial \tau_t}\frac{\partial \tau_t}{\partial T_t}$, remains unobservable. *What the econometrician observes is* $\frac{dy_t}{dT_t}$*, which incorporates adaptive responses to temperature but excludes adaptation costs, whereas the envelope theorem requires* $\frac{\partial y_t}{\partial T_t}$, which excludes both adaptive responses and adaptation costs.

Still, the result suggests once again that an unanticipated shock could make the partial variation $\frac{\partial y_t}{\partial T_t}$, and thus $\frac{dy_t^{net}}{dT_t}$, observable. Such shocks could arise either from genuinely unexpected events (see below) or from expected but random weather fluctuations (a conjecture I examine in Section 3).

Dynamic envelope theorem and social cost. I now turn to a dynamic version of the envelope theorem. The preceding discussion treated losses in net output as the object of interest. Here, I adopt a welfare-theoretic perspective and derive the observable object of interest explicitly. To keep the structure transparent, I restrict attention to the essential state variables—temperature and adaptation capital—while dropping standard capital (Section 5 extends to a more general setting). This dynamic setting is most naturally handled using the Bellman equation

$$V(T_t, A_t) = \max_{i_t} U(C_t, T_t, i_t, A_t) + \beta V(T_{t+1}, A_{t+1}).$$

The arguments of the present period utility $U(\cdot)$ capture the dependencies of the underlying production system. For the optimized problem, the social cost of a marginal temperature increase in welfare units is

$$\frac{\partial V(T_t, A_t)}{\partial T_t} = \overbrace{\frac{\partial U(C_t, T_t, i_t, A_t)}^{\text{observed: } \frac{dU}{dT}}}{\partial T_t} + \underbrace{\frac{\partial U(C_t, T_t, i_t, A_t)}{\partial i_t}\frac{\partial i_t^*}{\partial T_t}}_{=0 \text{ by the envelope theorem/FOC for adaptation investments}} + \beta\frac{\partial V(T_{t+1}, A_{t+1})}{\partial A_{t+1}}\frac{\partial A_{t+1}}{\partial i_t}\frac{\partial i_t}{\partial T_t}$$

$$+\beta\frac{\partial V(T_{t+1}, A_{t+1})}{\partial T_{t+1}}\frac{\partial T_{t+1}}{\partial T_t} \quad = \quad \sum_{l=t}^{\infty} \beta^{l-t}\frac{\partial U(C_l, T_l, i_l, A_l)}{\partial T_l}\frac{\partial T_l}{\partial T_t}\,, \tag{14}$$

where the last equality recursively substitutes the value function (assuming well-behaved convergence). The term $\frac{\partial T_l}{\partial T_t}$ abbreviates temperature dynamics and the persistence of a current temperature "shock". Equation (14) illustrates the same dilemma and hope as in the static case. The partial variation of utility in temperature, holding adaptation constant, would be enough to calculate the full welfare cost of a change in temperature $\frac{\partial V(T_t, A_t)}{\partial T_t}$. Unfortunately, in a deterministic world, we do not observe this partial variation because agents can adapt to temperature realizations.

Returning to the present model's functional forms, translating the marginal utility loss

into its observable measure, I find[19]

$$\begin{aligned}\left.\frac{\partial U(C_t,T_t,i_t,A_t)}{\partial T_t}\right|_{i_t=i_t^*(T_t,A_t)} &= -\left.\frac{\partial d(\tau_t,i_t,A_t)}{\partial \tau_t}\frac{\partial \tau_t}{\partial T_t}\right|_{i_t=i_t^*(T_t,A_t)} \\ &= \Big[\xi_0+\eta\frac{\lambda}{\chi}\Big(1-\beta(1-\delta_A)\Big)\Big]\xi_1\tau_t\end{aligned} \tag{15}$$

As suggested by the dynamic envelope theorem, the term in square brackets captures the social cost of a temperature increase. This same term also appears as the central damage component of the SCC in Proposition 2. When calculating the social cost of *carbon*, the proposition adds terms for temperature and carbon dynamics, and it translates the welfare loss from utility into consumption units (USD).[20] In connection with Proposition 2, the last line of equation (15) confirms that the envelope argument can, in principle, identify the correct (and non-zero) cost of adaptation. Crucially, however, equation (15) takes the partial derivative at a given adaptation level, *before* inserting the optimal adaptation level. Proposition 1 shows that this required variation is not observable: the econometrician only sees the total (post-adaptation) variation, not the partial variation needed to identify the separate components of the damage function. Within a deterministic setting, therefore, the required information cannot be recovered.

Surprise. An unanticipated shock—one the agent has not adapted to—offers a direct observation of the partial derivative of the damage function in equation (3), with A_t and i_t fix. Generalizing the reasoning at the end of Section 2.2, equations (14) and (15) suggest once more that the adaptation-cost-inclusive damages of climate change can be identified from a surprise weather shock. This "surprise identification strategy" does not require direct observation of adaptation expenditures: observed damages in a surprise reflect both the marginal impact under adaptation and the impact of not having adapted, the latter corresponding to the persistence-weighted optimal cost of adaptation.

Interestingly, *in a dynamic setting with persistent adaptation measures, it is only this indirect observation of adaptation costs that correctly identifies their contribution to the social cost of a temperature increase.* As shown in equation (13), directly observing adaptation expenditures misattributes their long-term benefits, thereby overstating their contribution to the social cost.

3 Uncertain Climate Evolution and Temperature Shocks

Up to this point, temperatures have evolved deterministically, allowing agents to adapt optimally to realized levels. The preceding section suggested that unexpected variations may help

[19]The first equality uses equation (C.1) from Appendix C.1.

[20]For the SCC, the multiplier $\tau_t = \exp[\xi_1 T_t]$ is effectively neutralized by the logarithmic temperature response to carbon concentrations (see the radiative forcing equation (29) in Appendix A.1).

identify adaptation-cost-inclusive damages.

In reality, agents anticipate random weather shocks and adapt to their statistical distribution. This section analyzes the implications of such anticipated fluctuations for adaptation and observed damages, while also introducing uncertainty about longer-term climate trends. Appendix A.2 summarizes the stochastic temperature dynamics, following Traeger (2021). Here, I introduce the objective function under uncertainty, outline the timing of events, and derive the implications for optimal adaptation investment and observed damages.

3.1 Objective Function and Timing of Decisions

Forward-looking behavior under uncertainty is naturally captured by Bellman's dynamic programming equation. Instead of optimizing over an infinite horizon directly, the problem reduces to finding the value function $V(\cdot)$, which summarizes future discounted welfare as a function of current economic and climatic states.

In the present model, these states include: exponential temperatures in the atmosphere and ocean layers $\boldsymbol{\tau}_t$, carbon concentrations $\boldsymbol{M}_t$, remaining resources $\boldsymbol{R}_t$, adaptation capital A_t, aggregate capital K_t, and informational state variables $\boldsymbol{I}_t$ (e.g. lagged shocks). Time t also enters as a state variable to allow for exogenous trends. The Bellman equation is

$$\begin{aligned} &V(K_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, A_t, \boldsymbol{I}_t, t) \\ &= \max_{\boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t, i_t} \mathbb{E}_t \Big(\log C_t + \beta V(K_{t+1}, \boldsymbol{\tau}_{t+1}, \boldsymbol{M}_{t+1}, \boldsymbol{R}_{t+1}, A_{t+1}, \boldsymbol{I}_{t+1}, t+1) \Big). \end{aligned} \tag{16}$$

For notational simplicity, I denote the atmospheric temperature realized at the end of period t by $\tau_t \equiv \boldsymbol{\tau}_{1,t+1}$, although it occurs after decisions at time t. This convention keeps damages expressed as functions of τ_t, consistent with the deterministic setting, and avoids cluttering expressions with t+1 indices. The decision maker chooses labor allocation $\boldsymbol{N}_t$, sectoral capital $\boldsymbol{K}_t$, energy inputs $\boldsymbol{E}_t$, and adaptation investment i_t.[21] Log-utility ($\log C_t$) captures the current consumption benefit, while $\beta V(\cdot)$ represents expected future welfare.

A key feature of equation (16) is the placement of the expectation operator $\mathbb{E}_t$. Here, expectations apply to the entire right-hand side and follow the maximization operator, reflecting that agents make their consumption and adaptation choices before the realization of temperature τ_t. Damages in equation (3) thus depend on stochastic temperatures, and agents adapt only to their distribution. By assuming decisions precede realizations, we capture the empirically observed costs of heat waves, mortality spikes, and productivity or crop losses when adaptation is imperfectly matched to weather realizations.

[21] Aggregate capital $K_t = \sum_i K_{i,t}$ determines investment and thus consumption. Appendix A.2 splits this into a consumption (and investment) rate and a sectoral capital distribution.

3.2 Optimal Investment and Observed Damages

The following proposition characterizes optimal adaptation investment and the resulting expected damages under uncertainty.

Proposition 3 *The optimal investment into adaptation capital is*

$$i_t^* = \frac{1}{\chi}\Big[\log(\rho) - \log\Big(\frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\Big) + \underline{\mathrm{CGF}_{\tau_t}(\lambda)}\Big] - A_t. \tag{17}$$

where $\mathrm{CGF}_{\tau_t}(\lambda)$ *denotes the cumulant generating function of the conditional distribution of* τ_{t+1}*. The resulting optimally adapted expected damages are*

$$\mathbb{E}_t\, d(\tau_t, i_t^*, A_t) = \xi_0(\mathbb{E}_t\,\tau_{t+1} - 1) + \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\ . \tag{18}$$

Optimal investment thus follows the same structure as under certainty, with one crucial difference: instead of responding directly to realized temperature, it depends on the entire distribution of future temperatures, summarized by the cumulant generating function

$$\mathrm{CGF}_{\tau_t}(\lambda) = \sum_{n=1}^{\infty} \frac{\kappa_n}{n!}\lambda^n = \lambda\,\mathbb{E}_t\,\tau_t + \frac{1}{2}\lambda^2\,\mathbb{V}\mathrm{ar}_t(\tau_t) + \frac{1}{6}\lambda^3\,\mathbb{S}\mathrm{kew}(\tau_t) + \ldots$$

where κ_n denotes the n-th cumulant of τ_t, conditional on information at t.[22] The first cumulant is the expected value, the second the variance, the third captures skewness, and higher-order cumulants (e.g. excess kurtosis) convey information about tail risk.

If higher-order moments are ignored, investment under uncertainty responds to expected temperature in the same way as in the deterministic case. But once uncertainty is taken seriously, variance, skewness, and fat tails of the climate distribution can significantly alter optimal adaptation, with direct implications for observed damages.

Equation (18) shows that optimally adapted expected damages under uncertainty take the same form as under certainty, with realized temperature replaced by its expected value. As in the deterministic model, this implies that expected damages exhibit no observable variation. This stark result arises from the specific functional forms chosen and is didactically helpful, though not general: the decision-maker offsets higher expected damages through greater adaptation, leaving adapted damages unchanged in expectation. Fortunately, we observe realized rather than expected temperatures.

Proposition 4 *Observed damages at a given temperature realization* τ_t *are*

$$d(\tau_t, i_t^*, A_t) = \xi_0(\tau_t - 1) + \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\exp\left(\lambda\tau_t - \mathrm{CGF}_{\tau_t}(\lambda)\right). \tag{19}$$

[22] The cumulant generating function is the logarithm of the moment generating function, $\mathrm{CGF}_{\tau_t}(\lambda) \equiv \log\left[\mathbb{E}_t\, e^{\lambda\tau_{t+1}}\right]$, which allows for a convenient expansion in terms of moments. Higher-order cumulants differ slightly in normalization from central moments.

Equation (19) highlights both the promise and the challenge of identification through surprise variation. Under certainty, the cumulant generating function simplifies to $\mathrm{CGF}_{\tau_t}(\lambda) = \lambda \mathbb{E}_t \tau_t = \lambda \bar{\tau}_t$, so equation (19) collapses to the deterministic benchmark in equation (9). This corresponds to the setting where the agent is unaware of weather fluctuations. By contrast, once agents anticipate fluctuations, an additional factor $\exp(-\lambda^2 \mathbb{V}\mathrm{ar}(\tau_t))$ raises adaptation investment in anticipation of occasional adverse realizations, and positive skewness or fat tails further increase adaptation, reducing observed damages.

Neglecting this anticipation—that is, treating agents as if they were unaware of fluctuations—would misinterpret equation (19) as equivalent to equation (9), ignoring the role of $\mathrm{CGF}_{\tau_t}(\lambda)$. The resulting inference would underestimate true costs of climate change by mistaking the (indirectly) observed adaptation costs for lower damages.

4 Identification of adaptation-inclusive damages

Excellent reviews of the empirical work identifying climate impacts include Hsiang (2016), Auffhammer (2018), Hogan & Schlenker (2024), Lemoine et al. (2025), and Hsiang (2025). This section develops a framework that connects econometric identification with the structural lens of the ACE model. The goal is to clarify what existing regressions capture, what remains hidden, and how observed variation in weather realizations can be used to identify adaptation-cost-inclusive damages.

4.1 Empirical approach in the literature

The most prominent strand of work seeking to incorporate unobserved adaptation costs follows an approach suggested by Butler & Huybers (2013) and Schlenker et al. (2013), and further developed by Carleton et al. (2022). This literature typically estimates versions of a panel regression of the form

$$\log Y_{jt} = \alpha_j + \underbrace{\left(\boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1 \bar{T}_j + \text{“further controls”}\right)}_{\equiv \boldsymbol{\beta}_j^\dagger} \boldsymbol{f}(T_{jt}) + \boldsymbol{\zeta} \boldsymbol{X}_{jt} + \delta_t + \epsilon_{jt} \tag{20}$$

where j indexes geographical units. I use log-output as the dependent variable, remaining consistent with the paper's focus on linking to the IAM literature and the SCC. The same reasoning extends to crop yields, mortality, and other outcomes examined in the climate impacts literature.

The specification in equation (20) includes location fixed effects α_j to capture time-invariant heterogeneity across regions, year dummies δ_t (or, in more refined settings, country-year fixed effects) to absorb common shocks, and a vector of controls $\boldsymbol{X}_{jt}$ for time-varying observables. The coefficient $\boldsymbol{\beta}_j^\dagger$ can thus be understood as a reduced-form measure of how outcomes respond to weather variation at location j, conditional on these controls. In this

sense, $\boldsymbol{\beta}_j^\dagger$ is the core empirical object summarizing the net effect of weather realizations on the outcome of interest. Many climate-economy panels pool a common weather response $\boldsymbol{\beta}^\dagger$. Auffhammer's (2022) CARE approach uses the above regression as a first stage to then decompose $\boldsymbol{\beta}_j^\dagger$ in the second stage.

To connect with adaptation, the literature decomposes $\boldsymbol{\beta}_j^\dagger$ into a baseline sensitivity $\boldsymbol{\gamma}_0$, an interaction with average climate $\boldsymbol{\gamma}_1 \bar{T}_j$, and additional controls not relevant for the present analysis. The interaction term $\boldsymbol{\gamma}_1$ is of particular interest, as it is used to identify systematic cross-sectional differences in adaptation based on differences in mean temperature. In this framework, T_{jt} is treated as a sufficient statistic for the weather realization over the relevant period (e.g. one year). Carleton et al. (2022) implement $\boldsymbol{f}(T_{jt})$ as a polynomial in daily average temperatures aggregated to the annual level, though other specifications are possible, such as counts of hot or cold days or temperature bins. The crucial point is that these realizations reflect *weather*, which is stochastic, whereas *climate* corresponds to the distribution of such realizations. The ultimate objective is to use observed weather impacts to infer how damages change as the climate distribution shifts.

The appropriate aggregation period depends on seasonality, data availability, and the timeframe of adaptation. For outcomes such as crop yields, AC adoption, or log output, annual weather statistics provide a natural basis.[23] Applying the regression framework to an economy governed by ACE's equations, the empirical specification seeks to recover

$$\log Y_{jt} = f_{jt} \overbrace{\underbrace{-\xi_0}_{\stackrel{?}{=}\gamma_{0,1}} \tau_{jt} \underbrace{-\frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\exp\big(-\operatorname{CGF}_{\tau_{jt}}(\lambda)\big)}_{\stackrel{?}{=}\gamma_{0,2}+\gamma_{1,2}\bar{T}_j}\exp(\lambda\tau_{jt})}^{\boldsymbol{\beta}_j^\dagger \boldsymbol{f}(T_{jt})}, \tag{21}$$

where f_{jt} collects production- and capital-related terms (not to be confused with the vector $\boldsymbol{f}$ describing weather), and where I propose using a two-dimensional weather vector to capture the two distinct damage slopes (see below).[24] The adaptable damage component corresponds to two terms: a component of $\boldsymbol{\gamma}_0$ measuring damages in the absence of adaptation, and a component of $\boldsymbol{\gamma}_1$ capturing the reduction in damages due to adaptation.

4.2 Conceptual identification

Unobserved components of f_{jt} are assumed independent of the regressors. In addition, I take two strong assumptions helping identification that I will re-evaluate and relax further below.

[23] Some adaptation measures respond to much shorter events: a handful of hot days or even hours can trigger costly behaviors such as staying indoors or seeking shelter, thereby reducing mortality. Using annual sufficient statistics can capture part of these shorter-term costs, but might fail to condition short-term variation information available at shorter time scales.

[24] The Appendix shows that $f_{jt} = \kappa \log K_t + \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) - \eta i_t - \xi_0$. This additive separability of income and capital follows from log utility and from damages being proportional to output.

Assumption 1 *The vector $\boldsymbol{f}(T)$ approximates the temperature nonlinearity correctly:*

$$\boldsymbol{f}(T) = \begin{bmatrix} \tau \\ \exp(\lambda\tau) \end{bmatrix} \text{ resulting in, e.g., } \quad \boldsymbol{\beta}_j^\dagger \boldsymbol{f}(T) = \beta_{j,1}^\dagger \tau + \beta_{j,2}^\dagger \exp(\lambda\tau) \ .$$

Assumption 1 allows the regression specification to capture the relevant nonlinearities in realized temperature effects. While selecting the appropriate functional form is empirically challenging, this assumption helps clarify a few conceptual points at the outset. As noted in Section 2, the τ-scale provides a convenient transformation of temperature on which marginal damages are constant. For now, I seek to recover precisely these constant marginal damages and, to keep the exposition focused, analyze variation in τ rather than in degrees Celsius. I will discuss varying temperatures in degrees Celsius in a remark further below.

Assumption 2 *Climate is constant over the estimation horizon or agents fail to adapt to ongoing climatic change.*

Assumption 2 is useful for understanding the approach taken in the present literature, but, as I will show, it is neither crucial nor particularly helpful for estimating climate damages.

Overall cost coefficient. Under Assumptions 1 and 2, the scalar product $\boldsymbol{\beta}_j^\dagger \boldsymbol{f}(T_{jt})$ captures the overbraced component of the regression equation (21). Because $\boldsymbol{f}(T_{jt})$ is nonlinear, essential information for interpreting the estimated damage coefficient $\boldsymbol{\beta}_j^\dagger$—for example, the steepness parameter λ—is embedded in the co-estimated function $\boldsymbol{f}$. Calculating (regional) expected damages from the resulting expression results in

$$\begin{aligned} -\mathbb{E}_j\, \boldsymbol{\beta}_j^\dagger \boldsymbol{f}(T_{jt}) &= \xi_0\, \mathbb{E}_j\, \tau_{jt} + \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big) \exp\big(-\operatorname{CGF}_{\tau_{jt}}(\lambda)\big)\, \mathbb{E}_j \exp(\lambda\tau_{jt}) \\ &= \xi_0\, \mathbb{E}_j\, \tau_{jt} + \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big) = d(\tau_t, i_t^*, A_t) + \xi_0, \end{aligned} \tag{22}$$

The empirical specification misses equation (19)'s constant term "$-\xi_0$," which equation (20) absorbs into the additive constant.[25] Fortunately, this missing constant will not matter once we process the regression coefficients to calculate the cost of climate *change*, as shown below and more generally in Section 5.

As noted in Section 3.2, it would be misleading to take the derivative of expected damages to obtain expected marginal damages: in equation (22), the adaptable damage component is independent of temperature and would drop out. Moreover, the steepness parameter λ, which is central to the SCC, does not appear in equation (22). The intuition that "surprise

[25] This issue is closely related to the fact that $\boldsymbol{\beta}_j^\dagger \boldsymbol{f}(T_{jt})$ is not invariant to transformations of the temperature scale, a topic I will revisit later. Here, τ_{jt} is normalized to unity at the preindustrial temperature. The regression would recover the correct expression if τ_{jt} were normalized to zero at the preindustrial temperature. Because the regression requires a constant to absorb confounding factors, the damage intercept is generally not identified.

variation" helps identification suggests taking derivatives before taking expectations:[26]

$$\frac{\partial \boldsymbol{\beta}_j^\dagger \boldsymbol{f}(T_{jt})}{\partial \tau_{jt}} = -\left[\xi_0 + \frac{\eta}{\chi}\lambda\big(1-\beta(1-\delta_A)\big)\exp\big(-\operatorname{CGF}_{\tau_{jt}}(\lambda)\big)\exp(\lambda\tau_{jt})\right]$$

$$\Rightarrow \mathbb{E}_{jt}\frac{\partial \boldsymbol{\beta}_j^\dagger \boldsymbol{f}(T_{jt})}{\partial \tau_{jt}} = -\left[\xi_0 + \frac{\eta}{\chi}\lambda\big(1-\beta(1-\delta_A)\big)\right]. \tag{23}$$

Taking the derivative w.r.t. temperature variation reintroduces the "missing" damage steepness parameter λ (first line). The parameter derives not from the regression coefficient itself but from the estimated nonlinear function $\boldsymbol{f}(T_{jt})$, highlighting the informational split between $\boldsymbol{\beta}_j^\dagger$ and $\boldsymbol{f}$. I emphasize that equation (23) is derived within each period. If Assumption 2 is satisfied, climate and adaptation remain constant over the estimation horizon, and the term $\exp\big(-\operatorname{CGF}_{\tau_{jt}}(\lambda)\big)$ reduces to a multiplicative constant. Yet climate has been changing rapidly in recent decades. *If agents adapt to these changes,* $\operatorname{CGF}_{\tau_{jt}}(\lambda)$ *itself will vary, making it essential to* ***control for*** *some of the* ***climate moments*** *in the regression, a point I emphasize here and illustrate numerically in the next section.*

In the second line of equation (23), taking expectations delivers the core damage expression of the SCC. Proposition 5 below confirms the central role of this expression in determining social costs, also for the stochastic world. Taking expectations eliminates the "non-surprise component" $\exp(-\operatorname{CGF}_{\tau_{jt}}(\lambda)) = \exp(-\sum_{n=2}^{\infty}\frac{\kappa_n^{jt}}{n!}\lambda^n)$. This finding is no coincidence: agents' adaptation decisions already account for the foreseeable part of stochastic variation. In the first-order condition, damage convexity augments expected damages by $\exp(\operatorname{CGF}_{\tau_{jt}}(\lambda)) = \mathbb{E}_{jt}\exp(\lambda\tau_{jt})$, which in turn augments adaptation and reduces damages. Here, these two uncertainty corrections cancel, so the regression can, in principle, identify adaptation-inclusive marginal damages. Most importantly, *a standard regression of log output on weather will generally identify* ***adaptation-inclusive*** *marginal damages—not because adaptation costs are observed, but because the regression captures variation that agents cannot adapt to within a single period.*

Remarks:

1. Whereas the expected marginal damages in equation (23) form the core of the social cost expressions, the full derivation of the social cost is generally more involved. Because marginal damages interact with evolving climate moments before expectations are taken (first line of equation 23), the resulting expressions often depend on correlations between marginal damages and climate change, especially under endogenous learning or persistent climate shocks. See Section 5 and Theorem 1 for a general discussion.[27]

2. A closely related argument applies when returning to a temperature variation T_{jt} measured

[26] Here the convenience of working on the τ-scale becomes apparent. The expression implicitly uses $\frac{\partial T_{jt}}{\partial \tau_{jt}} = \frac{\partial}{\partial \tau_{jt}}\frac{1}{\xi_t}\log(\tau_{jt}) = \frac{1}{\xi_t \tau_{jt}}$. Regarding the difference to deriving equation (22), I note that Leibniz' rule about interchanging expectation (integration) and derivatives does not apply to differentiation with respect to the random variable being integrated over.

[27] For the present case, Footnote 50 in Section 5.1 illustrates how the interaction between shifting climate

in degrees Celsius. The first line of equation (23) then picks up the scale transformation $\frac{\partial \tau_{jt}}{\partial T_{jt}} = \xi_1 \exp(\xi_1 T_{jt})$. The expectation of this scale transformation adds a risk premium through Jensen's inequality. Section 4.4's Proposition 5 characterizes this premium (denoted $\epsilon(\cdot)$).

3. Assumption 1 is particularly strong, as it requires that the nonlinearity parameter λ be correctly recovered. In a linear regression framework, this implies that the econometrician has included the correct functional form in the basis. Explicitly identifying λ, however, transforms the problem into a nonlinear regression, generally a more challenging task.

Singling out indirectly observed adaptation costs. For some applications, it is important to distinguish adaptation costs from overall damages/costs. For example, Carleton et al. (2022) analyze the differential impact of temperature on mortality across regions and Hultgren et al. (2025) examine indirectly observed adaptation costs in the agricultural sector. The remainder of this section examines the challenge of decomposing overall damages into its components and, in particular, separating out adaptation costs.

For this task, the literature analyzes the impact of mean temperature on the outcome variable, interpreting mean temperature as a proxy for the level of adaptation: on a (similary) hot day, damages in a generally hotter region will be lower due to better adaptation. I note one caveat with a theoretically suggested identification strategy which is bound to fail: first taking expectations over $\boldsymbol{\beta}_j^\dagger \boldsymbol{f}(T_{jt})$ and then differentiate the resulting expression with respect to mean temperature $\bar{T}_j$. As shown in equation (22), the expectation operator already eliminates the adaptation component. Consequently, the derivative does not recover adaptation costs but instead identifies only the non-adaptable component of damages.[28]

This theoretical caveat has not been central to empirical approaches such as Carleton et al. (2022), which decompose the regression coefficient $\boldsymbol{\beta}_j^\dagger$ into $\boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1 \bar{T}_j$. Before turning to this empirical strategy, I clarify that the correct procedure mirrors the logic used above to recover overall costs: first isolate variation in mean temperature (reflecting climate) and only then evaluate expectations,

$$\frac{\partial \boldsymbol{\beta}_j^\dagger \boldsymbol{f}(T_{jt})}{\partial \bar{\tau}_{jt}} = \frac{\eta}{\chi} \lambda (1 - \beta(1 - \delta_A)) \exp\big(-\operatorname{CGF}_{\tau_{jt}}(\lambda)\big) \exp(\lambda \tau_{jt}), \tag{24}$$

$$\Rightarrow \mathbb{E}_j \frac{\partial \boldsymbol{\beta}_j^\dagger \boldsymbol{f}(T_{jt})}{\partial \bar{\tau}_{j,t}} = \frac{\eta}{\chi} \lambda (1 - \beta(1 - \delta_A)).$$

Keeping a time index on the mean-temperature variation $\bar{\tau}_{jt}$ allows the climate to change over time. While the literature generally assumes a constant climate, the next section suggests exploiting time variation in climate for identification purposes. Under either the literature's

moments and the nonlinear marginal damages of the adaptable damage component generates an additional risk premium, denoted SCC^{ao} in Proposition 5's SCC formula.

[28] Taking the derivative w.r.t. $\bar{\tau}_j = \mathbb{E}_j \tau_{jt}$ on the right-hand side of equation (22) returns ξ_0. Differentiating instead w.r.t. $\bar{T}_j = \mathbb{E}_j T_{jt}$ approximately gives $\xi_0 \xi_1$, as can be seen from the Taylor expansion $-\mathbb{E}_{jt} \boldsymbol{\beta}_j^\dagger \boldsymbol{f}(T_{jt}) \approx \xi_0 \xi_1 \bar{T}_j + \xi_0 \xi_1^2 \operatorname{Var}(T_j) + \ldots + \frac{\eta}{\chi}(1 - \beta(1 - \delta_A))$.

somewhat restrictive Assumption 2, or by explicitly using time variation in climate, equation (24) promises to recover a crucial damage component associated with adaptation costs. Again, I effectively eliminated the risk premium by varying (mean) temperature on the τ-scale (see Remark 2 above).

From the optimal investment decision, we know that varying only the mean cannot capture the full marginal adaptation costs, since optimal investments also respond to temperature variance and higher moments (Proposition 3). Let $\Omega \equiv \frac{\eta}{\chi}(1-\beta(1-\delta_A))$ and let $\boldsymbol{m}_t = (m_{1,t}, m_{2,t}, m_{3,t}, \dots)$ be the vector of moments characterizing the climate (weather distribution). Using the analytic model, I can explicitly calculate the full marginal adaptation benefits under uncertainty experienced in period t:

$$\frac{\partial y_t}{\partial i_t^*(\cdot)\boldsymbol{m}_t}\frac{\partial i_t^*(\boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} = \Omega\Big(\lambda\bar{\tau}_{j,t} + \frac{\lambda^2}{2}\,\mathbb{V}\mathrm{ar}_t(\tau_{j,t}) + \frac{\lambda^3}{6}\,\mathbb{S}\mathrm{kew}(\tau_{j,t}) + \dots\Big)\exp\big(\lambda\tau_{jt} - \mathrm{CGF}_{\tau_{jt}}(\lambda)\big)\,.$$

Equation (24) recovers the benefits of adapting to the mean climate.[29]

The literature's practical approach identifies unobserved adaptation costs with $\boldsymbol{\gamma}_1\,\mathbb{E}\,\boldsymbol{f}(T_{jt})$ using equation (20)'s interaction coefficient. Neglecting higher-order moments and reducing climate to its mean, the moment generating function simplifies to $\mathrm{CGF}_{\tau_{jt}}(\lambda) \approx \lambda\bar{\tau}_{jt}$. As I will discuss in Section 5.3, this simplification could be viewed as a behavioral bias. In addition, for consistency with the SCC derivation, I analyze the case where I use $\bar{\tau}_{j,t}$ in equation (20) instead of $\bar{T}_j$. Using the τ-scale preserves the stronger response to warming at higher temperatures and simplifies the discussion, and preserving the time coefficient allows me to analyze the case of a changing climate. Then, the regression's linear specification in $\bar{\tau}_{jt}$ approximates the nonlinear structural response and the regression picks up the two components from equation (21)

$$\gamma_{1,2} = \frac{\mathrm{Cov}_\star\big(\bar{\tau}_{jt},\, -\Omega e^{-\lambda\bar{\tau}_{jt}}\big)}{\mathrm{Var}_\star(\bar{\tau}_{jt})}, \qquad \gamma_{0,2} = \mathbb{E}_\star\big[-\Omega e^{-\lambda\bar{\tau}_{jt}}\big] - \gamma_{1,2}\,\mathbb{E}_\star[\bar{\tau}_{jt}],$$

where $\mathbb{E}_\star[\cdot]$ and $\mathrm{Cov}_\star(\cdot,\cdot)$ are residualized moments after partialling out other regressors (Frisch–Waugh–Lovell). Because the true coefficient $\Omega e^{-\lambda\bar{\tau}_{jt}}$ is nonlinear, its curvature is projected onto the linear basis $\{1, \bar{\tau}_{jt}\}$, so the adaptation-cost component is generally split across both $\gamma_{0,2}$ and $\gamma_{1,2}$.

This finding can also be illustrated by a simple first-order expansion of $\Omega e^{-\lambda\bar{\tau}_{jt}}$ around a

[29] Footnote 50 in Section 5.1 shows how the additional terms also arise when evaluating climate change using the marginal temperature damage (first row of equation 22).

reference point $\bar{\tau}_{\text{ref}}$, which results in[30,31]

$$\Omega e^{-\lambda\bar{\tau}_{jt}} f_2(T_{jt}) \approx \underbrace{\Omega\, e^{-\lambda\bar{\tau}_{\text{ref}}}\big(1+\lambda\bar{\tau}_{\text{ref}}\big)}_{-\gamma_{0,2}} f_2(T_{jt}) - \underbrace{\lambda\,\Omega\, e^{-\lambda\bar{\tau}_{\text{ref}}}}_{\gamma_{1,2}}\, \bar{\tau}_{jt} f_2(T_{jt}).$$

Rewriting the final term as a coefficient on mean temperature yields the familiar expression $\gamma_{1,2} f_2(T_{jt}) = \lambda\frac{\eta}{\chi}\lambda(1-\beta(1-\delta_A))\exp(\lambda\bar{\tau}_{jt}-\bar{\tau}_{\text{ref}})$. Given the approximation $\text{CGF}_{\tau_{jt}}(\lambda) \approx \lambda\bar{\tau}_{jt}$ adopted above, the resulting expression for $\gamma_{1,2} f_2(T_{jt})$ matches equation (24). It correctly identifies how unobserved adaptation costs vary with mean temperature, evaluated at the realized temperature shocks $f_2(T_{jt}) = \exp(\lambda\tau_{jt})$ and the reference climate $\bar{\tau}_{\text{ref}}$.[32,33]

Because the regression equation (21) linearizes in mean temperature, $\gamma_{1,2} f_2(T_{jt})$ directly captures the *marginal* costs of adaptation. At the same time, it is crucial for the approach discussed here that the regression captures the nonlinear response in realized temperatures. If the regression were linear in realized temperature as well, the interaction coefficient would merely approximate the cross-partial derivative rather than the marginal costs of adaptation. Then, a substantial part of the marginal adaptation costs would be absorbed by the unidentified fixed effects of the panel regression.[34] Because it is difficult to estimate this nonlinearity correctly (see Remark 3), the next section explores an alternative approach that exploits time series variation in climate to identify unobserved adaptation costs in a simpler linear regression.

[30] A natural reference is the weighted mean of $\bar{\tau}_{jt}$ under the residualizing metric implied by Frisch–Waugh–Lovell. Assuming $\lambda \ll 1$ and modest dispersion of $\bar{\tau}_{jt}$ around $\bar{\tau}_{ref}$ implies $\Omega e^{-\lambda\bar{\tau}_{jt}} \approx \Omega e^{-\lambda\bar{\tau}_{ref}} - \lambda\Omega e^{-\lambda\bar{\tau}_{ref}}(\bar{\tau}_{jt}-\bar{\tau}_{ref})$.

[31] Alternatively, if I assume that the residualized climate mean is approximately normally distributed ($\bar{\tau}^{\star}_{jt} \sim \mathcal{N}(\mu_\star, \sigma^2_\star)$, Stein's lemma delivers similar expressions using the formulas above:

$$\gamma_{1,2} = \lambda\, E^{\star}\big[\Omega e^{-\lambda\bar{\tau}_{jt}}\big], \qquad \gamma_{0,2} \;=\; -\big(1+\lambda\, E^{\star}[\bar{\tau}_{jt}]\big)\; E^{\star}\big[\Omega e^{-\lambda\bar{\tau}_{jt}}\big].$$

[32] If the function f_2 were shifted to $f_2 = \exp(\lambda(\tau+c))$, e.g. to reflect a shifted temperature scale $\tau' = \tau + c$, the coefficient $\gamma_{1,2}$ would absorb the additional multiplier $\exp(\lambda c)$, and $\boldsymbol{\gamma}_1$ alone would no longer correspond exactly to $\lambda\Omega$. Nevertheless, the *joint* expression $\boldsymbol{\gamma}_1 \boldsymbol{f}(\cdot)$ remains invariant and continues to represent marginal adaptation costs at a given temperature exposure.

[33] The direct estimation of $\boldsymbol{\beta}^{\dagger}_j$ already nets out the reduction $\gamma_{1,2}$ from adaptation captured by $\gamma_{0,2}$ or, rather, the coefficient $\gamma_{0,2}$ in this approach is augmented by the adaptation component proportional to climate (mean temperature). This does not mean that $\boldsymbol{\beta}^{\dagger}_j$ misses adaptation, it picks up precisely the costs required for identifying the SCC.

[34] If the regression is linear in both $\bar{\tau}_{jt}$ and τ_{jt}, the above approximation becomes

$$\Omega e^{-\lambda\bar{\tau}_{jt}} f_2(T_{jt}) \approx \Omega + \Omega\lambda\tau_{jt} - \Omega\lambda\bar{\tau}_{jt} + \Omega\lambda^2\tau_{jt}\bar{\tau}_{jt},$$

where I expand the exponential function around zero for simplicity. In this case, the interaction coefficient does not generally recover marginal adaptation costs, and a major share of these costs is projected onto the term $\Omega\lambda\bar{\tau}_{jt}$. As explained in the next section, this term cannot be identified in panel regressions unless time variation is explicitly exploited.

4.3 Applied identification: numeric illustration

This section tests, within ACE's model world, how well standard regressions recover overall damages and whether they can isolate unobserved adaptation costs. I use a simple linear specification; the preceding section shows that the model-consistent nonlinear specification recovers the relevant components. I work with a single-region time series and directly estimate simulated observable damages d_t to focus on damage identification rather than production covariates; the additivity of damages in log-output justifies this simplification. Damages are generated from equation (19) using continental U.S. annual temperature averages (1975–2014) with Gaussian observation noise (NOAA 2025). Parameters are calibrated so that adaptation costs contribute roughly the same to the SCC as non-adaptable damages, i.e., $\frac{\eta}{\chi}\lambda(1-\beta(1-\delta_A)) \approx \xi_0 \approx 5\%$.[35] This calibration setup also yields $\lambda = 5\%$, suggesting that higher-order climate moments do not play a major role in this example, since their effects are proportional to higher powers of λ.

My main specifications are

$$
\begin{aligned}
&\text{(A)} && d_t = \alpha + \beta_T T_t && + \varepsilon_t,\\
&\text{(A}^*\text{)} && d_t = \alpha + \beta_T T_t + \beta_{\text{int}}\, T_t\mu_t && + \varepsilon_t,\\
&\text{(B)} && d_t = \alpha + \beta_T T_t + \beta_\mu \mu_t && + \varepsilon_t,\\
&\text{(B}^*\text{)} && d_t = \alpha + \beta_T T_t + \beta_\mu \mu_t + \beta_{\text{int}}\, T_t\mu_t && + \varepsilon_t,
\end{aligned}
$$

where $\mu_t \equiv \mathbb{E}[T_t \mid \mathcal{W}_t]$ denotes mean temperature at current climatic information $\mathcal{W}_t$ (a rolling-window climate statistic).

Models (A) and (A*) are linear versions of the standard specifications in the empirical literature. Model (A) corresponds to a simple version of equation (20), capturing weather shocks with coefficient β_T, which corresponds to $-\boldsymbol{\beta}_j^\dagger$ due to estimating damages rather than log-output. Model (A*) adds the interaction with mean temperature ("climate") to proxy adaptation costs, as done in the adaptation-inclusive literature cited at the beginning of this section. My models (B) and (B*) additionally control for mean temperature directly. These variants are well-identified only when climate varies over time; if μ_t were time-invariant in a panel, it would be absorbed by region fixed effects.

The literature sometimes argues that, conditional on realized temperature, mean climate μ_t should have no direct effect on output, collapsing B-type into A-type specifications: climate serves as a proxy for adaptation, which mitigates weather shocks through an interaction with realized temperature. Mérel et al. (2024) argue that μ_t can also carry level adaptation costs

[35] Following Traeger (2025), I scale the non-adaptable damage coefficient upward by 50% to account for omitted sectors, giving $\xi_0 = 0.0493$. With $\lambda = 0.05$, $\eta = 0.5$, $\chi = 0.1$, and $(1-\beta(1-\delta_A)) = 0.8$, the adaptation cost component becomes $\frac{\eta}{\chi}\lambda(1-\beta(1-\delta_A)) = 0.05$. I draw the measurement error from a normal distribution with $\sigma_\varepsilon = 0.0015$. The results are independent of the level multiplier ρ; for instance, $\rho = 4$ yields an equilibrium adaptation stock of about 25% of GDP at current temperatures.

that the standard approach would miss. The ACE framework offers a helpful perspective on these points. First, in the model, adaptation mitigates weather shocks only through the interaction term. Second, nevertheless, adaptation induces level costs. Yet, because adaptation responds to climate rather than to each weather realization, this level effect remains observable using weather variation.[36] The interaction term alone identifies the adaptation response to mean climate, provided the econometric specification matches the nonlinear damage function.

However, in simple linear regressions, or when the nonlinearity is misspecified, controlling for μ_t becomes essential. If the "interaction only" model (A*) represents the data under one temperature normalization (e.g., deviation from a preindustrial baseline), then under any other normalization (e.g., absolute °C) only model (B*) correctly represents the same world. Thus, absent the appropriate nonlinear specification, a climate control is required in linear projections to absorb the mean component induced by normalization; in the linear specifications above, the set of normalizations for which model (A*) alone suffices has Lebesgue measure zero. Finally, in panels where climate (or climate-induced adaptation) is constant over the sample, regional fixed effects absorb μ_t. Although adaptation costs are not separately identified in this case, overall costs remain identified from within-unit temperature variation, since the fixed effects serve as the relevant climate controls.

Figure 3 illustrates the issue of not controlling for mean climate in a weather-based regression. Solid circles plot damages under continuous adaptation, where climate is defined by the first four moments of the temperature distribution over a 30-year window.[37] The dashed blue line shows the perfectly adapted certainty benchmark. Actual damages under uncertainty fluctuate around it, and the fitted black dashed line through the solid circles has a much steeper slope, which the envelope theorem exploits to infer adaptation costs. The slope of the black dashed line corresponds to β_T in regression (A): 2.39% per °C. Translated to the τ-scale,[38] $\beta_\tau = 8.72\%$ implies an 8.72% increase in damages per unit of exponential temperature increase ($e^{0.231 \cdot T_{jt}}$). This slope is steeper than the certainty benchmark (ξ_0) characterizing the blue dashed line, but still below the full marginal damage

$$\beta_\tau^* \equiv \underbrace{\xi_0}_{4.93\%} + \underbrace{\frac{\eta}{\chi}\lambda(1-\beta(1-\delta_A))}_{5\%} = 9.93\%.$$

The gap arises because model (A) omits climate controls and thus climate-responsive adap-

[36] Where perfect adaptation to *weather* is feasible, the level effect is missed; but then μ_t does not capture the corresponding costs either.

[37] The window spans 20 past and 10 future years, assuming agents update expectations using projections; see Kelly & Kolstad (1999), Leach (2007), and Jensen & Traeger (2024) for structural learning models based on Bayesian updating. The figure omits negative temperature realizations. The calibration is reasonable for recent and future temperatures but becomes unrealistic near or below preindustrial levels, where the analytic extrapolation continues to imply rising output as temperatures fall. Negative damages as a percent of GDP correspond to output above that at the 1900 base temperature.

[38] The scale translation uses $\frac{\partial \tau_{jt}}{\partial T_{jt}} = \xi_1 \tau_{jt}$, evaluated at the mean temperature of the full time window. This gives $\xi_0 \tau_{\text{ref}} = 0.0584$, where $\tau_{\text{ref}} = \frac{1}{\xi_1}\log(\bar{T})$.

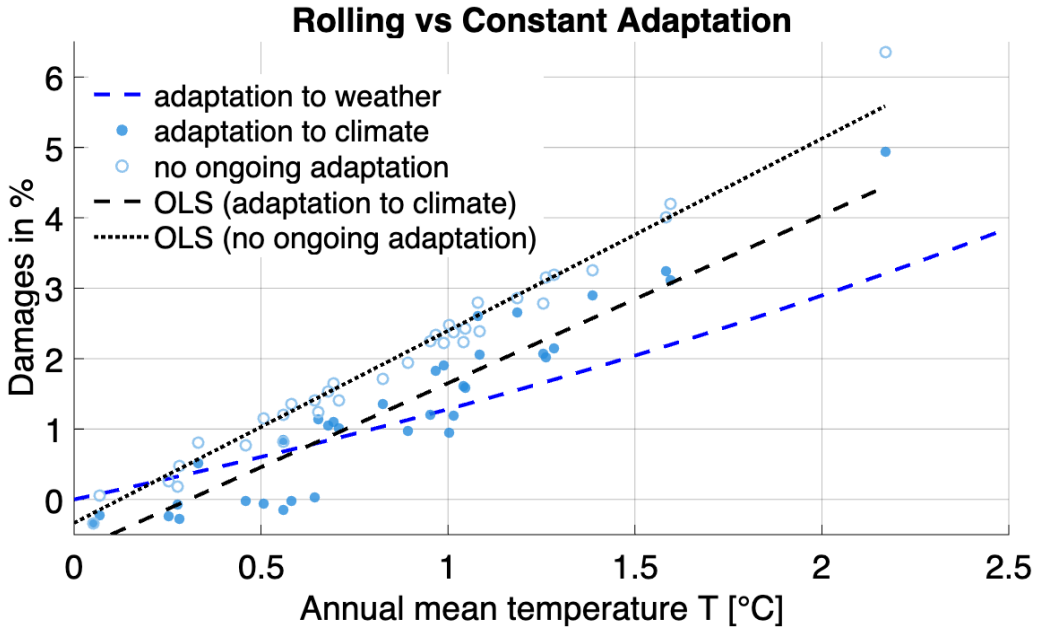


Figure 3: Model-based damages using continental US temperature 1975-2014 measured as increase over 1900 in degrees Celsius, under (i) optimal adaptation to evolving climate (30-year rolling window, solid circles) and (ii) fixed adaptation to initial climate (open circles). Black lines show the corresponding OLS regressions; the dashed blue line illustrates damages under certainty, where agents perfectly adapt to realized temperatures.

tation. To illustrate, the open circles hold adaptation fix at its initial level. Estimating model (A) in this no-adaptation case (black dotted line) yields $\beta_T = 2.73\%$ and, on the τ-scale, $\beta_\tau = 9.97\%$, essentially the true marginal damage despite underlying nonlinearity. Model (A) is therefore informative only when adaptation does not adjust over the estimation horizon; with ongoing adaptation it is biased. The comparison underscores the need to control for climatic moments in weather-based estimation. More broadly, *contrary to the common critique that weather-based estimation overstates damages by ignoring adaptation, using short-run temperature shocks without climate controls tends to understate the costs of climate change when adaptation adjusts with the climate.*[39]

Table 2 extends the regressions to account for climate responsive adaptation and to identify unobserved adaptation costs. Model (B) adds mean temperature as a control and yields an estimated slope $\beta_\tau = 10.2\%$, slightly above $\beta_\tau^* = 9.9\%$. As noted in Section 4.2's first remark and discussed in later sections, this small gap reflects a Jensen-style uncertainty premium. The coefficient on mean temperature, $\beta_\mu = 5.08\%$, aligns with the unobserved adaptation cost component; identifying it requires a changing climate with adaptive responses. Importantly, this is *not* the interaction term between weather and climate used in the literature, but the direct effect of climate; it is only identified because of the time series variation in climate.

[39]Sections 4.4 and 5.1 discuss in detail how marginal damages map into the costs of climate change. The key point is that failing to condition on climate understates the marginal damages from the weather shocks, which are central to this calculation.

Adding variance or skewness in models (C) and (D) does not further improve the estimates.

Table 2: Estimating simulated damages using different regression models

Model	β_T (°C^{-1})	β_τ	β_{μ_T} (°C^{-1})	β_{μ_τ}	β_{int} (°C^{-2})	β_{int}-norm	$\beta_{\text{int}} \times \mathbb{E}\,T$ (°C^{-1})
A: T only	0.023 86	0.087 16					
B: " $+\mu_T$	0.027 95	0.102 11	−0.013 91	−0.050 81			
C: " $+\sigma_T^2$	0.027 95	0.102 13	−0.013 82	−0.050 49			
D: " $+\text{skew}_T$	0.027 95	0.102 11	−0.013 81	−0.050 47			
With interaction $T \times \mu_T$, abbreviated as "..." below							
A*: A + ...	0.041 54	0.151 77			−0.001 20	−0.015 96	−0.013 75
B*: B + ...	−0.041 81	−0.152 73	−0.083 73	−0.305 90	0.006 11	0.081 50	0.070 18
C*: C + ...	−0.058 87	−0.215 09	−0.103 04	−0.376 46	0.007 59	0.101 29	0.087 22
D*: D + ...	−0.059 11	−0.215 95	−0.103 25	−0.377 20	0.007 61	0.101 54	0.087 44
With interactions $T \times \mu_T$ and $T \times \sigma_T$ and $\mu_T \times \sigma_T$, abbreviated as "..." below							
A +...	0.023 09	0.084 35	−0.000 39	−0.001 43	−0.000 39	−0.005 23	−0.004 51
B +...	−0.033 41	−0.122 06	−0.069 87	−0.255 26	0.005 05	0.067 36	0.058 00
C +...	−0.032 05	−0.117 07	−0.070 28	−0.256 75	0.004 93	0.065 75	0.056 62
D +...	−0.032 83	−0.119 95	−0.071 56	−0.261 44	0.005 00	0.066 79	0.057 52
With interactions $T \times \mu_T$ and $T \times \sigma_T$ and $\mu_T \times \sigma_T$ and $T \times \mu_T \times \sigma_T$, abbreviated as "..." below							
A +...	0.027 17	0.099 25	−0.000 74	−0.002 70	−0.000 74	−0.009 88	−0.008 50
B +...	−0.030 07	−0.109 87	−0.068 23	−0.249 25	0.004 75	0.063 36	0.054 56
C +...	0.016 37	0.059 79	−0.018 58	−0.067 88	0.000 54	0.007 19	0.006 20
D +...	0.043 63	0.159 41	0.009 61	0.035 10	−0.001 92	−0.025 59	−0.022 04

Translation to τ-scale uses $\tau_{\text{ref}} = 1.19$ resulting from $T_{\text{ref}} = \mathbb{E}\,T = 11.5$. Only the dark black coefficients are invariant to re-anchoring of the temperature scale.

Table 3: Marginal effects implied by Table 2's coefficient estimates

Model	$\partial D/\partial T$ (°C^{-1})	$\partial D/\partial \tau$	$\partial D/\partial \mu_T$ (°C^{-1})	$\partial D/\partial \mu_\tau$
B*	0.027 90	0.101 93	−0.013 55	−0.049 49
C*	0.027 75	0.101 39	−0.015 83	−0.057 82
D*	0.027 73	0.101 32	−0.015 81	−0.057 76
B*$+T \times \mu_T + \mu_T \times \sigma_T^2$	0.027 67	0.101 07	−0.015 12	−0.055 25
C*$+T \times \mu_T + \mu_T \times \sigma_T^2$	0.027 71	0.101 23	−0.015 47	−0.056 50
D*$+T \times \mu_T + \mu_T \times \sigma_T^2$	0.027 71	0.101 22	−0.015 53	−0.056 74
B* $+...+ T \times \mu_T \times \sigma_T^2$	0.027 70	0.101 20	−0.015 46	−0.056 48
C* $+...+ T \times \mu_T \times \sigma_T^2$	0.027 45	0.100 30	−0.015 08	−0.055 11
D* $+...+ T \times \mu_T \times \sigma_T^2$	0.027 30	0.099 73	−0.014 99	−0.054 78

Reported at $T_{\text{ref}} = 11.5$ ($\tau_{\text{ref}} = 1.19$). Only the dark black marginals are invariant to re-anchoring of the temperature scale.

The literature's model (A*) introduces the interaction between realized and mean temperature. Similarly, model (B*) introduces this interaction term while retaining mean temperature itself as a direct control, again relying on time-varying climate for identification. In Table 2,

coefficients invariant to temperature re-anchoring are shown in black; gray shading marks terms that shift with the reference point and thus cannot be interpreted as "per degree."[40] Instead, invariant marginal damages and adaptation costs must be computed explicitly as

$$\left.\frac{\partial d}{\partial T}\right|_{(T_{ref},\mu_{ref})} = \beta_T + \beta_{int}\,\mu_{ref}, \qquad \left.\frac{\partial d}{\partial \mu}\right|_{(T_{ref},\mu_{ref})} = -\,\beta_\mu - \beta_{int}\,T_{ref},$$

with $T_{ref} = \mathbb{E}[T_t]$ and $\mu_{ref} = \mathbb{E}[\mu_t]$, reported in Table 3. If climate is time-invariant or adaptation does not respond to change, the literature's model (A*) estimated on panel data becomes a hybrid between my models (A*) and (B*): the interaction coefficient remains scale-invariant since μ_t is absorbed by fixed effects, but the adaptation intercept and the marginal effect of mean temperature are not identified, which is why model (A*) does not appear in Table 3.

Model (B*) produces, once more, a slope $\frac{\partial d}{\partial T}$ of 10.2% on the τ-scale. The interaction coefficient alone (Table 2) is scale-invariant and negative, consistent with adaptation reducing damages, but larger in magnitude than the underlying 5% cost component, since it locally approximates a nonlinear effect. Computing $\frac{\partial d}{\partial \mu}$ recovers 4.95% (Table 3), close to the theoretical adaptation costs of 5%. Adding higher-order climate controls in models (C*) and (D*) slightly raises the estimated coefficient on mean temperature, consistent with the uncertainty premium discussed in the subsequent section, which is somewhat larger for the adaptation-cost component because of its stronger nonlinearity. The third block of Table 2 introduces interactions with variance, a second key climate moment. These terms enter the computation of marginal damages but yield only minor changes. The fourth block adds the triple interaction $T_t \times \mu_t \times \sigma_t^2$, providing the most refined specification.[41] Only at this stage does including skewness, as in model (D), slightly alter the estimates.

Given the central role of climate controls, I evaluate how well common empirical strategies proxy for them in the ACE time series setting. First, several papers use temperature deviations from a time trend. In my numerical experiment, where higher-order climate moments play a minor role, using the temperature deviation from a rolling climate mean effectively controls for climate and recovers the marginal-cost parameter ($\beta_\tau = 10.2\%$). Replacing the rolling climate mean with a linear or quadratic time trend yields the close approximation $\beta_\tau = 10.3\%$. Second, I ask how far time fixed effects can substitute for explicit climate controls. Because this section analyzes a single-region time series, I implement two exercises. Replacing local (U.S.) climate moments with global climate moments leaves the estimate unchanged at 10.2% for Model B's temperature coefficient and for Model B*'s invariant marginal-damage

[40] The regressions in the main text use temperature in degrees Celsius. Table 4 in Appendix B reports the corresponding estimates using temperature increase over preindustrial levels.

[41] In this specification, marginal damages are $\frac{\partial d}{\partial T} = \beta_T + \beta_{\text{int}}\,\mathbb{E}[\mu_t] + \beta_{T\times\sigma^2}\,\mathbb{E}[\sigma_t^2] + \beta_{T\times\mu\times\sigma^2}\,\mathbb{E}[\mu_t]\,\mathbb{E}[\sigma_t^2]$. Table 3 shows that marginal effects from model (B*) lose scale invariance once higher-order interactions are introduced, because regressions remain invariant only if they include all terms up to the highest interaction order.

estimate.[42] Alternatively, introducing biannual fixed effects in lieu of climate controls produces $\beta_\tau = 10.4\%$, somewhat higher but still close. A noise stress test reinforces this pattern: increasing measurement error tenfold materially worsens the (biannual) fixed-effects estimate to $\beta_\tau = 13.2\%$. By contrast, specifications that directly control for climate remain stable under this perturbation: using local or global climate moments yields 10.3% for Models B and B*; the deviation-from-rolling-climate-mean specification also returns $\beta_\tau = 10.3\%$; and linear and quadratic detrending deliver $\beta_\tau = 10.2\%$ and $\beta_\tau = 10.0\%$, respectively. In sum, explicit climate controls, whether local or global, are preferred. Detrending or time fixed effects can serve as proxies when other covariates are well controlled and measurement error is modest, but they are more sensitive to noise.

4.4 Social Cost of Carbon

Placing the applied identification exercise into context, this section discusses the SCC under adaptation and uncertainty. It is more complex than in a deterministic framework because uncertainty prevents perfect adaptation, introducing nonlinear climate damages even on the τ-scale. The proposition below relies on specifics of ACE's temperature dynamics that I relegated to Appendix A.2. The parameters c and δ_τ characterize the uncertainty; the parameter δ_τ scales the part of the uncertainty generated by climate change and c scales overall uncertainty governing temperature dynamics. The parameter γ^z characterizes the autoregressive nature of atmospheric temperature shocks. Finally, the function $\epsilon(c, \delta_\tau)$ ensures that shocks to τ do not distort expected temperature dynamics in degree Celsius. It is best thought of as a Jensen inequality correction deriving from temperature uncertainty when moving from T_t in degree Celsius to the more convenient $\tau_t = \exp(\xi_1 T_t)$. See Appendix A.2 and Traeger (2021) for details.[43]

Proposition 5 *The SCC under optimal adaptation and climate uncertainty is*

$$SCC = \underbrace{\frac{Y_t^{net}}{M_{pre}}\left(\xi_0 + \frac{\eta\lambda}{\chi}\big(1-\beta(1-\delta_A)\big)\right)\left(\sigma^{forc} + \right.}_{\text{as under certainty}} \underbrace{\frac{\epsilon(c,\delta_\tau)}{1-\beta\gamma^z}}_{\text{nonlinearity uncertainty correction}} \left.\right)$$
$$\big[(\mathbf{1}-\beta\boldsymbol{\sigma})^{-1}\big]_{1,1}\big[(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\big]_{1,1} + SCC^{ao}$$

with the additional adaptation-only contribution

$$SCC^{ao} = \frac{Y_t^{net}}{M_{pre}}\delta_\tau\frac{\eta\lambda}{\chi}\big(1-\beta(1-\delta_A)\big)\Big(\frac{\beta\gamma^z}{1-\beta\gamma^z}\frac{\lambda c}{1-\lambda c} - 1 - \frac{\log(1-\lambda c)}{\lambda c}\Big)\big[(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\big]_{1,1}.$$

[42] I use Berkeley Earth annual global time series using land plus air temperature above sea ice (Rohde & Hausfather 2020).

[43] The theoretical model assumes that transformed temperatures follow a shifted autoregressive gamma process, which Traeger (2025) calibrates for global temperatures. This calibration differs from the annual US data used in the preceding section, and the risk premia here would be expected to differ slightly from those I identified numerically in the previous section.

The first part of the expression coincides with the SCC in Proposition 2. The second part, labeled "nonlinearity uncertainty correction," arises from temperature uncertainty. Climate damages are a convex function of temperature. As a result, a symmetric shock to temperature causes a higher welfare loss than the expected temperature. This convexity (or "Jensen-inequality") correction is captured by the (numerically calibrated) function $\epsilon(c, \delta_\tau)$, which depends on the magnitude of the uncertainty.

Given that uncertainty governs the long-run future, this contribution to the SCC is higher for a larger discount factor (lower discount rate) and a larger persistence of climate uncertainty. A more patient decision-maker pays more attention to the persistent intertemporally correlated long-run impact of stochastic climate realizations.

Finally, Proposition 5 identifies a third contribution to the SCC that mirrors only the adaptable damage component. This contribution would vanish if the decision maker could adapt perfectly to temperature realizations. With uncertainty, however, adaptation is set for the distribution rather than each realization; deviations from the adapted level are steeply penalized: damages rise exponentially in τ_t (and thus double-exponentially in T_t). This contribution comes from the exponential term in equation (19) and reflects the influence of higher cumulants of the uncertainty distribution on the τ-scale; the next section formalizes this conjecture in the context of a general envelope theorem.

5 Generalizations and Extensions

This section abstracts from ACE's specifics and develops a general dynamic envelope framework with climate represented by its moments. Theorem 1 shows that the welfare effect of climate change equals the marginal impact of weather shocks at the realized state, multiplied by how climate moments evolve and propagate, including learning. A corollary maps the result back to ACE; subsequent subsections discuss identification, practical conditioning, and information frictions that delay belief updating.

5.1 General Dynamic Stochastic Envelope Theorem

The preceding sections showed how adaptation-inclusive damages can be identified from time series variation in weather shocks within a specific dynamic stochastic model. I now extend this analysis to a general stochastic dynamic setting using an envelope theorem argument. Let $P(T_t; \boldsymbol{m}_t)$ denote the probability density of temperature, characterized by the vector of moments $\boldsymbol{m}_t = (m_{1,t}, m_{2,t}, \ldots, m_{k,t})$ (mean, variance, skewness, etc.). For practical purposes, a finite number of moments suffices to approximate the distribution. I assume that the distribution is continuous and bounded from below. I denote the associated cumulative distribution function by $\mathit{CDF}(T_t; \boldsymbol{m}_t)$. To ease the presentation, I assume that small changes in $\boldsymbol{m}_t$ shift the CDF smoothly, and that such shifts occur only where the distribution already places

positive probability mass.[44]

As in Section 2.3, I specify utility directly in terms of the variables relevant for abatement and temperature impacts: $U(T_t, i_t, A_t, x_t, K_t)$, where x_t denotes the consumption/savings decision, which itself depends on climate. In contrast to the earlier presentation, I now allow for a fully general model; the value function contains the full set of states from Section 3.1's equation (16), including regular capital K_t. Damages in capital accumulation may differ from those in consumption. Starting with a general welfare result and limited structure, I denote the values of next-period generic capital in current utils by $p_K \equiv \beta \frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1})}{\partial K_{t+1}}$, and of abatement capital by $p_A \equiv \beta \frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1})}{\partial A_{t+1}}$,[45] before returning to a specific IAM structure further below. I focus on the basic setting and a few crucial assumptions, not discussing and ruling out all possible pathologies. For simplicity, I assume that utility and damages are bounded.[46]

I write the value function as $V(\boldsymbol{m}_t, A_t, K_t)$, focusing on the most essential arguments. The welfare effect of a marginal change in climate is given by the gradient $\frac{\partial V}{\partial \boldsymbol{m}_t}$, which represents the shadow values of the climate moments. Because climate evolves, a perturbation at time t also shifts future moments. I denote this propagation by the Jacobian $\frac{\partial \boldsymbol{m}_l}{\partial \boldsymbol{m}_t}$. In applications, the key perturbation is often the release of one ton of CO_2 (or another greenhouse gas). To keep notation light, I assume that relevant additional states characterizing, e.g., the carbon cycle, are also included in the climate vector $\boldsymbol{m}_t$. A change in current climate $\Delta \boldsymbol{m}_t$, where one component of the vector $\Delta \boldsymbol{m}_t$ could be a ton of CO_2 added to the atmosphere, induces the first-order welfare change $\Delta V_t \approx \frac{\partial V}{\partial \boldsymbol{m}_t} \Delta \boldsymbol{m}_t$, which aggregates the different components and implications of climatic shift over time.

Theorem 1 *For each period $s \geq t$, define the marginal welfare impact*

$$\boldsymbol{G}_s \equiv \int \left(\frac{\partial U\big(T, i_s^*, A_s, x_s^*, K_s\big)}{\partial T} + p_{A,s} \frac{\partial A_{s+1}}{\partial T} + p_{K,s} \frac{\partial K_{s+1}}{\partial T} \right) \frac{\partial \mathrm{CDF}\big(T; \boldsymbol{m}_s\big)}{\partial \boldsymbol{m}_s} \, dT,$$

and iteratively the matrix measure of future climate change[47]

$$\boldsymbol{M}_s \equiv \prod_{l=t}^{s-1} \left(\frac{\partial \boldsymbol{m}_{l+1}}{\partial \boldsymbol{m}_l} - \frac{\partial \boldsymbol{m}_{l+1}}{\partial T_l} \, \boldsymbol{r}_l(T_l) \right),$$

[44] Formally, let $\mu_t(dT) = P(T_t; \boldsymbol{m}_t)\, dT$ be the probability measure and $\boldsymbol{\nu}_t(dT) = \frac{\partial CDF(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} \, dT$ the signed measure obtained by differentiating the CDF with respect to $\boldsymbol{m}_t$. I assume $\boldsymbol{\nu}_t \ll \mu_t$, i.e. that $\boldsymbol{\nu}_t$ is absolutely continuous with respect to μ_t, ensuring the existence of the Radon–Nikodym derivative. In addition, I impose the usual technical conditions of measurability with respect to the filtration generated by the history of the temperature process, integrability, and stability so that the forward expansion converges. I note that the assumption that the density is bounded from below reflects that temperatures cannot fall below zero Kelvin.

[45] In a specific model, one can use the investment first-order condition to express these prices in terms of, e.g., marginal utility. Here, I keep the model structure open.

[46] The proof of Theorem 1 also spells out an alternative convergence assumption in the case of unbounded utility or damages.

[47] By convention, the empty product for $s = t$ gives $\boldsymbol{M}_t = \boldsymbol{I}$.

where the second term, a correction for anticipated endogenous learning, relies on the Radon–Nikodym derivative of $\frac{\partial \text{CDF}(T_l;\boldsymbol{m}_l)}{\partial \boldsymbol{m}_l}$ *w.r.t.* $P(T_l;\boldsymbol{m}_l)$*:* $\boldsymbol{r}_l(T_l) = \frac{\frac{\partial \text{CDF}(T_l;\boldsymbol{m}_l)}{\partial \boldsymbol{m}_l}}{P(T_l;\boldsymbol{m}_l)}$*. Then,*

$$\frac{\partial V(\boldsymbol{m}_t, A_t, K_t)}{\partial \boldsymbol{m}_t} = -\sum_{s=t}^{\infty} \beta^{s-t}\, \mathbb{E}_t[\boldsymbol{M}_s\, \boldsymbol{G}_s]\,, \tag{25}$$

gives the forward-looking welfare cost of a marginal change in climate. Each entry corresponds to the welfare impact of a change in the respective moment. $\mathbb{E}_t[\cdot]$ *denotes expectation conditional on time-t information.*

Note (absence of endogenous learning): *If the law of motion for climate moments does not depend on current temperature,* $\frac{\partial \boldsymbol{m}_{l+1}}{\partial T_s} = 0$ *for all* $l \geq t$*, then the endogenous learning correction involving the Radon-Nikodym derivative* $\boldsymbol{r}_l(T_l)$ *vanishes, and* $\boldsymbol{M}_s = \prod_{l=t}^{s-1} \frac{\partial \boldsymbol{m}_{l+1}}{\partial \boldsymbol{m}_l}$*.*

The theorem shows that the welfare costs of climate change can be decomposed into two components. The first, $\boldsymbol{G}_s$, collects the period-s welfare losses associated with a marginal change in the climate distribution. These losses arise from three channels: (i) direct losses within a given period through reduced consumption or direct harm, (ii) the effect of temperature on the accumulation of adaptation capital, and (iii) its effect on productive capital. Most importantly, each of these impact channels evaluates the partial derivative wrt. a temperature change, i.e., the effect of a weather fluctuation holding adaptation and other stock variables fixed. In more structured models and practical applications, the three components collect different market and potentially non-market measures that capture the impact of a temperature shock. The result thus extends the basic reasoning of the preceding sections to a general stochastic dynamic framework: *we can measure the impact of a temperature shock while keeping state variables and investment decisions fixed.*

To calculate the welfare loss from climate change, each impact channel is weighted by the CDF's change in the relevant climate moment. The most important changes typically involve a shift in the mean climate, an increase in climate volatility, and, in some cases, changes in downside risk or fat tails. Taken together, $\boldsymbol{G}_s$ captures how a shift in the probability distribution characterizing the climate translates into a period-s welfare loss.

The second component, $\boldsymbol{M}_s$, traces how a perturbation in the current climate propagates through the dynamics of the climate system. It maps today's marginal change into the predicted moments that determine future weather realizations. Multiplying $\boldsymbol{M}_s$ by $\boldsymbol{G}_s$ therefore gives the welfare effect of today's change as it unfolds through the climate system in period s. Summing this product forward in time, with discounting, yields the full forward-looking welfare cost of a marginal change in climate.

The note clarifies that, in the absence of anticipated learning updates, the term $\boldsymbol{M}_s$, which captures climate dynamics, reduces to the familiar Jacobian. When shocks are persistent and/or contain structural information affecting future beliefs, today's realized temperature also conveys information about tomorrow's climate. The Radon-Nikodym term $\boldsymbol{r}_l(T_l)$

reweights the distribution to reflect how such realizations alter beliefs about the future, and the factor $\frac{\partial \boldsymbol{m}_{l+1}}{\partial T_l}$ translates this informational update into changes in tomorrow's moments. The correction term thus explicitly incorporates the effect of (anticipated) endogenous learning from observed weather shocks into the welfare measure.

To link the general result to a concrete IAM, I now specialize to ACE's structure. This yields an explicit form of $\boldsymbol{G}_s$ for a general damage function and, with ACE's specific functional forms, a direct connection between the abstract theorem and the model's economic parameters.

Corollary 1 *Suppose the economy follows ACE's structure. For a general damage function $d(\tau_t, i_t, A_t)$, the welfare loss term $\boldsymbol{G}_s$ takes the form*

$$\begin{aligned}\boldsymbol{G}_s &= \frac{1}{x}\int \frac{\partial d(T_s, i_s^*, A_s)}{\partial T_s}\,\frac{\partial CDF(T_s;\boldsymbol{m}_s)}{\partial \boldsymbol{m}_s}\,dT_s \\ &= \frac{1}{x}\int d(T_s, i_s^*, A_s)\,\frac{\partial P(T_s;\boldsymbol{m}_s)}{\partial \boldsymbol{m}_s}\,dT_s,\end{aligned}$$

where x is the (optimal) consumption rate. For the specific damage function analyzed in the preceding sections (equation 3), this expression turns into

$$\boldsymbol{G}_s = \frac{1}{x}\Big[\xi_0\,\frac{\partial\,\mathbb{E}_t[\tau_t]}{\partial \boldsymbol{m}_t} + \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\,\frac{\partial\,\mathrm{CGF}_{\tau_t}(\lambda)}{\partial \boldsymbol{m}_t}\Big], \tag{26}$$

with τ_t denoting the exponential temperature transform and CGF_{τ_t} its cumulant generating function.

The specialized form of $\boldsymbol{G}_s$ shows how the general welfare loss expression translates into ACE's economic structure, representative of many IAMs.[48] Here, welfare losses are obtained by weighting marginal damages with the induced shift in climate. As noted in Section 4.2, empirical estimation often drops constant terms in the damage function. This first formulation shows that such omissions do not impair the ability to calculate the social cost of climate change. More generally, *the corollary shows that it suffices to identify marginal damages from weather shocks to characterize all moment-related welfare losses from climate change*. The second line uses integration by parts to present the result in a more intuitive form, where the change in the temperature probability density weights absolute damages.

The final expression ties these components back to ACE's particular damage function and parameters. The two terms inside the bracket correspond to the distinct economic channels of damages: the non-adaptable component, proportional to ξ_0, and the adaptation-cost component, proportional to $\frac{\eta}{\chi}(1-\beta(1-\delta_A))$. Both contain a component proportional to the change in expected temperatures, for the adaptable damage component, this is the first component of

[48] In ACE, damages reduce net production, which is then optimally split between consumption and investment, so a single damage function captures both effects.

the cumulant generating function. This component is responsible for the term labeled "nonlinearity uncertainty correction" in Proposition 5.[49] Given imperfect adaptation, the adaptable damage component, through its additional convexity, also depends on the higher moments of the cumulant generating function. Interacted with the climate change measure $\boldsymbol{M}_s$ in equation (25), these higher-order terms give rise to the second contribution (SCC^{ao}) in Proposition 5.[50]

5.2 Discussion

Sections 3 and 4 rely on particular functional forms to obtain closed-form solutions; in that setting, marginal damages on the τ-scale are independent of the system states. One might therefore worry that, in a more general setting, omitting capital controls (including adaptation capital) or other drivers of adaptation would create distortions analogous to those in Section 4 when the econometrician fails to control for climate moments. Theorem 1 and its corollary show that the core identification logic is more general. We must control for climate because changes in climate *simultaneously* shift installed adaptation and the distribution of realized weather. Once we condition on the relevant climate moments, the remaining variation in weather is—by construction—a forecast error with zero conditional mean and thus uncorrelated with pre-set adaptation and capital. The resulting slope gives an unbiased estimate of the impact of temperature shocks, evaluated at the state of the economy.

The envelope theorem implies that we do not need to pin down the full state vector of the economy to identify the relevant climate damages. Regions with more adaptation will naturally experience smaller losses, but that difference simply reflects the state at which the shock occurs and is not a source of bias once climate is held fixed. As clarified by Theorem 1 and its corollary, what matters for welfare is the marginal effect of a weather shock evaluated

[49] On the τ-scale, this first component merely reflects the expected change in temperature. If, however, the temperature distribution is expressed over T_t in degrees Celsius, then this expectation itself depends on all moments of the uncertainty distribution. In particular, this contribution will reflect that, under uncertainty, $\mathbb{E}_t\,\tau_t = \mathbb{E}_t \exp(\xi_1 T_t) > \exp(\xi_1\,\mathbb{E}_t\,T_t)$. The factor $\frac{1}{1-\beta\gamma^z}$ in Proposition 5's "nonlinearity uncertainty correction" derives from summing the contribution over the infinite time horizon.

[50] If we pick the moments characterizing climate to be the cumulants $\boldsymbol{\kappa}_t$ of the τ_t distribution, the expression simplifies to

$$\begin{aligned}\int d(\tau_t, i_t^*, A_t)\,\frac{\partial P(T_t;\boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\,dT_t &= \xi_0\,\frac{\partial}{\partial \boldsymbol{\kappa}_t}\;\mathbb{E}_t[\tau_t] \;+\; \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\,\frac{\partial}{\partial \boldsymbol{\kappa}_t}\left(\sum_{n=1}^{\infty}\frac{\kappa_{n,t}}{n!}\,\lambda^n\right)\\ &= \xi_0\,\boldsymbol{e}_1 \;+\; \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\,\big(\lambda,\;\tfrac{\lambda^2}{2},\;\tfrac{\lambda^3}{3!},\ldots,\tfrac{\lambda^n}{n!},\ldots\big),\end{aligned}$$

where $\boldsymbol{e}_1$ is the first unit vector. The factor $\frac{1}{x}$ enters because the corollary evaluates welfare, which is the log transformation of consumption. A 1% increase in damages reduces current consumption by 1% and also lowers future capital. The value of future capital happens to depend on the degree of homogeneity in capital κ and the discount rate in the same way that the consumption-investment trade-off is reflected in the consumption rate. The factor cancels when translating the welfare change back into consumption units.

at the realized state, holding all other determinants of the economy constant.

The same reasoning explains why we do not need to explicitly accommodate expectations about the future to measure marginal damages. Future expectations generally shape the current adaptation level. Yet, by the envelope theorem, we do not need to account for the endogenous response of agents to those expectations. In the proof of Theorem 1, the envelope theorem eliminates six forward-looking expectation terms involving agents' endogenous adjustments.[51]

Forecasting the costs of climate change is a different task from identifying current damages. In a forward-looking economy, installed adaptation reflects expectations about long-run conditions, shaped not only by today's temperature distribution but also by broader climate information (e.g., CO_2 concentrations, policy trajectories, and learning about physical parameters, technological progress, and potential scarcity rents in adaptation measures). None of this is needed to measure *current* damages: panel data identify the marginal effect of weather conditional on current climate. For projection, however, one must specify how climate moments will evolve and how adaptation and capital will adjust, either through explicit transition rules or scenario paths. That in turn calls for structural assumptions about technology, wealth accumulation, institutions, resources, migration, and policy, because these determine how adaptation capacity evolves and interacts with future climate. In short, identification of the local marginal is empirical; translating it into long-run costs is inherently model-based.

The envelope theorem assumes agents optimize in a continuous choice space, yet some adaptation measures are discrete. In most cases, aggregation smooths individual kinks or jumps at the aggregate level. The adaptation costs that the econometric approach will miss are those generated by actions that discretely and permanently eliminate damages. For example, if a dam removes flood risk and never fails over the sample, the econometric approach will miss the associated adaptation cost. By contrast, if the time series is long enough, or the number of dams large enough, for failures to occur with some regularity at the optimal investment level, the approach will recover those costs because damages will then, at the margin, incorporate the cost of adapting.

5.3 Wrong Beliefs and Imperfect Adaptation

Humans ordinarily strive to adapt to prevailing climate conditions: farmers tailor practices to local climate zones to raise yields, households invest to reduce mortality risk, and competitive pressures encourage firms to adjust production settings. At the same time, belief heterogeneity is pervasive and a sizable share of the U.S. population has, e.g., long been skeptical about climate change. Then, adaptation may be guided by misspecified beliefs rather than by the true climate distribution. This section uses the preceding results to shed some light on such a

[51] This count is for a model with one general capital stock and one adaptation stock; without the envelope theorem, the number of endogenous adjustments to track would grow overwhelmingly in realistic settings.

setting where agents optimize adaptation w.r.t. a misspecified moment vector, while outcomes are generated by the true climate.

An agent's adaptation response is based on their beliefs about the climate. These beliefs enter Theorem 1 and its corollary through the optimal investment level i_t^*. For the ACE example, equation (17) shows how optimal investment depends on the climate through the moments of the weather distribution (through $\mathrm{CGF}_{\tau_l}(\lambda)$). Let me distinguish the cumulant generating function $\mathrm{CGF}_{\tau_l}(\lambda)$ characterizing the agent's belief about climate from the true climate distribution characterized by the cumulant generating function $\mathrm{CGF}_{\boldsymbol{m}_t}(\lambda)$, which I explicitly index by the true moments $\boldsymbol{m}_t$ characterizing the actual climate. An intermediate step in the proof of Corollary 1 shows that, in this case,

$$\begin{aligned}&\int d(\tau_l, i_l^*, A_l) \frac{\partial P(T_l; \boldsymbol{m}_l)}{\partial \boldsymbol{m}_l}\, dT_l \\ &= \xi_0 \frac{\partial\, \mathbb{E}_{\boldsymbol{m}_t}[\tau_t]}{\partial \boldsymbol{m}_t} \;+\; \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\, \exp\big(\,\mathrm{CGF}_{\boldsymbol{m}_t}(\lambda) - \mathrm{CGF}_{\tau_l}(\lambda)\big)\, \frac{\partial\, \mathrm{CGF}_{\boldsymbol{m}_t}(\lambda)}{\partial \boldsymbol{m}_t}.\end{aligned}$$

If an agent denies or underestimates climate change, we expect $\mathrm{CGF}_{\tau_l}(\lambda) < \mathrm{CGF}_{\boldsymbol{m}_l}(\lambda)$, so the factor $\exp\big(\,\mathrm{CGF}_{\boldsymbol{m}_t}(\lambda) - \mathrm{CGF}_{\tau_l}(\lambda)\big) > 1$ magnifies expected damages from climate change relative to the rational, forward-looking benchmark in equation (26). In words, the agent adapts too little and damages are higher. This reasoning carries through to Theorem 1's SCC. When adaptation falls short, marginal damages per ton of CO_2 are larger, implying a higher social cost and stronger optimal abatement (higher optimal carbon taxes) than under optimal adaptation. If the structural model and agents' beliefs are known, the framework permits quantifying the wedge between belief-based and optimal adaptation, and the associated reduction in the social cost attainable under optimal adaptation.

Using the setup of Section 4.3, I can estimate adaptation-cost-inclusive damages when adaptation responds to a delayed perception of climate change. E.g., let agents' beliefs be formed based on the climate representing weather realizations from 35 to 5 years preceding the current period,[52] rather than on a real-time update of the current climate. Assuming that the econometrician identifies the correct (delayed) climate moments that drive the adaptation decision, regression B yields marginal damages of $\beta_\tau^{delay} \approx 10.4$, slightly exceeding the corresponding estimate $\beta_\tau \approx 10.2$ in Table 2; delayed adaptation increases marginal damages.[53] Another way to incorporate behavioral frictions in this framework is to modify the decision rule for optimal adaptation investment in equation (17), rather than agents' beliefs. One could, for instance, explore whether agents respond only to the first moment, or whether they

[52] This is merely an illustrative example. Actual empirical work would need to identify the periods and delays in climate perception that best explain observed behavior, a challenge I do not address here. My goal is simply to illustrate how such issues can be conceptualized within the model.

[53] A similar reasoning applies to the coefficient on (delayed) mean temperature. The estimate $\beta_{\mu_\tau}^{delay} \approx -5.5$, which characterizes the benefits of the marginal adaptation response, is slightly larger in magnitude than in the baseline setting of Table 2, suggesting that the marginal value of adaptation rises when adaptation levels lag behind.

also take into account weather fluctuations, as would seem natural, and possibly even adjust to downside risk. I leave these extensions and related challenges to future work and hope the present results help motivate such analyses.

6 Conclusion

This paper resolves a long-standing conceptual tension in empirical climate economics. Ricardian analyses exploit cross-sectional climate differences but struggle with omitted variables, whereas weather-based time series and panel approaches have been faulted for recovering only short-run responses to shocks rather than the welfare-relevant costs of climate change. By embedding forward-looking adaptation into a fully dynamic and stochastic framework, the paper shows that this perceived short-run limitation is in fact a strength. Once the distribution of weather is held fixed, weather shocks reveal the adaptation-inclusive marginal damages that matter for welfare and for the SCC. The result clarifies the precise variation and conditioning set under which reduced-form designs recover the correct object.

The key contribution is to formalize how forward-looking adaptation interacts with the distributional nature of climate in the context of climate damage estimation. Agents adapt to the statistical structure of climate rather than to each weather realization, so persistent climatic shifts induce systematic adjustments in adaptation, but short-lived deviations cannot be perfectly offset. This gap between adaptation to climate and inadaptability to shocks is what makes weather variation informative. A dynamic envelope logic shows that, once climate is held fixed, the marginal effect of a weather shock coincides with the welfare-relevant derivative of damages, even though adaptation costs themselves are not observed. In short, standard weather-based regressions recover adaptation-inclusive marginal damages because they isolate the partial variation that agents cannot neutralize within a single period. By articulating these mechanisms formally, the paper provides a unified framework that links forward-looking structural IAMs with empirical weather-based identification strategies.

The paper's dynamic decomposition clarifies how climate affects welfare through two distinct channels. The physical block captures how the moments of the climate distribution evolve, and how changes in mean temperature, variability, and higher moments propagate forward through the climate system. The economic block captures marginal damages from weather shocks, evaluated at the prevailing adaptation level. Together, these components show that the welfare cost of rising temperatures is the product of the marginal damages and the way climate is expected to change. Crucially, what matters for welfare is the marginal effect of a weather shock evaluated at the realized state, holding all other determinants of the economy constant, so we do not need to track the full adaptation state or agents' forward-looking expectations for the measurement of adaptation-inclusive damages.

The paper's closed-form solution for observed damages and the SCC provides a transparent setting to understand how forward-looking adaptation enters climate damages and how

the envelope logic must be applied in dynamic environments. It also shows why direct observation of adaptation spending is only partially informative: persistent investments protect an unspecified future and can be misattributed if one tries to infer social costs from expenditures alone. By contrast, indirect observation via weather shocks isolates the relevant partial variation at a fixed adaptation level, which is the object that enters the SCC. In particular, the closed form makes clear that patience- and persistence-weighted adaptation costs contribute to the SCC, contrary to earlier propositions and to the common practice of omitting them. In the structural example, optimal adaptation depends on the cumulant-generating function of future temperatures, highlighting how agents respond to the full distribution of expected weather outcomes. The stochastic extension adds a further layer: uncertainty interacts with nonlinear marginal damages to generate risk premia that reflect how shifts in climate moments correlate with adapted marginal damages. The adaptable component is likely more nonlinear under forward-looking adaptation, demanding larger risk premia, although this curvature is obscured in observed damages unless climate is properly controlled for. Taken together, the closed-form expressions clarify the channels through which adaptation shapes observed damages and the SCC.

The analysis clarifies what econometricians need to recover adaptation-inclusive damages. It is not necessary to estimate separate damage responses for each climate moment, but it is essential to condition on climate, at least through the mean temperature and, where relevant, through higher moments such as variance, skewness, or tail risks. When climate adaptation remains fixed over the time horizon, regional fixed effects that absorb static climate differences are not an impediment to identification, because the remaining weather variation delivers the correct marginal object. By contrast, when climate and adaptation change over time, explicit controls for climate moments become necessary to condition on the state to which agents have adapted. The stochastic dynamic framework developed in the paper provides a consistent way to do so. It also shows that time-series variation in climate has two advantages: it avoids the assumption that marginal damage coefficients coincide across regions, and it can reveal adaptation costs directly in settings where the standard panel approach cannot identify them fully. Finally, contrary to a common critique, weather-based regressions without climate controls tend to understate rather than overstate the welfare costs of climate change when adaptation adjusts to the evolving climate.

The framework also clarifies the strengths and limits of common empirical strategies. Interaction terms between weather and mean climate can recover how adaptation varies with climate, but only when the regression basis correctly captures the relevant nonlinearity in the temperature response. In linear specifications, or in bases that cannot represent this curvature properly, the interaction term cannot isolate adaptation costs and part of the information is absorbed by climate controls or fixed effects. Weather shocks, when combined with controls for climate moments, therefore provide a more reliable route to identification, especially when climate is changing and adaptation adjusts with it. The results also show that global climate

controls can practically substitute for US-specific controls, so tracking climate signals at the panel scale may be sufficient when the broad evolution of climate is aligned across units. Time fixed effects can play a similar role when covariates are well controlled and measurement error is modest, but they can lose traction when noise increases. In sum, the key requirement is conditioning on the weather distribution that agents adapt to, so that weather shocks can be read as the correct marginal object.

Finally, the paper is conceptual in nature, translating the theory into general results, closed-form examples, and econometric designs intended to support a broad empirical program. The framework applies to yields, mortality, production, and other outcomes; here I focus on a closed economy to sharpen ideas. Several limitations point to natural extensions. Identification requires weather shocks that outpace agents' ability to adjust within the period; discrete adaptation measures that permanently eliminate damages are not picked up unless failures occur at a regular frequency. A follow-up analysis further explores some of these limitations in the context of production dynamics and the open-economy setting with regional spill-overs. Additional extensions could further explore behavioral adaptation, heterogeneous or misspecified beliefs, and alternative information structures. The central empirical takeaway of the present paper remains simple: use weather variation, condition on the climate distribution, and the resulting estimates recover adaptation-inclusive marginal damages for policy analysis.

Appendix

A Summary of Remaining IAM Components

This section briefly summarizes the remaining components of the Analytic Climate Economy (Traeger 2023) and the stochastic temperature model (Traeger 2021).

A.1 Deterministic Base Model

Emissions and Resources. The first d entries of the energy vector $\boldsymbol{E}_t$ are fossil-fuel-based, measured by their carbon content, and summarized in the vector $\boldsymbol{E}_t^d$. Total emissions from these fossil fuels are $\sum_{i=1}^{I^d} E_{i,t}$, and exogenous emissions from land conversion, forestry, and agriculture are denoted by E_t^{exo}. Fossil fuel use reduces the resource stocks in the ground $\boldsymbol{R}_t \in \mathbb{R}_+^{I^d}$:

$$\boldsymbol{R}_{t+1} = \boldsymbol{R}_t - \boldsymbol{E}_t^d,$$

with initial stock levels $\boldsymbol{R}_0$ given. If a resource is scarce along the optimal path, I assume that its use is stretched over an infinite time horizon.

Capital Accumulation. The base model invests production left after climate damages and consumption

$$I_t = Y_t^{net} - C_t. \tag{27}$$

The full depreciation assumption by Golosov et al. (2014) would correspond to $K_{t+1} = I_t$. Such a full depreciation assumption has two main shortcomings: (i) it misses the remaining capital from previous periods, and (ii) it fails to account for how current investments increase future capital availability. The equation

$$K_{t+1} = I_t \left[\frac{1 + g_{k,t}}{\delta_k + g_{k,t}} \right], \tag{28}$$

provides a "quick fix" using a correction factor that is greater than one when the depreciation factor δ_k is below one. It enhances both the availability of capital in the next period and the payoff from current investments in terms of future capital. Traeger (2023) shows that equation (28) coincides with the standard capital accumulation equation

$$K_{t+1} = Y_t[1 - D(T_t)] - C_t + (1 - \delta_k)K_t.$$

if the exogenous capital growth approximation $g_{k,t} = \frac{K_{t+1}}{K_t} - 1$ is accurate, or if $\delta_k = 1$. This approach adjusts the capital dynamics and capital-output ratio to align with macroeconomic observations, making decision-makers aware of additional future returns from capital investments.

Carbon Cycle and Greenhouse Effect. Carbon cycles through atmospheric and non-atmospheric reservoirs, with atmospheric carbon $M_{1,t}$ and reservoirs $M_{2,t}, ..., M_{m,t}$ governed by:

$$\boldsymbol{M}_{t+1} = \boldsymbol{\Phi}\boldsymbol{M}_t + \boldsymbol{e}_1(\textstyle\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}).$$

Increased atmospheric carbon causes radiative forcing, modeled by

$$Q_t = \eta \frac{\log \frac{M_{1,t}+G_t}{M_{pre}}}{\log 2}. \tag{29}$$

where η defines the greenhouse effect's strength and G_t captures the radiative forcing from other, exogenous greenhouse gases in CO_2 equivalents.

Temperature Dynamics. Apart from atmospheric temperature T_t, the model uses a finite number of ocean layers $T_{i,t}, i \in \{2, ..., l\}, l \in \mathbb{N}$ to capture the atmosphere-ocean heat exchange that substantially cools our surface at the moment. Abbreviating the atmospheric equilibrium temperature resulting from the radiative forcing level Q_t by $T_{0,t} = \frac{s}{\eta}Q_t$, I let temperatures between different layers adjust using a generalized mean (see Traeger (2023) for details)

$$T_{i,t+1} = \mathfrak{M}_i^{\boldsymbol{\sigma}}(T_{i,t}, T_{i-1,t}, T_{i+1,t}) \text{ for } i \in \{1, ..., l\}.$$

The weight matrix $\boldsymbol{\sigma}$ defines heat flow. Using the transformation $\tau_{i,t} = \exp(\xi_1 T_{i,t})$, $i \in 1, ..., l$, already introduced in the main text for atmospheric temperatures, and stacking them into the vector $\boldsymbol{\tau}_t$, the transformed diffusion model becomes linear

$$\boldsymbol{\tau}_{t+1} = \boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t} + G_t}{M_{pre}}\boldsymbol{e}_1 \tag{30}$$

enabling an analytic solution.

A.2 Stochastic Temperature Dynamics

Temperature evolution is uncertain for many reasons. I focus on the uncertain temperature response of atmospheric CO_2 concentrations, also known as climate sensitivity. The underlying stochastic process is highly autoregressive because bad news about climate feedbacks today imply that climate feedbacks tomorrow are most likely higher than expected as well. This base model of stochastic climate feedbacks and its implications for the evalution of global average surface temperature is adopted from Traeger (2021), where it is calibrated to IPCC's (2013) and IPCC's (2021) transient climate sensitivity. I summarize this model below. The only extension in the present context is the addition of serially uncorrelated temperature shocks that represent weather and, in the next section, idiosyncratic regional shocks.

The adopted model uses a modified version of Gourieroux & Jasiak's (2006)'s autoregressive gamma process, denoted by y_t, and an expectation adjusted shift of this process, z_t. Incorporating this stochastic process into the equation of motion (30) for temperatures, see Appendix A.1, yields

$$\boldsymbol{\tau}_{t+1} = \boldsymbol{\sigma}\boldsymbol{\tau}_t + \Big(\underbrace{\sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}}}_{\text{deterministic forcing}} + \underbrace{z_{t+1}}_{\substack{\text{persistent}\\ \text{stochastic}\\ \text{deviation}}} \Big)\boldsymbol{e}_1. \tag{31}$$

The unit vector $\boldsymbol{e}_1$ ensures that feedbacks drive atmospheric temperature. Traeger (2023) summarizes how to transform a nonlinear temperature-ocean interaction as well as the standard nonlinear radiative forcing question into an equation that is linear in $\boldsymbol{\tau}_t$, and Appendix A.1 gives a brief summary. The shifted autoregressive process z_t is summarized by its conditional expectation and variance:

$$\mathbb{E}_t\, z_{t+1} \quad = \underbrace{\gamma^z z_t}_{\text{persistence}} + \underbrace{\epsilon(c,\delta_\tau)\left(\frac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)}_{\text{expectation correction}} \tag{32}$$

$$\mathbb{V}\text{ar}_t\, z_{t+1} = \mathbb{V}\text{ar}_t\, \tau_t = \mathbb{V}\text{ar}_t\, y_{t+1} = c\Big[\underbrace{2\gamma^z y_t}_{\substack{\text{variance}\\ \text{persistence}}} + \underbrace{\delta_\tau\left(\frac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)}_{\text{endogenous forcing uncertainty}} \Big] \tag{33}$$

where $\tau_t = \boldsymbol{\tau}_{1,t+1}$ as defined in Section 3.1. Equation (32) represents the expectation's deviation from the deterministic model. The first term shows the persistence of learning shocks, reflecting updates to our understanding of nature's response to CO_2. Stronger-than-expected feedbacks in the current period imply stronger feedbacks in the future, quantified by the autoregression coefficient γ^z. The final term aligns the expected temperature evolution with its deterministic counterpart. The atmospheric temperature in degrees Celsius, $T_{1,t+1} = \frac{1}{\xi_1}\log(\tau_t)$, is concave in τ_t. By Jensen's inequality, a mean zero shock to τ_t would reduce the expected temperature $T_{1,t+1}$. The "expectation correction" adjusts the stochastic process upward to account for this nonlinearity, ensuring the long-term deterministic and expected temperatures match.

Equation (33) describes the uncertainty in atmospheric temperature evolution, which increases with climate system perturbations ($\frac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau$) and is autoregressive. Greater forcings from fossil fuel burning and other greenhouse gases raise uncertainty about future temperatures. The parameter $\eta_\tau \in (0,1)$ ensures low uncertainty in the unperturbed preindustrial system ($M_{1,t} = M_{pre}$ and $G_t = 0$). The parameter δ_τ scales endogenous uncertainty from climate perturbations relative to the autoregressive component. The process y_t is the underlying autoregressive gamma process, with variance persistence characterized by γ^z. If uncertainty is high in one period, it likely remains high in subsequent periods. The parameter c scales the overall variance of the autoregressive gamma process. The expectation correction term in

(32) increases with uncertainty, requiring a larger correction due to temperature expectation nonlinearity. Thus, the correction term grows with the perturbation term $\frac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau$ and the uncertainty-scaling parameters c and δ_τ, described by the function $\epsilon(c, \delta_\tau)$.

New to the present paper is only the novel volatility term Δ_t which I assume to be iid. This stochastic process represents weather shock as compared to the persistent stochastic process governing global average temperatures. The novel term introduces the necessary distinction to explore how the adaptation and damage response differs between an uncertain persistent temperature evolution and expected but stochastic weather shocks.

B Estimations using temperature increase over preindustrial

This appendix re-estimates the specifications when temperature is measured as the increase over the preindustrial baseline, rather than in absolute °C. Tables 4 and 5 show how re-anchoring the temperature scale shifts the coefficients displayed in grey, both here and in Tables 2 and 3 in the main text. The economically meaningful objects of interest discussed in the main text, reported in black, remain unchanged. I note that model B*'s marginal damage estimate in Table 5 shifts with the temperature scale. Adding the interaction $\mu_T \times \sigma_T^2$ yields an incomplete basis of the corresponding order since the model lacks the variance control σ_T^2.

Table 4: Estimating simulated damages using temperature increase over preindustrial levels

Model	β_T (°C^{-1})	β_τ	β_{μ_T} (°C^{-1})	β_{μ_τ}	β_{int} (°C^{-2})	β_{int}-norm	$\beta_{\text{int}} \times \mathbb{E}\,T$ (°C^{-1})
A: T only	0.023 86	0.087 16					
B: " + μ_T	0.027 95	0.102 11	−0.013 91	−0.050 81			
C: " + σ_T^2	0.027 95	0.102 13	−0.013 82	−0.050 49			
D: " +skew$_T$	0.027 95	0.102 11	−0.013 81	−0.050 47			
With interaction $T \times \mu_T$, abbreviated as "..." below							
A*: A + ...	0.031 43	0.114 83			−0.008 79	−0.117 34	−0.006 45
B*: B + ...	0.023 90	0.087 31	−0.018 03	−0.065 86	0.006 11	0.081 50	0.004 48
C*: C + ...	0.022 78	0.083 22	−0.021 39	−0.078 15	0.007 59	0.101 29	0.005 57
D*: D + ...	0.022 75	0.083 11	−0.021 39	−0.078 14	0.007 61	0.101 54	0.005 58
With interaction $T \times \mu_T$ and $T \times \sigma_T$ and $\mu_T \times \sigma_T$, abbreviated as "..." below							
A +...	0.018 22	0.066 55	−0.000 27	−0.000 99	−0.000 27	−0.003 62	−0.000 20
B +...	0.020 90	0.076 36	−0.018 60	−0.067 95	0.003 84	0.051 30	0.002 82
C +...	0.020 96	0.076 57	−0.017 28	−0.063 11	0.004 93	0.065 75	0.003 61
D +...	0.021 01	0.076 76	−0.017 72	−0.064 73	0.005 00	0.066 79	0.003 67
With interaction $T \times \mu_T$ and $T \times \sigma_T$ and $\mu_T \times \sigma_T$ and $T \times \mu_T \times \sigma_T$, abbreviated as "..." below							
A +...	0.022 76	0.083 14	−0.010 21	−0.037 30	−0.010 21	−0.136 29	−0.007 49
C +...	0.022 17	0.080 98	−0.012 78	−0.046 69	0.000 54	0.007 19	0.000 40
D +...	0.023 00	0.084 03	−0.011 03	−0.040 28	−0.001 92	−0.025 59	−0.001 41

Normalizations: $\beta_\tau = \beta_T/(\xi_1\,\tau_{\text{ref}})$, $\beta_{\mu_\tau} = \beta_{\mu_T}/(\xi_1\,\tau_{\text{ref}})$, β_{int}-norm $= \beta_{\text{int}}/(\xi_1\,\tau_{\text{ref}})^2$. Here $\tau_{\text{ref}} = 1.185$ is computed from Mean(T) and $\overline{\mu_T} = 0.655$ is Mean(μ_T) over the sample.

Table 5: Marginal effects implied by Table 4's coefficient estimates
Uses temperature increase over preindustrial level.

Model	$\partial D/\partial T$ (°C^{-1})	$\partial D/\partial \tau$	$\partial D/\partial \mu_T$ (°C^{-1})	$\partial D/\partial \mu_\tau$
B*	0.027 90	0.101 93	−0.013 55	−0.049 49
C*	0.027 75	0.101 39	−0.015 83	−0.057 82
D*	0.027 73	0.101 32	−0.015 81	−0.057 76
B*+$T \times \mu_T$+$\mu_T \times \sigma_T^2$	0.027 80	0.101 57	−0.015 51	−0.056 67
C*+$T \times \mu_T$+$\mu_T \times \sigma_T^2$	0.027 71	0.101 23	−0.015 47	−0.056 50
D*+$T \times \mu_T$+$\mu_T \times \sigma_T^2$	0.027 71	0.101 22	−0.015 53	−0.056 74
B* +...+ $T \times \mu_T \times \sigma_T^2$	0.028 04	0.102 42	−0.015 90	−0.058 07
C* +...+ $T \times \mu_T \times \sigma_T^2$	0.027 45	0.100 30	−0.015 08	−0.055 11
D* +...+ $T \times \mu_T \times \sigma_T^2$	0.027 30	0.099 73	−0.014 99	−0.054 78

Reported at $T_{\text{ref}} = 0.734$ ($\tau_{\text{ref}} = 1.185$) and $\overline{\mu_T} = 0.655$. These marginals are invariant to linear re-anchoring of the temperature scale.

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Technical Appendix

C Derivations

Appendix C derives the closed-form solution results stated in the paper.

C.1 Setting up the optimization problem

Let the production inputs be $\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t \in \mathbb{R}_+^I$ and let the first $I^d \leq I$ energy sources $E_1, ..., E_{I^d}$ be fossil fuels generating emissions with stock levels $\boldsymbol{R}_t \in \mathbb{R}_+^{I_d}$ left in the ground (all measured in terms of their CO_2 content). Let $\boldsymbol{M}_t \in \mathbb{R}_+^m$ characterize carbon in the different reservoirs and $\boldsymbol{T}_t \in \mathbb{R}_+^l$ comprise atmospheric global average temperature $T_{1,t}$ and temperature in the different ocean layers. The components of the normalized vector $\boldsymbol{\mathcal{K}}_t \equiv \frac{\boldsymbol{K}_t}{K_t}$ satisfy $\sum_{i=1}^I \mathcal{K}_{i,t} = 1$ and I define log-capital as $k_t \equiv \log K_t = \log\left(\sum_{i=1}^I K_{i,t}\right)$. I work with transformed exponential temperature states $\tau_{i,t} \equiv \exp(\xi_i T_{i,t})$, $i \in \{1, ..., L\}$ that I collect in the vector $\boldsymbol{\tau}_t \in \mathbb{R}^L$. Anticipating that optimal consumption and investment will (in most cases) be proportional to net production, I define the consumption rate $x_t = \frac{C_t}{Y_t^{net}}$. I refer to DACE on how to derive equation (30) for the temperature system.

The assumption that production is homogeneous of some degree κ in capital implies the existence of $\kappa \in (0, 1)$ such that

$$Y_t^{gross} = F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t) = K_t^\kappa F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t).$$

Expressing utility in terms of the consumption rate results in

$$\begin{aligned} u(x_t) \equiv \log C_t &= \log x_t + \log Y_t^{gross} - \eta i_t + \log[1 - D_t(T_{1,t})] \\ &= \log x_t + \kappa \log K_t + \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) - \eta i_t - d(\tau_t, i_t, A_t). \end{aligned} \tag{C.1}$$

The Bellman is

$$\begin{aligned} V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, A_t, t) = \max_{x_t, \boldsymbol{N_t}, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E_t}, i_t} &\log x_t + \kappa k_t + \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) - \eta i_t \\ &- d(\tau_t, i_t, A_t) + \beta V(k_{t+1}, \boldsymbol{\tau}_{t+1}, \boldsymbol{M}_{t+1}, \boldsymbol{R}_{t+1}, A_{t+1}, t+1) \end{aligned} \tag{C.2}$$

subject to the following linear equations of motion

$$
\begin{aligned}
k_{t+1} &= \kappa k_t + \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) - \eta i_t - d(\tau_t, i_t, A_t) + \log(1-x_t) \\
&\quad + \log[1+g_{k,t}] - \log[\delta_k + g_{k,t}] && \text{(C.3)} \\
\boldsymbol{M_{t+1}} &= \boldsymbol{\Phi} \boldsymbol{M}_t + \left(\textstyle\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}\right) \boldsymbol{e}_1 && \text{(C.5)} \\
\boldsymbol{\tau}_{t+1} &= \boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc} \frac{M_{1,t} + G_t}{M_{pre}} \boldsymbol{e}_1 && \text{(C.6)} \\
\boldsymbol{R}_{t+1} &= \boldsymbol{R}_t - \boldsymbol{E}_t^d && \text{(C.7)} \\
A_{t+1} &= (1-\delta_A)(A_t + i_t)
\end{aligned}
$$

and constraints

$$
\textstyle\sum_{i=0}^{I} N_{i,t} = N_t,\ N_{i,t} \geq 0,\ \sum_{i=1}^{I_K} \mathcal{K}_{i,t} = 1,\ \mathcal{K}_{i,t} \geq 0,\ \boldsymbol{R}_t \geq 0, \tag{C.8}
$$

and the initial states given. Following DACE, I assumes that the optimal labor and capital allocation across sectors has an interior solution and that potentially scarce resources are stretched over the infinite time horizon along the optimal path, avoiding boundary value complications.

C.2 Solving the deterministic dynamic programming problem – Proof of Propositions 1 and 2

The following derivation extends DACE's solution approach by adaptation decisions and adaptation capital.

i. Bellman equation. I show that the following (affine) value function solve the Bellman equation (C.2)

$$
V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, A_t, t) = \varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_A A_t + \varphi_t. \tag{C.9}
$$

Inserting the value function's trial solution (equation C.9) and the next period states (equations C.3-C.7) into the (deterministic) Bellmann equation (C.2) delivers

$$
\begin{aligned}
\varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_A A_t + \varphi_t = \max_{x_t, \boldsymbol{N_t}, \boldsymbol{\mathcal{K}_t}, \boldsymbol{E_t}, i_t} & \log x_t + \beta \varphi_k \log(1-x_t) && \text{(C.10)} \\
& + (1+\beta\varphi_k)\kappa k_t + (1+\beta\varphi_k)\log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \underline{-(1+\beta\varphi_k)(\eta i_t + d(\tau_t, i_t, A_t))} \\
& + \beta\varphi_k(\log[1+g_{k,t}] - \log[\delta_k + g_{k,t}]) + \lambda_t^{\mathcal{K}}\big(1 - \textstyle\sum_{i=1}^{I_K} \mathcal{K}_{i,t}\big) + \lambda_t^N\big(N_t - \textstyle\sum_{i=1}^{I_N} N_{i,t}\big) \\
& + \beta \boldsymbol{\varphi}_{R,t+1}^\top \left(\boldsymbol{R}_t - \boldsymbol{E}_t^d\right) + \underline{\beta\varphi_A\left((1-\delta_A)(A_t + i_t)\right)} + \beta\varphi_{t+1} \\
& + \beta \boldsymbol{\varphi}_M^\top \left(\boldsymbol{\Phi}\boldsymbol{M}_t + \big(\textstyle\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}\big)\boldsymbol{e}_1\right) + \beta \boldsymbol{\varphi}_\tau^\top \left(\boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc}\tfrac{M_{1,t}+G_t}{M_{pre}}\boldsymbol{e}_1\right), && \text{(C.11)}
\end{aligned}
$$

where I underline the novel terms generated or altered by adaptation. Maximizing the right hand side of the Bellman equation over the first four controls/control vectors delivers as in DACE

$$x^* = \frac{1}{1+\beta\varphi_k} \tag{C.12}$$

$$\mathcal{K}_{i,t} = \frac{\sigma_{Y,\mathcal{K}_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}{\sum_{i=1}^{I_K} \sigma_{Y,\mathcal{K}_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)} \tag{C.13}$$

$$N_{i,t} = \frac{\sigma_{Y,N_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}{\sum_{i=1}^{I_N} \sigma_{Y,N_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)} \tag{C.14}$$

$$E_{i,t} = \frac{(1+\beta\varphi_k)\sigma_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}{\beta(\varphi_{R,i,t+1} - \varphi_{M,1})} \tag{C.15}$$

where the shadow cost (Hotelling rent) of non-scarce or non-fossil energy inputs is $\varphi_{R,i,t+1} = 0$ and I defined the elasticities $\sigma_{Y,\mathcal{K}_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \equiv \frac{\partial F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}{\partial \mathcal{K}_{i,t}} \frac{\mathcal{K}_{i,t}}{F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}$, $\sigma_{Y,N_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \equiv \frac{\partial F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}{\partial N_{i,t}} \frac{N_{i,t}}{F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}$, and $\sigma_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \equiv \frac{\partial F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}{\partial E_{i,t}} \frac{E_{i,t}}{F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}$

ii. FOCs & effective damages. The new first order conditions governing adaptation investment i_t are

$$\frac{\partial}{\partial i_t}\left[-(1+\beta\varphi_k)(\eta i_t + d(\tau_t, i_t, A_t)) + \beta\varphi_A\left((1-\delta_A)(A_t + i_t)\right)\right] = 0 \tag{C.16}$$

with $d(\tau_t, i_t, A_t) = \xi_0(\tau_{1,t} - 1) + \rho\exp(\lambda\tau_{1,t})\exp(-\chi(A_t + i_t))$ implying

$$\begin{aligned}
&-(1+\beta\varphi_k)\eta - (1+\beta\varphi_k)\big(-\rho\chi\exp(\lambda\tau_{1,t})\exp(-\chi(A_t+i_t))\big) + \beta\varphi_A(1-\delta_A) = 0 \\
\Rightarrow\ &-(1+\beta\varphi_k)\big(-\rho\chi\exp(\lambda\tau_{1,t})\exp(-\chi(A_t+i_t))\big) = (1+\beta\varphi_k)\eta - \beta\varphi_A(1-\delta_A) \\
\Rightarrow\ &i_t^* = \frac{1}{\chi}\Big[\log(\rho\chi) - \log\left(\eta - \frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\right) + \lambda\tau_{1,t}\Big] - A_t.
\end{aligned} \tag{C.17}$$

It is straightforward to verify that the second-order derivative is negative, and I assumed that the parametric choices yield an interior well-defined solution (in particular, positivity of the arguments of the logarithms). For any given set of shadow values, we can solve the system of equations (C.13-C.15) for the optimal controls $\boldsymbol{N}_t^*(\boldsymbol{A}_t, \varphi_k, \boldsymbol{\varphi}_M, \boldsymbol{\varphi}_{R,t+1})$, $\boldsymbol{\mathcal{K}}_t^*(\boldsymbol{A}_t, \varphi_k, \boldsymbol{\varphi}_M, \boldsymbol{\varphi}_{R,t+1})$, and $\boldsymbol{E}_t^*(\boldsymbol{A}_t, \varphi_k, \boldsymbol{\varphi}_M, \boldsymbol{\varphi}_{R,t+1})$. After noting that these solutions are independent of the state variables I will suppress the arguments. I can now plug these solutions, x^*, and equation (C.17) for the optimal adaptation investment back into the Bellman equation (1). For this purpose, I first simplify some of the resulting expressions. Given optimal adaptation levels i_t^*,

the damage function becomes

$$
\begin{aligned}
d(\tau_t, i_t^*, A_t) &= \xi_0(\tau_{1,t}-1) + \rho\exp(\lambda\tau_{1,t})\exp\Big(\\
&\qquad -\chi\Big(A_t + \frac{1}{\chi}\Big[\log(\rho\chi) - \log\Big(\eta - \frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\Big) + \lambda\tau_{1,t}\Big] - A_t\Big)\Big) \\
&= \xi_0(\tau_{1,t}-1) + \rho\exp(\lambda\tau_{1,t})\exp\Big(-\Big[\log(\rho\chi) - \log\Big(\eta - \frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\Big) + \lambda\tau_{1,t}\Big]\Big) \\
&= \xi_0(\tau_{1,t}-1) + \frac{1}{\chi}\left(\eta - \frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\right). \qquad \text{(C.18)}
\end{aligned}
$$

iii. Bellman equation evaluated at optimal controls. Equipped with equations (C.17) and (C.18) the novel (underlined) terms in the Bellman equation (C.10) become

$$
\begin{aligned}
&-(1+\beta\varphi_k)(\eta i_t^* + d(\tau_t, i_t^*, A_t)) + \beta\varphi_A\left((1-\delta_A)(A_t + i_t^*)\right) \\
&= \left(\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta\right)\left(\frac{1}{\chi}\Big[\log(\rho\chi) - \log\Big(\eta - \frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\Big) + \lambda\tau_{1,t}\Big] - A_t\right) \\
&\quad -(1+\beta\varphi_k)\left(\xi_0(\tau_{1,t}-1) + \frac{1}{\chi}\left(\eta - \frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\right)\right) + \beta\varphi_A(1-\delta_A)A_t \\
&= \left(\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta\right)\frac{1}{\chi}\Big[\log(\rho\chi) - \log\Big(\eta - \frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\Big) - 1\Big] + (1+\beta\varphi_k)\xi_0 \\
&\quad +\Big(\left(\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta\right)\frac{\lambda}{\chi} - (1+\beta\varphi_k)\xi_0\Big)\tau_{1,t} \\
&\quad +\Big(\beta\varphi_A(1-\delta_A) - \left(\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta\right)\Big)A_t \qquad \text{(C.19)}
\end{aligned}
$$

With this preparation, I plug the solutions for the control variables back into the Bellman equation (C.10) and sort by states

$$
\begin{aligned}
&\big(\varphi_k - (1+\beta\varphi_k)\kappa\big)k_t + \big(\boldsymbol{\varphi}_{R,t}^\top - \beta\boldsymbol{\varphi}_{R,t+1}^\top\big)\boldsymbol{R}_t + \big(\boldsymbol{\varphi}_M^\top - \beta\boldsymbol{\varphi}_M^\top\boldsymbol{\Phi} - \beta\varphi_{\tau,1}\frac{\sigma^{forc}}{M_{pre}}\boldsymbol{e}_1^\top\big)\boldsymbol{M}_t \\
&+\Big(\boldsymbol{\varphi}_\tau^\top - \beta\boldsymbol{\varphi}_\tau^\top\boldsymbol{\sigma} + \Big((1+\beta\varphi_k)\xi_0 - \underline{\left(\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta\right)\frac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top\Big)\boldsymbol{\tau}_t \\
&+\underline{\Big(\varphi_A - \big(\beta\varphi_A(1-\delta_A) - \left(\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta\right)\big)\Big)A_t} \\
&+\varphi_t \;=\; \beta\varphi_{t+1} \qquad \text{(C.20)} \\
&\quad +\;\log x_t^*(\varphi_k) + \beta\varphi_k\log(1-x_t^*(\varphi_k)) + (1+\beta\varphi_k)\xi_0 + (1+\beta\varphi_k)\log F(\boldsymbol{A}_t, \boldsymbol{N}_t^*, \boldsymbol{\mathcal{K}}_t^*, \boldsymbol{E}_t^*) \\
&\quad +\beta\varphi_k(\log[1+g_{k,t}] - \log[\delta_k + g_{k,t}]) - \beta\boldsymbol{\varphi}_{R,t+1}^\top\boldsymbol{E}_t^{d*} + \beta\varphi_{M,1}\left(\textstyle\sum_{i=1}^{I^d} E_{i,t}^* + E_t^{exo}\right) \\
&\quad +\beta\varphi_{\tau,1}\frac{\sigma^{forc}}{M^{pre}}G_t + \underline{\left(\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta\right)\tfrac{1}{\chi}\Big[\log(\rho\chi) - \log\Big(\eta - \tfrac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\Big) - 1\Big]}
\end{aligned}
$$

where I continue to underline the novel terms with respect to DACE.[54] This equation has to hold for all levels of the state variables, which requires that the coefficients in front of the state variables vanish.

iv. Shadow values. Thus, the shadow values have to satisfy as already in DACE

$$\varphi_k - (1+\beta\varphi_k)\kappa = 0 \qquad \Rightarrow \varphi_k = \frac{\kappa}{1-\beta\kappa} \tag{C.21}$$

$$\boldsymbol{\varphi}_M^\top - \beta\boldsymbol{\varphi}_M^\top\boldsymbol{\Phi} - \beta\varphi_{\tau,1}\frac{\sigma^{forc}}{M_{pre}}\boldsymbol{e}_1^\top = 0 \Rightarrow \boldsymbol{\varphi}_M^\top = \frac{\beta\varphi_{\tau,1}\sigma^{forc}}{M_{pre}}\boldsymbol{e}_1^\top(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1} \tag{C.22}$$

$$\boldsymbol{\varphi}_{R,t}^\top - \beta\boldsymbol{\varphi}_{R,t+1}^\top = 0 \qquad \Rightarrow \boldsymbol{\varphi}_{R,t} = \beta^{-t}\boldsymbol{\varphi}_{R,0}\ .$$

and for the new and altered equations

$$\begin{aligned}
&\varphi_A - \beta\varphi_A(1-\delta_A) + \beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta = 0\\
&\Rightarrow \varphi_A = (1+\beta\varphi_k)\eta = \frac{\eta}{(1-\beta\kappa)}\ , \qquad\qquad (C.23)\\
&\boldsymbol{\varphi}_\tau^\top - \beta\boldsymbol{\varphi}_\tau^\top\boldsymbol{\sigma} + \Big((1+\beta\varphi_k)\xi_0 - \underline{(\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta)\,\tfrac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top = 0\\
&\Rightarrow \boldsymbol{\varphi}_\tau^\top = -\Big((1+\beta\varphi_k)\xi_0 - \underline{(\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta)\,\frac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}\\
&\qquad = -\frac{1}{1-\beta\kappa}\Big(\xi_0 + \underline{\eta\frac{\lambda}{\chi}\,(1-\beta(1-\delta_A))}\Big)\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}\ . \qquad\qquad (C.24)
\end{aligned}$$

Inserting the shadow value of atmospheric temperature (first entry of $\boldsymbol{\tau}_t$-vector in equation C.24) into equation (C.22) delivers

$$\boldsymbol{\varphi}_M^\top = -\frac{\beta\sigma^{forc}}{M_{pre}}\frac{1}{1-\beta\kappa}\Big(\xi_0 + \underline{\eta\frac{\lambda}{\chi}\Big(1-\beta(1-\delta_A)\Big)}\Big)\boldsymbol{e}_1^\top(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}.$$

As in DACE, the Bellman equation (C.20) turns into a condition for the affine constant of the value function of the form $\varphi_t - \beta\varphi_{t+1} = ...$ which can always be met by appropriately choosing the sequence of shadadow values $\varphi_0, \varphi_1, \varphi_2, ...$.[55] By equations (C.12) and (C.21) the optimal consumption rate is $x^* = 1-\beta\kappa$.

v. SCC and propositions. The SCC is the shadow cost of atmospheric carbon in money-measured consumption units. Translating the shadow value from utils into consumption equivalents uses

[54] Equation (C.19) also includes the terms $(1+\beta\varphi_k)\xi_0\tau_{1,t}$ and $(1+\beta\varphi_k)\xi_0$ in addition to the ones underlined above. These are terms that derive from the original (non-adaptable) part of the damage function which is also part of DACE.

[55] See DACE for details. In case of a positive Hotelling rent, $\boldsymbol{\varphi}_{R,0}^\top$ is pinned down by eventually exhausting the scarce resource (otherwise it is always zero). The Bellman equation (C.20) does not directly pin down the initial value of the affine constant φ_0. The additional condition $\lim_{t\to\infty}\beta^t V(\cdot) = 0 \Rightarrow \lim_{t\to\infty}\beta^t\varphi_t = 0$ pins down this initial value φ_0, ensuring that the value function is normalized just as the infinite sum of optimized utility (Stokey & Lucas 1989, chapter 4.1). The discussion in this paper does not depend on these constants.

$du_t = \frac{1}{C_t} dC_t = \frac{1}{x^* Y_t^{net}} dC_t \Rightarrow dC_t = (1-\beta\kappa) Y_t^{net} du_t$ and results in

$$
\begin{aligned}
SCC &= -(1-\beta\kappa) Y_t^{net} \varphi_{M,1} \\
&= Y_t^{net} \frac{\beta \sigma^{forc}}{M_{pre}} \Big(\xi_0 + \eta \frac{\lambda}{\chi} \Big(1 - \beta(1-\delta_A) \Big) \Big) \left[(\mathbf{1} - \beta \mathbf{\Phi})^{-1} \right]_{1,1} \left[(1 - \beta \boldsymbol{\sigma})^{-1} \right]_{1,1},
\end{aligned}
$$

delivering the proof of Proposition 2.

Finally, I use equations (C.21) and (C.23) to eliminate the shadow values in equations (C.17) and (C.18). Observing that $\frac{\beta \varphi_A}{(1+\beta\varphi_k)} = \beta\eta$ I find

$$
\begin{aligned}
i_t^* \qquad &= \frac{1}{\chi} \Big[\log(\rho\chi) - \log\left(\eta - \eta\beta(1-\delta_A)\right) + \lambda \tau_{1,t} \Big] - A_t \\
d(\tau_t, i_t^*, A_t) &= \xi_0(\tau_{1,t} - 1) + \frac{1}{\chi} \left(\eta - \eta\beta(1-\delta_A) \right)
\end{aligned}
$$

delivering Proposition 1.

C.3 Stochastic model and proofs of Propositions 3, 4, and 5

Under uncertainty, all equations of motion are stochastic. Formally, the state variables become functions on a probability space, whose filtration $\mathcal{F}$ is generated by the stochastic climate and damage evolution. My notation suppresses this dependence. All processes are adapted to the filtration $\mathcal{F}$: we learn only what we have already observed. Formally, conditioning on time t information means conditioning expectations on the filtration at time t: $\mathbb{E}_t(\cdot) \equiv \mathbb{E}(\cdot \,|\, \mathcal{F}_t)$.

Following Traeger (2025), I model global average surface temperature using an expectation-corrected autoregressive gamma process. Introduced by Gourieroux & Jasiak (2006), it is a Poisson mixture of gamma distributions

$$
\frac{y_{t+1}}{c} | (Z, X_t) \sim gamma(\nu_t + Z), \text{ where } Z | y_t \sim Poisson\left(\frac{\gamma y_t}{c} \right) \tag{C.25}
$$

for $c, \gamma, \nu_t > 0$ in all periods. The autoregressive gamma process y_t draws from a standard gamma distribution with scale c and modulates the shape parameter drawing from a Poisson distribution. The one-step-ahead expectation of the process are

$$
\begin{aligned}
\mathbb{E}_t(y_{i,t+1} | y_t) \quad &= \nu_t c + \gamma y_t \\
\mathbb{V}\mathrm{ar}_t(y_{i,t+1} | y_t) &= \nu_t c^2 + 2c\gamma y_t
\end{aligned}
$$

and its moment generating function is

$$
\mathrm{MGF}_{y_{t+1}}(u) \equiv \mathbb{E}\left(\exp(u y_{t+1}) | y_t \right) = \exp\big(-\nu_t \log(1-uc) \big) \exp\left(\tfrac{u}{1-uc} \gamma y_t \right) .
$$

The subsequent implementation is based on Traeger's (2021) expectation-corrected shifted gamma process. For this purpose, I pick

$$\nu_t = \frac{\delta_\tau}{c}\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right) ,$$

which results in the expectation and variance

$$\begin{aligned}\mathbb{E}\, y_{t+1} &= \gamma^z y_t + \delta_\tau\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right) \\ \mathbb{V}\mathrm{ar}\, y_{t+1} &= c\Big[2\gamma^z y_t + \delta_\tau\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\Big]\end{aligned} \tag{C.26}$$

and moment generating function

$$\mathrm{MGF}_{y_{t+1}}(u) = \mathbb{E}\left(\exp(uy_{t+1})|y_t\right) = \exp\left(-\tfrac{\delta_\tau}{c}\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\log(1-uc) + \tfrac{u}{1-uc}\gamma^z y_t\right). \tag{C.27}$$

The following deterministic process neutralizes the temperature expectations to those of the deterministic model

$$y^o_{t+1} = \gamma^z y^o_t + \left(\delta_\tau - \epsilon(c,\delta_\tau)\right)\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right), \tag{C.28}$$

where $\epsilon(c,\delta_\tau)$ is a given function that Traeger (2021) calibrates together with the parameters c and δ_τ based on the IPCC's (2013) RCP scenarios and IPCC's (2021) transient climate response.

The expectation adjusted process $z_t \equiv y_t - y^o_t$ has the expectation and variance

$$\begin{aligned}\mathbb{E}_t\, z_{t+1} &= \gamma^z z_t + \epsilon(c,\delta_\tau)\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right) \\ \mathbb{V}\mathrm{ar}_t\, z_{t+1} &= \mathbb{V}\mathrm{ar}_t\, y_{t+1} = c\Big[2\gamma^z y_t + \delta_\tau\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\Big].\end{aligned}$$

Global average temperature dynamics follows

$$\boldsymbol{\tau}_{t+1} = \boldsymbol{\sigma}\boldsymbol{\tau}_t + \left(\sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}} + z_{t+1}\right)\boldsymbol{e}_1. \tag{C.29}$$

with expectations

$$\begin{aligned}\mathbb{E}_t\, \boldsymbol{\tau}_{t+1} &= \boldsymbol{\sigma}\boldsymbol{\tau}_t + \left(\sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}} + \gamma^z z_t + \epsilon(c,\delta_\tau)\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right)\boldsymbol{e}_1 \\ \mathbb{V}\mathrm{ar}_t\, \tau_{1,t+1} &= \mathbb{V}\mathrm{ar}_t\, z_{t+1} = \mathbb{V}\mathrm{ar}_t\, y_{t+1} \\ &= c\Big[2\gamma^z y_t + \delta_\tau\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\Big].\end{aligned} \tag{C.30}$$

I maintain for notational convenience the definition $\tau_t \equiv \tau_{1,t+1}$ introduced in the main text.

i. Stochastic Bellman equation. Using a generic information vector, the Bellman equation is

$$V(K_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, A_t, \boldsymbol{I}_t, t) = \max_{x_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t, i_t} \mathbb{E}_t \Big(\log x_t + \kappa k_t - \eta i_t - d(\tau_t, i_t, A_t) + \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) + \beta V(K_{t+1}, \boldsymbol{\tau}_{t+1}, \boldsymbol{M}_{t+1}, \boldsymbol{R}_{t+1}, A_{t+1}, \boldsymbol{I}_{t+1}, t+1) \Big)$$

which is subject to the following equations. The equations of motion (C.4) for carbon and (C.7) for the resource stock are the same as before. The equation of motion for temperature changes to equation (C.29), which, in addition, relies on the equations of motion (C.25) and (C.28) for the stochastic temperature updates. The constraints (C.8) remain.

Once again, I use an affine trial solution for the value function, adding the new informational state variables y_t and y_t^o

$$V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, A_t, y_t, y_t^o, t) = \varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_A A_t + \varphi_{y^o}^\tau y_t^o + \varphi_y^\tau y_t + \varphi_t. \tag{C.31}$$

The changed equation of motion and the novel terms change the Bellman equation (C.11) to the form

$$\begin{aligned}
&\varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_A A_t + \varphi_{y^o}^\tau y_t^o + \varphi_y^\tau y_t + \varphi_t = \\
&\max_{x_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t, i_t} \log x_t + \beta\varphi_k \log(1-x_t) + (1+\beta\varphi_k)\kappa k_t + (1+\beta\varphi_k)\log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \\
&\quad + \beta\varphi_k(\log[1+g_{k,t}] - \log[\delta_k + g_{k,t}]) + \lambda_t^{\mathcal{K}}\big(1 - \textstyle\sum_{i=1}^{I_K} \mathcal{K}_{i,t}\big) + \lambda_t^N\big(N_t - \textstyle\sum_{i=1}^{I_N} N_{i,t}\big) \\
&\quad + \beta\boldsymbol{\varphi}_{R,t+1}^\top \big(\boldsymbol{R}_t - \boldsymbol{E}_t^d\big) + \underline{\beta\varphi_A\left((1-\delta_A)(A_t + i_t)\right)} + \beta\varphi_{t+1} \\
&\quad + \beta\boldsymbol{\varphi}_M^\top \Big(\boldsymbol{\Phi}\boldsymbol{M}_t + \big(\textstyle\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}\big)\boldsymbol{e}_1\Big) + \beta\boldsymbol{\varphi}_\tau^\top \Big(\boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc}\tfrac{M_{1,t}+G_t}{M_{pre}}\boldsymbol{e}_1\Big) \\
&\quad - \mathbb{E}_t\,\underline{(1+\beta\varphi_k)(\eta i_t + d(\tau_t, i_t, A_t))} \\
&\quad + (-\beta\varphi_{\tau,1} + \beta\varphi_{y^o}^\tau)\Big(\gamma^z y_t^o + (\delta_\tau - \epsilon(c, \delta_\tau))\Big(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\Big)\Big) \quad \text{from } \tau_{1,t+1} \text{ and } y_{t+1}^o \\
&\quad + \mathbb{E}_t(\beta\varphi_{\tau,1} + \beta\varphi_y^\tau) y_{t+1} \quad \text{from } \tau_{1,t+1} \text{ and } y_{t+1}
\end{aligned} \tag{C.33}$$

where the last two lines collect the novel terms result from the stochastic setting.

ii. Damage expectations. The damage function terms now appear in the argument of an expected value operator

$$\mathbb{E}_t(1+\beta\varphi_k)d(\tau_t, i_t, A_t) = \mathbb{E}_t(1+\beta\varphi_k)\xi_0(\tau_{1,t+1} - 1) + \mathbb{E}_t(1+\beta\varphi_k)\rho\exp(\lambda\tau_{1,t+1})\exp(-\chi(A_t + i_t))\ .$$

I evaluate these two damage terms one after the other. By equation (C.30) for next period

temperature, the non-adaptable damage term is

$$\mathbb{E}_t(1+\beta\varphi_k)\xi_0(\tau_{1,t+1}-1) = (1+\beta\varphi_k)\underbrace{\xi_0\Big(\boldsymbol{\sigma}_{1\cdot}\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}}\Big)}_{\equiv\tilde{\xi}_0(\boldsymbol{\tau}_t, M_{1,t})} - (1+\beta\varphi_k)\xi_0$$
$$+(1+\beta\varphi_k)\xi_0\left(\gamma^z z_t + \epsilon(c,\delta_\tau)\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right),$$

where $\boldsymbol{\sigma}_{1\cdot}$ denotes the first row vector of the heat transfer matrix $\boldsymbol{\sigma}$. The adaptable damage term caused by $\tau_{1,t+1}$ is

$$\mathbb{E}_t(1+\beta\varphi_k)\rho\exp(\lambda\tau_{1,t+1})\exp(-\chi(A_t+i_t)) \tag{C.34}$$
$$= (1+\beta\varphi_k)\underbrace{\rho\exp\left(\lambda\Big(\boldsymbol{\sigma}_{1\cdot}\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}}\Big)\right)}_{\equiv\tilde{\rho}(\boldsymbol{\tau}_t, M_{1,t})}\mathbb{E}_t\exp\big(\lambda(z_{t+1})\big)\exp(-\chi(A_t+i_t))$$
$$= (1+\beta\varphi_k)\tilde{\rho}(\boldsymbol{\tau}_t, M_{1,t})\ \exp(-\chi(A_t+i_t))\ \exp\big(-\lambda y^o_{t+1}\big)\,\mathbb{E}_t\exp\big(\lambda(y_{t+1})\big)$$
$$= (1+\beta\varphi_k)\underbrace{\tilde{\rho}(\boldsymbol{\tau}_t, M_{1,t})\ \exp\left(-\lambda\left(\gamma^z y^o_t + (\delta_\tau - \epsilon(c,\delta_\tau))\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right)\right)}_{\equiv\hat{\rho}(\boldsymbol{\tau}_t, M_{1,t}, y^o_t)}\exp(-\chi(A_t+i_t))$$
$$\mathbb{E}_t\exp\big(\lambda(y_{t+1})\big)$$
$$= (1+\beta\varphi_k)\,\hat{\rho}(\boldsymbol{\tau}_t, M_{1,t}, y^o_t)\ \exp(-\chi(A_t+i_t))\ \mathrm{MGF}_{y_{t+1}}(\lambda),$$

where $\mathrm{MGF}_{y_{t+1}}(\lambda)$ is the moment generating function in equation (C.27) evaluated at λ.

iii. Bellman equation with expected damages. Plugging these results into the Bellman

equation (C.32) delivers

$$
\begin{aligned}
&\varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_A A_t + \varphi_{y^o}^\tau y_t^o + \varphi_y^\tau y_t + \varphi_t = \\
&\max_{x_t, \boldsymbol{N_t}, \boldsymbol{\mathcal{K}_t}, \boldsymbol{E_t}, i_t} \ \log x_t + \beta\varphi_k \log(1-x_t) + (1+\beta\varphi_k)\kappa k_t + (1+\beta\varphi_k)\log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \\
&\quad + \beta\varphi_k(\log[1+g_{k,t}] - \log[\delta_k + g_{k,t}]) + \lambda_t^{\mathcal{K}}\big(1 - \textstyle\sum_{i=1}^{I_K} \mathcal{K}_{i,t}\big) + \lambda_t^N\big(N_t - \textstyle\sum_{i=1}^{I_N} N_{i,t}\big) \\
&\quad + \beta\boldsymbol{\varphi}_{R,t+1}^\top \big(\boldsymbol{R}_t - \boldsymbol{E}_t^d\big) + \underline{\beta\varphi_A\left((1-\delta_A)(A_t+i_t)\right)} \\
&\quad + \beta\boldsymbol{\varphi}_M^\top \Big(\boldsymbol{\Phi}\boldsymbol{M}_t + \big(\textstyle\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}\big)\boldsymbol{e}_1\Big) + \beta\boldsymbol{\varphi}_\tau^\top \Big(\boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc}\tfrac{M_{1,t}+G_t}{M_{pre}}\boldsymbol{e}_1\Big) \\
&\quad \underline{-(1+\beta\varphi_k)(\eta i_t)} + \beta\varphi_{t+1} \\
&\quad \underline{-(1+\beta\varphi_k)\tilde{\xi}_0(\boldsymbol{\tau}_t, M_{1,t}) + (1+\beta\varphi_k)\xi_0} \\
&\quad\quad \underline{-(1+\beta\varphi_k)\xi_0\left(\gamma^z z_t + \epsilon(c,\delta_\tau)\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right)} \\
&\quad \underline{-(1+\beta\varphi_k)\,\hat{\rho}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)\,\exp(-\chi(A_t+i_t))\,\mathrm{MGF}_{y_{t+1}}(\lambda)} \\
&\quad + (-\beta\varphi_{\tau,1} + \beta\varphi_{y^o}^\tau)\left(\gamma^z y_t^o + (\delta_\tau - \epsilon(c,\delta_\tau))\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right) \\
&\quad + \mathbb{E}_t(\beta\varphi_{\tau,1} + \beta\varphi_y^\tau) y_{t+1}\ .
\end{aligned}
\tag{C.35}
$$

iv. FOCs. The first order condition for the abatement investment i_t now has the form

$$
\begin{aligned}
\frac{\partial}{\partial i_t}\Big[\ &\beta\varphi_A(1-\delta_A)i_t - (1+\beta\varphi_k)\eta i_t \\
&-(1+\beta\varphi_k)\,\hat{\rho}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)\,\exp(-\chi(A_t+i_t))\,\mathrm{MGF}_{y_{t+1}}(\lambda)\Big] = 0.
\end{aligned}
\tag{C.36}
$$

This equation is equivalent to the previous first order condition equation (C.16), which contains the term $-(1+\beta\varphi_k)\rho\exp(\lambda\tau_{1,t})\exp(-\chi(A_t+i_t))$, when replacing

$$\rho\exp(\lambda\tau_{1,t}) \quad \to \quad \hat{\rho}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)\,\mathrm{MGF}_{y_{t+1}}(\lambda)$$

or, equivalently,

$$\log(\rho) + \lambda\tau_{1,t} \quad \to \quad \log(\hat{\rho}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)) + \log(\mathrm{MGF}_{y_{t+1}}(\lambda))$$

yielding by analogy to equation (C.17)

$$
\begin{aligned}
i_t^* = \frac{1}{\chi}\Big[& \log(\hat{\rho}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)) + \log(\chi) - \log\left(\eta - \frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\right) \\
& + \log(\mathrm{MGF}_{y_{t+1}}(\lambda))\Big] - A_t \qquad \text{(C.37)} \\
= \frac{1}{\chi}\Bigg[& \log(\rho) + \left(\lambda\left(\boldsymbol{\sigma}_1.\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}}\right)\right) \\
& + \left(- \lambda\left(\gamma^z y_t^o + (\delta_\tau - \epsilon(c,\delta_\tau))\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right)\right) \\
& + \log(\chi) - \log\left(\eta - \frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\right) \\
& + \left(- \tfrac{\delta_\tau}{c}\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\log(1-\lambda c) + \tfrac{\lambda}{1-\lambda c}\gamma^z y_t\right)\Bigg] - A_t
\end{aligned}
$$

where the final transformation uses equation (C.27) to evaluate $\log(\mathrm{MGF}_{y_{t+1}}(\lambda))$ and reinserts the definition

$$
\begin{aligned}
\hat{\rho}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o) = \rho \exp & \left(\lambda\left(\boldsymbol{\sigma}_1.\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}}\right)\right) \\
& \exp\left(- \lambda\left(\gamma^z y_t^o + (\delta_\tau - \epsilon(c,\delta_\tau))\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right)\right)
\end{aligned}
$$

The changes in the equations will generally lead to updates in the shadow values, see below, but it does not change the mechanics of solving the system of equations for the optimal controls $\boldsymbol{N}_t^*(\boldsymbol{A}_t, \varphi_k, \boldsymbol{\varphi}_M, \boldsymbol{\varphi}_{R,t+1})$, $\boldsymbol{\mathcal{K}}_t^*(\boldsymbol{A}_t, \varphi_k, \boldsymbol{\varphi}_M, \boldsymbol{\varphi}_{R,t+1})$, and $\boldsymbol{E}_t^*(\boldsymbol{A}_t, \varphi_k, \boldsymbol{\varphi}_M, \boldsymbol{\varphi}_{R,t+1})$. Most importantly, these solutions remain independent of the state variables, and hence, I will suppress their arguments.

v. Effective damages and Bellman equation evaluated at optimal controls. Before plugging the solution back into the Bellman equation (C.35), I first simplify the terms corresponding to adaptable damages after inserting the optimal investment levels (the last two underlined terms in equation C.35). First, for the adaptable component of temperature-driven

damage, I find using the first line of equation (C.37)

$$
\begin{aligned}
&\underline{-(1+\beta\varphi_k)\,\hat{\rho}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)\,\exp(-\chi(A_t+i_t^*))\,\mathrm{MGF}_{y_{t+1}}(\lambda)} \\
&= -(1+\beta\varphi_k)\hat{\rho}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)\,\mathrm{MGF}_{y_{t+1}}(\lambda)\exp\Big(-\chi\Big(A_t+\frac{1}{\chi}\Big[\log(\hat{\rho}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)) \\
&\quad +\log(\chi)-\log\left(\eta-\frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\right)+\log(\mathrm{MGF}_{y_{t+1}}(\lambda))+\log()\Big]-A_t\Big)\Big) \\
&= -(1+\beta\varphi_k)\hat{\rho}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)\,\mathrm{MGF}_{y_{t+1}}(\lambda)\exp\Big(-\Big(\log(\hat{\rho}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)) \\
&\quad +\log(\chi)-\log\left(\eta-\frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\right)+\log(\mathrm{MGF}_{y_{t+1}}(\lambda))\Big)\Big) \\
&= -(1+\beta\varphi_k)\exp\Big(-\log(\chi)+\log\left(\eta-\frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\right)\Big)\Big) \\
&= -(1+\beta\varphi_k)\frac{1}{\chi}\left(\eta-\frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\right), \qquad \text{(C.38)}
\end{aligned}
$$

which coincides with the corresponding term in the deterministic model's equation (C.18). Returning to the Bellman equation, I rearrange the first two underlined terms in equation (C.35) as follows

$$
\begin{aligned}
&\underline{+\beta\varphi_A\left((1-\delta_A)(A_t+i_t^*)\right)-(1+\beta\varphi_k)(\eta i_t^*)} \\
&= \underline{+\beta\varphi_A(1-\delta_A)A_t}+\underline{(\beta\varphi_A(1-\delta_A)-(1+\beta\varphi_k)\eta)i_t^*},
\end{aligned}
$$

Plugging the solutions for the optimized control variables back into the Bellman equation (C.35)

delivers

$$
\begin{aligned}
&\varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_A A_t + \varphi_A^\Delta A_t^\Delta + \varphi_{y^o}^\tau y_t^o + \varphi_y^\tau y_t + \varphi_t = \\
&\quad \log x_t + \beta\varphi_k \log(1-x_t) + (1+\beta\varphi_k)\kappa k_t + (1+\beta\varphi_k)\log F(\boldsymbol{A}_t^*, \boldsymbol{N}_t^*, \boldsymbol{\mathcal{K}}_t^*, \boldsymbol{E}_t^*) \\
&\quad + \beta\varphi_k(\log[1+g_{k,t}] - \log[\delta_k + g_{k,t}]) + \lambda_t^{\mathcal{K}}\big(1 - \textstyle\sum_{i=1}^{I_K} \mathcal{K}_{i,t}^*\big) + \lambda_t^N\big(N_t - \textstyle\sum_{i=1}^{I_N} N_{i,t}^*\big) \\
&\quad + \beta\boldsymbol{\varphi}_{R,t+1}^\top \left(\boldsymbol{R}_t - \boldsymbol{E}_t^{\boldsymbol{d}*}\right) + \underline{\beta\varphi_A(1-\delta_A)A_t} \\
&\quad + (\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta)\Bigg(\frac{1}{\chi}\Bigg[\log(\rho) + \lambda\Bigg(\boldsymbol{\sigma}_{1\cdot}\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}}\Bigg) \\
&\qquad\quad + \underline{\Bigg(-\lambda\left(\gamma^z y_t^o + (\delta_\tau - \epsilon(c,\delta_\tau))\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right)\Bigg)} \\
&\qquad\quad + \underline{\log(\chi) - \log\left(\eta - \frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\right)} \\
&\qquad\quad + \underline{\Bigg(-\tfrac{\delta_\tau}{c}\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\log(1-\lambda c) + \tfrac{\lambda}{1-\lambda c}\gamma^z y_t\Bigg)\Bigg] - A_t\Bigg) \\
&\quad + \beta\varphi_{t+1} \\
&\quad + \beta\boldsymbol{\varphi}_M^\top\left(\boldsymbol{\Phi}\boldsymbol{M}_t + \big(\textstyle\sum_{i=1}^{I^d} E_{i,t}^* + E_t^{exo}\big)\boldsymbol{e}_1\right) + \beta\boldsymbol{\varphi}_\tau^\top\left(\boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc}\tfrac{M_{1,t}+G_t}{M_{pre}}\boldsymbol{e}_1\right) \\
&\quad \underline{-(1+\beta\varphi_k)\xi_0\Big(\boldsymbol{\sigma}_{1\cdot}\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}}\Big) + (1+\beta\varphi_k)\xi_0} \\
&\qquad \underline{-(1+\beta\varphi_k)\xi_0\left(\gamma^z(y_t - y_t^o) + \epsilon(c,\delta_\tau)\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right)} \\
&\quad \underline{-(1+\beta\varphi_k)\frac{1}{\chi}\left(\eta - \frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)}\right)} \\
&\quad + (-\beta\varphi_{\tau,1} + \beta\varphi_{y^o}^\tau)\left(\gamma^z y_t^o + (\delta_\tau - \epsilon(c,\delta_\tau))\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right) \\
&\quad + (\beta\varphi_{\tau,1} + \beta\varphi_y^\tau)\left(\gamma^z y_t + \delta_\tau\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right).
\end{aligned}
$$

where the last line uses equation (C.26) to evaluate $\mathbb{E}_t\, y_{t+1}$. Most importantly, the Bellman equation above is linear in the state variables. As a result, I can match the coefficients of the states on both sides of the equation and the affine trial solution for the value function indeed solves the dynamic programming problem. The expressions that are independent of the states can be matched as before using an appropriate sequence of the value functions time varying constants $\varphi_0, \varphi_1, ...$ as before.

vi. Shadow values. I obtain the states' shadow values by carrying out the coefficient matching one step at a time. I first not that the terms containing the resource stock $\boldsymbol{R}_t$, the overall capital k_t and the abatement capital A_t are identical to those appearing in the deterministic Bellman equation (C.20) and still return

$$
\varphi_k = \frac{\kappa}{1-\beta\kappa}\ ; \qquad \boldsymbol{\varphi}_{R,t} = \beta^{-t}\boldsymbol{\varphi}_{R,0}\ ; \qquad \varphi_A = (1+\beta\varphi_k)\eta = \frac{\eta}{(1-\beta\kappa)} \tag{C.39}
$$

and the temperature shadow value experiences a minor modification based on the slightly different timing owed to uncertainty (instead of $\boldsymbol{e}_1^\top$):

$$\boldsymbol{\varphi}_\tau^\top = -\Big((1+\beta\varphi_k)\xi_0 - (\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta)\,\tfrac{\lambda}{\chi}\Big)\boldsymbol{\sigma}_{1\cdot}\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]$$

The remaining equations matching the state coefficients and determining their shadow values become

$$\begin{aligned}
\boldsymbol{\varphi}_M^\top = \beta\boldsymbol{\varphi}_M^\top\boldsymbol{\Phi} + \Bigg(& \beta\tfrac{\sigma^{forc}}{M_{pre}}\varphi_{\tau,1} + \underbrace{\tfrac{\beta}{M_{pre}}(\varphi_{y^o}^\tau - \varphi_{\tau,1})(\delta_\tau - \epsilon(c,\delta_\tau)) + \tfrac{\beta}{M_{pre}}(\varphi_y^\tau + \varphi_{\tau,1})\delta_\tau}_{= \tfrac{\beta}{M_{pre}}\left[(\varphi_{\tau,1} - \varphi_{y^o}^\tau)\epsilon(c,\delta_\tau) + (\varphi_y^\tau + \varphi_{y^o}^\tau)\delta_\tau\right]} \\
& + (\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta)\tfrac{1}{\chi}\left[\lambda\sigma^{forc} - \lambda(\delta_\tau - \epsilon(c,\delta_\tau))\right]\tfrac{1}{M_{pre}} \\
& - (\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta)\tfrac{1}{\chi}\tfrac{\delta_\tau}{c}\log(1-\lambda c)\tfrac{1}{M_{pre}} \\
& - (1+\beta\varphi_k)\xi_0\sigma^{forc}\tfrac{1}{M_{pre}} - (1+\beta\varphi_k)\xi_0\epsilon(c,\delta_\tau)\tfrac{1}{M_{pre}}\Bigg)\boldsymbol{e}_1^\top \\
= \beta\boldsymbol{\varphi}_M^\top\boldsymbol{\Phi} + \tfrac{1}{M_{pre}}\Bigg(& \beta\sigma^{forc}\varphi_{\tau,1} + \beta\left[(\varphi_{\tau,1} - \varphi_{y^o}^\tau)\epsilon(c,\delta_\tau) + \beta(\varphi_y^\tau + \varphi_{y^o}^\tau)\delta_\tau\right] \\
& + (\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta)\tfrac{1}{\chi}\Big(\lambda\sigma^{forc} - \lambda(\delta_\tau - \epsilon(c,\delta_\tau)) - \tfrac{\delta_\tau}{c}\log(1-\lambda c)\Big) \\
& - (1+\beta\varphi_k)\xi_0\Big(\sigma^{forc} + \epsilon(c,\delta_\tau)\Big)\Bigg)\boldsymbol{e}_1^\top \\
\varphi_{y^o}^\tau = -(\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta)\tfrac{1}{\chi}\lambda\gamma^z + (1+\beta\varphi_k)\xi_0\gamma^z + (\beta\varphi_{y^o}^\tau - \beta\varphi_{\tau,1})\gamma^z \\
\varphi_y^\tau = (\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta)\frac{1}{\chi}\frac{\lambda}{1-\lambda c}\gamma^z - (1+\beta\varphi_k)\xi_0\gamma^z + (\beta\varphi_{\tau,1} + \beta\varphi_y^\tau)\gamma^z
\end{aligned}$$

Ultimately, my focus is on the shadow value of atmospheric carbon. It depends on all preceding shadow values and I solve the expressions successively:

$$\begin{aligned}
\varphi_{y^o}^\tau &= \tfrac{1}{1-\beta\gamma^z}\Big(-(\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta)\tfrac{\lambda}{\chi}\gamma^z + (1+\beta\varphi_k)\xi_0\gamma^z - \beta\gamma^z\varphi_{\tau,1}\Big) \\
\varphi_y^\tau &= \tfrac{1}{1-\beta\gamma^z}\Big((\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta)\tfrac{1}{\chi}\tfrac{\lambda}{1-\lambda c}\gamma^z - (1+\beta\varphi_k)\xi_0\gamma^z + \beta\gamma^z\varphi_{\tau,1}\Big)
\end{aligned}$$

Using the abbreviation

$$\begin{aligned}
\Omega \equiv (\beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta) &= \beta(1-\delta_A)(1+\beta\varphi_k)\eta - (1+\beta\varphi_k)\eta \\
&= (1+\beta\varphi_k)\eta\,(\beta(1-\delta_A) - 1) = \frac{\eta}{1-\beta\kappa}\,(\beta(1-\delta_A) - 1)
\end{aligned}$$

I calculate the combinations

$$\begin{aligned}
\varphi_{y^o}^{\tau} + \varphi_{y}^{\tau} &= \frac{1}{1-\beta\gamma^z}\Omega\frac{\lambda}{\chi}\frac{-1+\lambda c+1}{1-\lambda c}\gamma^z = \frac{1}{1-\beta\gamma^z}\Omega\frac{\lambda}{\chi}\frac{\lambda c}{1-\lambda c}\gamma^z \\
\varphi_{1,\tau} - \varphi_{y^o}^{\tau} &= \frac{1}{1-\beta\gamma^z}\left(\Omega\frac{\lambda}{\chi}\gamma^z - (1+\beta\varphi_k)\xi_0\gamma^z + \beta\gamma^z\varphi_{\tau,1} + (1-\beta\gamma^z)\varphi_{1,\tau}\right) \\
&= \frac{}{1-\beta\gamma^z}\left(\Omega\frac{\lambda}{\chi}\gamma^z - (1+\beta\varphi_k)\xi_0\gamma^z + \varphi_{1,\tau}\right) \\
&= \frac{}{1-\beta\gamma^z}\left(\Omega\frac{\lambda}{\chi} - (1+\beta\varphi_k)\xi_0\right)\left(\gamma^z + \boldsymbol{\sigma}_{1\cdot}\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]\right)
\end{aligned}$$

which feed into the shadow value of carbon

$$\begin{aligned}
\boldsymbol{\varphi}_M^{\top} = \beta\boldsymbol{\varphi}_M^{\top}\boldsymbol{\Phi} + \tfrac{1}{M_{pre}}\Bigg[&\left(\Omega\tfrac{\lambda}{\chi} - (1+\beta\varphi_k)\xi_0\right)\times \\
&\left(\beta\sigma^{forc}\boldsymbol{\sigma}_{1\cdot}\left[(1-\beta\boldsymbol{\sigma})^{-1}\right] + \tfrac{\beta}{1-\beta\gamma^z}\left(\gamma^z + \boldsymbol{\sigma}_{1\cdot}\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]\right)\epsilon(c,\delta_\tau)\right) \\
&+\Omega\tfrac{\lambda}{\chi}\left(\tfrac{\beta}{1-\beta\gamma^z}\tfrac{\lambda c}{1-\lambda c}\gamma^z\delta_\tau + \left(\sigma^{forc} - (\delta_\tau - \epsilon(c,\delta_\tau)) - \tfrac{\delta_\tau}{\lambda c}\log(1-\lambda c)\right)\right) \\
&-(1+\beta\varphi_k)\xi_0\left(\sigma^{forc} + \epsilon(c,\delta_\tau)\right)\Bigg]\boldsymbol{e}_1^{\top} \\
= \beta\boldsymbol{\varphi}_M^{\top}\boldsymbol{\Phi} + \tfrac{1}{M_{pre}}\Bigg[&\left(\Omega\tfrac{\lambda}{\chi} - (1+\beta\varphi_k)\xi_0\right)\times \\
&\left(\beta\left(\sigma^{forc} + \tfrac{\epsilon(c,\delta_\tau)}{1-\beta\gamma^z}\right)\boldsymbol{\sigma}_{1\cdot}\left[(1-\beta\boldsymbol{\sigma})^{-1}\right] + \tfrac{\beta\gamma^z\epsilon(c,\delta_\tau)}{1-\beta\gamma^z} + \sigma^{forc} + \epsilon(c,\delta_\tau)\right) \\
&+\Omega\tfrac{\lambda}{\chi}\delta_\tau\left(\tfrac{\beta\gamma^z}{1-\beta\gamma^z}\tfrac{\lambda c}{1-\lambda c} - 1 - \tfrac{1}{\lambda c}\log(1-\lambda c)\right)\Bigg)\Bigg]\boldsymbol{e}_1^{\top} \\
= \beta\boldsymbol{\varphi}_M^{\top}\boldsymbol{\Phi} + \tfrac{1}{M_{pre}}\Bigg[&\left(\Omega\tfrac{\lambda}{\chi} - (1+\beta\varphi_k)\xi_0\right)\left(\sigma^{forc} + \tfrac{\epsilon(c,\delta_\tau)}{1-\beta\gamma^z}\right)\left(1 + \beta\boldsymbol{\sigma}_{1\cdot}\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]\right) \\
&+\Omega\tfrac{\lambda}{\chi}\left(\tfrac{\beta}{1-\beta\gamma^z}\tfrac{\lambda c}{1-\lambda c}\gamma^z\delta_\tau - \delta_\tau - \tfrac{\delta_\tau}{\lambda c}\log(1-\lambda c)\right)\Bigg)\Bigg]\boldsymbol{e}_1^{\top}\ .
\end{aligned}$$

The j^{th} element of the temperature dynamics term simplifies as follows

$$\begin{aligned}
\sum_{k=1}^{n}\left(1 + \beta\,\sigma_{1k}\left[(I-\beta\boldsymbol{\sigma})^{-1}\right]_{kj}\right) &= 1 + \sum_{k=1}^{n}\beta\,\sigma_{1k}\sum_{m=0}^{\infty}\beta^m(\sigma^m)_{kj} \\
= 1 + \sum_{m=0}^{\infty}\beta^{m+1}(\sigma^{m+1})_{1j} &= \sum_{m=0}^{\infty}\beta^m(\sigma^m)_{1j} = \left[(I-\beta\boldsymbol{\sigma})^{-1}\right]_{1j},
\end{aligned}$$

resulting in the shadow value of carbon

$$\begin{aligned}\boldsymbol{\varphi}_M^\top = \tfrac{1}{M_{pre}}\Bigg[&\Big(\Omega\tfrac{\lambda}{\chi} - (1+\beta\varphi_k)\xi_0\Big)\Big(\sigma^{forc} + \tfrac{\epsilon(c,\delta_\tau)}{1-\beta\gamma^z}\Big)\left[(I-\beta\boldsymbol{\sigma})^{-1}\right]_{1\cdot}\\ &+\Omega\tfrac{\lambda}{\chi}\Big(\tfrac{\beta}{1-\beta\gamma^z}\tfrac{\lambda c}{1-\lambda c}\gamma^z\delta_\tau - \delta_\tau - \tfrac{\delta_\tau}{\lambda c}\log(1-\lambda c)\Big)\Big)\Bigg]\left[(1-\beta\boldsymbol{\Phi})^{-1}\right]_{1\cdot}\\ = \tfrac{1}{1-\beta\kappa}\,\tfrac{1}{M_{pre}}\Bigg[&-\Big(\eta\tfrac{\lambda}{\chi}(1-\beta(1-\delta_A)) + \xi_0\Big)\Big(\sigma^{forc} + \tfrac{\epsilon(c,\delta_\tau)}{1-\beta\gamma^z}\Big)\left[(I-\beta\boldsymbol{\sigma})^{-1}\right]_{1\cdot}\\ &-\eta\tfrac{\lambda}{\chi}\tfrac{1}{1-\beta\kappa}\,(1-\beta(1-\delta_A))\Big(\tfrac{\beta}{1-\beta\gamma^z}\tfrac{\lambda c}{1-\lambda c}\gamma^z\delta_\tau - \delta_\tau - \tfrac{\delta_\tau}{\lambda c}\log(1-\lambda c)\Big)\Big)\Bigg]\left[(1-\beta\boldsymbol{\Phi})^{-1}\right]_{1\cdot}.\end{aligned}$$

vii. SCC and propositions. Once again, the SCC follows from the shadow value of carbon as

$$\begin{aligned}SCC &= -(1-\beta\kappa)Y_t^{net}\varphi_{M,1}\\ &= \frac{Y_t^{net}}{M_{pre}}\Bigg(\Big(\eta\frac{\lambda}{\chi}(1-\beta(1-\delta_A)) + \xi_0\Big)\Big(\sigma^{forc} + \frac{\epsilon(c,\delta_\tau)}{1-\beta\gamma^z}\Big)\left[(I-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}\\ &\quad+\eta\tfrac{\lambda}{\chi}\tfrac{1}{1-\beta\kappa}\,(1-\beta(1-\delta_A))\Big(\tfrac{\beta}{1-\beta\gamma^z}\tfrac{\lambda c}{1-\lambda c}\gamma^z\delta_\tau - \delta_\tau - \tfrac{\delta_\tau}{\lambda c}\log(1-\lambda c)\Big)\Big)\Bigg)\left[(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\right]_{1,1}\end{aligned}$$

delivering Proposition 5 by splitting off the second part of the expression.

By equations (C.33) and (C.36), i_t^* solves the FOC

$$\begin{aligned}&\frac{\partial}{\partial i_t}\Big[\beta\varphi_A(1-\delta_A)i_t - (1+\beta\varphi_k)\eta i_t - (1+\beta\varphi_k)\rho\,\mathbb{E}_t\exp(\lambda\tau_{1,t+1})\exp(-\chi(A_t+i_t))\Big] = 0.\\ &\Rightarrow \beta\varphi_A(1-\delta_A) - (1+\beta\varphi_k)\eta = -(1+\beta\varphi_k)\rho\chi\,\mathbb{E}_t\exp(\lambda\tau_{1,t+1})\exp(-\chi(A_t+i_t))\end{aligned}$$

Using $\frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)} = \beta\eta(1-\delta_A)$, I find the optimal adaptation investment

$$\Rightarrow i_t^* = \frac{1}{\chi}\Big[\log(\rho) + \log(\chi) - \log\Big(\eta\big(1-\beta(1-\delta_A)\big)\Big) + \mathrm{CGF}_{\tau_{t+1}}(\lambda)\Big] - A_t,$$

stated as equation (17) in Proposition 3. I note that equation (C.37)'s expression for optimal investment breaks the cumulant generating function down further based on the composition of the stochastic process governing τ_{t+1}. Proposition 3's expected damage expression derives from equation (C.38) once more using $\frac{\beta\varphi_A(1-\delta_A)}{(1+\beta\varphi_k)} = \beta\eta(1-\delta_A)$:

$$\begin{aligned}\mathbb{E}_t\,d(\tau_t, i_t^*, A_t) &= \mathbb{E}_t\,\xi_0(\tau_{1,t+1}-1) + \mathbb{E}_t\,\rho\exp(\lambda\tau_{1,t+1})\exp(-\chi(A_t+i_t^*))\\ &= \xi_0(\mathbb{E}_t\,\tau_{1,t+1} - 1) + \frac{1}{\chi}\left(\eta - \eta\beta(1-\delta_A)\right)\ .\end{aligned}$$

Proposition 4 inserts the optimal investment stated above into $d(\tau_t, i_t^*, A_t)$, without taking expectations, resulting in

$$\begin{aligned} d(\tau_t, i_t^*, A_t) &= \xi_0(\tau_{1,t+1} - 1) + \rho \exp(\lambda \tau_{1,t+1}) \exp(-\chi(A_t + i_t^*)) \\ &= \xi_0(\tau_{t+1} - 1) + \frac{\eta}{\chi}\big(1 - \beta(1 - \delta_A)\big) \exp\left(\lambda \tau_{t+1} - \mathrm{CGF}_{\tau_{t+1}}(\lambda)\right) \ . \end{aligned}$$

D Proof of Theorem 1 and Corollary 1

Proof of Propositions 1. Differentiating the value function with respect to the vector of moments $\boldsymbol{m}_t$ results in the vector of changes

$$\begin{aligned} \frac{\partial V(\boldsymbol{m}_t, A_t, K_t)}{\partial \boldsymbol{m}_t} &= \frac{\partial}{\partial \boldsymbol{m}_t} \mathbb{E}_t \left[U(T_t, i_t^*(\boldsymbol{m}_t, A_t, K_t), A_t, x_t^*(\boldsymbol{m}_t, A_t, K_t), K_t) \right] \\ &\quad + \beta \frac{\partial}{\partial \boldsymbol{m}_t} \mathbb{E} \left[V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1}) \right]. \end{aligned} \tag{D.1}$$

The first order conditions for investment into adaptation and regular capital deliver

$$\mathbb{E}\left[\frac{\partial U(T_t, i_t, A_t, x_t, K_t)}{\partial i_t} + \beta \frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1})}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial i_t} + \beta \frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1})}{\partial K_{t+1}} \frac{\partial K_{t+1}}{\partial i_t} \right] = 0,$$

$$\mathbb{E}\left[\frac{\partial U(T_t, i_t, A_t, x_t, K_t)}{\partial x_t} + \beta \frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1})}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial x_t} + \beta \frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1})}{\partial K_{t+1}} \frac{\partial K_{t+1}}{\partial x_t} \right] = 0.$$

Spelling out the different contributions on the right of equation (D.1) in detail, canceling those adding up to zero based on the first order conditions (envelope theorem), delivers

$$
\begin{aligned}
\frac{\partial V(\boldsymbol{m}_t, A_t, K_t)}{\partial \boldsymbol{m}_t} = {} & \cancel{\mathbb{E}_t \left[\frac{\partial U(T_t, i_t^*(\boldsymbol{m}_t, A_t, K_t), A_t, x_t^*(\boldsymbol{m}_t, A_t, K_t), K_t)}{\partial i_t} \right] \frac{\partial i_t^*(\boldsymbol{m}_t, A_t, K_t)}{\partial \boldsymbol{m}_t}} \\
& \cancel{+\mathbb{E}_t \left[\frac{\partial U(T_t, i_t^*(\boldsymbol{m}_t, A_t, K_t), A_t, x_t^*(\boldsymbol{m}_t, A_t, K_t), K_t)}{\partial x_t} \right] \frac{\partial x_t^*(\boldsymbol{m}_t, A_t, K_t)}{\partial \boldsymbol{m}_t}} \\
& + \int U(T_t, i_t^*(\boldsymbol{m}_t, A_t, K_t), K_t, x_t^*(\boldsymbol{m}_t, A_t, K_t), A_t) \frac{\partial P(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} \, dT_t \\
& + \beta \, \mathbb{E}_t \left[\frac{\partial V(\boldsymbol{m}_{t+1}(T_t), A_{t+1}(T_t), K_{t+1}(T_t))}{\partial \boldsymbol{m}_{t+1}} \frac{\partial \boldsymbol{m}_{t+1}}{\partial \boldsymbol{m}_t} \right] \\
& \cancel{+\beta \, \mathbb{E}_t \left[\frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1})}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial i_t} \right] \frac{\partial i_t^*(\boldsymbol{m}_t, A_t, K_t)}{\partial \boldsymbol{m}_t}} \\
& \cancel{+\beta \, \mathbb{E}_t \left[\frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1})}{\partial K_{t+1}} \frac{\partial K_{t+1}}{\partial i_t} \right] \frac{\partial i_t^*(\boldsymbol{m}_t, A_t, K_t)}{\partial \boldsymbol{m}_t}} \\
& \cancel{+\beta \, \mathbb{E}_t \left[\frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1})}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial x_t} \right] \frac{\partial x_t^*(\boldsymbol{m}_t, A_t, K_t)}{\partial \boldsymbol{m}_t}} \\
& \cancel{+\beta \, \mathbb{E}_t \left[\frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1})}{\partial K_{t+1}} \frac{\partial K_{t+1}}{\partial i_t} \right] \frac{\partial x_t^*(\boldsymbol{m}_t, A_t, K_t)}{\partial \boldsymbol{m}_t}} \\
& + \beta \int V(\boldsymbol{m}_{t+1}(T_t), A_{t+1}(T_t), K_{t+1}(T_t)) \frac{\partial P(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} \, dT_t. \qquad \text{(D.2)}
\end{aligned}
$$

The first non-vanishing term quantifies how changes in the probability density function governing temperatures alter the representative agent's expected utility in the current period keeping investments fix (no adaptation).

The second non-vanishing term captures the effect on future welfare when a shift in the current climate influences the future climate. Typically, we are not concerned with arbitrary climate change but rather with anthropogenic shifts driven by greenhouse gas emissions. These emissions determine both the specific change $d\boldsymbol{m}_t$ and its evolution over time, given by $\frac{\partial \boldsymbol{m}_{t+1}}{\partial \boldsymbol{m}_t}$.

The third and final non-vanishing term captures how current temperature realizations influence the next period's state variables and, consequently, future value. Specifically, shifts in the temperature distribution affect not only current consumption but also the evolution of capital. A higher likelihood of adverse temperature realizations can reduce the accumulation of capital and lower future value. Furthermore, the moments $\boldsymbol{m}_{t+1}(T_t)$ can depend on the temperature realization. Such a dependency arises in cases where endogenous learning occurs through temperature observations. In such scenarios, a high-temperature realization may

update our beliefs about the future climate.

I have assumed that temperature's probability distribution is continuous and bounded from below. Without loss of generality (WLOG), I denote the lower boundary by 0. Then, using integration by parts,

$$\begin{aligned}
&\int_0^\infty \underbrace{U(T_t, i_t^*(\boldsymbol{m}_t, A_t, K_t), A_t, x_t^*(\boldsymbol{m}_t, A_t, K_t), K_t)}_{\equiv u} \underbrace{\frac{\partial P(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}}_{\equiv v'} \, dT_t \\
&= \underbrace{U(T_t, i_t^*(\boldsymbol{m}_t, A_t, K_t), A_t, x_t^*(\boldsymbol{m}_t, A_t, K_t), K_t)}_{u} \underbrace{\left[\int_0^{T_t} \frac{\partial P(\tilde{T}_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} \, d\tilde{T}_t \right]}_{v} \Bigg|_0^\infty \qquad \text{(D.3)} \\
&- \int_0^\infty \underbrace{\frac{\partial U(T_t, i_t^*(\boldsymbol{m}_t, A_t, K_t), A_t, x_t^*(\boldsymbol{m}_t, A_t, K_t), K_t)}{\partial T_t}}_{u'} \underbrace{\frac{\partial CDF(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}}_{v} \, dT_t,
\end{aligned}$$

where I used Leibniz's rule to exchange the order of differentiation and integration in

$$\frac{\partial CDF(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} = \frac{\partial}{\partial \boldsymbol{m}_t} \int_0^{T_t} P(\tilde{T}_t; \boldsymbol{m}_t) \, d\tilde{T}_t = \int_0^{T_t} \frac{\partial P(\tilde{T}_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} \, d\tilde{T}_t$$

and where the first term on the right in equation (D.3) vanishes because

$$\begin{aligned}
&\int_0^\infty P(T_t; \boldsymbol{m}_t) \, dT_t = CDF(T_t; \boldsymbol{m}_t)\big|_0^\infty = 1 \qquad \text{for all } \boldsymbol{m}_t \\
\Rightarrow \quad &\frac{\partial}{\partial \boldsymbol{m}_t} \int_0^\infty P(T_t; \boldsymbol{m}_t) \, dT_t = \int_0^\infty \frac{\partial P(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} \, dT_t = 0
\end{aligned}$$

and $CDF(0; \boldsymbol{m}_t) = 0 \;\Rightarrow\; \frac{\partial CDF(0; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} = 0$. In the main text, I assumed bounded utility to ensure that the products in equation (D.3) vanish when evaluated at zero and infinity. A weaker convergence condition eliminating the term is the following dominated tail condition

$$\lim_{T_t \to \infty} \left| U(T_t, i_t^*(\boldsymbol{m}_t, A_t, K_t), A_t, x_t^*(\boldsymbol{m}_t, A_t, K_t), K_t) \right| \int_{T_t}^\infty \left\| \frac{\partial P(s; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} \right\| \, ds = 0 \qquad \text{(D.4)}$$

using, e.g., the Euclidean norm.[56]

[56] I note that the only reasonable potential worry about the utility diverging with an infinite temperature increase would be that $\lim_{T_t \to \infty} U(T_t, i_t^*(\boldsymbol{m}_t, A_t, K_t), A_t, x_t^*(\boldsymbol{m}_t, A_t, K_t), K_t) = -\infty$, not ∞.

Applying the same reasoning to the final part of equation (D.2) yields

$$
\begin{aligned}
&\int V(\boldsymbol{m}_{t+1}(T_t), A_{t+1}(T_t), K_{t+1}(T_t)) \frac{\partial P(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\, dT_t \\
&= V(\boldsymbol{m}_{t+1}(T_t), A_{t+1}(T_t), K_{t+1}(T_t)) \left[\int_0^{T_t} \frac{\partial P(\tilde{T}_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\, d\tilde{T}_t \right] \Bigg|_0^{\infty} \\
&\quad - \int_0^{\infty} \frac{dV(\boldsymbol{m}_{t+1}(T_t), A_{t+1}(T_t), K_{t+1}(T_t))}{dT} \frac{\partial CDF(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} \\
&= -\int_0^{\infty} \left[\frac{\partial V(\boldsymbol{m}_{t+1}(T_t), A_{t+1}(T_t), K_{t+1}(T_t))}{\partial \boldsymbol{m}_{t+1}} \frac{\partial \boldsymbol{m}_{t+1}(T_t)}{\partial T} \right. \\
&\quad + \frac{\partial V(\boldsymbol{m}_{t+1}(T_t), A_{t+1}(T_t), K_{t+1}(T_t))}{\partial A_{t+1}} \frac{\partial A_{t+1}(T_t)}{\partial T} \\
&\quad \left. + \frac{\partial V(\boldsymbol{m}_{t+1}(T_t), A_{t+1}(T_t), K_{t+1}(T_t))}{\partial K_{t+1}} \frac{\partial K_{t+1}(T_t)}{\partial T} \right] \frac{\partial CDF(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} dT_t
\end{aligned}
$$

where the value function is finite as a result of bounded utility and discounting.[57] Then, equation (D.2) becomes

$$
\begin{aligned}
&\frac{\partial V(\boldsymbol{m}_t, A_t, K_t)}{\partial \boldsymbol{m}_t} = -\int_0^{\infty} \frac{\partial U(T_t, i_t^*, A_t, x_t^*, K_t)}{\partial T_t} \frac{\partial CDF(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\, dT_t \\
&-\beta \int_0^{\infty} \left[\frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1})}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial T_t} + \frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1})}{\partial K_{t+1}} \frac{\partial K_{t+1}}{\partial T_t} \right] \frac{\partial CDF(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} dT_t \\
&-\beta \int_0^{\infty} \frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1}}{\partial \boldsymbol{m}_{t+1}} \frac{\partial \boldsymbol{m}_{t+1}}{\partial T} \frac{\partial CDF(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} dT_t \\
&+\beta \int_0^{\infty} \left[\frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1})}{\partial \boldsymbol{m}_{t+1}} \right] \frac{\partial \boldsymbol{m}_{t+1}}{\partial \boldsymbol{m}_t} P(T_t; \boldsymbol{m}_t) dT_t
\end{aligned}
$$

To rewrite the integral in the second-to-last line w.r.t. the probability measure $P(T_t; \boldsymbol{m}_t)$ let $\mu_t(dT) \equiv P(T_t; \boldsymbol{m}_t)\, dT$ and define the signed measure $\boldsymbol{\nu}_t(dT) \equiv \frac{\partial \mathrm{CDF}(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\, dT$ (a row vector). Given the assumption of absolute continuity, $\boldsymbol{\nu}_t \ll \mu_t$, the Radon–Nikodym derivative

$$
\boldsymbol{r}_t(T_t) \equiv \frac{d\boldsymbol{\nu}_t}{d\mu_t}(T_t) = \frac{\frac{\partial \mathrm{CDF}(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}}{P(T_t; \boldsymbol{m}_t)}.
$$

exists and $d\boldsymbol{\nu}_t = \boldsymbol{r}_t\, d\mu_t$. I define

$$
\begin{aligned}
&\boldsymbol{G}_t(m_t) \equiv \int_0^{\infty} \left(\frac{\partial U(T_t, i_t^*, A_t, x_t^*, K_t)}{\partial T_t} + p_{A,t} \frac{\partial A_{t+1}}{\partial T_t} + p_{K,t} \frac{\partial K_{t+1}}{\partial T_t} \right) \frac{\partial CDF(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\, dT_t \\
&\tilde{\boldsymbol{M}}_t(m_t, T_t) \equiv \frac{\partial \boldsymbol{m}_{t+1}}{\partial \boldsymbol{m}_t} - \frac{\partial \boldsymbol{m}_{t+1}}{\partial T_t}\, \boldsymbol{r}_t(T_t),
\end{aligned}
$$

[57] In the case of using the dominated tail condition above, it is still intuitive that the value function remains finite, but it requires more assumptions on the modeling details.

noting only their climate dependencies in the arguments. Using the definitions $p_{K,t}$ and $p_{A,t}$ from the main text, the original equation can be written as

$$\frac{\partial V(\boldsymbol{m}_t, A_t, K_t)}{\partial \boldsymbol{m}_t} = -\boldsymbol{G}_t(m_t) + \beta\, \mathbb{E}_t\left[\tilde{\boldsymbol{M}}_t(m_t, T_t)\, \frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1})}{\partial \boldsymbol{m}_{t+1}}\right], \tag{D.5}$$

where $\mathbb{E}_t[\cdot] \equiv \int(\cdot)\, \mu_t(dT)$ denotes expectations under the probability density $P(T_t; \boldsymbol{m}_t)$.

Iterating (D.5) forward in time and applying the tower property $\mathbb{E}_t[\mathbb{E}_{t+1}[\cdot]] = \mathbb{E}_t[\cdot]$ gives the forward-looking representation

$$\frac{\partial V(\boldsymbol{m}_t, A_t, K_t)}{\partial \boldsymbol{m}_t} = -\sum_{s=t}^{\infty} \beta^{\,s-t}\, \mathbb{E}_t[\boldsymbol{M}_s\, \boldsymbol{G}_s]\,,$$

with the convention that the empty product equals the identity matrix for $s = 0$ and the definition

$$\boldsymbol{M}_s \;\equiv\; \prod_{l=t}^{s-1}\left(\frac{\partial \boldsymbol{m}_{l+1}}{\partial \boldsymbol{m}_l} - \frac{\partial \boldsymbol{m}_{l+1}}{\partial T_l}\, \boldsymbol{r}_l(T_l)\right).$$

Note: I note that, applying the change in measure also to $\boldsymbol{G}_t(T_t)$ itself, equation (D.5) can alternatively be written as

$$\frac{\partial V(\boldsymbol{m}_t, A_t, K_t)}{\partial \boldsymbol{m}_t} = -\,\mathbb{E}_t[\boldsymbol{g}_t(T_t)\, \boldsymbol{r}_t(T_t)] + \beta\, \mathbb{E}_t\left[\tilde{\boldsymbol{M}}_t(T_t)\, \frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1}, K_{t+1})}{\partial \boldsymbol{m}_{t+1}}\right]$$

with

$$\boldsymbol{g}_t(T_t) \equiv \frac{\partial U(T_t, i_t^*, A_t, x_t^*, K_t)}{\partial T_t} + p_{A,t}\frac{\partial A_{t+1}}{\partial T_t} + p_{K,t}\frac{\partial K_{t+1}}{\partial T_t}$$

iterating forward to the form

$$\frac{\partial V(\boldsymbol{m}_t, A_t, K_t)}{\partial \boldsymbol{m}_t} = -\sum_{s=0}^{\infty} \beta^s\, \mathbb{E}_t[\boldsymbol{M}_{t+s}(T_{t+s})\boldsymbol{g}_{t+s}(T_{t+s})\, \boldsymbol{r}_{t+s}(T_{t+s})]\,.$$

Proof of Proposition 1. This proof applies Proposition 1's general statement to the structure of ACE. Based on equation (4), ACE leaves adaptation capital unaffected by temperature shocks so that $\frac{\partial A_{t+1}}{\partial T_t} = 0$. Thus, in ACE,

$$\boldsymbol{g}_t(T_t) = \frac{\partial U(T_t, i_t^*, A_t, x_t^*, K_t)}{\partial T_t} + \cancel{p_{A,t}\frac{\partial A_{t+1}}{\partial T_t}} + p_{K,t}\frac{\partial K_{t+1}}{\partial T_t},$$

Based on equation (C.1) for utility

$$\frac{\partial U(T_t, i_t^*, A_t, x_t^*, K_t)}{\partial T_t} = -\frac{\partial d(\tau_t, i_t^*, A_t)}{\partial \tau_t}\frac{\partial \tau_t}{\partial T_t}.$$

The price $p_{K,t}$ is the discounted shadow value of next period capital (see also equation C.31). Equation (C.39) states the shadow value of log-capital. Transforming it into the shadow value of absolute levels of capital, I obtain

$$\varphi_k \frac{\partial k_{t+1}}{\partial K_{t+1}} = \frac{\kappa}{1-\beta\kappa}\Big(Y_t^{net}(1-x_t)\Big[\frac{1+g_{k,t}}{\delta_k+g_{k,t}}\Big]\Big)^{-1},$$

which uses the next period capital stock obtained from inserting equation (27) into equation (28), and the consumption rate is x. Furthermore using the definition of Y_t^{net},

$$\begin{aligned}\frac{\partial K_{t+1}}{\partial T_t} &= \frac{\partial}{\partial T_l}\left(F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{K}_t,\boldsymbol{E}_t)\exp[-d(\tau_t(T_t)]\exp(-\eta\ i_t)\left[\frac{1+g_{k,t}}{\delta_k+g_{k,t}}\right](1-x_t)\right)\\ &= -\frac{\partial d(\tau_t,i_t^*,A_t)}{\partial \tau_l}\frac{\partial \tau_l}{\partial T_l}\Big(Y_t^{net}(1-x_t)\Big[\frac{1+g_{k,t}}{\delta_k+g_{k,t}}\Big]\Big).\end{aligned}$$

Together, I obtain the contribution

$$\begin{aligned}\boldsymbol{g}_t(T_t) &= -\frac{\partial d(\tau_t,i_t^*,A_t)}{\partial \tau_l}\frac{\partial \tau_l}{\partial T_l} + \beta\frac{\kappa}{1-\beta\kappa}\frac{Y_t^{net}(1-x_t)\left[\frac{1+g_{k,t}}{\delta_k+g_{k,t}}\right]}{Y_t^{net}(1-x_t)\left[\frac{1+g_{k,t}}{\delta_k+g_{k,t}}\right]}\left(-\frac{\partial d(\tau_t,i_t^*,A_t)}{\partial \tau_l}\frac{\partial \tau_l}{\partial T_l}\right)\\ &= -\frac{\partial d(\tau_t,i_t^*,A_t)}{\partial \tau_l}\frac{\partial \tau_l}{\partial T_l}\left[1+\frac{\beta\kappa}{1-\beta\kappa}\right] = -\frac{\partial d(\tau_t,i_t^*,A_t)}{\partial \tau_l}\frac{\partial \tau_l}{\partial T_l}\frac{1}{1-\beta\kappa}.\end{aligned}$$

Using the result that the optimal consumption rate is $x = 1-\beta\kappa$, I obtain the first equation stated in the second part of the Proposition.

The second line follows from partial integration

$$\begin{aligned}\boldsymbol{G}_t(T_t) &= -\frac{1}{x}\int_0^\infty \frac{\partial d(\tau_t,i_t^*,A_t)}{\partial T_t}\frac{\partial CDF(T_t;\boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\\ &= -\frac{1}{x}\,d(\tau_t,i_t^*,A_t)\frac{\partial CDF(T_t;\boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\Big|_0^\infty + \frac{1}{x}\int_0^\infty d(\tau_t,i_t^*,A_t)\frac{\partial^2 CDF(T_t;\boldsymbol{m}_t)}{\partial T_t \partial \boldsymbol{m}_t}dT_t\\ &= \frac{1}{x}\int_0^\infty d(\tau_t,i_t^*,A_t)\frac{\partial P(T_t;\boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\,dT_t,\end{aligned}$$

where the first expression on the second line vanishes as a result of the same argument used in the proof of Proposition 1. Here as well, the assumption of finite damages could be replaced by a dominated tail condition analogous to equation (D.4).

Finally, I obtain equation (26) as follows[58]

$$
\begin{aligned}
&\int d(\tau_t, i_t^*, A_t) \frac{\partial P(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\, dT_t \\
&= \int \xi_0 \tau_t + \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big) \exp\left(\lambda\tau_t - \mathrm{CGF}_{\tau_l}(\lambda)\right) \frac{\partial P(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\, dT_t \\
&= \lim_{\Delta \boldsymbol{m}_t \to 0} \left[\int \xi_0 \tau_t P(T_t; \boldsymbol{m}_t + \boldsymbol{\Delta m})\, dT_t - \int \xi_0 \tau_t P(T_t; \boldsymbol{m}_t)\, dT_t \right] \oslash \boldsymbol{\Delta m} \\
&\quad + \lim_{\Delta \boldsymbol{m}_t \to 0} \left[\int \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big) \exp\left(\lambda\tau_t - \mathrm{CGF}_{\tau_l}(\lambda)\right) P(T_t; \boldsymbol{m}_t + \boldsymbol{\Delta m})\, dT_t \right. \\
&\quad \left. - \int \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big) \exp\left(\lambda\tau_t - \mathrm{CGF}_{\tau_l}(\lambda)\right) P(T_t; \boldsymbol{m}_t)\, dT_t \right] \oslash \boldsymbol{\Delta m}
\end{aligned}
$$

where $\oslash$ defines component-wise division of the two vectors, and I kept the index τ_l on the cumulant generating function to emphasize that it results from the fixed adaptation decision and does not vary with the temperature realization T_t. Continuing

$$
\begin{aligned}
&= \lim_{\Delta \boldsymbol{m}_t \to 0} \left[\xi_0\, \mathbb{E}_{\boldsymbol{m}_t + \boldsymbol{\Delta m}}\, \tau_t - \xi_0\, \mathbb{E}_{\boldsymbol{m}_t}\, \tau_t\right] \oslash \boldsymbol{\Delta m} \\
&\quad + \lim_{\Delta \boldsymbol{m}_t \to 0} \left[\frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big) \exp\left(\mathrm{CGF}_{\boldsymbol{m}_t + \boldsymbol{\Delta m}}(\lambda) - \mathrm{CGF}_{\tau_l}(\lambda)\right) \right. \\
&\quad \left. - \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big) \exp\left(\mathrm{CGF}_{\boldsymbol{m}_t}(\lambda) - \mathrm{CGF}_{\tau_l}(\lambda)\right) \right] \oslash \boldsymbol{\Delta m} \\
&= \xi_0 \frac{\partial}{\partial \boldsymbol{m}_t}\, \mathbb{E}_{\boldsymbol{m}_t}[\tau_t] \;+\; \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\, \exp(\mathrm{CGF}_{\boldsymbol{m}_t}(\lambda) - \mathrm{CGF}_{\tau_l}(\lambda))\, \frac{\partial}{\partial \boldsymbol{m}_t}\, \mathrm{CGF}_{\boldsymbol{m}_t}(\lambda).
\end{aligned}
$$

Here, I label expectations and CGF by the moment vector of the underlying distribution rather than by time or the integration variable, to avoid confusion, distinguishing it from the cumulant generating function $\mathrm{CGF}_{\tau_l}(\lambda)$ induced by the agent's adaptation response reflecting beliefs. Under full information, beliefs coincide with the actual climate, and the optimized investment decision implies $\mathrm{CGF}_{\tau_l}(\lambda) = \mathrm{CGF}_{\boldsymbol{m}_t}(\lambda)$.[59] Then

$$
\begin{aligned}
\int d(\tau_t, i_t^*, A_t)\, \frac{\partial P(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\, dT_t &= \xi_0 \frac{\partial}{\partial \boldsymbol{m}_t}\, \mathbb{E}_{\boldsymbol{m}_t}[\tau_t] \;+\; \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\, \frac{\partial}{\partial \boldsymbol{m}_t}\, \mathrm{CGF}_{\boldsymbol{m}_t}(\lambda) \\
&= \xi_0 \frac{\partial}{\partial \boldsymbol{m}_t}\, \mathbb{E}_t[\tau_t] \;+\; \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\, \frac{\partial}{\partial \boldsymbol{m}_t}\, \mathrm{CGF}_{\tau_t}(\lambda),
\end{aligned}
$$

[58] Given that I am dealing with an integration with respect to a change of the probability measure, constants will aggregate to zero. Thus, I simply keep $\xi_0 \tau_{t+1}$ of the original $\xi_0(\tau_{t+1} - 1)$ in the damage function, which is the part directly recovered (up to a constant) by inverting the integration by parts again starting from the marginal damage function.

[59] Because of the envelope theorem (equivalently, the first-order condition for adaptation investment), we do not need to track how i_t^* responds to a marginal change in climate; these terms cancel in equation (D.2).

where in the second line I return to the original notation for expectations and the cumulant generating function. This delivers the final expression stated in the proposition.

If, in addition, the moment vector coincides with the cumulants $\boldsymbol{\kappa}_t$ of the τ_t distribution, the expression simplifies to

$$\begin{aligned}\int d(\tau_t, i_t^*, A_t)\, \frac{\partial P(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\, dT_t &= \xi_0\, \frac{\partial}{\partial \boldsymbol{\kappa}_t}\, \mathbb{E}_t[\tau_t] \;+\; \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\, \frac{\partial}{\partial \boldsymbol{\kappa}_t}\left(\sum_{n=1}^{\infty} \frac{\kappa_{n,t}}{n!}\, \lambda^n\right) \\ &= \xi_0\, \boldsymbol{e}_1 \;+\; \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\, \big(\lambda,\; \tfrac{\lambda^2}{2},\; \tfrac{\lambda^3}{3!}, \ldots, \tfrac{\lambda^n}{n!}, \ldots\big),\end{aligned}$$

where $\boldsymbol{e}_1$ is the first unit vector.