Implementing Ramsey Taxes Without Debt Rescheduling: A Dynamic Program*

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March 25, 2026

*We thank Andrew Atkeson, Fernando Alvarez, Marco Bassetto, Roberto Chang, Lars Peter Hansen, Zhiguo He, Andrea Lanteri, Ben Moll, Robert Shimer, Nancy Stokey, Pierre-Olivier Weill, Pierre Yared, and seminar participants at London School of Economics, Princeton University, and the University of Chicago for helpful comments. Wei Jiang acknowledges support from the Hong Kong Research Grants Council (Grant No.).

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Abstract

This paper formulates a continuous-time version of Lucas and Stokey's (1983) optimal tax problem as a dynamic program that we cast in terms of a state variable $X_t = \Pi_t U_{C,t}$, where Π_t denotes the government's cumulative liability in the absence of debt rescheduling and $U_{C,t}$ denotes the marginal utility of consumption. A Ramsey planner's value function satisfies a Hamilton-Jacobi-Bellman equation that takes an additively separable form $V(X_t, t) = -\Phi_0 X_t + H(t; \Phi_0)$. The dynamic program tells how to implement the Ramsey tax plan with a sequence of continuation planners who simply automatically issue the instantaneous debt that is required to avoid rescheduling the tail of the initial term-debt structure. Under a timing protocol that allows a time-t government to set a time-$(t + dt)$'s government's tax rate, our implementation supports a Ramsey tax plan. Our implementation avoids all of the recurrent rescheduling of term debt that Debortoli, Nunes, and Yared (2021) showed can imperil Lucas and Stokey's implementation. We provide quantitative examples featuring exponential and cyclical debt-service processes, as well as cyclical government spending processes.

Keywords: Ramsey planner, continuation Ramsey planner, implementability, instantaneous debt, local commitment, dynamic programming.

JEL Classification: E3, E6, C7

"Finding the state is an art."

Fernando Alvarez, 1999, 2000, ...

1 Introduction

This paper reformulates Lucas and Stokey's (1983) Ramsey tax problem as a dynamic program that invites us to think about implementing the Ramsey plan sequentially from a perspective that was not available to Lucas and Stokey. To focus on essential aspects of the problem, we assume that all economic fundamentals are not random.[1]

Our paper is a practical exercise in the "partial commitment" tradition of Lucas and Stokey (1982, Sec. 6), a topic that strains the adage that "finding the state is an art." "Partial commitments" appear in our problem because of how the term structure of government debt influences the state vector that faces continuation Ramsey planners and the timing protocols required to support implementation of a Ramsey plan by a sequence of government decision makers. The difficult relationship between dynamic programming and partial commitment problems dates back at least to Lucas and Stokey (1983), whose model of an optimal term structure of government debt rested on a timing protocol that imparted "partial commitments" to a sequence of policy makers. (For an enlightening discussion of partial commitments that referees and editors of Lucas and Stokey (1983) insisted be removed from the published version, see the discussion of Alexander Hamilton's 1795 second report on public credit[2] in Lucas and Stokey (1982, Sec. 6)).[3] To construct their theory of debt management, Lucas and Stokey used a Lagrangian formulation of a Ramsey planner's problem instead of the dynamic programming machinery that in the previous 15 years Lucas had used in a string of pioneering papers.[4]

Section 2 describes the environment. A representative household supplies labor and consumes a nonstorable good. The government finances an exogenous stream of expenditures $\{G_t; t \geqslant 0\}$ and services an exogenous initial debt structure $\{b_{0-,t}; t \geqslant 0\}$ by levying flat taxes on labor income. We define competitive equilibrium and derive an implementability constraint that restricts competitive equilibrium allocations. This constraint follows from combining the household's first-order conditions with the government's budget constraint.

[1] As we note in Section 7, it is possible to extend our results to settings in which fundamentals are described by autonomous stochastic differential equations.

[2] That report is available here: Hamilton (1973).

[3] For recent work on "partial commitment", see, e.g., Clymo and Lanteri (2020) and Chari et al. (2020).

[4] For example, see Lucas and Prescott (1971) and Lucas (1978) and Stokey and Lucas Jr (1989).

Section 3 formulates the Ramsey problem using a Lagrangian. We attach a Lagrange multiplier Λ_0 to the implementability constraint and derive the first-order condition for optimal consumption $C(\Lambda_0; b_{0-,t}, G_t)$. The Lagrange multiplier Λ_0^* minimizes the Lagrangian while satisfying the implementability constraint. This section establishes existence of a Ramsey plan.

Section 4 formulates the Ramsey problem as a dynamic program that we cast in terms of a state variable $X_t = \Pi_t U_{C,t}$, where $U_{C,t}$ is the household's marginal utility and Π_t equals the government's cumulative liability if it never reschedules the tail of an initial term debt debt stream $\{b_{0-,s}; s \geqslant t\}$. We show that X_t evolves according to a differential equation and that the planner's value function satisfies a Hamilton-Jacobi-Bellman equation that takes an additively separable form $V(X_t, t) = -\Phi_0 X_t + H(t; \Phi_0)$. The scalar Φ_0 is chosen to satisfy the single implementability constraint that restricts competitive equilibrium allocations in Lucas and Stokey's model, with initial condition $X_0 = 0$. Proposition 5 establishes that the Lagrangian method and the Bellman equation generate identical Ramsey tax plans.

Section 5 presents quantitative examples under additively separable utility in consumption and labor. We compute Ramsey plans for three specifications: exponential debt service, cyclical debt service, and cyclical government spending. The examples illustrate how tax rates, interest rates, primary surpluses, and the state variable evolve over time.

Section 6 uses our section 4 dynamic programming formulation to revisit the heart of Lucas and Stokey's (1983) paper in which they derived a theory of optimal debt management from their sequential implementation of the Ramsey plan under partial commitment. Lucas and Stokey (1983) proposed a debt-management strategy that entailed continuous rescheduling of term debt. Debortoli, Nunes, and Yared (2021) show that this strategy may not implement the Ramsey plan in discrete time. We propose an alternative implementation that uses instantaneous debt and does not reschedule term debt. At each date t, the government inherits the tail of the initial debt service stream $\{b_{0-,s}; s \geqslant t\}$ and adjusts the balance of instantaneous debt to equal the cumulative liability Π_t^*. We introduce a timing protocol under which the government at time $t - dt$ sets the tax rate at time t. This timing protocol grants an infinitesimal increment of commitment. Theorem 1 establishes that under this protocol, continuation Ramsey planners confirm the Ramsey plan. The formulation in Section 4 enables this recursive characterization of implementation.

Our work is related to Jiang et al. (2025), who also provide a short-term-debt-based implementation strategy in response to Debortoli et al.'s counterexample, which may invalidate term-debt-based implementation of a Ramsey plan proposed in Lucas and Stokey

(1983). Our work differs from Jiang et al. (2025) in three key aspects. First, we identify a state variable and provide a recursive formulation of the original Lucas and Stokey's problem. Our approach characterizes a well-defined history-dependent Ramsey plan as a tractable recursive problem for any admissible term debt structure, government spending process, and time-additive separable preferences. Second, our recursive formulation reveals that the key to successfully implementing a Ramsey plan is the preservation of the 'purchasing power' of the household's net worth. A short-term-debt-based implementation strategy always works, provided that a Ramsey plan exists, because it preserves this purchasing power by enabling continuation planners to inherit the state variable X_t. Economically, short-term debt effectively addresses continuation planners' incentives to manipulate government debt prices, thereby eliminating equilibrium debt overhang. Third, our continuous-time formulation clearly distinguishes instantaneous debt, which matures immediately, from debt with very short maturities. Expanding the contractible asset space to include instantaneous debt is crucial, as it allows us to always implement a Ramsey plan whenever one exists. This addition of instantaneous debt significantly expands our tool boxes, allowing us to take advantage of the recursive formulation. Without instantaneous debt, some implementation strategy will fail when debt burden is high.

Appendix A provides technical details and proofs. Appendix B derives analytical properties of the cyclical Ramsey plans in Section 5. Appendix C shows that discrete-time Ramsey plans converge to the continuous-time Ramsey plan as the time increment shrinks to zero.

2 Environment

Time $t \in [0, \infty)$ is continuous. A representative household and a benevolent government participate in a complete set of perfectly competitive markets. The government finances an exogenous stream of expenditures with a stream of distorting flat tax rates. There is no uncertainty.

A first subscript denotes the time at which a variable is chosen and a second subscript denotes the time at which it is realized. The representative household supplies a labor process $\{N_{0,t};\, t \geqslant 0\}$ that produces a flow of a single nonstorable good that can be divided between a consumption flow $\{C_{0,t};\, t \geqslant 0\}$ and a government expenditure flow $\{G_t;\, t \geqslant 0\}$:

$$C_{0,t} + G_t = N_{0,t}, \quad \text{for all } t \geqslant 0. \tag{1}$$

Assumption 1. *Exogenous flows of government expenditures* $\{G_t;\, t \geqslant 0\}$ *and initial debt-service payouts* $\{b_{0-,t};\, t \geqslant 0\}$ *are continuous functions of time.*

At time 0, the representative household orders consumption and labor supply streams $\{C_{0,t};\, t \geqslant 0\}$ and $\{N_{0,t};\, t \geqslant 0\}$ according to

$$E_0 \int_0^\infty e^{-\rho t} U(C_{0,t}, N_{0,t})\, dt, \tag{2}$$

where $\rho > 0$ and $U(\cdot, \cdot)$ is strictly increasing in consumption C, strictly decreasing in labor supply N, globally concave, and continuously differentiable. Consumption and labor decisions are convex controls and the household derives utility $U(C_{0,t}, N_{0,t})\, dt$ over a small time interval dt, so we focus on policies that we call *admissible*, meaning that $\{C_{0,t};\, t \geqslant 0\}$ and $\{N_{0,t};\, t \geqslant 0\}$ are continuous processes. As in Debortoli, Nunes, and Yared (2021), we assume that labor supply $N_{0,t}$ has no upper bound.

Let $\tau_{0,t}$ denote a tax rate at t chosen at time 0 and $q_{0,t}$ be the time-0 value of a zero-coupon bond with a unit payoff at t. The representative household faces a single intertemporal budget constraint:

$$\int_0^\infty C_{0,t}\, q_{0,t}\, dt \leqslant \int_0^\infty b_{0-,t}\, q_{0,t}\, dt + \int_0^\infty (1 - \tau_{0,t}) N_{0,t}\, q_{0,t}\, dt. \tag{3}$$

The left side of (3) is the present value of the household's consumption stream $\{C_{0,t};\, t \geqslant 0\}$ and the right side is the sum of the household's financial wealth and its human wealth in the form of the present value of after-tax labor income. Given tax rate process $\{\tau_{0,t};\, t \geqslant 0\}$ and bond price process $\{q_{0,t};\, t \geqslant 0\}$, the household chooses $\{C_{0,t}, N_{0,t};\, t \geqslant 0\}$ to maximize (2) subject to constraint (3), an optimization problem brings the following first-order necessary conditions:

$$1 - \tau_{0,t} = -\frac{U_{N,t}}{U_{C,t}} \quad \text{and} \quad q_{0,t} = e^{-\rho t} \frac{U_{C,t}}{U_{C,0}}. \tag{4}$$

The time-0 government finances an initial debt and its exogenous spending stream by taxing labor income:

$$\int_0^\infty b_{0-,t}\, q_{0,t}\, dt \leqslant \int_0^\infty S_{0,t}\, q_{0,t}\, dt, \tag{5}$$

where

$$S_{0,t} = \tau_{0,t} N_{0,t} - G_t \tag{6}$$

is the primary surplus. The left side of (5) is time-0 value of payouts on government debt

$\{b_{0-,t};\, t \geqslant 0\}$ and the right side of (5) is time-0 value of $\{S_{0,t};\, t \geqslant 0\}$.

Definition 1. *Given the government's initial debt term structure* $\{b_{0-,t};\, t \geqslant 0\}$*, spending flow process* $\{G_t;\, t \geqslant 0\}$*, and flat-rate tax process* $\{\tau_{0,t};\, t \geqslant 0\}$*, a competitive equilibrium is a feasible allocation* $\{C_{0,t}, N_{0,t};\, t \geqslant 0\}$ *and a bond price process* $\{q_{0,t};\, t \geqslant 0\}$ *for which*

- *the government's budget constraint* (5) *is satisfied, and*
- *the allocation* $\{C_{0,t}, N_{0,t};\, t \geqslant 0\}$ *solves the household's optimization problem.*

We follow Lucas and Stokey (1983) and use first-order conditions (4) together with feasibility constraint (1) to eliminate tax rates and bond prices from the government budget constraint. We thereby obtain the following constraint on competitive equilibrium allocations $\{C_{0,t}, N_{0,t}\}_{t=0}^{\infty}$:

$$\int_0^\infty e^{-\rho t} b_{0-,t}\, U_{C,t}\, dt \leqslant \int_0^\infty e^{-\rho t} \big(C_{0,t}\, U_{C,t} + (C_{0,t} + G_t)\, U_{N,t} \big)\, dt. \tag{7}$$

Proposition 1. *Given an initial term-debt structure* $\{b_{0-,t};\, t \geqslant 0\}$ *and government spending path* $\{G_t;\, t \geqslant 0\}$*, an allocation* $\{C_{0,t}, N_{0,t};\, t \geqslant 0\}$ *is a competitive equilibrium allocation if and only if it satisfies* (1) *for all* $t \geqslant 0$ *and* (7) *at* $t = 0$*.*

2.1 A Ramsey problem

At time 0, a Ramsey planner selects the best competitive equilibrium. According to the first-order approach used by Lucas and Stokey and many others, the Ramsey planner can compute the allocation associated with the best competitive equilibrium by finding a continuous process $\{C_{0,t};\, t \geqslant 0\}$ that maximizes

$$\int_0^\infty e^{-\rho t} U(C_{0,t}, C_{0,t} + G_t)\, dt \tag{8}$$

subject to constraint (7).[5] With the maximizing process $\{C_{0,t};\, t \geqslant 0\}$ in hand, we can then use first-order conditions (4) to compute a tax rate process and bond price system that supports it, meaning that when it confronts this tax plan and price system, the representative agent's best responses confirm the associated competitive equilibrium allocation.

[5] In the spirit of Lucas and Stokey (1983), this is the single "implementability constraint" that a competitive equilibrium places on a Ramsey allocation.

3 A Lagrangian

Attach a multiplier Λ_0 to implementability constraint (7) and form

$$L_0 \;=\; \int_0^\infty e^{-\rho t}\Big[U(C_{0,t}, C_{0,t}+G_t) + \Lambda_0\big(C_{0,t}\,U_{C,t} + (C_{0,t}+G_t)\,U_{N,t} - b_{0-,t}\,U_{C,t}\big)\Big]\,dt. \quad (9)$$

The Ramsey planner maximizes the right side of (9) over consumption plans $\{C_{0,t}\}_{t=0}^\infty$ and minimizes over the nonnegative multiplier Λ_0. Call extremizing values $\{C^*_{0,t}\}_{t=0}^\infty$ and Λ_0^*. For a given Λ_0, the optimal consumption plan satisfies[6]

$$C(\Lambda_0;\, b_{0-,t}, G_t) := \arg\max_{C_{0,t}}\big[U(C_{0,t}, C_{0,t}+G_t)+\Lambda_0(C_{0,t}\,U_C+(C_{0,t}+G_t)\,U_N-b_{0-,t}\,U_C)\big]. \quad (10)$$

Substituting (10) into implementability condition (7), we deduce that Λ_0 satisfies:

$$\int_0^\infty e^{-\rho t}\big[C(\Lambda_0;\, b_{0-,t}, G_t)\,U_C + (C(\Lambda_0;\, b_{0-,t}, G_t)+G_t)\,U_N\big]\,dt = \int_0^\infty e^{-\rho t} b_{0-,t}\,U_C\,dt. \quad (11)$$

Substituting $C(\Lambda_0;\, b_{0-,t}, G_t)$ into the right side of (9) lets us express L_0 as a function $L_0(\Lambda_0)$. The non-negative root of equation (11) that minimizes $L_0(\Lambda_0)$ is the optimal Λ_0^*. Thus, the Ramsey allocation is $C^*_{0,t} = C(\Lambda_0^*;\, b_{0-,t}, G_t)$ and $N^*_{0,t} = C^*_{0,t} + G_t = N(\Lambda_0^*;\, b_{0-,t}, G_t)$.

Proposition 2. *Under regularity conditions provided in Appendix A, there exists a non-negative root Λ_0^* of equation* (11) *that solves*

$$\Lambda_0^* = \arg\min_{\Lambda_0} L_0(\Lambda_0),$$

where the Ramsey allocation, tax rate, and bond price process satisfy

$$C^*_{0,t} = C(\Lambda_0^*;\, b_{0-,t}, G_t), \qquad N^*_{0,t} = C^*_{0,t} + G_t,$$
$$\tau^*_{0,t} = 1 + \frac{U_{N,t}(C^*_{0,t}, N^*_{0,t})}{U_{C,t}(C^*_{0,t}, N^*_{0,t})}, \qquad q^*_{0,t} = e^{-\rho t}\frac{U_{C,t}(C^*_{0,t}, N^*_{0,t})}{U_{C,0}(C^*_{0,0}, N^*_{0,0})}.$$

The value of the Ramsey plan is $L_0^* = \int_0^\infty e^{-\rho t} U(C^*_{0,t}, C^*_{0,t}+G_t)\,dt$.

[6] We impose a regularity condition that implies that, for a given Λ_0, the optimal consumption plan for problem (10) is unique.

4 A Dynamic Program

To formulate the Ramsey problem recursively, we proceed in four steps. 1. For a given consumption process, we define a process, $\{\Pi_t;\, t \geqslant 0\}$ that tracks the government's cumulative liabilities in the absence of any debt restructuring. 2. From the $\{\Pi_t;\, t \geqslant 0\}$ process, we construct a state variable process $\{X_t;\, t \geqslant 0\}$. 3. We frame a dynamic program with $\{X_t;\, t \geqslant 0\}$ as a state variable and a time-t value function, $V(X_t, t)$ that is additively separable: $V(X_t, t) = -\Phi_0\, X_t + H(t;\, \Phi_0)$, ans Φ_0 is a scalar that must be computed. 4. We compute Φ_0.

4.1 Cumulative Liabilities with no Debt Restructuring

Given initial debt service flow $\{b_{0-,s},\, s \geqslant 0\}$, in the absence of any debt restructuring the government's time t liabilities are

$$\Pi_t = \int_0^t \frac{q_{0,s}}{q_{0,t}} \big(b_{0-,s} + G_s - \tau_{0,s}\, N_{0,s}\big)\, ds, \qquad \Pi_0 = 0. \tag{12}$$

Here $(b_{0-,s} + G_s - \tau_{0,s}\, N_{0,s})\, ds$ is the (flow) increase of the government's liability over a small time interval ds; the ratio $q_{0,s}/q_{0,t}$ compounds this flow's contribution to the government's time-t liability stock.[7]

We have assumed that $\{G_t\}$ and $\{b_{0-,t}\}$ are continuous processes, so optimal allocations $\{C_{0,t}, N_{0,t};\, t \geqslant 0\}$ are also continuous. That implies that the zero-coupon bond price $q_{0,t}$ is continuous in t and that $dq_{0,t} = -r_{0,t}\, q_{0,t}\, dt$, where $\{r_{0,t}\}$ is the continuously compounded risk-free interest-rate process. Differentiating (12) with respect to t, we obtain:

$$d\Pi_t = r_{0,t}\, \Pi_t\, dt + (b_{0-,t} + G_t - \tau_{0,t}\, N_{0,t})\, dt. \tag{13}$$

4.2 A State Variable X_t

Define a scalar state variable

$$X_t = \Pi_t\, U_{C,t}. \tag{14}$$

Evidently, X_t encodes information about both the household's wealth Π_t and its marginal utility of consumption $U_{C,t}$. Because Π_t and $U_{C,t}$ are both continuous, X_t is continuous as

[7] When the interest rate is a constant r over (s, t), $q_{0,s}/q_{0,t} = e^{r(t-s)} > 1$.

a function of t. Differentiating both sides of (14) with respect to t and using (13) together with FOCs given in (4), we obtain:

$$\begin{aligned} dX_t &= U_{C,t}\, d\Pi_t + \Pi_t\, d\big(e^{\rho t} q_{0,t}\, U_{C,0}\big) \\ &= U_{C,t}\left[-\Pi_t \frac{dq_{0,t}}{q_{0,t}} + (b_{0-,t} + G_t - \tau_{0,t}\, N_{0,t})\, dt\right] + \Pi_t\left(q_{0,t}\, U_{C,0}\, \rho\, e^{\rho t}\, dt + e^{\rho t}\, U_{C,0}\, q_{0,t} \frac{dq_{0,t}}{q_{0,t}}\right) \\ &= -X_t \frac{dq_{0,t}}{q_{0,t}} + (b_{0-,t}\, U_{C,t} - C_{0,t}\, U_{C,t} - N_{0,t}\, U_{N,t})\, dt + \rho\, X_t\, dt + X_t \frac{dq_{0,t}}{q_{0,t}} \\ &= \big(\rho\, X_t - C_{0,t}\, U_{C,t} - N_{0,t}\, U_{N,t} + b_{0-,t}\, U_{C,t}\big)\, dt. \end{aligned} \tag{15}$$

Thus, we have verified

Lemma 1. *X_t defined in* (14) *is continuous in t and evolves according to* (15).

In general, the initial debt structure $b_{0-,t}$ and government spending process G_t are time dependent. Consequently, the Ramsey problem is time inhomogeneous, which implies that t appears in the value function along with X_t. We formulate the Ramsey problem in two steps. First, in Subsection 4.3, we pose an appropriate dynamic program (DP). In Subsection 4.4, we then characterize the Ramsey plan.

4.3 HJB Equations

Definition 2 (DP Problem)**.** *Let*

$$V(X_0, 0) = \max_{\{C_{0,s};\, s \geqslant 0\}} \int_0^\infty e^{-\rho s} U(C_{0,s}, C_{0,s} + G_s)\, ds, \tag{16}$$

where maximization is subject to law of motion (15).

To compute $V(X_0, 0)$, first pose the time-t optimization problem on the right side of the following equation:

$$V(X_t, t) = \max_{\{C_{0,s};\, s \geqslant t\}} \int_t^\infty e^{-\rho(s-t)} U(C_{0,s}, C_{0,s} + G_s)\, ds \tag{17}$$

subject to law of motion (15). For all $t \geqslant 0$, value function $V(X_t, t)$ satisfies the following HJB equation:

$$\rho\, V = \frac{\partial V}{\partial t} + \max_{C_{0,t}} U(C_{0,t}, C_{0,t} + G_t) + \frac{\partial V}{\partial X_t}\big(\rho\, X_t - C_{0,t}\, U_{C,t} - (C_{0,t} + G_t)\, U_{N,t} + b_{0-,t}\, U_{C,t}\big). \tag{18}$$

Guess that $V(X_t, t)$ is additively separable in X_t and t:

$$V(X_t, t) = -\Phi_0 X_t + H(t;\, \Phi_0), \qquad t \geqslant 0, \tag{19}$$

where Φ_0 is an object to be chosen by the Ramsey planner. Substituting (19) into (18), we obtain the following differential equation for all $t \geqslant 0$:

$$\rho\, H(t;\, \Phi_0) = H'(t;\, \Phi_0) + \max_{C_{0,t}} U(C_{0,t}, C_{0,t} + G_t) + \Phi_0\big(C_{0,t}\, U_{C,t} + (C_{0,t} + G_t)\, U_{N,t} - b_{0-,t}\, U_{C,t}\big). \tag{20}$$

Let $C(\Phi_0;\, b_{0-,t}, G_t)$ be the function that solves the maximum problem on the right side of equation (20):

$$C(\Phi_0;\, b_{0-,t}, G_t) = \arg\max_{C_{0,t}} U(C_{0,t}, C_{0,t} + G_t) + \Phi_0\big(C_{0,t}\, U_{C,t} + (C_{0,t} + G_t)\, U_{N,t} - b_{0-,t}\, U_{C,t}\big). \tag{21}$$

Proposition 3. *Optimal choices of $\{C_{0,t}\}$ for the DP Problem indexed by $\Phi_0 \geqslant 0$ satisfy* (21) *and attain the value function:*

$$V(X_t, t;\, \Phi_0) = -\Phi_0 X_t + H(t;\, \Phi_0), \tag{22}$$

where $H(t;\, \Phi_0)$ is given by

$$\begin{aligned} H(t;\, \Phi_0) = \int_t^\infty e^{-\rho(s-t)} \Big[& U\big(C(\Phi_0;\, b_{0-,s}, G_s),\, C(\Phi_0;\, b_{0-,s}, G_s) + G_s\big) \\ & + \Phi_0\big(C(\Phi_0;\, b_{0-,s}, G_s)\, U_{C,s} + (C(\Phi_0;\, b_{0-,s}, G_s) + G_s)\, U_{N,s} - b_{0-,s}\, U_{C,s}\big) \Big]\, ds \end{aligned} \tag{23}$$

and $N(\Phi_0;\, b_{0-,t}, G_t) = C(\Phi_0;\, b_{0-,t}, G_t) + G_t$.

4.4 Characterizing the Ramsey Plan

We must now determine the scalar Φ_0 that appears in the Ramsey planner's value function defined in equation (16). The consumption stream $\{C_{0,t}\}$ is a continuous process: $C_{0,0} = \lim_{s \downarrow 0} C_{0,s} = C(\Phi_0;\, b_{0-,0}, G_0)$, which implies that $C_{0,0}$ is a function of Φ_0. Subject to implementability constraint (7), the Ramsey planner chooses Φ_0 by solving the problem on the right side of

$$W = \max_{\Phi_0} V(X_0, 0;\, \Phi_0), \tag{24}$$

where $V(X_0, 0;\, \Phi_0)$ is given in (19). The Ramsey planner takes the initial debt term structure $\{b_{0-,t};\, t \geqslant 0\}$ as given. The planner also sets $X_0 = 0$, which by virtue of definition (14) imposes that $\Pi_0 = 0$. Implementability constraint (7) requires that Φ_0 satisfies $I_0(\Phi_0) = 0$, where

$$I_0(\Phi_0) = \int_0^\infty e^{-\rho t}\big(C(\Phi_0;\, b_{0-,t}, G_t)\, U_{C,t} + (C(\Phi_0;\, b_{0-,t}, G_t) + G_t)\, U_{N,t} - b_{0-,t}\, U_{C,t}\big)\, dt. \quad (25)$$

Let $\mathcal{A} = \{\Phi_0 \geqslant 0 : I_0(\Phi_0) = 0\}$ denote the admissible set for Φ_0. Using (22), the representative household's value under a Ramsey plan is

$$W = \max_{\Phi_0 \in \mathcal{A}} V(0,0) = \max_{\Phi_0 \in \mathcal{A}} \int_0^\infty e^{-\rho t} U\big(C(\Phi_0, b_{0-,t}, G_t),\, C(\Phi_0, b_{0-,t}, G_t) + G_t\big)\, dt.$$

Proposition 4. *If one exists, a Ramsey plan satisfies*

$$\begin{aligned} C^*_{0,t} &= C(\Phi^*_0;\, b_{0-,t}, G_t), \qquad & N^*_{0,t} &= C^*_{0,t} + G_t, \\ \tau^*_{0,t} &= 1 + \frac{U_{N,t}(C^*_{0,t}, N^*_{0,t})}{U_{C,t}(C^*_{0,t}, N^*_{0,t})}, \qquad & q^*_{0,t} &= e^{-\rho t}\frac{U_{C,t}(C^*_{0,t}, N^*_{0,t})}{U_{C,0}(C^*_{0,0}, N^*_{0,0})} \end{aligned}$$

where $\Phi^*_0 = \arg\max_{\Phi_0 \in \mathcal{A}} V(0,0)$. *Under a Ramsey plan,* $X_t = X^*_t$ *where*

$$X^*_t = \int_t^\infty e^{-\rho(s-t)}\big[C^*_{0,s}\, U_{C,s}(C^*_{0,s}, N^*_{0,s}) + N^*_{0,s}\, U_{N,s}(C^*_{0,s}, N^*_{0,s}) - b_{0-,s}\, U_{C,s}(C^*_{0,s}, N^*_{0,s})\big]\, ds \quad (26)$$

and the representative household's value is

$$W = \int_0^\infty e^{-\rho t} U(C^*_{0,t}, C^*_{0,t} + G_t)\, dt. \quad (27)$$

Proposition 5. *The Lagrangian in Proposition 2 and the Bellman equation in Proposition 4 generate identical Ramsey plans.*

Remark 2. *Showing that a Ramsey plan has a value function* $V(X_t, t)$ *that satisfies HJB* (18) *with state variable* X_t *paves the way to formulating a debt-management strategy that implementing a Ramsey plan whenever it exists, something that Debortoli, Nunes, and Yared (2021) showed is not always possible with Lucas and Stokey's recommended debt-management policy.*

5 Quantitative Examples

To construct some quantitative examples, we assume the same the instantaneous utility function $U(C, N) = \ln C - N$ utilized by Debortoli et al. (2021). Under this specification, the first-order condition (4) implies $\tau^*_{0,t} = 1 - C^*_{0,t}$.

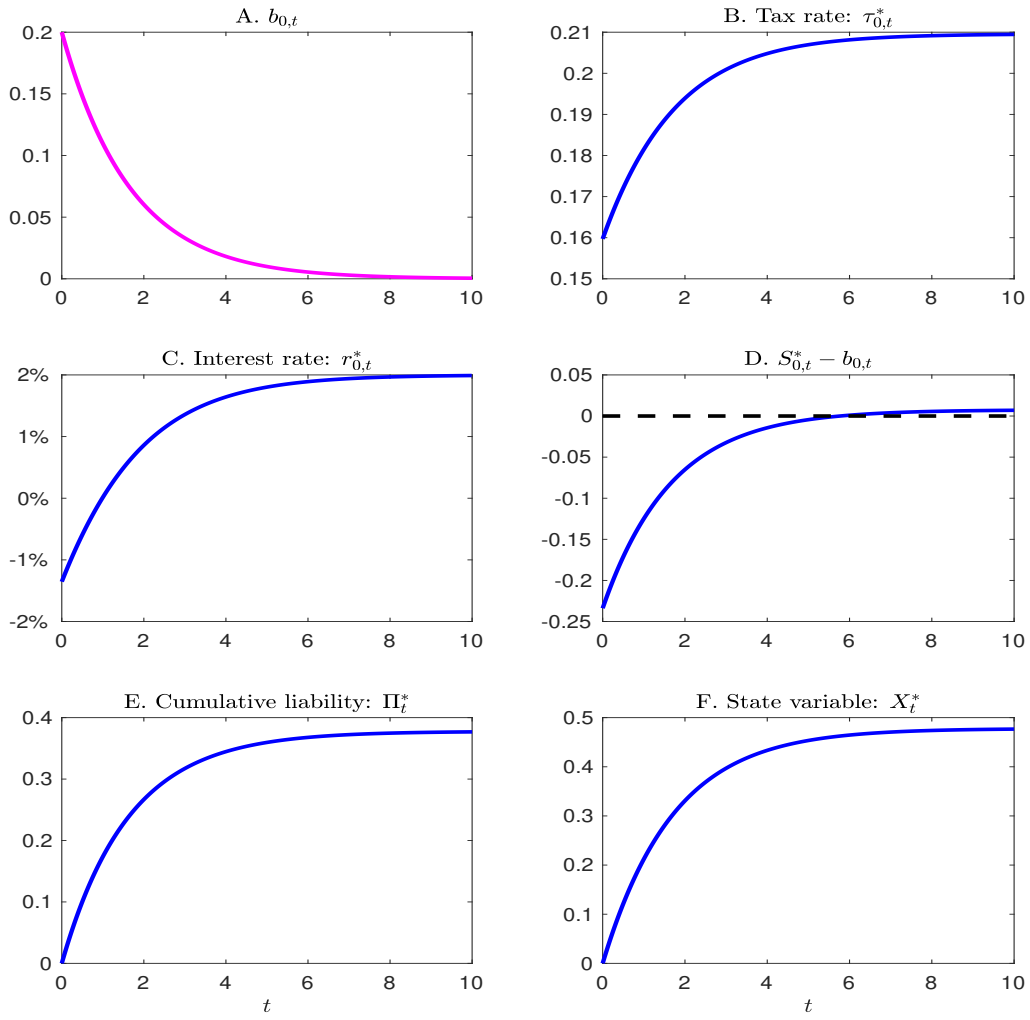


Figure 1: Ramsey plan for exponentially decaying $b_{0,t} = b_0 e^{-\theta t}$. Parameter values: $b_0 = 0.2$, $\theta = 0.6$, $G_t = 0.2, \rho = 0.02$.

5.1 Exponential $\{b_{0-,t}\}$

Figure 1 shows a Ramsey plan when $b_{0-,t} = b_0 e^{-\theta t}$, where $b_0 = 0.2$ and $\theta = 0.6$, and that $G_t = 0.2$ for all t. The Ramsey tax rate $\tau^*_{0,t}$ increases over time (panel B), which implies that consumption $C^*_{0,t} = 1 - \tau^*_{0,t}$ and the labor supply $N^*_{0,t} = C^*_{0,t} + G_t$ both decrease over time. The equilibrium interest rate is $r^*_{0,t} = \rho + g^*_{0,t}$ where $g^*_{0,t} = \dot{C}^*_{0,t}/C^*_{0,t}$ is the equilibrium consumption growth rate, which is negative:[8] it is less than $-\rho$ at $t = 0$, then gradually increases over time, eventually approaching zero. Coincidentally, the interest rate is negative at $t = 0$, increases over time, and eventually approaches $\rho = 0.02$ (see panel C).

The Ramsey planner front-loads consumption so heavily that increases in output $N^*_{0,t} = C^*_{0,t} + G_t$ dominate decreases in the tax rate $\tau^*_{0,t} = 1 - C^*_{0,t}$, making the primary surplus $S^*_{0,t} = \tau^*_{0,t} N^*_{0,t} - G_t = (1 - C^*_{0,t})(C^*_{0,t} + G_t) - G_t$ increase over time. Since $b_{0-,t}$ decreases over time, the wedge $S^*_{0,t} - b_{0-,t}$ is negative initially, turns positive at $t = 5.8$, and ultimately converges to 0.7% (see panel D).

By inducing an increasing equilibrium $r^*_{0,t}$ path (panel C), the planner lowers the time-0 value of its wedge $S^*_{0,t} - b_{0-,t}$ shown in panel D. As a result, the cumulative liability $\Pi^*_t = \int_0^t e^{\int_0^u r^*_{0,v}\, dv}(S^*_{0,u} - b_{0-,u})\, du$ increases (panel E). Since Π^*_t and $U^*_{C,t}$ both increase over time, the state variable $X^*_t = \Pi^*_t\, U^*_{C,t}$ also increases over time (panel F).

5.2 Cyclical $\{b_{0-,t}\}$

Figure 2 shows a Ramsey plan when $b_{0-,t} = b_0 + \theta_b \sin(\frac{\pi}{2} t)$ with $b_0 = 0.1$ and $\theta_b = 0.01$ for $t \geqslant 0$. The tax rate $\tau^*_{0,t}$ is cyclical and peaks when $\{b_{0-,t}\}$ reaches a trough (see panel B). $C^*_{0,t} = 1 - \tau^*_{0,t}$, so consumption process shares peaks and troughs with the $\{b_{0-,t}\}$ process.[9] The equilibrium interest rate is $r^*_{0,t} = \rho + g^*_{0,t}$ where $g^*_{0,t} = \dot{C}^*_{0,t}/C^*_{0,t}$ is the equilibrium consumption growth rate. It is also cyclical but attains different peaks and troughs at different times that do $b_{0-,t}$ and $\tau^*_{0,t}$: we mark peaks/troughs for $r^*_{0,t}$ with triangles. They evidently differ from the peaks/troughs in panels A, B, and D for $b_{0-,t}$, $\tau^*_{0,t}$ and with the wedge between primary surplus and $b_{0-,t}$: $S^*_{0,t} - b_{0-,t}$.[10] Panel D also shows that the primary surplus exceeds $b_{0-,t}$ when $b_{0-,t} < 0.1$ and otherwise when $b_{0-,t} > 0.1$.

[8] $C^*_{0,t} = 2b_0 e^{-\theta t}\Phi^*_0/(\sqrt{1 + 4b_0 e^{-\theta t}\Phi^*_0(1 + \Phi^*_0)} - 1)$ and $\Phi^*_0 = 0.2652$. Therefore, $g^*_{0,t} = dC^*_{0,t}/dt/C^*_{0,t} < 0$ because $dC^*_{0,t}/dt = -b_0\Phi^*_0\theta e^{-\theta t}/\sqrt{1 + 4b_0 e^{-\theta t}\Phi^*_0(1 + \Phi^*_0)} < 0$.

[9] $C^*_{0,t} = \left(\sqrt{1 + 4(b_0 + \theta_b \sin(\frac{\pi}{2} t))\Phi^*_0(1 + \Phi^*_0)} - 1\right)/(2(1 + \Phi^*_0))$, which is cyclical with the same period as $b_{0-,t}$. Lagrangian multiplier $\Phi^*_0 = 0.7256$.

[10] The wedge shares peaks and troughs with the tax rate $\{\tau^*_{0,t}\}$ process.

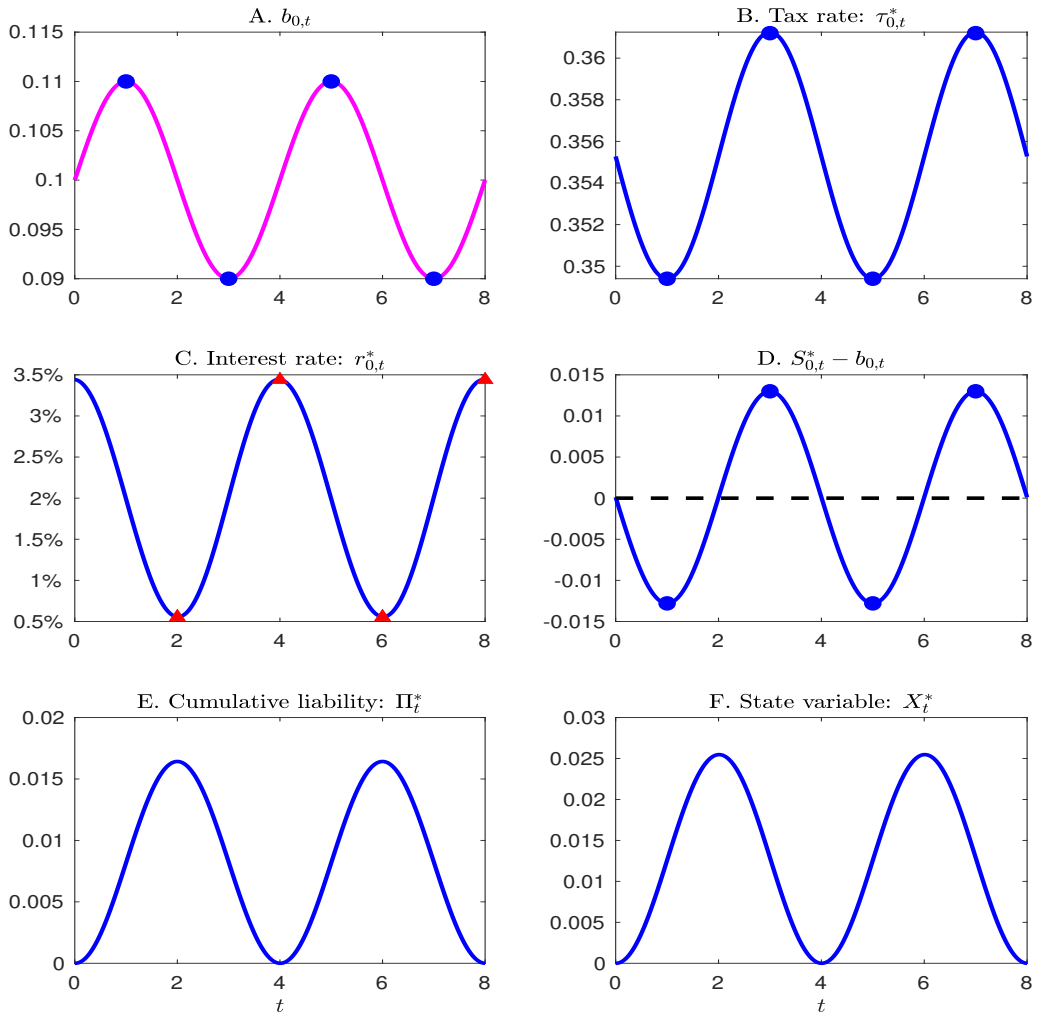


Figure 2: Ramsey plan for cyclical $b_{0,t} = b_0 + \theta_b \sin\left(\frac{\pi}{2}t\right)$. Parameter values: $b_0 = 0.1, \theta_b = 0.01$, $G_t = 0.2, \rho = 0.02$.

The tax rate and the wedge $S^*_{0,t} - b_{0-,t}$ processes move in an opposite direction from the $b_{0-,t}$ process because the marginal utility (and hence bond price) is low when $\{b_{0-,t}\}$ is high. Such covariation minimizes the value of the government's cumulative liability. Panels E and F show that the implied cumulative liability Π^*_t and the state variable $X_t = \Pi^*_t U^*_{C,t}$ are also cyclical but with different peaks/troughs than the $b_{0-,t}$ process.

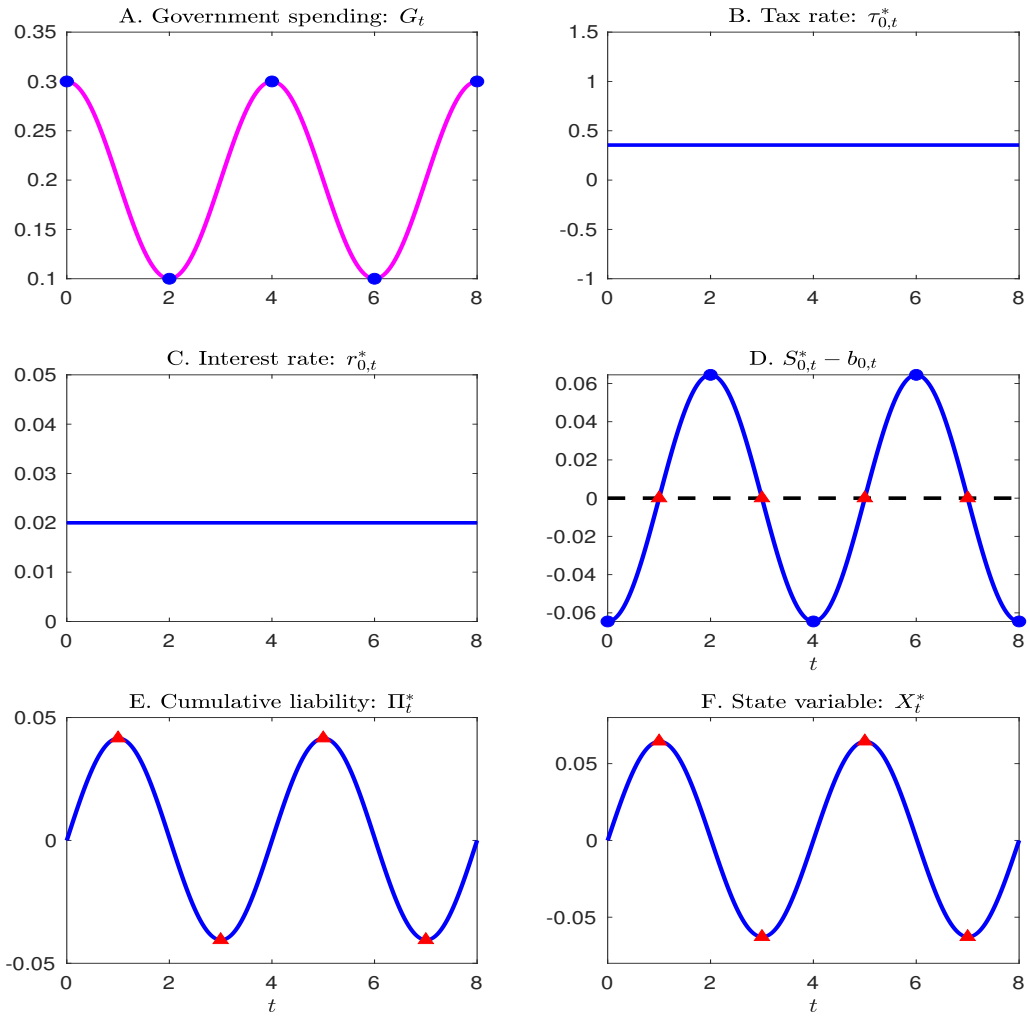


Figure 3: Ramsey plan for cyclical spending $G_t = G_0 + \theta_G \cos\left(\frac{\pi}{2}t\right)$. Parameter values: $G_0 = 0.2, \theta_G = 0.1$, $b_{0,t} = 0.1, \rho = 0.02$.

5.3 Cyclical $\{G_t\}$

In figure 3, the Ramsey planner finances a cyclical government spending process and a constant $b_{0-,t}$ process. The planner chooses a constant tax rate process because when $b_{0-,t}$ is constant, the maximand in (21) yields an optimal C that is independent of G_t. Since $C^*_{0,t} = 0.64$, $\tau^*_{0,t} = 1 - C^*_{0,t} = 0.36$ for all t. This echoes Barro's (1979) finding that intertemporal properties of an optimal tax rate are disconnected from intertemporal properties of

the government expenditure process that the Ramsey planner must finance. Since consumption $C^*_{0,t}$ is constant, the equilibrium interest rate process $r^*_{0,t} = \rho + \dot{C}^*_{0,t}/C^*_{0,t} = \rho$ (see panel C). Panel D shows that the primary surplus process $S^*_{0,t}$ is cyclical, reaching a peak (trough) when the G_t process reaches a trough (peak). Solid dots indicate troughs/peaks in panels A and D. Triangles in panel D indicate points where $S^*_{0,t} = b_{0-,t} = 0.1$. Panel E plots cumulative liability $\Pi^*_t = \int_0^t e^{\rho u}(S^*_{0,u} - b_{0-,u})\, du$, which reaches a peak (trough) when $S^*_{0,t} = b_{0-,t} = 0.1$.[11] The state variable $X^*_t = \Pi^*_t U^*_{C,t}$ cyclical and, because constant consumption implies constant $U^*_{C,t}$, shares peaks and troughs as Π^*_t (see the triangles in panels D, E, and F).

6 Implementing a Ramsey Plan

We now answer the question that concerned Lucas and Stokey (1983) and Debortoli, Nunes, and Yared (2021): how to implement the Ramsey plan sequentially under a timing protocol that attributes "partial commitment" to a sequence of government decision makers. We adopt a timing protocol that adds a speck of partial commitment to a continuation Ramsey planner's choice of a tax policy. This lets us implement Ramsey plans that Lucas and Stokey can not.

6.1 Debt Management

Under a Ramsey plan, the equilibrium bond price $\{q^*_{0,t};\, t \geqslant 0\}$ satisfies $q^*_{s,t} = e^{-\int_s^t r^*_{0,u}\, du}$. Let Π^*_t denote time-t cumulative liabilities in the absence of term-debt adjustments, defined in equation (12), evaluated under the Ramsey plan. Let B_t denote the government's instantaneous debt balance at t. Over an infinitesimal time interval $[t, t + dt)$, the government issues instantaneous debt and term debt to service its debts and to finance the continuation of its cumulative liabilities. Consequently

$$d\Pi^*_t - r^*_{0,t}\Pi^*_t dt = dB_t - r^*_{0,t} B_t dt + \left(\int_{s>t} q^*_{t,s}\, \partial_t\, b_{t,s}\, ds\right) dt - \left(\int_{s<t} \partial_s\, b_{s,t}\, ds\right) dt\,. \tag{28}$$

[11] This follows from $d\Pi^*_t/dt = e^{\rho t}(S^*_{0,t} - b_{0-,t})$.

Equation (13) implies that the left side of (28), which is the net change of Π_t^*, equals $(b_{0-,t} + G_t - \tau_{0,t}^* N_{0,t}^*)dt$. Substituting this into (28) and re-arranging yields:

$$\underbrace{(G_t - \tau_{0,t}^* N_{0,t}^*)dt}_{\text{primary deficits}} + \underbrace{\left(b_{0-,t} + \int_{s<t} \partial_s\, b_{s,t}\, ds\right) dt}_{\text{maturing term debt repayments}}$$

$$= \underbrace{dB_t - r_{0,t}^* B_t dt}_{\text{net instantaneous-debt issuance}} + \underbrace{\left(\int_{s>t} q_{t,s}^*\, \partial_t\, b_{t,s}\, ds\right) dt}_{\text{gross term-debt issuance}}\,. \tag{29}$$

Relation (29) equates uses and sources of government funds. Uses include primary deficits (the first term) and repayments for all maturing debts (the second term), the latter of which includes the initial term debt $b_{0-,t}$ as well as all other term debts due at t.[12] Sources include (net) issuance of instantaneous debt, $dB_t - r_{0,t}^* B_t dt$, and term debts of all possible maturities issued at t: $\int_{s>t} q_{t,s}^*\, \partial_t\, b_{t,s}\, ds$.

Next, we describe a debt-management strategy that always implement a Ramsey plan, provided that it exists.

Managing Π_t^* with instantaneous debt. To implement a Ramsey plan, we show that it is sufficient to restrict a continuation Ramsey planner's debt restructuring policies to a set in which $\partial_t\, b_{t,s} = 0$ so that $b_{t,s} = b_{0-,s}$ for all $s \geqslant t$, $t \geqslant 0$, so that no term debt is issued at any $t \geqslant 0$. That in turn means that the instantaneous debt balance B_t equals Π_t^* for all $t \geqslant 0$. Thus, it is sufficient to only use instantaneous debt to finance the cumulative liabilities that are governed so that:

$$dB_t = r_{0,t}^*\, B_t\, dt + (b_{0-,t} + G_t - \tau_{0,t}^*\, N_{0,t}^*)\, dt. \tag{30}$$

Remark 3. *Lucas and Stokey (1983) use a coupon flow management policy of* $\{\partial_t\, b_{t,s};\ s \geqslant t,\ t \geqslant 0\}$ *and* $\{B_t = 0;\ t \geqslant 0\}$ *that satisfies the dynamic budget constraint:*

$$d\Pi_t^* - r_{0,t}^* \Pi_t^* dt = \left(\int_{s>t} q_{t,s}^*\, \partial_t\, b_{t,s}\, ds\right) dt - \left(\int_{s<t} \partial_s\, b_{s,t}\, ds\right) dt\,. \tag{31}$$

Their rescheduling policy requires continuous adjustments to the term debt, a requirement that makes it challenging, if not impossible, to formulate their government's problem recur-

[12] Here $\partial_s\, b_{s,t}$ denotes the flow rate of coupon obligations issued at time u and maturing at time t, so $(\int_{s<t} \partial_s\, b_{s,t}\, ds)\, dt$ represents all outstanding term debts that were issued after $t = 0-$ and are due at time t.

sively.

We propose what we think is a better way to implement the Ramsey plan. When the government can issue instantaneous debt, it gains nothing by rescheduling term debt. Ultimately, this is because instantaneous debt balance B_t perfectly tracks the cumulative deficits Π_t^* and adequately summarizes the key price and quantity information about the Ramsey plan. Furthermore, being the shortest maturity debt possible, the value of instantaneous debt is least vulnerable to manipulation by future governments' deviating from the Ramsey tax plan.

6.2 Slightly More Commitment

In order to commit an immediate successor government to the current government's planned tax, we infinitesimally constrain Lucas and Stokey's (1983) governments in the following way.[13] Let $\tau_{t-,t}$ be a time-t tax rate that is preferred by a time-$t-$ government.

Assumption 2 (Instantaneous partial commitment)**.** *A time-$t-$ government sets a time-t government's current period tax rate $\tau_{t,t}$:*

$$\tau_{t,t} = \tau_{t-,t}. \tag{32}$$

Thus, the time t tax rate that applies to taxable income over a short time interval $(t, t + dt)$ is set by the instantaneous predecessor of the time-t government, also known as the continuation planner at time $t-$. Via the household's first-order necessary condition, this timing protocol induces

$$C_{t,t} = C_{t-,t}, \tag{33}$$

where $C_{t-,t}$ is the time-t consumption effectively chosen by the time-$t-$ government. To support this outcome, the government at time $t-$ hands over to the time-t government,a debt structure that includes a balance of instantaneous debt B_t due at t and a stream of term debt services $\{b_{t-,s};\ s \geqslant t\}$ that satisfies

$$B_t = \Pi_t^* \quad \text{and} \quad b_{t-,s} = b_{0-,s}; \quad s \geqslant t. \tag{34}$$

Thus, the term debt service flow inherited by the time-t government equals the tail $\{b_{0-,s};\ s \geqslant t\}$ of the initial term debt services stream and government's instantaneous debt B_t adjusts

[13] Jiang, Sargent, and Wang (2025) analyze a discrete-time counterpart of this timing protocol.

accordingly. Managing government debt in this way ensures that X_t defined in equation (14) is continuous. If a Ramsey plan has been implemented until time $t-$, then $C_{s,s} = C^*_{0,s}$ and $N_{s,s} = N^*_{0,s}$ for $0 < s < t$. Under Assumption 2 and that the Ramsey plan is implemented up to time $t-$, i.e., $C_{t-,t} = C^*_{0,t}$, then

$$B_t\, U_{C,t}(C_{t,t}, N_{t,t}) = \Pi^*_t\, U_{C,t}(C^*_{0,t}, N^*_{0,t}) = X^*_t, \tag{35}$$

where X^*_t is the value of state variable X_t under the Ramsey plan. Assumption 2 forces a time t continuation planner to preserve the purchasing power of the instantaneous debt it has inherited.

We now state the key implementation result.

Theorem 4. *Under our Assumption 2 timing protocol, a Ramsey plan can be implemented by accepting the tail of initial debt service stream* $\{b_{0-,t}\}$ *and adjusting instantaneous debt according to* (34) .

Proof. We proceed by induction. The key step is to show that if the Ramsey plan has been implemented up to time $t-$, then the continuation Ramsey planner at time t will confirm the continuation of the Ramsey plan. This requires showing that decisions of the time t continuation Ramsey planner imply, first, that $X_t = X^*_t$ and, second, that $\Phi^*_t = \Phi^*_0$.

Under the assumption that the Ramsey plan is implemented up to time $t-$, it is sufficient to prove that the Ramsey plan is also implemented at time t.[14] We can show that, confronting $B_t = \Pi^*_t$ and $b_{t-,s} = b_{0-,s}$ for all $s \geqslant t$ at time t, the continuation Ramsey planner will confirm the continuation of the Ramsey plan by choosing $C_{t,s} = C^*_{0,s}$ and $N_{t,s} = N^*_{0,s}$ for all $s \geqslant t$. We establish this by subdividing the time-t continuation Ramsey planner's problem into two subproblems.

First, the time-t continuation Ramsey planner chooses $\{C_{t,s},\, s \geqslant t\}$ to attain

$$\max_{\{C_{t,s};\, s \geqslant t\}} \int_t^\infty e^{-\rho(s-t)} U(C_{t,s}, C_{t,s} + G_s)\, ds, \tag{36}$$

where maximization is subject to $X_t = B_t\, U_{C,t}(C_{t,t}, N_{t,t}) = X^*_t$ and the following dynamics for $s \geqslant t$:

$$dX_s = \left(\rho\, X_s - C_{t,s}\, U_{C,s} - N_{t,s}\, U_{N,s} + b_{0-,s}\, U_{C,s}\right) ds. \tag{37}$$

As in our Section 4 analysis, we can show that the value function for (36), denoted as

[14] At $t = 0$, the Ramsey planner is also a time-0 continuation Ramsey planner.

$\widehat{V}(X_t, t; \Phi_t)$, is additively separable: $\widehat{V}(X_t, t; \Phi_t) = -\Phi_t X_t + \widehat{H}(t; \Phi_t)$, where $\widehat{H}(t; \Phi_t)$ solves the following differential equation[15]

$$\rho \widehat{H}(s; \Phi_t) = \widehat{H}'(s; \Phi_t) + \max_{C_{t,s}} U(C_{t,s}, C_{t,s} + G_s) + \Phi_t \big(C_{t,s} U_{C,s} + (C_{t,s} + G_s) U_{N,s} - b_{0-,s} U_{C,s} \big), \tag{38}$$

where $C_{t,s} = C(\Phi_t; b_{0-,s}, G_s)$, and $C(\Phi_t; b_{0-,s}, G_s)$, the same function presented in (21).

Under the new timing protocol given in Assumption 2, the time-t continuation Ramsey planner optimally chooses $C_{t,t} = C_{t-,t} = C^*_{0,t}$ to confirm the continuation of the Ramsey plan. Because $B_t = \Pi^*_t$ and $B_{t-} = \Pi^*_{t-}$ under our implementation, we have $X_t = B_t U_{C,t}(C_{t,t}, N_{t,t}) = \Pi^*_t U_{C,t}(C^*_{0,t}, N^*_{0,t}) = X^*_t$. So we have proved $X_t = X^*_t = \Pi^*_t U_{C,t}(C^*_{0,t}, N^*_{0,t})$.

We now turn to establishing that $\Phi^*_t = \Phi^*_0$. We start by proving that the time-t continuation Ramsey planner chooses $C_{t,s} = C^*_{0,s}$ for all $s > t$. Because the optimal continuation plan $\{C^*_{t,s};\, s > t\}$ satisfies $C^*_{t,s} = C(\Phi^*_t; b_{0-,s}, G_s)$ for some $\Phi^*_t > 0$ and the Ramsey plan satisfies $C^*_{0,s} = C(\Phi^*_0; b_{0-,s}, G_s)$, we need to verify that $\Phi^*_t = \Phi^*_0$. This we do by contradiction. Suppose that there exists a $\Phi^*_t \neq \Phi^*_0$ such that the associated continuation Ramsey plan yields a higher value than the tail of the Ramsey plan, meaning that

$$\int_t^\infty e^{-\rho(s-t)} U(C^*_{t,s}, C^*_{t,s} + G_s)\, ds > \int_t^\infty e^{-\rho(s-t)} U(C^*_{0,s}, C^*_{0,s} + G_s)\, ds, \tag{39}$$

where $C^*_{t,s} = C(\Phi^*_t; b_{0-,s}, G_s)$. Recall that the time-$t$ continuation Ramsey planner chooses an allocation $\{C^*_{t,s};\, s > t\}$ that satisfies the following continuation implementability condition:

$$\int_t^\infty e^{-\rho(s-t)} \big(C^*_{t,s} U_{C,s} + (C^*_{t,s} + G_s) U_{N,s} \big)\, ds \geqslant \int_t^\infty e^{-\rho(s-t)} b_{0-,s} U_{C,s}\, ds + B_t U_{C,t}. \tag{40}$$

Construct a plan $\{\widetilde{C}_{0,s};\, s \geqslant 0\}$ in which the allocation from 0 to t follows the Ramsey plan, $\widetilde{C}_{0,s} = C^*_{0,s}$ for $s \in [0, t]$, but the allocation from time t onward follows the time-t continuation Ramsey plan: $\widetilde{C}_{0,s} = C^*_{t,s}$ for $s \in (t, \infty)$. Using (40) and $B_t = \Pi^*_t$, we can verify that $\{\widetilde{C}_{0,s};\, s \geqslant 0\}$ satisfies the implementability condition (7) for the Ramsey plan. Next, using (39), we can show this proposed plan $\{\widetilde{C}_{0,s};\, s \geqslant 0\}$ yields a value for the household

[15] See more technical details in Appendix A.

that is higher than the value under the Ramsey plan:

$$\begin{aligned}\int_0^\infty e^{-\rho s}U(\widetilde{C}_{0,s},\widetilde{C}_{0,s}+G_s)\,ds &= \int_0^t e^{-\rho s}U(\widetilde{C}_{0,s},\widetilde{C}_{0,s}+G_s)\,ds + \int_t^\infty e^{-\rho s}U(\widetilde{C}_{0,s},\widetilde{C}_{0,s}+G_s)\,ds\\ &= \int_0^t e^{-\rho s}U(C^*_{0,s},C^*_{0,s}+G_s)\,ds + \int_t^\infty e^{-\rho s}U(C^*_{t,s},C^*_{t,s}+G_s)\,ds\\ &> \int_0^t e^{-\rho s}U(C^*_{0,s},C^*_{0,s}+G_s)\,ds + \int_t^\infty e^{-\rho s}U(C^*_{0,s},C^*_{0,s}+G_s)\,ds\\ &= \int_0^\infty e^{-\rho s}U(C^*_{0,s},C^*_{0,s}+G_s)\,ds.\end{aligned}$$

This contradicts that $\{C^*_{0,s}\}$ is the Ramsey plan. We thus conclude $\Phi^*_t = \Phi^*_0$ and that $C^*_{t,s} = C(\Phi^*_t;\, b_{0-,s}, G_s) = C(\Phi^*_0;\, b_{0-,s}, G_s) = C^*_{0,s}$ for all $s \geqslant t$. □

7 Concluding Remarks

We have formulated a continuous-time Ramsey taxation problem as a dynamic program with state variable $X_t = \Pi_t U_{C,t}$. The value function satisfies a Hamilton-Jacobi-Bellman equation and takes the additively separable form $V(X_t, t) = -\Phi_0 X_t + H(t; \Phi_0)$. This formulation yields the same Ramsey allocation as the Lagrangian method.

Our dynamic programming formulation underlies a recursive implementation of the Ramsey plan via a debt-management strategy that differs markedly from Lucas and Stokey's. We show that a government can implement the Ramsey plan by never adjusting the tail of the initial term-debt structure that it has inherited and simply setting its instantaneous debt equal to its cumulative liability Π^*_t. Under a timing protocol that grants the government infinitesimal commitment, continuation Ramsey planners confirm the Ramsey plan. The state variable X_t encodes the purchasing power of government liabilities, which depends on the government's cumulative liability Π_t as well as on the household's marginal utility of consumption $U_{C,t}$.

Much of our analysis will extend to settings with discontinuous debt service processes. Appendix C establishes that discrete-time Ramsey plans converge to continuous-time Ramsey plans as the time increment vanishes. The results apply immediately to economies with processes for initial government debt and spending streams that are right-continuous and possess left limits.

It will also be feasible to extend our analysis to settings in which fundamentals — i.e.,

government expenditures and an initial debt service stream — are stochastic processes. An HJB equation arising from a construction of a martingale and an application of the martingale representation theorem will emerge from such a setting.

Appendices

A Technical Details

Regularity Conditions for Proposition 2. We provide two regularity conditions that guarantee that a Ramsey plan exists. First, for a given initial debt structure, the admissible allocation set is not empty, so there exist allocations $\{C_{0,t};\, t \geqslant 0\}$ that satisfy the implementability condition (7). Second, $(CU_C + (C+G)U_N - bU_C)$ is concave for $G > 0$, $b \geqslant 0$.

For economies analyzed in Debortoli, Nunes, and Yared (2021) and this paper, these two regularity conditions are satisfied when b_0 is not too high.

Proof of Proposition 3. Given Φ_0, consumption at t, $C_{0,t} = C(\Phi_0;\, b_{0-,t}, G_t)$, solves

$$\max_{C_{0,t}} U(C_{0,t}, C_{0,t} + G_t) + \Phi_0\big(C_{0,t}\, U_{C,t} + (C_{0,t} + G_t)\, U_{N,t} - b_{0-,t}\, U_{C,t}\big). \tag{41}$$

First we can verify that under the preceding regularity conditions a solution $C_{0,t} > 0$ exists for (41). Second, we can show that $C_{0,t} = C(\Phi_0;\, b_{0-,t}, G_t)$ satisfies the following first-order necessary condition for problem (41):

$$\begin{aligned}(1+\Phi_0)(U_{C,t} + U_{N,t}) \\ + \Phi_0\big[(C_{0,t} - b_{0-,t})(U_{CC,t} + U_{CN,t}) + (C_{0,t} + G_t)(U_{NC,t} + U_{NN,t})\big] = 0.\end{aligned} \tag{42}$$

The associated labor supply is $N(\Phi_0;\, b_{0-,t}, G_t) = C(\Phi_0;\, b_{0-,t}, G_t) + G_t$.

Recall $V(X_t, t) = -\Phi_0\, X_t + H(t;\, \Phi_0)$ (see (19) in Subsection 4.3). Now solve ODE (20)

for $H(t;\,\Phi_0)$. Using $C(\Phi_0;\,b_{0-,t},G_t)$ and $N(\Phi_0;\,b_{0-,t},G_t)$, we can rewrite (20) as

$$d(-e^{-\rho t}H(t;\,\Phi_0)) = e^{-\rho t}\Big[U\big(C(\Phi_0;\,b_{0-,t},G_t),N(\Phi_0;\,b_{0-,t},G_t)\big) + \Phi_0\big(C(\Phi_0;\,b_{0-,t},G_t)\,U_{C,t}+N(\Phi_0;\,b_{0-,t},G_t)\,U_{N,t}-b_{0-,t}\,U_{C,t}\big)\Big]\,dt.$$

Integrating the above equation from t to ∞ and using the transversality condition $\lim_{t\to\infty} e^{-\rho t}H(t;\,\Phi_0) = 0$, we obtain

$$H(t;\,\Phi_0) = \int_t^\infty e^{-\rho(s-t)}\Big[U\big(C(\Phi_0;\,b_{0-,s},G_s),N(\Phi_0;\,b_{0-,s},G_s)\big) + \Phi_0\big(C(\Phi_0;\,b_{0-,s},G_s)\,U_{C,s}+N(\Phi_0;\,b_{0-,s},G_s)\,U_{N,s}-b_{0-,s}\,U_{C,s}\big)\Big]\,ds.$$

Therefore, the value function $V(X_t,t)$ defined for the DP problem in Definition 2 satisfies

$$\begin{aligned} V(X_t,t) &= -\Phi_0\,X_t + H(t;\,\Phi_0)\\ &= -\Phi_0\,\Pi_t\,U_{C,t} + \int_t^\infty e^{-\rho(s-t)}U\big(C(\Phi_0;\,b_{0-,s},G_s),N(\Phi_0;\,b_{0-,s},G_s)\big)\,ds \qquad (43)\\ &\quad + \Phi_0\int_t^\infty e^{-\rho(s-t)}\big(C(\Phi_0;\,b_{0-,s},G_s)\,U_{C,s}+N(\Phi_0;\,b_{0-,s},G_s)\,U_{N,s}-b_{0-,s}\,U_{C,s}\big)\,ds\\ &= \int_t^\infty e^{-\rho(s-t)}U\big(C(\Phi_0;\,b_{0-,s},G_s),N(\Phi_0;\,b_{0-,s},G_s)\big)\,ds. \qquad (44)\end{aligned}$$

Equation (44) follows from equation (43) because the time-0 budget constraint and the definition of Π_t in (12) together imply that the sum of the first and third terms in (43) equals zero. □

Technical Details for Theorem 4. Here we describe some details involved in solving the time-t continuation Ramsey planner's dynamic problem. The value function $\widehat{V}(X_s,s;\,\Phi_t)$ at s for $s\geqslant t$ satisfies the Bellman equation:

$$\rho\,\widehat{V} = \frac{\partial\widehat{V}}{\partial s} + \max_{C_{t,s}} U(C_{t,s},C_{t,s}+G_s) + \frac{\partial\widehat{V}}{\partial X_s}\big(\rho\,X_s - C_{t,s}\,U_{C,s} - (C_{t,s}+G_t)\,U_{N,s} + b_{0-,s}\,U_{C,s}\big). \quad (45)$$

Note that it is Φ_t not Φ_s that appears in $\widehat{V}(X_s,s;\,\Phi_t)$. We guess and verify that $\widehat{V}(X_s,s;\,\Phi_t)$

is additively separable in X_s and s:

$$\widehat{V}(X_s, s;\, \Phi_t) = -\Phi_t\, X_s + \widehat{H}(s;\, \Phi_t), \qquad s \geqslant t. \tag{46}$$

Substituting (46) into (45), we obtain the following for $\widehat{H}(s;\, \Phi_t)$ where $s \geqslant t$:

$$\rho\, \widehat{H}(s;\, \Phi_t) = \widehat{H}'(s;\, \Phi_t) + \max_{C_{t,s}} U(C_{t,s}, C_{t,s} + G_s) + \Phi_t\big(C_{t,s}\, U_{C,s} + (C_{t,s} + G_s)\, U_{N,s} - b_{0-,s}\, U_{C,s}\big). \tag{47}$$

Integrating (47) from t to ∞ and using the transversality condition, we obtain:

$$\begin{aligned}\widehat{H}(s;\, \Phi_t) = \int_s^{\infty} e^{-\rho(u-s)} \Big[& U\big(C(\Phi_t;\, b_{0-,u}, G_u),\, C(\Phi_t;\, b_{0-,u}, G_u) + G_u\big) \\ & + \Phi_t\big(C(\Phi_t;\, b_{0-,u}, G_u)\, U_{C,u} + (C(\Phi_t;\, b_{0-,u}, G_u) + G_u)\, U_{N,u} - b_{0-,u}\, U_{C,u}\big)\Big]\, du. \end{aligned} \tag{48}$$

B Cyclical Ramsey Plans

B.1 Cyclical $b_{0-,t}$ in Subsection 5.2

When $b_{0-,t} = b_0 + \theta_b \sin(\frac{\pi}{2} t)$, G_t is constant, and the utility function $U(C, N) = \log C - N$, $C^*_{0,t}$ under the Ramsey plan given by (21) can be simplified to

$$C^*_{0,t} = \frac{\sqrt{1 + 4(b_0 + \theta_b \sin(\frac{\pi}{2} t))\Phi^*_0(1 + \Phi^*_0)} - 1}{2(1 + \Phi^*_0)}, \tag{49}$$

which is periodic with the same peaks, troughs, and period (four) as the initial debt structure $b_{0-,t}$. The tax rate $\tau^*_{0,t} = 1 - C^*_{0,t}$ also has period four but is countercyclical to $b_{0-,t}$. When G_t is constant over time, the primary surplus $S(C^*_{0,t}) = \tau^*_{0,t}\, N^*_{0,t} - G_t = C^*_{0,t}(1 - C^*_{0,t} - G_t)$ is a periodic function with the same periodicity as $\tau^*_{0,t}$. This follows from

$$\frac{d(S^*_{0,t} - b_{0-,t})}{dt} = (1 - 2C^*_{0,t} - G)\frac{dC^*_{0,t}}{dt} - \frac{db_{0-,t}}{dt}$$

and $dC^*_{0,t}/dt = 0$ if and only if $db_{0-,t}/dt = 0$.

Table 1: Peaks and troughs in Figure 2.

	$t=1$	$t=3$
Initial debt $b_{0-,t}$	Peak	Trough
Consumption $C^*_{0,t}$	Peak	Trough
Tax rate $\tau^*_{0,t}$	Trough	Peak
$S(C^*_{0,t}) - b_{0-,t}$	Trough	Peak

B.2 Cyclical G_t in Subsection 5.3

When $G_t = G_0 + \theta_G \cos(\frac{\pi}{2}t)$, $b_{0-,t} = b_0$ is constant, and the utility function $U(C,N) = \log C - N$, $C^*_{0,t}$ under the Ramsey plan (21) is constant:

$$C^*_{0,t} = \frac{\sqrt{1 + 4b_0\Phi^*_0(1+\Phi^*_0)} - 1}{2(1+\Phi^*_0)}.$$

The equilibrium interest rate $r^*_{0,t}$ equals ρ for all $t \geqslant 0$.

Peaks, troughs, and inflection points of the primary surplus process $S^*_{0,t} - b_{0-,t}$ coincide with those of G_t because

$$\frac{dG_t}{dt} = -\frac{\pi}{2}\theta_G \sin\Big(\frac{\pi}{2}t\Big), \qquad \frac{d^2G_t}{dt^2} = -\Big(\frac{\pi}{2}\Big)^2 \theta_G \cos\Big(\frac{\pi}{2}t\Big),$$

and

$$\frac{d(S^*_{0,t} - b_{0-,t})}{dt} = -C_{0,t}\frac{dG_t}{dt}, \qquad \frac{d^2(S^*_{0,t} - b_{0-,t})}{dt^2} = -C_{0,t}\frac{d^2G_t}{dt^2}.$$

Since under a constant interest rate $r^*_{0,t} = \rho$ cumulative liabilities Π^*_t equal the present value of the net-of-interest surplus $S(C^*_{0,t}) - b_{0-,t}$, it is also a periodic function of time t. But Π^*_t peaks and troughs at $S^*_{0,t} = b_{0-,t}$, since

$$\frac{d\Pi^*_t}{dt} = 0 \iff (b_{0-,t} - S^*_{0,t}) = 0. \tag{50}$$

Because $U_{C,t}$ is constant over time, the state variable $X^*_t = \Pi^*_t U^*_{C,t}$ is also a periodic function. Its peaks and troughs coincide with those of Π^*_t.

Table 2: Peaks and troughs in Figure 3.

	$t=0$	$t=2$
Government spending G_t	Peak	Trough
Tax rate $\tau^*_{0,t}$	constant	
Interest rate $r^*_{0,t}$	constant	
$S(C^*_{0,t}) - b_{0-,t}$	Trough	Peak

C Discrete-time Counterpart

In this section, we recast the continuous-time formulation developed in Section 2 in a discrete-time setting with a time increment of Δ. We then show that the Ramsey plan in this discrete-time setting converges to that in the continuous-time formulation, characterized in Section 4, as Δ shrinks to 0. A key takeaway of this section is that our results in preceding sections continue to hold in discrete time. However, mathematically, our analysis is technically more convenient as we are able to work with differential equations.

Consider a discrete-time setting with a time increment $\Delta > 0$: $\{t_i = i\Delta,\, i = 0, 1, \ldots\}$. The representative household supplies a labor supply sequence $\{N^{(\Delta)}_{0,i}\,\Delta\}^{\infty}_{i=0}$ that produces a sequence of a single nonstorable good that must be allocated between a consumption sequence $\{C^{(\Delta)}_{0,i}\,\Delta\}^{\infty}_{i=0}$ and an exogenous government expenditure sequence $\{\tilde{G}^{(\Delta)}_{i}\,\Delta\}^{\infty}_{i=0}$:

$$C^{(\Delta)}_{0,i\Delta} + \tilde{G}^{(\Delta)}_{i\Delta} = N^{(\Delta)}_{0,i\Delta}, \quad \text{for all } i \geqslant 0, \tag{51}$$

where a first subscript indicates a time that a variable is chosen and a second subscript indicates a time that a variable is realized.[16]

The representative household orders consumption and labor supply streams $\{C^{(\Delta)}_{0,i}\}^{\infty}_{i=0}$ and $\{N^{(\Delta)}_{0,i}\}^{\infty}_{i=0}$ according to

$$\sum_{i=0}^{\infty} e^{-\rho i\Delta}\, U\big(C^{(\Delta)}_{0,i}, N^{(\Delta)}_{0,i}\big)\,\Delta, \tag{52}$$

where $\rho > 0$ and $U(\cdot,\cdot)$ are the same as those given in Section 2.

In period 0, the government inherits an initial term-debt structure $\{b^{(\Delta)}_{-1,i\Delta}\}^{\infty}_{i=0}$ that it must service: the government must pay its creditors an amount $b^{(\Delta)}_{-1,i\Delta}$ during time interval $[i\Delta, (i+1)\Delta)$. Let $\tau^{(\Delta)}_{0,i}$ denote a period-i flat tax rate that the government sets in period 0 and let $q^{(\Delta)}_{0,i}$ be the period-0 value of a zero-coupon bond with a unit payoff at time i. The

[16] To make the dependence of equilibrium objects, e.g., consumption and labor supply, on the size of time step Δ explicit, we add a superscript (Δ) for these equilibrium objects.

representative household faces a single intertemporal budget constraint:

$$\sum_{i=0}^{\infty}\big(C_{0,i\Delta}^{(\Delta)}\,\Delta\big)\cdot q_{0,i}^{(\Delta)} \leqslant \sum_{i=0}^{\infty}\big(b_{-1,i\Delta}^{(\Delta)}\big)\cdot q_{0,i}^{(\Delta)} + \sum_{i=0}^{\infty}\big((1-\tau_{0,t_i}^{(\Delta)})\,N_{0,i}^{(\Delta)}\,\Delta\big)\cdot q_{0,i}^{(\Delta)}. \tag{53}$$

Given tax-rate sequence $\{\tau_{0,i}^{(\Delta)}\}_{i=0}^{\infty}$ and bond-price sequence $\{q_{0,i}^{(\Delta)}\}_{i=0}^{\infty}$, the household chooses $\{C_{0,i\Delta}^{(\Delta)}, N_{0,i\Delta}^{(\Delta)}\}_{i=0}^{\infty}$ to maximize (2) subject to constraint (53). First-order necessary conditions for $N_{0,i}^{(\Delta)}$ and $C_{0,i}^{(\Delta)}$ are

$$1-\tau_{0,i}^{(\Delta)} = -\frac{U_{N,i}}{U_{C,i}} \tag{54}$$

$$q_{0,i}^{(\Delta)} = e^{-\rho i\Delta}\frac{U_{C,i}}{U_{C,0}}. \tag{55}$$

Similarly, we use first-order conditions (54) and (55) together with feasibility constraint (51) to eliminate tax rates and bond prices from the household budget constraint (53) and thereby to deduce the following implementability constraint on competitive equilibrium allocations $\{C_{0,i\Delta}^{(\Delta)}, N_{0,i\Delta}^{(\Delta)}\}_{i=0}^{\infty}$:

$$\sum_{i=0}^{\infty} e^{-\rho i\Delta}\big[C_{0,i}^{(\Delta)}\,U_{C,i} + (C_{0,i}^{(\Delta)} + \tilde{G}_i^{(\Delta)})\,U_{N,i} - b_{-1,i}^{(\Delta)}\,U_{C,i}\big]\cdot\Delta \geqslant 0. \tag{56}$$

In this discrete-time setting, a Ramsey plan is obtained by choosing a sequence $\{C_{0,i\Delta}^{(\Delta)}\}_{i=0}^{\infty}$ that maximizes (2) subject to resource constraint (51) and implementability constraint (56). Given that $\{C_{0,i\Delta}^{(\Delta)}\}_{i=0}^{\infty}$ sequence, we can use first-order necessary conditions (54) and (55) to compute associated equilibrium tax rate and bond price sequences.

Following Jiang et al. (2025), we can characterize the Ramsey plan using a Lagrange method. Let $\{\tilde{C}_{0,i}^{(\Delta)*}, \tilde{N}_{0,i}^{(\Delta)*}, \tilde{\tau}_{0,i}^{(\Delta)*}, \tilde{q}_{0,i}^{(\Delta)*}\}_{i=0}^{\infty}$ denote the Ramsey plan. Then construct a $\{C_{0,t}^{(\Delta)*}\}_{t\geqslant 0}$ process, where $C_{0,t}^{(\Delta)*} = \tilde{C}_{0,\lfloor t/\Delta\rfloor}^{(\Delta)*}$ and $\lfloor t/\Delta\rfloor$ is the integer part of t/Δ, that is, the unique integer i such that $t\in[i\Delta,(i+1)\Delta)$. We similarly construct processes $\{N_{0,t}^{(\Delta)*}\}_{t\geqslant 0}$, $\{\tau_{0,t}^{(\Delta)*}\}_{t\geqslant 0}$, and $\{q_{0,t}^{(\Delta)*}\}_{t\geqslant 0}$.

In addition, we also construct a debt payment process $\{b_{0-,t}^{(\Delta)}\}_{t\geqslant 0}$ and a spending process $\{G_t^{(\Delta)}\}_{t\geqslant 0}$. The following proposition states that, when $\{b_{0-,t}^{(\Delta)}\}_{t\geqslant 0}$ converges to $\{b_{0-,t}\}_{t\geqslant 0}$ and $\{G_t^{(\Delta)}\}_{t\geqslant 0}$ converges to $\{G_t\}_{t\geqslant 0}$ as $\Delta\to 0$, the Ramsey allocations in discrete time converge to those in the continuous-time setting analyzed in Section 4.

Proposition 6. *If* $\{b_{0-,t}^{(\Delta)}\}_{t\geqslant 0}$ *converges to* $\{b_{0-,t}\}_{t\geqslant 0}$ *and* $\{G_t^{(\Delta)}\}_{t\geqslant 0}$ *converges to* $\{G_t\}_{t\geqslant 0}$

as Δ goes to 0, the processes $\{C_{0,t}^{(\Delta)*}, N_{0,t}^{(\Delta)*}, \tau_{0,t}^{(\Delta)*}, q_{0,t}^{(\Delta)*}\}$ converge to $\{C_{0,t}^{*}, N_{0,t}^{*}, \tau_{0,t}^{*}, q_{0,t}^{*}\}$, respectively.

Proof. First, we impose two regularity conditions: (1) The utility function $U(\cdot,\cdot)$ and the associated function $C(\,\cdot\,;\,\cdot\,,\,\cdot\,)$ given in (21) are continuous in all arguments; (2) as functions of time t, $U(C(\Phi_0;\, b_{0-,t}^{(\Delta)}, G_t^{(\Delta)}), C(\Phi_0;\, b_{0-,t}^{(\Delta)}, G_t^{(\Delta)}) + G_t^{(\Delta)})$ and $(C(\Phi_0;\, b_{0-,t}^{(\Delta)}, G_t^{(\Delta)})\, U_{C,t} + (C(\Phi_0;\, b_{0-,t}^{(\Delta)}, G_t^{(\Delta)}) + G_t^{(\Delta)})\, U_{N,t} - b_{0-,t}^{(\Delta)}\, U_{C,t})$ are bounded by an integrable function of time t.

Recall that Proposition 4 in our Section 4 and Proposition 2 in Jiang et al. (2025) characterize the Ramsey plans for continuous-time and discrete-time settings, respectively, when they exist. Thus, for $\forall\Delta$, there exist $\Phi_0^{(\Delta)*} \geqslant 0$ and $\Phi_0^{*} \geqslant 0$ such that $C_{0,t}^{(\Delta)*} = C(\Phi_0^{(\Delta)*};\, b_{0-,t}^{(\Delta)}, G_t^{(\Delta)})$ and $C_{0,t}^{*} = C(\Phi_0^{*};\, b_{0-,t}, G_t)$. Via contradiction we can prove that $\lim_{\Delta\to 0} \Phi_0^{(\Delta)*} = \Phi_0^{*}$, which implies approximation

$$\lim_{\Delta\to 0} C_{0,t}^{(\Delta)*} = C_{0,t}^{*}, \qquad \lim_{\Delta\to 0} N_{0,t}^{(\Delta)*} = N_{0,t}^{*}, \qquad t \geqslant 0. \tag{57}$$

Suppose that for a sequence $\{\Delta_1 > \Delta_2 \cdots > \Delta_j > \cdots\}$ a non-negative limit of $\Phi_0^{(\Delta_j)*}$ exists that does not equal Φ_0^{*}: $\lim_{j\to\infty} \Phi_0^{(\Delta_j)*} = \Phi_0' \neq \Phi_0^{*}$. We can then construct a sequence of consumption allocations $\{C^{(\Delta_j)} = C(\Phi_0^{*};\, b_{0-,t}^{(\Delta_j)}, G_t^{(\Delta_j)});\, t \geqslant 0\}$ for $j = 1, 2, \ldots$.

First, we show that $\{C^{(\Delta_j)};\, t \geqslant 0\}$ is an admissible consumption policy. According to the definition of $C_{0,t}^{(\Delta)*}$, we have

$$\int_0^{\infty} e^{-\rho\lfloor t/\Delta\rfloor\Delta}\big[C_{0,t}^{(\Delta)*}\, U_{C,t} + (C_{0,t}^{(\Delta)*} + G_t^{(\Delta)})\, U_{N,t} - b_{0-,t}^{(\Delta)}\, U_{C,t}\big]\cdot dt \geqslant 0. \tag{58}$$

Using the definition of $C(\,\cdot\,;\,\cdot\,,\,\cdot\,)$ given in (21), for $j \geqslant 1$, we have

$$\begin{aligned} &U(C_{0,t}^{(\Delta_j)}, C_{0,t}^{(\Delta_j)} + G_t^{(\Delta_j)}) + \Phi_0^{*}\big(C_{0,t}^{(\Delta_j)}\, U_{C,t} + (C_{0,t}^{(\Delta_j)} + G_t^{(\Delta_j)})\, U_{N,t} - b_{0-,t}^{(\Delta_j)}\, U_{C,t}\big) \\ &\geqslant U(C_{0,t}^{(\Delta_j)*}, C_{0,t}^{(\Delta_j)*} + G_t^{(\Delta_j)}) + \Phi_0^{*}\big(C_{0,t}^{(\Delta_j)*}\, U_{C,t} + (C_{0,t}^{(\Delta_j)*} + G_t^{(\Delta_j)})\, U_{N,t} - b_{0-,t}^{(\Delta_j)}\, U_{C,t}\big). \end{aligned} \tag{59}$$

Multiplying both sides of (59) by $e^{-\rho\lfloor t/\Delta\rfloor\Delta}$ and integrating, we obtain

$$\begin{aligned} &\int_0^{\infty} e^{-\rho\lfloor t/\Delta\rfloor\Delta}\big[U(C_{0,t}^{(\Delta_j)}, C_{0,t}^{(\Delta_j)} + G_t^{(\Delta_j)}) + \Phi_0^{*}(C_{0,t}^{(\Delta_j)}\, U_{C,t} + (C_{0,t}^{(\Delta_j)} + G_t^{(\Delta_j)})\, U_{N,t} - b_{0-,t}^{(\Delta_j)}\, U_{C,t})\big] \\ &\geqslant \int_0^{\infty} e^{-\rho\lfloor t/\Delta\rfloor\Delta}\big[U(C_{0,t}^{(\Delta_j)*}, C_{0,t}^{(\Delta_j)*} + G_t^{(\Delta_j)}) + \Phi_0^{*}(C_{0,t}^{(\Delta_j)*}\, U_{C,t} + (C_{0,t}^{(\Delta_j)*} + G_t^{(\Delta_j)})\, U_{N,t} - b_{0-,t}^{(\Delta_j)}\, U_{C,t})\big]. \end{aligned} \tag{60}$$

Because a policy $C^{(\Delta_j)}$ yields a smaller household utility than the policy $C^{(\Delta_j)*}$, we have

$$\int_0^\infty e^{-\rho\lfloor t/\Delta\rfloor\Delta} U(C_{0,t}^{(\Delta_j)*}, C_{0,t}^{(\Delta_j)*} + G_t^{(\Delta_j)}) > \int_0^\infty e^{-\rho\lfloor t/\Delta\rfloor\Delta} U(C_{0,t}^{(\Delta_j)}, C_{0,t}^{(\Delta_j)} + G_t^{(\Delta_j)}). \tag{61}$$

Combining (58), (60) and (61), we obtain

$$\int_0^\infty e^{-\rho\lfloor t/\Delta\rfloor\Delta} \big(C_{0,t}^{(\Delta_j)}\, U_{C,t} + (C_{0,t}^{(\Delta_j)} + G_t^{(\Delta_j)})\, U_{N,t} - b_{0-,t}^{(\Delta_j)}\, U_{C,t}\big) > 0. \tag{62}$$

Therefore, consumption process $\{C^{(\Delta_j)} = C(\Phi_0^*;\, b_{0-,t}^{(\Delta_j)}, G_t^{(\Delta_j)});\, t \geqslant 0\}$ for $j = 1, 2, \ldots$ satisfies the implementability condition and is admissible.

Second, under our regularity conditions, using $\lim_{j\to\infty} \Phi_0^{(\Delta_j)*} = \Phi_0'$, $\lim_{j\to\infty} b_{0-,t}^{(\Delta_j)} = b_{0-,t}$, $\lim_{j\to\infty} G_t^{(\Delta_j)} = G_t$, we have

$$\lim_{j\to\infty} C(\Phi_0^*;\, b_{0-,t}^{(\Delta_j)}, G_t^{(\Delta_j)}) = C(\Phi_0^*;\, b_{0-,t}, G_t), \quad \text{and} \quad \lim_{j\to\infty} C(\Phi_0^{(\Delta_j)*};\, b_{0-,t}^{(\Delta_j)}, G_t^{(\Delta_j)}) = C(\Phi_0';\, b_{0-,t}, G_t).$$

Applying the dominated convergence theorem to (61), we obtain:

$$\begin{aligned} &\int_0^\infty e^{-\rho t} U\big(C(\Phi_0';\, b_{0-,t}, G_t), C(\Phi_0';\, b_{0-,t}, G_t) + G_t\big)\, dt \\ &\quad \geqslant \int_0^\infty e^{-\rho t} U\big(C(\Phi_0^*;\, b_{0-,t}, G_t), C(\Phi_0^*;\, b_{0-,t}, G_t) + G_t\big)\, dt. \end{aligned} \tag{63}$$

The implementability condition under the Ramsey plan is

$$\int_0^\infty e^{-\rho\lfloor t/\Delta\rfloor\Delta} \big(C_{0,t}^{(\Delta_j)*}\, U_{C,t} + (C_{0,t}^{(\Delta_j)*} + G_t^{(\Delta_j)})\, U_{N,t} - b_{0-,t}^{(\Delta_j)}\, U_{C,t}\big) \geqslant 0. \tag{64}$$

Applying the dominated convergence theorem to (64), we obtain:

$$\int_0^\infty e^{-\rho t} \big(C(\Phi_0';\, b_{0-,t}, G_t)\, U_{C,t} + (C(\Phi_0';\, b_{0-,t}, G_t) + G_t)\, U_{N,t} - b_{0-,t}\, U_{C,t}\big)\, dt \geqslant 0. \tag{65}$$

Therefore, $\{C(\Phi_0';\, b_{0-,t}, G_t);\, t \geqslant 0\}$ is an admissible policy for the initial debt structure $\{b_{0-,t};\, t \geqslant 0\}$ and government spending process $\{G_t;\, t \geqslant 0\}$. However, the optimality of Φ_0^* requires that

$$\int_0^\infty e^{-\rho t} U\big(C(\Phi_0^*;\, b_{0-,t}, G_t), C(\Phi_0^*;\, b_{0-,t}, G_t) + G_t\big)\, dt$$

$$> \int_0^\infty e^{-\rho t} U\big(C(\Phi_0';\, b_{0-,t}, G_t), C(\Phi_0';\, b_{0-,t}, G_t) + G_t\big)\, dt, \tag{66}$$

which contradicts (63). □

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