Fiscal and Monetary Unions*

Rafael Berriel † Eugenia Gonzalez-Aguado ‡ Patrick Kehoe§

Elena Pastorino¶

February 2026

Abstract

How should fiscal policies be delegated within a monetary union, and how does enlargement affect the case for centralization? Two forces compete. The *fiscal federalism* view favors decentralization: local authorities hold superior information about local conditions. The *macroeconomic* view favors centralization: decentralized governments issuing nominal debt ignore the inflationary spillovers their borrowing imposes on others, generating an endogenous fiscal externality. We model a monetary union with a finite number of countries—rather than the standard continuum—which produces nontrivial debt dynamics requiring new viscosity solution techniques. Our key result: the fiscal externality grows with union size while the informational advantage of decentralization remains constant, yielding a threshold. Small unions prefer decentralization; large ones prefer centralization. Highly indebted unions hit this threshold sooner. Adding many small countries worsens free-riding and tilts toward centralization. Our framework speaks directly to the fiscal governance debate as the European Monetary Union expands.

Keywords: Fiscal externalities, Time inconsistency, EU expansion

JEL classification: E6, E61, E62, E65.

*The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

†Stanford University

‡Toulouse School of Economics

§Stanford University and the Federal Reserve Bank of Minneapolis

¶Hoover Institution and Stanford University.

How does expanding a monetary union affect the tradeoff between centralized and decentralized fiscal authority? This question has deep historical roots. Sargent (2012) analyzes precisely this tension in the evolution of the United States: fiscal policy was decentralized under the Articles of Confederation, then greatly centralized under the Constitution, a shift Sargent argues was critical as the union grew larger. The same debate is alive today. As the European Monetary Union expands, does enlargement strengthen the case for a union-wide fiscal authority over the current decentralized arrangement? Tabellini (2003) argues that the optimal delegation of fiscal decisions—whether to centralize them within a union-wide authority or leave them to member states—is among the most pressing questions facing the European Union. This paper studies exactly that question.

Existing work falls into two starkly different views. The fiscal federalism view, developed in the seminal work of Oates (1999), argues that "*individual local governments are presumably much closer to the people and geography of their respective jurisdictions; they possess knowledge of both local preferences and cost conditions that a central agency is unlikely to have.*" Under commitment and without direct externalities, Oates's Decentralization Theorem states that decentralization is always optimal.

The macroeconomic view, developed in Chari and Kehoe (2008) and Aguiar et al. (2015) among others, starts from the opposite premise: with commitment, centralization and decentralization are equivalent.[1] The key friction is that without commitment, governments issuing nominal debt create an ex-post incentive for the monetary authority to inflate it away. Decentralized fiscal authorities ignore the inflationary spillovers their borrowing imposes on other union members, generating an *endogenous fiscal externality*. Centralization is always optimal.[2] These polar views illustrate the competing forces but are silent on how the tradeoff evolves as the union grows. We build a model that captures both.

Our model is a tractable continuous-time monetary union with a finite number of countries. To capture Oates's case for decentralization, local fiscal authorities have an *information advantage*: they observe their citizens' taste shocks—which govern the preference for private versus public goods—perfectly, while the centralized authority observes them imperfectly.

[1]Both papers discuss implementing centralized allocations with limits on debts and deficits. Since experience has shown that such limits are in practice not credible, we focus on the more extreme case of centralization, by which we mean a centralized authority makes fiscal decisions for all countries.

[2]Beetsma and Uhlig (1997) reach the same conclusion from a political-economy model.

To capture the macroeconomic case for centralization, a union-wide monetary authority sets inflation but lacks commitment, creating a temptation to erode the real value of nominal debt. However, increasing inflation has a direct utility cost.[3] Under decentralization, each country independently sets its own fiscal policy: consumption of private goods, local public goods, labor, and nominal debt. Under centralization, a single fiscal authority makes these choices for all countries. Risk-neutral foreign lenders price government bonds at a nominal rate that, in equilibrium, reflects their real opportunity cost plus expected inflation.

For sufficiently high nominal debt the temptation outweighs the costs and the monetary authority inflates whereas for sufficiently low nominal debt the monetary authority choses zero inflation. These forces lead both the decentralized and centralized regime to feature two regions in equilibrium: a low-debt, zero-inflation region and a high-debt, high-inflation region. In both regions the real interest rate is the same, since the nominal rate adjusts for the difference in expected inflation rates. Hence, in equilibrium, the monetary authority cannot erode the real value of the debt, and inflation is only a cost born by a highly indebted union.[4] In the high-debt, high-inflation region, fiscal authorities in both regimes run a fiscal surplus—an *austerity policy*—reducing consumption and increasing labor supply in order to reduce the total debt of the union and, hence, eventually induce the monetary authority to choose zero inflation.

The key distinction between regimes emerges here. The centralized authority weighs the austerity costs borne by a given country against the benefits—lower inflation—enjoyed by all members. The decentralized authority weighs the same costs against benefits accruing only to itself, failing to internalize the gains its austerity delivers to others. This is the source of the endogenous fiscal externality: decentralization produces insufficient austerity and prolonged inflation. Critically, this externality grows with union size— as more countries join, each decentralized authority internalizes a shrinking share of the union-wide benefits of its fiscal discipline.

A novel feature of our analysis is modeling a finite number of countries rather than

[3]Aguiar et al. (2015) draw on evidence from Reinhart and Rogoff (2009) to argue that this cost can also be interpreted as a partial default technology.

[4]This parallels the Barro-Gordon model of time inconsistency: given the initial price level, the monetary authority takes nominal wages as fixed and inflates to push unemployment below its natural rate. In equilibrium, wage-setters anticipate this, unemployment remains at its natural rate, and society bears the cost of excessive inflation.

assuming a continuum. This distinction is critical. With a continuum, as in Aguiar et al. (2015), no individual country can affect union-wide debt, so none try—each simply holds debt constant under decentralization. The fiscal externality exists but is invariant to union size, making the continuum assumption ill-suited to our question. Moving to a finite number of countries introduces strategic power for each member, pushing the model outside the standard mean-field game framework. Such finite-player games, even symmetric ones, are notoriously intractable analytically (Nutz, 2018). The key assumption in our model that restores tractability is linear disutility of labor. Under this assumption, each country's best response in the decentralized equilibrium depends only on the union averages of debt, consumption, and output—a sufficient reduction in the state space that allows a full analytical characterization.

Our main result characterizes how union size affects the desirability of centralization. In the high-inflation region, the macroeconomic case for centralization grows with the number of countries, as the fiscal externality worsens with each new member. The fiscal federalism case for decentralization, by contrast, is invariant to union size—the informational advantage of local authorities is constant.[5]

When the information advantage is not so overwhelming as to always favor decentralization, these two forces yield a cutoff result: decentralization is optimal for unions below a critical size, centralization for those above it. Highly indebted unions hit this cutoff at fewer members, since larger initial debt amplifies the fiscal externality for any given number of countries. In the zero-inflation region, the fiscal externality is absent while the informational advantage remains, so decentralization is always preferred there.

We then extend the analysis to unions expanding by adding members that differ from existing ones. Three results stand out. First, the higher the average debt of incoming members relative to existing ones, the fewer new members the union can absorb before centralization becomes preferred. Higher-debt entrants raise union-wide average debt, amplifying fiscal externalities and strengthening the macroeconomic case for centralization.

Second, we consider adding countries with the same debt per capita as existing mem-

[5]We maintain that the informational disadvantage of central authority is constant in union size. If it shrank with size—say, through increasing returns to information gathering—centralization would be optimal for even smaller unions. If it grew faster than the fiscal externality, centralization would require a larger union, or may never be optimal. We focus on the constant disadvantage case as a natural baseline.

bers, but varying in size, to reach a fixed total population target. Adding many small countries is worse for decentralization than adding few large ones. Large countries internalize a greater share of the union-wide inflation costs and therefore impose stricter austerity on themselves. Small countries free-ride on this discipline, implementing weaker austerity programs. For a given total population, the smaller each new entrant, the more severe this free-rider problem—and the stronger the case for centralization. This analysis suggests that adding four small countries, Estonia, Latvia, Lithuania, and Slovakia would make the case for centralization stronger than would adding a larger country, Czechia, a similar total population.

Third, when considering adding a single country, centralization is less desirable for either a small country or a large country rather than a middle-sized country. This occurs because a small country free rides more than existing ones but has a negligible effect on the union. Likewise, a large country runs a more severe austerity plan reducing the fiscal externality.

Related literature. Closest to our paper is Aguiar et al. (2015), who study the fiscal externality in a monetary union.[6] We depart in two key dimensions: we give decentralized authorities an informational advantage, and we model a finite number of countries. Nicolini (1998), Diaz-Gimenez et al. (2008), and Afrouzi et al. (2023) also study Markov perfect equilibria without monetary commitment but focus on optimal policy in a one-country closed economy, not on the desirability of fiscal centralization. These papers motivate the cost of inflation through a cash-in-advance constraint or sticky-price dispersion; we instead follow Aguiar et al. (2015) in assuming a direct utility cost that preserves analytical tractability.

Halac and Yared (2018, 2022) design fiscal rules for present-biased, privately informed governments; we share the private-information feature, but the fiscal distortion in our model stems from the inflationary externality of nominal debt. Farhi and Werning (2017), Kehoe and Pastorino (2017), and Orlandi and Piguillem (2025) design fiscal unions with cross-country transfers but do not consider the strategic overborrowing externality central to our analysis. All of these papers model a continuum of countries; our finite-country framework is

[6]Our earlier working paper, Berriel et al. (2023), grew into two distinct papers: Berriel et al. (2024) and the current paper. Berriel et al. (2024) examined a two-period real model with fiscal commitment and a standard static public goods externality (Oates, 1972), abstracting from the time-inconsistency forces central to the monetary framework studied here. The current paper recasts the analytically intractable two-period monetary model without commitment from Berriel et al. (2023) into a tractable continuous-time setup building on Aguiar et al. (2015).

what allows us to study how the optimal fiscal constitution varies with union size. Relatedly, Cooper and Kempf (2004) show that a monetary union can reduce welfare when fiscal policy remains decentralized, owing to a fiscal externality closely related to ours, but they do not study the optimal delegation of fiscal authority. Cooper, Kempf, and Peled (2010) discuss multiple equilibria when the monetary authority can bailout out regional governments by printing money. See Nicolini et al. (2002) for discussion of the problems that decentralized fiscal authority generated in Argentina.

Our model also connects to the literature on self-fulfilling debt crises. In Cole and Kehoe (2000), Conesa and Kehoe (2017), and Aguiar et al. (2022), the level of government debt partitions the state space into distinct equilibrium regions. Our model shares this structure: initial debt conditions place the union in either a zero-inflation or a high-inflation region, with austerity in the latter eventually driving a transition to the former. In those papers, sunspot realizations can trigger crises that abruptly move the economy across regions.

We also make a technical contribution. As in Aguiar et al. (2015), the two inflation regions create discontinuities in debt dynamics and flow utilities at their boundary, so the value function is non-differentiable there and the Hamilton–Jacobi–Bellman equation must be interpreted in the viscosity sense. Aguiar et al. (2015) establish existence and uniqueness via Bressan and Hong (2007), which requires that each fiscal authority can locally control every country's debt—a condition that fails in our finite-country setting. We instead apply Barles et al. (2014, 2023), which requires only that each authority can move average union debt by an arbitrarily small amount at the boundary—a condition our model satisfies. We thus extend the class of economic environments amenable to tractable analytical characterization.

1 Overview of Our Economies

We consider a continuous-time currency union of I countries with at most $\overline{I} < \infty$ countries incorporating fiscal federalism in the spirit of Oates (1972) by giving each local fiscal authority an *informational advantage* over the centralized one about its citizens' tastes. Each country i is populated by a measure 1 of identical consumers with preferences

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t}[\theta_{it} u(c_{it}) + (1-\theta_{it}) h(g_{it}) - l_{it} - \psi \pi_t] dt. \tag{1}$$

where c_{it}, g_{it}, l_{it}, and π_t denote private consumption, local public goods, labor, and union-wide inflation. The taste shock θ_{it}, i.i.d. across countries, governs the relative preference for public versus private goods.[7] Each local fiscal authority observes its own θ_{it} perfectly; the centralized authority observes each country's shock only at a Poisson rate. Following Aguiar et al. (2015), inflation disutility is linear in inflation. See that paper for a discussion of the interpretation and the usefulness of this assumption.

We proceed in two parts. First, we set $\theta_{it} = 1$ for all i and t, eliminating the informational advantage of decentralization. Second, we restore private information and characterize when centralization or decentralization is optimal. The novel technical challenge arises in the first part: with finitely many countries, each fiscal authority recognizes its ability to influence union-wide debt.

Our contributions relative to the literature are summarized in the following table.

Optimality of Centralization (C) vs Decentralization (D)

	Public Info	*Private Info* (∞)	*Private Info (Finite)*
Commit	$C \sim D$	$C \prec D$	$C \prec D$
No Commit	$C \succ D$	Doesn't depend on size	Depends on size

Consider first the top row. Under commitment, the monetary authority optimally sets inflation to zero, eliminating the endogenous fiscal externality. With public information, decentralization offers no informational advantage, so the two regimes are equivalent ($C \sim D$). With private information, decentralization has an informational advantage and is preferred ($C \prec D$). This *commitment–private information* case is, effectively, the focus of the fiscal federalism literature.

Consider next the bottom row. Without commitment, the fiscal externality is internalized by the centralized authority but not by decentralized ones, giving centralization an inherent advantage. Hence, absent fiscal federalism considerations, centralization is preferred ($C \succ D$). This *no-commitment–public information* case is the focus of the macroeconomic literature, with Aguiar et al. (2015) as a prime example.

In both literatures—the top-right and bottom-left cells of the table —there is no meaningful tension between regimes: one is always preferred by assumption. To capture the tradeoffs

[7] We have chosen these two goods to be private and public goods to capture the intuition of Oates (1972), but we could equivalently consider differential information about the desirability of any two types of goods.

emphasized by Tabellini (2003) and others as unions expand, we proceed as follows. First, we introduce an inherent tension between regimes by studying the no-commitment–private information case. Second, to trace how the tradeoff between centralization and decentralization varies with union size, we focus on the novel case of a finite number of countries.

Handling the finiteness assumption in the decentralized regime is by far the most technically challenging part of the analysis. For clarity, we first develop these results under public information and then show that incorporating private information is straightforward.

2 Economy without Fiscal Federalism

The union of I countries is small in the world economy. With $\theta_{it} = 1$, there is a single freely traded consumption good with international price normalized to 1. Each country $i = 1, ..., I$ has a linear technology that converts one unit of labor into one unit of consumption. The nominal price in the union in units of its currency relative to the world price at t is $P_t = P_0 \exp(\int_0^t \pi_s \, ds)$, where π_t is the rate of inflation at t. Preferences of country i reduce to

$$U_i = \int_0^\infty e^{-\rho t} \left[u(c_{it}) - l_{it} - \psi \pi_t \right] dt \tag{2}$$

with $u' > 0$, $u'' < 0$, $u''' > 0$, $\lim_{c \to 0} u'(c) = \infty$. Inflation is restricted to be non-negative and have an upper bound of $\bar{\pi}$ as in Aguiar et al. (2015) so $\pi_t \in [0, \overline{\pi}]$; both bounds sometimes bind. Consumption satisfies $c \in [\underline{c}, \bar{c}]$ with $\underline{c} \geq \varepsilon$ for some small $\varepsilon > 0$ and $\bar{c}$ arbitrarily large; neither bound binds in equilibrium. Labor satisfies $l_{it} \in [\underline{l}, \bar{l}]$.

Bonds Country i issues short-term nominal uncontingent bonds. Let B_{it} denote the outstanding stock and $b_{it} = B_{it}/P_t$ its real value. Real debt satisfies $b_{it} \in [\underline{b}, \bar{b}]$. Since P_0 is given and P_t is a continuous function of time, b_{it} serves as a state variable.[8]

Bonds are sold to international lenders with real opportunity cost ρ, who charge nominal rate $r_{it}(\boldsymbol{b}_t)$ where $\boldsymbol{b}_t = (b_{1t}, ..., b_{It})$. Since countries cannot default and share a common inflation rate, all countries pay the same rate r_t in equilibrium, which will clear the bond market.

[8] Taking P_0 as a primitive avoids complications from a large initial price-level jump, analogous to bounding the initial capital levy in a Ramsey taxation program. It also distinguishes our environment from the fiscal theory of the price level, in which the initial price level adjusts to equate real liabilities to the discounted stream of fiscal surpluses. We instead take P_0 as given and solve for the equilibrium path of surpluses and inflation.

The real value of country i's debt evolves as

$$\dot{b}_{it} = c_{it} - l_{it} + [r_t(b_t) - \pi_t]b_{it} \tag{3}$$

making clear that, for a given nominal interest rate $r_t(\boldsymbol{b}_t)$, higher inflation reduces the real value of debt and that cross-country transfers are excluded. The intertemporal budget constraint in any period s requires

$$b_{is} = \int_{t=s}^{\infty} e^{-(r_t(b_t)-\pi_t)(t-s)}(l_{it} - c_{it})\, dt \tag{4}$$

so that debt at s equals the present value of future primary surpluses $l_{it} - c_{it}$. For a given initial debt b_{i0}, the intertemporal budget constraint is the (4) at $s = 0$.

Monetary Authority The monetary authority chooses the inflation path π_t with preferences

$$U^m = \frac{1}{I}\sum_i \int_0^{\infty} e^{-\rho t}\left[u(c_{it}) - l_{it} - \psi\pi_t\right] dt. \tag{5}$$

Fiscal Regimes In the *centralized* regime, a centralized fiscal authority chooses (c_{it}, l_{it}) for all countries, with preferences

$$U^f = \frac{1}{I}\sum_i \int_0^{\infty} e^{-\rho t}\left[u(c_{it}) - l_{it} - \psi\pi_t\right] dt.$$

In the *decentralized* regime, each country i's fiscal authority chooses (c_{it}, l_{it}) with the preferences of its own citizens, as in (2).

Foreign Investors Perfectly competitive foreign investors purchase each country's nominal bonds at a real opportunity cost of ρ.

2.1 Markov Perfect Equilibria

Throughout we maintain two assumptions. First, the bounds on labor satisfy:

Assumption 1. *$\underline{l} < c^* + \rho\underline{b}$, for c^* defined as $u'(c^*) = 1$.*

Here c^*, the *optimal static consumption*, equates the marginal rate of substitution between

labor and consumption to its marginal rate of transformation. Later, this assumption will ensure that agents can consume c^* while working above their lower bound ($l_i > \underline{l}$) and keeping debt constant. Second, real bonds satisfy

Assumption 2. *$b_{it} \in [\underline{b}, \bar{b}] = B$, where $\bar{b} = (\bar{l} - c^*)/\rho > \psi$ and $\underline{b}/\bar{I} + (\bar{I} - 1)\bar{b}/\bar{I} < \psi$.*

Here $\bar{b}$ is the largest debt level such that, under the fiscal policy $(c^*, \bar{l})$, debt decreases for any $b_{it} < \bar{b}$, since under it, $\dot{b}_{it} = c^* - \bar{l} + \rho b_{it} < c^* - \bar{l} + \rho\bar{b} = 0$. Next, $\bar{b} > \psi$ implies that cost of inflation is sufficiently small that for high enough debt there is high inflation. The condition $\underline{b}/\bar{I} + (\bar{I} - 1)\bar{b}/\bar{I} < \psi$ ensures that for a given $\underline{b}$, even when the union is at its maximal size $I = \bar{I}$ each country is large enough such that if the other $\bar{I} - 1$ countries hold the maximum debt $\bar{b}$ and the last country holds $\underline{b}$, average union debt falls below ψ. This will guarantee that when inflation is high, all countries have an incentive to reduce their debt. Under Assumptions 1 and 2, we can analytically solve for the optimum in both regimes.

The aggregate state is the vector of outstanding real debts $\boldsymbol{b} = (b_1, \ldots, b_I)$. We focus on Markov perfect equilibria with the largest *no-inflation* set $A \subset [\underline{b}, \bar{b}]^I$ such that the monetary authority does not inflate for debt vectors in A.[9] In both regimes, the policy rules (c, l, π) and the equilibrium nominal interest rate r are functions of the debt vector $\boldsymbol{b}$.

Foreign Investors Taking as given the inflation rule $\pi(\boldsymbol{b})$ and nominal interest rate schedule $r(\boldsymbol{b})$, each foreign investor chooses debt supply $\tilde{b}$ to maximize

$$\max_{\tilde{b}} \left(r(\boldsymbol{b}) - \pi(\boldsymbol{b}) - \rho\right) \tilde{b}, \tag{6}$$

where ρ is their opportunity cost. Perfect competition in the bond market implies zero profits in equilibrium, so the real rate satisfies $r(\boldsymbol{b}) - \pi(\boldsymbol{b}) = \rho$.

Monetary Authority The sequence problem of the monetary authority is as follows. Taking as given the nominal interest-rate function $r(\boldsymbol{b})$ and the fiscal policies $(c_i(\boldsymbol{b}), l_i(\boldsymbol{b}))$ for $i = 1, \ldots, I$ under either regime, the monetary authority chooses the inflation path (π_t) from the set of admissible debt-inflation paths $\mathcal{T}_{\boldsymbol{b}}^M$, starting from $\boldsymbol{b_0} = \boldsymbol{b}$. Here, $\mathcal{T}_{\boldsymbol{b}}^M$ is the

[9] Aguiar et al. (2015) restrict attention to Markov equilibria in which the inflation rule is monotonically increasing in debt, and show that multiple equilibria exist. We focus on the equilibrium with the largest zero-inflation set, which features a monotonic inflation rule. Applied to their model, our selection criterion would single out the equilibrium with the largest zero-inflation set from those characterized in that work.

set of debt-inflation paths under which $\boldsymbol{b}_t$ is Lipschitz continuous in t and $\pi_t \in [0, \overline{\pi}]$, and an additional requirement, discussed in the Appendix, that convexifies the set of feasible debt dynamics. Formally:

$$J_M(\boldsymbol{b}) = \max_{(\boldsymbol{b}_t, \pi_t) \in \mathcal{T}_{\boldsymbol{b}}^M} \frac{1}{I} \int_0^\infty e^{-\rho t} \sum_i \left[u(c_i(\boldsymbol{b}_t)) - l_i(\boldsymbol{b}_t) - \psi \pi_t \right] dt \text{ s.t.} \tag{7}$$

$$\dot{b}_{it} = c_i(\boldsymbol{b}_t) - l_i(\boldsymbol{b}_t) + [r(\boldsymbol{b}_t) - \pi_t] b_{it} \quad \text{for } i = 1, \ldots, I. \tag{8}$$

In the centralized regime, the fiscal policies taken as given are the centralized fiscal policies; in the decentralized regime, the decentralized ones. We let J_M^C and J_M^D denote the monetary authority's value function in each regime, dropping the superscript where no confusion arises.

The Hamilton-Jacobi-Bellman (HJB) equation for the monetary authority is

$$\rho J_M(\boldsymbol{b}) = \max_{\pi \in [0, \overline{\pi}]} \sum_i \frac{u(c_i(\boldsymbol{b})) - l_i(\boldsymbol{b})}{I} - \psi \pi + \sum_i \frac{\partial J_M(\boldsymbol{b})}{\partial b_i} [c_i(\boldsymbol{b}) - l_i(\boldsymbol{b}) + (r(\boldsymbol{b}) - \pi) b_i] \tag{9}$$

at any point $\boldsymbol{b}$ where J_M is differentiable. Throughout, we consider the viscosity solution of Barles (2014). The optimal inflation rule is

$$\pi(\boldsymbol{b}) = \begin{cases} 0 & \text{if} \quad -\sum_i \frac{\partial J_M(\boldsymbol{b})}{\partial b_i} b_i < \psi, \\ [0, \overline{\pi}] & \text{if} \quad -\sum_i \frac{\partial J_M(\boldsymbol{b})}{\partial b_i} b_i = \psi, \\ \overline{\pi} & \text{if} \quad -\sum_i \frac{\partial J_M(\boldsymbol{b})}{\partial b_i} b_i > \psi. \end{cases} \tag{10}$$

In (7), the monetary authority trades off the marginal cost of inflation, ψ, against the marginal gain from eroding real debt, $-\sum_i \frac{\partial J_M(\boldsymbol{b})}{\partial b_i} b_i > 0$, which is positive because higher debt lowers utility.[10] Since the monetary authority takes the nominal rate $r(\boldsymbol{b})$ as given when contemplating different inflation rates π_t, it views itself as changing the real rate $r(\boldsymbol{b}) - \pi_t$ and thus real debt payments $[r(\boldsymbol{b}_t) - \pi_t] b_{it}$. In equilibrium, the real rate adjusts to equal lenders' opportunity cost, $r(\boldsymbol{b}) - \pi(\boldsymbol{b}) = \rho$.

The tradeoff the monetary authority faces arises from its inability to commit ex-ante to an inflation policy, and this lack of commitment is at the heart of our analysis. With

[10] Here, inflation is tempting because it erodes the real value of debt owed to foreigners. The same force would operate if countries borrowed nominally from domestic citizens and relied on distortionary taxes: surprise inflation would act as a lump-sum levy, substituting for costlier tax instruments. Such a model would deliver similar conclusions but at the cost of analytical tractability.

commitment, the monetary authority would choose $\pi_t = 0$ at time zero, understanding that its choices affect equilibrium interest rates, yielding $r_t = \rho$ and Ramsey outcomes under both fiscal regimes. To build intuition for the time inconsistency problem, consider a discrete-time approximation with periods $t = 0, \Delta, 2\Delta, \ldots$ that converges to our continuous-time model as $\Delta \to 0$. At each t, the monetary authority issues nominal bonds $B_{t+\Delta}$ at a continuously compounded nominal rate $i_{t+\Delta}$, with a real payoff at $t + \Delta$ of $B_{t+\Delta}/P_{t+\Delta}$. At $t + \Delta$ the monetary authority monetary authority chooses inflation $\pi_{t+\Delta} \in [0, \bar{\pi}]$ so that the price level at $t + \Delta$ is $P_{t+\Delta} = P_t(1 + \pi_{t+\Delta}\Delta)$. Lenders, expecting inflation $\pi^e_{t+\Delta}$, price bonds to earn an expected real rate $1 + i_{t+\Delta}\Delta = (1 + \rho\Delta)(1 + \pi^e_{t+\Delta}\Delta)$ which is the discrete Fisher equation. Since the nominal rate is locked in at t, the monetary authority at $t+\Delta$ can generate surprise inflation $\pi_{t+\Delta} > \pi^e_{t+\Delta}$, reducing the real value of outstanding debt by

$$\frac{B_{t+\Delta}}{P_{t+\Delta}} - \frac{B_{t+\Delta}}{P^e_{t+\Delta}} = \frac{B_{t+\Delta}}{(1 + \pi_{t+\Delta}\Delta)P_t} - \frac{B_{t+\Delta}}{(1 + \pi^e_{t+\Delta}\Delta)P_t} \geq 0$$

with strict inequality when $\pi_{t+\Delta} > \pi^e_{t+\Delta}$. This debt-erosion benefit is weighed against the disutility of inflation at $t + \Delta$ of $\psi\pi_{t+\Delta}$. Taking logs of the Fisher equation, dividing by Δ, and letting $\Delta \to 0$ yields the continuous-time Fisher equation $i_t = \rho + \pi^e_t$, where $\pi^e_t \equiv \lim_{\Delta\to 0}(\log P^e_{t+\Delta} - \log P_t)/\Delta$.[11] Note that assuming that at time t the monetary authority commits to the price level at $P_{t+\Delta} = P_t(1 + \pi_{t+\Delta}\Delta)$ yields the commitment outcomes, since in this timing there can be no surprise inflation on the bond that pays off at $t + \Delta$.

2.2 Decentralized Regime

The fiscal authority of country i takes as given the inflation rule $\pi(\boldsymbol{b})$, the nominal interest rate $r(\boldsymbol{b})$, the fiscal rules $(c_j(\boldsymbol{b_t}), l_j(\boldsymbol{b_t}))$ for $j \neq i$, and initial debt $\boldsymbol{b_0} = \boldsymbol{b}$, and chooses fiscal rules (c_{it}, l_{it}) from the set of admissible debt-fiscal paths $\mathcal{T}^D_{\boldsymbol{bi}}$, in which $\boldsymbol{b}_t$ is Lipschitz continuous, $c_{it}, l_{it} \in [\underline{c}, \bar{c}] \times [\underline{l}, \bar{l}]$, and an additional convexification requirement holds. Formally:

$$V_i(\boldsymbol{b}) = \max_{(\boldsymbol{b}_t, c_{it}, l_{it}) \in \mathcal{T}^D_{\boldsymbol{bi}}} \int_0^\infty e^{-\rho t} \left[u(c_{it}) - l_{it} - \psi\pi(\boldsymbol{b_t})\right] dt \text{ s.t.}$$

[11] Note that we could have defined new variables by $e^{\tilde{i}_{t+\Delta}\Delta} \equiv 1 + i_{t+\Delta}$, $e^{\tilde{\rho}\Delta} \equiv 1 + \rho\Delta$, and $e^{\tilde{\pi}^e_{t+\Delta}\Delta} \equiv 1 + \pi^e_{t+\Delta}$ which substituting and taking logs of the discrete Fisher equations gives $\tilde{i}_{t+\Delta} = \tilde{\rho} + \tilde{\pi}^e_{t+\Delta}$. We chose not to do it because it might lead the reader to think that time is evolving in continuous subperiods between t and $t + \Delta$ but agents only make decisions at discrete points t and $t + \Delta$, and so on.

$$\dot{b}_{it} = c_{it} - l_{it} + \rho b_{it} \text{ and } \dot{b}_{jt} = c_j(\boldsymbol{b_t}) - l_j(\boldsymbol{b_t}) + \rho b_{jt} \tag{11}$$

Markov perfection requires the deviating country to evaluate the entire sequence of future deviations induced by its initial deviation. In general, this strategic interaction would make the analysis intractable. However, under Assumptions 1 and 2, linearity of utility in labor will implies that all decentralized fiscal authorities choose constant consumption levels in the high-inflation region that do not depend on other countries' debt. This feature eliminates the need for each authority to forecast how others adjust their behavior in response to its own strategy, rendering the problem tractable. The HJB for country i's fiscal authority is

$$\rho V_i(\boldsymbol{b}) = \max_{c_i \in [\underline{c}, \bar{c}], l_i \in [\underline{l}, \bar{l}]} u(c_i) - l_i - \psi \pi(\boldsymbol{b}) + \frac{\partial V_i(\boldsymbol{b})}{\partial b_i}(c_i - l_i + \rho b_i) + \sum_{j \neq i} \frac{\partial V_i(\boldsymbol{b})}{\partial b_j}(c_j(\boldsymbol{b}) - l_j(\boldsymbol{b}) + \rho b_j), \tag{12}$$

at any point $\boldsymbol{b}$ where V_i is differentiable. The first-order condition for c_i yields

$$u'(c_i(\boldsymbol{b})) = -\frac{\partial V_i(\boldsymbol{b})}{\partial b_i}, \tag{13}$$

so the decentralized authority trades off the cost of reducing c_i against the benefit to *country i alone* of reducing its own debt $b_i$where we assumed the bounds on c_i are slack. The first-order condition for l_i implies

$$l_i(\boldsymbol{b}) = \begin{cases} \underline{l} & \text{if} \quad -\frac{\partial V_i(\boldsymbol{b})}{\partial b_i} < 1, \\ \left[\underline{l}, \bar{l}\right] & \text{if} \quad -\frac{\partial V_i(\boldsymbol{b})}{\partial b_i} = 1, \\ \bar{l} & \text{if} \quad -\frac{\partial V_i(\boldsymbol{b})}{\partial b_i} > 1. \end{cases} \tag{14}$$

A *Markov perfect equilibrium for the decentralized regime* is an interest rate schedule $r(\boldsymbol{b})$, fiscal policy functions $(c_i(\boldsymbol{b}), l_i(\boldsymbol{b}))$, an inflation policy function $\pi(\boldsymbol{b})$, and value functions $(V_i(\boldsymbol{b}), J_M^D(\boldsymbol{b}))$ such that: *i)* zero profits in the bond market, $r(\boldsymbol{b}) = \rho + \pi(\boldsymbol{b})$; *ii)* given $r(\boldsymbol{b})$ and $(c_i(\boldsymbol{b}), l_i(\boldsymbol{b}))$ for $i = 1, \ldots, I$, $(\pi(\boldsymbol{b}), J_M^D(\boldsymbol{b}))$ satisfies the monetary authority HJB (9); and *iii)* given $r(\boldsymbol{b})$, $\pi(\boldsymbol{b})$, and $(c_j(\boldsymbol{b}), l_j(\boldsymbol{b}))$ for $j \neq i$, $(c_i(\boldsymbol{b}), l_i(\boldsymbol{b}), V_i(\boldsymbol{b}))$ satisfies the decentralized fiscal authority HJB (12) for country i.

An immediate implication is that the monetary authority's value function equals the average of the fiscal authorities' value functions: $J^D(\boldsymbol{b}) = \sum_i V_i(\boldsymbol{b})/I$. We drop the M superscript

and write $J^D(\boldsymbol{b})$ for both. Under our assumptions, all value functions are continuous.

Given fiscal rules (c_i, l_i) and initial debt $\boldsymbol{b}_0$, the associated *debt path* $(\boldsymbol{b}_t)_{t=0}^{\infty}$ is

$$b_{it}(\boldsymbol{b}_0) = e^{\rho t} b_{i0} + e^{\rho t} \int_0^t e^{-\rho s} [c_i(\boldsymbol{b}_s(\boldsymbol{b}_0)) - l_i(\boldsymbol{b}_s(\boldsymbol{b}_0))] ds, \qquad \text{for all } i. \tag{15}$$

obtained by integrating (11) from 0 to t. The associated fiscal, inflation, and interest rate paths are $c_{it} = c_i(\boldsymbol{b}_t(\boldsymbol{b_0}))$, $l_{it} = l_i(\boldsymbol{b}_t(\boldsymbol{b_0}))$, $\pi_t = \pi(\boldsymbol{b}_t(\boldsymbol{b_0}))$, and $r_t = \rho + \pi_t$.

2.3 Centralized Regime

In the sequence problem, the centralized fiscal authority takes as given the inflation rule $\pi(\boldsymbol{b})$, the nominal interest rate $r(\boldsymbol{b}) = \rho + \pi(\boldsymbol{b})$, and initial debt $\boldsymbol{b_0} = \boldsymbol{b}$, and chooses fiscal rules (c_{it}, l_{it}) for all countries from the set of admissible debt-fiscal paths $\mathcal{T}_{\boldsymbol{b}}^C$, in which $\boldsymbol{b}_t$ is Lipschitz continuous, $c_{it}, l_{it} \in [\underline{c}, \overline{c}] \times [\underline{l}, \overline{l}]$ for each i, and a convexification requirement holds. Formally:

$$J^C(\boldsymbol{b}) = \max_{(\boldsymbol{b}_t, \{c_{it}, l_{it}\}) \in \mathcal{T}_{\boldsymbol{b}}^C} \frac{1}{I} \sum_i \int_0^{\infty} e^{-\rho t} \left[u(c_{it}) - l_{it} - \psi \pi(\boldsymbol{b_t}) \right] dt \text{ s.t} \tag{16}$$

$$\dot{b}_{it} = c_{it} - l_{it} + \rho b_{it} \text{ for } i = 1, \ldots, I, \tag{17a}$$

where we used the equilibrium condition $r(\boldsymbol{b}) = \rho + \pi(\boldsymbol{b})$ in (3). The constraints on consumption never bind in equilibrium and are henceforth ignored. The HJB for the centralized fiscal authority is

$$\rho J^C(\boldsymbol{b}) = \max_{(c_i, l_i) \text{ s.t } c_i \in [\underline{c}, \overline{c}], l_i \in [\underline{l}, \overline{l}]} \sum_i \frac{u(c_i) - l_i}{I} - \psi \pi(\boldsymbol{b}) + \sum_i \frac{\partial J^C(\boldsymbol{b})}{\partial b_i} (c_i - l_i + \rho b_i), \tag{18}$$

at any point $\boldsymbol{b}$ where J^C is differentiable. The first-order condition for c_i is

$$\frac{u'(c_i(\boldsymbol{b}))}{I} = -\frac{\partial J^C(\boldsymbol{b})}{\partial b_i}. \tag{19}$$

The centralized authority trades off the cost of reducing country i's consumption to the union, $u'(c_i(\boldsymbol{b}))/I$, against the benefit to *all* I *countries* of reducing the rate of country i's

debt accumulation, $-\partial J^C(\boldsymbol{b})/\partial b_i > 0$. The first-order condition for l_i implies

$$l_i(\boldsymbol{b}) = \begin{cases} \underline{l} & \text{if} \quad -\frac{\partial J^C(\boldsymbol{b})}{\partial b_i} < \frac{1}{I}, \\ \left[\underline{l}, \overline{l}\right] & \text{if} \quad -\frac{\partial J^C(\boldsymbol{b})}{\partial b_i} = \frac{1}{I}, \\ \overline{l} & \text{if} \quad -\frac{\partial J^C(\boldsymbol{b})}{\partial b_i} > \frac{1}{I}. \end{cases} \tag{20}$$

where the authority trades off the utility cost of increasing labor in country i against the benefit to all I countries of reducing country i's debt.

A *Markov perfect equilibrium for the centralized regime* is an interest rate schedule $r(\boldsymbol{b})$, fiscal policy functions $(c_i(\boldsymbol{b}), l_i(\boldsymbol{b}))$, an inflation policy function $\pi(\boldsymbol{b})$, and value functions $(J^C(\boldsymbol{b}), J^C_M(\boldsymbol{b}))$ such that: *i)* zero profits in the bond market, $r(\boldsymbol{b}) = \rho + \pi(\boldsymbol{b})$; *ii)* given $r(\boldsymbol{b})$ and $(c_i(\boldsymbol{b}), l_i(\boldsymbol{b}))$, $(\pi(\boldsymbol{b}), J^C_M(\boldsymbol{b}))$ satisfies the monetary authority HJB (9); and *iii)* given $r(\boldsymbol{b})$ and $\pi(\boldsymbol{b})$, $(c_i(\boldsymbol{b}), l_i(\boldsymbol{b}), J^C(\boldsymbol{b}))$ satisfies the centralized fiscal authority HJB (18). Since both the fiscal and monetary authorities share the same objective, their maximized values coincide: $J^C(\boldsymbol{b}) = J^C_M(\boldsymbol{b})$ for all $\boldsymbol{b}$. We drop the M subscript henceforth.

2.4 Preliminary Results

We now characterize the equilibrium inflation rule, which is identical across regimes, and present preliminary results on optimal fiscal rules. We begin by constructing the largest set of debts under which inflation is zero—the *zero-inflation* set A—and use it to build the rest of the equilibrium. For any $\boldsymbol{b} \in A^C$, the complement of the zero-inflation set, each fiscal authority pursues an *austerity plan*—reducing consumption and raising labor—so that each country's debt decreases over time until it reaches the frontier of A.

The debt state space B^I thus divides into two regions: the zero-inflation set A, where fiscal authorities keep debt constant; and its complement A^C, where inflation is $\bar{\pi}$ and fiscal authorities run down debt until inflation stops, after which debt remains constant.

2.4.1 The Zero Inflation Set and the Optimal Inflation Rule

We first derive fiscal rules for an arbitrary set of debts A' under which inflation is zero, then use them to characterize the equilibrium zero-inflation set A. Linearity of utility in inflation then implies that for any debt outside A, inflation is at its maximum $\overline{\pi}$. The partition into A, where $\pi = 0$, and A^C, where $\pi = \overline{\pi}$, fully characterizes the equilibrium inflation rule $\pi(\boldsymbol{b})$.

Lemma 1. *(Fiscal best response in the zero inflation set) Let $A \subset B^I$ and suppose that $\pi(\boldsymbol{b}) = 0$ for all $\boldsymbol{b} \in A$. Then, for all $\boldsymbol{b} \in A$, the following fiscal rules are optimal under both regimes*

$$\dot{b}_{it}(\boldsymbol{b}) = 0, \quad l_i(\boldsymbol{b}) = c^* + \rho b_i, \quad c_i(\boldsymbol{b}) = c^* \tag{21}$$

where c^* is defined as $u'(c^*) = 1$. Furthermore, for any $\boldsymbol{b} \in A$, the value functions are

$$\rho V_i(\boldsymbol{b}) = u(c^*) - c^* - \rho b_i \text{ and } \rho J^C(\boldsymbol{b}) = \rho J^D(\boldsymbol{b}) = \frac{1}{I}\sum_i [u(c^*) - c^* - \rho b_i]. \tag{22}$$

Proof. In the zero-inflation set A, countries are connected only indirectly through the effect of their debt on inflation. Since inflation is zero throughout A, consumption and labor in country i are independent of other countries' debt, so the same argument applies to both regimes.

Since the discount rate ρ equals the real interest rate and leaving the zero-inflation region is costly, it is never optimal to exit A, and consumption smoothing implies c_{it} is constant. For labor and debt: since labor enters utility linearly and ρ equals the real rate, the fiscal authority is indifferent among labor paths with the same present value, provided debt stays in A. It is therefore without loss to set l_{it} constant and $\dot{b}_{it} = 0$, reducing the problem to

$$\rho V_i(\boldsymbol{b}) = \max_{\underline{l} \le l_i \le \bar{l}} u(l_i - \rho b_i) - l_i. \tag{23}$$

The unconstrained solution is $u'(c^*) = 1$ and $l = c^* + \rho b$. Assumptions 1 and 2 imply $\underline{l} < c^* + \rho b \le \bar{l}$, so the bounds are slack. □

We now characterize the largest zero-inflation set.

Proposition 1. *(Equilibrium inflation rule) The largest set A such that inflation is zero is $A = \left\{\boldsymbol{b} \in B^I : \sum_i b_i/I \le \psi\right\}$. The corresponding inflation rule is*

$$\pi(\boldsymbol{b}) = \begin{cases} 0 & \text{if } \boldsymbol{b} \in A, \\ \bar{\pi} & \text{if } \boldsymbol{b} \in A^C \end{cases}. \tag{24}$$

Proof. Substituting (22) into (10) yields (24), where we include the indifference point

$\sum_i b_i/I = \psi$ in A. The boundary of the zero-inflation set is the hyperplane

$$\mathcal{H} = \left\{ \boldsymbol{b} : \frac{1}{I} \sum_i b_i = \psi \right\}. \tag{25}$$

at which average debt equals the marginal cost of inflation ψ.

To show that no larger zero-inflation set exists, suppose by contradiction that $\boldsymbol{b} \in A^C$ but inflation is zero in equilibrium. Then Lemma 1 implies $-\sum_i \frac{\partial J^C(\boldsymbol{b})}{\partial b_i} b_i = -\sum_i \frac{\partial J^D(\boldsymbol{b})}{\partial b_i} b_i = \sum_i b_i/I > \psi$, where the inequality follows from $\boldsymbol{b} \in A^C$. But then (10) implies $\pi(\boldsymbol{b}) = \overline{\pi}$, a contradiction. □

Figure 1: Set A

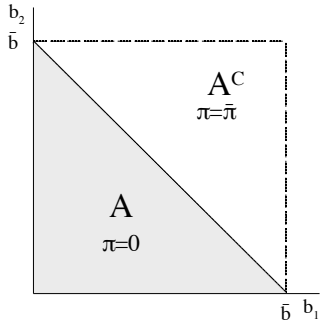


Figure 1 considers the case $I = 2$, so that $\mathcal{H}$ is a line, and depicts the no-inflation set A and the high-inflation set A^C.

2.4.2 Some Properties of Equilibrium Fiscal Rules

We establish preliminary results about fiscal rules in the high-inflation region A^C. We show: (i) under constant fiscal rules, we can calculate the boundary point $\boldsymbol{b}^* \in \mathcal{H}$ reached and the time $\tau(\boldsymbol{b})$ to reach it; (ii) the centralized fiscal authority pursues an austerity program that runs down debt until the no-inflation region is reached; (iii) it optimally does so with constant fiscal rules at maximal labor; (iv) in the decentralized regime, if all authorities but one choose constant fiscal rules, so does the last one, also at maximal labor; and (v) fiscal authorities adopt austerity plans with consumption below its statically optimal level, $c_i < c^*$.

Two preliminary remarks. First, given initial debt b_{i0} and fiscal plan (c_{it}, l_{it}), the date-0

budget constraint and the value of debt at s are

$$b_{i0} = \int_0^\infty e^{-\rho t}(l_{it} - c_{it})dt \text{ and } b_{is} = \int_s^\infty e^{-\rho(t-s)}(l_{it} - c_{it})dt \tag{26}$$

so that

$$b_{i0} = \int_0^s e^{-\rho t}(l_{it} - c_{it})dt + e^{-\rho s}b_{is} \text{ so } b_{is} = e^{\rho s}b_0 - \int_0^s e^{-\rho(t-s)}(l_{it} - c_{it})dt \tag{27}$$

Second, any two plans (c_t, l_{1t}) and (c_t, l_{2t}) with the same consumption that satisfy the date-0 budget constraint and remain in the same region (high- or zero-inflation) yield the same utility. The budget constraint forces equal present values of labor, and since utility is linear in labor with the consumer discount rate equal to the real rate, the labor component of utility is the same; since both plans stay in the same region, the inflation disutility is also the same.

Define $\tau(\boldsymbol{b}_0) = \inf\{t \geq 0 : \boldsymbol{b}_t \in A\}$ and let $c_m = \sum_i c_i/I$, $l_m = \sum_i l_i/I$, $b_{mt} = \sum_i b_{it}/I$ denote union averages. The following lemma takes constant policies as given and therefore applies to both regimes.

Lemma 2. *If $\boldsymbol{b} \in A^C$ and the fiscal rules (c_i, l_i) are constant with $\dot{b}_{it} < 0$ until $\tau(\boldsymbol{b})$ when debt reaches the zero-inflation set, the boundary point $\boldsymbol{b}^* = (b_i^*(\boldsymbol{b})) \in \mathcal{H}$ and the time to reach it are*

$$b_{i0} = \frac{(1 - e^{-\rho\tau})}{\rho}(l_i - c_i) + e^{-\rho\tau}b_i^*, \quad \textit{for all } i, \textit{ and} \tag{28}$$

$$e^{\rho\tau(b_m)} = \frac{l_m - c_m - \rho\psi}{l_m - c_m - \rho b_{m0}}. \tag{29}$$

Proof. Imposing constant fiscal rules for $t < \tau(\boldsymbol{b})$ in (27) gives (28). Averaging both sides of (28), imposing $\sum_i b_i^*/I = \psi$ at τ, and solving for $e^{\rho\tau}$ gives (29). □

Several results below draw on two lemmas about fiscal policy. Consider two fiscal rules for country i: both start from debt b_0, end at b_s in period s, share the same labor plan $(l_t)_{t=0}^s$, and stay in the high-inflation region throughout. Rule 1 has time-varying consumption $(c_{1t})_{t=0}^s$; rule 2 has constant consumption c_2.

Lemma 3. *The rule with time-varying consumption has strictly lower utility than the rule with constant consumption.*

Proof. Since both rules share the same labor plan and terminal debt, (26) implies equal present values of consumption: $\int_0^s e^{-\rho t} c_{1t}\, dt = \int_0^s e^{-\rho t} c_2\, dt$. The welfare difference reduces to showing

$$u(c_2) > \int_0^s \omega_t u(c_{1t}) dt \text{ with } \omega_s = e^{-\rho s} / \int_0^s e^{-\rho t} dt. \tag{30}$$

Since ω_t are positive weights integrating to 1 and c_{1t} is non-constant, (30) follows from strict concavity of u. □

Lemma 4. *Under centralization, for any $\boldsymbol{b} \in A^C$, it is optimal to reach the no-inflation region at some finite time τ with $l_t = \bar{l}$.*

Proof. By Lemma 3, any optimal consumption rule has $c_{it} = c_i$ for all i until the no-inflation set is reached. By our second remark, ignoring the benefits of eliminating inflation, the central fiscal authority is indifferent among all plans $(c_i, l_{it})_{t=0}^{\infty}$ satisfying each country's date-0 budget constraint. We now argue that, counting the benefits of eliminating inflation, reaching A in finite time is strictly preferred.

Consider first the case where some country j has $b_j < \bar{b}$. Suppose by contradiction that staying in A^C forever is optimal. Among such plans, $c_i = c^*$ for all i gives the highest utility (by the same logic as Lemma 1: equating the marginal rate of substitution $1/u'(c)$ to the marginal rate of transformation 1). By Assumption 2, $\dot{b}_{jt} = c^* - \bar{l} + \rho b_{jt} < 0$ when $b_j < \bar{b}$. Since $I \leq \bar{I}$ and $b_{it} \leq \bar{b}$ for $i \neq j$, the second part of Assumption 2 ensures that mean debt reaches ψ in finite time τ—at most the time for country j to reach $\underline{b}$, since the worst case is when all other countries have debt $\bar{b}$. This generates a welfare gain of $\int_\tau^\infty e^{-\rho t} \psi \bar{\pi}\, dt > 0$, a contradiction.

When $b_i = \bar{b}$ for all i, we modify the plan by having one country j set $c_j = c^* - \varepsilon$ with $l_{jt} = \bar{l}$. We show in the Appendix that for small enough $\varepsilon > 0$, the increase in country j's value is $\psi \overline{\pi} \lim_{\varepsilon \to 0} \frac{\partial (e^{-\rho\tau})}{\partial \varepsilon} > 0$, where $e^{-\rho\tau}$ is defined in (29). Intuitively, reducing consumption shortens the time to reach the zero-inflation region, and for small ε this benefit exceeds the cost. All countries $i \neq j$ also gain, so this plan is preferred by the central authority.

It remains to show $l_t = \bar{l}$ is optimal. Among feasible plans reaching A at τ and using zero-inflation policies thereafter, by the second remark above, the plan with $l_t = \bar{l}$ is tied with any alternative having the same consumption path but reaches A strictly faster than any other budget-feasible plan. Counting the benefits from eliminating inflation, it is therefore strictly preferred. □

Figure 2: Simulated paths under the centralized allocation.

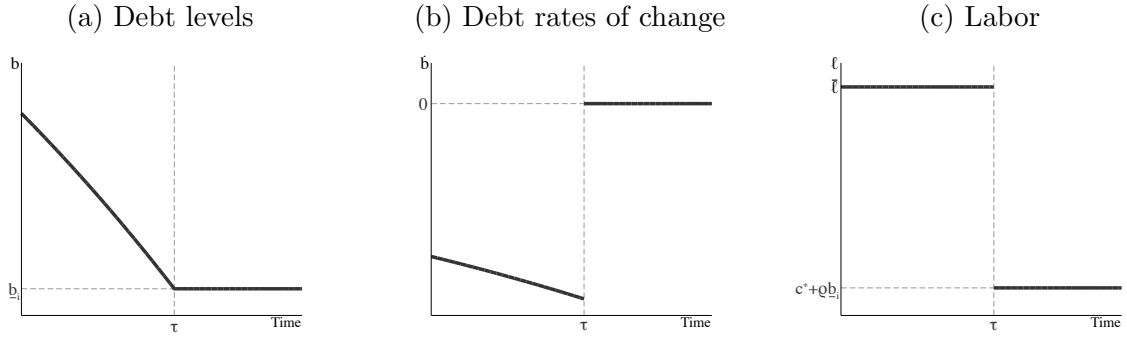


Figure 2 considers an initial debt vector $\boldsymbol{b} \in A^C$. It plots the debt, debt change, and labor paths for a country i following a constant $(c_i, \bar{l})$ rule until debt reaches $\boldsymbol{b}^* = (b_i^*) \in A$ at time $\tau(\boldsymbol{b})$. For $t \geq \tau(\boldsymbol{b})$, the country consumes c^* and works $l_i = c^* + \rho b_i^*$. Under constant fiscal rules in the high-inflation region, as debt falls, the rate of decline increases.

We now turn to the decentralized regime and focus on a class of equilibria of similar form to those in the centralized regime. The logic of Lemmas 3 and 4 extends directly: if all authorities but i choose constant fiscal rules $(c_j, l_j)_{j \neq i}$ in A^C, then country i's best response is also a constant rule with maximal labor.

Lemma 5. *In the decentralized regime, suppose all authorities but i choose constant fiscal rules $(c_j, l_j)_{j \neq i}$ in A^C. If $\boldsymbol{b}_0 \in A^C$, the best response for country i is to adopt constant fiscal rules $\left(c_i, \bar{l}\right)$ until debt reaches the zero-inflation set.*

Country i is allowed to adopt any rule in the admissible class $\mathcal{T}_{bi}^D$; the lemma shows that if all others choose constant rules, so does country i. Finally, austerity in the high-inflation region always involves consumption below the static optimum:

Lemma 6. *For any $\boldsymbol{b} \in A^C$, in both regimes the optimal fiscal rule satisfies $c_i(\boldsymbol{b}) \leq c^*$.*

Proof. If $c > c^*$, then $u'(c) < 1$, and reducing both consumption and labor by ε leaves debt unchanged but increases flow utility by $(1 - u'(c))\varepsilon > 0$, a contradiction. □

2.5 Characterization of the Regimes

Having characterized fiscal rules in the zero-inflation region, we now turn to the high-inflation region by revisiting the sequence formulation of both problems with our preliminary results applied. Recall: in the zero-inflation region, both regimes set $c_i = c^*$ and $l_i = c^* + \rho b_i$ (Lemma 1); in the high-inflation region, both set c_i constant and $l_i = \bar{l}$ (Lemmas 3 and 5). Under such rules, the time τ in the high-inflation region is given by (29) and the boundary point $\boldsymbol{b}^*$ by (28).

Decentralized. Since labor supply is known in each region, the problem of country i's fiscal authority in the high-inflation region reduces to choosing the constant c_i to solve

$$V_i(\boldsymbol{b}) = \max_{c_i} \left\{ \int_0^{\tau} e^{-\rho\tau}[u(c_j) - \bar{l} - \psi\bar{\pi}]dt + \int_{\tau}^{\infty} e^{-\rho\tau}[u(c^*) - (c^* + \rho b_i^*)] \right\} \tag{31}$$

subject to (29) and (28), which using $\int_0^{\tau} e^{-\rho t}dt = (1 - e^{-\rho\tau})/\rho$ and $\int_{\tau}^{\infty} e^{-\rho t}dt = e^{-\rho\tau}/\rho$ gives

$$\rho V_i(\boldsymbol{b}) = \max_{c_i} \ (1 - e^{-\rho\tau})[u(c_i) - \bar{l} - \psi\bar{\pi}] + e^{-\rho\tau}[u(c^*) - (c^* + \rho b_i^*)], \tag{32}$$

subject to the same constraints. The first-order condition for c_i, using $\frac{\partial e^{-\rho\tau}}{\partial c_i} = -\rho e^{-\rho\tau}\frac{\partial \tau}{\partial c_i}$, is

$$(1 - e^{-\rho\tau})u'(c_i) = \rho e^{-\rho\tau}\frac{\partial \tau}{\partial c_i}\Delta U(c_i, b_i^*) + \rho e^{-\rho\tau}\frac{\partial b_i^*}{\partial c_i} = 0 \tag{33}$$

The left side of (33) is the cost of lowering c_i during the τ periods in the high-inflation regime. The right side captures two benefits. First, greater austerity reduces τ through $\partial\tau/\partial c_i$, shifting time from the high-inflation to the zero-inflation region with a flow-utility gain per unit time of

$$\Delta U(c_i, b_i^*) = [u(c^*) - (c^* + \rho b_i^*)] - \left[u(c_i) - \bar{l} - \psi\bar{\pi}\right] > 0. \tag{34}$$

Second, greater austerity lowers country i's debt at the boundary of the zero-inflation set, reducing the interest payments it must service for $t \geq \tau$, captured by the positive term $\rho\,\partial b_i^*/\partial c_i$. To see how these tradeoffs change with I, holding mean debt b_m fixed, differenti-

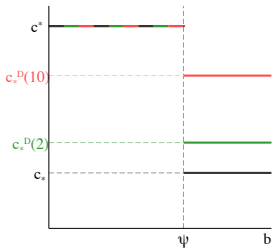
ate (29) and (28):

$$\frac{\partial \tau}{\partial c_i} = e^{-\rho\tau}\frac{1}{I}\frac{(b_m - \psi)}{\left(c - \bar{l} + \rho b_m\right)^2} \quad \text{and} \quad \rho\frac{\partial b_i^*}{\partial c_i} = \rho e^{\rho\tau}\frac{\partial \tau}{\partial c_i}(c_i - l_i + \rho b_i) + e^{\rho\tau} - 1 \tag{35}$$

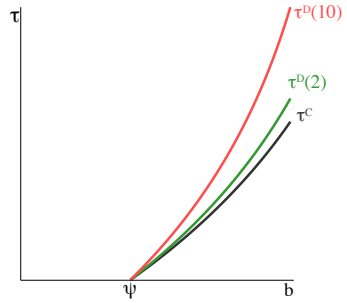
Both benefits in (33)—reducing the time in high inflation and reducing the debt country i must service once inflation stops—decrease with I. Hence, under decentralization the optimal degree of fiscal austerity decreases with the number of countries.

Figure 3: Optimal policies as a function of debt b

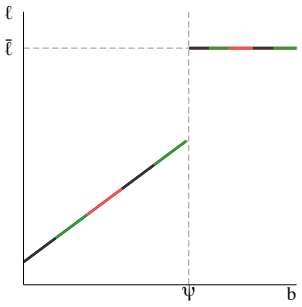
(a) Consumption



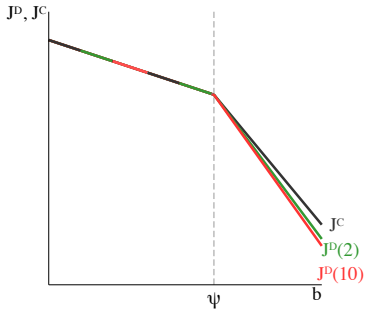
(b) Time in high inflation



(c) Labor



(d) Value function w/o fiscal federalism



We characterize the resulting policies formally in the next proposition. Figure 3 illustrates them for $I = 2$ and $I = 10$, assuming equal initial debt. In Figure 3a in the zero-inflation region, consumption is c^* regardless of I; in the high-inflation region, decentralized consumption $c_*^D(I)$ increases with I, so larger unions adopt less austerity. In Figure 3b, accordingly, larger unions take longer to reach the zero-inflation set. In Figure 3c, labor is at its maximum $\bar{l}$ under austerity, and once inflation stops, it is lower the lower is terminal debt. In Figure 3d, the value function in the high-inflation region is decreasing in debt and decreases more sharply for larger unions.

Despite having different initial and terminal debt, all fiscal authorities adopt the same austerity consumption level: there is a unique solution to (34) of the form $c = \underline{c}^D(b_m, I)$ for all i. From (17a), countries with lower initial debt end up with lower final debt and work less for $t > \tau$ than countries with higher initial debt.

Centralized. The corresponding problem for the central fiscal authority is choosing (c_i) to solve

$$J^C(\boldsymbol{b}) = \max_{(c_i)} \frac{1}{I} \sum_i \left\{ \int_0^\tau e^{-\rho\tau}[u(c_i) - \bar{l} - \psi\bar{\pi}]dt + \int_\tau^\infty e^{-\rho\tau}[u(c^*) - (c^* + \rho b_i^*)] \right\} \tag{36}$$

subject to (29) and (28), which similarly simplifies to

$$J^C(\boldsymbol{b}) = \max_{(c_i)} (1 - e^{-\rho\tau}) \frac{1}{I} \sum_i [u(c_i) - \bar{l} - \psi\bar{\pi}] + e^{-\rho\tau} \frac{1}{I} \sum_i [u(c^*) - c^* - \rho b_i^*].$$

Using ΔU_j from (34), the first-order condition is

$$(1 - e^{-\rho\tau})u'(c_i) = \rho e^{-\rho\tau} \frac{\partial \tau}{\partial c_i} \sum_j \Delta U(c_j, b_j^*) + \rho e^{-\rho\tau} \sum_j \frac{\partial b_j^*}{\partial c_i} = 0 \tag{37}$$

The cost of increasing austerity for country i—the left side of (37)—is the same as in the decentralized case. The key difference lies in the benefits: the central authority considers the gain to all I countries, whereas the country i authority considers only its own. Specifically, the central authority values the faster transition to zero inflation for all countries. It also considers the induced change in the sum of all countries' debts at the boundary, but this effect, $\sum_j \frac{\partial b_j^*}{\partial c_i}$, is zero: any point on the boundary $\mathcal{H}$ of the zero-inflation set satisfies $\frac{1}{I}\sum_j b_j^* = \psi$, so mean debt is constant. Hence, if i's debt falls, the debt of countries $j \neq i$ rises to compensate.

Since utility is concave in c_i and τ depends only on average consumption c_m, all consumptions are equal, $c_i = c_m = c$ for all i. Combined with $\sum_j \frac{\partial b_j^*}{\partial c_i} = 0$, the FOC (37) reduces to

$$(1 - e^{-\rho\tau})u'(c) = \rho e^{-\rho\tau} \frac{\partial \tau}{\partial c_i} I \Delta U(c, \psi). \tag{38}$$

In stark contrast to the decentralized case, these tradeoffs do not vary with I. The effect of

country i's austerity on the duration of high inflation, $\partial\tau/\partial c_i$ from (35), still falls with I. However, because the central authority cares about all I members, the gain to the union as a whole—$\frac{\partial\tau}{\partial c_i} I\Delta U(c,\psi)$—is unchanged with I, and hence so is the optimal degree of fiscal austerity.

We now solve these first-order conditions. Substituting $e^{-\rho\tau}/(1-e^{-\rho\tau})$ from (29) and the expressions for $\partial\tau/\partial c_i$ and $\partial b_i^*/\partial c_i$ from (35) into the decentralized and centralized first order conditions (33) and (38) yields the fiscal rules (39) and (41) in the following proposition. Substituting these rules into the HJB equations gives the value functions (40) and (42), which we rewrite in terms of the relevant minimal sufficient statistics derivable from the I dimensional vector $\boldsymbol{b}$, such as (b_i, b_m, I) and b_m, where $b_m = \frac{1}{I}\sum_{i=1}^{I} b_i$ is the mean debt.

Proposition 2. *(Fiscal Rules in the Two Regimes) If $\boldsymbol{b} \in A^C$, in a decentralized regime, labor supply is $\bar{l}$, and the unique constant consumption $c = c_*^D(I) < c^*$ solves*

$$\Delta U(c,\psi) = -u'(c)(c - \bar{l} + \rho\psi) + (u'(c^*) - u'(c))(I-1)(c - \bar{l} + \rho\psi) \tag{39}$$

and increases in I, and $\rho V_i(\boldsymbol{b}, I)$ decreases with I, where, using as shorthand $c_*^D = c_*^D(I)$,

$$\rho V_i(b_i, b_m, I) = u(c_*^D) - \bar{l} - \psi\overline{\pi} - u'(c^*)(c_*^D - \bar{l} + \rho b_i) - (u'(c^*) - u'(c_*^D))I(c_*^D - \bar{l} + \rho b_m) \tag{40}$$

If $\boldsymbol{b} \in A^C$, in a centralized regime labor supply is $\bar{l}$, and consumption $c = c_* < c^*$ solves

$$\Delta U(c,\psi) = -u'(c)(c - \bar{l} + \rho\psi). \tag{41}$$

The value function is

$$\rho J^C(b_m) = u(c_*) - \bar{l} - \psi\overline{\pi} - u'(c_*)\left(c_* - \bar{l} + \rho b_m\right). \tag{42}$$

The average welfare under decentralization with I countries and debt vector $\boldsymbol{b}$ is

$$J^D(b_m, I) = u(c_*^D) - \bar{l} - \psi\overline{\pi} - u'(c^*)(c_*^D - \bar{l} + \rho b_m) + (u'(c^*) - u'(c_*^D))I(c_*^D - \bar{l} + \rho b_m) \tag{43}$$

which depends only on mean debt $b_m = \frac{1}{I}\sum_{i=1}^{I} b_i$ and the number of countries in the union.

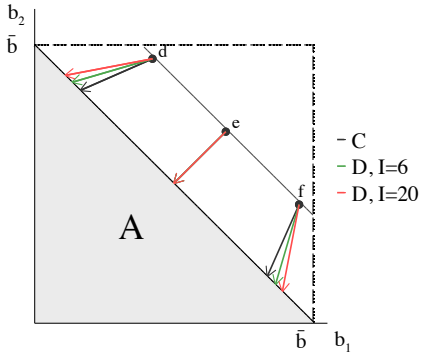
Corollary 1. *If* $\boldsymbol{b} \in A^C$, $J^D(b_m, I)$ *is decreasing in* b_m *and* I.

Figure 4a illustrates three configurations of initial debt in both regimes. At point d, half the countries start with $b_1 = b_L$ and half with $b_2 = b_H$ and at point f, the assignment is reversed. At point e, all countries share the same debt equal to the mean of b_L and b_H. Points d and f highlight two findings. First, under decentralization countries with higher initial debt have terminal debt b^{*D} that increases in I. Second, under centralization, the terminal debt does not vary with I, and for countries with b_H, this terminal debt is lower than under decentralization for both $I = 6$ and $I = 20$.

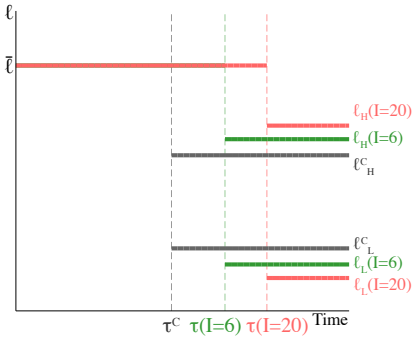
Figure 4b shows the labor paths from point d and highlights two additional findings. First, in both regimes and for any I, countries with higher initial debt work more once the zero-inflation region is reached, reflecting their higher terminal debt. Second, the gap in labor supply between high- and low-debt countries is smallest under centralization and grows progressively larger under decentralization as I increases.

Figure 4: Set A, debt, and labor paths

(a) Set A and debt paths



(b) Labor paths



3 Economy with Fiscal Federalism

We now reintroduce fiscal federalism considerations in the spirit of Oates (1972) by specializing preferences (1) to log utility: $u(\cdot) = \log(\cdot)$ and $h(\cdot) = \log(\cdot)$, so that

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t}[\theta_{it} \log(c_{it}) + (1 - \theta_{it}) \log(g_{it}) - l_{it} - \psi \pi_t] dt. \tag{44}$$

The Poisson taste shock $\theta_{it} \in \{\theta_L, \theta_H\}$, with $1 > \theta_H > \theta_L > 0$, switches between its two values with intensity λ, independently across countries. In the decentralized regime, each

country i's fiscal authority observes θ_{it} in every period t. In the centralized regime, the central fiscal authority learns θ_{it} only when an independent Poisson signal with intensity ϕ arrives; it does observe the initial values $\boldsymbol{\theta}_0 = \{\theta_{i0}\}$. The parameter ϕ thus captures the information disadvantage of centralization: $1/\phi$ is the average time between information arrivals, and as $\phi \to \infty$ this disadvantage vanishes. The monetary authority receives no private signals about shocks.

The log specification simplifies the analysis in two ways. First, it makes the dynamic decisions for debt accumulation identical to those already characterized without fiscal federalism, confining the benefits of more precise information to a purely static allocation. Second, it eliminates intertemporal effects of the idiosyncratic shocks θ_{it}.

To see why, consider first the decentralized regime. Since the country i fiscal authority observes θ_{it} without noise, its static problem is to allocate total spending x_{it} between c_{it} and g_{it} conditional on θ_{it}:

$$v(\theta_{it}, x_{it}) = \max_{c,g} \ \theta_{it} \log c_{it} + (1 - \theta_{it}) \log g_{it}, \quad \text{s.t. } c_{it} + g_{it} = x_{it}.$$

Substituting the solution $c_{it} = \theta_{it} x_{it}$ and $g_{it} = (1 - \theta_{it}) x_{it}$ yields

$$v(\theta_{it}, x_{it}) = \log x_{it} + \kappa(\theta_{it}),$$

where $\kappa(\theta) = \theta \log \theta + (1 - \theta) \log(1 - \theta)$. Preferences (44) then become

$$U_i = \int_0^\infty e^{-\rho t} \left[\log x_{it} - l_{it} - \psi \pi_t \right] dt + \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \kappa(\theta_{it}) dt \right]. \tag{45}$$

The right-most term, capturing all effects of taste shocks, is additively separable, so the optimal choices of (x_t, l_t, π_t) are independent of θ_{it}. Under decentralization, the optimal policies for (x, l, π) are therefore the same as those without fiscal federalism, with log preferences for consumption.

A similar logic applies to the centralized regime. The static problem of allocating x_{it} is

$$\max_{c,g} \ \mathbb{E}_{it} \left[\theta_{it} \log c_{it} + (1 - \theta_{it}) \log g_{it} \,|\, \phi \right], \quad \text{s.t. } c_{it} + g_{it} = x_{it},$$

where $\mathbb{E}_{it}[\cdot|\phi]$ denotes the expectation conditional on information available at t. Since the objective is linear in θ_{it}, it suffices to use the central authority's belief $\hat{\theta}_{it} \equiv \mathbb{E}_{it}[\theta_{it}|\phi]$, giving $v(\hat{\theta}_{it}, x_{it}) = \log x_{it} + \kappa(\hat{\theta}_{it})$. The central authority's preferences for country i can therefore be written as

$$U_i = \int_0^\infty e^{-\rho t} \left[\log x_{it} - l_{it} - \psi \pi_t\right] dt + \mathbb{E}_0 \int_0^\infty e^{-\rho t} \kappa(\hat{\theta}_{it})\, dt. \tag{46}$$

The right-most terms of (45) and (46) represent the *value of information* in the two regimes. In the centralized regime this value is

$$f(\theta_i|\phi) = \mathbb{E}_0 \int_0^\infty e^{-\rho t} \kappa(\hat{\theta}_{it})\, dt. \tag{47}$$

In the decentralized regime this value is $f(\theta_i|\infty)$, corresponding to the case $\hat{\theta}_{it} = \theta_{it}$ almost surely.

As in the economy without fiscal federalism, we focus on Markov perfect equilibria in which each country's fiscal rule is constant in the high-inflation region A^C with respect to $\boldsymbol{b}$. Let $\tilde{J}^D(\boldsymbol{b}, \boldsymbol{\theta})$ denote mean welfare across countries under decentralization, and $\tilde{J}^C(\boldsymbol{b}, \hat{\boldsymbol{\theta}})$ denote welfare under centralization, with beliefs $\hat{\boldsymbol{\theta}}$ replacing the unobserved $\boldsymbol{\theta}$. We let $J^C(\boldsymbol{b})$ and $J^D(\boldsymbol{b}) = \sum_i V_i(\boldsymbol{b}, I)/I$ for the corresponding values of welfare when $u(c) = \log c$ and $\theta_{it} = 1$ for all t, that is without fiscal federalism. For general $u(c)$, they are characterized in Lemma 1 and Proposition 2.

Because the first terms in (45) and (46) coincide with those in the economy without fiscal federalism ($\theta_{it} = 1$ for all i, t) and the information terms are additively separable, the optimal policies for (x, l, π) under fiscal federalism are identical to those without it. The values with fiscal federalism therefore decompose into two parts: the values without fiscal federalism and the value of information.

Under preferences (44), the following decomposition holds:

$$\tilde{J}^C(\boldsymbol{b}, \hat{\boldsymbol{\theta}}) = J^C(\boldsymbol{b}) + \frac{1}{I} \sum_i f(\hat{\theta}_i|\phi) \quad \text{and} \quad \tilde{J}^D(\boldsymbol{b}, \boldsymbol{\theta}) = J^D(\boldsymbol{b}) + \frac{1}{I} \sum_i f(\theta_i|\infty). \tag{48}$$

We turn next to the value of information in the two regimes, $f(\theta_i|\phi)$ and $f(\theta_i|\infty)$. The lower is ϕ, the harder it is for the central authority to allocate (c, g) efficiently. This relative

disadvantage in resource allocation is the key implication of our fiscal federalism model. Since $f(\theta_i|\phi)$ captures this disadvantage, one expects it to be increasing in ϕ. The next lemma confirms this.

Lemma 7. *The value of information $f(\theta|\phi)$ is strictly increasing in ϕ.*

Proof. From (47), $f(\theta_i|\phi) = \mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\kappa(\hat{\theta}_{it})\,dt\right]$, where $\hat{\theta}_{it} = \mathbb{E}_t[\theta_{it}|\phi]$. Let $\phi > \phi'$. Since ϕ is more informative than ϕ'—it generates a finer filtration—the Law of Iterated Expectations gives $\mathbb{E}_t[\theta_{it}|\phi'] = \mathbb{E}_t(\mathbb{E}_t[\theta_{it}|\phi] \,|\, \phi')$. Thus,

$$\mathbb{E}_0\left\{\kappa\left(\mathbb{E}_t[\theta_{it}|\phi']\right)\right\} = \mathbb{E}_0\left\{\kappa\left(\mathbb{E}_t(\mathbb{E}_t[\theta_{it}|\phi] \,|\, \phi')\right)\right\} < \mathbb{E}_0\left\{\mathbb{E}_t(\kappa\left(\mathbb{E}_t[\theta_{it}|\phi]\right) \,|\, \phi')\right\} = \mathbb{E}_0\left\{\kappa\left(\mathbb{E}_t[\theta_{it}|\phi]\right)\right\},$$

where the inequality uses Jensen's Inequality and the strict convexity of κ, and the last equality uses the Law of Iterated Expectations. Multiplying by $e^{-\rho t}$, integrating over time, using that $\kappa(\cdot)$ is bound and applying Fubini's theorem yields $f(\theta_i|\phi') < f(\theta_i|\phi)$. □

We can now state our main result. If debt is in the zero-inflation region, decentralization is always preferred. If debt is in the high-inflation region, either decentralization is always preferred or there exists a cutoff in the number of union members I below which decentralization is preferred and above which centralization is; this cutoff decreases in members' debt. For the next proposition, we assume that adding enough countries—each with debt equal to that of existing members—eventually makes centralization preferred:

$$f(\theta_0|\infty) - f(\theta_0|\phi) < J^C(b) - J^D(b, \bar{I}). \tag{49}$$

Proposition 3. *(Centralization is relatively more desirable the larger the union). Let initial taste shocks and debt be symmetric: $\theta_{i0} = \theta$, $b_{i0} = b$, and (49) hold. If $b \leq \psi$, decentralization is preferred. If $b > \psi$, either decentralization is always preferred, or there exists a cutoff $\mathcal{I}(b, \phi)$ such that centralization is preferred if and only if $I > \mathcal{I}(b, \phi)$. This cutoff decreases as initial debt b increases and the informativeness ϕ in the centralized regime increases, so $\mathcal{I}(b', \phi) \leq \mathcal{I}(b, \phi)$ for $b' > b$ and $\mathcal{I}(b, \phi') \leq \mathcal{I}(b, \phi)$ for $\phi' > \phi$.*

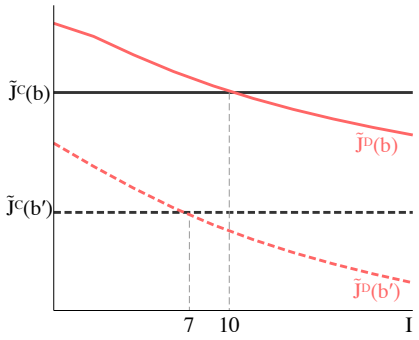
When condition (49) fails, the information advantage of decentralization is so strong that it is preferred regardless of the number of countries.

The logic is straightforward. The fiscal externality makes the deterministic component $J^D(\boldsymbol{b}) = \sum_i V(b_m, I)/I$ decreasing in I, while $J^C(\boldsymbol{b})$ is independent of I. Meanwhile, the value of information under decentralization, $f(\theta_i|\infty)$, strictly exceeds that under centralization, $f(\theta_i|\phi)$, for all $\phi < \infty$, and their cross-country means $\frac{1}{I}\sum_i f(\theta_i|\cdot)$ do not depend on I. Since decentralization is preferred with one country and, under (49), centralization is preferred at $I = \bar{I}$, a cutoff of the stated form must exist.

Figures 5 and 6 illustrate. Figure 5a shows that increasing initial debt from b to b' lowers the cutoff from 10 to 7 countries. Figure 5b shows that improving information under centralization by raising the arrival rate from ϕ to ϕ', lowers it from 10 to 3. Figure 6 plots value functions for the centralized and decentralized regimes with $I = 6$ and $I = 10$ as a function of mean debt. In the high-inflation region, $b_m > \psi$, the decentralized value falls more steeply with 10 countries than with 6, reflecting the worsening fiscal externality, and consequently crosses the centralized value at lower debt levels.

Figure 5: Cutoff rules in the number of countries.

(a) Effect of increasing b: $b' > b$



(b) Effect of increasing ϕ: $\phi' > \phi$

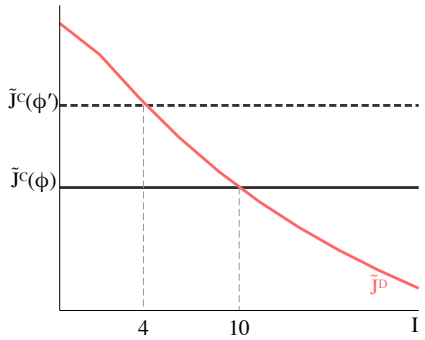
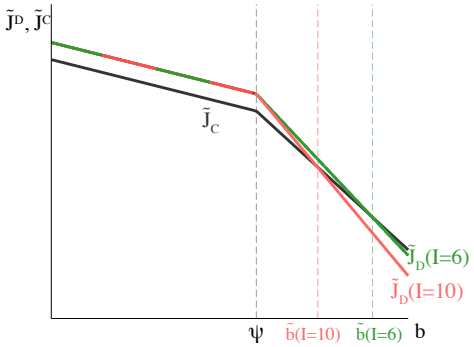


Figure 6: Value functions under centralized and decentralized regimes.



A more general version of Proposition 3 follows from Corollary 1: if centralization is preferred for some (b_m, I), it is also preferred for any $b'_m \geq b_m$ and $I' \geq I$; conversely, if decentralization is preferred, it remains so for $b'_m \leq b_m$ and $I' \leq I$. These monotonicity properties are visible in Figure 6.

4 Adding Heterogeneous Countries

Our main result considers adding members with the same debt and population as existing ones. Here we extend it to asymmetric new members. Throughout, we start from an initial union of I_0 countries with average debt $b_{m,I_0} = \frac{1}{I_0}\sum_i^{I_0} b_i > \psi$, each with population 1, and taste shock $\theta_{i0} = \theta$, for which decentralization is optimal. We first add countries with different debt, then countries with different population. In both cases the decomposition of Proposition 3 continues to hold, but the value without fiscal federalism (the first component) differs. We use the utilitarian welfare criterion throughout.

4.1 Adding new countries with different debt

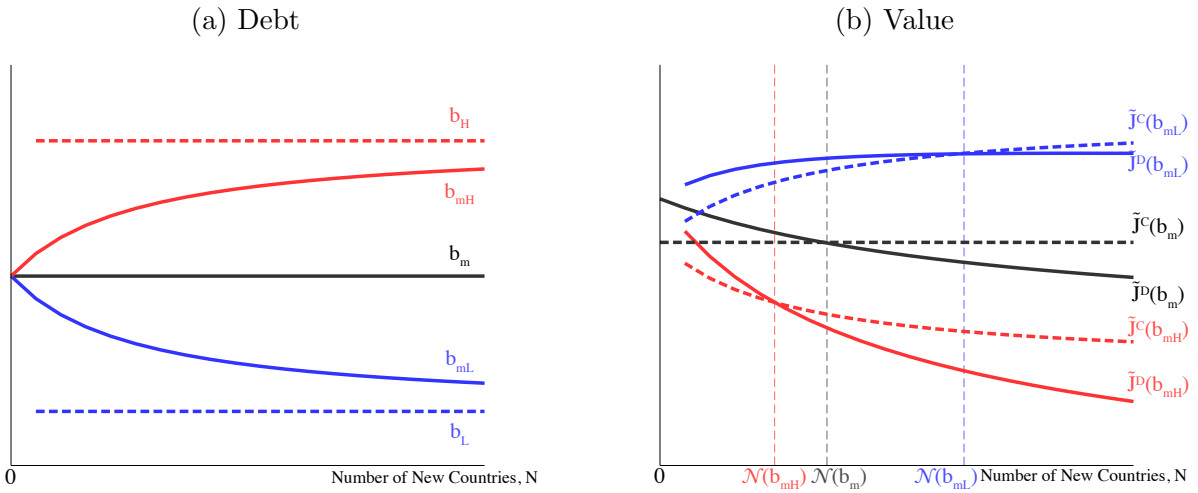
Suppose N new countries join the union, each with debt b' that may differ from the mean debt of the original members. Mean debt for all $I = I_0 + N$ countries is $b_{m,I} = (I_0 b_{m,I_0} + Nb')/I$. Let $\mathcal{N}(b')$ denote the cutoff number of *new members* above which centralization is preferred. An immediate implication of Proposition 3 is:

Corollary 2. *Consider a union with I_0 members, mean debt in the high-inflation region $b_{m,I_0} > \psi$, for which decentralization is preferred. If new members have higher debt, $b' > b_{m,I_0}$, the cutoff falls, $\mathcal{N}(b') \leq \mathcal{N}(b_{m,I_0})$ and if $b' < b_{m,I_0}$, it rises, $\mathcal{N}(b') \geq \mathcal{N}(b_{m,I_0})$.*

The cutoff on the total number of countries for switching to centralization at mean debt b_{m,I_0} is $\mathcal{I}(b_{m,I_0})$, so $\mathcal{N}(b_{m,I_0}) = \mathcal{I}(b_{m,I_0}) - I_0$. Adding countries with $b' > b_{m,I_0}$ raises mean debt, which lowers $\mathcal{I}$ and hence $\mathcal{N}$. The reverse holds for $b' < b_{m,I_0}$.

Figure 7a plots mean debt as a function of $I = I_0 + N$ starting from $I_0 = 5$ original members. The solid red line shows mean debt b_{mH} when new members have debt $b_H > b_{m,I_0}$ (dashed red); the solid blue line shows b_{mL} when new members have $b_L < b_{m,I_0}$ (dashed blue). Figure 7b plots the corresponding welfare in both regimes. Relative to the benchmark of identical debt (black lines), adding higher-debt members lowers the cutoff $\mathcal{N}$, while adding lower-debt members raises it.

Figure 7: Effect of adding countries with higher ($b_H > b$) or lower ($b_L < b$) debt levels.



4.2 Allowing for unequal populations

We now allow arbitrary heterogeneity in both debt and population. Country i has debt b_i and population L_i, with $L = \sum_i L_i$ and population share $\omega_i = L_i/L$, rather than $\omega_i = 1/I$ as earlier. The zero-inflation set now depends on population-weighted average debt, $A = \{\boldsymbol{b} : \sum_i \omega_i b_i \leq \psi\}$. If $\boldsymbol{b} \in A$, policies are as before. We focus on $\boldsymbol{b} \in A^C$ with $\sum_i \omega_i b_i > \psi$. We maintain our earlier assumptions on taste shocks and utility, and restrict attention to combinations of (ω_i) and $\underline{b}$ satisfying $\omega_i \underline{b} + \sum_{j \neq i} \omega_j \overline{b} < \psi$.

The decomposition result continues to hold with the value of information unchanged from (47), so the comparison between regimes differs only through the values without fiscal federalism. The fiscal rules and value functions are as follows.

Proposition 4. *If $\boldsymbol{b} \in A^C$, in the decentralized regime labor supply is $\bar{l}$ and the constant consumption rules $c_i = c^D_{*i}(\boldsymbol{\omega}) < c^*$ for all i, solve the system of I equations*

$$\Delta U(c_i, \psi) = -u'(c_i)(c_i - \bar{l} + \rho\psi) + (u'(c^*) - u'(c_i)) \sum_{j \neq i} \frac{\omega_j}{\omega_i} \left(c_j - \bar{l} + \rho\psi\right), \tag{50}$$

and, using shorthand notation $c^D_{*i} = c^D_{*i}(\boldsymbol{\omega})$, the value function without fiscal federalism is

$$\rho V_i(\boldsymbol{b}, \boldsymbol{\omega}) = u(c_{i*}) - \bar{l} - \psi\overline{\pi} - u'(c_{i*})\left(c_{i*} - \bar{l} + \rho b_i\right) + (u'(c^*) - u'(c_{i*})) \sum_{j \neq i} \frac{\omega_j}{\omega_i} \left(c^D_{*j} - \bar{l} + \rho b_j\right). \tag{51}$$

In the centralized regime, labor supply is $\bar{l}$ *and consumption* $c_i = c_* < c^*$ *for all* i*, where* c_* *solves* $\Delta U(c_*, \psi) = -u'(c_*)(c_* - \bar{l} + \rho\psi)$*,and the union value function without fiscal federalism is*

$$\rho J^C(\boldsymbol{b}, \boldsymbol{\omega}) = u(c_*) - \bar{l} - \psi\overline{\pi} - u'(c_*)\left(c_* - \bar{l} + \rho \sum_i \omega_i b_i\right). \tag{52}$$

In the austerity region, overall welfare under centralization is $\rho\tilde{J}^C(\boldsymbol{b}, \boldsymbol{\omega}, \theta) = \rho J^C(\boldsymbol{b}, \boldsymbol{\omega}) + \rho f(\theta|\phi)$, and country i's welfare and union welfare under decentralization are $\rho V_i(\boldsymbol{b}, \boldsymbol{\omega}, \theta) = \rho V_i(\boldsymbol{b}, \boldsymbol{\omega}) + \rho f(\theta|\infty)$ and $\tilde{J}^D(\boldsymbol{b}, \boldsymbol{\omega}, \theta) = \sum_i \omega_i V_i(\boldsymbol{b}, \boldsymbol{\omega}, \theta)$.Earlier, in unions with equal populations, Proposition 2) showed the decentralized fiscal rule $c_*^D(I)$ was identical across countries and depended only on I. Here, (50) shows that each country i's rule depends on its own share and the full vector $(\omega_j)_{j\neq i}$, because each country cares about the severity of austerity in every other member. As earlier, the austerity rules in both regimes are independent of debt, but initial debt affects the value functions in both regimes analogously.

We consider two experiments highlighting the new forces that arise when countries of different size join the union.

4.2.1 Adding one country of a different population size.

Consider expanding the union from I_0 members, each with population 1, to $I_0 + 1$ by adding a country of population L. We choose I_0 so that decentralization is preferred in the original union but centralization is preferred when one country of size 1 is added.

Figure 8: Add 1 country of size L.

(a) Value (b) Consumption (c) Time in high-inflation

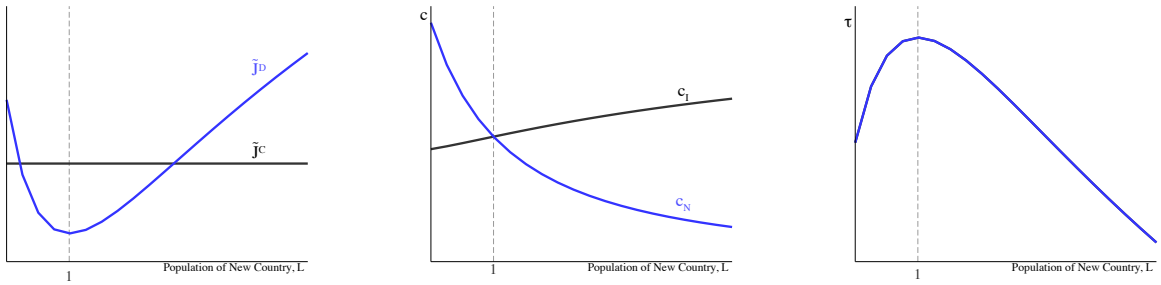


Figure 8a reveals a non-monotonicity: if the new country is sufficiently small or sufficiently large, decentralization is preferred, but when it is the same size as existing members, central-

ization is preferred. The intuition is as follows. When the new country is small, it chooses consumption c_N with low austerity, free-riding on existing members, Figure 8b. Yet when it is sufficiently small, this free-riding has little effect on the key variable—the time τ before inflation ends, Figure 8c. As the new country grows toward the size of original members, 1, its free-riding diminishes and c_N converges to the existing members' austerity plan c_I, echoing our earlier result: more countries reduce the relative desirability of decentralization.

When the new country becomes larger than existing members, $L > 1$, the pattern reverses. The original members now free-ride on the large newcomer, which adopts an increasingly stringent austerity program while existing members become increasingly lenient. The time until inflation ends, τ, monotonically declines in this phase, so the desirability of decentralization rises and is eventually preferred for a sufficiently large new member.

4.2.2 Adding smaller or larger countries to achieve given total population size

Consider expanding a union of I_0 countries of size 1 to total population $I_0 + L$ by adding N countries of size L/N each, all with debt b. We set L so that adding $N^* = \mathcal{N}(b)$ countries of size 1 leaves the union indifferent between regimes.

Figure 9: Total population L is split into N new countries.

(a) Value (b) Consumption (c) Time in high-inflation

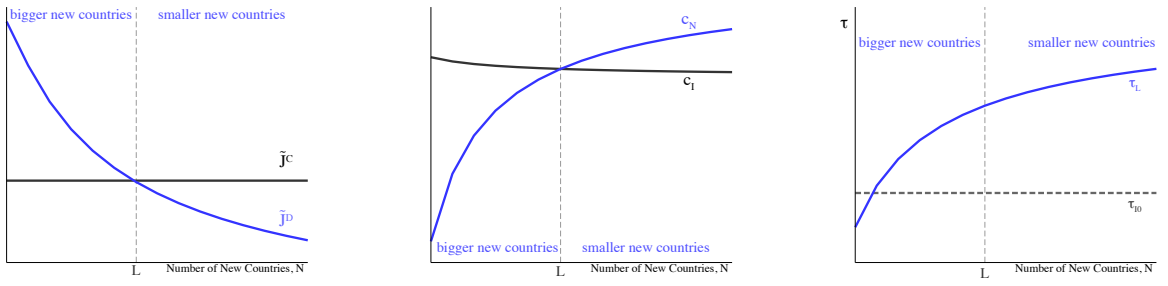


Figure 9a shows welfare in both regimes. Adding fewer but larger countries, N small, preserves the preference for decentralization, while adding more but smaller countries, N large, tips the balance toward centralization. Figures 9b and 9c provide intuition: the larger each new member, the more austere its plan and the shorter the time under high inflation. Indeed, as the dashed line τ_{I_0} in Figure 9c shows, sufficiently large new members adopt such severe austerity that the time under high inflation falls below that of the original union.

5 Conclusion

How should fiscal authority be delegated within a monetary union, and how does enlargement affect the case for centralization? We develop a novel dynamic framework to capture both fiscal externalities and information advantages in a finite-country monetary union without commitment. Unlike the mean game approach used in Aguiar et al. (2015), our model explicitly accounts for strategic interactions among a discrete number of member states, allowing us to study how union size affects optimal fiscal delegation.

Our analysis yields a new set of principles for optimal delegation. The traditional macroeconomic view favors centralization to internalize fiscal externalities; the fiscal federalism view favors decentralization due to local information advantages. We show that optimal delegation depends critically on union characteristics: centralization is optimal if and only if the union is sufficiently large. The cutoff at which centralization becomes preferred depends on three factors: the quality of information available to the central authority, and the debt and population levels of prospective members.

These findings carry direct implications for EMU expansion. First, eastward enlargement strengthens, not weakens, the case for centralization. Second, the accession of high-debt countries makes a regime shift from decentralized to centralized fiscal governance more likely, so expansion should be accompanied by stronger central oversight when new members carry significant fiscal burdens. Third, adding many small countries is more likely to trigger a regime shift than adding a few large ones suggesting that the composition of expansion matters as much as its scale.

6 Appendix

When we defined our economy, we set up the problems of the monetary and fiscal authorities as sequential problems. A classical HJB solution does not exist here because our value functions are non-differentiable when the state crosses the hyperplane $\mathcal{H} = \left\{ b : \frac{1}{I} \sum_{i=1}^{I} b_i = \psi \right\}$. We therefore consider *viscosity solutions* to the HJB.

A viscosity solution replaces pointwise differentiability with inequalities against smooth test functions ϕ, called *test functions*, that locally touch the value function from above or below, allowing the HJB to be formulated even at kinks. Our objective is to show that the

value functions in the body are viscosity solutions to the HJB in the sense of Barles, Briani, and Chasseigne (2014), hereafter *BBC*, and therefore solve the original sequential problem.

We briefly flag what is nonstandard relative to the standard Ishii (1989) viscosity framework and why BBC is needed. Since our value function has a concave kink at the hyperplane $\mathcal{H}$, no smooth function defined on $\mathbb{R}^I$ can touch the value function from below on $\mathcal{H}$, and multiple (Ishii) solutions may exist. BBC restores uniqueness by adding a tangential Hamiltonian on $\mathcal{H}$, defined using test functions that live only on the hyperplane $\mathcal{H}$, rather than on the entire space $\mathbb{R}^I$.

We also use BBC rather than Bressan and Hong (2007) because the latter requires a controllability condition that each authority can locally move the entire state vector in any direction.[12] This does not hold for the decentralized fiscal authority of country i, which controls only (c_i, l_i), nor for the monetary authority, which moves only the average debt. BBC requires only local control of the *average* debt on $\mathcal{H}$, which our environment satisfies.[13] Here, we focus on the decentralized equilibrium, which is the difficult and novel part.

6.1 State Space, Preferences and Debt Dynamics

Our state space is $\boldsymbol{b} = (b_1, \ldots, b_I) \in [\underline{b}, \bar{b}]^I = B^I$. The BBC results are stated in $\mathbb{R}^I$, so we extend our domain from B^I to $\mathbb{R}^I$, but impose a penalty high enough in $\mathbb{R}^I \setminus B^I$, such that starting from $\boldsymbol{b} \in B^I$, it is never optimal to leave B^I.[14] We divide the state space $\mathbb{R}^I$ into the high-inflation region A_1, the interior of the no-inflation region A_2 and the hyperplane $\mathcal{H}$ that divides A_1, A_2, so $\mathbb{R}^I = A_1 \cup A_2 \cup \mathcal{H}$ and

$$A_1 = \left\{ \boldsymbol{b} : \sum_i \frac{b_i}{I} > \psi \right\}, \quad A_2 = \left\{ \boldsymbol{b} : \sum_i \frac{b_i}{I} < \psi \right\}, \quad \text{and} \quad \mathcal{H} = \left\{ \boldsymbol{b} : \sum_i \frac{b_i}{I} = \psi \right\}$$

The authorities face different problems, but share the same discontinuity at $\mathcal{H}$. We therefore write them in a common notation. Let x denote the control and X its compact set. For the monetary authority, $x = \pi$, $X = [0, \overline{\pi}]$; for the decentralized fiscal authority, $x = (c, l)$ and $X = [\underline{c}, \overline{c}] \times [\underline{l}, \bar{l}]$, with $\underline{c} > 0$ small and $\overline{c} > 0$ large enough, so that neither binds.

[12] Aguiar et al. (2015) uses the methodology from Bressan and Hong (2007) in a related setting.

[13] Technically, for the monetary authority to satisfy the BBC assumption, we first need to extend its controls in a way that does not affect the equilibrium.

[14] This extension is standard in the literature; see Bressan and Hong (2007) for a detailed discussion.

For $k = 1, 2$, let $\overline{A}_k$ be the closure of A_k, which includes $\mathcal{H}$. The *flow utility* is $u_k : \overline{A}_k \times X \to \mathbb{R}$ and the *debt flow dynamics* is $f_k : \overline{A}_k \times X \to \mathbb{R}^I$. Throughout, $\{x_i\}$ refers to the vector of variables $(x_1, \ldots, x_I)$, where x_i may itself be a vector as well.

Monetary authority. For the monetary authority in the *decentralized regime*, the debt dynamics and utility flow in the two regions $\overline{A}_1$ and $\overline{A}_2$ are[15]

$$f_1(\boldsymbol{b}, \pi) = \{c_*^D(I) - \bar{l} + (\rho + \overline{\pi} - \pi)b_i\}, \qquad f_2(\boldsymbol{b}, \pi) = \{-\pi b_i\}$$
$$u_1(\boldsymbol{b}, \pi) = \frac{1}{I}\sum_i [c_*^D(I) - \bar{l} - \psi\pi], \qquad u_2(\boldsymbol{b}, \pi) = \frac{1}{I}\sum_i [u(c^*) - c^* - \rho b_i - \psi\pi].$$

Decentralized fiscal authority. For the decentralized fiscal authority i, dropping the i subscript, $(c_i, l_i) = (c, l)$, taking as given the constant austerity rules $(c_j, l_j)_{j \neq i}$ for the other authorities and denoting $f_k = \{f_{ik}\}$, the debt dynamics and flow utilities are[16]

$$f_{i1}(\boldsymbol{b}, c, l) = c - l + \rho b_i, \quad f_{j1}(\boldsymbol{b}, c, l) = c_j - l_j + \rho b_j, \quad \text{for } j \neq i \tag{53}$$
$$f_{i2}(\boldsymbol{b}, c, l) = c - l + \rho b_i, \quad f_{j2}(\boldsymbol{b}, c, l) = 0, \quad \text{for } j \neq i \tag{54}$$
$$u_1(\boldsymbol{b}, c, l) = u(c) - l - \psi\overline{\pi}, \quad u_2(\boldsymbol{b}, c, l) = u(c) - l. \tag{55}$$

Convexified controls in $\mathcal{H}$. At $\mathcal{H}$, the authority can stay on $\mathcal{H}$ or move into either region. Fast alternation between the two regions effectively generates a convex combination of their dynamics and payoffs, so we define the convexified objects $f_{\mathcal{H}}, u_{\mathcal{H}} : \mathcal{H} \times (X^2 \times [0, 1]) :\to \mathbb{R}^I$[17]

$$f_{\mathcal{H}}(\boldsymbol{b}, x_1, x_2, \mu) = \mu f_1(\boldsymbol{b}, x_1) + (1 - \mu) f_2(\boldsymbol{b}, x_2),$$
$$u_{\mathcal{H}}(\boldsymbol{b}, x_1, x_2, \mu) = \mu u_1(\boldsymbol{b}, x_1) + (1 - \mu) u_2(\boldsymbol{b}, x_2).$$

We can also define the set of controls $X_{\mathcal{H}}(\boldsymbol{b})$ under which the debt vector $\boldsymbol{b}$ stays in $\mathcal{H}$

$$X_{\mathcal{H}}(\boldsymbol{b}) = \left\{ (x_1, x_2, \mu) \in X \times X \times [0, 1] : \sum_i f_{i\mathcal{H}}(\boldsymbol{b}, x_1, x_2, \mu) = 0 \right\}.$$

[15]The first equation for debt dynamics uses that, in the high inflation region A_1, $i(\boldsymbol{b}) = \rho + \overline{\pi}$, and the second uses that, in the no-inflation region A_2, $i(\boldsymbol{b}) = \rho$ and $c_i - l_i + \rho b_i = 0$.

[16]Note that $f_{j1}(\boldsymbol{b}, c, l) = c_j - l_j + \rho b_j$ captures that the debt dynamics of any country $j \neq i$ do not depend on the fiscal policy (c, l) of country i.

[17]We can interpret the weights $(\mu, 1 - \mu)$, with $\mu \in [0, 1]$, as the fraction of time the authority spends randomizing from $\mathcal{H}$ into the high-inflation region and into the zero-inflation region, respectively.

In our model, neither authority optimally randomizes, so this convexification is merely a technical device to ensure the problem is well-posed.

Regularity Assumptions. We next state the assumptions needed to apply BBC.

Assumption 3. *$X \subset \mathbb{R}^n$ is compact and $f_k : \overline{A}_k \times X \to \mathbb{R}^I$ is continuous and bounded. There exists an $L > 0$ such that $|f_k(\boldsymbol{b}, x) - f_k(\boldsymbol{b'}, x)| \leq L |\boldsymbol{b} - \boldsymbol{b'}|$, for $\boldsymbol{b}, \boldsymbol{b'} \in \overline{A}_k, x \in X$.*

Assumption 4. *For $k = 1, 2$, $u_k : \overline{A}_k \times X \to \mathbb{R}$ is uniformly continuous, bounded and there exists $L > 0$ such that $|u_k(\boldsymbol{b}, x) - u_k(\boldsymbol{b'}, x)| \leq L |\boldsymbol{b} - \boldsymbol{b'}|$ for $\boldsymbol{b}, \boldsymbol{b'} \in \overline{A}_k, x \in X$.*

Assumption 5. *For $\boldsymbol{b} \in \overline{A}_k$, the set $\{(f_k(\boldsymbol{b}, x), u_k(\boldsymbol{b}, x)) : x \in X\}$ is closed and convex.*

Assumption 6. *There exist $\delta > 0$ such that, for any $\boldsymbol{b} \in \mathcal{H}$, $(-\delta, \delta) \subset \{\sum_i f_{ik}(\boldsymbol{b}, x) : x \in X\}$.*

Assumption 6 is the key departure in BBC from the stronger condition often imposed in the viscosity literature: that there exists $\delta > 0$ such that, for any $\boldsymbol{b} \in A_k$, one has $\{\boldsymbol{z} : |\boldsymbol{z}| < \delta\} \subset \{f_k(\boldsymbol{b}, x) : x \in X\}$. This requires local control of the *entire state vector* in every direction, which does not hold in our decentralized regime, where country i controls only its own debt. BBC instead requires only local control over the *average* debt on $\mathcal{H}$, which holds here.

6.2 Admissible Set of Dynamics

Let $\mathcal{X} = L^\infty(\mathbb{R}_+, X \times X \times [0, 1])$. Given $\boldsymbol{b_0} \in \mathbb{R}^I$, debt trajectories $\boldsymbol{b}(t) \in \mathbb{R}^I$ are Lipschitz continuous functions such that there exist controls $(x_1(\cdot), x_2(\cdot), \mu(\cdot)) \in \mathcal{X}$ that satisfy a.e.

$$\dot{\boldsymbol{b}}(t) = f_1(\boldsymbol{b}(t), x_1(t)) 1_{A_1}(\boldsymbol{b}) + f_{\mathcal{H}}(\boldsymbol{b}(t), x_1(t), x_2(t), \mu(t)) 1_{\mathcal{H}}(\boldsymbol{b}) + f_2(\boldsymbol{b}(t), x_2(t)) 1_{A_2}(\boldsymbol{b}), \quad (56)$$

and we denote the right-side of this equation as $f(\boldsymbol{b}(t), x_1(t), x_2(t), \mu(t))$. Let $Lip(\mathbb{R}_+, \mathbb{R}^I)$ be the set of Lipschitz continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^I$. Define the feasible set of debt dynamics and controls starting from some debt vector $\boldsymbol{b_0}$ to be

$$\mathcal{T}_{\boldsymbol{b_0}} = \left\{(\boldsymbol{b}(\cdot), x_1(\cdot), x_2(\cdot), \mu(\cdot)) \in Lip(\mathbb{R}_+, \mathbb{R}^I) \times \mathcal{X} : (56) \text{ holds a.e. and } \boldsymbol{b}(0) = \boldsymbol{b_0}\right\}. \quad (57)$$

Specializing this generic set to each authority gives the set $\mathcal{T}^M_{\boldsymbol{b_0}}$ and $\mathcal{T}^D_{\boldsymbol{b_0}}$, used in the body. Given initial debt $\boldsymbol{b}$, the corresponding objective is

$$V(\boldsymbol{b}) = \sup_{(\boldsymbol{b},x_1,x_2,\mu)(t)\in\mathcal{T}_{\boldsymbol{b}}} \int_0^\infty e^{-\rho t} u(\boldsymbol{b}_t, x_1(t), x_2(t), \mu(t))dt, \tag{58}$$

where $u(\boldsymbol{b}, x_1, x_2, \mu) = u_1(\boldsymbol{b}, x_1)1_{A_1}(\boldsymbol{b}) + u_{\mathcal{H}}(\boldsymbol{b}, x_1, x_2, \mu)1_{\mathcal{H}}(\boldsymbol{b}) + u_2(\boldsymbol{b}, x_2)1_{A_2}(\boldsymbol{b})$.

6.3 Hamiltonians and Viscosity Solutions

We now define the Hamiltonians on $A_1, A_2, \mathcal{H}$. For $k = 1, 2$ let $H_k : \overline{A}_k \times \mathbb{R}^I \to \mathbb{R}$ be

$$H_k(\boldsymbol{b}, \boldsymbol{p}) = \sup_{x_k\in X} u_k(\boldsymbol{b}, x_k) + \boldsymbol{p}\cdot f_k(\boldsymbol{b}, x_k), \quad k = 1, 2.$$

Next, let $T\mathcal{H} = \{\boldsymbol{b} : \sum_i b_i = 0\}$. The *tangential Hamiltonian* $H_T : \mathcal{H} \times T\mathcal{H} \to \mathbb{R}$ is

$$H_T(\boldsymbol{b}, \boldsymbol{p}) = \sup_{(x_1,x_2,\mu)\in X_{\mathcal{H}}(\boldsymbol{b})} u_{\mathcal{H}}(\boldsymbol{b}, x_1, x_2, \mu) + \boldsymbol{p}\cdot f_{\mathcal{H}}(\boldsymbol{b}, x_1, x_2, \mu)$$

Our Hamiltonian is $H(\boldsymbol{b}, \boldsymbol{p}) = H_1(\boldsymbol{b}, \boldsymbol{p})1_{A_1}(\boldsymbol{b}) + H_T(\boldsymbol{b}, \boldsymbol{p})1_{\mathcal{H}}(\boldsymbol{b}) + H_2(\boldsymbol{b}, \boldsymbol{p})1_{A_2}(\boldsymbol{b})$. To ease notation, we write $H(\boldsymbol{b}, Dv)$ as short-hand for $H(\boldsymbol{b}, Dv(\boldsymbol{b}))$. We define next viscosity solutions.

Definition 1. *A continuous, bounded function $v : \mathbb{R}^I \to \mathbb{R}$ is a viscosity solution of H if: for any $\phi \in C^1(\mathbb{R}^I)$ such that $v - \phi$ has a local maximum at $\boldsymbol{b} \in \mathbb{R}^I$, one has* [18]

$$\rho v(\boldsymbol{b}) \leq H_k(\boldsymbol{b}, D\phi) \text{ on } A_k,\ k = 1, 2 \tag{59}$$

$$\rho v(\boldsymbol{b}) \leq \max\{H_1(\boldsymbol{b}, D\phi), H_2(\boldsymbol{b}, D\phi)\} \text{ on } \mathcal{H}. \tag{60}$$

for any $\phi \in C^1(\mathbb{R}^I)$ such that $v - \phi$ has a local minimum at $\boldsymbol{b} \in \mathbb{R}^I$, one has

$$\rho v(\boldsymbol{b}) \geq H_k(\boldsymbol{b}, D\phi) \text{ on } A_k,\ k = 1, 2 \tag{61}$$

$$\rho v(\boldsymbol{b}) \geq \min\{H_1(\boldsymbol{b}, D\phi), H_2(\boldsymbol{b}, D\phi)\} \text{ on } \mathcal{H}. \tag{62}$$

[18] Since BBC studies minimization problems, and we study maximization problems, these inequalities and the next are reversed relative to BBC.

and for any $\phi \in C^1(\mathcal{H})$ *such that* $v_{\mathcal{H}} - \phi$ *has a local minimum at* $\boldsymbol{b} \in \mathcal{H}$*, one has*

$$\rho v_{\mathcal{H}}(\boldsymbol{b}) \geq H_T(\boldsymbol{b}, D_{\mathcal{H}}\phi) \quad \text{on} \quad \mathcal{H}, \tag{63}$$

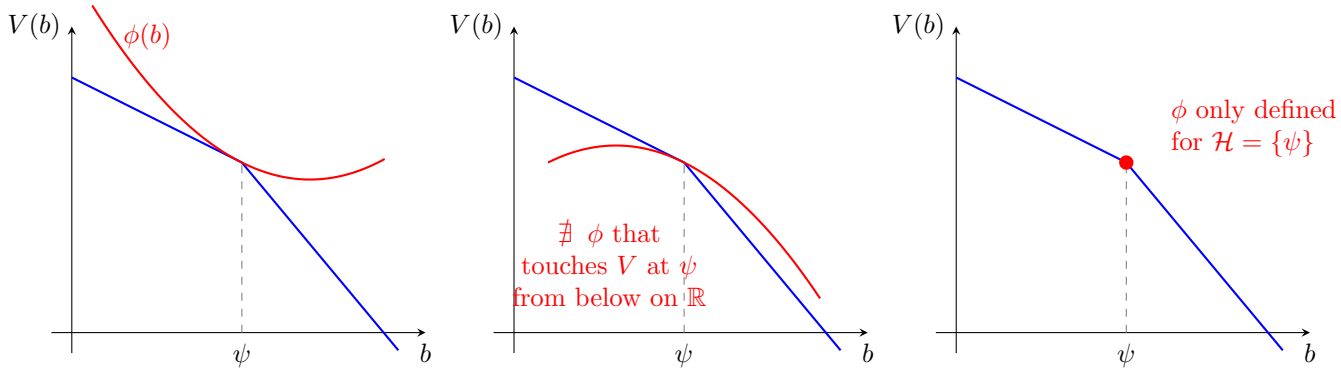
The following proposition is the infinite-horizon counterpart of BBC's Theorem 4.4.

Proposition 5. *Assume 3, 4, 5 and 6. Then, there exists a unique viscosity solution to* H*, it is continuous and coincides with the sequential value function* V *defined in (58).*

Without loss, we can restrict attention to test functions ϕ with $\phi(\boldsymbol{b}) = v(\boldsymbol{b})$. We say that ϕ *touches* v *from above (below) at* $\boldsymbol{b}$ if $v - \phi$ has a local maximum (minimum) at $\boldsymbol{b}$.

Relative to Ishii (1989), BBC adds the tangential condition (63). This is essential here because condition (62) holds vacuously: no smooth test function on $\mathbb{R}^I$ touches V from below at $\mathcal{H}$ exists. This lack of test functions is at the heart of the multiplicity problem.[19] BBC shows that adding (63) selects the unique function that solves the sequential problem.

Figure 10: Test functions for the viscosity conditions on $\mathcal{H}$



Left panel: A smooth test function ϕ (red) touches the value function V (blue) from above at the kink point $\boldsymbol{b} = \psi$. Such functions always exist at a concave kink. *Center panel:* When V has a concave kink, no smooth function defined on all of $\mathbb{R}$ can touch V from below at ψ—any potential candidate would cross V on one side of the kink. *Right panel:* By restricting attention to test functions defined only on the hyperplane, here $\mathcal{H} = \{\psi\}$, one *can* find a ϕ that touches $V|_{\mathcal{H}}$ from below.

The key idea behind this extra requirement is that, by restricting the value function to $\mathcal{H}$, and looking for test functions defined *only* on $\mathcal{H}$, one obtains test functions that touch $v|_{\mathcal{H}}$ from below at $\boldsymbol{b} \in \mathcal{H}$, even if there are no test functions that touch v from below in the extended domain $\mathbb{R}^I$ — see Figure 10 for an illustration with $I = 1$ countries.

[19] See BBC for examples of problems that have multiple solutions satisfying (59)-(62).

6.4 Verifying Assumptions and Viscosity Conditions

We now show that each authority's problem satisfies Assumptions 3, 4, 5 and 6, and that their respective value functions are viscosity solutions in the BBC sense. We assume that the debt vector $\boldsymbol{b}$ lies in $B^I = [\underline{b}, \overline{b}]^I$, rather than $\mathbb{R}^I$, see the discussion in Section 6.1. The only non-trivial Assumption is 6, which we verify next.

6.4.1 Verifying Assumptions

Decentralized fiscal authority. For country i's decentralized fiscal authority, we have[20]

$$\begin{aligned}\left\{\sum_i f_{i1}(\boldsymbol{b},x) : x \in X\right\} &= \left(\sum_{j\neq i}(c_j - l_j - \rho b_j) + \underline{c} - \overline{l} - \rho b_i, \sum_{j\neq i}(c_j - l_j - \rho b_j) + \overline{c} - \underline{l} - \rho b_i\right)\\ \left\{\sum_i f_{i2}(\boldsymbol{b},x) : x \in X\right\} &= \left(\underline{c} - \overline{l} - \rho b_i, \overline{c} - \underline{l} - \rho b_i\right),\end{aligned}$$

In the high-inflation region, this authority takes as given that other fiscal authorities implement austerity plans, so $\sum_{j\neq i}(c_j - l_j - \rho b_j) < 0$. By Assumption 1, $\underline{c} - \overline{l} - \rho b_i < 0$, so the lower-bound on the first interval is negative, while the interval in the upper-bound is positive for $\overline{c}$ large. Analogous arguments apply to the second equality. Hence, there exists $\delta > 0$ such that Assumption 6 holds.

Monetary authority. For the monetary authority, Assumption 6 does not directly hold.[21] As a purely technical matter, we now artificially expand the monetary authority's controls to $(\pi, k) \in [0, \overline{\pi}] \times [-K, K]$, $K > 0$ large. The new debt dynamics and flow utilities are

$$f_1(\boldsymbol{b},\pi,k) = \{c - \overline{l} + (\rho + \overline{\pi} - \pi)b_i + k\}, \quad f_2(\boldsymbol{b},\pi,k) = \{-\pi b_i + k\},$$

$$u_1(\boldsymbol{b},\pi) = \sum_i \frac{u(c) - \overline{l} - \psi\pi}{I} - M|k|, \quad u_2(\boldsymbol{b},\pi) = \sum_i \frac{u(c^*) - c^* - \rho b_i - \psi\pi}{I} - M|k|.$$

Clearly, for $K > 0$ large, Assumption 6 holds, and for M sufficiently large, the monetary authority will never choose $k \neq 0$. We drop the extended control k from now on.

[20] These equalities follow from (53) and (54) together with $(c, l) \in [\underline{c}, \overline{c}] \times [\underline{l}, \overline{l}]$.

[21] Since $\pi \in [0, \overline{\pi}]$, the monetary authority cannot increase the average debt through deflation.

6.4.2 Viscosity Conditions

To apply Proposition 5, it suffices to show that that the candidate value functions are viscosity solutions to their respective Hamiltonians. Since our candidate value function is differentiable and it was constructed to satisfy $\rho v = H_k(\boldsymbol{b}, Dv)$, (59) and (61) follow immediately.[22] Since each value function has a concave kink for $\boldsymbol{b} \in \mathcal{H}$, (62) holds vacuously.

Thus, it remains only to check (60) and (63) for $\boldsymbol{b} \in \mathcal{H}$. For both authorities, the proof structure is the same: *i*) for (60), use that $H_2 \leq \max\{H_1, H_2\}$, we then show that $\rho v \leq H_2$; *ii*) for (63), we show that the optimal policy along $\mathcal{H}$ is to stay there at zero inflation, so $\rho v = H_T$. We proceed to verify these two conditions.

Decentralized Fiscal Authority The candidate value function is

$$\rho V_i(\boldsymbol{b}) = \begin{cases} u(c^*) - c^* - \rho b_i & \boldsymbol{b} \in A_2 \cup \mathcal{H}, \\ u(c_*^D) - \bar{l} - \psi\overline{\pi} - u'(\underline{c}^D)(c_*^D - \bar{l} + \rho b_i) + (u'(c^*) - u'(c_*^D)) \sum_{j \neq i}(c_j - \bar{l} + \rho b_j) & \boldsymbol{b} \in A_1, \end{cases}$$

Consider first (60). Let $\phi \in C^1$ be a test function that touches V_i from above at $\boldsymbol{b} \in \mathcal{H}$. Denote $\boldsymbol{p} = D\phi$. It is enough to show that $\rho V_i(\boldsymbol{b}) \leq H_2(\boldsymbol{b}, \boldsymbol{p})$, where

$$H_2(\boldsymbol{b}, \boldsymbol{p}) = \sup_{(c_i, l_i) \in [\underline{c}, \overline{c}] \times [\underline{l}, \bar{l}]} \frac{u(c_i) - l_i}{I} + p_i(c_i - l_i + \rho b_i).$$

A lower bound to the right side is to set $c_i = c^*$ and $l_i = c^* + \rho b_i$, but then, using the formula for $V_i(\boldsymbol{b})$, we obtain that this lower bound yields exactly $\rho V_i(\boldsymbol{b})$, and taking the maximum over all possible strategies, we conclude that $\rho V_i(\boldsymbol{b}) \leq H_2(\boldsymbol{b}, \boldsymbol{p})$, and thus (60) holds.

Consider next (63). Let $\boldsymbol{b} \in \mathcal{H}$ and $\phi \in C^1(\mathcal{H})$ a test function that touches $V_{i,\mathcal{H}}$ from below at $\boldsymbol{b}$. To derive $D_{\mathcal{H}}\phi$, use that $\rho V_{i,\mathcal{H}}(\boldsymbol{b}) = u(c^*) - c^* - \rho b_i$ and without loss taking $i = I$, we obtain that $\boldsymbol{p} \equiv D_{\mathcal{H}}\phi(\boldsymbol{b}) = D_{\mathcal{H}} V_{i,\mathcal{H}}(\boldsymbol{b}) = (1/I, 1/I, \ldots, 1/I, -(I-1)/I)$.

Letting $X = [\underline{c}, \overline{c}] \times [\underline{l}, \bar{l}]$, the tangential Hamiltonian H_T is

$$\begin{aligned} H_T(\boldsymbol{b}, \boldsymbol{p}) \;=\; & \max_{(x_1, x_2, \mu) \in X_{\mathcal{H}}} \mu \left[u(c_1) - l_1 - \psi\overline{\pi} + (1/I, \ldots, 1/I, -(I-1)/I) \cdot f_1(\boldsymbol{b}, x_1) \right] \\ & + (1-\mu) \left[u(c_2) - l_2 + (1/I, \ldots, 1/I, -(I-1)/I) \cdot f_2(\boldsymbol{b}, x_2) \right], \end{aligned}$$

[22] To see why, note that for $\boldsymbol{b} \in A_k$, each candidate value function v is differentiable and was constructed to satisfy $\rho v = H_k(\boldsymbol{b}, Dv)$. Since each v is differentiable, any test function candidate $\phi \in C^1$ satisfies $v(\boldsymbol{b}) = \phi(\boldsymbol{b})$ and $Dv(\boldsymbol{b}) = D\phi(\boldsymbol{b})$, and thus (59) and (61) hold.

where $X_{\mathcal{H}}(\boldsymbol{b}) = \{(x_1, x_2, \mu) \in X \times X \times [0,1] : (1/I, \ldots, 1/I, 1/I) \cdot f_{\mathcal{H}}(\boldsymbol{b}, x_1, x_2, \mu) = 0\}$.

Rearranging H_T, using the definition of $f_{\mathcal{H}}$, that the controls lie in $X_{\mathcal{H}}(\boldsymbol{b})$, we obtain that

$$H_T(\boldsymbol{b}, \boldsymbol{p}) = \max_{(x_1,x_2,\mu)\in X_{\mathcal{H}}} \mu\left[u(c_1) - l_1 - \psi\overline{\pi} - (c_1 - l_1 + \rho b_I)\right] + (1-\mu)\left[u(c_2) - l_2 - (c_2 - l_2 + \rho b_I)\right].$$

The solution is $\mu = 0$, $c_2 = c^*$ and $l_2 = c^* + \rho b_I$. Thus $H_T(\boldsymbol{b}, \boldsymbol{p}) = u(c^*) - c^* - \rho b_I = \rho V_i(\boldsymbol{b})$, which concludes the proof that V_i satisfies (63).

Monetary Authority The proof for (60) is nearly identical to the decentralized fiscal authority case. The non-trivial step is to show that (63) holds. To do so, note that $J_{\mathcal{H}}(\boldsymbol{b}) = (u(c^*) - c^*)/\rho - \psi$, so that $\boldsymbol{p} = D_{\mathcal{H}}\phi = (0, \ldots, 0)$ and the tangent Hamiltonian is

$$H_T(\boldsymbol{b}, \boldsymbol{p}) = \max_{(\pi_1,\pi_2,\mu)\in X_{\mathcal{H}}(\boldsymbol{b})} \mu\left[u(c_*^D(I)) - \bar{l} - \psi\pi_1\right] + (1-\mu)\left[u(c^*) - c^* - \rho\psi - \psi\pi_2\right],$$

and $X_{\mathcal{H}}(\boldsymbol{b}) = \left\{(\pi_1, \pi_2, \mu) \in [0, \overline{\pi}] \times [0, \overline{\pi}] \times [0,1] : \mu\left(c_*^D(I) - \bar{l} + (\rho + \overline{\pi} - \pi_1)\psi - (1-\mu)\pi_2\psi\right) = 0\right\}$. Substituting out the constraint on controls, the tangent Hamiltonian becomes

$$H_T(\boldsymbol{b}, \boldsymbol{p}) = \max_{\mu\in[0,1]} \mu\left[u(c) - c - \rho\psi - \psi\overline{\pi}\right] + (1-\mu)\left[u(c^*) - c^* - \rho\psi\right].$$

Since $u'(c^*) = 1$, we have that $u(c^*) - c^* > u(c_*^D(I)) - c_*^D(I)$, and thus the solution is $\mu = 0$, yielding $\rho J_{\mathcal{H}}(\boldsymbol{b}) = H_T(\boldsymbol{b}, \boldsymbol{p})$, finishing the proof.

References

Afrouzi, H., Halac, M., Rogoff, K.S. and Yared, P., 2023. *Monetary policy without commitment* (No. w31207). National Bureau of Economic Research.

Aguiar, M., Amador, M., Farhi, E. and Gopinath, G., 2015. *Coordination and crisis in monetary unions*. The Quarterly Journal of Economics, 130(4), pp.1727-1779.

Aguiar, M., Chatterjee, S., Cole, H. and Stangebye, Z., 2022. *Self-fulfilling debt crises, revisited*. Journal of Political Economy, 130(5), pp.1147-1183.

Barles, G., Briani, A., Chasseigne, E., 2014. *A Bellman approach for two-domains optimal control problems in* $\mathbb{R}^N$. SIAM Journal on Control and Optimization, 52(3), pp.1712-1744.

Barles, G. and Chasseigne, E., 2023. *An illustrated guide of the modern approaches of Hamilton-Jacobi equations and control problems with discontinuities.* arXiv preprint arXiv:1812.09197.

Beetsma, R.M., Uhlig, H. and Tilburg University. Center for Economic Research, 1997. *An analysis of the Stability Pact*. Centre for Economic Policy Research.

Berriel, R., Gonzalez-Aguado, E., Kehoe, P.J. and Pastorino, E., 2023. *Fiscal Federalism and Monetary Unions*. NBER Working Paper

Berriel, R., Gonzalez-Aguado, E., Kehoe, P.J. and Pastorino, E., 2024. *Is a fiscal union optimal for a monetary union?*. Journal of Monetary Economics, 141, pp.157-177.

Bressan, A. and Hong, Y., 2007. *Optimal control problems on stratified domains*. Networks and Heterogeneous Media, 2(2), p.313.

Chari, V.V. and Kehoe, P.J., 2008. *Time inconsistency and free-riding in a monetary union*. Journal of Money, Credit and Banking, 40(7), pp.1329-1356.

Cole, H.L. and Kehoe, T.J., 2000. *Self-fulfilling debt crises*. The Review of Economic Studies, 67(1), pp.91-116.

Conesa, J.C. and Kehoe, T.J., 2017. *Gambling for redemption and self-fulfilling debt crises*. Economic Theory, 64(4), pp.707-740.

Cooper, R. and Kempf, H., 2004. *Overturning Mundell: Fiscal policy in a monetary union*. The Review of Economic Studies, 71(2), pp.371-396.

Díaz-Giménez, J., Giovannetti, G., Marimon, R., Teles, P., 2008. *Nominal debt as a burden on monetary policy*. Review of Economic Dynamics, 11(3), 493–514.

Farhi, E. and Werning, I., 2017. *Fiscal unions*. American Economic Review, 107(12), pp.3788-3834.

Halac, M. and Yared, P., 2018. *Fiscal rules and discretion in a world economy*. American Economic Review, 108(8), pp.2305-2334.

Halac, M. and Yared, P., 2022. *Fiscal rules and discretion under limited enforcement*. Econometrica, 90(5), pp.2093-2127.

Ishii, H., 1989. *A boundary value problem of the Dirichlet type for Hamilton-Jacobi equations*. Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 16(1), pp.105-135.

Kehoe, P.J. and Pastorino, E., 2017. *Fiscal unions redux*. Economic Theory, 64(4), pp.741-776.

Nicolini, J.P., 1998. *More on the time consistency of monetary policy*. Journal of Monetary Economics, 41(2), pp.333-350.

Nicolini, J.P., Posadas, J., Sanguinetti, J., Sanguinetti, P. and Tommasi, M., 2002. *Decen-*

tralization, fiscal discipline in sub-national governments and the bailout problem: The case of Argentina.

Nutz, M., 2018. *A mean field game of optimal stopping*. SIAM Journal on Control and Optimization, 56(2), pp.1206-1221.

Oates, W.E., 1972. *Fiscal federalism*. New York, 1.

Oates, W.E., 1999. *An essay on fiscal federalism*. Journal of economic literature, 37(3), pp.1120-1149.

Orlandi, F., Piguillem, F., 2025. *Fiscal unions with transfers*. Working paper.

Sargent, T.J., 2012. *Nobel lecture: United States then, Europe now*. Journal of Political Economy, 120(1), 1–40.

Tabellini, G., 2003. *Principles of policymaking in the European Union: an economic perspective*. CESifo Economic Studies, 49(1), pp.75-102.

7 Online Appendix: Details of Proofs

7.1 Details for Lemma 4

Here we show the details for proving that when $b_j = \overline{b}$ for all countries in the union, it is also optimal to reach the zero-inflation set at a finite time τ. The argument is similar to that for the case with $b_j < \overline{b}$ for at least one country j that we discussed in the body. Here we construct a feasible but possibly suboptimal policy that the central authority can follow that strictly improves welfare for all the countries in the union. Now, however, we assume country j follows the rule $(c_j, \bar{l})$, with $c_j = c^* - \varepsilon$, for ε small and positive as long as debt is in the high-inflation region. Under this rule, the debt of country j, b_{jt}, monotonically falls towards $\underline{b}$. Now, regardless of what all other countries do, their debt is at most $\overline{b}$, hence an upper bound for the mean debt along this path is

$$\bar{b}_{mt} = \frac{1}{I} b_{jt} + \frac{(I-1)}{I} \overline{b}. \tag{64}$$

Since $\dot{b}_{jt} < 0$ and b_{jt} falls monotonically towards $\underline{b}$, this upper bound falls as b_{jt} falls to

$$\bar{b}_{m} = \frac{1}{I} \underline{b} + \frac{(I-1)}{I} \overline{b}, \tag{65}$$

and reaches it in some finite time $\tau^*(\varepsilon)$ where from (29)

$$e^{-\rho \tau^*(\varepsilon)} = \frac{\bar{l} - (c^* - \varepsilon) - \rho \overline{b}}{\bar{l} - (c^* - \varepsilon) - \rho \underline{b}} \tag{66}$$

Since $I \leq \bar{I}$, by Assumption 2 which implies that $\underline{b}/\bar{I} + (I-1)\overline{b}/\bar{I} < \psi$, the upper bound for the average debt of the union, $\bar{b}_{mt}$, eventually falls below ψ at some time $\tau < \tau^*(\varepsilon)$, at which point inflation falls to zero.

We argue that there exists $\varepsilon > 0$ small enough such that our lower bound on the value of country j under the plan $(c^* - \varepsilon, \bar{l})$ is higher than it is under the plan $(c^*, \bar{l})$ and we will conclude that it is optimal to reach the zero inflation set in finite time in this case. Here a lower bound for country j's utility is . The lower bound

$$\rho V_j(\boldsymbol{b}, \varepsilon) = \left(u(c^* - \varepsilon) - \bar{l} - \psi \overline{\pi}\right)\left(1 - e^{-\rho \tau^*(\varepsilon)}\right) + e^{-\rho \tau^*(\varepsilon)} \left[u(c^*) - \left(c^* + \rho b_j^*(\varepsilon)\right)\right] \tag{67}$$

where using (28) we have

$$\rho\bar{b} = (1 - e^{-\rho\tau^*(\varepsilon)})(\bar{l} - c^* + \varepsilon) + \rho e^{-\rho\tau^*(\varepsilon)} b_j^*(\varepsilon).$$

Substituting for $\rho e^{-\rho\tau} b_j^*(\varepsilon)$ from (**??**) into (67) gives

$$\begin{aligned}\rho V_j(\boldsymbol{b}, \varepsilon) = &\left(u(c^* - \varepsilon) - \bar{l} - \psi\overline{\pi}\right)\left(1 - e^{-\rho\tau^*(\varepsilon)}\right) + e^{-\rho\tau^*(\varepsilon)}\left[u(c^*) - c^*\right] \\ &- \left[\rho\bar{b} - (1 - e^{-\rho\tau^*(\varepsilon)})(\bar{l} - c^* + \varepsilon)\right] \\ = &\left(u(c^* - \varepsilon) + \varepsilon - \psi\overline{\pi}\right)\left(1 - e^{-\rho\tau^*(\varepsilon)}\right) + e^{-\rho\tau^*(\varepsilon)} u(c^*) - (c^* + \rho\bar{b})\end{aligned}$$

Differentiating with respect to ε yields

$$\rho\frac{\partial V_j(\boldsymbol{b}, \varepsilon)}{\partial\varepsilon} = \left[1 - u'(c^* - \boldsymbol{\varepsilon})\right]\left(1 - e^{-\rho\tau^*(\varepsilon)}\right) - \left[u(c^* - \varepsilon) + \varepsilon - \psi\overline{\pi} - u(c^*)\right]\frac{\partial(e^{-\rho\tau^*(\varepsilon)})}{\partial\varepsilon}.$$

Then, taking the limit as ε converges to 0 and using that $u'(c^*) = 1$, yields

$$\rho\lim_{\varepsilon\to 0}\frac{\partial V_j(\boldsymbol{b}, \varepsilon)}{\partial\varepsilon} = \psi\overline{\pi}\lim_{\varepsilon\to 0}\frac{\partial(e^{-\rho\tau})}{\partial\varepsilon} \tag{68}$$

Next, we show that $\lim_{\varepsilon\to 0}\partial(e^{-\rho\tau})/\partial\varepsilon > 0$. To see why the inequality holds, differentiating (66) with respect to ε

$$\frac{\partial(e^{-\rho\tau^*(\varepsilon)})}{\partial\boldsymbol{\varepsilon}} = \frac{\rho\left(\bar{b} - \underline{b}\right)}{\left(\bar{l} - (c^* - \varepsilon) - \rho\underline{b}\right)^2} > 0$$

That is, a decrease in consumption of country j by ε, leads to a decrease in the time it takes to reach the zero-inflation region. Moreover, $\lim_{\varepsilon\to 0}\partial(e^{-\rho\tau})/\partial\varepsilon > 0$, and hence, from (68), $\rho\lim_{\varepsilon\to 0}\partial V_j(\boldsymbol{b}, \varepsilon)/\partial\varepsilon > 0$. This implies that $V_j(\boldsymbol{b}; \varepsilon)$ is strictly increasing for $\boldsymbol{\varepsilon} > 0$ small enough. So country j is better off under this plan and so are all countries $i \neq j$ because there is a welfare gain of having inflation stop for $t \geq \tau(\varepsilon)$. Therefore, for any number of countries $I \leq \bar{I}$ and any $\boldsymbol{b} \in A^C$, the centralized authority can strictly improve welfare for the union by following an austerity plan that leads inflation to stop in finite time. *Q.E.D.*

Details for Proposition 2

Here, we provide the details for the characterization of the solution for centralized and decentralized regimes and the derivation of (39)-(42) from the paper. Throughout this section we will use the following expressions for the time it takes to reach the zero-inflation region, τ, which follow from (29)

$$e^{-\rho\tau} = \frac{\bar{l} - c_m - \rho b_m}{\bar{l} - c_m - \rho\psi} \qquad \text{so} \qquad 1 - e^{-\rho\tau} = \frac{\rho(b_m - \psi)}{\bar{l} - c_m - \rho\psi} \tag{69}$$

and, their derivative with respect to consumption in country i

$$\frac{\partial e^{-\rho\tau}}{\partial c_i} = -\frac{1}{I}\frac{\rho\left(b_m - \psi\right)}{\left(\bar{l} - c_m - \rho\psi\right)^2} \qquad \text{and} \qquad \frac{\partial e^{\rho\tau}}{\partial c_i} = \frac{1}{I}\frac{\rho\left(b_m - \psi\right)}{\left(\bar{l} - c_m - \rho b_m\right)^2} \tag{70}$$

Centralized problem

The problem of the central fiscal authority is

$$J^C(b_m) = \max_{(c_i)} \frac{1}{I}\sum_i \left\{ \int_{t=0}^{\tau} e^{-\rho t}[u(c_i) - \bar{l} - \psi\bar{\pi}]dt + \int_{t=\tau}^{\infty} e^{-\rho t}[u(c^*) - (c^* + \rho \underline{b}_i)] \right\}$$

which can be written as

$$I\rho J^C(b_m) = \max_{(c_i)} \sum_i \left\{ (1 - e^{-\rho\tau})[u(c_i) - \bar{l} - \psi\bar{\pi}] + e^{-\rho\tau}[u(c^*) - (c^* + \rho b_i^*)] \right\}$$

Taking FOC with respect to c_i

$$(1-e^{-\rho\tau})u'(c_i) - \rho e^{-\rho\tau}\frac{\partial\tau}{\partial c_i}\sum_j \left\{ [u(c^*) - (c^* + \rho b_j^*)] - [u(c_j) - \bar{l} - \psi\bar{\pi}] \right\} - e^{-\rho\tau}\rho\sum_j \frac{\partial b_j^*}{\partial c_i} = 0 \tag{71}$$

Let

$$\Delta U(c_i, b_i^*) \equiv [u(c^*) - (c^* + \rho b_i^*)] - [u(c_i) - \bar{l} - \psi\bar{\pi}] \tag{72}$$

so that (71) is

$$(1-e^{-\rho\tau})u'(c_i) = \rho e^{-\rho\tau}\frac{\partial\tau}{\partial c_i}\sum_j \Delta U(c_j, b_j^*) + \rho e^{-\rho\tau}\sum_j \frac{\partial b_j^*}{\partial c_i} = 0$$

Then, using symmetry in consumption, that is $c_i = c$ for all i, and $\sum_j b_j^* = I\psi$, we have that $\sum_j \Delta U(c_j, b_j^*) = I\Delta U(c,\psi)$. Together with $\sum_j \frac{\partial b_j^*}{\partial c_i} = 0$, it implies that (71) becomes

$$(1-e^{-\rho\tau})u'(c) = \rho e^{-\rho\tau}\frac{\partial\tau}{\partial c_j} I\Delta U(c,\psi).$$

Now, to fully characterize the solution c_*, use from (70) that

$$-\frac{\partial e^{-\rho\tau}}{\partial c_i} = \rho e^{-\rho\tau}\frac{\partial\tau}{\partial c_j} = \frac{1}{I}\frac{\rho\left(b_m-\psi\right)}{\left(\bar{l}-c-\rho\psi\right)^2}$$

and (69), to get,

$$u'(c)\left(\bar{l}-c-\rho\psi\right) = \Delta U(c,\psi).$$

Decentralized problem

The problem of the decentralized fiscal authority is

$$V_i(b_m) = \max_{c_i}\left\{\int_{t=0}^{\tau} e^{-\rho t}[u(c_i)-\bar{l}-\psi\bar{\pi}]dt + \int_{t=\tau}^{\infty} e^{-\rho t}[u(c^*)-(c^*+\rho b_i^*)]\right\}$$

which can be written as

$$\rho V_i(b_m) = \max_{c_i}\left\{(1-e^{-\rho\tau})[u(c_i)-\bar{l}-\psi\bar{\pi}] + e^{-\rho\tau}[u(c^*)-(c^*+\rho b_i^*)]\right\}$$

Taking FOC with respect to c_i

$$(1-e^{-\rho\tau})u'(c_i) - \rho e^{-\rho\tau}\frac{\partial\tau}{\partial c_i}\left\{[u(c^*)-(c^*+\rho b_i^*)] - [u(c_i)-\phi\bar{l}-\psi\bar{\pi}]\right\} - e^{-\rho\tau}\rho\frac{\partial b_i^*}{\partial c_i} = 0 \quad (73)$$

Using the definition in (72) for ΔU, we can write it as [this is the intuitive foc as in paper]

$$(1-e^{-\rho\tau})u'(c_i) - \rho e^{-\rho\tau}\frac{\partial \tau}{\partial c_i}\Delta U(c_i, b_i^*) - e^{-\rho\tau}\rho\frac{\partial b_i^*}{\partial c_i} = 0$$

To fully characterize the solution $c_*^D(I)$, first note that

$$\rho b_i^* = \bar{l} - c_i - e^{\rho\tau}(\bar{l} - c_i - \rho b_i) \tag{74}$$

so, taking the derivative with respect to c_i we get,

$$\rho\frac{\partial b_i^*}{\partial c_i} = -\frac{\partial e^{\rho\tau}}{\partial c_i}(\bar{l} - c_i - \rho b_i) + e^{\rho\tau} - 1.$$

Substituting it in previous expression

$$(1-e^{-\rho\tau})u'(c_i) + \frac{\partial e^{-\rho\tau}}{\partial c_i}\Delta U(c_i, b_i^*) + e^{-\rho\tau}\frac{\partial e^{\rho\tau}}{\partial c_i}(\bar{l} - c_i - \rho b_i) - \left(1 - e^{-\rho\tau}\right) = 0$$

and using (69) and (70)

$$\begin{aligned}\frac{\rho(b_m - \psi)}{\bar{l} - c_m - \rho\psi}u'(c_i) - \frac{1}{I}\frac{\rho\left(b_m - \psi\right)}{\left(\bar{l} - c_m - \rho\psi\right)^2}\Delta U(c_i, b_i^*) \\ + \frac{\bar{l} - c_m - \rho b_m}{\bar{l} - c_m - \rho\psi}\frac{1}{I}\frac{\rho\left(b_m - \psi\right)}{\left(\bar{l} - c_m - \rho b_m\right)^2}(\bar{l} - c_i - \rho b_i) - \frac{\rho(b_m - \psi)}{\bar{l} - c_m - \rho\psi} = 0\end{aligned}$$

which implies

$$u'(c_i) - \frac{1}{I}\frac{\Delta U(c_i, b_i^*)}{\left(\bar{l} - c_m - \rho\psi\right)} + \frac{1}{I}\frac{(\bar{l} - c_i - \rho b_i)}{\left(\bar{l} - c_m - \rho b_m\right)} - 1 = 0$$

Next, using (74) together with (69) we have that

$$(\bar{l} - c_i - \rho b_i) = \frac{\bar{l} - c_m - \rho b_m}{\bar{l} - c_m - \rho\psi}\left(\bar{l} - c_i + \rho b_i^*\right)$$

or,

$$\frac{\bar{l} - c_i - \rho b_i}{\bar{l} - c_m - \rho b_m} = \frac{\bar{l} - c_i + \rho b_i^*}{\bar{l} - c_m - \rho\psi}$$

which substituted in the previous expression yields

$$u'(c_i) - \frac{1}{I}\frac{\Delta U(c_i, b_i^*)}{\left(\bar{l} - c_m - \rho\psi\right)} + \frac{1}{I}\frac{\bar{l} - c_i - \rho b_i^*}{\bar{l} - c_m - \rho\psi} - 1 = 0$$

$$u'(c_i)\left(\bar{l} - c_m - \rho\psi\right) - \frac{1}{I}\Delta U(c_i, b_i^*) + \frac{1}{I}\left(\bar{l} - c_i - \rho b_i^*\right) - 1 = 0.$$

Finally, we use that $\Delta U(c_i, b_i^*) = \Delta U(c_i, \psi) - \rho(b_i^* - \psi)$, where ΔU is defined by (72), to get the final expression

$$Iu'(c_i)\left(\bar{l} - c_m - \rho\psi\right) - \Delta U(c_i, \psi) - (I-1)\left(\bar{l} - c_m - \rho\psi\right) = 0$$