Uncertainty and Disagreement: Theory and Macroeconomic Applications*

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Abstract

Disagreement across forecasters is widely used as a proxy for forecast uncertainty, but the mapping between these objects is poorly understood. We develop a general theorem linking disagreement, subjective uncertainty, and realized forecast errors in common-prior Bayesian learning environments, and use it to construct two novel diagnostic statistics: the disagreement/uncertainty ratio and disagreement about uncertainty (the dispersion of posterior variances). We show that expected disagreement is always bounded above by expected posterior variance and that whenever disagreement exceeds uncertainty conditionally, subjective uncertainty understates average squared forecast errors. Specializing the framework to canonical noisy- and sticky-information models yields closed-form expressions for these metrics and reveals a sharp theoretical contrast: in Gaussian noisy-information models, disagreement is bounded by uncertainty period by period, so the ratio is tightly constrained to be at most one, whereas sticky-information models lack such pointwise bounds and generically imply substantial disagreement about uncertainty. We then apply these diagnostics to point and density forecasts for inflation and unemployment from the U.S. Survey of Professional Forecasters. Empirically, the disagreement/uncertainty ratio is almost always below one date by date, with only modest, short-lived violations in high-volatility episodes, and cross-sectional disagreement about uncertainty is extremely small. These facts are difficult to reconcile with sticky-information models but are qualitatively more consistent with noisy-information benchmarks.

Keywords: Uncertainty, disagreement, information frictions, expectations, survey forecasts
JEL Classification: D82, D84, E17, E37

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1 Introduction

In a 2003 speech, Alan Greenspan remarked that "uncertainty is not just an important feature of the monetary policy landscape; it is the defining characteristic of that landscape." [1] It is perhaps unsurprising, thus, that the measurement of economic uncertainty has been a prominent theme in contemporary macroeconomic research and policy debates. However, there is still considerable contention about how to define and measure uncertainty. In principle, survey data capturing the beliefs of economic agents should provide the most direct measure of subjective expectations and perceived uncertainty. However, many macroeconomic surveys only require respondents to provide point predictions, rather than reporting full subjective probability distributions for different macroeconomic variables. In such scenarios, it is often important to develop indirect measures of forecast uncertainty, and a measure commonly used in empirical practice is the cross-sectional dispersion, or disagreement, of point forecasts across survey respondents. [2]

However, there is relatively little theoretical foundation for the use of disagreement as a proxy for uncertainty. As Zarnowitz and Lambros (1987) suggest, the relationship between uncertainty and disagreement can be conceptualized within a two-by-two matrix. Intuitively, low uncertainty and low disagreement jointly occur when forecasters have precise signals and similar information sets. High uncertainty and low disagreement occur together when forecasters have similar information sets, but their signals are relatively uninformative. High uncertainty and high disagreement can occur either when forecasters possess diffuse information sets or when their priors are sufficiently heterogeneous. The most atypical case is that of high disagreement but low uncertainty, which would typically be inconsistent with the common prior doctrine, under which differences in forecasts solely occur due to differences

[1] "Monetary Policy under Uncertainty" (August 29, 2003).

[2] Disagreement is also commonly used as an empirical proxy for uncertainty in other disciplines, including accounting (e.g. Barron and Stuerke, 1998; Barron et al., 1998); finance (e.g. Diether et al., 2002; Zhang, 2006) and operations management (e.g. Gaur et al., 2007). This literature has provided support for disagreement as a proxy for uncertainty by establishing empirical relationships between forecast errors and dispersion, but has devoted comparatively less attention to the theoretical foundations of this relationship.

in information.

Formalizing these themes, this paper investigates, both theoretically and empirically, the relationship between disagreement and uncertainty. We begin by providing rigorous definitions for disagreement and uncertainty for arbitrary informational structures and show in our main theorem that with a common prior and rational expectations, expected disagreement cannot exceed expected uncertainty. The projection arguments employed in the proof also illustrate that the ratio of expected disagreement to expected uncertainty is a quantity that is highly amenable to both theoretical and empirical analysis. We show that the ratio of expected disagreement to expected uncertainty is generally decreasing in the degree of informational frictions faced by forecasters and that this ratio is a powerful statistic in discriminating between competing models of informational frictions. While the main theorem captures the unconditional relationship between disagreement and uncertainty, it does not rule out the possibility of disagreement exceeding uncertainty in some states. We show, however, that when disagreement exceeds uncertainty conditionally, uncertainty must underestimate the average realized forecast error. We also illustrate that the bound holds pointwise when uncertainty is signal-invariant (that is, when there is no "disagreement about uncertainty" across forecasters).

Next, to gain a richer understanding of the conditional dynamics of uncertainty and disagreement, we focus on two canonical models of informational frictions that are widely used in macroeconomics, namely, the noisy information model of Woodford (2003) and Sims (2003) and the sticky information model of Mankiw and Reis (2002). We provide novel analytical characterizations of disagreement, uncertainty and the disagreement/uncertainty ratio under noisy and sticky information in both static and dynamic environments, deriving a rich array of comparative statics with respect to the model primitives. We build on the general theoretical results developed in the previous section to illustrate the diagnostic power of the disagreement/uncertainty ratio in discriminating between the noisy and sticky information models.

Finally, we relate the theoretical findings to empirical data from the Survey of Professional Forecasters (SPF), one of the few major macroeconomic surveys that collects both point and density forecasts. Based on novel arguments pertaining to the dynamics of the disagreement/uncertainty ratio and heterogeneity in posterior uncertainty, we find that the noisy information model outperforms the sticky information model in fitting the data. In essence, the noisy information model is characterized by homogeneity across forecasters in posterior uncertainty and the disagreement/uncertainty ratio remaining pointwise below unity. Conversely, in the sticky information model, there is substantial cross-sectional heterogeneity in uncertainty (disagreement about uncertainty), which stems from the stochastic staggered updating that is emblematic of the model, leading to the possibility of disagreement exceeding uncertainty in some states of the world. In the empirical data, cross-sectional disagreement about uncertainty is very low, and the disagreement/uncertainty ratio seldom exceeds one. However, we also show that some modifications to the standard noisy information model, namely, stochastic volatility and irreducible uncertainty, are required. The irreducible uncertainty concept appears to be new to the literature, generalizing the noisy information model to the (admittedly more realistic) case where signals are generated based on a part of the true state, rather than the full state. Even with these modifications, we find that challenges remain in jointly explaining the spikes in the disagreement/uncertainty ratio during periods of heightened macroeconomic volatility in conjunction with the seasonal dynamics of disagreement and uncertainty for fixed-event forecasts. We suggest that this difficulty arises from the excess seasonal stickiness of uncertainty, which may imply that some relaxations of the common prior/rational expectations paradigm may be required.

This paper is closely linked to several strands of the literature. The first body of work relates to the development of metrics to measure macroeconomic uncertainty. A wide range of approaches have been proposed, including text-based methods (Baker et al., 2016), purely statistical measures (Jurado et al., 2015) and disagreement-based proxies, as in Bachmann et al. (2013) and an earlier draft of Baker et al. (2016). This paper provides a complementary

perspective by analytically characterizing the conditions under which disagreement is a good or poor proxy for uncertainty. The notion of irreducible uncertainty that we develop also has some important similarities to the fundamental uncertainty concept developed by Jurado et al. (2015).

Second, the paper naturally builds on prior theoretical work exploring the relationship between uncertainty and disagreement. While valuable theoretical contributions about the behavior of disagreement and uncertainty have been made, for example, by Patton and Timmermann (2010) and Lahiri and Sheng (2010), these papers have typically invoked restrictive assumptions about the statistical properties of forecasts or the informational regimes that forecasters face. For example, Lahiri and Sheng (2010) focus on the static Gaussian noisy information model, under which disagreement never exceeds uncertainty in any state of the world and uncertainty is signal and state-invariant. However, this is not true under other assumptions about the signal generating process. By allowing for a very general class of informational primitives, both in the presence and absence of dynamic state persistence, we obtain a richer array of theoretical results and challenge some of the conclusions highlighted in the previous literature. In particular, we bring a novel focus to the theoretical and empirical properties of the disagreement/uncertainty ratio and use these results to discriminate between competing macroeconomic models of imperfect information.

Third, the paper is situated within a rich body of literature that has leveraged survey data to study the dynamics of uncertainty and/or disagreement, including Mankiw et al. (2003), Giordani and Söderlind (2003), Lahiri and Liu (2006), D'Amico and Orphanides (2008), Lahiri and Sheng (2010), Rich et al. (2012), Abel et al. (2016), Andrade et al. (2016), and Gambetti et al. (2023). The literature has broadly found that disagreement and uncertainty are positively correlated, but there is comparatively much less consensus about the strength, robustness and temporal stability of this relationship. To this end, there is still a great deal of contention about how to interpret and theoretically model the dynamics of disagreement and uncertainty. For example, Patton and Timmermann (2010) suggest that the observed

characteristics of disagreement in the SPF cannot be rationalized without heterogeneity in priors or models, while Coibion and Gorodnichenko (2015) dispute this conclusion, arguing that "heterogeneity about long-run means is not playing a quantitatively significant role in accounting for the predictability of inflation forecast errors." In this paper, we do not take a strong positive stand on the "correct" way of modeling informational rigidities, but we invoke the common prior assumption throughout. In this sense, our results can be thought of as a valuable theoretical benchmark in the absence of the additional possibility that forecasters "agree to disagree".

Fourth, we also contribute to the literature on testable implications of models of informational frictions, including Mankiw and Reis (2002), Mankiw et al. (2003), Coibion and Gorodnichenko (2012), Coibion and Gorodnichenko (2015), Andrade et al. (2016), and Bordalo et al. (2020). There is considerable contention in the literature about the "right" model of informational frictions, with some authors, such as Coibion and Gorodnichenko (2012) advocating for the noisy information model, and others, such as Mankiw and Reis (2002) and Andrade et al. (2016) suggesting that the sticky information model or a hybrid of the two standard models provides a better fit to the data. Using novel arguments based on the theory of the disagreement/uncertainty ratio developed in this paper, we concur with the finding of Coibion and Gorodnichenko (2012) that the noisy information model outperforms the sticky information model empirically, but we also elucidate potential modifications to the standard noisy information model that improve its performance. These findings should certainly not be interpreted as a statement that there is no stickiness in information updating, but they are indicative of the standard noisy information model providing a very useful theoretical baseline.

Fifth, the findings are also connected to the literature assessing the consistency between point and density forecasts, including Diebold et al. (1997) and Del Negro et al. (2022). An important contribution of this paper is to illustrate a connection between the disagreement/uncertainty nexus and the rationality of expectations about forecast uncertainty.

2 Empirical Motivation

To motivate the theoretical analysis, we use data from the Survey of Professional Forecasters (SPF), which is a quarterly survey conducted by the Federal Reserve Bank of Philadelphia, where a panel of professional forecasters from Wall Street financial firms, banks, economic consulting firms and large corporations are tasked with providing forecasts of a range of macroeconomic variables at different horizons. A unique feature of the SPF is that participants are asked not only to provide point forecasts, but also subjective probability estimates for some of these variables, that is, their perceived probabilities of these variables landing in specific ranges at a given horizon. The SPF is one of the few major macroeconomic surveys that provide subjective probability distributions, making it an ideal choice for this analysis.

However, some data limitations must be acknowledged, in that probability estimates are only available for quite a small number of variables (real GDP growth, core CPI inflation, core PCE inflation and the unemployment rate). Moreover, the horizons and levels of aggregation at which the participants are required to provide probability estimates often do not coincide with those for which they report point forecasts, something that is particularly evident in the case of real GDP. To that end, we focus on the latter three variables, namely, core CPI inflation (CCPI), core PCE inflation (CPCE) and the unemployment rate (UNEMP).

For CCPI and CPCE, participants report their perceived probabilities of the fourth-quarter-over-fourth-quarter changes in these variables falling into ten different ranges, both for the current year and the next year. These ranges have remained constant for the entire history of this survey question. For UNEMP, the same style of question is asked for the average of the unemployment rate over the four quarters of a given year, but participants are tasked with reporting their probability estimates for four years (the current year and the next three years). The value ranges (bins), however, have changed over time, and three vintages of these bins exist in the data. For the purposes of this paper, we focus solely on current-year forecasts. It is also important to note that within a year, forecasters are tasked with producing *fixed-event* forecasts, in the sense that they are forecasting the

same quantity over all four quarters of a year. Some previous work, such as Giordani and Söderlind (2003), has only focused on data from a single quarter in each year, but we find that the seasonal dynamics of fixed-event forecasts provide additional evidence to discriminate between different models of imperfect information.

Given that survey participants do not report a full continuous probability density forecast (rather, we have discrete bins), it is important to come up with a consistent protocol to interpolate within bins to synthesize a conditional probability distribution. Here, we follow the common practice of interpolating a uniform distribution within each bin. In essence, we assume that each bin is drawn with the probability specified by the forecaster and that conditional on drawing a given bin, the variable is distributed uniformly. This approach is beneficial in that it enables us to tractably compute moments of the implied distribution, such as the mean and the variance. The uniform interpolation strategy is evidently not the only way to proceed, but is quite standard in the literature (see Zarnowitz and Lambros (1987) for a discussion). Indeed, with N bins and given bin probabilities $p_1, p_2, ..., p_N$, and left and right endpoints $L_1, R_1, L_2, R_2, ..., L_N, R_N$, the implied mean μ can be computed as $\sum_{i=1}^{N} \frac{1}{2}(R_i - L_i)p_i$ and the implied variance can be computed using the law of total variance. The conditional variance in each bin i is $V_i = \frac{1}{12}(R_i - L_i)^2$ and the conditional expectation in each bin is $E_i = \frac{1}{2}(R_i - L_i)$. We compute the implied variance (which corresponds to a forecaster's uncertainty) as $\mathrm{E}(V|.) + \mathrm{V}(E|.) = \sum_{i=1}^{N} V_i p_i + \sum_{i=1}^{N} E_i^2 p_i - \mu^2$. For each variable, we can then compute the average uncertainty across forecasters at time t, which we call U_t.

We also explore alternative interpolation protocols that are based on parametric assumptions about the entire probability distribution. The conditionally uniform interpolation method can exhibit poor performance when substantial probability mass is located in either the top or bottom bins, which are specified to be unbounded. For much of the sample, negligible mass is located in these bins, so we can just make the simplifying assumption that the top and bottom bins have the same width as all other bins. However, the treatment of these bins becomes problematic during periods such as the pandemic or the post-pandemic infla-

tion, during which the bins were not adjusted adequately to handle the extreme outcomes that were observed. Two common parametric distributions that are used for subjective probability forecasts are the generalized beta distribution, as developed by Engelberg et al. (2009), and the normal distribution. While the procedure of Engelberg et al. (2009) is widely used in the recent literature, it cannot be used in cases where only two bins have positive probability mass, which makes it unsuited to handle the aforementioned issues with inadequately granular top and bottom bins. Therefore, we opt to fit a normal distribution, which has the dual benefit of leaving only a single free standard deviation parameter to estimate (since the mean can be pre-specified as a forecaster's point forecast) as well as coinciding with the theoretical assumptions made later in the paper when investigating canonical noisy and sticky information models in macroeconomics. We choose the standard deviation to minimize a loss function given by the sum of the squared deviations between the reported and distribution-implied probabilities over all bins.

It is comparatively straightforward to compute disagreement, which we obtain by taking the cross-sectional variance of point predictions for a particular variable at a given time t across forecasters. We now show some graphs and summary statistics that capture the dynamics and relationships between uncertainty and disagreement for the variables of interest.

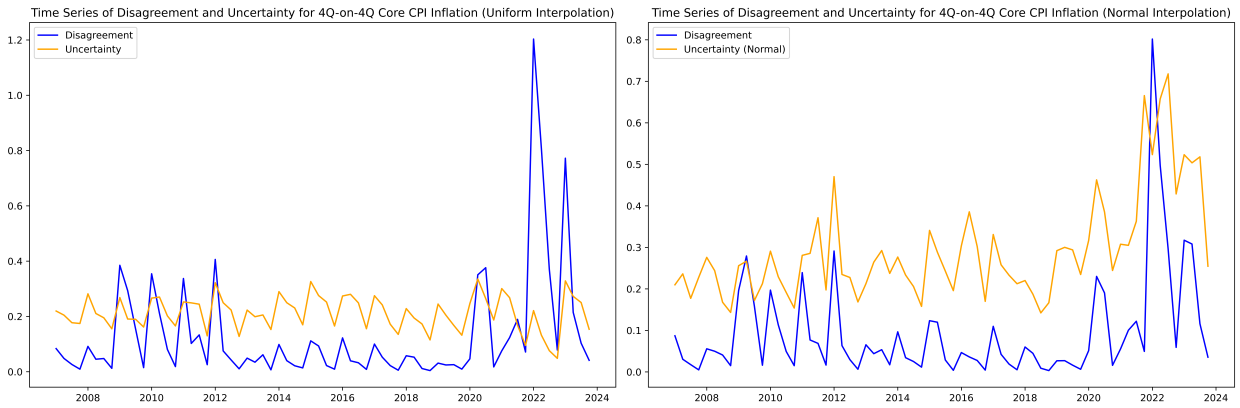


Figure 1: Dynamics of uncertainty and disagreement for core CPI inflation forecasts in the SPF

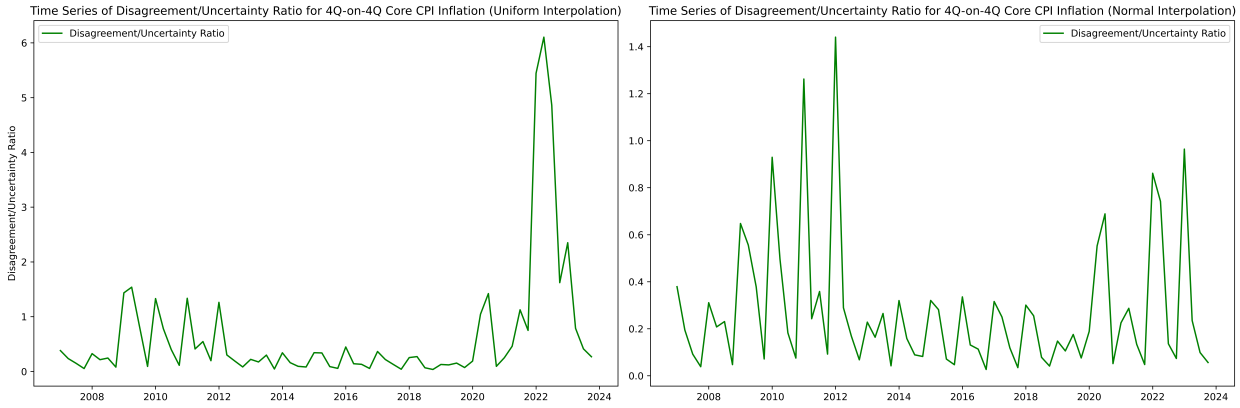


Figure 2: Dynamics of disagreement/uncertainty ratio for core CPI inflation forecasts in the SPF

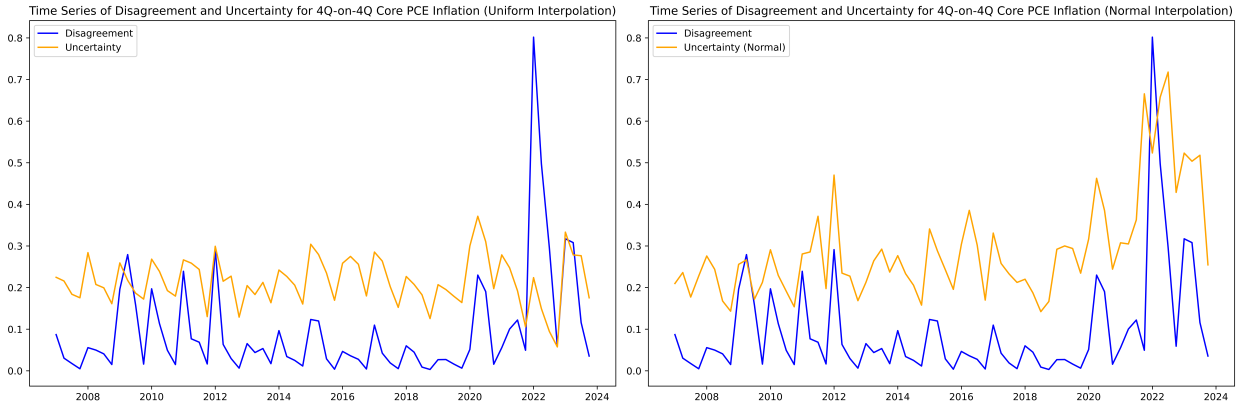


Figure 3: Dynamics of uncertainty and disagreement for core PCE inflation forecasts

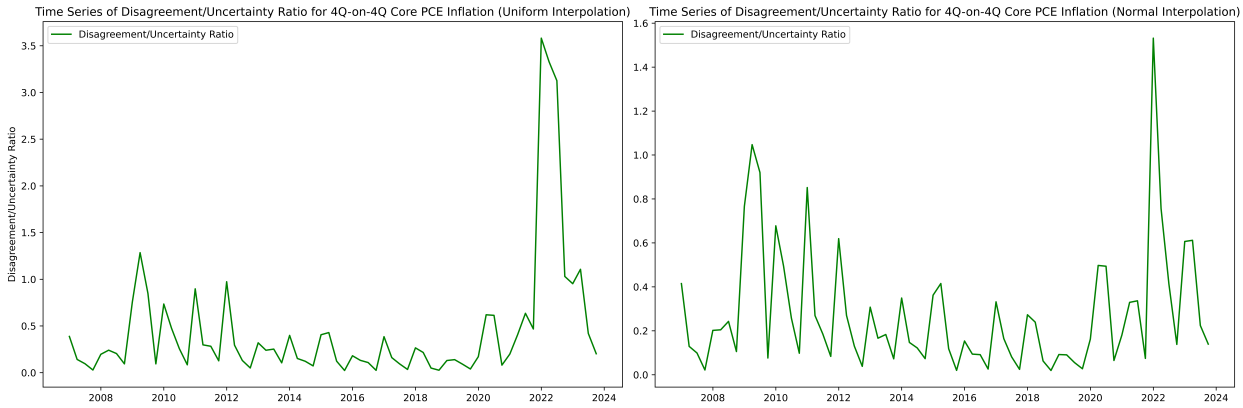


Figure 4: Dynamics of disagreement/uncertainty ratio for core PCE inflation forecasts

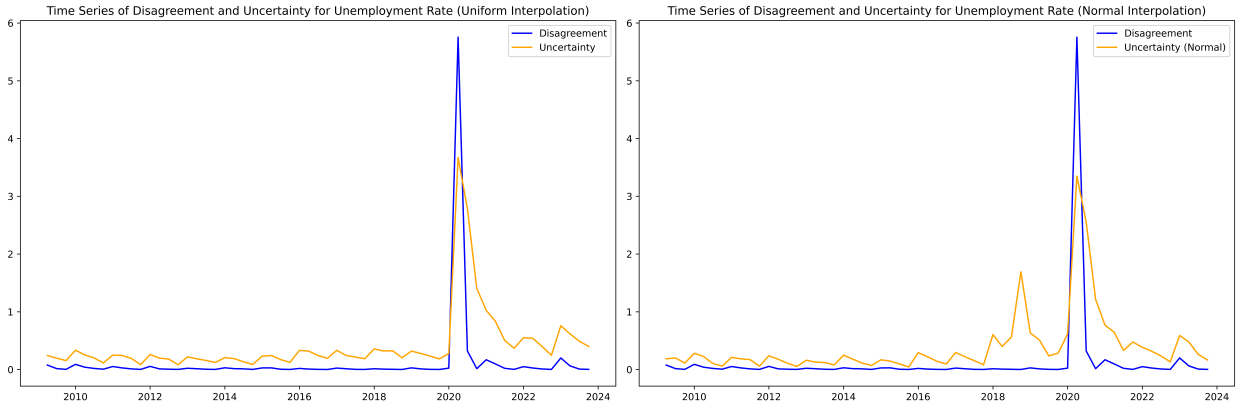


Figure 5: Dynamics of uncertainty and disagreement for unemployment rate forecasts

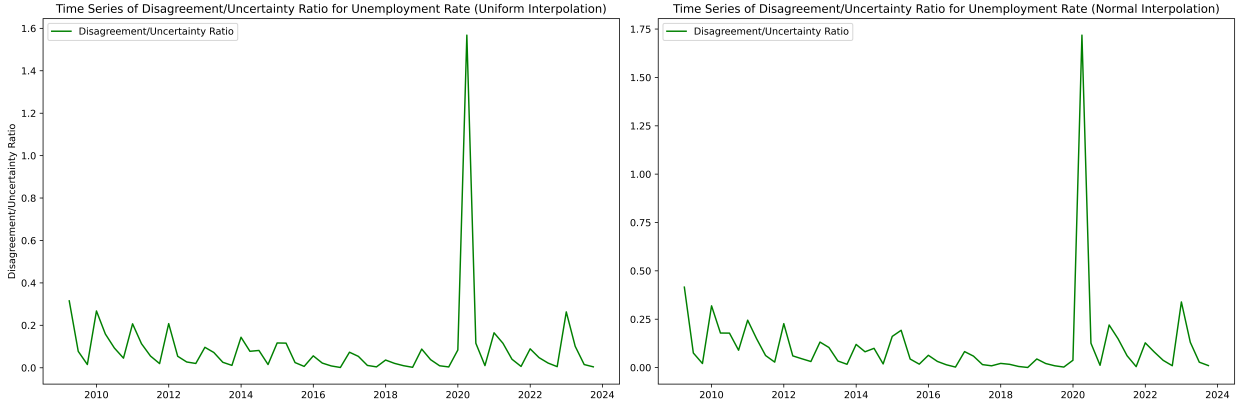


Figure 6: Dynamics of disagreement/uncertainty ratio for unemployment rate forecasts

Barring the rather atypical COVID-19 period, some general observations from Figures 1 to 6 are that the disagreement/uncertainty ratio is generally less than unity, and even when it occasionally surpasses one (which typically appears to occur during periods of heightened macroeconomic volatility, such as the Great Recession or the COVID-19 pandemic), it does not do so by a large margin. The time series under the two interpolation methods are also very similar, except during the pandemic (indeed, the extreme effect of the COVID inflation on the disagreement/uncertainty ratio is sharply attenuated when the normal interpolation method is used instead of the uniform method). Moreover, disagreement and uncertainty exhibit quite strong comovement (the pre-COVID correlation is 0.55 for core CPI inflation, 0.60 for core PCE inflation, and 0.44 for unemployment), findings that are in line with those

of previous authors, such as Zarnowitz and Lambros (1987), Giordani and Söderlind (2003) and Rich et al. (2012). There is also a striking seasonal pattern in uncertainty, disagreement and the uncertainty/disagreement ratio, which is natural given that the forecasters are tasked with making fixed-event forecasts over the four quarters of the year. Indeed, all of these three quantities fall between Q1 and Q4 (that is, disagreement falls much more in proportional terms than uncertainty). Motivated by these empirical observations, we will derive theoretical properties of uncertainty and disagreement and subsequently explore how well standard theoretical models of informational frictions in macroeconomics can rationalize these findings.

3 Theoretical Preliminaries

The simplest setting that captures the theoretical ideas in this paper is a static model, where the true state is given by the random variable X and the observed stochastic signal is given by the random variable Y. Consider an infinite continuum of forecasters who draw signals independently according to this probability distribution. A forecaster's uncertainty is simply the posterior variance of the state given the signal, that is, $\mathbb{V}(X|Y)$, and expected uncertainty is thus given by $\mathbb{E}[\mathbb{V}(X|Y)]$. Disagreement (which is synonymous with dispersion for the purposes of this paper), on the other hand, corresponds to the cross-sectional variance of forecasters' point forecasts conditional on the true state, that is, $\mathbb{V}[\mathbb{E}(X|Y)|X]$. Hence, expected disagreement is given by $\mathbb{E}[\mathbb{V}(\mathbb{E}(X|Y)|X)]$.

We begin to examine the relationship between these quantities by deriving a novel variance decomposition that captures the contributions of disagreement and uncertainty to the total variance of the true state variable. We first obtain from the law of total variance that

$$\mathbb{V}(X) = \underbrace{\mathbb{E}[\mathbb{V}(X|Y)]}_{\text{expected uncertainty}} + \mathbb{V}[\mathbb{E}(X|Y)]$$

The important point to note here is that the second term is *not* a suitable measure of disagreement, since the variance is unconditional. To see this, consider, for example, the scenario in which forecasters observe the true state noiselessly but there is variability in the true state. In this scenario, there would clearly be no disagreement across forecasters, but the second term in the variance decomposition would be positive. Instead, we will proceed by decomposing the second term with another application of the law of total variance, conditioning on the true state:

$$\mathbb{V}(X) = \underbrace{\mathbb{E}\left[\mathbb{V}(X|Y)\right]}_{\text{expected uncertainty}} + \underbrace{\mathbb{E}\left[\mathbb{V}(\mathbb{E}(X|Y)|X)\right]}_{\text{expected disagreement}} + \underbrace{\mathbb{V}\left[\mathbb{E}(\mathbb{E}(X|Y)|X)\right]}_{\text{forecast variability}}$$

In essence, the conditioning on the true state sequence is necessary to avoid misattributing unconditional movement in the forecasts as disagreement. While this simple setup encompasses many standard models, including the one-period noisy and sticky information models in macroeconomics, it is not well suited to handle more complex signal structures. For example, forecasters might form expectations based on the full history of signals observed thus far (as in the dynamic noisy information model), or might be stochastically stuck with stale information (as in the dynamic sticky information model). To handle a far more general class of informational structures, we proceed as follows.

Let $X \in L^2$ be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and suppose $\mathcal{G}$ and $\mathcal{H}$ are σ-algebras such that $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$ and $\sigma(X) \subseteq \mathcal{G}$. Here, $\mathcal{G}$ represents shared/predetermined information across forecasters (even though they may not know this information), while $\mathcal{H}$ represents forecasters' private information. We make the natural assumption that $\sigma(X) \subseteq \mathcal{G}$ (in essence, the true state is "stochastically predetermined" prior to the generation of private information sets).

In the subsequent sections, we now proceed to analyze the relative magnitudes of uncertainty and disagreement under different informational regimes. In particular, in light of the empirical evidence, we explore whether it is possible for disagreement to be greater than

uncertainty, as well as what conditions lead to temporal and state-dependent variation in these quantities.

4 Main Theorem

In this section, we present the key result of this paper, a theorem that states that expected disagreement is always less than or equal to expected uncertainty when forecasters have a common prior and form rational forecasts based on private signals.

Theorem 1. *Given an L^2-integrable random variable X and σ-algebras $\mathcal{G}$ and $\mathcal{H}$ on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if $\sigma(X) \subseteq \mathcal{G}$, we have that $\mathbb{E}[\mathbb{V}[\mathbb{E}(X|\mathcal{H})|\mathcal{G}]] \leq \mathbb{E}[\mathbb{V}(X|\mathcal{H})]$.*

This theorem provides a rigorous formulation of the previously discussed concepts of uncertainty and disagreement using the language of σ-algebras. To prove the result, we begin by proving an auxiliary lemma.

Lemma 1. *For any L^2-integrable random variable X and σ-algebra $\mathcal{G}$ on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{V}[X - \mathbb{E}(X|\mathcal{G})] = \mathbb{E}[\mathbb{V}(X|\mathcal{G})]$. Furthermore, $\mathbb{V}[X - \mathbb{E}(X|\mathcal{G})] \leq \mathbb{V}(X)$.*

Proof. Let $U = X - \mathbb{E}(X|\mathcal{G})$. For the first part of the result, we proceed as follows:

$$\begin{aligned}
\mathbb{V}(U) &= \mathbb{E}[\mathbb{V}(U|\mathcal{G})] + \mathbb{V}[\mathbb{E}(U|\mathcal{G})] \text{ (by the law of total variance)} \\
&= \mathbb{E}[V(X|\mathcal{G})] + 0 \\
&= \mathbb{E}[V(X|\mathcal{G})]
\end{aligned}$$

Now, for the second part, we have that:

$$\begin{aligned}
\mathbb{V}(X) &= \mathbb{E}[\mathbb{V}(X|\mathcal{G})] + \mathbb{V}[\mathbb{E}(X|\mathcal{G})] \text{ (by the law of total variance)} \\
&= \mathbb{V}(U) + \mathbb{V}[\mathbb{E}(X|\mathcal{G})] \text{ (by the first part of the lemma)} \\
&\geq \mathbb{V}(U)
\end{aligned}$$

$$= \mathbb{V}[X - \mathbb{E}(X|\mathcal{G})]$$

□

We now proceed to the proof of the main theorem:

Proof.

$$\begin{aligned}
\mathbb{E}[\mathbb{V}[\mathbb{E}(X|\mathcal{H})|\mathcal{G}]] &= \mathbb{V}\left[\mathbb{E}(X|\mathcal{H}) - \mathbb{E}[\mathbb{E}(X|\mathcal{H})|\mathcal{G}]\right] \text{ (by the first part of Lemma 1)} \\
&= \mathbb{V}\left[X - \mathbb{E}(X|\mathcal{H}) - \mathbb{E}[X - \mathbb{E}(X|\mathcal{H})|\mathcal{G}]\right] \text{ (since } \sigma(X) \subseteq \mathcal{G}) \\
&\leq \mathbb{V}[X - \mathbb{E}(X|\mathcal{H})] \text{ (by the second part of Lemma 1)} \\
&= \mathbb{E}[\mathbb{V}(X|\mathcal{H})] \text{ (by another application of the first part of Lemma 1)}
\end{aligned}$$

□

A useful (but admittedly incomplete) visual representation of the ideas in the proof is shown below in Figure 7.

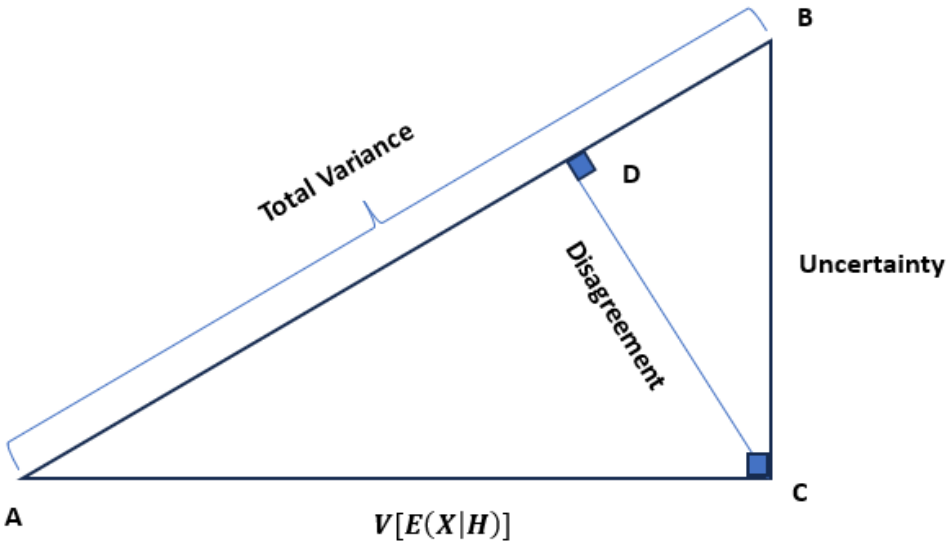


Figure 7: Geometric visualization of proof of main theorem

In essence, line segment CD is the common orthogonal rejection of $\mathbb{E}(X|\mathcal{H})$ and $X - \mathbb{E}(X|\mathcal{H})$ on $\mathcal{G}$, which enables us to prove the desired result. However, it is not necessarily true in the general case that the projections on $\mathcal{G}$ are collinear, as suggested by the figure (this appears to be only true when we restrict attention to univariate linear projections). Nonetheless, the visualization offers much intuition into the nature of the proof strategy.

Indeed, a further implication of the figure is that triangles BDC and BCA are similar, which implies that $\frac{D}{U} = \frac{V-U}{V} = 1 - \frac{U}{V}$, where D represents expected disagreement, U represents expected uncertainty and V represents the total variance of X. In other words, one might conjecture that the ratio of expected disagreement to expected uncertainty is decreasing in the degree of informational frictions that forecasters face. While this is of course far from a rigorous argument, it is interesting that this underlying intuition indeed turns out to hold in the standard macroeconomic models of noisy information and sticky information that we explore later in the paper. What can be shown rigorously, however, is that the right-hand side, $1 - \frac{U}{V}$, is an upper bound for the disagreement/uncertainty ratio.

Proposition 1. *Under the same conditions as in the main theorem,*

$$\frac{\mathbb{E}[\mathbb{V}[\mathbb{E}(X|\mathcal{H})|\mathcal{G}]]}{\mathbb{E}[\mathbb{V}(X|\mathcal{H})]} \leq 1 - \frac{\mathbb{E}[\mathbb{V}(X|\mathcal{H})]}{\mathbb{V}(X)}.$$

Proof. From the tower property of conditional expectation, we have that:

$$\mathbb{E}[X\,\mathbb{E}[X \mid \mathcal{H}]] = \mathbb{E}\big[\mathbb{E}[X\,\mathbb{E}[X \mid \mathcal{H}] \mid \mathcal{H}]\big] = \mathbb{E}\big[\mathbb{E}[X \mid \mathcal{H}]\,\mathbb{E}[X \mid \mathcal{H}]\big] = \mathbb{E}\big[\mathbb{E}[X \mid \mathcal{H}]^2\big],$$

and $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{H}]]$. Hence,

$$\mathrm{Cov}(X, \mathbb{E}[X \mid \mathcal{H}]) = \mathbb{E}[\mathbb{E}[X \mid \mathcal{H}]^2] - \mathbb{E}[\mathbb{E}[X \mid \mathcal{H}]]^2 = \mathbb{V}(\mathbb{E}[X \mid \mathcal{H}]).$$

By the law of total variance,

$$\mathbb{V}(\mathbb{E}[X \mid \mathcal{H}]) = \mathbb{E}[\mathbb{V}(\mathbb{E}[X \mid \mathcal{H}] \mid \mathcal{G})] + \mathbb{V}(\mathbb{E}[\mathbb{E}[X \mid \mathcal{H}] \mid \mathcal{G}]) \,.$$

Moreover, by the orthogonality property of conditional expectation and since $\sigma(X) \subseteq \mathcal{G}$,

$$\mathrm{Cov}\big(X, \mathbb{E}[X \mid \mathcal{H}] - \mathbb{E}[\mathbb{E}[X \mid \mathcal{H}] \mid \mathcal{G}]\big) = 0,$$

so $\mathrm{Cov}(X, \mathbb{E}[\mathbb{E}[X \mid \mathcal{H}] \mid \mathcal{G}]) = \mathrm{Cov}(X, \mathbb{E}[X \mid \mathcal{H}])$. The Cauchy-Schwarz inequality and the previously derived equations then yield:

$$\begin{aligned}
\mathbb{V}(\mathbb{E}[X|\mathcal{H}])^2 &= \mathrm{Cov}(X, \mathbb{E}[X|\mathcal{H}])^2 \\
&= \mathrm{Cov}(X, \mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}]])^2 \\
&\leq \mathbb{V}(X)\, \mathbb{V}(\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}]]) = \mathbb{V}(X)\, (\mathbb{V}(\mathbb{E}[X|\mathcal{H}]) - \mathbb{E}[\mathbb{V}(\mathbb{E}[X|\mathcal{H}]|\mathcal{G})]) \,,
\end{aligned}$$

Invoking the law of total variance again, we have that:

$$\mathbb{V}(X)\mathbb{E}[\mathbb{V}(\mathbb{E}[X|\mathcal{H}]|\mathcal{G})] \leq \mathbb{V}(\mathbb{E}[X|\mathcal{H}])(\mathbb{V}(X) - \mathbb{V}(\mathbb{E}[X|\mathcal{H}]) = (\mathbb{V}(X) - \mathbb{E}\left[\mathbb{V}(X|\mathcal{H})\right])\mathbb{E}\left[\mathbb{V}(X|\mathcal{H})\right],$$

from which the result immediately follows. Equality holds if and only if equality holds in the Cauchy-Schwarz inequality, that is, when $\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}]]$ is an affine function of X. $\square$

It is also valuable to consider which assumptions can and cannot be relaxed while still preserving the key result. For example, suppose that there are some public signals in addition to the existing private signals that forecasters observe. Then we replace the statement of Theorem 1 by $\mathbb{E}[\mathbb{V}[\mathbb{E}(X|\mathcal{G}^{\mathcal{C}})|\mathcal{F}^{\mathcal{C}}]] \leq \mathbb{E}[\mathbb{V}(X|\mathcal{G}^{\mathcal{C}})]$, where the superscript $\mathcal{C}$ denotes a refinement of the original σ-algebra with the additional information encoded by the public signals. Since $\sigma(X) \subseteq \mathcal{G} \subseteq \mathcal{G}^{\mathcal{C}}$, the theorem still holds in this case. Moreover, the theorem allows for heterogeneity in forecasters' signal distributions, since we are effectively consider-

ing the forecasters as an infinite population and taking cross-sectional averages in computing disagreement and average uncertainty.

Furthermore, the theorem also accounts for settings where there is *irreducible uncertainty*, that is, when even the union of forecasters' information sets is insufficient to identify the true state. For example, suppose forecasters obtained noisy signals about a latent state, where the true state is this latent state plus some independent noise. The σ-algebra $\mathcal{G}$ could encode both the latent and true states, preserving the validity of the theorem.

However, it is easy to see that the theorem need not hold if expectations are non-rational. If, say, forecasters always reported their noisy signal as their best guess, where this signal is the true state plus some random noise, then expected disagreement could certainly exceed expected uncertainty. In some sense, the problem is not even well-posed without imposing further structure on the informational environment, since we would need to define how the posterior uncertainty of forecasters should be computed.

One canonical model of non-rational expectations is the *diagnostic expectations* model developed by Bordalo et al. (2018), under which the perceived conditional distribution of the signal is the true signal distribution taken to some power γ. Denote the true probability measure by P and the diagnostic measure by Q^γ. When $\gamma > 1$, this corresponds to overconfidence or overprecision, while when $\gamma < 1$, this corresponds to underconfidence or underprecision. It is natural to posit that greater diagnosticity of expectations would raise disagreement and lower uncertainty. However, to formally prove this, we must put some parametric structure on the information model. In the literature, the diagnostic expectations model is conventionally used in conjunction with the assumption that both the state and the signal are Gaussian. Here, we can generalize this treatment somewhat to the broader class of conjugate single-parameter exponential family distributions.

Suppose that we have a univariate signal Y following an exponential family distribution with canonical parameter θ, where the true signal likelihood $p_\theta(y) = h(y)\exp\{y\theta - A(\theta)\}$ Then it is well known that the mean of the distribution is $\mu(\theta) := A'(\theta)$, and the variance is

$V(\mu(\theta)) := A''(\theta)$. We will let the state X be $\mu(\theta)$. Now, following Diaconis and Ylvisaker (1979), we assign a conjugate prior for θ, namely, $\pi(\theta) \propto \exp\{\kappa[m\theta - A(\theta)]\},\ \kappa > 0$. The diagnostic posterior for $\gamma \geq 1$ is then given by $\pi_\gamma(\theta \mid y) \propto \exp\{(\kappa m + \gamma y)\theta - (\kappa + \gamma)A(\theta)\}$. Define $t(\gamma) := \kappa + \gamma$, $m_\gamma(y) := \frac{\kappa m + \gamma y}{\kappa + \gamma} = m + \lambda_\gamma(y - m)$, and $\lambda_\gamma := \frac{\gamma}{\kappa+\gamma} \in [0, 1)$. Under mild integrability conditions, we also have the following standard results, for which short proofs are provided.

(F1) For $p_{t,m}(\theta) \propto e^{t[m\theta - A(\theta)]}$, $\mathbb{E}_{p_{t,m}}[\mu] = m$.

Proof: By the score identity, $0 = \mathbb{E}_{p_{t,m}}\big[\frac{d}{d\theta}\log p_{t,m}(\theta)\big] = \mathbb{E}_{p_{t,m}}[\,t(m - \mu(\theta))\,] = t\big(m - \mathbb{E}_{p_{t,m}}[\mu]\big)$

$\Rightarrow\ \mathbb{E}_{p_{t,m}}[\mu] = m.$

(F2) For $p_{t,m},\ \mathbb{E}_{p_{t,m}}[V(\mu)] = t\,\mathbb{V}_{p_{t,m}}(\mu)$.

Proof: $\displaystyle\int A''p = \big[\mu p\big]_{\partial\Theta} - \int \mu p' = -\int \mu\, t(m - \mu)p = t(\mathbb{E}[\mu^2] - m\mathbb{E}[\mu]) = t\ \mathrm{Var}(\mu).$ $\square$

Proposition 2. *Let $X := \mu(\theta)$ and define*

$$D_\gamma := \mathbb{E}^P\Big(\mathbb{V}^P\big(\,\mathbb{E}^{Q_\gamma}[X \mid Y]\,\big|\,X\big)\Big), \qquad D_1 := \mathbb{E}^P\Big(\mathbb{V}^P\big(\,\mathbb{E}^{P}[X \mid Y]\,\big|\,X\big)\Big).$$

Then for every $\gamma \geq 1$, $D_\gamma \geq D_1$, with strict inequality if $\mathbb{V}^P(Y) > 0$.

Proof. By (F1) with $(t, m) = (t(\gamma), m_\gamma(Y))$, $\mathbb{E}^{Q_\gamma}[\mu \mid Y] = m_\gamma(Y) = m + \lambda_\gamma(Y - m)$. Conditioning on $X = \mu$:

$$\mathbb{V}^P\big(\mathbb{E}^{Q_\gamma}[\mu \mid Y]\ \big|\ \mu\big) = \mathbb{V}^P\big(m + \lambda_\gamma(Y - m)\ \big|\ \mu\big) = \lambda_\gamma^2\,\mathbb{V}^P(Y \mid \mu) = \lambda_\gamma^2\,V(\mu).$$

Taking $\mathbb{E}^P$ in μ, $D_\gamma = \lambda_\gamma^2\,\mathbb{E}^P[V(\mu)]$. Likewise $D_1 = \lambda_1^2\,\mathbb{E}^P[V(\mu)]$ with $\lambda_1 = \frac{1}{\kappa+1}$. Since $\lambda_\gamma = \frac{\gamma}{\kappa+\gamma}$ has derivative $\frac{d}{d\gamma}\lambda_\gamma = \frac{\kappa}{(\kappa+\gamma)^2} > 0$, we have $\lambda_\gamma^2 \geq \lambda_1^2$. If $\mathbb{V}^P(Y) > 0$, then $\mathbb{E}^P[V(\mu)] > 0$ and the inequality is strict. $\square$

Proposition 3. *Assume, in addition, that the exponential family distribution exhibits a*

concave quadratic variance function (QVF); that is, $V(\mu) = a\mu^2 + b\mu + c$ *with* $a \le 0$*. We also assume that* $t(\gamma) - a = \kappa + \gamma - a > 0$ *to ensure the existence of the posterior variance. Let*

$$U_\gamma := \mathbb{E}^P\big(\mathbb{V}^{Q_\gamma}(\mu \mid Y)\big), \qquad U_1 := \mathbb{E}^P\big(\mathbb{V}^P(\mu \mid Y)\big).$$

Then for every $\gamma \ge 1$*,* $U_\gamma \le U_1$*, with strict inequality if* $a < 0$ *and* $\mathbb{V}^P(Y) > 0$*.*

Proof. From (F2) under Q_γ, $\mathbb{E}^{Q_\gamma}[V(\mu) \mid Y] = t(\gamma)\,\mathbb{V}^{Q_\gamma}(\mu \mid Y)$. Also, by the QVF property and (F1),

$$\begin{aligned}\mathbb{E}^{Q_\gamma}[V(\mu) \mid Y] &= a\,\mathbb{E}^{Q_\gamma}[\mu^2 \mid Y] + b\,\mathbb{E}^{Q_\gamma}[\mu \mid Y] + c \\ &= a\big(\mathbb{V}^{Q_\gamma}(\mu \mid Y) + m_\gamma(Y)^2\big) + b\,m_\gamma(Y) + c \\ &= a\,\mathbb{V}^{Q_\gamma}(\mu \mid Y) + V(m_\gamma(Y)).\end{aligned}$$

Equating gives

$$\big(t(\gamma) - a\big)\mathbb{V}^{Q_\gamma}(\mu \mid Y) = V(m_\gamma(Y)) \quad \Rightarrow \quad \mathrm{Var}^{Q_\gamma}(\mu \mid Y) = \frac{V(m_\gamma(Y))}{\kappa + \gamma - a}.$$

Taking $\mathbb{E}^P$ on both sides, we obtain:

$$U_\gamma = \frac{\mathbb{E}^P\big[V(m_\gamma(Y))\big]}{\kappa + \gamma - a}.$$

Let $Z := Y - m$ (so $\mathbb{E}^P[Z] = 0$) and define $g(\lambda) := \mathbb{E}^P[V(m + \lambda Z)]$. Then $\mathbb{E}^P[V(m_\gamma(Y))] = g(\lambda_\gamma)$ with $\lambda_\gamma = \frac{\gamma}{\kappa+\gamma}$. Because $V'' \equiv 2a \le 0$ and $\mathbb{E}^P[Z^2] = \mathbb{V}^P(Y) < \infty$,

$$g''(\lambda) = \mathbb{E}^P\big[Z^2 V''(m + \lambda Z)\big] = 2a\,\mathbb{V}^P(Y) \le 0, \quad g'(0) = \mathbb{E}^P[Z]\,V'(m) = 0,$$

so g is concave with $g'(0) = 0$ and hence nonincreasing on $[0, \infty)$. Since λ_γ increases in γ while $\kappa + \gamma - a$ increases strictly in γ, the ratio $U_\gamma = g(\lambda_\gamma)/(\kappa + \gamma - a)$ is nonincreasing in γ. If $a < 0$ and $\mathbb{V}^P(Y) > 0$, then g is strictly concave and λ_γ strictly increases, making the

numerator strictly decrease; the denominator strictly increases, so U_γ strictly decreases. $\square$

An additional point to note is that the main theorem only considers disagreement and uncertainty in expectation. It does not say that disagreement is always less than uncertainty for any given state of the world. Indeed, since they focused attention on the static noisy information model, Lahiri and Sheng (2010) suggested that this inequality should also be true pointwise. However, this only follows immediately for informational environments where disagreement and uncertainty are invariant with respect to the state of the world. In linear-Gaussian models, this condition essentially coincides with the absence of cross-sectional heterogeneity in forecast uncertainty, which is a very stringent restriction on the admissible class of informational models. A simple counterexample can be found in the model of sticky information that we will consider later in the paper, under which disagreement can exceed uncertainty when large shocks occur. Nevertheless, the theorem is suggestive that disagreement exceeding uncertainty is an atypical outcome, an intuition that we will formalize for some specific settings later in the paper.

However, it is possible to derive a conditional (or "pointwise") version of the theorem based on realized forecasts and forecast errors. If we let Z denote the random forecast (which could be rational or irrational), we can derive that:

$$\underbrace{\mathbb{E}[(X-Z)^2|\mathcal{G}]}_{\text{expected realized error}} = \underbrace{\mathbb{E}[(X-\mathbb{E}(Z|\mathcal{G}))^2|\mathcal{G}]}_{\text{consensus error}} + \underbrace{\mathbb{E}[(\mathbb{E}(Z|\mathcal{G})-Z)^2|\mathcal{G}]}_{\text{disagreement}}$$

In this result, the realized error always exceeds disagreement for any state of the world, which implies that if uncertainty is conditionally less than disagreement in a given state, then it must be an underestimate of the expected realized forecast error. This is not incompatible with rational expectations due to the informational frictions that forecasters face. For example, if forecasters updated their information in a staggered manner (as in the sticky information model) and a large shock hit in the current period, one could potentially observe very high disagreement but low uncertainty. However, based on the theorem, this must be

a sufficiently unlikely event, formalizing the intuition of Zarnowitz and Lambros (1987).

Naturally, to fully characterize the dynamics of disagreement and uncertainty, we require parametric assumptions on the stochastic processes governing the true state and the signals observed by forecasters. Therefore, we now proceed to investigate special cases of this theorem with macroeconomic applications, namely, variants of Woodford (2003)'s model of noisy information and Mankiw and Reis (2002)'s model of sticky information. With the specific assumptions of these models, we can derive other complementary comparative statics results and ascertain a broad suite of theoretical implications of these models for the dynamics of disagreement and uncertainty. We will then explore the relative ability of these two models to fit the empirical behavior of disagreement and uncertainty in the SPF data.

5 Macroeconomic Applications: Static Models

In this section, we analyze simple versions of standard macroeconomic models with informational frictions where there is no persistence in the state variable. Suppose that in each period t, the true state $X_t = \mu_X + \epsilon_t$, where $\epsilon_t \sim N(0, \sigma_\epsilon^2)$ and ϵ_t is independent and identically distributed over time. Without loss of generality, we may normalize $\mu_X = 0$. Under the first informational regime that we study, namely, *noisy information*, each forecaster observes a noisy signal of the true state, $Y_{it} = X_t + \nu_{it}$, where $\nu_{it} \sim N(0, \sigma_\nu^2)$ and is independent across forecasters and over time. Since we consider forecasters as an "infinite population", we may proceed to drop the i subscript, with the understanding that the realization is for an arbitrary forecaster.

We are interested in deriving expressions for uncertainty and disagreement, as defined in the previous section, in this environment. Applying standard results pertaining to the multivariate normal distribution and noting that forecasters need only condition on Y_t since there is no persistence in the state, we find that the conditional variance of X_t given Y_t, which we define by U_t (since it denotes forecast uncertainty) is $(1-\rho^2)\sigma_\epsilon^2$, where $\rho = Corr(X_t, Y_t) =$

$\frac{\sigma_\epsilon^2}{\sigma_\epsilon\sqrt{\sigma_\epsilon^2+\sigma_\nu^2}} = \frac{\sigma_\epsilon}{\sqrt{\sigma_\epsilon^2+\sigma_\nu^2}}$. Hence, uncertainty, which is the conditional variance, is given by

$$U_t = \frac{\sigma_\epsilon^2\sigma_\nu^2}{\sigma_\epsilon^2+\sigma_\nu^2}.$$

Note that if we fix the underlying state variance σ_ϵ^2, we find that as the noise variance σ_ν^2 tends to 0, the conditional variance also tends to 0. Moreover, uncertainty is increasing in σ_ν^2 and tends to σ_ϵ^2 as σ_ν^2 tends to ∞. The symmetric argument applies if we fix σ_ν^2 and vary σ_ϵ^2, in that uncertainty is increasing in σ_ϵ^2 and tends to σ_ν^2 as σ_ϵ^2 tends to ∞ . Another interesting but intuitive finding is that uncertainty is upper bounded by the maximum of the signal variance and the noise variance.

Now, let us examine disagreement across forecasters in their point forecasts, which, following the previous section, is defined as the variance of the posterior mean conditional on the true sequence of states $\{X_t\}$. Again, because of the absence of any persistence in the state, we need only condition on the current state X_t. Again using standard properties of the multivariate normal distribution, we find that the posterior mean is $\frac{\rho\sigma_\epsilon}{\sqrt{\sigma_\epsilon^2+\sigma_\nu^2}}Y_t = \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2+\sigma_\nu^2}Y_t = \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2+\sigma_\nu^2}(X_t+\nu_t)$. Conditional on X_t, the stochastic component is $\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2+\sigma_\nu^2}\nu_t$. Thus, the cross-sectional forecast disagreement is given by

$$D_t = \frac{\sigma_\epsilon^4\sigma_\nu^2}{(\sigma_\epsilon^2+\sigma_\nu^2)^2}.$$

Moreover, we have that $(\sigma_\epsilon^2-\sigma_\nu^2)^2 = (\sigma_\epsilon^2+\sigma_\nu^2)^2-4\sigma_\epsilon^2\sigma_\nu^2 \geq 0$, so that $\frac{\sigma_\epsilon^2\sigma_\nu^2}{(\sigma_\epsilon^2+\sigma_\nu^2)^2} \leq \frac{1}{4}$. Hence, for a fixed σ_ϵ^2, we have that $D_t \leq \frac{\sigma_\epsilon^2}{4}$, and this bound is attained with equality when $\sigma_\nu^2 = \sigma_\epsilon^2$. Thus, disagreement is maximized when the signal and noise are equally variable. This formalizes the intuition discussed in the introduction, where we highlighted that disagreement can be low both when individual signals are very precise (thereby essentially revealing the true state) as well as when individual signals are extremely noisy (whereupon forecasters essentially ignore their private signals and pool at the prior distribution).

A few useful observations can be made at this point. First, note that the disagree-

ment/uncertainty ratio is given by

$$\frac{D_t}{U_t} = \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_\nu^2},$$

implying that disagreement cannot exceed uncertainty, which accords with the finding of the general theorem. Equality holds if and only if we have the degenerate case of $\sigma_\nu^2 = 0$, that is, when the signals observed by the forecasters are noiseless. Second, neither uncertainty nor disagreement are time-varying or dependent on the realizations of either the systematic shock ϵ_t or the idiosyncratic noise shock ν_t.

We can also illustrate the general result pertaining to the effect of diagnostic expectations in this context. In essence, under diagnostic expectations with exponent $\gamma > 1$, a forecaster perceives the variance of the signal as $\frac{\sigma_\nu^2}{\gamma}$. It is then straightforward to show that the diagnostic uncertainty is $\frac{1}{\frac{1}{\sigma_\epsilon^2}+\frac{\gamma}{\sigma_\nu^2}}$, which is decreasing in γ. Moreover, the diagnostic forecast is $\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2+\frac{\sigma_\nu^2}{\gamma}} Y_t$, so that disagreement is given by $\frac{\sigma_\epsilon^4 \sigma_\nu^2}{\left(\sigma_\epsilon^2+\frac{\sigma_\nu^2}{\gamma}\right)^2}$, which is increasing in γ. It is straightforward to see that the disagreement/uncertainty ratio can sharply exceed 1 when γ is sufficiently large; indeed, uncertainty intuitively tends to zero as the perceived diagnosticity of the signal becomes large.

In addition, we can briefly examine the effects of heterogeneous prior distributions. Suppose that conditional on the state, each forecaster draws a random perceived prior mean in addition to the independent draw of their signal. We know that the posterior mean is a variance-weighted linear combination of the prior mean and the signal, and since these are conditionally independent, the introduction of prior heterogeneity must raise disagreement. However, in contrast to the case of diagnostic expectations, uncertainty will remain unchanged. Formally, if μ_t denotes the random variable representing forecasters' heterogeneous prior means (which we assume to be independent of the signal draws), then the posterior mean is $\frac{\sigma_\nu^2}{\sigma_\epsilon^2+\sigma_\nu^2}\mu_t + \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2+\sigma_\nu^2}(X_t + \nu_t)$, so that disagreement now becomes $\frac{\sigma_\nu^4 \mathbb{V}(\mu_t)+\sigma_\epsilon^4\sigma_\nu^2}{(\sigma_\epsilon^2+\sigma_\nu^2)^2}$, which exceeds disagreement in the common prior case by a *prior heterogeneity* component

$\frac{\sigma_\nu^4 \mathbb{V}(\mu_t)}{(\sigma_\epsilon^2+\sigma_\nu^2)^2}$.

Now, let us consider the second canonical model of information frictions discussed in the macroeconomics literature, namely, *sticky information*. Under this informational regime, forecasters' observations of the true state are not contaminated with noise, but instead, with probability $1-\lambda$, they update their information set at each time point t and with complementary probability λ, their information remains stale. Thus, λ serves as an index of informational rigidities. Alternatively, we can also view λ as dictating heterogeneity in forecasters' signal precision, where forecasters have either a perfectly precise or a completely imprecise signal.

Let us begin by examining uncertainty in this sticky information environment. Observe that all forecasters who updated their information set in the current period have zero uncertainty (a mass of $1-\lambda$), while the uncertainty of forecasters with stale information (a mass of λ) is given by σ_ϵ^2. Immediately, we see that unlike the setting with noisy information, there is cross-sectional heterogeneity in forecasters' uncertainty. Averaging across forecasters, the mean uncertainty is given by $U_t = \lambda\sigma_\epsilon^2$. Observe that this quantity is not time-varying, since we are considering the forecasters at a population level.

Next, let us look at inter-forecaster disagreement, that is, the dispersion of point forecasts across forecasters. Indeed, the point forecast of all forecasters who update their information set in the current period (a mass of $1-\lambda$) is ϵ_t (recall that we normalized $\mu_X = 0$), while the point forecast of all other forecasters (the mass λ with stale information) is 0. Hence, disagreement, which is the conditional variance of the point forecast, is given by $D_t = \lambda(1-\lambda)\epsilon_t^2$. Thus, we see that cross-sectional forecast disagreement is time-varying and increasing in the magnitude of the shock under sticky information. Moreover, the unconditional expectation of disagreement is $D_u = \lambda(1-\lambda)\sigma_\epsilon^2$, where the u subscript denotes that we are looking at the unconditional version of this metric.

Therefore, we find that $\frac{D_u}{U_t} = 1-\lambda$, so it emerges that in an unconditional sense, disagreement again cannot be greater than uncertainty under sticky information in this static

environment, which is again an illustration of the main theorem. Uncertainty and disagreement are equal if and only if $\lambda = 0$, that is, when forecasters have perfect information. Thus, there are qualitatively similar relationships between uncertainty and disagreement in the cases of noisy and sticky information, but the underlying mechanisms are, of course, very different. Under both noisy information and sticky information, uncertainty and disagreement are closest in magnitude when information frictions are at their lowest, that is, when observations are noiseless or when there is no informational stickiness. In both cases, the disagreement-uncertainty ratio is decreasing in the degree of informational frictions (as we predicted from the non-rigorous geometric argument in the previous section). However, it is worth emphasizing an important difference between the sticky and noisy information environments. Due to state-dependent variation in cross-sectional forecast disagreement, it is possible under sticky information for disagreement to exceed uncertainty at a given point in time. Indeed, the ratio of conditional disagreement to uncertainty is $\frac{(1-\lambda)\epsilon_t^2}{\sigma_\epsilon^2}$. Therefore, disagreement exceeds uncertainty if and only if $\epsilon_t^2 \geq \frac{\sigma_\epsilon^2}{1-\lambda}$.

6 Macroeconomic Applications: Dynamic Models

We now consider a more complex dynamic setting where the underlying state follows a stationary AR(1) process. Therefore, we have that $X_t = \phi X_{t-1} + \epsilon_t$, where $\epsilon_t \sim N(0, \sigma_\epsilon^2)$ is independent and identically distributed over time and $0 \leq \phi < 1$. As in the previous section, we begin by considering the case of noisy information, where each forecaster observes the noisy signal $Y_{it} = X_t + \nu_{it}$ and $\nu_{it} \sim N(0, \sigma_\nu^2)$ is independent across forecasters and over time. Again, since we consider the forecasters as an infinite population, we will proceed to suppress the i subscript.

In this environment, forecasters apply the Kalman filter to generate posterior distributions for the current state given the observed data. At each time step t, the Kalman filter

algorithm performs the following steps:

$$\hat{X}_{t|t-1} = \phi \hat{X}_{t-1|t-1}$$

$$P_{t|t-1} = \phi^2 P_{t-1} + \sigma_\epsilon^2$$

$$K_t = \frac{P_{t|t-1}}{P_{t|t-1} + \sigma_\nu^2}$$

$$\hat{X}_{t|t} = \hat{X}_{t|t-1} + K_t(Y_t - \hat{X}_{t|t-1})$$

$$P_t = (1 - K_t)P_{t|t-1}$$

These equations are simply the standard Kalman filter equations adapted to our univariate setting. $\hat{X}_{t|t}$ is the posterior mean of the state at time t given the information set at time t, $\hat{X}_{t|t-1}$ is the posterior mean of the state at time t given the information set at time $t-1$, P_t is the posterior variance of the state at time t given the information set at time t, $P_{t|t-1}$ is the posterior variance of the state at time t given the information set at time $t-1$, and K_t is the *Kalman gain*, which specifies the relative weight that should be put on the new information at time t.

From Hamilton (1994), we know that with our given assumptions, the Kalman filter converges to a steady state, which we find by solving the *Riccati equation* $P_{t-1} = P_t = P_\infty$, replacing K_t with the steady-state Kalman gain K_∞. Hence, $K_\infty = \frac{\phi^2 P_\infty + \sigma_\epsilon^2}{\phi^2 P_\infty + \sigma_\epsilon^2 + \sigma_\nu^2}$ and $P_\infty = \frac{\sigma_\nu^2(\phi^2 P_\infty + \sigma_\epsilon^2)}{\phi^2 P_\infty + \sigma_\epsilon^2 + \sigma_\nu^2}$. The last equality yields the following quadratic equation in P_∞:

$$\phi^2 P_\infty^2 + \left[\sigma_\epsilon^2 + \sigma_\nu^2(1 - \phi^2)\right] P_\infty - \sigma_\epsilon^2 \sigma_\nu^2 = 0$$

Clearly, we must take the positive root, so that

$$P_\infty = \frac{-\sigma_\epsilon^2 - \sigma_\nu^2(1 - \phi^2) + \sqrt{\left[\sigma_\epsilon^2 + \sigma_\nu^2(1 - \phi^2)\right]^2 + 4\phi^2 \sigma_\epsilon^2 \sigma_\nu^2}}{2\phi^2}$$

It is precisely P_∞ that is the steady-state uncertainty of a given forecaster, and once again, it is time and state-invariant.

Next, we look at the cross-sectional disagreement of point forecasts under the noisy information regime in this dynamic environment. Analogous to the simpler static environment, it is natural to define this as the steady-state variance of the posterior distribution conditional on the true sequence of states $\{X_t\}$. However, the dynamic case is somewhat more complicated as forecasters need to genuinely condition on the full sequence of states.

To proceed, consider the update rule of the Kalman filter in steady state:

$$\hat{X}_{t|t} = \phi\hat{X}_{t-1|t-1} + K_\infty(Y_t - \phi\hat{X}_{t-1|t-1}) = \phi(1-K_\infty)\hat{X}_{t-1|t-1} + K_\infty(X_t + \nu_t)$$

From the convergence of the Kalman filter, we have that $\mathbb{V}(\hat{X}_{t|t}|\{X_t\})$ converges in the ergodic regime to $\mathbb{V}_\infty$, which we find by solving $\mathbb{V}(\hat{X}_{t|t}|\{X_t\}) = \mathbb{V}(\hat{X}_{t-1|t-1}|\{X_t\}) = \mathbb{V}(\hat{X}_{t-1|t-1}|\{X_{t-1}\}) = \mathbb{V}_\infty$. We can then substitute this into the update rule and take the conditional variance of both sides with respect to $\{X_t\}$ (noting that X_t is in the conditioning set) to obtain $\mathbb{V}_\infty = \phi^2(1-K_\infty)^2\mathbb{V}_\infty + K_\infty^2\sigma_\nu^2$. Therefore, we have that

$$\mathbb{V}_\infty = \frac{K_\infty^2\sigma_\nu^2}{1-\phi^2(1-K_\infty)^2}$$

Since $\mathbb{V}_\infty$ corresponds to disagreement and P_∞ corresponds to uncertainty, it is convenient to instead define $D_\infty \equiv \mathbb{V}_\infty$ and $U_\infty \equiv P_\infty$. Note that in the static case, we have $\phi = 0$ and $K_\infty = \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2+\sigma_\nu^2}$, so that we indeed recover that $D_\infty = \frac{\sigma_\epsilon^4\sigma_\nu^2}{(\sigma_\epsilon^2+\sigma_\nu^2)^2}$

We now proceed to plot $R \equiv \frac{D_\infty}{U_\infty}$ for different values of ϕ, σ_ϵ^2 and σ_ϵ^2. As Figures 8 and 9 show, we see that $\frac{D_\infty}{U_\infty}$ appears to be increasing in ϕ, decreasing in σ_ν^2 and increasing in σ_ϵ^2. Moreover, the disagreement-uncertainty ratio appears to tend to 1 as σ_ϵ^2 becomes large, as well as when σ_ν^2 tends to 0, but it seems that disagreement never exceeds uncertainty. As it turns out, it is in fact possible to analytically prove all of these properties.

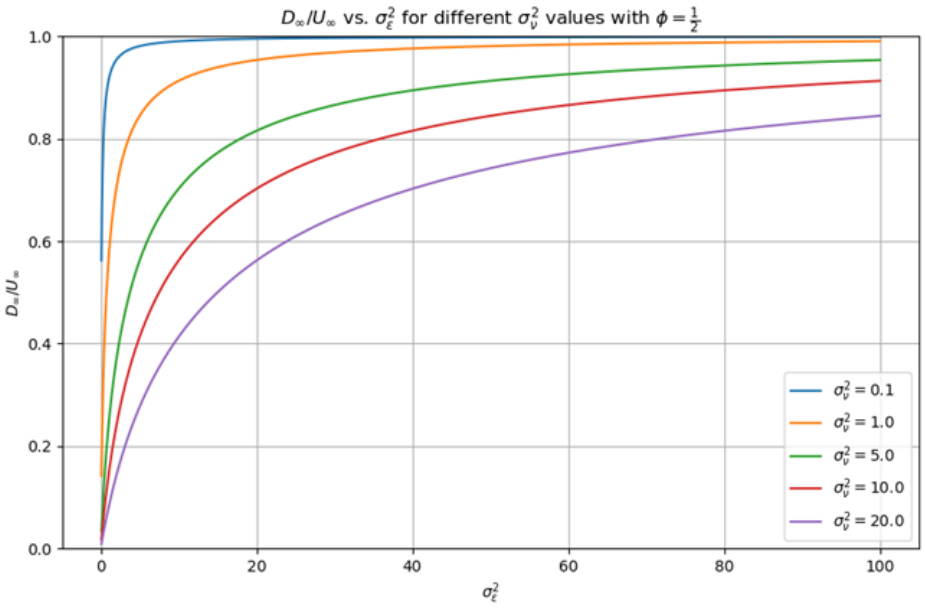


Figure 8: Comparative statics for disagreement/uncertainty ratio with respect to σ_ϵ^2 and σ_ν^2 under noisy information

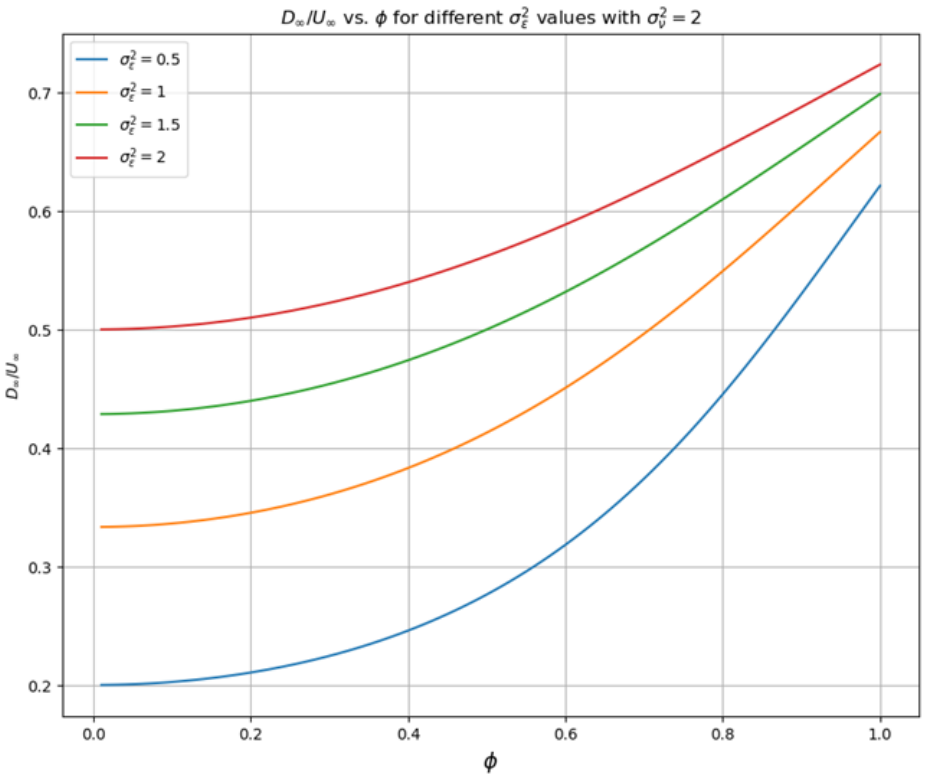


Figure 9: Comparative statics for disagreement/uncertainty ratio with respect to ϕ under noisy information

Proposition 4. *With noisy information,* $R \equiv \frac{D_\infty}{U_\infty}$ *is increasing in* ϕ*, decreasing in* σ_ν^2 *and increasing in* σ_ϵ^2*. Moreover,* $0 \leq R \leq 1$ *and* $\lim_{\sigma_\epsilon^2 \to \infty} R = \lim_{\sigma_\nu^2 \to 0} R = 1$*.*

Proof. After some algebraic manipulation, we find that:

$$\frac{dR}{d\sigma_\epsilon^2} = \frac{\sigma_\nu^2 \left(\sigma_\epsilon^2 + \sigma_\nu^2(1-\phi^2)\right)}{\left(4\phi^2\sigma_\epsilon^2\sigma_\nu^2 + \left(\sigma_\epsilon^2 + \sigma_\nu^2 - \phi^2\sigma_\nu^2\right)^2\right)^{3/2}} > 0$$

$$\frac{dR}{d\sigma_\nu^2} = -\frac{\sigma_\epsilon^2 \left(\sigma_\epsilon^2 + \sigma_\nu^2 - \phi^2\sigma_\nu^2\right)}{\left(\sigma_\epsilon^4 + 2(1+\phi^2)\sigma_\epsilon^2\sigma_\nu^2 + (-1+\phi^2)^2\sigma_\nu^4\right)^{3/2}} < 0$$

$$\frac{dR}{d\phi} = \frac{4\phi\sigma_\epsilon^2\sigma_\nu^4}{\left(4\phi^2\sigma_\epsilon^2\sigma_\nu^2 + \left(\sigma_\epsilon^2 + \sigma_\nu^2 - \phi^2\sigma_\nu^2\right)^2\right)^{3/2}} > 0$$

Thus, the first three properties are directly proven. To show the asymptotic behavior, first observe that $\lim_{\sigma_\epsilon^2 \to \infty} K_\infty = 1$, so that $\lim_{\sigma_\epsilon^2 \to \infty} D_\infty = \sigma_\nu^2$. All that remains is to characterize the asymptotic behavior of U_∞. Indeed, we have:

$$\begin{aligned}
U_\infty &= \frac{-\sigma_\epsilon^2 - \sigma_\nu^2(1-\phi^2) + \sqrt{\left[\sigma_\epsilon^2 + \sigma_\nu^2(1-\phi^2)\right]^2 + 4\phi^2\sigma_\epsilon^2\sigma_\nu^2}}{2\phi^2} \\
&= \frac{-\sigma_\epsilon^2 - \sigma_\nu^2(1-\phi^2) + \sqrt{\sigma_\epsilon^4 + \sigma_\nu^4(1-\phi^2)^2 + 2\sigma_\epsilon^2\sigma_\nu^2(1-\phi^2) + 4\phi^2\sigma_\epsilon^2\sigma_\nu^2}}{2\phi^2} \\
&\sim \frac{-\sigma_\epsilon^2 - \sigma_\nu^2(1-\phi^2) + \sqrt{\sigma_\epsilon^4 + 2\sigma_\epsilon^2\sigma_\nu^2(1-\phi^2) + 4\phi^2\sigma_\epsilon^2\sigma_\nu^2}}{2\phi^2} \\
&= \frac{-\sigma_\epsilon^2 - \sigma_\nu^2(1-\phi^2) + \sqrt{\sigma_\epsilon^4 + 2\sigma_\epsilon^2\sigma_\nu^2(1+\phi^2)}}{2\phi^2} \\
&= \frac{-\sigma_\epsilon^2 - \sigma_\nu^2(1-\phi^2) + \sigma_\epsilon^2\sqrt{1 + 2\frac{\sigma_\nu^2}{\sigma_\epsilon^2}(1+\phi^2)}}{2\phi^2} \\
&\sim \frac{-\sigma_\epsilon^2 - \sigma_\nu^2(1-\phi^2) + \sigma_\epsilon^2\left(1 + \frac{\sigma_\nu^2}{\sigma_\epsilon^2}(1+\phi^2)\right)}{2\phi^2} \\
&= \sigma_\nu^2
\end{aligned}$$

Here, $\sim$ denotes asymptotic equivalence, and the second-last line follows from a first-order Taylor series expansion of the function $f(x) = \sqrt{1+x}$. Using this result, we find that indeed $\frac{D_\infty}{U_\infty}$ tends to 1 as σ_ϵ^2 tends to ∞. Moreover, monotonicity with respect to σ_ϵ^2 and this

limiting behavior imply that D_∞ never exceeds U_∞. Finally, the same asymptotic behavior for $P = D_\infty$ and U_∞ occurs as σ_ν^2 tends to 0, so we also have that $\frac{D_\infty}{U_\infty}$ tends to 1 as σ_ν^2 tends to 0, as required. □

Now, let us examine the case of sticky information. The point forecast of an agent who last updated their information set s periods ago is $\phi^s X_{t-s}$, and the concomitant steady-state mass of agents is $\lambda^s(1-\lambda)$.

Therefore, the conditional steady-state disagreement given the realized sequence of states is:

$$D_t = \sum_{s=0}^{\infty} \lambda^s(1-\lambda)\phi^{2s} X_{t-s}^2 - \left[\sum_{s=0}^{\infty} \lambda^s(1-\lambda)\phi^s X_{t-s}\right]^2$$

This is admittedly a complicated and time/state-dependent quantity that, unlike in the static case, exhibits dependence on the full sequence of previously realized states. Now, we will proceed to take the unconditional expectation to find the unconditional steady-state cross-sectional disagreement D_∞. Note first from the moments of the AR(1) process that $\mathbb{E}(X_t) = 0$, $\mathbb{V}(X_t) = \mathbb{E}(X_t^2) = \frac{\sigma_\epsilon^2}{1-\phi^2}$ and $\text{Cov}(X_{t-s}, X_{t-r}) = \mathbb{E}(X_{t-s}X_{t-r}) = \frac{\phi^{|s-r|}\sigma_\epsilon^2}{1-\phi^2}$. Now, the first term of D_∞ simplifies to:

$$\sum_{s=0}^{\infty} \lambda^s(1-\lambda)\phi^{2s}\frac{\sigma_\epsilon^2}{1-\phi^2} = \frac{(1-\lambda)\sigma_\epsilon^2}{(1-\lambda\phi^2)(1-\phi^2)}$$

The second term of D_∞ simplifies to:

$$\begin{aligned}
\frac{\sigma_\epsilon^2(1-\lambda)^2}{1-\phi^2}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\lambda^{s+r}\phi^{s+r}\phi^{|s-r|} &= \frac{\sigma_\epsilon^2(1-\lambda)^2}{1-\phi^2}\left(2\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}\lambda^{r+s}\phi^{2s} + \sum_{s=0}^{\infty}\lambda^{2s}\phi^{2s}\right) \\
&= \frac{\sigma_\epsilon^2(1-\lambda)^2}{1-\phi^2}\left(2\sum_{r=0}^{\infty}\frac{(\lambda\phi^2)^{r+1}\lambda^r}{1-\lambda\phi^2} + \frac{1}{1-\lambda^2\phi^2}\right) \\
&= \frac{\sigma_\epsilon^2(1-\lambda)^2}{1-\phi^2}\left(2\lambda\phi^2\sum_{r=0}^{\infty}\frac{(\lambda\phi)^{2r}}{1-\lambda\phi^2} + \frac{1}{1-\lambda^2\phi^2}\right) \\
&= \frac{\sigma_\epsilon^2(1-\lambda)^2}{1-\phi^2}\left(\frac{2\lambda\phi^2}{(1-\lambda\phi^2)(1-\lambda^2\phi^2)} + \frac{1}{1-\lambda^2\phi^2}\right)
\end{aligned}$$

$$= \frac{\sigma_\epsilon^2(1-\lambda)^2(1+\lambda\phi^2)}{(1-\phi^2)(1-\lambda\phi^2)(1-\lambda^2\phi^2)}$$

Hence,

$$\begin{aligned} D_\infty &= \frac{(1-\lambda)\sigma_\epsilon^2}{(1-\lambda\phi^2)(1-\phi^2)}\left(1 - \frac{(1-\lambda)(1+\lambda\phi^2)}{1-\lambda^2\phi^2}\right) \\ &= \frac{(1-\lambda)\sigma_\epsilon^2\lambda(1-\phi^2)}{(1-\lambda\phi^2)(1-\phi^2)(1-\lambda^2\phi^2)} \\ &= \frac{\lambda(1-\lambda)\sigma_\epsilon^2}{(1-\lambda\phi^2)(1-\lambda^2\phi^2)} \end{aligned}$$

Moreover, the individual uncertainty of a forecaster who last updated their information s periods ago is $\sum_{i=0}^{s-1}\phi^{2i}\sigma_\epsilon^2 = \sigma_\epsilon^2\frac{1-\phi^{2s}}{1-\phi^2}$, and as we noted earlier, the associated mass of forecasters is $\lambda^s(1-\lambda)$. Hence, the average uncertainty across forecasters is:

$$\begin{aligned} U_\infty &= (1-\lambda)\sigma_\epsilon^2\sum_{s=0}^{\infty}\lambda^s\frac{1-\phi^{2s}}{1-\phi^2} \\ &= (1-\lambda)\sigma_\epsilon^2\left[\frac{1}{(1-\lambda)(1-\phi^2)} - \frac{1}{(1-\lambda\phi^2)(1-\phi^2)}\right] \end{aligned}$$

Simplifying the latter expression, we have

$$U_\infty = \frac{\lambda\sigma_\epsilon^2}{(1-\lambda\phi^2)}$$

Hence,

$$R \equiv \frac{D_\infty}{U_\infty} = \frac{1-\lambda}{1-\lambda^2\phi^2}$$

The following proposition is now a straightforward corollary since $\frac{dR}{d\lambda} = \frac{-1+\phi^2(2\lambda-\lambda^2)}{(1-\lambda^2\phi^2)^2} < 0$:

Proposition 5. *With sticky information, $R \equiv \frac{D_\infty}{U_\infty}$ is increasing in ϕ and decreasing in λ. Moreover, $0 < R < 1$.*

Once again, we can plot $\frac{D_\infty}{U_\infty}$ for different values of ϕ and λ.

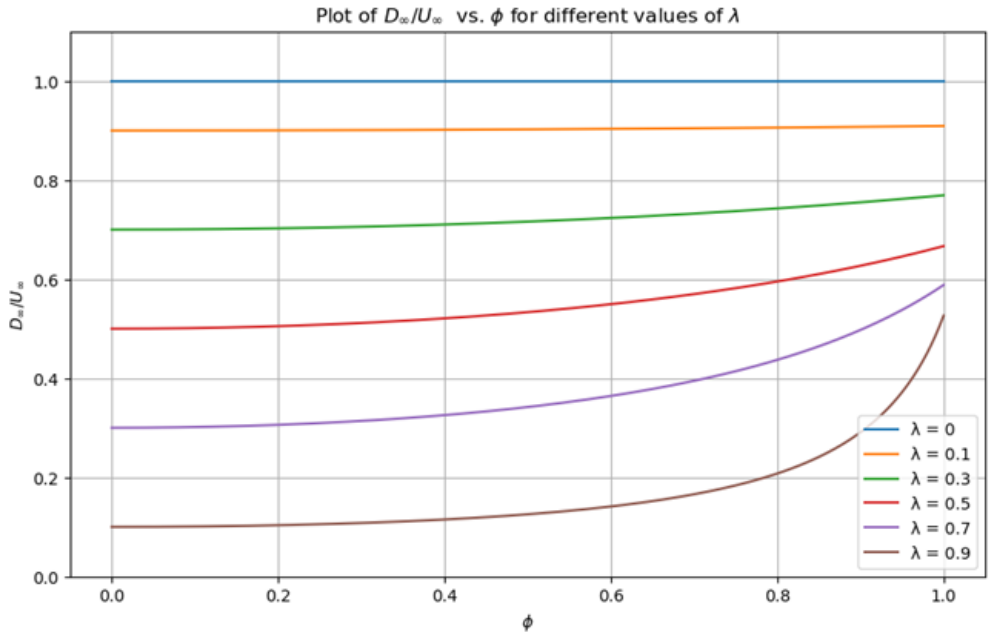


Figure 10: Comparative statics for disagreement/uncertainty ratio under sticky information

7 Discussion

At this point, it is useful to recap some of the key results that we have derived, especially in terms of the similarities and differences in the disagreement/uncertainty ratio under noisy and sticky information. Indeed, the steady-state disagreement/uncertainty ratio is upper-bounded by unity, and is decreasing in the degree of informational frictions (σ_ν^2 in the case of noisy information and λ in the case of sticky information). Moreover, the ratio is invariant with respect to shocks under noisy information but is time and state-dependent under sticky information (it is increasing in the absolute magnitude of shocks). Under both noisy and sticky information, the ratio is increasing in the degree of state persistence (ϕ). A salient difference, however, is that the steady-state disagreement/uncertainty ratio is increasing in fundamental volatility (σ_ϵ^2) under noisy information, but is invariant with respect to σ_ϵ^2 under sticky information.

To roughly fit the observed time series properties of the disagreement/uncertainty ratio (excluding the COVID-19 period, which appears to be somewhat anomalous), there are

two ways in which one could generate time variation in the disagreement/uncertainty ratio. The first approach would be to assume sticky information, whereby movement in the ratio would be driven by shocks. However, this explanation is somewhat unsatisfactory for two key reasons. First, the disagreement/uncertainty ratio generally lies below one, and even on the few occasions where this does not hold, the ratio does not greatly exceed unity. The *pointwise* bound of unity is a natural property of the noisy information model, but not the sticky information model. Second, the periods in which the ratio exceeds unity appear to be times of high fundamental volatility, such as the Great Recession, an observation that is much more concordant with the noisy information than with the sticky information hypothesis. Under sticky information, it is large shocks relative to fundamental volatility that produce transitory variation in the disagreement/uncertainty ratio, which would imply that spikes in the ratio should occur at a homogeneous rate through time and that shocks to fundamental volatility should not affect the steady-state value of ratio. This is a complementary finding to that of Coibion and Gorodnichenko (2012), who suggest that disagreement responds quite negligibly to shocks in general, although it is worth noting that we present a novel argument here that combines information from measures of both disagreement and uncertainty.

There is another equivalent way to capture how this disagreement/uncertainty ratio test is able to disambiguate between the noisy and sticky information models. Indeed, we showed that the Gaussian noisy information model is "special", in the sense that there is no cross-sectional heterogeneity across forecasters in their uncertainty. In contrast, the sticky information model is an extreme example of cross-sectional heterogeneity in uncertainty. In fact, in the static sticky information model, if we measure uncertainty by the standard deviation as opposed to the variance, a fraction λ of forecasters have uncertainty σ_ϵ and the complementary fraction $1-\lambda$ have uncertainty 0. If we then compute the variance of this distribution, we find that this quantity is $\lambda\sigma_\epsilon^2 - \lambda^2\sigma_\epsilon^2 = \lambda(1-\lambda)\sigma_\epsilon^2$. This quantity, which is essentially a measure of *disagreement about uncertainty*, is then equal in expectation to disagreement itself. Empirically, however, as Figure 11 suggests, disagreement about uncertainty is very

small, does not move substantially over time, and is appreciably lower than disagreement itself. The key insight here relative to prior work is that that the sticky information model is distinguished not so much by time-varying disagreement as by cross-sectional heterogeneity in subjective uncertainty.

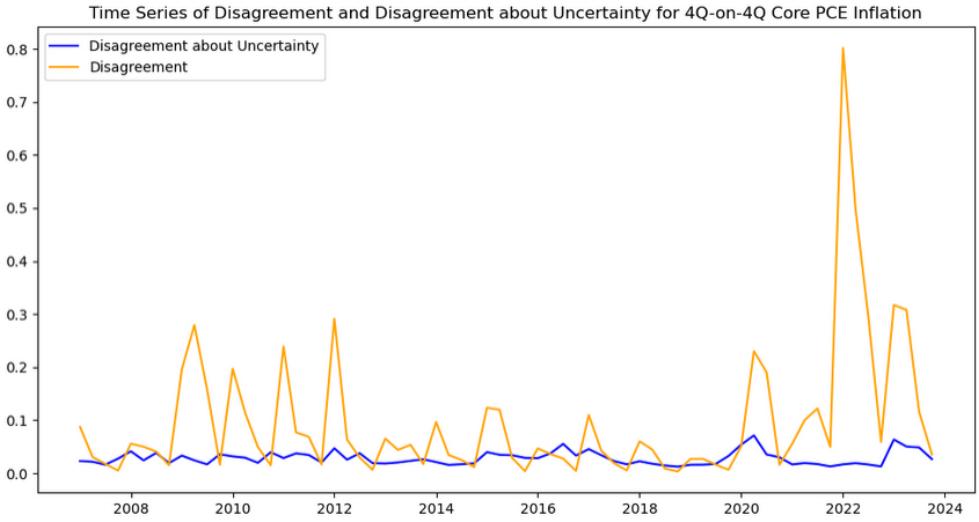


Figure 11: Dynamics of disagreement and disagreement about uncertainty

To this end, we conclude that the noisy information model, coupled with stochastic volatility of fundamentals, provides a superior explanation of the time series properties of the disagreement/uncertainty ratio. It is rather difficult to analyze stochastic volatility in the fully dynamic setting, since the Kalman filter would no longer converge to a meaningful steady state, so we just discuss the intuition in the static case. Indeed, the empirical dynamics of core PCE inflation (we focus on this metric here as an empirical case study) are well described by a random walk, and assuming that forecasters are predicting the incremental changes in inflation relative to the previous year, the static model appears to be entirely reasonable in capturing this scenario. Since the disagreement/uncertainty ratio is increasing in σ_ϵ^2 and tends to 1 as fundamental volatility becomes large, the canonical noisy information model is a natural candidate to rationalize the observed spikes in the disagreement/uncertainty ratio during periods of heightened macroeconomic volatility.

One difficulty lies in explaining the seasonal dynamics of the disagreement/uncertainty

ratio. We would naturally expect that the noisiness of forecasters' signals diminishes over the four quarters of the year, which would imply that the disagreement/uncertainty ratio should *rise* between Q1 and Q4. In contrast, the precise opposite occurs in the data. Nonetheless, this limitation can be overcome by introducing a degree of *irreducible uncertainty*; that is, we posit that forecasters only obtain noisy signals about a latent state, where the true state is the latent state plus some independent noise. Denoting this irreducible variance by C, uncertainty now becomes $\frac{\sigma_\epsilon^2\sigma_\nu^2}{\sigma_\epsilon^2+\sigma_\nu^2} + C$, and the disagreement/uncertainty ratio need not be monotonically decreasing in σ_ν^2 anymore, as shown in Figure 12.

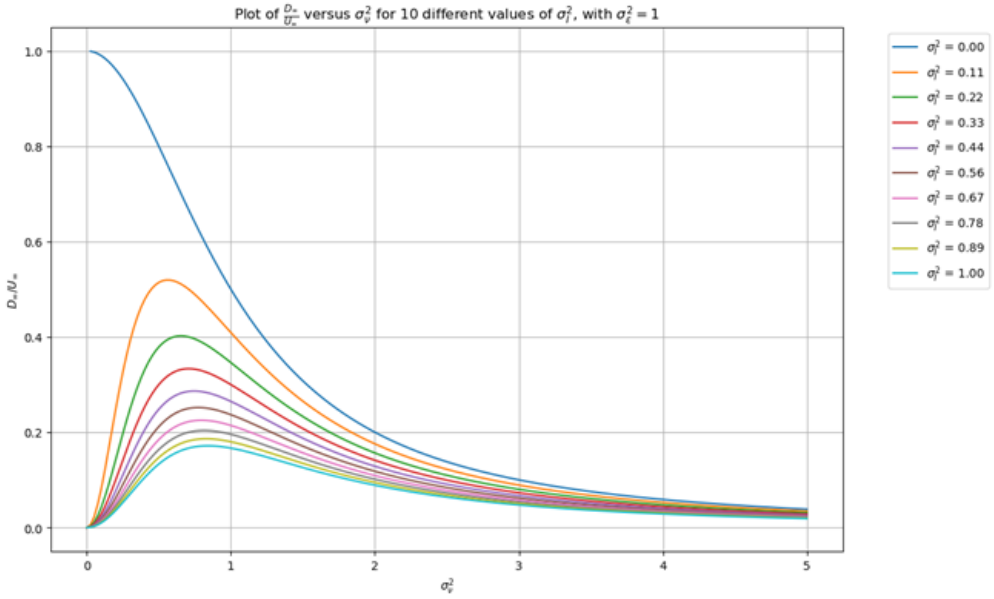


Figure 12: Non-monotonicity of disagreement/uncertainty ratio with irreducible uncertainty

While the concept of irreducible uncertainty has been relatively unexplored in previous literature relating to models of imperfect information in macroeconomics, it certainly appears to be a useful theoretical construct to capture residual uncertainty even when signals are perfectly revealing. However, this assumption does not come without loss, since the introduction of irreducible uncertainty implies that the disagreement/uncertainty ratio no longer tends to unity as fundamental volatility becomes large. To preserve the limiting behavior, we would require σ_ϵ^2 and σ_ν^2 to both increase, but the degree of increase in σ_ϵ^2 should be an order of magnitude greater than that of σ_ν^2.

To illustrate these ideas, we perform a rough calibration of the static noisy information model to fit the observed time series of disagreement, uncertainty and the disagreement/uncertainty ratio for core PCE inflation, where the noise variance exhibits seasonal variation, with the identifying assumption that the noise variance is monotonically decreasing over the four quarters of the year. We essentially perform a simulated method of moments estimation protocol, where the objective function minimizes the sum of squared deviations between the simulated and empirical means of uncertainty and the disagreement/uncertainty ratio in each of the four quarters of the year. Indeed, the estimated parameters of $C = 0.08$, $\sigma^2_{\nu,1} = 0.24$, $\sigma^2_{\nu,2} = 0.18$, $\sigma^2_{\nu,3} = 0.08$, $\sigma^2_{\nu,4} = 0.016$ (where the numerical subscripts denote the quarter under consideration) allow us to closely match the empirical moments. In Figures 13 and 14, we show the simulated time series of disagreement, uncertainty and the disagreement/uncertainty ratio, where the baseline parameters are overlaid with occasional random spikes in σ^2_ϵ and σ^2_ν.

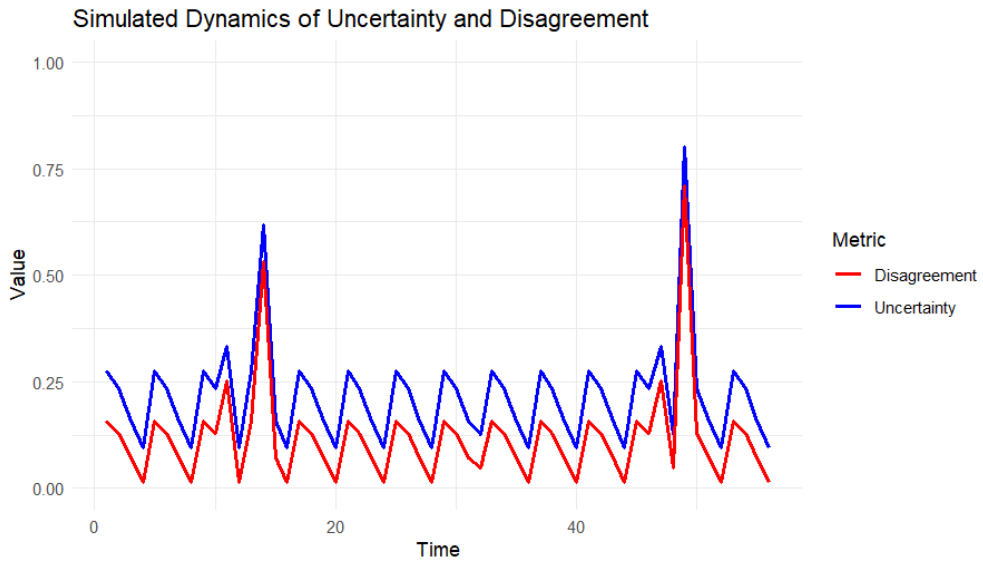


Figure 13: Simulated dynamics of uncertainty and disagreement in noisy information model with stochastic volatility and irreducible uncertainty

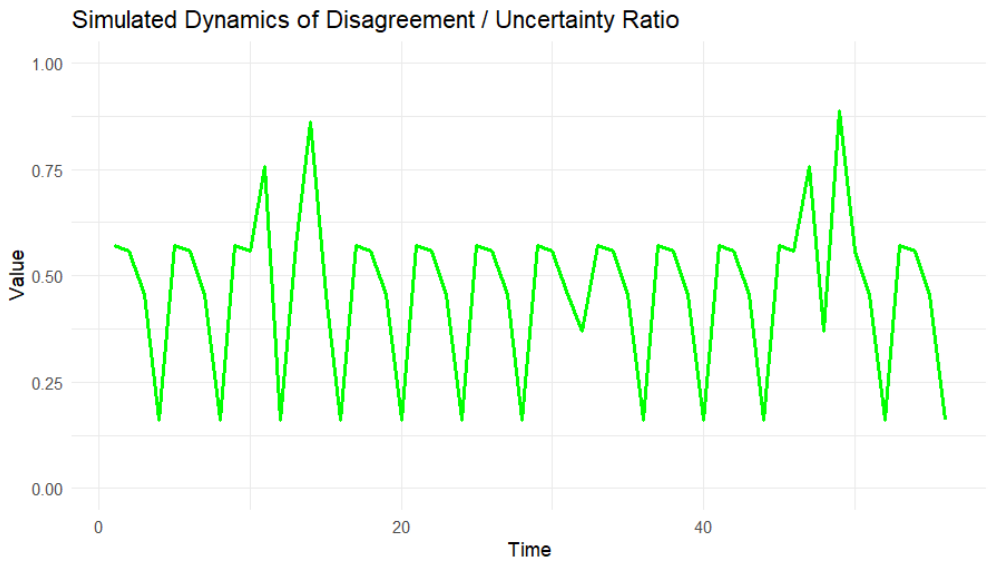


Figure 14: Simulated dynamics of disagreement/uncertainty ratio in noisy information model with stochastic volatility and irreducible uncertainty

The primary issue, however, with the calibration is that very large increases in σ_ϵ^2 are required to yield increases in the disagreement/uncertainty ratio toward 1 (for illustrative purposes in generating the figures, we multiply σ_ϵ^2 by 50 and σ_ν^2 by 3 during designated "crisis periods"). This results in excessively large spikes in uncertainty relative to what is observed in the empirical data. Thus, the benefits of irreducible uncertainty in accounting for the seasonal dynamics comes at the cost of making it significantly more difficult to rationalize the spikes in the disagreement/uncertainty ratio.

As suggested by the earlier theoretical analysis, it is also worthwhile to examine the behavior of realized forecast errors in parallel with uncertainty and disagreement. It is interesting to note, as Figure 15 shows, that there are systematic deviations between implied uncertainty and realized forecast errors over the four quarters of the year (forecasters systematically underestimate their uncertainty in Q1 and overestimate their uncertainty in Q4). In essence, uncertainty implied by probability forecasts exhibits excess stickiness, perhaps due to asymmetric loss functions as in Capistrán and Timmermann (2009), and the observed residual uncertainty is at least in part an artifact of this phenomenon.

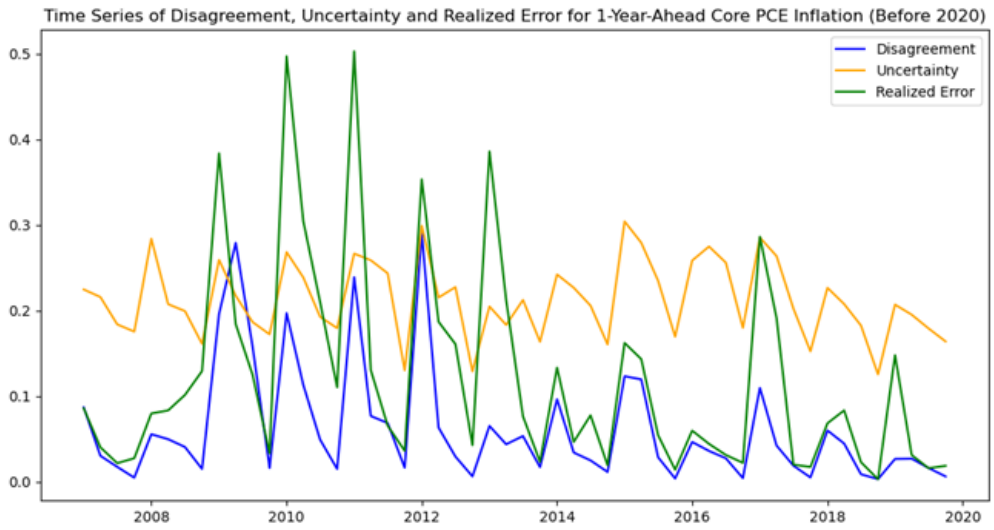


Figure 15: Disagreement, uncertainty and realized forecast errors for core PCE inflation

Naturally, it is likely that there are elements of truth to different models, and one example of a situation where it is difficult to rationalize either the noisy or sticky information models is during the post-COVID inflation. Here, the disagreement/uncertainty ratio is anomalously large, suggesting that other forces, such as differences in priors or perceived models across forecasters, may be playing an important role. Nonetheless, we concur with the finding in Coibion and Gorodnichenko (2012) and Coibion and Gorodnichenko (2015) that the noisy information model does well in capturing the behavior of forecasters during "normal times", though we also identify some caveats that necessitate modifications to the standard model.

8 Conclusions

While disagreement has been widely used as an empirical proxy for uncertainty, comparatively little is known about the theoretical soundness of this practice. In this paper, motivated by observed time series and cross-sectional properties of disagreement and uncertainty in forecasts taken from the Survey of Professional Forecasters, we derive a novel suite of theoretical results to fill this void.

We first derive a general theorem showing that under a common prior with rational

expectations, expected disagreement cannot exceed expected uncertainty, even when information structures are arbitrary, formalizing the intuition of Zarnowitz and Lambros (1987) about the atypical nature of periods with high disagreement but low uncertainty. We also highlight that when disagreement exceeds uncertainty conditionally, it must be the case that uncertainty understates the expected forecast error. We then proceed to derive additional theoretical properties of disagreement, uncertainty and the disagreement/uncertainty ratio in setups based on the noisy information models of Woodford (2003) and Sims (2003), as well as the sticky information models in the style of Mankiw and Reis (2002). An important auxiliary contribution of our paper is to analytically characterize both disagreement and uncertainty under noisy information with state persistence by solving for fixed points of appropriate Riccati equations. We derive a rich set of comparative statics of the disagreement/uncertainty ratio under both informational regimes with respect to fundamental volatility, the extent of noise (in the case of noisy information), the degree of informational rigidity (in the case of sticky information) and the amount of persistence in the state variable.

Based on these theoretical results, we investigate the ability of the noisy and sticky information models to fit the observed dynamics of disagreement and uncertainty in data from the SPF. We concur with Coibion and Gorodnichenko (2012) and Coibion and Gorodnichenko (2015) that the noisy information model outperforms the sticky information model in fitting the data, but we show that some modifications to the standard model, namely, stochastic volatility and irreducible uncertainty, are required to reproduce some of the observed empirical patterns. Indeed, even with these modifications, some challenges remain in jointly fitting the seasonal patterns in uncertainty and disagreement and the rise in the disagreement/uncertainty ratio during periods of elevated volatility, a phenomenon that appears to be partly attributable to "excess stickiness" in subjective uncertainty.

A simultaneous benefit and limitation of the analysis in this paper is the focus on common prior environments. While this assumption permits analytical tractability and indeed appears to be a valuable empirical benchmark in most situations, it is certainly true that

heterogeneity in priors and models cannot always be neglected. Though we make some progress in this regard, a natural extension of the analysis in this paper would be to consider a much broader range of informational and behavioral frictions, though it is quite likely that clear-cut and general theoretical results of the kind derived here will be significantly more difficult to obtain. We leave this interesting and important avenue for future work.

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